Calabi type functionals for coupled Kähler–Einstein metrics

Satoshi Nakamura[1]

Abstract

We introduce the coupled Ricci–Calabi functional and the coupled H-functional which measure how far a Kähler metric is from a coupled Kähler–Einstein metric in the sense of Hultgren–Witt Nyström. We first give corresponding moment weight type inequalities which estimate each functional in terms of algebraic invariants. Secondly, we give corresponding Hessian formulas for these functionals at each critical point, which have an application to a Matsushima type obstruction theorem for the existence of a coupled Kähler–Einstein metric.

1 Introduction

Hultgren–Witt Nyström [28] introduced the notion of a coupled Kähler–Einstein metric on a compact complex manifold of general type or a Fano manifold. In this paper, we mainly focus on the Fano case. Let X be an n-dimensional Fano manifold. *A decomposition* of the first Chern class $2\pi c_1(X)$ is a sum

$$2\pi c_1(X) = \alpha_1 + \cdots + \alpha_N$$

where each α_i is a Kähler class for X. Let $\theta_i \in \alpha_i$ be a reference Kähler metric, and let V_i be the volume $\int_X \theta_i^n$ of α_i. We define the set of tuples of Kähler potentials by

$$\mathcal{M} := \prod_{i=1}^{N} \mathcal{M}_i := \prod_{i=1}^{N} \left\{ \phi_i \in C^\infty(X; \mathbb{R}) \,\middle|\, \omega_{\phi_i} := \theta_i + \sqrt{-1}\partial\bar{\partial}\phi_i > 0 \right\},$$

and identify a Kähler metric ω_{ϕ_i} with its potential ϕ_i. A tangent space of $\mathcal{M}$ is identified with $(C^\infty(X; \mathbb{R}))^N$. For any tuple of Kähler metrics $\Phi = (\phi_i)_{i=1}^N \in \mathcal{M}$, since the Ricci form $\mathrm{Ric}(\omega_{\phi_i}) := -\sqrt{-1}\partial\bar{\partial}\log\omega_{\phi_i}^n$ and the sum $\sum_{i=1}^N \omega_{\phi_i}$ represent $2\pi c_1(X)$, there exists

✉ Satoshi Nakamura
  s.nakamura@math.titech.ac.jp

[1] Department of Mathematics, Tokyo Institute of Technology, 2-12-1, Ookayama, Meguro-ku, Tokyo 152-8551, Japan

a unique smooth real function $f_i(\Phi)$ satisfying

$$\mathrm{Ric}(\omega_{\phi_i}) - \sum_{j=1}^{N} \omega_{\phi_j} = \sqrt{-1}\partial\overline{\partial}f_i(\Phi) \quad \text{and} \quad \int_X (1 - e^{f_i(\Phi)})\omega_{\phi_i}^n = 0. \tag{1}$$

In this paper, we call the tuple $(f_i(\Phi))_{i=1}^{N}$ *the Ricci potential* for $\Phi = (\phi_i)_{i=1}^{N}$. Then, the tuple $\Phi = (\phi_i)_{i=1}^{N}$ is *a coupled Kähler–Einstein metric* for the decomposition $(\alpha_i)_{i=1}^{N}$ if every $f_i(\Phi)$ vanishes, that is,

$$\mathrm{Ric}(\omega_{\phi_1}) = \cdots = \mathrm{Ric}(\omega_{\phi_N}) = \sum_{i=1}^{N} \omega_{\phi_i}.$$

Coupled Kähler–Einstein metrics were studied extensively in recent years [13, 14, 19–22, 27, 28, 31, 32, 36, 37]. One of the motivation to study comes from algebro-geometric stabilities. Indeed, Hultgren–Witt Nyström [28] introduced the notion of called K-polystability for $(X, (\alpha_i)_{i=1}^{N})$ and showed that the existence of a coupled Kähler–Einstein metric implies it. Datar-Pingali [13] introduced a framework of geometric invariant theory for a coupled constant scalar curvature Kähler metric which is a generalization of a coupled Kähler–Einstein metric.

The well-known Calabi functional [6, 7], which is the L^2-norm of a scalar curvature, plays an important role for studies of a Kähler–Einstein metric and a constant scalar curvature Kähler metric. In this paper, we introduce two Calabi type functionals which measure how Φ is far from a coupled Kähler–Einstein metric. We first discuss moment weight type inequalities which give lower bounds of these functionals in terms of algebro-geometric stability invariants. Secondly, we discuss Hessians for these functionals at each critical point to obtain various corollaries including a new proof of a Matsushima type obstruction theorem for the existence of a coupled Kähler–Einstein metric.

Let us introduce two Calabi type functionals as follows.

$$R_c(\Phi) = \sum_{i=1}^{N} \frac{1}{V_i} \int_X (1 - e^{f_i(\Phi)})^2 \omega_{\phi_i}^n \quad \text{and} \quad H_c(\Phi) = \sum_{i=1}^{N} \frac{1}{V_i} \int_X f_i(\Phi) e^{f_i(\Phi)} \omega_{\phi_i}^n.$$

In this paper, we call R_c *the coupled Ricci–Calabi functional* and H_c *the coupled H-functional*. These are non-negative functionals in $\mathcal{M}$ whose zeros are coupled Kähler–Einstein metrics (see the inequality (8)). When $N = 1$, these functional are written as R and H, respectively, and are called the Ricci–Calabi functional and the H-functional, respectively. Functionals R and H were studied in [2, 15, 17, 23, 25, 30, 38–40] and, in particular, play important roles in the context of optimal destabilizers for a Fano manifold admitting no Kähler–Einstein metric.

1.1 Moment weight type inequalities

The Calabi functional for a polarized manifold has a lower bound in terms of the Donaldson–Futaki invariant [16]. Such an inequality is called the moment weight inequality since it already appears in geometric invariant theory as an inequality between the squared norm of a moment map and a Hilbert–Mumford weight. The Ricci–Calabi functional R and the H-Functional H satisfy a corresponding moment weight inequality [2, 15, 23–25]. The first results in this paper are two moment weight type inequalities for R_c and H_c which generalize these inequalities.

Theorem 1.1 *We have*

$$\inf_{\Phi \in \mathcal{M}} R_c(\Phi)^{1/2} \geq \sup_{(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)} \frac{-D_c(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)}{\|(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)\|_2} \quad and$$

$$\inf_{\Phi \in \mathcal{M}} H_c(\Phi) \geq \sup_{(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)} -H_c(\mathcal{X},(\mathcal{L}_i)_{i=1}^N).$$

Here $(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)$ in the above supremums runs through arbitrary test configuration of the decomposition $(X,(\alpha_i)_{i=1}^N)$ introduced in [28], $\|(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)\|_2$ is the L^2-norm, $D_c(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)$ is the coupled Ding invariant, and $H_c(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)$ is the coupled H-invariant. These notions are introduced in Sect. 2.

As a direct consequence of Theorem 1.1, a Fano manifold admitting a coupled Kähler–Einstein metric satisfies algebraic (semi-)stability conditions.

Corollary 1.2 *Suppose the existence of a coupled Kähler–Einstein metric for the decomposition $(\alpha_i)_{i=1}^N$ of $2\pi c_1(X)$. Then, we have*

$$D_c(\mathcal{X},(\mathcal{L}_i)_{i=1}^N) \geq 0 \quad and \quad H_c(\mathcal{X},(\mathcal{L}_i)_{i=1}^N) \geq 0$$

for any test configuration $(\mathcal{X},(\mathcal{L}_i)_{i=1}^N)$ for $(X,(\alpha_i)_{i=1}^N)$.

When $N = 1$, the equalities in Theorem 1.1 in fact hold. Dervan-Székelyhidi [15] showed the moment weight equality for H by applying the Kähler–Ricci flow together with deep results in [8, 10]. Hisamoto [25] showed corresponding equalities for R and H by using the inverse Monge–Ampère flow [12] and the Kähler–Ricci flow, respectively, together with a technique for multiplier ideal sheaves. When $N > 1$, in order to establish the equality in Theorem 1.1, it is natural to consider the generalization of these flow, that is, the coupled inverse Monge–Ampère flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_i(t) = 1 - e^{f_i(\Phi(t))} \quad (i = 1, \ldots, N) \tag{2}$$

and the coupled Kähler–Ricci flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_i(t) = -f_i(\Phi(t)) \quad (i = 1, \ldots, N). \tag{3}$$

However, little is known for these flows at present. For instance, the short time existence for each flow is true since they are parabolic. However, the long time existence is not established. In Sect. 3, we see that each flow is a gradient flow for R_c and H_c, respectively (Corollary 3.5). They will present not only some applications to establish the equalities in Theorem 1.1 but also some interesting problems in geometric analysis.

Remark 1.3 Very recently, Hashimoto [22] introduced a different framework of test configurations for a decomposition $(\alpha_i)_{i=1}^N$ where $\alpha_i = 2\pi c_1(L_i)$ for a line bundle $L_i \to X$. The author expects that corresponding moment weight type inequalities hold in his framework.

1.2 Hessian formulas for functionals and its application to a Matsushima type obstruction theorem

In this paper, we call a critical point of R_c *a coupled Mabuchi soliton* (cf. [26, 29, 40]) and call a critical point of H_c *a coupled Kähler–Ricci soliton* (cf. [23]). In Sect. 3, we show that

a tuple $\Phi = (\phi_i)_{i=1}^N \in \mathcal{M}$ is a coupled Mabuchi soliton if and only if the vector fields $\mathrm{grad}_{\phi_i} e^{f_i(\Phi)}$ are holomorphic and $\mathrm{grad}_{\phi_1} e^{f_1(\Phi)} = \cdots = \mathrm{grad}_{\phi_N} e^{f_N(\Phi)}$. Similarly, we show that $\Phi = (\phi_i)_{i=1}^N \in \mathcal{M}$ is a coupled Kähler–Ricci soliton if and only if the vector fields $\mathrm{grad}_{\phi_i} f_i(\Phi)$ are holomorphic and $\mathrm{grad}_{\phi_1} f_1(\Phi) = \cdots = \mathrm{grad}_{\phi_N} f_N(\Phi)$.

Examples of coupled Mabuchi solitons and coupled Kähler–Ricci solitons on Fano manifolds with large symmetry have already appeared in [14] (see also [27]). However, in that paper, the conditions $\mathrm{grad}_{\phi_1} e^{f_1(\Phi)} = \cdots = \mathrm{grad}_{\phi_N} e^{f_N(\Phi)}$ and $\mathrm{grad}_{\phi_1} f_1(\Phi) = \cdots = \mathrm{grad}_{\phi_N} f_N(\Phi)$ are not required for each definition. Their motivation is the construction of such metrics by proving C^0-estimate to a class of coupled Monge–Ampère equation, which is independent of the Calabi type functionals.

The second result in this paper shows that each critical metric is in fact a local minimum of the corresponding functional by giving the Hessian formulas at each critical point. Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi^f$ be Hermitian inner products on $(C^\infty(X; \mathbb{C}))^N$ defined by

$$\langle\!\langle \boldsymbol{u}, \boldsymbol{v} \rangle\!\rangle_\Phi = \sum_{i=1}^N \frac{1}{V_i} \int_X u_i \overline{v_i} \omega_{\phi_i}^n \quad \text{and} \quad \langle\!\langle \boldsymbol{u}, \boldsymbol{v} \rangle\!\rangle_\Phi^f = \sum_{i=1}^N \frac{1}{V_i} \int_X u_i \overline{v_i} e^{f_i(\Phi)} \omega_{\phi_i}^n.$$

Let P_Φ and P_Φ^f be the operators acting on $T_\Phi \mathcal{M}$ defined by (9) and (10) in Sect. 3.1.

Theorem 1.4 *At a coupled Mabuchi soliton $\Phi_R \in \mathcal{M}$, the Hessian of the coupled Ricci–Calabi functional R_c is written as*

$$\mathrm{Hess}(R_c)(\delta\Phi_1, \delta\Phi_2) = 2\langle\!\langle P_{\Phi_R}^f \overline{P_{\Phi_R}^f} \delta\Phi_1, \delta\Phi_2 \rangle\!\rangle_{\Phi_R} = 2\langle\!\langle \overline{P_{\Phi_R}^f} P_{\Phi_R}^f \delta\Phi_1, \delta\Phi_2 \rangle\!\rangle_{\Phi_R} \quad (4)$$

for any variations $\delta\Phi_1, \delta\Phi_2 \in T_{\Phi_R}\mathcal{M}$. At a coupled Kähler–Ricci soliton $\Phi_H \in \mathcal{M}$, the Hessian of the coupled H-functional is written as

$$\mathrm{Hess}(H_c)(\delta\Phi_1, \delta\Phi_2) = \langle\!\langle P_{\Phi_R} \overline{P_{\Phi_R}} \delta\Phi_1, \delta\Phi_2 \rangle\!\rangle_{\Phi_R}^f = \langle\!\langle \overline{P_{\Phi_R}} P_{\Phi_R} \delta\Phi_1, \delta\Phi_2 \rangle\!\rangle_{\Phi_R}^f \quad (5)$$

for any variations $\delta\Phi_1, \delta\Phi_2 \in T_{\Phi_H}\mathcal{M}$.

Here the operator $\overline{P_{\Phi_R}}$ (resp. $\overline{P_{\Phi_R}^f}$) in the above theorem is the complex conjugate of P_{Φ_R} (resp. $P_{\Phi_R}^f$). Since the operator P_Φ^f (resp. P_Φ) is self-adjoint and non-negative with respective to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi$ (resp. $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi^f$) (Proposition 3.1), it turns out the following.

Corollary 1.5 *Operators $P_{\Phi_R}^f$ and $\overline{P_{\Phi_R}^f}$ (resp. P_{Φ_H} and $\overline{P_{\Phi_H}}$) are commutative. As a result, the composition $P_{\Phi_R}^f \overline{P_{\Phi_R}^f}$ (resp. $P_{\Phi_H} \overline{P_{\Phi_H}}$) is a self-adjoint non-negative operator with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\Phi_R}$ (resp. $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\Phi_H}^f$). In particular, each Hessian of R_c and H_c is non-negative at each critical point.*

When $N = 1$, the Hessian formula of the Ricci–Calabi functional at a Mabuchi soliton is obtained by the author [30]. On the other hand, Fong [17] gives the Hessian formula of the H-functional at any point by a tensor calculus. It seems to be technically difficult to apply the Fong's tensor calculus for our case where $N > 1$. A unifying technique which generalizes the author's one in [30] gives the Hessian formulas for R_c and H_c in Theorem 1.4.

By Corollary 1.5, operators P_Φ^f and $\overline{P_\Phi^f}$ (resp. P_Φ and $\overline{P_\Phi}$) are commutative at a coupled Mabuchi soliton (resp. at a coupled Kähler–Ricci soliton). Applying this commutativity, we show a Matsushima type obstruction theorem for the existence of a coupled Mabuchi soliton and a coupled Kähler–Ricci soliton.

Theorem 1.6 *Let X be a Fano manifold and $\mathfrak{h}(X)$ be the space of holomorphic vector fields. If X admits a coupled Mabuchi soliton $\Phi = (\phi_i)_{i=1}^N$, then $\mathfrak{h}(X)$ is, as a vector space, the direct sum*

$$\mathfrak{h}(X) = \sum_{\lambda \geq 0} \mathfrak{h}_\lambda(X)$$

where $\mathfrak{h}_\lambda(X)$ is the λ-eigenspace of the adjoint action of the holomorphic vector field $\mathrm{grad}_{\phi_1}(1 - e^{f_1(\Phi)}) = \cdots = \mathrm{grad}_{\phi_N}(1 - e^{f_N(\Phi)})$. If X admits a coupled Kähler–Ricci soliton $\Phi = (\phi_i)_{i=1}^N$, then $\mathfrak{h}(X)$ has the same decomposition as above where $\mathfrak{h}_\lambda(X)$ is the λ-eigenspace of the adjoint action of the holomorphic vector field $-\mathrm{grad}_{\phi_1} f_1(\Phi) = \cdots = -\mathrm{grad}_{\phi_N} f_N(\Phi)$. Furthermore, in both cases, $\mathfrak{h}_0(X)$ coincides with the complexification of the Lie algebra of Killing vector fields for every ω_{ϕ_i}. In particular, $\mathfrak{h}_0(X)$ is reductive.

The above theorem gives a new proof of a Matsushima type obstruction theorem for the existence of a coupled Kähler–Einstein metric which is already proved by Hultgren–Witt Nyström [28] and Futaki-Zhang [20].

Corollary 1.7 *Let X be a Fano manifold admitting a coupled Kähler–Einstein metric for a decomposition of $2\pi c_1(X)$. The holomorphic automorphism group $\mathrm{Aut}(X)$ is reductive.*

Organization

This paper is organized as follows. In Sect. 2, we introduce the notion of test configurations for a decomposition and some algebraic invariants to prove the moment weight type inequalities for R_c and H_c. Some energy functionals and its slope formulas at infinity play an important role for the proof. In Sect. 3, we give the Hessian formulas of R_c and H_c to see that each critical point is a local minimum. As an application of the Hessian formulas, Matsushima type obstruction theorems for the existence of coupled Mabuchi solitons and coupled Kähler–Ricci solitons are proved.

2 Moment weight type inequalities

2.1 Test configurations and invariants

Following [28], we define the notion of test configurations for a decomposition $(\alpha_i)_{i=1}^N$ of $2\pi c_1(X)$.

Definition 2.1 Let L be an ample line bundle over a projective manifold Y. A test configuration $(\mathcal{Y}, \mathcal{L})$ for (Y, L) is a normal scheme $\mathcal{Y}$, a flat surjective morphism $\mathcal{Y} \to \mathbb{C}$ and a relatively ample line bundle $\mathcal{L}$ together with a $\mathbb{C}^*$-action on $\mathcal{L}$ compatible with the standard $\mathbb{C}^*$-action on $\mathbb{C}$, such that the fiber over $1 \in \mathbb{C}$ is equal to (Y, L).

An $\mathbb{R}$-line bundle is understood as a formal linear combination over $\mathbb{R}$ of line bundles. The following definition is based on the fact that any Kähler class α on a Fano manifold can be written as the first Chern class of an $\mathbb{R}$-line bundle, that is, $\alpha = c_1(\sum_j r_j L_j) := \sum_j r_j c_1(L_j)$ for an ample line bundle L_j and a positive real number r_j.

Definition 2.2 Let α be a Kähler class on a Fano manifold X. A test configuration $(\mathcal{X}, \mathcal{L})$ for (X, α) is defined as a test configuration in the sense of Definition 2.1 satisfying the following.

(1) The scheme $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein.
(2) There exists an $\mathbb{R}$-line bundle $L = \sum_j r_j L_j$ over X satisfying $\alpha = c_1(L)$, where each L_j is ample and $r_j > 0$.
(3) The line bundle $\mathcal{L}$ is written as $\sum_j r_j \mathcal{L}_j$ where $\mathcal{L}_j$ is a line bundle over $\mathcal{X}$ such that $(\mathcal{X}, \mathcal{L}_j)$ is a test configuration for (X, L_j) in the sense of Definition 2.1.

Definition 2.3 Let $(\alpha_i)_{i=1}^N$ be a decomposition of $2\pi c_1(X)$. A test configuration $(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N)$ for $(X, (\alpha_i)_{i=1}^N)$ is defined as follows.

(1) The scheme $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein.
(2) For each i, the $\mathbb{R}$-line bundle $\mathcal{L}_j$ over $\mathcal{X}$ defines a test configuration $(\mathcal{X}, \mathcal{L}_j)$ for (X, α_i) in the sense of Definition 2.2.
(3) The sum $\sum_{i=1}^N \mathcal{L}_i$ defines a test configuration $(\mathcal{X}, \sum_{i=1}^N \mathcal{L}_i)$ for the Fano manifold $(X, -K_X)$ in the sense of Definition 2.1.

Now we define some invariants appearing in the right hand side of moment weight type inequalities. Gluing each $(\mathcal{X}, \mathcal{L}_i)$ with the trivial family, we have the unique $\mathbb{C}^*$-equivalent family $(\bar{\mathcal{X}}, \bar{\mathcal{L}}_i)$ over $\mathbb{P}^1$. This gives a compactification of a test configuration $(\mathcal{X}, (\mathcal{L})_{i=1}^N)$ for $(X, (\alpha_i)_{i=1}^N)$. Set

$$E(\mathcal{X}, \mathcal{L}_i) = \frac{\bar{\mathcal{L}}_i^{n+1}}{(n+1)V_i}$$

to define *the coupled Ding invariant* (cf. [2]) using the log canonical threshold

$$D_c\left(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N\right) = L\left(\mathcal{X}, \sum_{i=1}^N \mathcal{L}_i\right) - \sum_{i=1}^N E(\mathcal{X}, \mathcal{L}_i)$$

$$:= \mathrm{lct}_{(\bar{\mathcal{X}}, \mathcal{B})}(\mathcal{X}|_{t=0}) - 1 - \sum_{i=1}^N \frac{\bar{\mathcal{L}}_i^{n+1}}{(n+1)V_i}$$

where $\mathcal{X}|_{t=0}$ is the fiber over $0 \in \mathbb{P}^1$ and $\mathcal{B}$ is the boundary divisor uniquely determined by the properties $\mathcal{B} \sim_{\mathbb{Q}} -K_{\bar{\mathcal{X}}/\mathbb{P}^1} - \sum_{i=1}^N \bar{\mathcal{L}}_i$ and $\mathrm{supp}\mathcal{B} \subset \mathcal{X}|_{t=0}$. We can consider the $\mathbb{C}^*$-action on the fiber over $t = 0$ to describe $E(\mathcal{X}, \mathcal{L}_i)$ in terms of the weight $\lambda_{i,1}, \ldots, \lambda_{i,N_{i,k}}$ for the action on $H^0(\mathcal{X}|_{t=0}, k\mathcal{L}_i|_{t=0})$ for fixed positive integer k, where $N_{i,k} := \dim H^0(\mathcal{X}|_{t=0}, k\mathcal{L}_i|_{t=0})$. It is well known that we have

$$E(\mathcal{X}, \mathcal{L}_i) = \lim_{k \to \infty} \frac{\sum_{j=1}^{N_{i,k}} \lambda_{i,j}}{k N_{i,k}}.$$

In view of this formula, the equality $E(\mathcal{X}, \mathcal{L}_i + c\mathcal{X}|_{t=0}) = E(\mathcal{X}, \mathcal{L}_i) + c$ holds under the constant replacing $\mathcal{L}_i \mapsto \mathcal{L}_i + c\mathcal{X}|_{t=0}$. Setting $\hat{\lambda}_i = N_{i,k}^{-1}(\lambda_{i,1} + \cdots + \lambda_{i,N_{i,k}})$, we can define the L^p-norm

$$\|(\mathcal{X}, \mathcal{L}_i)\|_p = \lim_{k \to \infty} \left[\frac{\sum_{j=1}^{N_{i,k}} |\lambda_{i,j} - \hat{\lambda}_i|^p}{k^p N_{i,k}} \right]^{1/p}$$

which is invariant under the replacing $\mathcal{L}_i \mapsto \mathcal{L}_i + c\mathcal{X}|_{t=0}$. We define the L^p-norm of $(\mathcal{X}, (\mathcal{L})_{i=1}^N)$ by

$$\|(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N)\|_p = \left[\sum_{i=1}^N \|(\mathcal{X}, \mathcal{L}_i)\|_p^p \right]^{1/p}.$$

Finally, we define *the coupled H-invariant* (cf. [15])

$$H_c\Big(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N\Big) = L\Big(\mathcal{X}, \sum_{i=1}^N \mathcal{L}_i\Big) - \sum_{i=1}^N F(\mathcal{X}, \mathcal{L}_i)$$

$$:= L\Big(\mathcal{X}, \sum_{i=1}^N \mathcal{L}_i\Big) - \sum_{i=1}^N \lim_{k\to\infty} \Big[-\log \frac{1}{N_{i,k}} \sum_{j=1}^{N_{i,k}} e^{-\frac{\lambda_{i,j}}{k}} \Big].$$

Note that Jensen's inequality shows $H_c\Big(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N\Big) \geq D_c\Big(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N\Big)$.

2.2 Energy functionals and geodesic rays

We define some energy functionals on $\mathcal{M}$. For $\Phi = (\phi_i)_{i=1}^N \in \prod_{i=1}^N(\text{PSH}(X, \theta_i) \cap L^\infty)$, *the Monge–Ampère energy* is defined by

$$E_{\theta_i}(\phi_i) = \frac{1}{(n+1)V_i} \sum_{k=0}^n \int_X \phi_i \omega_{\phi_i}^k \wedge \theta_i^{n-k}.$$

The functional E_{θ_i} satisfies $E_{\theta_i}(\phi_i + C) = E_{\theta_i}(\phi_i) + C$ for any $C \in \mathbb{R}$. For $\Phi = (\phi_i)_{i=1}^N \in \prod_{i=1}^N(\text{PSH}(X, \theta_i) \cap L^\infty)$, set

$$L(\Phi) = -\log \int_X e^{-\sum_{i=1}^N \phi_i} \theta_0^n$$

where θ_0 is a Kähler metric satisfying $\text{Ric}(\theta_0) = \sum_{i=1}^N \theta_i$ and $\int_X \theta_0^n = 1$. Hultgren–Witt Nyström [28] introduced *the coupled Ding functional*

$$D_c(\Phi) = L(\Phi) - \sum_{i=1}^N E_{\theta_i}(\phi_i).$$

For any smooth $\Phi \in \mathcal{M}$, by using the equality of probability measures

$$\frac{e^{-\sum_i \phi_i} \theta_0^n}{\int_X e^{-\sum_i \phi_i} \theta_0^n} = \frac{e^{f_1(\Phi)} \omega_{\phi_1}^n}{V_1} = \cdots = \frac{e^{f_N(\Phi)} \omega_{\phi_N}^n}{V_N}, \tag{6}$$

we have the first variation formula

$$\delta D_c(\delta\Phi) = -\sum_{i=1}^N \frac{1}{V_i} \int_X \delta\phi_i \Big(1 - e^{f_i(\Phi)}\Big) \omega_{\phi_i}^n \tag{7}$$

which shows that a coupled Kähler–Einstein metric is a critical point of D_c.

In order to relate the invariants of test configurations and the energy functional, we introduce the notion of geodesic rays on the space of Kähler metrics.

Definition 2.4 Let θ is a Kähler form on X and Δ^* be the punctured unit disc in $\mathbb{C}$. We identify θ with its lift to $X \times \Delta^*$. Let $\phi(x, \tau)$ be an upper-semicontinuous, locally integrable, S^1-invariant function on $X \times \Delta^*$ and $\Omega(x, \tau)$ be the $(1,1)$-form on $X \times \Delta^*$ defined by $\theta + \sqrt{-1}\partial\bar{\partial}\phi$. Then, $\phi^t(x) := \phi(x, \tau) \in \text{PSH}(X, \theta)$ with $t = -\log|\tau|^2$ is *a subgeodesic ray* if the restriction of Ω to $X \times \{\tau\}$ is semipositive for all τ. Moreover, $\phi^t(x)$ is *a weak geodesic ray* if it is a subgeodesic ray satisfying $\Omega^{n+1} = 0$ on $X \times \Delta^*$.

The optimal $C^{1,1}$-regularity of the weak geodesic ray is proved by [11] (see also [34]).

For each locally bounded weak geodesic ray $\phi_i^t \in \text{PSH}(X, \theta_i)$, the function $t \mapsto E_{\theta_i}(\phi_i^t)$ is affine [3]. Since $\sum_{i=1}^N \phi_i^t$ is a locally bounded subgeodesic ray in $\text{PSH}(X, \sum_{i=1}^N \theta_i)$, the function $t \mapsto L(\sum_{i=1}^N \phi_i^t)$ is convex [4]. Thus, the coupled Ding functional D_c is convex along $(\phi_i^t)_{i=1}^N$.

In view of [9, 33, 35], a test configuration $(\mathcal{X}, \mathcal{L})$ for a Fano manifold X with an ample line bundle L in the sense of Definition 2.1 defines a weak geodesic ray starting from a given Kähler potential. Note that it is equivalent to the rays constructed in [9, 33, 35], since it is known the uniqueness theorem for the completely degenerate complex Monge–Ampère equation [34]. It follows from an argument in [28, page 6786] that a test configuration $(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N)$ for $(X, (\alpha_i)_{i=1}^N)$ in the sense of Definition 2.3 and a collection of given Kähler potentials $(\phi_i^0)_{i=1}^N \in \mathcal{M}$ induce a collection of weak geodesic rays $(\phi_i^t)_{i=1}^N \in \prod_{i=1}^N \text{PSH}(X, \theta_i)$ for $t \geq 0$.

The slopes at infinity of energy functionals along $(\phi_i^t)_{i=1}^N$ play an important role in the moment weight type inequalities we are interested in. Results of Berman [2] showed

$$E(\mathcal{X}, \mathcal{L}_i) = \lim_{t \to \infty} \frac{E_{\theta_i}(\phi_i^t)}{t} \quad \text{and} \quad D_c(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N) = \lim_{t \to \infty} \frac{D_c((\phi_i^t)_{i=1}^N)}{t}.$$

By the $C^{1,1}$-regularity of the weak geodesic ray ϕ_i^t, the existence of the time derivative $\dot{\phi}_i^t$ is guaranteed. Berndtsson [5] showed that the push forward probability measure $\text{DH}(\mathcal{X}, \mathcal{L}_i) := (\dot{\phi}_i^t)_*(V_i^{-1}\omega_{\phi_i^t}^n)$ on $\mathbb{R}$ is independent of t. Hisamoto [24] showed that the weak convergence of the spectral measure

$$\frac{1}{N_{i,k}} \sum_{j=1}^{N_{i,k}} \delta_{\frac{\lambda_{i,j}}{k}} \to \text{DH}(\mathcal{X}, \mathcal{L}_i)$$

as $k \to \infty$ to obtain the equality

$$\|(\mathcal{X}, \mathcal{L}_i)\|_p = \left[\frac{1}{V_i} \int_X \left| \dot{\phi}_i^t - E(\mathcal{X}, \mathcal{L}_i) \right|^p \omega_{\phi_i^t}^n \right]^{1/p}.$$

Consider the "virtual slope"

$$F(\dot{\phi}_i^t) = -\log \frac{1}{V_i} \int_X e^{-\dot{\phi}_i^t} \omega_{\phi_i}^n = -\log \int_{\mathbb{R}} e^{-x} \text{DH}(\mathcal{X}, \mathcal{L}_i)$$

to obtain the slope formula

$$H_c\left(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N\right) = \lim_{t \to \infty} \left[\frac{L(\sum_{i=1}^N \phi_i^t)}{t} - \sum_{i=1}^N F(\dot{\phi}_i^t) \right].$$

2.3 Proof of Theorem 1.1

Proof of the moment weight type inequality for R_c. Fix any $\Phi = (\phi_i)_{i=1}^N \in \mathcal{M}$ and any test configuration $(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N)$ for a decomposition $(\alpha_i)_{i=1}^N$ of $2\pi c_1(X)$. Take weak geodesic rays $(\phi_i^t)_{i=1}^N$ for $t \geq 0$ starting from $(\phi_i)_{i=1}^N$ associated with $(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N)$.

By the convexity of the coupled Ding functional,

$$-D(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N) = \lim_{t \to \infty} \frac{-D_c((\phi_i^t)_{i=1}^N)}{t} \leq -\frac{d}{dt}\bigg|_{t=0} D_c((\phi_i^t)_{i=1}^N).$$

By the equality (7), the normalization of the Ricci potentials and the Schwartz inequality,

$$
-\frac{d}{dt}\Big|_{t=0} D_c((\phi_i^t)_{i=1}^N) = \sum_{i=1}^N \frac{1}{V_i} \int_X \dot{\phi}_i^0 \Big(1 - e^{f_i(\Phi)}\Big) \omega_{\phi_i}^n
$$

$$
= \sum_{i=1}^N \frac{1}{V_i} \int_X \Big(\dot{\phi}_i^0 - E(\mathcal{X}, \mathcal{L}_i)\Big)\Big(1 - e^{f_i(\Phi)}\Big) \omega_{\phi_i}^n
$$

$$
\leq \Big[\sum_{i=1}^N \frac{1}{V_i} \int_X \big|\dot{\phi}_i^0 - E(\mathcal{X}, \mathcal{L}_i)\big|^2 \omega_{\phi_i^0}^n\Big]^{1/2}\Big[\sum_{i=1}^N \frac{1}{V_i} \int_X \big(1 - e^{f_i(\Phi)}\big)^2 \omega_{\phi_i}^n\Big]^{1/2}
$$

$$
= \|(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N)\|_2 R_c(\Phi)^{1/2}.
$$

This completes the proof.

Before a proof of the moment weight type inequality for the coupled H-functional H_c, we give some remarks. For two probability measures μ and ν on X, the relative entropy is defined by

$$
H(\mu|\nu) = \int_X \log\Big(\frac{\mu}{\nu}\Big)\mu.
$$

In this terminology, $H_c(\Phi)$ is written as $\sum_{i=1}^N H(\mu_\Phi|\nu_{\phi_i})$ where μ_Φ is one of the probability measures in the equality 6), that is,

$$
H_c(\Phi) = H\Big(\frac{e^{f_1(\Phi)}\omega_{\phi_1}^n}{V_1}\Big|\nu_{\phi_1}\Big) + \cdots + H\Big(\frac{e^{f_N(\Phi)}\omega_{\phi_N}^n}{V_N}\Big|\nu_{\phi_N}\Big)
$$

and where $\nu_{\phi_i} = \omega_{\phi_i}^n/V_i$. Note that the Csiszár–Kullback–Pinsker inequality yields the inequality

$$
\sqrt{2H\Big(\frac{e^{f_i(\Phi)}\omega_{\phi_i}^n}{V_i}\Big|\nu_{\phi_i}\Big)} \geq \frac{1}{V_i} \int_X \big|1 - e^{f_i(\Phi)}\big|\omega_{\phi_i}^n \tag{8}
$$

which shows that a zero point of H_c is a coupled Kähler–Einstein metric. Note also that the relative entropy has an expression in terms of the Legendre duality as follows [1].

$$
H(\mu|\nu) = \sup_{f \in C^0(X;\mathbb{R})} \Big(\int_X f\mu - \log\int_X e^f\nu\Big).
$$

Proof of the moment weight type inequality for H_c. We use the same notation as in the previous proof. By using the Legendre duality expression and the convexity of the function $L(\sum_{i=1}^N \phi_i^t)$ for $t \geq 0$, we have

$$
H_c(\Phi) \geq \sum_{i=1}^N \Big(\int_X -\dot{\phi}_i^0\mu_\Phi - \log\int_X e^{-\dot{\phi}_i^0}\nu_{\phi_i}\Big)
$$

$$
= -\frac{d}{dt}\Big|_{t=0} L\Big(\sum_{i=1}^N \phi_i^t\Big) + \sum_{i=1}^N F(\dot{\phi}_i^0)
$$

$$
\geq -\frac{L(\sum_{i=1}^N \phi_i^t)}{t} + \sum_{i=1}^N F(\dot{\phi}_i^t).
$$

Taking $t \to \infty$, we get $H_c(\Phi) \geq -H_c(\mathcal{X}, (\mathcal{L}_i)_{i=1}^N)$. This completes the proof.

3 Hessian formulas and its application

3.1 Hessian formulas

We fix notations to obtain Hessian formulas for the coupled Ricci–Calabi functional R_c and the coupled H-functional H_c. For any $\Phi = (\phi_i)_{i=1}^N \in \mathcal{M}$, we write one of the probability measures in the equality (6) as μ_Φ. Let Δ_{ϕ_i} be the negative Laplacian of the metric ϕ_i, and let P_Φ and P_Φ^f be operators acting on $(C^\infty(X; \mathbb{C}))^N$ defined by

$$P_\Phi(\boldsymbol{u}) = \left(-\Delta_{\phi_i} u_i - \langle \overline{\partial} u_i, \overline{\partial} f_i(\Phi) \rangle_{\phi_i} - \sum_{j=1}^N u_j + \int_X \sum_{j=1}^N u_j \mu_\Phi \right)_{i=1}^N \tag{9}$$

and

$$P_\Phi^f(\boldsymbol{u}) = \left(\left(-\Delta_{\phi_i} u_i - \langle \overline{\partial} u_i, \overline{\partial} f_i(\Phi) \rangle_{\phi_i} - \sum_{j=1}^N u_j + \int_X \sum_{j=1}^N u_j \mu_\Phi \right) e^{f_i(\Phi)} \right)_{i=1}^N \tag{10}$$

for $\boldsymbol{u} = (u_i)_{i=1}^N \in (C^\infty(X; \mathbb{C}))^N$. Their complex conjugates are defined by

$$\overline{P_\Phi}(\boldsymbol{u}) = \overline{P_\Phi(\overline{\boldsymbol{u}})} \quad \text{and} \quad \overline{P_\Phi^f}(\boldsymbol{u}) = \overline{P_\Phi^f(\overline{\boldsymbol{u}})}.$$

Recall the Hermitian inner products $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi^f$ on $(C^\infty(X; \mathbb{C}))^N$ are defined by

$$\langle\!\langle \boldsymbol{u}, \boldsymbol{v} \rangle\!\rangle_\Phi = \sum_{i=1}^N \int_X u_i \overline{v_i} \frac{\omega_{\phi_i}^n}{V_i} \quad \text{and} \quad \langle\!\langle \boldsymbol{u}, \boldsymbol{v} \rangle\!\rangle_\Phi^f = \sum_{i=1}^N \int_X u_i \overline{v_i} \mu_\Phi.$$

The followings are basic properties for the operator P_Φ and the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi^f$.

Proposition 3.1 [37, Proposition 2.4]

(1) *The operator P_Φ is self-adjoint with respect to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi^f$.*
(2) *The operator P_Φ is non-negative, and the Kernel $\mathrm{Ker}\, P_\Phi$ is equal to*

$$\left\{ (u_i)_{i=1}^N \in (C^\infty(X; \mathbb{C}))^N \;\middle|\; \mathrm{grad}_{\phi_1} u_1 = \cdots = \mathrm{grad}_{\phi_N} u_N =: V \text{ and } V \text{ is holomorphic} \right\},$$

where $\mathrm{grad}_{\phi_i} u_i$ is a type $(1, 0)$ gradient vector field on X defined by

$$i_{(\mathrm{grad}_{\phi_i} u_i)} \omega_{\phi_i} = \sqrt{-1} \overline{\partial} u_i.$$

Note that the same properties as in the above proposition holds for P_Φ^f and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi$.

We give the first variation formula of the Ricci potential to obtain that of R_c and H_c. Note that the variation $\delta\Phi$ of $\Phi \in \mathcal{M}$ is in $T_\Phi \mathcal{M}$ and it is identified with an element in $(C^\infty(X; \mathbb{R}))^N$.

Lemma 3.2 *The first variation in the Ricci potential at $\Phi = (\phi_i)_{i=1}^N \in \mathcal{M}$ is given by*

$$\delta f_i(\delta\Phi) = -\Delta_{\phi_i} \delta\phi_i - \sum_{j=1}^N \delta\phi_j + \int_X \sum_{j=1}^N \delta\phi_j \mu_\Phi \tag{11}$$

for any variation $\delta\Phi = (\delta\phi_i)_{i=1}^N \in T_\Phi \mathcal{M}$.

Proof The derivation of the first equation in (1) shows $\delta f_i(\delta\Phi) = -\Delta_{\phi_i}\delta\phi_i - \sum_{j=1}^{N}\delta\phi_j + C$ for some constant C. The constant C is equal to $\int_X \sum_{j=1}^{N}\delta\phi_j\mu_\Phi$ since

$$0 = \int_X \delta(e^{f_i(\Phi)}\omega_{\phi_i})/V_i = \int_X \delta f_i(\delta\Phi)\mu_\Phi + \int_X \Delta_{\phi_i}\delta\phi_i\mu_\Phi.$$

$\square$

Lemma 3.3 *The first variations in R_c and H_c at $\Phi \in \mathcal{M}$ are given by*

$$\delta R_c(\delta\Phi) = 2\langle\!\langle P_\Phi^f(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)}), \delta\Phi\rangle\!\rangle_\Phi = 2\langle\!\langle \overline{P_\Phi^f}(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)}), \delta\Phi\rangle\!\rangle_\Phi$$

and

$$\delta H_c(\delta\Phi) = \langle\!\langle P_\Phi(f_1(\Phi), \ldots, f_N(\Phi)), \delta\Phi\rangle\!\rangle_\Phi^f = \langle\!\langle \overline{P_\Phi}(f_1(\Phi), \ldots, f_N(\Phi)), \delta\Phi\rangle\!\rangle_\Phi^f$$

for any variation $\delta\Phi \in T_\Phi\mathcal{M}$.

Remark 3.4 In Lemma 3.3, the first variation in the coupled Ricci–Calabi functional is also expressed as $2\langle\!\langle P_\Phi(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)}), \delta\Phi\rangle\!\rangle_\Phi^f$. However, the expression in Lemma 3.3 is technically crucial for the proof of Theorem 1.4.

Proof For any variation $\delta\Phi = (\delta\phi_i)_{i=1}^N$, direct computations together with Lemma 3.2 show

$$\delta\int_X (1 - e^{f_i(\Phi)})^2\frac{\omega_{\phi_i}^n}{V_i} = \int_X 2e^{f_i(\Phi)}\delta f_i(\Phi)\mu_\Phi + \int_X e^{f_i(\Phi)}(\Delta_{\phi_i}\delta\phi_i)\mu_\Phi$$

$$= -\int_X e^{f_i(\Phi)}\Delta_{\phi_i}\delta\phi_i\mu_\Phi - 2\int_X e^{f_i(\Phi)}\sum_{j=1}^{N}\delta\phi_j\mu_\Phi$$

$$+2\int_X e^{f_i(\Phi)}\mu_\Phi\int_X\sum_{j=1}^{N}\delta\phi_j e^{f_i(\Phi)}\mu_\Phi$$

and

$$\delta\int_X f_i(\Phi)e^{f_i(\Phi)}\frac{\omega_{\phi_i}^n}{V_i} = \int_X \delta f_i(\delta\Phi)\mu_\Phi + \int_X f_i(\Phi)\delta f_i(\delta\Phi)\mu_\Phi + \int_X f_i(\Phi)(\Delta_{\phi_i}\delta\phi_i)\mu_\Phi$$

$$= -\int_X \Delta_{\phi_i}\delta\phi_i\mu_\Phi - \int_X f_i(\Phi)\sum_{j=1}^{N}\delta\phi_j\mu_\Phi + \int_X f_i(\Phi)\mu_\Phi\int_X\sum_{j=1}^{N}\delta\phi_j\mu_\Phi.$$

By the integration by parts, we have

$$\int_X e^{f_i(\Phi)}\Delta_{\phi_i}\delta\phi_i\mu_\Phi = 2\int_X \delta\phi_i(\Delta_{\phi_i}e^{f_i(\Phi)} + \langle\bar{\partial}e^{f_i(\Phi)}, \bar{\partial}f_i(\Phi)\rangle_{\phi_i})\mu_\Phi$$

to show that

$$\delta R_c(\delta\Phi) = 2\sum_{i=1}^{N}\int_X \delta\phi_i(-\Delta_{\phi_i}e^{f_i(\Phi)} - \langle\bar{\partial}e^{f_i(\Phi)}, \bar{\partial}f_i(\Phi)\rangle_{\phi_i})\mu_\Phi$$

$$-2\int_X\sum_{j=1}^{N}e^{f_j(\Phi)}\sum_{i=1}^{N}\delta\phi_i\mu_\Phi + 2\int_X\sum_{j=1}^{N}e^{f_j(\Phi)}\mu_\Phi\int_X\sum_{i=1}^{N}\delta\phi_i\mu_\Phi$$

$$= 2\langle\!\langle P_\Phi^f(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)}), \delta\Phi\rangle\!\rangle_\Phi.$$

On the other hand, by the integration by parts, we have

$$\int_X \Delta_{\phi_i} \delta\phi_i \mu_\Phi = \int_X \delta\phi_i (\Delta_{\phi_i} f_i(\Phi) + \langle \overline{\partial} f_i(\Phi), \overline{\partial} f_i(\Phi) \rangle_{\phi_i}) \mu_\Phi,$$

to show that

$$\delta H_c(\delta\Phi) = \sum_{i=1}^{N} \int_X \delta\phi_i (-\Delta_{\phi_i} f_i(\Phi) - \langle \overline{\partial} f_i(\Phi), \overline{\partial} f_i(\Phi) \rangle) \mu_\Phi$$

$$- \int_X \sum_{j=1}^{N} f_j(\Phi) \sum_{i=1}^{N} \delta\phi_i \mu_\Phi + \int_X \sum_{j=1}^{N} f_j(\Phi) \mu_\Phi \int_X \sum_{i=1}^{N} \delta\phi_i \mu_\Phi$$

$$= \langle\!\langle P_\Phi(f_1(\Phi), \ldots, f_N(\Phi)), \delta\Phi \rangle\!\rangle_\Phi^f.$$

Similarly, we have

$$\int_X e^{f_i(\Phi)} \Delta_{\phi_i} \delta\phi_i \mu_\Phi = 2 \int_X \delta\phi_i (\Delta_{\phi_i} e^{f_i(\Phi)} + \overline{\langle \overline{\partial} e^{f_i(\Phi)}, \overline{\partial} f_i(\Phi) \rangle_{\phi_i}}) \mu_\Phi$$

and

$$\int_X \Delta_{\phi_i} \delta\phi_i \mu_\Phi = \int_X \delta\phi_i (\Delta_{\phi_i} f_i(\Phi) + \overline{\langle \overline{\partial} f_i(\Phi), \overline{\partial} f_i(\Phi) \rangle_{\phi_i}}) \mu_\Phi$$

which yield the equalities

$$\delta R_c(\delta\Phi) = 2 \overline{\langle\!\langle P_\Phi^f(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)}), \delta\Phi \rangle\!\rangle_\Phi}$$

and

$$\delta H_c(\delta\Phi) = \langle\!\langle \overline{P_\Phi}(f_1(\Phi), \ldots, f_N(\Phi)), \delta\Phi \rangle\!\rangle_\Phi^f.$$

This completes the proof.   □

Therefore, Proposition 3.1 and Lemma 3.3 show that a pair $\Phi = (\phi_i)_{i=1}^{N} \in \mathcal{M}$ is a coupled Mabuchi soliton if and only if the vector fields $\mathrm{grad}_{\phi_i} e^{f_i(\Phi)}$ are holomorphic and $\mathrm{grad}_{\phi_1} e^{f_1(\Phi)} = \cdots = \mathrm{grad}_{\phi_N} e^{f_N(\Phi)}$. Similarly, $\Phi = (\phi_i)_{i=1}^{N} \in \mathcal{M}$ is a coupled Kähler–Ricci soliton if and only if the vector fields $\mathrm{grad}_{\phi_i} f_i(\Phi)$ are holomorphic and $\mathrm{grad}_{\phi_1} f_1(\Phi) = \cdots = \mathrm{grad}_{\phi_N} f_N(\Phi)$.

Now we prove the Hessian formulas. The following argument generalizes the authors's one in [30].

Proof of Theorem 1.4 We first compute the variation $(\delta_\Phi P_\Phi^f)(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)})$ at a coupled Mabuchi soliton $\Phi = (\phi_i)_{i=1}^{N} \in \mathcal{M}$ to obtain the Hessian of the coupled Ricci–Calabi functional

$$\frac{1}{2} \mathrm{Hess}(R_c)(\delta\Phi, \delta\Psi) = \langle\!\langle \delta_\Phi(P_\Phi^f(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)})), \delta\Psi \rangle\!\rangle_\Phi$$

$$= \langle\!\langle (\delta_\Phi P_\Phi^f)(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)}), \delta\Psi \rangle\!\rangle_\Phi$$

$$+ \langle\!\langle P_\Phi^f(\delta_\Phi(e^{f_1(\Phi)}), e^{f_2(\Phi)}, \ldots, e^{f_N(\Phi)}), \delta\Psi \rangle\!\rangle_\Phi$$

$$+ \cdots$$

$$+ \langle\!\langle P_\Phi^f(e^{f_1(\Phi)}, \ldots, e^{f_{N-1}(\Phi)}, \delta_\Phi(e^{f_N(\Phi)})), \delta\Psi \rangle\!\rangle_\Phi$$

where δ_Φ stands for the variation along $\delta\Phi = (\delta\phi_i)_{i=1}^N$ at Φ and $\delta\Psi$ is another variation at Φ. Now we have the holomorphic vector field

$$Z_R := \mathrm{grad}_{\phi_1} e^{f_1(\Phi)} = \cdots = \mathrm{grad}_{\phi_N} e^{f_N(\Phi)},$$

since $\Phi = (\phi_i)_{i=1}^N$ is a coupled Mabuchi soliton. Note that Z_R is also expressed as

$$Z_R = \mathrm{grad}_{\phi_1 + t\delta\phi_1}(e^{f_1(\Phi)} + t Z_R(\delta\phi_1)) = \cdots = \mathrm{grad}_{\phi_N + t\delta\phi_N}(e^{f_1(\Phi)} + t Z_R(\delta\phi_N)).$$

Indeed, the equality $i_{Z_R}(\omega_{\phi_i} + t\sqrt{-1}\partial\bar\partial\delta\phi_i) = \sqrt{-1}\partial(e^{f_i(\Phi)} + t Z_R(\delta\phi_i))$ holds for each i and for any small $t \in (-\varepsilon, \varepsilon)$. Set $\Phi_t = (\phi_i + t\delta\phi_i)_{i=1}^N \in \mathcal{M}$ as a perturbation of Φ. Since $Z_R \in \mathrm{Ker}\, P_{\Phi_t}^f$ by Proposition 3.1, we then have

$$P_{\Phi_t}^f(e^{f_1(\Phi)} + t Z_R(\delta\phi_1), \ldots, e^{f_N(\Phi)} + t Z_R(\delta\phi_N)) = 0.$$

Thus, the derivative of the above equation at $t = 0$ yields the equality

$$\begin{aligned}
(\delta_\Phi P_\Phi^f)(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)}) &= -P_\Phi^f(Z_R(\delta\phi_1), e^{f_2(\Phi)}, \ldots, e^{f_N(\Phi)}) \\
&\quad - \cdots \\
&\quad - P_\Phi^f(e^{f_1(\Phi)}, \ldots, e^{f_{N-1}(\Phi)}, Z_R(\delta\phi_N)).
\end{aligned} \tag{12}$$

Therefore, by using Eq. (12), the formula $Z_R(\delta\phi_i) = \overline{\langle\bar\partial\delta\phi_i, \bar\partial(e^{f_i(\Phi)})\rangle}_{\phi_i}$ and the formula of the derivative of $e^{f_i(\Phi)}$ in Lemma 3.2, we obtain

$$\begin{aligned}
&\delta_\Phi(P_\Phi^f(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)})) \\
&= P_\Phi^f(-\overline{\langle\bar\partial\delta\phi_1, \bar\partial(e^{f_1(\Phi)})\rangle}_{\phi_1} + \delta_\Phi(e^{f_1(\Phi)}), \ldots, -\overline{\langle\bar\partial\delta\phi_N, \bar\partial(e^{f_N(\Phi)})\rangle}_{\phi_N} + \delta_\Phi(e^{f_N(\Phi)})) \\
&= P_\Phi^f \overline{P_\Phi^f}(\delta\Phi).
\end{aligned}$$

Similarly, $\delta_\Phi(\overline{P_\Phi^f}(e^{f_1(\Phi)}, \ldots, e^{f_N(\Phi)})) = \overline{P_\Phi^f} P_\Phi^f(\delta\Phi)$. This completes the proof of the Hessian formula for the coupled Ricci–Calabi functional.

In order to prove the Hessian formula for the coupled H-functional H_c, we follow the same argument as above to obtain

$$\begin{aligned}
(\delta_\Phi P_\Phi)(f_1(\Phi), \ldots, f_N(\Phi)) &= -P_\Phi(Z_H(\delta\phi_1), f_2(\Phi), \ldots, f_N(\Phi)) \\
&\quad - \cdots \\
&\quad - P_\Phi(f_1(\Phi), \ldots, f_{N-1}(\Phi), Z_H(\delta\phi_N))
\end{aligned} \tag{13}$$

where $\Phi = (\phi_i)_{i=1}^N \in \mathcal{M}$ is a coupled Kähler–Ricci soliton, Z_H is the holomorphic vector field $\mathrm{grad}_{\phi_1} f_1(\Phi) = \cdots = \mathrm{grad}_{\phi_N} f_N(\Phi)$ and $\delta\Phi := (\delta\phi_i)_{i=1}^N$ is a variation at Φ. By equation (13), the formula $Z_H(\delta\phi_i) = \overline{\langle\bar\partial\delta\phi_i, \bar\partial(f_i(\Phi))\rangle}_{\phi_i}$ and Lemma 3.2, we have

$$\begin{aligned}
\mathrm{Hess}(R_c)(\delta\Phi, \delta\Psi) &= \langle\!\langle(\delta_\Phi P_\Phi)(f_1(\Phi), \ldots, f_N(\Phi)), \delta\Psi\rangle\!\rangle_\Phi^f \\
&\quad + \langle\!\langle P_\Phi(\delta_\Phi f_1(\Phi), f_2(\Phi), \ldots, f_N(\Phi)), \delta\Psi\rangle\!\rangle_\Phi^f \\
&\quad + \cdots \\
&\quad + \langle\!\langle P_\Phi(f_1(\Phi), \ldots, f_{N-1}(\Phi), \delta_\Phi f_N(\Phi)), \delta\Psi\rangle\!\rangle_\Phi^f \\
&= \langle\!\langle P_\Phi \overline{P_\Phi}(\delta\Phi), \delta\Psi\rangle\!\rangle_\Phi^f,
\end{aligned}$$

where $\delta\Psi$ is another variation at Φ. Similarly $\mathrm{Hess}(H_c)(\delta\Phi, \delta\Psi) = \langle\!\langle\overline{P_\Phi} P_\Phi(\delta\Phi), \delta\Psi\rangle\!\rangle_\Phi^f$. This completes the proof of Theorem 1.4.

Proof of Corollary 1.5 This is a consequence of Theorem 1.4 and the non-negativity of P_Φ (resp. P_Φ^f) with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi^f$ (resp. $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Phi$) in Proposition 3.1.

To end this subsection, we discuss the coupled flows introduced in Sect. 1. Lemma 3.3 shows the following.

Corollary 3.5 *The coupled Ricci–Calabi functional R_c is monotonically decreasing along a coupled inverse Monge–Ampére flow in the sense of* (3). *The coupled H-functional H_c is monotonically decreasing along a coupled Kähler–Ricci flow in the sense of* (2).

Proof Set $F_R = F_R(t) = (e^{f_i(\Phi_R(t))})_{i=1}^N$ where $\Phi_R = \Phi_R(t) \in \mathcal{M}$ is a coupled inverse Monge–Ampére flow. Set $F_H = F_H(t) = (f_i(\Phi_H(t))_{i=1}^N$ where $\Phi_H = \Phi_H(t) \in \mathcal{M}$ is a coupled Kähler–Ricci flow. Lemma 3.3 shows

$$\frac{\mathrm{d}}{\mathrm{d}t} R_c(\Phi_R) = -2\langle\!\langle P_{\Phi_R}^f(F_R), F_R \rangle\!\rangle_{\Phi_R} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} H_c(\Phi_H) = -\langle\!\langle P_{\Phi_H}(F_H), F_H \rangle\!\rangle_{\Phi_H}^f$$

which are both non-positive by Proposition 3.1. This completes the proof. □

Since a coupled Mabuchi soliton is a self-similar solution of a coupled inverse Monge–Ampére flow $\Phi_R(t)$, Corollary 1.5 suggests that for a Fano manifold admitting a coupled Mabuchi soliton, this flow $\Phi_R(t)$ starting from any metric converges to a coupled Mabuchi soliton in some sense. The same statement for a coupled Kähler–Ricci flow and a coupled Kähler–Ricci soliton is expected to hold.

3.2 An application to Matsushima type obstruction theorem

As an application of the commutativity of the operators in Corollary 1.5, we prove Theorem 1.6.

Proof of Theorem 1.6 We first fix a coupled Mabuchi soliton $\Phi = (\phi_i)_{i=1}^N$. Since operators P_Φ^f and $\overline{P_\Phi^f}$ are commutative by Corollary 1.5, $\overline{P_\Phi^f} \in \mathrm{End}(\mathrm{Ker}\, P_\Phi^f)$. Let E_λ be the λ-eigenspace of $\overline{P_\Phi^f}$ in $\mathrm{Ker}\, P_\Phi^f$. Then, we have

$$\mathrm{Ker}\, P_\Phi^f = \sum_{\lambda \geq 0} E_\lambda. \tag{14}$$

Note that, by Proposition 3.1, every $V \in \mathfrak{h}(X)$ is written as $V = \mathrm{grad}_{\phi_1} u_1 = \cdots = \mathrm{grad}_{\phi_N} u_N$ for some $(u_i)_{i=1}^N \in \mathrm{Ker}\, P_\Phi^f$ which are unique up to additive constants. Thus, setting

$$\mathfrak{h}_\lambda(X) = \left\{ V \in \mathfrak{h}(X) \;\middle|\; V = \mathrm{grad}_{\phi_1} u_1 = \cdots = \mathrm{grad}_{\phi_N} u_N \quad \text{and} \quad (u_i)_{i=1}^N \in E_\lambda \right\}$$

and using the relation (14), we have the decomposition

$$\mathfrak{h}(X) = \sum_{\lambda \geq 0} \mathfrak{h}_\lambda(X). \tag{15}$$

Here we claim that $\mathfrak{h}_\lambda(X)$ is the λ-eigenspace of the adjoint action of $Z_R := \mathrm{grad}_{\phi_1}(1 - e^{f_1(\Phi)}) = \cdots = \mathrm{grad}_{\phi_N}(1 - e^{f_N(\Phi)})$. To see this, fix an element $V = \mathrm{grad}_{\phi_1} u_1 = \cdots = \mathrm{grad}_{\phi_N} u_N$ in $\mathfrak{h}_\lambda(X)$ where $(u_i)_{i=1}^N \in E_\lambda$ and observe

$\lambda(u_1, \ldots, u_N)$

$$= \overline{P_\Phi^f}(u_1, \ldots, u_N)$$
$$= (\overline{P_\Phi^f} - P_\Phi^f)(u_1, \ldots, u_N)$$
$$= (-\overline{\langle \overline{\partial} \overline{u}_1, \overline{\partial} e^{f_1(\Phi)} \rangle}_{\phi_1} + \langle \overline{\partial} u_1, \overline{\partial} e^{f_1(\Phi)} \rangle_{\phi_1}, \ldots, -\overline{\langle \overline{\partial} \overline{u}_N, \overline{\partial} e^{f_N(\Phi)} \rangle}_{\phi_N} + \langle \overline{\partial} u_N, \overline{\partial} e^{f_N(\Phi)} \rangle_{\phi_N})$$
$$= (\{e^{f_1(\Phi)}, u_1\}_{\phi_1}, \ldots, \{e^{f_N(\Phi)}, u_N\}_{\phi_N})$$

where $\{\cdot, \cdot\}_{\phi_i}$ is the Poisson bracket defined by $\{u, v\}_{\phi_i} = (\text{grad}_{\phi_i} v)u - (\text{grad}_{\phi_i} u)v$. Since the map $u \to -\text{grad}_{\phi_i} u$ is a complex Lie algebra homomorphism from $(C^\infty(X; \mathbb{C}), \{\cdot, \cdot\}_{\phi_i})$ to $(\Gamma(TX), [\cdot, \cdot])$ where $[\cdot, \cdot]$ is the Lie bracket defined by $[Z, W] = ZW - WZ$, it follows that

$$\lambda V = \text{grad}_{\phi_i} \lambda u_i$$
$$= \text{grad}_{\phi_i} \{e^{f_i(\Phi)}, u\}_{\phi_i}$$
$$= [-\text{grad}_{\phi_i} e^{f_i(\Phi)}, \text{grad}_{\phi_i} u]$$
$$= [Z_R, V].$$

This shows $\mathfrak{h}_\lambda(X)$ is the λ-eigenspace of the adjoint action of Z_R.

We next focus on $\mathfrak{h}_0(X)$. Since $E_0 = \text{Ker}\, P_\Phi^f \cap \text{Ker}\, \overline{P_\Phi^f}$, then both the real part $(\text{Re}\, u_i)_{i=1}^N$ and the imaginary part $(\text{Im}\, u_i)_{i=1}^N$ of $(u_i)_{i=1}^N \in E_0$ are in E_0 again. It follows that

$$E_0 = \left\{ (u_i)_{i=1}^N \in (\sqrt{-1} C^\infty(X; \mathbb{R}))^N \,\middle|\, \text{grad}_{\phi_1} u_1 = \cdots = \text{grad}_{\phi_N} u_N \in \mathfrak{h}(X) \right\} \otimes \mathbb{C}$$

and thus

$$\mathfrak{h}_0(X) = \left\{ V + \overline{V} \,\middle|\, (u_i)_{i=1}^N \in (\sqrt{-1} C^\infty(X; \mathbb{R}))^N \quad \text{and} \quad V = \text{grad}_{\phi_1} u_1 = \cdots = \text{grad}_{\phi_N} u_N \right\} \otimes \mathbb{C}.$$

According to [18, Lemma 2.3.8], the vector filed $V + \overline{V} = \text{grad}_{\phi_i} u_i + \overline{\text{grad}_{\phi_i} u_i}$ as above is killing with respect to the Kähler metric ω_{ϕ_i}. Therefore, $\mathfrak{h}_0(X)$ is the complexification of the Lie algebra of Killing vector fields for ω_{ϕ_i}. This completes the proof for the case when $\Phi = (\phi_i)_{i=1}^N$ is a coupled Mabuchi soliton.

When $\Phi = (\phi_i)_{i=1}^N$ is a coupled Kähler–Ricci soliton, we follow the same argument as above to obtain the decomposition

$$\mathfrak{h}(X) = \sum_{\lambda \geq 0} \mathfrak{h}_\lambda(X),$$

where $\mathfrak{h}_\lambda(X) = \left\{ V \in \mathfrak{h}(X) \,\middle|\, V = \text{grad}_{\phi_1} u_1 = \cdots = \text{grad}_{\phi_N} u_N \quad \text{and} \quad (u_i)_{i=1}^N \in E_\lambda \right\}$ and E_λ is the λ-eigenspace of $\overline{P_\Phi}$ in $\text{Ker}\, P_\Phi$. In order to finish the proof, we only check that $\mathfrak{h}_\lambda(X)$ is the λ-eigenspace of the adjoint action of $Z_H := -\text{grad}_{\phi_1} f_1(\Phi) = \cdots = -\text{grad}_{\phi_N} f_N(\Phi)$. For any $V = \text{grad}_{\phi_1} u_1 = \cdots = \text{grad}_{\phi_N} u_N$ in $\mathfrak{h}_\lambda(X)$ where $(u_i)_{i=1}^N \in E_\lambda$, we have

$$\lambda(u_1, \ldots, u_N)$$
$$= \overline{P_\Phi}(u_1, \ldots, u_N)$$
$$= (\overline{P_\Phi} - P_\Phi)(u_1, \ldots, u_N)$$
$$= (-\overline{\langle \overline{\partial} \overline{u}_1, \overline{\partial} f_1(\Phi) \rangle}_{\phi_1} + \langle \overline{\partial} u_1, \overline{\partial} f_1(\Phi) \rangle_{\phi_1}, \ldots, -\overline{\langle \overline{\partial} \overline{u}_N, \overline{\partial} f_N(\Phi) \rangle}_{\phi_N} + \langle \overline{\partial} u_N, \overline{\partial} f_N(\Phi) \rangle_{\phi_N})$$
$$= (\{f_1(\Phi), u_1\}_{\phi_1}, \ldots, \{f_N(\Phi), u_N\}_{\phi_N}).$$

This yields the equality $\lambda V = [Z_H, V]$. The remaining proof is very similar to the case of coupled Mabuchi solitons.

References

1. Berman, R.: A thermodynamical formalism for Monge–Ampère equations, Moser–Trudinger inequalities and Kähler–Einstein metrics. Adv. Math. **248**, 1254–1297 (2013)
2. Berman, R.: K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler–Einstein metrics. Invent. Math. **203**(3), 973–1025 (2016)
3. Berman, R., Boucksom, S., Guedj, V., Zeriahi, A.: A variational approach to complex Monge–Ampère equations. Publ. Math. Inst. Hautes Études Sci. **117**, 179–245 (2013)
4. Berndtsson, B.: A Brunn–Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. Invent. Math. **200**(1), 149–200 (2015)
5. Berndtsson, B.: Probability measures associated to geodesics in the space of Kähler metrics, Algebraic and analytic microlocal analysis. Springer Proc. Math. Stat., 269, pp. 395–419. Springer, Cham (2018)
6. Calabi, E.: Extremal Kähler metrics, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, pp. 259–290. Princeton University Press, Princeton (1982)
7. Calabi, E.: Extremal Kähler metrics II. In: Chavel, I., Farkas, H.M. (eds.) Differential Geometry and Complex Analysis, pp. 95–114. Springer, Berline (1985)
8. Chen, X., Sun, S., Wang, B.: Kähler–Ricci flow, Kähler–Einstein metric, and K-stability. Geom. Topol. **22**, 3145–3173 (2018)
9. Chen, X.X., Tang, Y.: Test configuration and geodesic rays. Astérisque **321**, 139–167 (2008)
10. Chen, X., Wang, B.: Space of Ricci flows (II). J. Differential Geom. **116**, 1–123 (2020)
11. Chu, J., Tossatti, V., Weikove, B.: $C^{1,1}$-regularity for degenerate complex Monge-Ampère equations and geodesic rays. Comm. Partial Differential Equations **43**(2), 292–312 (2018)
12. Collins, T., Hisamoto, T., Takahashi, R.: The inverse Monge–Ampère flow and application to Kähler–Einstein metrics. J. Differential Geom. **120**, 51–95 (2022)
13. Datar, V., Pingali, V.P.: On coupled constant scalar curvature Kähler metrics. J. Symplectic Geom. **18**(4), 961–994 (2020)
14. Delcroix, T., Hultgren, J.: Coupled complex Monge–Ampère equation on Fano horosymmetric manifolds. J. Math. Pures Appl. **9**(153), 281–315 (2021)
15. Dervan, R., Székelyhidi, G.: The Kähler–Ricci flow and optimal degenerations. J. Differential Geom. **116**, 187–203 (2020)
16. Donaldson, S.K.: Lower bounds on the Calabi functional. J. Differential Geom. **70**(3), 453–472 (2005)
17. Fong, F. T.: Boltzmann's entropy and Kähler–Ricci solitons, arXiv:1605.08019v1
18. Futaki, A.: Kähler–Einstein metrics and integral invariants, Lecture Notes in Math., 1314. Springer, Berlin (1988)
19. Futaki, A.: Moment polytopes on Sasaki manifolds and volume minimization. arXiv:2201.10832
20. Futaki, A., Zhang, Y.: Coupled Sasaki–Einstein metrics. Sci. China Math. **64**, 1447–1462 (2021)
21. Futaki, A., Zhang, Y.: Residue formula for an obstruction to Coupled Kähler–Einstein metrics. J. Math. Soc. Japan **73**, 389–401 (2021)
22. Hashimoto, Y.: Anticanonically balanced metrics and the Hilbert–Mumford criterion for the δ_m-invariant of Fujita–Odaka. arXiv:2104.12346
23. He, W.: Kähler-Ricci soliton and H-functional. Asian J. Math. **20**, 645–664 (2016)
24. Hisamoto, T.: On the limit of spectral measures associated to a test configuration of a polarized Kähler manifold. Crelles J. **713**, 129–148 (2016)
25. Hisamoto, T.: Geometric flow, Multiplier ideal sheaves and Optimal destabilizer for a Fano manifold. arXiv:1901.08480

26. Hisamoto, T.: Mabuchi's soliton metric and relative D-stability. arXiv:1905.05948v2
27. Hultgren, J.: Coupled Kähler–Ricci solitons on toric manifolds. Anal. PDE **12**(8), 2067–2094 (2019)
28. Hultgren, J., Witt Nyström, D.: Coupled Kähler-Einsten metrics. Int. Math. Res. Not. **298**, 6765–6796 (2018)
29. Mabuchi, T.: Kähler Einstein metrics for manifolds with non vanishing Futaki character. Tohoku Math. J. **2**(53), 171–182 (2001)
30. Nakamura, S.: Hessian of the Ricci Calabi functional. Proc. Amer. Math. Soc. **147**, 1247–1254 (2019)
31. Nakamura, S.: Deformation for coupled Kähler–Einstein metrics. J. Math. Soc. Japan **73**(3), 933–947 (2021)
32. Pingali, V. P.: Existence of coupled Kähler–Einstein metrics using the continuity method. Internat. J. Math. **29**, 1850041, 8pp (2018)
33. Phong, D.H., Sturm, J.: Test configurations for K-stability and geodesic rays. J. Symplectic Geom. **5**(2), 221–247 (2007)
34. Phong, D.H., Sturm, J.: Regularity of geodesic rays and Monge–Ampère equations. Proc. Amer. Math. Soc. **138**(10), 3637–3650 (2010)
35. Ross, J., Witt-Nyström, D.: Analytic test configurations and geodesic rays. J. Symplectic Geom. **12**(1), 125–169 (2014)
36. Takahashi, R.: Ricci iteration for coupled Kähler–Einstein metrics. Int. Math. Res. Not. **20**, 15850–15872 (2021)
37. Takahashi, R.: Geometric quantization of coupled Kähler–Einstein metrics. Anal. PDE **14**(6), 1817–1849 (2021)
38. Tian, Gang, Zhang, Shijin, Zhang, Zhenlei, Zhu, Xiaohua: Perelman's entropy and Kähler–Ricci flow on a Fano manifold. Trans. Amer. Math. Soc. **365**(12), 6669–6695 (2013)
39. Xia, M.: On sharp lower bounds for Calabi type functionals and destabilizing properties of gradient flows. Anal. PDE **14**(6), 1951–1976 (2021)
40. Yao, Y.: Mabuchi metrics and relative Ding stability of toric Fano varieties. Int. Math. Res. Not, IMRN (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.