Families of degenerating Poincaré–Einstein metrics on $\mathbb{R}^4$

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Abstract

We provide the first example of continuous families of Poincaré–Einstein metrics developing cusps on the trivial topology $\mathbb{R}^4$. We also exhibit families of metrics with unexpected degenerations in their conformal infinity only. These are obtained from the Riemannian version of an ansatz of Debever and Plebański–Demiański. We additionally indicate how to construct similar examples on more complicated topologies.

Keywords Poincaré–Einstein metrics · Cusp formation · Dimension 4 · Topology

Introduction

An *Einstein metric* satisfies for some real number Λ:

$$\text{Ric}(g) = \Lambda g. \tag{1}$$

This is a central equation in Geometry and in several instances of Physics, especially in dimension 4. A *Poincaré–Einstein metric* is a noncompact Einstein metric with a specific asymptotic behavior giving rise to a *conformal boundary metric* at infinity, the simplest example being the Poincaré model for hyperbolic space whose conformal infinity is the round sphere. Poincaré–Einstein metrics were first notably used to construct a number of conformal invariants of the boundary geometry; see [15, 16]. More recently, they have also played an important role in the physics literature in relationship with AdS/CFT correspondence; see [7, 31].

From several perspectives, dimension 4 is a threshold dimension in topology and geometry. In this dimension, there are three ways for compact Einstein or Poincaré–Einstein metrics on a given manifold to degenerate: orbifold singularity formation, collapsing and cusp formation.

Orbifold formation has been widely studied and is now reasonably understood. Numerous examples of *curves* of such degenerations have been produced in the Kähler and Poincaré–Einstein settings, see [8, 9, 25]. *All* such degenerations have moreover been reconstructed by gluing-perturbation [27, 28].

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Despite deep general results such as [12], the collapsing and cusp formation remain comparatively mysterious. The collapsing situation has received a lot of attention, and many examples of *curves* of Einstein metrics collapsing have been produced on K3 surfaces, see for instance [17, 21]. The third situation of cusp formation has, however, never been observed except from "trivial" examples of (warped) products of degenerating surfaces and from sequences of metrics requiring infinitely many different topologies [5, 6]. More concretely, the following question was left open:

Question 0.1 [4] *Another interesting open question is whether cusps can actually form within a given or fixed component of [the moduli space of Poincaré–Einstein metrics], on a fixed manifold M.*

A simple but not so appealing example showing that this exists is the so-called *topological black hole* metric. The metric is $V(r)^{-1}dr^2 + V(r)d\theta^2 + r^2 g_N$ for $V(r) := -1 + r^2 - 2m/r^2$ with m large enough and g_N the metric of a hyperbolic surface. Letting g_N degenerate creates a cusp that extends to the conformal infinity. This naive example answers Anderson's question but, to the authors' knowledge, does not seem to have been mentioned before. This is still a two-dimensional behavior, and we provide many more interesting examples here.

Another intriguing question is whether cusp formation requires some topology—like orbifold degeneration requires nontrivial 2-homology. Anderson conjectured that it was the case:

Question 0.2 [4] *It would also be very interesting to know if the possible formation of cusps is restricted by the topology of the ambient manifold M. [...] One might conjecture for instance that on the 4-ball cusp formation is not possible.*

We instead provide explicit examples of continuous families of smooth Poincaré–Einstein metrics on $\mathbb{R}^4$ developing different kinds of cusps. We moreover find curves of metrics *without* any degeneration in the bulk but forming various *conical*, *cusp* or *naked* singularities in their conformal infinity.

Debever and Plebański–Demiański's local family of metrics

In this article, we study families of Poincaré–Einstein metrics exhibiting the above three types of degenerations focusing on the least understood case of cusp formation. These examples are surprisingly *explicitly given in coordinates* and are found in the families of Einstein metrics whose Lorentzian counterparts were discovered by Debever [14] and which were given in more convenient coordinates by Plebański–Demiański [29]. These metrics are known in the physics literature as Plebański–Demiański metrics (PD metrics). PD metrics are algebraically special of Petrov *type D* meaning (in the Riemannian setting) that at every point the selfdual and anti-selfdual parts of the Weyl curvature have repeated eigenvalues. This is also equivalent to the *ambiKähler* condition of [1]: the metric is *conformally Kähler* or *Hermitian* in both orientations. This curvature condition forces toric symmetry by [18].

The metrics of the PD family have a remarkably compact form (2) and depend solely on two related *quartic polynomials P and Q* of one variable. Still, despite their simplicity and their discovery in the early 70's, these explicit metrics, once extended to the Riemannian setting contain in some limits most known examples of Einstein metrics ($\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$, Fubini–Study, Page's metric, Taub-NUT, Taub–Bolt, Eguchi–Hanson, Schwarzschild, Kerr and their AdS counterparts.) that were often discovered much later with complicated ansatz, see [24] where smooth Ricci-flat and compact Einstein PD metrics are conjecturally classified. Extensions

of these families more generally solve the Einstein–Maxwell equations and include known metrics such as LeBrun's earliest family of scalar-flat Kähler metrics [23].

This family also contains families developing *orbifold* singularities in the so-called AdS-Taub–Bolt family. It moreover contains continuous families of metrics exhibiting global *collapsing* bubbling out (Ricci-flat) Taub-NUT or Schwarzschild metrics in the so-called AdS-Taub-NUT (or Pedersen's) metrics or AdS-Schwarzschild families. We will focus on cusp formation here.

Families of Poincaré–Einstein metrics forming cusps

Degeneration in the family of AdS *C*-metrics

It is now classical in the physics literature that a limit "without rotation or twisting" of the PD metrics leads to the well-known AdS *C*-metrics whose Ricci-flat versions were found by Levi-Civita [22] and Weyl [30] in the 1910 s(!). In this family, we first find a two-dimensional moduli space of smooth Poincaré–Einstein metrics on $\mathbb{R}^4$ containing the hyperbolic 4-metric and whose limiting behaviors include metrics forming one or two cusps. A significant asymptotic quantity of Poincaré–Einstein metrics is the *renormalized volume* defined in [20]. Despite the drastic degenerations presented in this article, the renormalized volume stays bounded.

Theorem A (Sect. 2) *There exists a smooth family of smooth Poincaré–Einstein metrics on* $\mathbb{R}^4$ *parametrized by an open region* Ω *in* $\mathbb{R}^2$. *Approaching some points at the boundary* $\partial\Omega$, *the metrics converge smoothly to the hyperbolic space or degenerate forming one or two codimension* 2 *cusps. These cusps have asymptotic behaviors:*

$$dr_1^2 + ae^{-r_1}d\theta_1^2 + dr_2^2 + bd\theta_2^2 \quad for \quad r_1 \in [0, +\infty),\ r_2 \in [0, 1],\ \theta_1, \theta_2 \in [0, 2\pi],$$

for $a, b > 0$ *in the bulk of the manifold, and* $dr^2 + ae^{-r}d\theta_1^2 + bd\theta_2^2$ *at conformal infinity with* $r \in [0, +\infty)$. *These examples have uniformly bounded renormalized volume.*

An important question left open is the following one.

Question 0.3 *Does there exist a continuous family of Poincaré–Einstein metrics forming cusps separating the manifold into a complete* finite volume *piece and another complete Poincaré–Einstein metric?*

Remark 0.4 Unfortunately, this is impossible in our family of metrics and there is little hope to find such a family of metrics explicitly given in coordinates. Indeed, in our case, one limit of such a degeneration has to be an Einstein metric with negative Ricci curvature and with at least one Killing vector field with finite length, which is impossible by Bochner's formula; see [32] for instance.

Degeneration in the Carter–Plebański family of metrics

The limits "without acceleration" of the PD metrics constitute the *Carter–Plebański* family of metrics. In Sect. 3, we exhibit a subfamily of smooth Poincaré–Einstein metrics with topology $\mathbb{CP}^2 \setminus D^4$ forming cusp in some limits and discuss how other topologies may be reached.

Degeneration in the full Plebański–Demiański family of metrics

In the full family of PD metrics, we also obtain cusps as in Theorem A which are this time "twisted" as in (5). We additionally find families of smooth Poincaré–Einstein metrics on $\mathbb{R}^4$ where *only* the conformal infinity degenerates in some limit.

Theorem B (Section 4) *There exists a smooth family of Poincaré–Einstein metrics on $\mathbb{R}^4$ whose conformal infinity approaches one of the following behaviors in some limit: for $a, b > 0$*

- *A conical (edge) singularity: $dr^2 + a^2 r^2 d\theta_1^2 + b^2 d\theta_2^2$ on $(r, \theta_1, \theta_2) \in [0, 1] \times [0, 2\pi] \times [0, 2\pi]$,*
- *A naked singularity: $dr^2 + a^2 r^6 d\theta_1^2 + b^2 d\theta_2^2$ on $(r, \theta_1, \theta_2) \in [0, 1] \times [0, 2\pi] \times [0, 2\pi]$, or*
- *A cusp end: $dr^2 + a^2 e^{-4r} d\theta_1^2 + b^2 d\theta_2^2$ on $(r, \theta_1, \theta_2) \in [0, +\infty] \times [0, 2\pi] \times [0, 2\pi]$.*

While approaching these behaviors at conformal infinity, the metrics converge smoothly in the bulk metric in the pointed Cheeger–Gromov sense. These examples have uniformly bounded renormalized volume.

These degenerations can occur in various limits that we describe in Sect. 4.

1 Regularity and asymptotics of the Plebański–Demiański family of metrics

A "Euclideanized" Plebański–Demiański (PD) metric has the following form

$$
g_{PD} = \frac{1}{(x-y)^2} \left[\frac{Q(y)}{1 - a^2 x^2 y^2} (d\psi - ax^2 d\varphi)^2 + \frac{1 - a^2 x^2 y^2}{Q(y)} dy^2 \right.
$$
$$
\left. + \frac{P(x)}{1 - a^2 x^2 y^2} (d\varphi - ay^2 d\psi)^2 + \frac{1 - a^2 x^2 y^2}{P(x)} dx^2 \right] \tag{2}
$$

where $Q(y)$ and $P(x)$ are polynomials of degree 4 which can be chosen depending on the value of $a \in \mathbb{R}$, physically understood as a *rotation parameter*, so that g_{PD} is an Einstein metric with $\mathrm{Ric}_{g_{PD}} = -3g_{PD}$. This follows the Riemannian version of the computations in [29], see also [24]. We will ensure that the metric has the right Riemannian signature by choosing adapted ranges for the coordinates (x, y) where $Q(y) > 0$ and $P(x) > 0$. Up to rescaling, we can assume $a \in \{0, 1\}$.

Let us first consider the larger family with $a = 1$ from which the other ones can be obtained from various limiting procedures. The Einstein condition (1) with $\Lambda = -3$ is equivalent to P and Q having the form

$$
P(x) = bx^4 + cx^3 + dx^2 + ex + b + 1 \text{ and}
$$
$$
-Q(y) = (b+1)y^4 + cy^3 + dy^2 + ey + b, \tag{3}
$$

for $b, c, d, e \in \mathbb{R}$ where we note the identity $-Q(y) = P(y) + y^4 - 1$. The *local* metric (2) is then Einstein *and Riemannian* on ranges depending on roots of P and Q. When "closing-up" at roots of P and Q, it may have codimension 2 cone-edge singularities (which we will avoid) or cusp ends which are discussed in Sect. 1.3

These metrics are moreover Poincaré–Einstein (when they are smooth) since they are conformal to a metric with boundary: the boundary is given by $\{x = y\}$ and the conformal

factor is $\frac{1}{(x-y)^2}$. The *conformal infinity* of these metrics is the conformal class of the metric induced on $\{x = y\}$ by $(x - y)^2 g_{PD}$. We will see in different instances that these conformal infinities may degenerate. The possible such degenerations are collected in Sect. 4.

Without loss of generality, we can write $c = k_+ + k_-$, $e = k_+ - k_-$ in (3), in which the set of eigenvalues of the $\pm$-selfdual part of Weyl curvature $W_{g_{PD}}$ is proportional to $\frac{k_\pm}{(1 \pm xy)^3}(2, -1, -1)$, see the computations of [29] and the Riemannian version of [24]. The pointwise norm of the Riemannian tensor of g_{PD} is then computed as

$$\| \mathrm{Rm}_{g_{PD}} \|^2 = 24 + 24(x - y)^6 \left(\frac{k_+^2}{(1 + xy)^6} + \frac{k_-^2}{(1 - xy)^6} \right).$$

The volume element in these coordinates is $\frac{-1+x^2 y^2}{(x-y)^4} dx dy d\varphi d\psi$, and one checks that $\|W_{g_{PD}}\|_{L^2(g_{PD})}$ is finite for the domains we consider, hence, by [3], the renormalized volume is controlled for our examples.

Let us describe here the possible asymptotics of our metrics and give the regularity conditions. The regularity conditions obtained for toric metrics are classical, and we focus on ruling our conical singularities.

1.1 At a simple root x_1 of P and a generic point y

As defined for instance in [2], a metric with cone-edge singularity of angle $2\pi\beta > 0$ along a codimension 2 submanifold Σ has the following asymptotic at Σ: for a 2π-periodic θ and a 1-form ω on Σ

$$dr^2 + \beta^2 r^2 (d\theta + \omega)^2 + g_\Sigma + \mathcal{O}(r^{1+\epsilon}), \text{ for } \epsilon > 0. \tag{4}$$

Lemma 1.1 *At a simple root x_1 of P and a generic point $y \notin \{-1/x_1, 1/x_1\}$, our metric (2) with $a \in \{0, 1\}$ has a cone-edge singularity whose angle is the period of $\frac{|P'(x_1)|}{2(1+a^2 x_1^4)}\theta_1$, where $\theta_1 := \varphi + ax_1^2\psi$.*

Proof To do this, we first note that close to its root x_1, we have $P(x) = P'(x_1)(x - x_1) + O((x - x_1)^2)$ at first order. Close to the root $x = x_1$ and at $y \neq \pm x_1^{-1}$ which is not a root of Q, the metric therefore reads:

$$g_{PD} = \frac{1}{(x_1 - y)^2} \left[\frac{Q(y)}{1 - a^2 x_1^2 y^2}(d\psi - ax_1^2 d\varphi)^2 + \frac{1 - a^2 x_1^2 y^2}{Q(y)}dy^2 \right.$$
$$\left. + \frac{P'(x_1)(x - x_1)}{1 - a^2 x_1^2 y^2}(d\varphi - ay^2 d\psi)^2 + \frac{1 - a^2 x_1^2 y^2}{P'(x_1)(x - x_1)}dx^2 \right] + \mathcal{O}((x - x_1)^2).$$

Our codimension 2 submanifold Σ is given by $\{x = x_1\}$, hence $dx = 0$ and $\theta_1 := \varphi + ax_1^2\psi = cst$, that is $d\theta_1 = d\varphi + ax_1^2 d\psi = 0$ (this is chosen as the orthogonal of the 1-form $d\psi - ax_1^2 d\varphi$). The local coframe on Σ we will use is therefore dy and $\omega_1 := d\psi - ax_1^2 d\varphi$. With these notations, the metric becomes:

$$g_{PD} = \frac{1}{(x_1 - y)^2} \left[\frac{Q(y)}{1 - a^2 x_1^2 y^2}\omega_1^2 + \frac{1 - a^2 x_1^2 y^2}{Q(y)}dy^2 \right.$$
$$\left. + (1 - a^2 x_1^2 y^2) \left(\frac{P'(x_1)(x - x_1)}{(1 + a^2 x_1^4)^2}(d\theta_1 + f(y, x_1)\omega_1)^2 + \frac{dx^2}{P'(x_1)(x - x_1)} \right) \right] + \mathcal{O}((x - x_1)^2).$$

where $f(y, x_1) = \frac{x_1^2 + y^2}{1 - x_1^2 y^2}$ if $a = 1$ and $f(y, x_1) = 0$ if $a = 0$. The regularity of the metric

close to $x = x_1$ therefore reduces to the regularity of $\frac{P'(x_1)(x - x_1)}{(1 + x_1^4)^2} (d\theta_1 + f(y, x_1)\omega_1)^2 +$

$\frac{1}{P'(x_1)(x - x_1)} dx^2$.

Considering a change of variables $x = x_1 + \frac{P'(x_1)}{4} r^2$, we find the conical singularity metric:

$$\frac{|P'(x_1)|^2}{4(1 + a^2 x_1^4)^2} r^2 (d\theta_1 + f(y, x_1)\omega_1)^2 + dr^2,$$

of angle the period of $\frac{|P'(x_1)|}{2(1 + a^2 x_1^4)}\theta_1$ by comparison with (4). It will be smooth if and only if this period is 2π. $\qquad\square$

The case of a simple root y_2 of Q is treated similarly and yields a cone-edge singularity whose angle is given by the period of $\frac{|Q'(y_2)|}{2(1 + a^2 y_2^4)}\theta_2$, where θ_2 is $\psi + y_2^2\varphi$.

1.2 At x_1 simple root of P and y_2 simple root of Q

Let us now consider $y = y_2$, a simple root of Q, and still assume that $x_1^2 \neq y_2^{-2}$ if $a = 1$.

Lemma 1.2 *Assume that x_1 is a simple root of P, and y_2 is a simple root of Q, and $1 - a^2 x_1^2 y_2^2 \neq 0$. Then, the metric (2) is smooth at (x_1, y_2) if and only if both $\frac{|P'(x_1)|}{2(1 + a^2 x_1^4)}\theta_1$ and $\frac{|Q'(y_2)|}{2(1 + a^2 y_2^4)}\theta_2$ are 2π-periodic, where $\omega_1 = d\psi - a x_1^2 d\varphi$ and $\omega_2 = d\varphi - a y_2^2 d\psi$.*

Proof Expanding the metric near $p = x_1$ and $q = y_2$, a first-order development of the metric gives:

$$g_{PD} = \frac{1}{(x_1 - y_2)^2} \left[\frac{Q'(y_2)(y - y_2)}{1 - a^2 x_1^2 y_2^2} (d\theta_2 + \tilde{\omega}_2)^2 + \frac{1 - a^2 x_1^2 y_2^2}{Q'(y_2)(y - y_2)} dy^2 \right.$$

$$\left. + (1 - a^2 x_1^2 y_2^2) \left(\frac{P'(x_1)(x - x_1)}{(1 + a^2 x_1^4)^2} (d\theta_1 + \tilde{\omega}_1)^2 + \frac{1}{P'(x_1)(x - x_1)} dx^2 \right) \right]$$

$$+ \mathcal{O}((x - x_1)^2 + (y - y_2)^2).$$

for some 1-forms $\tilde{\omega}_1 = f_1(y, x_1)\omega_1$ and $\tilde{\omega}_2 = f_2(x, y_2)\omega_2$ for some smooth functions f_1 and f_2 whose explicit value does not affect the regularity ($f_1 = f_2 = 0$ if $a = 0$), and where $\omega_1 = d\psi - a x_1^2 d\varphi$ and $\omega_2 = d\varphi - a y_2^2 d\psi$.

The same change of variables as in Sect. 1.1 in both x and y ensures that the metric is smooth at (x_1, y_2) if and only if $\frac{|P'(x_1)|}{2(1 + a^2 x_1^4)}\theta_1$ and $\frac{|Q'(y_2)|}{2(1 + a^2 y_2^4)}\theta_2$ are 2π-periodic. $\qquad\square$

We conclude with the following regularity proposition.

Proposition 1.3 *Let P and Q be polynomials such that $P > 0$ and $Q > 0$ on $(x_1, y_2) \subset \mathbb{R}$ and assume that: x_1 is a simple root of P, y_2 is a simple root of Q, and $1 - a^2 x^2 y^2 \neq 0$ for $x, y \in [x_1, y_2]$.*

Then, the metric (2) is smooth if and only if the variables $\frac{|P'(x_1)|}{2(1 + a^2 x_1^4)}\theta_1$ and $\frac{|Q'(y_2)|}{2(1 + a^2 y_2^4)}\theta_2$ are 2π-periodic where $\theta_1 := \varphi + a x_1^2 \psi$ and $\theta_2 := \psi + a y_2^2 \varphi$.

1.3 At a double root x_1 of P and a generic y: separating cusp

Similarly, close to x_1 a double root of the polynomial, one has $P(x) \approx P''(x_1)(x - x_1)^2/2$. As in Sect. 1.1, we find that close to the same codimension 2 submanifold Σ, the metric is asymptotic to:

$$(1 - a^2 x_1^2 y^2) \left(\frac{P''(x_1)(x - x_1)^2}{2(1 + a^2 x_1^4)^2} (d\theta_1 + f(y, x_1)\omega_1)^2 + \frac{2dx^2}{P''(x_1)(x - x_1)^2} \right) + g_\Sigma \quad (5)$$

for some smooth $y \mapsto f(y, x_1)$ vanishing when $a = 0$ which is an asymptotically cuspidal metric. This is a smooth complete metric, but it adds a cuspidal end to the manifold.

1.4 Approaching a cuspidal end

Let us now explain how one can approach a codimension 2 cuspidal end by smooth metrics. Assume that $x_1 \pm i\epsilon$ are two complex conjugate roots of P_ϵ for $\epsilon > 0$ that we will send to 0. This time, we have the following second-order approximation for $P_\epsilon(x)$ for x close to x_1: $P_\epsilon(x) \approx P''_\epsilon(x_1) \left((x - x_1)^2 + \epsilon^2 \right) /2 + \mathcal{O}((x - x_1)^3)$.

This implies that the metric is approximately

$$(1 - a^2 x_1^2 y^2) \left(\frac{P''_\epsilon(x_1) \left((x - x_1)^2 + \epsilon^2 \right)}{2(1 + a^2 x_1^4)^2} (d\theta_1 + f(y, x_1)\omega_1)^2 \right.$$
$$\left. + \frac{2dx^2}{P''_\epsilon(x_1) \left((x - x_1)^2 + \epsilon^2 \right)} \right) + g_\Sigma. \quad (6)$$

This is a smooth metric, but along $\{x = x_1\}$ it is close to a thin cylinder, see the left picture of Fig. 1. As $\epsilon \to 0$, close to any $x \neq x_1$, the metric Cheeger–Gromov converges to the cuspidal metric (5) on compact sets, see the right image in Fig. 1.
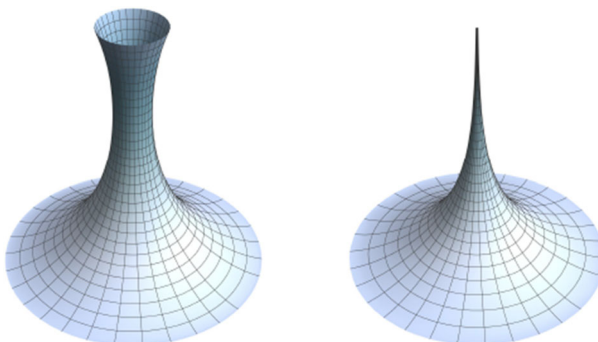

Fig. 1 Stages of cusp formation. The cusp on the right is infinitely long

2 Degenerations of AdS C-metrics

2.1 Non-rotating limit: AdS C-metrics

For this section, we will follow [10] and will adopt their notation. Our study and goals are purely geometric and differ from theirs. The AdS C-metrics are obtained from the general Plebański–Demiański family (2) by taking the *non-rotating limit* $a \to 0$. These metrics are the Riemannian analogues of the metrics considered in [11] and have the form

$$g_C = \frac{1}{(x-y)^2} \left[Q(y)d\psi^2 + \frac{dy^2}{Q(y)} + \frac{dx^2}{P(x)} + P(x)d\varphi^2 \right] \tag{7}$$

where we will assume that Q and P are parametrized by two variables μ, ν as

$$P(x) = (1+x)\left(1 + \nu x + \mu x^2\right), \text{ and}$$
$$-Q(y) = y\left[1 + \nu + (\mu + \nu)y + \mu y^2\right], \tag{8}$$

Note the identity $-Q(y) = P(y) - 1$. This ensures Einstein condition (1) is satisfied and the pointwise norm of the Riemannian tensor of g_C is given by

$$\| \operatorname{Rm}_{g_C} \|_{g_C}^2 = 24 + 12(x-y)^6 \mu^2.$$

More precisely, from the computations of [10], one has $\operatorname{Ric}(g_C) = -3g_C$ and the eigenvalues of both the selfdual and anti-selfdual parts of W_{g_C} are equal to $\frac{\mu}{4}(y-x)^3(2, -1, -1)$. As expected, these eigenvalues go to zero as $x \to y$ and when $\mu = 0$, the metric is *locally hyperbolic*. A direct computation ensures again that $\|W_{g_C}\|_{L^2(g_C)}^2$ is bounded. In particular, from [3], these examples have bounded *renormalized volume*.

2.2 Proof of Theorem A

In this section, we study a specific 2-dimensional family of AdS C-metrics on $\mathbb{R}^4$ forming one or two cusps in different limits. The cusps forming here effectively separate the manifold into two or three Poincaré–Einstein metrics with cusps ends in their bulk and their conformal infinities. We prove Theorem A.

As in Sect. 2.1, we consider the metric (7) where $-Q(y) = y\left[1 + \nu + (\mu + \nu)y + \mu y^2\right]$ and $P(x) = (1+x)\left(1 + \nu x + \mu x^2\right)$. The roots of P and Q, respectively, are as follows

$$x_0 = -1, \quad x_\pm = \frac{-\nu \pm \sqrt{\nu^2 - 4\mu}}{2\mu}, \text{ and}$$

$$y_0 = 0, \quad y_\pm = \frac{-(\mu + \nu) \pm \sqrt{(\mu + \nu)^2 - 4\mu(1+\nu)}}{2\mu}. \tag{9}$$

In order to approach metrics with cusp ends in this family by smooth metrics, we consider the case when $x_\pm, y_\pm$ are complex conjugate roots which we will let approach a real double root—leading to a cusp degeneration by Sect. 1.4. In the (μ, ν) plane, this condition means that (μ, ν) lies in the region bounded by the curves $\nu = 2\sqrt{\mu}$ and $\nu = \mu - 2\sqrt{\mu}$.

We then consider $-1 < x < y < 0$ where the conformal infinity is at $\{x = y\}$, see Fig. 2b. For the metric to be smooth, we require that $\frac{1-\nu+\mu}{2}\varphi$ and $\frac{1+\nu}{2}\psi$ be 2π-periodic, see Proposition 1.3. We further impose that $\mu > \max(\nu/2, -\nu)$. This corresponds to forcing the real part of $x_\pm$ and $y_\pm$ to be in $(-1, 0)$, this way the double root degeneration (when
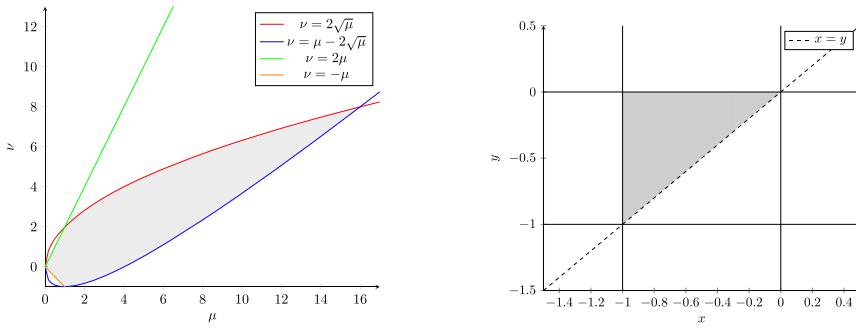
(a) Admissible (μ, ν) parameters (shaded).

(b) Range of coordinates (x, y) (shaded), the range for φ, ψ is a torus determined by Proposition 1.3

Fig. 2 Parameter ranges considered. The dashed $\{x = y\}$ is the conformal infinity

the imaginary part of the roots tends to zero) happens where the metric is defined and is geometrically meaningful. We end up with the region $D4$ in [10] shaded in Fig. 2a in the μ, ν plane bounded by the curves $\nu = 2\sqrt{\mu}$, $\nu = \mu - 2\sqrt{\mu}$, $\nu = 2\mu$ and $\nu = -\mu$.

Remark 2.1 In the limit $(\mu, \nu) \to 0$ from our region shaded in Fig. 2a, our metrics converge smoothly to the hyperbolic 4-space. Indeed, the metric is already locally hyperbolic by our curvature computations and the change of variables $x = -\sin^2((u - \pi)/2)$, at the conformal infinity $\{x = y\}$, the restriction of the metric $(x - y)^2 g_C$ with $\mu = \nu = 0$ takes the form

$$du^2 + \cos\left(\frac{u - \pi}{2}\right)^2 d\varphi^2 + \sin\left(\frac{u - \pi}{2}\right)^2 d\psi^2.$$

Thus, we recover the metric of the round 3-sphere in Hopf's coordinates since φ and ψ are 4π-periodic. This in particular ensures that the topology we consider is $\mathbb{R}^4$.

From (9), we see that, for (μ, ν) in the shaded region in Fig. 2a, if one of P, Q has a double root, then (μ, ν) lies on at least one of the boundary curves $\nu = 2\sqrt{\mu}$ or $\nu = \mu - 2\sqrt{\mu}$, respectively, in blue and red in Fig. 2, see the first two columns of Fig. 3 for the associated polynomials and geometric representation. The intersection of these curves, $(\mu, \nu) = (16, 8)$, is the unique case when P and Q have double roots at $x = \frac{-1}{4}$ and $y = \frac{-3}{4}$) as described in Fig. 3c, f, respectively, leading to two cusps dividing the manifold in three regions, while the point $(\mu, \nu) = (0, 0)$ corresponds to hyperbolic 4-space from Remark 2.1. The possible double roots of P and Q, respectively, lie in the intervals $(-1, \frac{-1}{4}]$ and $[\frac{-3}{4}, 0)$.

This completes the proof of Theorem A.

3 Degenerations in Carter–Plebański family of metrics

3.1 Non-accelerating limit: Carter–Plebański metrics

The Carter–Plebański family of metrics is a special limit of the Plebański–Demiański family of metrics (2) after a change of coordinates. To do this, start from (2) in the coordinates of [24] and perform a rescaling by $b > 0$ (acceleration parameter) of coordinates as in [19,
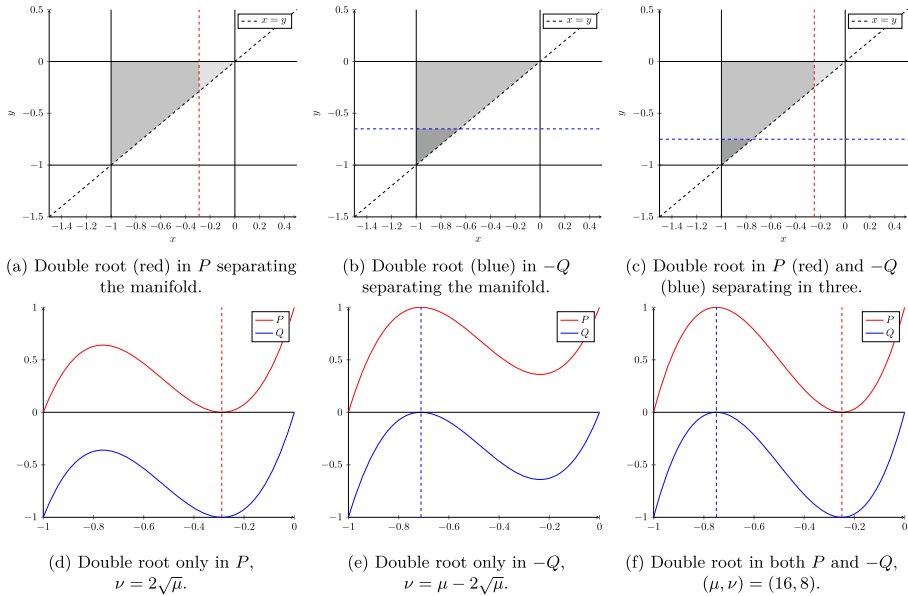
(a) Double root (red) in P separating the manifold.

(b) Double root (blue) in $-Q$ separating the manifold.

(c) Double root in P (red) and $-Q$ (blue) separating in three.

(d) Double root only in P, $\nu = 2\sqrt{\mu}$.

(e) Double root only in $-Q$, $\nu = \mu - 2\sqrt{\mu}$.

(f) Double root in both P and $-Q$, $(\mu, \nu) = (16, 8)$.

Fig. 3 Different configurations of double roots

Section 2.2], which yields the following metric:

$$
g_{PD} = \frac{1}{(1 - bpq)^2} \left[\frac{p^2 - q^2}{\mathcal{P}_b(p)} dp^2 + \frac{p^2 - q^2}{\mathcal{Q}_b(q)} dq^2 + \frac{\mathcal{P}_b(p)}{p^2 - q^2} \left(d\tau + q^2 d\sigma \right)^2 \right.
$$
$$
\left. + \frac{\mathcal{Q}_b(q)}{p^2 - q^2} \left(d\tau + p^2 d\sigma \right)^2 \right]
\tag{10}
$$

for polynomials $\mathcal{P}_b$ and $\mathcal{Q}_b$ depending on $b > 0$ chosen to satisfy (1) with $\Lambda = -3$. Taking the "no acceleration limit" $b \to 0$ as in [19, Section 5], we obtain from (10) the *Carter–Plebański* metric

$$
g_{CP} := \frac{p^2 - q^2}{\mathcal{P}(p)} dp^2 + \frac{p^2 - q^2}{\mathcal{Q}(q)} dq^2 + \frac{\mathcal{P}(p)}{p^2 - q^2} \left(d\tau + q^2 d\sigma \right)^2 + \frac{\mathcal{Q}(q)}{p^2 - q^2} \left(d\tau + p^2 d\sigma \right)^2,
\tag{11}
$$

where the limiting polynomials $\mathcal{P}$ and $\mathcal{Q}$ are of the form:

$$
\mathcal{P}(p) = p^4 + E^2 p^2 - 2Np + \alpha \text{ and}
$$
$$
\mathcal{Q}(q) = -q^4 - E^2 q^2 + 2Mq - \alpha
\tag{12}
$$

following the notations of [26] for some real numbers E, M, N and α.

We will consider intervals where $\mathcal{P}(p) \leqslant 0$ and $\mathcal{Q}(q) \leqslant 0$. This time, the range in p will be compact of the form $[p_-, p_+]$ for $p_\pm$ roots of $\mathcal{P}$ and the range in q will be of the form $[q_+, +\infty)$ for q_+ root of $\mathcal{Q}$.

This metric is Poincaré–Einstein and as $q \to +\infty$ (the infinity in these coordinates), the metric looks like

$$
g_{CP} \approx \frac{dq^2}{q^2} + q^2 \left(-\frac{dp^2}{\mathcal{P}(p)} - \mathcal{P}(p) d\sigma^2 + (d\tau + p^2 d\sigma)^2 \right)
\tag{13}
$$
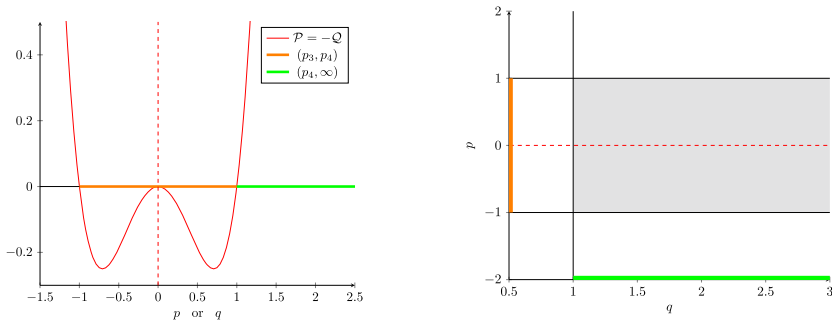
(a) Example of double root in $\mathcal{P}$ with $N = M$. Intervals where $\mathcal{P}$ and $\mathcal{Q}$ are nonpositive are highlighted.

(b) Double root in $\mathcal{P}$ (in orange) extending infinity $q \to +\infty$. Range in (p, q) shaded.

Fig. 4 Polynomial and range of coordinates

so the metric at conformal infinity is $-\frac{dp^2}{\mathcal{P}(p)} - \mathcal{P}(p)d\sigma^2 + (d\tau + p^2 d\sigma)^2$.

3.2 An example with different topology

In this section, we indicate how to find families of metrics forming cusps with different topologies. We take the simplest example here on $\mathbb{CP}^2 \backslash D^4$ with conformal infinity $\mathbb{S}^3$. We follow [26, Sections 2.1, 2.2 and 2.3] for our regularity conditions: we impose τ and σ to be as [26, Sections 2.13 and 2.17]. This also requires $N = M$, which is equivalent to the metric being *self-dual*, and forcing $\mathcal{P} = -\mathcal{Q}$.

We will moreover parametrize our polynomial by the roots and looking for a metric with a cusp, we will consider a polynomial with a double root: for $p_3, p_4, p_0 \in \mathbb{R}$ (following notations of [26]):

$$\mathcal{P}(p) = (p - p_3)(p - p_4)(p - p_0)^2. \tag{14}$$

We will then consider the range $(p, q) \in [p_3, p_4] \times [p_4, +\infty]$, where the associated metric is indeed Riemannian (Fig. 4).

Remark 3.1 Recall that from Remark 0.4, we cannot have a double root in $\mathcal{Q}$ on $(p_4, +\infty)$. All we will find instead is a double root of $\mathcal{P}$ on (p_3, p_4) corresponding to a cusp in the manifold extending to infinity.

We need our double root p_0, to lie in (p_3, p_4) so that it is reflected in our metric. Since the sum of the roots is 0 (the cubic coefficient of the polynomial is zero), $p_0 = -\frac{p_3 + p_4}{2}$ and so $p_0 \in (p_3, p_4)$ imposes

$$p_3 < 0 < p_4, \quad \text{and} \quad \frac{1}{3}|p_3| < |p_4| < 3|p_3|. \tag{15}$$

We can find this polynomial (14) as a limit of polynomials with two complex conjugate roots: for $\epsilon \geqslant 0$

$$\mathcal{P}_\epsilon(p) = (p - p_3)(p - p_4)\left(\left(p + \frac{1}{2}(p_3 + p_4)\right)^2 + \epsilon^2\right) \tag{16}$$

where we get the double root mentioned above when $\epsilon \to 0$ and we also let $-\mathcal{Q}_\epsilon = \mathcal{P}_\epsilon$ to satisfy the above regularity condition of [26]. Since the roots of the polynomials are the same, the intervals in which these are defined stay the same. Geometrically, in the limit $\epsilon \to 0$,

the metrics (11) associated with $-\mathcal{Q}_\epsilon = \mathcal{P}_\epsilon$ develop a cusp along $\{p = -(p_3 + p_4)/2\}$ separating the manifold in two parts by an argument similar to Sect. 1.3.

The topology of the manifold is that of $\mathbb{CP}^2$ minus a ball and the conformal infinity is $\mathbb{S}^3$. The "bolt" of the metric is reached at $[p_3, p_4] \times \{q = 1\}$ which is a codimension 2 submanifold (a 2-sphere), see [26].

Remark 3.2 It is likely possible to obtain infinitely many different topologies from the Carter–Plebański family of metrics by having a larger and larger "self-intersection" for the 2-sphere while obtaining a conformal infinity $\mathbb{S}^3/\mathbb{Z}_k$ for $\mathbb{Z}_k$ a cyclic subgroup of $SU(2)$ acting freely on $\mathbb{S}^3$. See [13, Section 5.1] for a discussion of the regularity conditions and possible topologies. In the larger Plebański–Demiański family of metrics, we believe that there is also a large class of additional possible topologies, with two "bolts" (and a "NUT"). The conformal infinity, could this time be an arbitrary lens space. See [13, Section 5.2] for a discussion of the regularity conditions and possible topologies.

4 Degenerations in the Plebański–Demiański family of metrics

We will now turn to the general PD family of metrics. The above degenerations of Sects. 2 and 3 can be found in the full family of Plebański–Demiański, but we focus on exhibiting new behaviors of complete metrics whose conformal infinities develop unexpected types of singularities. We prove Theorem B.

In this section, we consider a subfamily of metrics in (2) with $a = 1$, parametrizing our polynomials as

$$P_\infty(x) = C_\infty(x - \alpha_1)((x - 1 + \alpha_2)^2 + \alpha_3)(x - \alpha_4) \tag{17}$$

with $C_\infty = (-1 + \alpha_1\alpha_2^2\alpha_4 + \alpha_1\alpha_4 - 2\alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4)^{-1}$ and $-Q_\infty(y) = P_\infty(y) + y^4 - 1$. These metrics satisfy the Einstein Condition (1) with $\Lambda = -3$ for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$.

As for the bulk metric, the conformal infinity $\{x = y\}$ has different possible asymptotic behaviors close to roots of P_∞ or Q_∞. We consider (2), whose conformal metric at infinity is:

$$g_{bdry} = (1 - a^2x^4)\left(\frac{1}{P_\infty(x)} + \frac{1}{Q_\infty(x)}\right)dx^2 + \frac{Q_\infty(x)}{1 - a^2x^4}(d\psi - ax^2d\varphi)^2$$

$$+ \frac{P_\infty(x)}{1 - a^2x^4}(d\varphi - ax^2d\psi)^2. \tag{18}$$

We will moreover assume that the regularity conditions for the bulk of Proposition (1.3) are satisfied whenever applicable. Simpler arguments than Sects. 1.1 and 1.3 imply the following result.

Proposition 4.1 *Under the assumptions of Proposition 1.3, the conformal metric is smooth. Moreover, if P_∞ (or Q_∞) has a double root at $-1 < x_0 < 1$, then the conformal boundary metric of (18) has a codimension 2 separating cusp as described in (5).*

We will see that allowing the roots to be at ± 1 leads to different degenerate behavior for the conformal infinity alone. Setting $\alpha_2 = \alpha_3 = 0$ and choosing distinct $\alpha_1, \alpha_4 \in \mathbb{R}$ in (17), the polynomial P_∞ has a double root only at $x = 1$ while Q_∞ has a simple root at 1. This will correspond to a *Naked Singularity* in the metric, which we describe in Sect. 4.1 once we ensure that our metric is smooth and Riemannian. We first need to verify that we

have the correct signs $P_\infty > 0$ and $Q_\infty > 0$ on the region $\alpha_1 \leqslant x < y \leqslant 1$. Assume that $\alpha_1 < 1 \leqslant \alpha_4$, then the inequality $P'_\infty(\alpha_1) = \frac{(\alpha_1-1)^2(\alpha_1-\alpha_4)}{\alpha_1\alpha_4-1} > 0$, which is satisfied whenever $\alpha_1 < \alpha_4^{-1}$, guarantees that $P_\infty > 0$ on $(\alpha_1, 1)$. To guarantee that Q_∞ has the right sign, it is enough to impose $-1 < \alpha_1 < 0$.

Remark 4.2 This is true on a larger range of values of α_1 which we do not attempt to describe.

Lastly, we assume that φ and ψ satisfy the periodicity conditions imposed in Proposition 1.3 to ensure that we find smooth metrics.

4.1 A naked singularity

Assuming that 1 is a double root of P_∞ and a simple root of Q_∞, at $x = 1$, the metric (18) approaches

$$\frac{C_1}{(1-x)}dx^2 + C_2\theta_1(x)^2 + C_3(x-1)^3\theta_2(x)^2 \tag{19}$$

for $\theta_1(x) \to d\varphi - d\psi$, $\theta_2(x) \to d\varphi + d\psi$ as $x \to 1$ and where $C_1 = \frac{8}{P''_\infty(1)}$, $C_2 = \frac{Q'_\infty(1)}{4}$ and $4C_2C_3 = \frac{P''_\infty(1)}{2}Q'_\infty(1)$ so $C_3 = \frac{P''_\infty(1)}{2}$. A change of variables $r = 2\sqrt{1-x}$ in (19)

Fig. 5 Naked singularity, **not** infinitely long
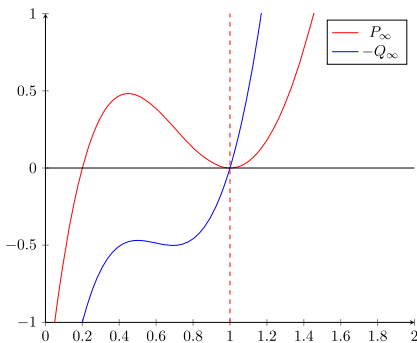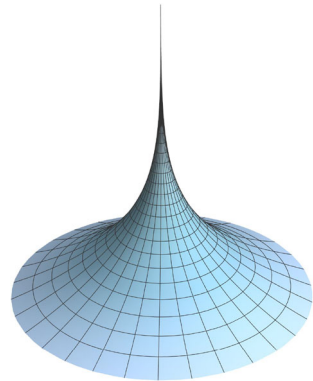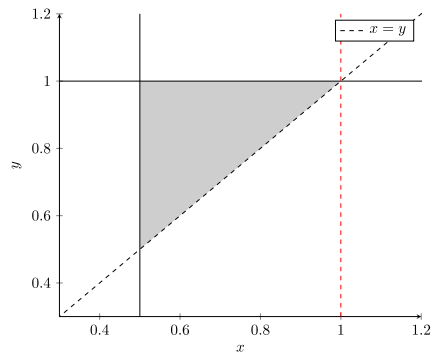
(a) $\alpha_2 = \alpha_3 = 0$, $\alpha_1 = 0.2$ and $\alpha_4 = 3$ (b) Double Root (red) only at infinity

Fig. 6 Example polynomials and region for metric with naked singularity at infinity

(a) $\alpha_3 > 0$ and $\alpha_2 = 0$.   (b) $\alpha_2 < 0$ and $\alpha_3 = 0$.
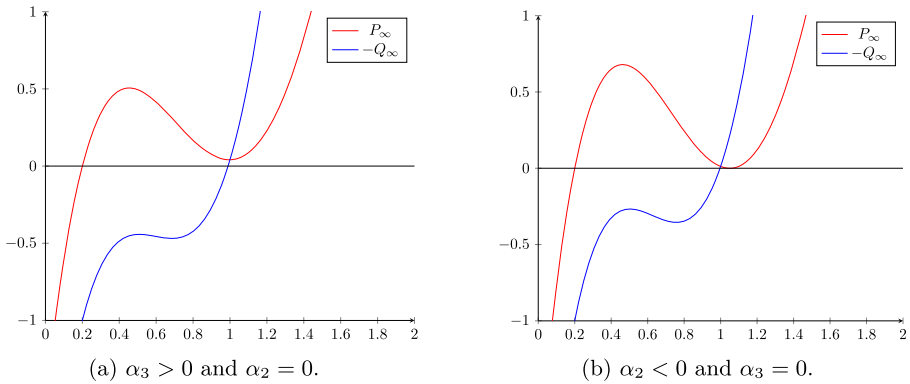
Fig. 7 Smooth metric to naked singularity

yields the *naked singularity* metric: close to $r = 0$,

$$C_1 dr^2 + C_2 \theta_1(x)^2 + \frac{C_3}{4} r^6 \theta_2(x)^2. \tag{20}$$

The metrics obtained in this way can be approached by perturbing the parameters α_2 and α_3 in various ways around $(0, 0)$. This gives the following different types of degenerations, which we describe below (Figs. 5, 6).

4.2 Degeneration 1: from a smooth metric to a naked singularity.

By taking $\alpha_3 > 0$ and keeping $\alpha_2 = 0$, the double root of P_∞ at 1 is replaced with two complex roots, see Fig. 7a. Taking the limit $\alpha_3 \to 0$ yields the above naked singularity. Similarly, by taking $\alpha_2 < 0$ and $\alpha_3 = 0$, the double root of P_∞ is moved past the conformal infinity $y = x$, see Fig. 7b. Taking the limit $\alpha_2 \to 0$ yields the above naked singularity.

Both of these situations yield a smooth metric at conformal infinity by Proposition 4.1. Indeed, P_∞ does not have any root close to the root of Q_∞.

4.3 Degeneration 2: from a conical singularity to a naked singularity.

Let us assume that 1 is a simple root of both P_∞ and Q_∞ for the metric (18). The case of -1 is treated similarly. As $x \to 1$, we obtain that the metric (18) is asymptotic to

$$C_1 dx^2 + C_2 \theta_1(x)^2 + C_3 (x - 1)^2 \theta_2(x)^2$$

where $\theta_1(x) \to d\varphi - d\psi, \theta_2(x) \to d\varphi + d\psi$ as $x \to 1$, and $C_1 = 4\left(\frac{1}{P'_\infty(1)} + \frac{1}{Q'_\infty(1)}\right), C_2 = \frac{P'_\infty(1) + Q'_\infty(1)}{4}$ and $C_3 = \frac{P'_\infty(1) Q'_\infty(1)}{P'_\infty(1) + Q'_\infty(1)} = \left(\frac{1}{P'_\infty(1)} + \frac{1}{Q'_\infty(1)}\right)^{-1}$. This yields a codimension 2 cone-edge singularity of angle $\frac{-P'_\infty(1) Q'_\infty(1)}{2(P'_\infty(1) + Q'_\infty(1))}$.

By taking $\alpha_3 < 0$ and $\alpha_2 = 0$, see Fig. 8a, the double root of P_∞ at 1 is split in two real roots $x_- < 1 < x_+$. This changes the topology and creates a codimension 2 cone-edge singularity along $\{x = x_-\}$ by Lemma 1.1, extending to the conformal infinity $\{x = y\}$. As $\alpha_3 \to 0$, the angle tends to zero and a *naked singularity* appears while the singularities in the bulk are "sent to infinity".

(a) $\alpha_3 < 0$ and $\alpha_2 = 0$.   (b) $\alpha_3 < 0$ and $\alpha_2 = -\sqrt{-\alpha_3}$.
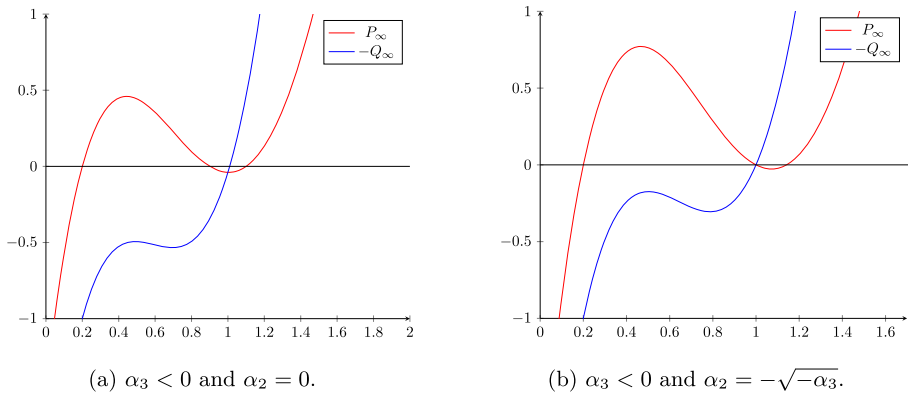
Fig. 8 Conical singularity to naked singularity

By setting $\alpha_3 < 0$ and $\alpha_2 = -\sqrt{-\alpha_3}$ as in Fig. 8b, the double root of P_∞ at 1 is split into a single root at 1 and a root larger than 1. This gives a conical singularity in the metric at the conformal infinity *only* this time. As $\alpha_3 \to 0$, the angle tends to zero and a naked singularity appears in the limit.

4.4 Degeneration 3: from cusp to naked singularity

By taking $\alpha_2 > 0$ and $\alpha_3 = 0$, the double root $1 - \alpha_2$ of P_∞ is moved to the left of the root in Q_∞. This creates a cusp in the bulk metric as well as in its infinity by Sect. 1.3 and Proposition 4.1.

As in Sect. 2.1, this cusp at $\{x = 1 - \alpha_2\}$ separates the manifold in two regions infinitely far apart, and the conformal infinity in two finite volume manifolds with cusp ends.

When $\alpha_2 \to 0$, the volume of $\{1 - \alpha_2 < x = y < 1\}$ tends to zero and the region disappears, and the metric on $\{\alpha_1 < x = y < 1 - \alpha_2\}$ has infinite diameter for $\alpha_2 > 0$ but finite diameter in the limit $\alpha_2 \to 0$ (these remarks do not depend on the representative of the conformal class) (Fig. 9).

This is a manifestation of cusp degenerations in the bulk manifold comparable to those of Sect. 2.1. Indeed, in the family of metrics obtained from (17), there is a four-dimensional

Fig. 9 Cusp to naked singularity
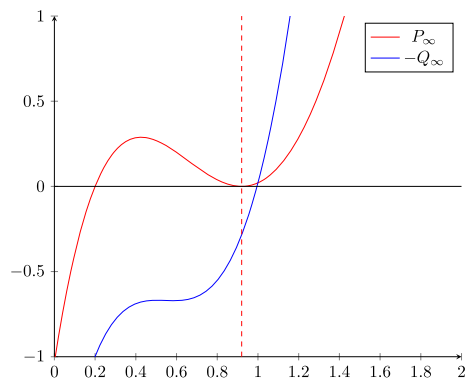
family of smooth Poincaré–Einstein metrics with a (three-dimensional) boundary constituted of metrics with one cusp separating the manifold in two set, and a two-dimensional family with two cusps separating the manifold in three.

Remark 4.3 There are important differences with Sect. 2.1. The cusps from (2) for $a = 1$ are "twisted" (see (5)) and do not look like mere products of surfaces in the limit. Moreover, as described above, as $\alpha_2 \to 0$ the cusps "escapes" to infinity creating the above unexpected naked singularity at infinity. This was impossible in the family (7) because the double root in P could not approach 0 and the double root in Q could not approach -1.

4.5 Two cusps at conformal infinity only

We finally assume that 1 is a triple root of P_∞ and a simple root of Q_∞. The metric (18) is asymptotic to

$$\frac{C_1}{(1-x)^2}dx^2 + C_2\theta_1(x)^2 + C_3(x-1)^4\theta_2(x)^2 \tag{21}$$

where again, $\theta_1(x) \to d\varphi - d\psi$, $\theta_2(x) \to d\varphi + d\psi$ as $x \to 1$ and where $C_1 = \frac{24}{P_\infty^{(3)}(1)}$, $C_2 = \frac{Q_\infty'(1)}{4}$ and $C_3 = \frac{P_\infty^{(3)}(1)}{6}$. A change of variables $r = -\log(1-x)$ in (21) yields the *cusp end* metric: for r close to $+\infty$,

$$C_1 dr^2 + C_2\theta_1(x)^2 + C_3 e^{-4r}\theta_2(x)^2. \tag{22}$$

This time, we exhibit a metric with codimension 2 cusps ends at the conformal infinity *only*— in particular, the conformal infinity is not compact. Unlike the example of Sect. 2, these cusps do not cut the manifold in different pieces. Consider

$$P_{2,\infty}(x) = -\frac{1}{2}(x-1)^3(x+1) \text{ and}$$

$$Q_{2,\infty}(y) = -\frac{1}{2}(y-1)(y+1)^3$$

which are limit of the polynomials in (17) for $\alpha_1 = -1, \alpha_2 = 0, \alpha_3 = 0$ and $\alpha_4 = 1$. These polynomials have the desired signs on the region $-1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1$ making the

Fig. 10 Two cusps at conformal infinity *only*
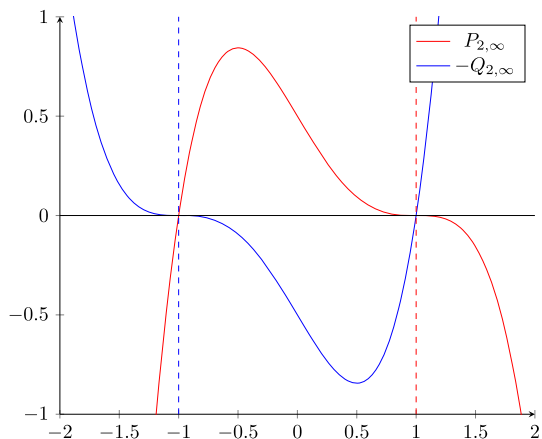
metric (2) with $a = 1$ Riemannian. Its infinity has *two* cusp ends at the points $(-1, -1)$ and $(1, 1)$ thanks to (21). This is a limiting case for all the previous degenerations as well as a limit of naked singularities at either 1 or -1 (Fig. 10).

Remark 4.4 From Sect. 1, one moreover notices that this metric is *anti-selfdual* since the linear and cubic coefficients of $P_{2,\infty}$ are opposite.

Declarations

Conflict of interest The authors declare no competing interests.

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