On the Steklov spectrum of covering spaces and total spaces

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Abstract

We show the existence of a natural Dirichlet-to-Neumann map on Riemannian manifolds with boundary and bounded geometry, such that the bottom of the Dirichlet spectrum is positive. This map regarded as a densely defined operator in the L^2-space of the boundary admits Friedrichs extension. We focus on the spectrum of this operator on covering spaces and total spaces of Riemannian principal bundles over compact manifolds.

1 Introduction

During the last years, the Steklov spectrum of compact Riemannian manifolds has been extensively studied, and analogues of various results on the Dirichlet and the Neumann spectrum have been established (cf. for instance the survey [13], or [11] and the references therein). However, the Steklov spectrum of non-compact Riemannian manifolds has not attracted that much attention yet. This is reasonable, since even the definition of Dirichlet-to-Neumann maps is quite more complicated in this case. Indeed, there may exist compactly supported smooth functions on the boundary of such a manifold which do not admit unique harmonic extension even under constraints, such as square-integrability or boundedness, or the normal derivative of the harmonic extension does not satisfy integrability conditions to give rise to an operator in a Hilbert space. For instance, on the half-line, we have that nonzero, constant functions on the boundary do not admit square-integrable harmonic extensions, while on a half-open, bounded interval, a constant function on the boundary admits different bounded and square-integrable harmonic extensions.

In this paper, we focus on a certain Dirichlet-to-Neumann map on Riemannian manifolds with boundary and bounded geometry, in the sense of [15, 25]. For a Riemannian manifold M with boundary, we denote by ν the outward pointing unit normal to the boundary, and

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by $\lambda_0^D(M)$ the bottom of the Dirichlet spectrum of M. The basis of our discussion is that if $\lambda_0^D(M) > 0$, then there exists a natural Dirichlet-to-Neumann map on M, as illustrated in the following.

Theorem 1.1 *Let M be a connected Riemannian manifold with boundary and bounded geometry with $\lambda_0^D(M) > 0$. Then any $f \in C_c^\infty(\partial M)$ admits a unique square-integrable harmonic extension $\mathcal{H}f \in C^\infty(M)$. Moreover, this extension satisfies $\nu(\mathcal{H}f) \in L^2(\partial M)$.*

The main ingredient in the preceding theorem is that the square-integrable harmonic extension actually belongs to $H^2(M)$. This relies on elliptic estimates on manifolds with boundary and bounded geometry, which were recently proved in [15]. In view of Theorem 1.1, we may consider the Dirichlet-to-Neumann map

$$\Lambda \colon C_c^\infty(\partial M) \subset L^2(\partial M) \to L^2(\partial M), \ f \mapsto \nu(\mathcal{H}f).$$

This linear operator admits a unique Friedrichs extension, being densely defined, symmetric and nonnegative definite. The spectrum of this self-adjoint operator is called the *Steklov spectrum* of M. It is worth to point out that if M is compact, then this definition coincides with the standard one in the literature. The first part of this paper is devoted to the study of some basic properties of this operator.

In the second part of the paper, we focus on the behavior of the Steklov spectrum under Riemannian coverings. The philosophy of such results is that some properties of the fundamental group of a compact manifold are reflected in the geometry of its universal covering space. A classic result in this direction is due to Brooks [5] asserting that the fundamental group of a closed (that is, compact and without boundary) Riemannian manifold is amenable if and only if the bottom of the spectrum of the Laplacian on its universal covering space is zero. The analogous results for manifolds with boundary involving the Dirichlet and the Neumann spectrum have been recently established in [20]. It is also worth to mention that according to [21], if the fundamental group of the manifold is amenable then its spectrum is contained in the spectrum of its universal covering space.

It is noteworthy that any covering space of a compact manifold with boundary has bounded geometry, and the bottom of its Dirichlet spectrum is positive. Hence, we may define the Steklov spectrum of the covering space as above. In this setting, we prove the analogue of Brooks' result for manifolds with boundary involving the Steklov spectrum.

Theorem 1.2 *Let M be a connected, compact Riemannian manifold with boundary and denote by $\tilde{M}$ its universal covering space. Then the following are equivalent:*

(i) $\pi_1(M)$ is amenable,
(ii) the Steklov spectra satisfy the inclusion $\sigma(M) \subset \sigma(\tilde{M})$,
(iii) the bottom of the Steklov spectrum of $\tilde{M}$ is zero.

It seems interesting that the topology of the boundary does not play any role in the preceding theorem, taking into account that we consider operators acting on functions defined on the boundary (which is a significant difference from the Dirichlet and the Neumann analogues of Brooks' result). For instance, the fundamental group of any boundary component may be non-amenable, while the fundamental group of the manifold is amenable. Furthermore, the fundamental group of any boundary component may be amenable (or even trivial), while the fundamental group of the manifold is non-amenable.

In the third part of the paper, we consider the Steklov spectrum of total spaces of Riemannian principal bundles over compact manifolds. The notion of Riemannian submersion has

been introduced in 1960 s as a tool to describe the geometry of a manifold in terms of simpler components (the base space and the fibers). Hence, it is natural to express the spectrum of the total space in terms of the geometry and the spectrum of the base space and the fibers. There are various results in this direction involving compact (cf. for instance the survey [4]) or non-compact manifolds (see for example [3, 6, 22, 23]). Our discussion is motivated by the recent paper [23], which focuses on non-compact total spaces of Riemannian principal bundles (see also [7] for the compact case).

To set the stage, let G be a connected Lie group acting freely, smoothly and properly via isometries on a Riemannian manifold M_2. Then the quotient $M_1 = M_2/G$ is a Riemannian manifold and the projection $p \colon M_2 \to M_1$ is a Riemannian submersion. We then say that p *arises from the action of G*. Similarly to the case of Riemannian coverings, we are interested in how properties of G are reflected in the spectrum of M_2. As indicated in [23], it is natural to compare the spectrum of the Laplacian on M_2 with the spectrum of the Schrödinger operator

$$S = \Delta + \frac{1}{4} \| p_* H \|^2 - \frac{1}{2} \operatorname{div} p_* H$$

on M_1, where H is the mean curvature vector field of the fibers. More precisely, according to [23, Theorem 1.3], if M_1 is compact, then G is unimodular and amenable if and only if $\lambda_0^D(M_2) = \lambda_0^D(S)$. It is noteworthy that if G is unimodular, then S is intertwined with the symmetric diffusion operator

$$L = \Delta + p_* H$$

regarded in $L^2_{\sqrt{V}}(M_1)$, where V is a function expressing the volume element of the fiber. If M_1 is compact, then we have that $\lambda_0^D(L) > 0$, which yields that any $f \in C^\infty(\partial M_1)$ has a unique L-harmonic extension $\mathcal{H}_L f \in C^\infty(M_1)$ (that is, $L(\mathcal{H}_L f) = 0$). Hence, we may consider the Dirichlet-to-Neumann map

$$\Lambda_L \colon C^\infty(\partial M_1) \subset L^2_{\sqrt{V}}(\partial M_1) \to L^2_{\sqrt{V}}(\partial M_1), \ f \mapsto \nu(\mathcal{H}_L f).$$

We denote the spectrum of the Friedrichs extension of this operator by $\sigma_L(M_1)$.

Returning to our discussion on the Steklov spectrum, we begin by pointing out that if $p \colon M_2 \to M_1$ is a Riemannian submersion arising from the action of a connected Lie group G, where M_1 is compact, then M_2 has bounded geometry and $\lambda_0^D(M_2) > 0$. (The latter one is a consequence of the recent [23, Theorem 1.1].) Therefore, we may define the Steklov spectrum of M_2 as above. In this setting, we establish the following analogue of [23, Theorem 1.3].

Theorem 1.3 *Let $p \colon M_2 \to M_1$ be a Riemannian submersion arising from the action of a connected Lie group G, where M_1 is connected and compact with boundary. Then the following are equivalent:*

  (i) *G is unimodular and amenable,*
 (ii) *G is unimodular and the Steklov spectra satisfy the inclusion $\sigma_L(M_1) \subset \sigma(M_2)$,*
(iii) *the bottom of the Steklov spectrum of M_2 is zero.*

The paper is organized as follows: In Sect. 2, we give some preliminaries involving functional analysis, the spectrum of Laplace-type operators, Sobolev spaces on manifolds with bounded geometry, amenable coverings, and Lie groups. Section 3 is devoted to the proof of Theorem 1.1 and the discussion of some properties of the Steklov spectrum. In Sect. 4, we focus on Riemannian coverings and establish an extension of Theorem 1.2. In Sect. 5, we study Riemannian submersions and give the proof of Theorem 1.3.

2 Preliminaries

We begin by recalling some basic facts from functional analysis, which may be found for instance in [16, 18]. Throughout this section, let H be a separable Hilbert space over $\mathbb{R}$.

The *spectrum* of a self-adjoint operator $T : \mathcal{D}(T) \subset H \to H$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{R} : T - \lambda : \mathcal{D}(T) \subset H \to H \text{ is not bijective}\}.$$

In general, the spectrum of a self-adjoint operator does not consist only of eigenvalues, but consists of approximate eigenvalues, as the following proposition indicates.

Proposition 2.1 *Let* $T : \mathcal{D}(T) \subset H \to H$ *be a self-adjoint operator and consider* $\lambda \in \mathbb{R}$. *Then* $\lambda \in \sigma(T)$ *if and only if there exists* $(v_n)_{n \in \mathbb{N}} \subset \mathcal{D}(T)$ *such that* $\|v_n\| = 1$ *and* $(T - \lambda)v_n \to 0$ *in* H.

The infimum of $\sigma(T)$ is called the *bottom* of the spectrum of T and is denoted by $\lambda_0(T)$. A particularly useful expression for $\lambda_0(T)$ is provided by Rayleigh's theorem, which asserts that

$$\lambda_0(T) = \inf_v \frac{\langle Tv, v \rangle}{\|v\|^2},$$

where the infimum is taken over all nonzero $v \in \mathcal{D}(T)$.

Consider now a densely defined, symmetric linear operator $T : \mathcal{D}(T) \subset H \to H$. We say that T is *bounded from below* if there exists $c \in \mathbb{R}$ such that $\langle Tv, v \rangle \geq c\|v\|^2$ for any $v \in \mathcal{D}(T)$. Fix such a c and define the inner product

$$\langle v, w \rangle_T = \langle Tv, w \rangle + (1 - c)\langle v, w \rangle$$

on $\mathcal{D}(T)$. Denoting by H_T the completion of $\mathcal{D}(T)$ with respect to this inner product, it is easy to see that the inclusion $\mathcal{D}(T) \hookrightarrow H$ extends to an injective map $H_T \to H$. Using this map, we regard H_T as a subspace of H. The *Friedrichs extension* T^F of T is the restriction of the adjoint T^* to the space $\mathcal{D}(T^F) = H_T \cap \mathcal{D}(T^*)$. The Friedrichs extension T^F is a self-adjoint extension of T, and Rayleigh's theorem implies the following expression for the bottom of its spectrum.

Proposition 2.2 *The bottom of the spectrum of* T^F *is given by*

$$\lambda_0(T^F) = \inf_v \frac{\langle Tv, v \rangle}{\|v\|^2},$$

where the infimum is taken over all nonzero $v \in \mathcal{D}(T)$.

2.1 Laplace type operators

Throughout this paper manifolds are assumed to be connected. However, their boundaries may be non-connected. Moreover, the term "manifold with boundary" refers to a manifold with non-empty, smooth boundary.

Let M be a Riemannian manifold with possibly empty boundary. A *symmetric Laplace type operator* L on M is an operator of the form $L = \Delta - 2\operatorname{grad}\ln\varphi + V$, where Δ is the Laplacian, $\varphi, V \in C^\infty(M)$ and $\varphi > 0$. This means that

$$Lf = \Delta f - 2\langle \operatorname{grad}\ln\varphi, \operatorname{grad} f \rangle + Vf$$

for any $f \in C^\infty(M)$. In the case where $\varphi = 1$, L is a *Schrödinger operator*, while L is called a *diffusion operator* if $V = 0$. Denote by $L^2_\varphi(M)$ the L^2-space of M with respect to the measure $\varphi^2 d\,\mathrm{vol}$, where $d\,\mathrm{vol}$ is the measure induced by the Riemannian metric. It is worth to point out that the isometric isomorphism $m_\varphi \colon L^2_\varphi(M) \to L^2(M)$ defined by $m_\varphi f = \varphi f$, intertwines L with the Schrödinger operator

$$S = m_\varphi \circ L \circ m_\varphi^{-1} = \Delta + V - \frac{\Delta \varphi}{\varphi}.$$

Given a nonzero $f \in C^\infty_c(M)$, set

$$\mathcal{R}_L(f) = \frac{\int_M (\|\operatorname{grad} f\|^2 + V f^2)\varphi^2}{\int_M f^2 \varphi^2}.$$

In the case of the Laplacian (that is, $\varphi = 1$ and $V = 0$), we denote this quantity by $\mathcal{R}(f)$.

If M does not have boundary, then

$$L \colon C^\infty_c(M) \subset L^2_\varphi(M) \to L^2_\varphi(M)$$

is densely defined and symmetric. This is the reason that L is called a symmetric Laplace type operator, although it is not symmetric in $L^2(M)$ in general. If it is bounded from below, we denote by $\lambda_0(L)$ the bottom of the spectrum of its Friedrichs extension. From Proposition 2.2 and the divergence formula, we obtain the following expression for $\lambda_0(L)$.

Proposition 2.3 *The bottom of the spectrum of L is given by*

$$\lambda_0(L) = \inf_f \frac{\langle Lf, f \rangle_{L^2_\varphi(M)}}{\|f\|^2_{L^2_\varphi(M)}} = \inf_f \mathcal{R}_L(f),$$

where the infimum is taken over all nonzero $f \in C^\infty_c(M)$.

For the rest of this subsection, suppose that M has non-empty boundary. We begin our discussion with the Dirichlet spectrum of L. The operator

$$L \colon \{f \in C^\infty_c(M) : f = 0 \text{ on } \partial M\} \subset L^2_\varphi(M) \to L^2_\varphi(M) \tag{1}$$

is densely defined and symmetric. If it is bounded from below, we denote by L^D its Friedrichs extension. It is noteworthy that if M is complete, then this operator is essentially self-adjoint (cf. for instance [2, Theorem A.24]); that is, L^D is the closure of this operator and actually coincides with the adjoint of this operator. The spectrum of L^D is called the *Dirichlet spectrum* of L. The following expression for the bottom $\lambda_0^D(L)$ of the Dirichlet spectrum is an immediate consequence of Proposition 2.2 and the divergence formula.

Proposition 2.4 *The bottom of the Dirichlet spectrum of L is given by*

$$\lambda_0^D(L) = \inf_f \frac{\langle Lf, f \rangle_{L^2_\varphi(M)}}{\|f\|^2_{L^2_\varphi(M)}} = \inf_f \mathcal{R}_L(f),$$

where the infimum is taken over all nonzero $f \in C^\infty_c(M)$ with $f = 0$ on ∂M.

It is not difficult to verify that the bottom of the Dirichlet spectrum of L coincides with the bottom of the spectrum of L considered on the interior of M.

We are also interested in the *Neumann spectrum* of L, which is the spectrum of the Friedrichs extension L^N of the operator

$$L \colon \{f \in C_c^\infty(M) : \nu(f) = 0 \text{ on } \partial M\} \subset L_\varphi^2(M) \to L_\varphi^2(M),$$

in the case where this operator is bounded from below, where ν stands for the outward pointing unit normal to ∂M. The following expression for the bottom $\lambda_0^N(L)$ of the Neumann spectrum may be found for instance in [20, Proposition 3.2] in the case of Schrödinger operators. This readily extends to symmetric Laplace type operators, by passing to the corresponding Schrödinger operator as described in the beginning of this subsection.

Proposition 2.5 *The bottom of the Neumann spectrum of L is given by*

$$\lambda_0^N(L) = \inf_f \mathcal{R}_L(f),$$

where the infimum is taken over all nonzero $f \in C_c^\infty(M)$.

It should be emphasized that the test functions in the preceding proposition are not required to satisfy any boundary condition.

To simplify our notation, in the case of the Laplacian, we set $\lambda_0(M) = \lambda_0(\Delta)$, $\lambda_0^D(M) = \lambda_0^D(\Delta)$ and $\lambda_0^N(M) = \lambda_0^N(\Delta)$.

2.2 Sobolev spaces on manifolds with bounded geometry

Let M be a Riemannian manifold with boundary and ν the outward pointing unit normal to ∂M. Denote by R the curvature tensor of M and by α the second fundamental form of ∂M. The following definition may be found in [15, 25].[1]

Definition 2.6 We say that M has *bounded geometry* if the following hold:

(i) there exists $r > 0$ such that

$$\exp \colon \partial M \times [0, r) \to M, \ (x, t) \mapsto \exp_x(-t\nu)$$

  is a diffeomorphism onto its image,
(ii) the injectivity radius of ∂M (as a manifold endowed with the induced Riemannian metric) is positive,
(iii) there is $r_0 > 0$ such that for any $x \in M \smallsetminus B(\partial M, r_0)$, the restriction of $\exp_x$ to $B(0, r_0) \subset T_x M$ is a diffeomorphism onto its image,
(iv) for any $k \in \mathbb{N} \cup \{0\}$ there exists $C_k > 0$ such that $\|\nabla^k R\| \le C_k$ and $\|\nabla^k \alpha\| \le C_k$.

We begin our discussion on such manifolds with the following observation.

Lemma 2.7 *Let M be a Riemannian manifold with boundary and bounded geometry. Then the outward pointing unit normal ν to ∂M can be extended to a bounded smooth vector field N on M with ∇N bounded.*

Proof Denoting by $d_{\partial M}$ the distance to ∂M, using the diffeomorphism from (i), extend ν to the vector field

$$V = \exp_* \left(-\frac{\partial}{\partial t} \right) = -\operatorname{grad} d_{\partial M}$$

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[1] The definition in [15] is equivalent to the one in [25], as pointed out in [15, p. 12].

defined in $B(\partial M, r)$. It follows from standard comparison theorems (cf. for instance [8]) that there exists $\delta > 0$ such that ∇V is bounded in $B(\partial M, \delta)$, keeping in mind that the sectional curvature of M is bounded and so are the principal curvatures of ∂M. Consider now a smooth $\chi : [0, \infty) \to \mathbb{R}$ with $\chi(x) = 1$ for $x \leq \delta/4$ and $\chi(x) = 0$ for $x \geq \delta/2$. It is immediate to verify that $N = (\chi \circ d_{\partial M})V$ is a bounded smooth vector field on M with ∇N bounded, which coincides with ν on ∂M. $\square$

The next theorem is essentially a special version of [15, Theorem 1.1], where we point out a difference in the form of the elliptic estimates, in the case where the bottom of the Dirichlet spectrum of the Laplacian is positive.

Theorem 2.8 *Let M be a Riemannian manifold with boundary and bounded geometry, such that $\lambda_0^D(M) > 0$. Then $\mathcal{D}(\Delta^D) = H_0^1(M) \cap H^2(M)$ and there exists $C > 0$ such that*

$$\|f\|_{H^2(M)} \leq C \|\Delta^D f\|_{L^2(M)}$$

for any $f \in \mathcal{D}(\Delta^D)$.

Proof From the definition of the Friedrichs extension, it is straightforward to verify that $H_0^1(M) \cap H^2(M) \subset \mathcal{D}(\Delta^D)$. Bearing in mind that the Laplacian on M regarded as in (1) is essentially self-adjoint, we readily see that for any $f \in \mathcal{D}(\Delta^D)$ there exists $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M)$ with $f_n = 0$ on ∂M such that $f_n \to f$ and $\Delta f_n \to \Delta^D f$ in $L^2(M)$. Therefore, it suffices to establish the asserted estimate for $f \in C_c^\infty(M)$ with $f = 0$ on ∂M. Indeed, this gives that $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $H^2(M)$, and thus, converges to f in $H^2(M)$, which also gives that $H_0^1(M) \cap H^2(M) = \mathcal{D}(\Delta^D)$.

It follows from [15, Theorem 1.1] that there exists $C > 0$ such that

$$\|f\|_{H^2(M)} \leq C(\|\Delta f\|_{L^2(M)} + \|f\|_{H^1(M)})$$

for any $f \in C_c^\infty(M)$ with $f = 0$ on ∂M. If, in addition, f is not identically zero, we readily see from Proposition 2.4 that

$$\|\Delta f\|_{L^2(M)} \geq \frac{\langle \Delta f, f \rangle_{L^2(M)}}{\|f\|_{L^2(M)}} = \mathcal{R}_\Delta(f)\|f\|_{L^2(M)} \geq \lambda_0^D(M)\|f\|_{L^2(M)}.$$

Since $\lambda_0^D(M) > 0$, this yields that

$$\int_M \|\operatorname{grad} f\|^2 = \langle \Delta f, f \rangle_{L^2(M)} \leq \|\Delta f\|_{L^2(M)}\|f\|_{L^2(M)} \leq \lambda_0^D(M)^{-1}\|\Delta f\|_{L^2(M)}^2.$$

The proof is completed by combining the above inequalities. $\square$

We will also exploit the following trace theorem, which is a special version of [15, Theorem 3.15].

Theorem 2.9 *Let M be a Riemannian manifold with boundary and bounded geometry. Then the restriction to the boundary* res$: C_c^\infty(M) \to C_c^\infty(\partial M)$ *extends to a continuous* res$: H^1(M) \to L^2(\partial M)$.

2.3 Amenable coverings

Consider a right action of a finitely generated discrete group Γ on a countable set X. We say that this action is *amenable* if there exists an invariant mean on $\ell^\infty(X)$; that is, a linear functional $\mu : \ell^\infty(X) \to \mathbb{R}$ such that:

(i) $\inf f \leq \mu(f) \leq \sup f$,

(ii) $\mu(f \circ r_g) = \mu(f)$, where $r_g(x) = xg$ for any $x \in X$,

for any $f \in \ell^\infty(X)$ and $g \in \Gamma$.

The group Γ is called amenable if the right action of Γ on itself is amenable. Standard examples of amenable groups are solvable groups and groups of subexponential growth. It is not difficult to verify that if Γ is amenable, then any right action of Γ is amenable. A particularly useful characterization of amenability is the following proposition due to Følner in the case of groups [10, Main Theorem and Remark], extended to actions by Rosenblatt [24, Theorems 4.4 and 4.9].

Proposition 2.10 *Let Γ be a finitely generated group and fix a finite, symmetric generating set G. Then the right action of Γ on a countable set X is amenable if and only if for any $\varepsilon > 0$ there exists a non-empty finite subset P of X such that*

$$|Pg \smallsetminus P| < \varepsilon |P|$$

for any $g \in G$.

Let $p: M_2 \to M_1$ be a Riemannian covering and choose $x \in M_1$ as a base point for $\pi_1(M_1)$. Given $y \in p^{-1}(x)$ and $g \in \pi_1(M_1)$, consider a representative loop γ of g based at x. Lift γ to a path $\tilde{\gamma}$ starting at y and denote its endpoint by yg. In this way, we obtain a right action of $\pi_1(M_1)$ to $p^{-1}(x)$, which is called the *monodromy action*. We say that the covering is *amenable* if the monodromy action is amenable. It is worth to mention that a normal Riemannian covering is amenable if and only if its deck transformation group is amenable. In particular, the universal covering of a manifold is amenable if and only if its fundamental group is amenable.

If M_1 is compact with boundary, then $\pi_1(M_1)$ is finitely generated. More specifically, the finite and symmetric set

$$S_r = \{g \in \pi_1(M_1) : g \text{ has a representative loop of length less than } r\}$$

generates $\pi_1(M_1)$ for $r > 0$ sufficiently large. The next elementary lemma provides a description of the monodromy action in terms of this set.

Lemma 2.11 *Given $y_1, y_2 \in p^{-1}(x)$, there exists $g \in S_r$ such that $y_2 = y_1 g$ if and only if $d(y_1, y_2) < r$.*

Proof Suppose that $y_2 = y_1 g$ for some $g \in S_r$. Then there exists a representative loop γ of g based at x of length less than r. Since the endpoint of its lift starting at y_1 is y_2, it is clear that $d(y_1, y_2) < r$. Conversely, if $d(y_1, y_2) < r$, consider a curve c from y_1 to y_2 of length less than r. Denoting by g the class of $p \circ c$ in $\pi_1(M_1)$, we readily see that $y_2 = y_1 g$ and $g \in S_r$, $p \circ c$ having length less than r. □

Amenability of a covering is intertwined with the preservation of the bottom of the spectrum, as Brooks' result illustrates. In the sequel, we will exploit the corresponding result involving the bottom of the Neumann spectrum.

Theorem 2.12 *([20, Theorem 1.1]) Let $p: M_2 \to M_1$ be a Riemannian covering, where M_1 is compact with boundary. Then p is amenable if and only if $\lambda_0^N(M_2) = 0$.*

2.4 Lie groups

A connected Lie group G is called *amenable* if there exists a left-invariant mean on $L^\infty(G)$; that is a linear functional $\mu \colon L^\infty(G) \to \mathbb{R}$ such that

(i) $\operatorname{ess\,inf} f \leq \mu(f) \leq \operatorname{ess\,sup} f$,
(ii) $\mu(f \circ L_x) = \mu(f)$,

for any $f \in L^\infty(G)$ and $x \in G$, where $L_x \colon G \to G$ stands for multiplication from the left with $x \in G$. Here, we consider $L^\infty(G)$ with respect to the Haar measure of G, which is just a constant multiple of the volume element of G induced by a left-invariant metric. For more details, see [14]. It is well known that a connected Lie group G is amenable if and only if it is a compact extension of a solvable group (cf. for example [19, Lemma 2.2]).

A Lie group G is called *unimodular* if its Haar measure is right-invariant. It is noteworthy that a connected Lie group G is unimodular if and only if $\operatorname{tr}(\operatorname{ad} X) = 0$ for any X in the Lie algebra of G (cf. for instance [17, Proposition 1.2]). Standard examples of unimodular and amenable Lie groups are connected, nilpotent Lie groups.

Even though the above properties are group theoretic, they are reflected in the spectrum of the Laplacian. The following characterization has been established for simply connected Lie groups in [17, Theorem 3.8] and extended to connected Lie groups in [21, Theorem 2.10].

Theorem 2.13 *A connected Lie group G is unimodular and amenable if and only if $\lambda_0(G) = 0$ for some/any left-invariant metric on G.*

3 The Steklov spectrum of manifolds with bounded geometry

In this section, we define the Steklov spectrum of manifolds satisfying the assumptions of Theorem 1.1, and discuss some basic properties of it. Throughout this section, we consider a Riemannian manifold M with boundary and bounded geometry such that $\lambda_0^D(M) > 0$, and we denote by ν the outward pointing unit normal to ∂M. It is worth to mention that compact manifolds are not excluded from our discussion.

Proposition 3.1 *Any $f \in C_c^\infty(\partial M)$ has a unique square-integrable harmonic extension $\mathcal{H}f \in C^\infty(M)$. In addition, this extension belongs to $H^2(M)$ and is written as $\mathcal{H}f = F + h$, where $F \in C_c^\infty(M)$ is an extension of f and $h \in \mathcal{D}(\Delta^D)$.*

Proof Let $f \in C_c^\infty(\partial M)$ and consider an extension $F \in C_c^\infty(M)$ of it. We readily see from the definition of the spectrum that the assumption that $\lambda_0^D(M) > 0$ means that

$$\Delta^D \colon \mathcal{D}(\Delta^D) \subset L^2(M) \to L^2(M)$$

is bijective. In particular, there exists $h \in \mathcal{D}(\Delta^D)$ such that $\Delta^D h = -\Delta F$. Bearing in mind that $F \in C^\infty(M)$, we derive from elliptic regularity that $h \in C^\infty(M)$ with $h = 0$ on ∂M, and $\Delta h = -\Delta F$. It is evident that $h + F$ is a square-integrable harmonic extension of f.

Let $F_1, F_2 \in C^\infty(M)$ be square-integrable harmonic extensions of a given $f \in C_c^\infty(\partial M)$. Then $h = F_1 - F_2 \in C^\infty(M) \cap L^2(M)$ is harmonic and vanishes on ∂M. Given $g \in C_c^\infty(M)$ with $g = 0$ on ∂M, we compute

$$\langle h, \Delta g \rangle_{L^2(M)} = \int_M \langle \operatorname{grad} h, \operatorname{grad} g \rangle - \int_{\partial M} h\nu(g) = \int_{\partial M} \nu(h)g = 0,$$

where we used that h is harmonic. This shows that $h \in \mathcal{D}(\Delta^*)$ with $\Delta^* h = 0$. Since M is complete, the Laplacian Δ regarded as in (1) is essentially self-adjoint, and hence, $\Delta^* = \Delta^D$. We deduce that h belongs to the kernel of Δ^D and thus, $h = 0$, due to the fact that $\lambda_0^D(M) > 0$.   □

Proof of Theorem 1.1 We know from Proposition 3.1 that any $f \in C_c^\infty(\partial M)$ admits a unique square-integrable harmonic extension $\mathcal{H} f \in H^2(M)$. In view of Lemma 2.7, we may extend ν to a bounded smooth vector field N on M with ∇N bounded. Then we have that $N(\mathcal{H} f) \in H^1(M)$, and its restriction $\nu(\mathcal{H} f)$ to the boundary is square-integrable, by virtue of Theorem 2.9.   □

It is now clear that the Dirichlet-to-Neumann map

$$\Lambda \colon C_c^\infty(\partial M) \subset L^2(\partial M) \to L^2(\partial M), \ f \mapsto \nu(\mathcal{H} f)$$

is well-defined. It is also evident that this linear operator is densely defined.

Lemma 3.2 *For any $f, h \in C_c^\infty(\partial M)$, we have that*

$$\langle \Lambda f, h \rangle_{L^2(\partial M)} = \langle f, \Lambda h \rangle_{L^2(\partial M)} = \int_M \langle \operatorname{grad}(\mathcal{H} f), \operatorname{grad}(\mathcal{H} h) \rangle.$$

Proof We know from Proposition 3.1 that $\mathcal{H} f = F + h_1$ and $\mathcal{H} h = H + h_2$, where $F \in C_c^\infty(M)$ is an extension of f, $H \in C_c^\infty(M)$ is an extension of h, and $h_1, h_2 \in \mathcal{D}(\Delta^D)$. Since $\mathcal{D}(\Delta^D) = H_0^1(M) \cap H^2(M)$, we derive from the divergence formula that

$$\int_M \langle \operatorname{grad} h_1, \operatorname{grad}(\mathcal{H} h) \rangle = \int_M h_1 \Delta H + \int_M h_1 \Delta^D h_2 = 0,$$

and therefore,

$$\int_M \langle \operatorname{grad}(\mathcal{H} f), \operatorname{grad}(\mathcal{H} h) \rangle = \int_M \langle \operatorname{grad} F, \operatorname{grad}(\mathcal{H} h) \rangle = \int_{\partial M} f \nu(\mathcal{H} h),$$

as we wished.   □

Hence, the aforementioned Dirichlet-to-Neumann map admits Friedrichs extension, being symmetric and bounded from below by zero. The spectrum $\sigma(M)$ of its Friedrichs extension is called the *Steklov spectrum* of M, and its bottom is denoted by $\sigma_0(M)$.

Proposition 3.3 *The bottom of the Steklov spectrum of M is given by*

$$\sigma_0(M) = \inf_f \frac{\int_M \| \operatorname{grad} f \|^2}{\int_{\partial M} f^2},$$

where the infimum is taken over all $f \in C_c^\infty(M)$ which are not identically zero on ∂M.

Before proceeding to the proof of this proposition, we need the following remark.

Lemma 3.4 *For any extension $F \in C_c^\infty(M)$ of a function $f \in C_c^\infty(\partial M)$, we have that*

$$\int_M \| \operatorname{grad}(\mathcal{H} f) \|^2 \leq \int_M \| \operatorname{grad} F \|^2.$$

Proof Keeping in mind that $h = \mathcal{H}f - F \in H_0^1(M)$, it is immediate to verify that

$$\int_M \|\operatorname{grad} F\|^2 = \int_M \|\operatorname{grad}(\mathcal{H}f)\|^2 + \int_M \|\operatorname{grad} h\|^2 - 2\int_M \langle \operatorname{grad}(\mathcal{H}f), \operatorname{grad} h \rangle,$$

while

$$\int_M \langle \operatorname{grad}(\mathcal{H}f), \operatorname{grad} h \rangle = \int_{\partial M} \nu(\mathcal{H}f)h = 0.$$

This establishes the asserted inequality. □

From the proof, it is clear that the equality in the preceding lemma holds if and only if M is compact and $F = \mathcal{H}f$.

Proof of Proposition 3.3 We know from Proposition 2.2 and Lemma 3.2 that

$$\sigma_0(M) = \inf_f \frac{\langle \Lambda f, f \rangle_{L^2(\partial M)}}{\|f\|^2_{L^2(\partial M)}} = \inf_f \frac{\int_M \|\operatorname{grad}(\mathcal{H}f)\|^2}{\int_{\partial M} f^2},$$

where the infimum is taken over all nonzero $f \in C_c^\infty(\partial M)$. This is clearly greater or equal to the asserted infimum, and the equality follows by virtue of Lemma 3.4. □

A straightforward consequence of Proposition 3.3 is the following relation between the bottom of the Steklov and the Neumann spectrum.

Theorem 3.5 *If $\sigma_0(M) = 0$, then $\lambda_0^N(M) = 0$.*

Proof We readily see from Proposition 3.3 that there exists $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M)$ such that $\|f_n\|_{L^2(\partial M)} = 1$ and

$$\int_M \|\operatorname{grad} f_n\|^2 \to 0.$$

Assume to the contrary that $\lambda_0^N(M) > 0$. Then Proposition 2.5 implies that

$$\|f_n\|^2_{L^2(M)} \le \lambda_0^N(M)^{-1} \int_M \|\operatorname{grad} f_n\|^2 \to 0.$$

This means that $f_n \to 0$ in $H^1(M)$ and thus, $f_n \to 0$ in $L^2(\partial M)$, by virtue of Theorem 2.9, which is a contradiction. □

Recall that according to Proposition 2.1, the spectrum of a self-adjoint operator consists of the approximate eigenvalues of the operator. In general, given $f \in C_c^\infty(\partial M)$ and $\lambda \in \mathbb{R}$, it may be quite complicated to estimate the quantity $\|\Lambda f - \lambda f\|_{L^2(\partial M)}$. The next observation allows us to substitute the harmonic extension with any compactly supported smooth extension, and the error term is controlled in terms of the Laplacian of the chosen extension.

Proposition 3.6 *There exists $C > 0$ such that*

$$\|\Lambda f - \lambda f\|_{L^2(\partial M)} \le \|\nu(F) - \lambda f\|_{L^2(\partial M)} + C\|\Delta F\|_{L^2(M)}$$

for any extension $F \in C_c^\infty(M)$ of any $f \in C_c^\infty(\partial M)$.

Proof Let $F \in C_c^\infty(M)$ be an extension of a given $f \in C_c^\infty(\partial M)$. Extending ν to a bounded smooth vector field N on M with ∇N bounded (as in Lemma 2.7), we obtain from Theorem 2.9 that there exists $C_1 > 0$ such that

$$\|\nu(F) - \nu(\mathcal{H}f)\|_{L^2(\partial M)} \leq C_1 \|N(F) - N(\mathcal{H}f)\|_{H^1(M)} \leq C_1 C_2 \|F - \mathcal{H}f\|_{H^2(M)},$$

where C_2 is a constant depending on N. In view of Proposition 3.1, it is apparent that $F - \mathcal{H}f \in \mathcal{D}(\Delta^D)$. Therefore, we derive from Theorem 2.8 that there exists $C_3 > 0$ such that

$$\|F - \mathcal{H}f\|_{H^2(M)} \leq C_3 \|\Delta(F - \mathcal{H}f)\|_{L^2(M)} = C_3 \|\Delta F\|_{L^2(M)}.$$

We conclude that

$$\begin{aligned}
\|\Lambda f - \lambda f\|_{L^2(\partial M)} &\leq \|\nu(F) - \lambda f\|_{L^2(\partial M)} + \|\nu(F) - \nu(\mathcal{H}f)\|_{L^2(\partial M)} \\
&\leq \|\nu(F) - \lambda f\|_{L^2(\partial M)} + C_1 C_2 C_3 \|\Delta F\|_{L^2(M)},
\end{aligned}$$

as we wished. $\qquad\square$

4 Steklov spectrum under Riemannian coverings

Throughout this section, we consider a Riemannian covering $p\colon M_2 \to M_1$ where M_1 is compact with boundary. It is not difficult to verify that M_2 has bounded geometry. Indeed, it is easily checked that properties (ii)-(iv) of Definition 2.6 are satisfied, after noticing that $B(\partial M_2, r) = p^{-1}(B(\partial M_1, r))$ for any $r > 0$, while the validity of (i) is explained for instance in [20, Lemma 4.2]. It is also important for our discussion that $\lambda_0^D(M_2) > 0$. This follows from [1, Theorem 1.3], keeping in mind that $\lambda_0^D(M_2)$ coincides with the bottom of the spectrum of the Laplacian on the interior of M_2. Hence, we may define the Steklov spectrum of M_2 as in the previous section. The aim of this section is to establish the following extension of Theorem 1.2.

Theorem 4.1 *Let $p\colon M_2 \to M_1$ be a Riemannian covering, where M_1 is compact with boundary. Then the following are equivalent:*

- *(i) p is amenable,*
- *(ii) the Steklov spectra satisfy the inclusion $\sigma(M_1) \subset \sigma(M_2)$,*
- *(iii) the bottom of the Steklov spectrum is preserved; that is, $\sigma_0(M_2) = 0$.*

Let $\tilde{M}$ be the universal covering space of M_1, consider the Riemannian coverings $p_i\colon \tilde{M} \to M_i$, and denote by Γ_i the deck transformation group of p_i, $i = 1, 2$. It should be noticed that $p \circ p_2 = p_1$.

We begin by introducing the fundamental domains that will be used in the sequel. For their definition we will exploit the deck transformations group of the universal covering. In the case where p is normal, an analogous construction is possible using the deck transformations of p, without having to pass to the universal covering space. It is worth to point out that the Dirichlet fundamental domains used in [1, 21] do not seem appropriate for our purposes, since we have to deal with integrals over the boundary.

Fix a finite, smooth triangulation of M_1 that induces a triangulation of ∂M_1, and for each full-dimensional simplex choose a lift on $\tilde{M}$, so that the union F of their images is connected. The set F is called a *finite sided fundamental domain* of the covering p_1. We readily see that $\mathrm{Vol}(\partial F) = \mathrm{Area}(\partial \tilde{M} \cap \partial F) = 0$ and the translates gF of F with $g \in \Gamma_1$ cover $\tilde{M}$. Here,

∂F stands for the boundary of F as a subset of $\tilde{M}$, and similarly below. It is also immediate to verify that

$$\int_F (f \circ p_1) = \int_{M_1} f \text{ and } \int_{\partial \tilde{M} \cap F} (f \circ p_1) = \int_{\partial M_1} f$$

for any $f \in C^\infty(M_1)$.

Choose $\tilde{x} \in F^\circ$ and set $x = p_1(\tilde{x})$. Given $y \in p^{-1}(x)$, there exists $g \in \Gamma_1$ such that $g\tilde{x} \in p_2^{-1}(y)$. Set $F_y = p_2(gF)$ and observe that it does not depend on the choice of g, since if $g'\tilde{x} \in p_2^{-1}(y)$ for some $g' \in \Gamma_1$, then $g'g^{-1} \in \Gamma_2$. It is evident that the domains F_y with $y \in p^{-1}(x)$ cover M_2 and $\mathrm{diam}(F_y) \leq \mathrm{diam}(F)$ for any $y \in p^{-1}(x)$. It is not hard to check that $\partial F_y \subset p_2(g\partial F)$, which implies that $\mathrm{Vol}(\partial F_y) = \mathrm{Area}(\partial M_2 \cap \partial F_y) = 0$. From the fact that $p_1 \colon gF \to M_1$ and $p_1 \colon \partial \tilde{M} \cap F \to \partial M_1$ are isometries up to sets of measure zero, we readily see that so are the restrictions $p \colon F_y \to M_1$ and $p \colon \partial M_2 \cap F_y \to \partial M_1$ for any $y \in p^{-1}(x)$. This yields that

$$\int_{F_y} (f \circ p) = \int_{M_1} f \text{ and } \int_{\partial M_2 \cap F_y} (f \circ p) = \int_{\partial M_1} f \tag{2}$$

for any $f \in C^\infty(M_1)$ and $y \in p^{-1}(x)$. For any $y, z \in p^{-1}(x)$ with $y \neq z$, using that $F_y \cap F_z \subset \partial F_y$, we derive that

$$\mathrm{Vol}(F_y \cap F_z) = \mathrm{Area}(\partial M_2 \cap F_y \cap F_z) = 0. \tag{3}$$

We now construct a partition of unity on M_2, which will be used to obtain cut-off functions. To this end, we will exploit the following.

Lemma 4.2 *Given $r > 0$, there exists $k(r) \in \mathbb{N}$ such that for any $z \in M_2$ the open ball $B(z, 1)$ intersects at most $k(r)$ of the closed balls $C(y, r)$ with $y \in p^{-1}(x)$.*

Proof Suppose that $B(z, 1)$ intersects the closed balls $C(y_i, r)$ with $y_i \in p^{-1}(x)$ pairwise different and let γ_i be a minimizing geodesic from z to y_i, $i = 1, \ldots, k$. Then the concatenations $(p \circ \gamma_i) * (p \circ \gamma_1^{-1})$ are pairwise non-homotopic loops based at x of length less than $2r + 2$. We conclude that $k \leq |S_{2r+2}|$. $\qquad \square$

Consider $r > 0$ such that $F \subset B(\tilde{x}, r)$ and S_r generates $\pi_1(M_1)$. Fix a nonnegative $\psi \in C_c^\infty(\tilde{M})$ with $\psi = 1$ in $B(\tilde{x}, r)$ and $\mathrm{supp}\,\psi \subset B(\tilde{x}, r + 1)$. Given $y \in p^{-1}(x)$, choose $g \in \Gamma_1$ such that $g\tilde{x} \in p_2^{-1}(y)$ and set

$$\psi_y(z) = \sum_{w \in p_2^{-1}(z)} (\psi \circ g^{-1})(w) = \sum_{g' \in \Gamma_2} (\psi \circ g^{-1} \circ g')(w_0), \tag{4}$$

for some fixed $w_0 \in p_2^{-1}(z)$. Since ψ is compactly supported, we readily see that ψ_y is well-defined, smooth, nonnegative, $\psi_y \geq 1$ in $B(y, r)$ and $\mathrm{supp}\,\psi_y \subset B(y, r+1)$. It follows from Lemma 4.2 (applied to the universal covering p_1) that there exists $\tilde{k} \in \mathbb{N}$ such that locally at most $\tilde{k}$ terms in the right-hand side of (4) are nonzero. Thus, we deduce that there exists $C_1 > 0$ such that

$$\| \mathrm{grad}\,\psi_y \| \leq C_1 \text{ and } |\Delta\psi_y| \leq C_1$$

for any $y \in p^{-1}(x)$.

According to Lemma 4.2, there exists $k \in \mathbb{N}$ such that at most k of the supports of ψ_y with $y \in p^{-1}(x)$ intersect any ball of radius one in M_2. Therefore, the function

$$\psi = \sum_{y \in p^{-1}(x)} \psi_y$$

is well-defined, smooth, and greater or equal to one. Moreover, we obtain that ψ, grad ψ and $\Delta \psi$ are bounded.

Consider now the smooth partition of unity on M_2 consisting of the functions

$$\varphi_y = \frac{\psi_y}{\psi},$$

with $y \in p^{-1}(x)$. It is clear that supp $\varphi_y \subset B(y, r+1)$, $\varphi_y > 0$ in $B(y, r)$, while grad φ_y and $\Delta \varphi_y$ are uniformly bounded for all $y \in p^{-1}(x)$. Given a non-empty, finite subset P of $p^{-1}(x)$, define the function $\chi_P \in C_c^\infty(M_2)$ by

$$\chi_P = \sum_{y \in P} \varphi_y$$

and the sets

$$Q_+ = \{y \in p^{-1}(x) : \chi = 1 \text{ in } F_y\},$$
$$Q_- = \{y \in p^{-1}(x) : 0 < \chi(z) < 1 \text{ for some } z \in F_y\}.$$

Since at most k of the supports of φ_y interest any ball of radius one in M_2, it is easy to see that exists $C_2 > 0$ such that

$$\| \text{grad } \chi_P \| \leq C_2 \text{ and } |\Delta \chi_P| \leq C_2 \tag{5}$$

for any finite subset P of $p^{-1}(x)$. Furthermore, the sets Q_+ and Q_- are finite, χ_P being compactly supported

The following proposition illustrates how amenability of the covering is related to our construction.

Proposition 4.3 *If $p \colon M_2 \to M_1$ is amenable, then for any $\varepsilon > 0$ there exists a non-empty, finite subset P of $p^{-1}(x)$ such that $|Q_-| < \varepsilon |Q_+|$.*

Proof Set $d = 2(r + 1 + \text{diam}(F))$. We know from Proposition 2.10 that for any $\varepsilon > 0$ there exists a non-empty, finite subset P of $p^{-1}(x)$ such that

$$|Pg \smallsetminus P| < \varepsilon |P|$$

for any $g \in S_d$. Consider the corresponding function χ_P and let $y \in P$ such that $yg \in P$ for any $g \in S_d$. Let $z \in F_y$ and $y' \in p^{-1}(x)$ such that $z \in \text{supp } \varphi_{y'}$. Then $d(z, y') < r + 1$ and thus, $d(y, y') < d/2$, because $\text{diam}(F_y) \leq \text{diam}(F)$. Lemma 2.11 shows that there exists $g \in S_d$ such that $y' = yg \in P$. Since $\{\varphi_{y'}\}_{y' \in p^{-1}(x)}$ is a partition of unity on M_2, we deduce that $y \in Q_+$. This yields that

$$|P \cap (\cap_{g \in S_d} Pg)| \leq |Q_+|,$$

and thus,

$$|Q_+| \geq |P| - |\cup_{g \in S_d} (P \smallsetminus Pg)| \geq (1 - \varepsilon |S_d|)|P|,$$

where we used that S_d is symmetric.

Consider now $y \in Q_-$ and $z \in F_y$ such that $0 < \chi_P(z) < 1$. Then there exist $y_1 \in P$ and $y_2 \in p^{-1}(x) \smallsetminus P$ such that $\varphi_i(z) > 0$, $i = 1, 2$. This implies that $d(z, y_i) < r + 1$, and hence, $d(y, y_i) < d/2$, $i = 1, 2$. In particular, we obtain that $d(y_1, y_2) < d$ and there exists $g \in S_d$ such that $y_1 = y_2 g \in P \smallsetminus Pg$, in view of Lemma 2.11. Since $d(y, y_1) < d/2$, we derive from Lemma 2.11 that for such a y_1 there exists at most $|S_{d/2}|$ such y. This gives the estimate

$$|Q_-| \le \sum_{g \in S_d} |S_{d/2}||P \smallsetminus Pg| < \varepsilon |S_{d/2}||S_d||P|.$$

Combining the above, for $\varepsilon > 0$ sufficiently small, we conclude that

$$\frac{|Q_-|}{|Q_+|} < \frac{\varepsilon |S_{d/2}||S_d|}{1 - \varepsilon |S_d|},$$

which completes the proof. $\square$

Proof of Theorem 4.1 Suppose first that the covering $p \colon M_2 \to M_1$ is amenable. Since M_1 is compact, its Steklov spectrum is discrete. Therefore, for any $\lambda \in \sigma(M_1)$, there exists a harmonic $f \in C^\infty(M_1)$ with $\|f\|_{L^2(\partial M_1)} = 1$ such that $\nu_1(f) = \lambda f$ on ∂M_1, where ν_1 is the outward pointing unit normal to ∂M_1. Denote by $\tilde{f} = f \circ p$ the lift of f on M_2 and, given a non-empty, finite subset P of $p^{-1}(x)$, consider the function $f_P = \chi_P \tilde{f} \in C_c^\infty(M_2)$.

For $y \in Q_+$, keeping in mind (2) and that $f_P = \tilde{f}$ in F_y, we readily see that

$$\int_{F_y} (\Delta f_P)^2 = 0, \quad \int_{\partial M_2 \cap F_y} (\nu_2(f_P) - \lambda f_P)^2 = 0 \text{ and } \int_{\partial M_2 \cap F_y} f_P^2 = 1,$$

where ν_2 is the outward pointing unit normal to ∂M_2. Fix now $y \in Q_-$. Using (2) and that $\tilde{f}$ is harmonic, we compute

$$
\begin{aligned}
\int_{F_y} (\Delta f_P)^2 &= \int_{F_y} (\tilde{f} \Delta \chi_P - 2\langle \operatorname{grad} \tilde{f}, \operatorname{grad} \chi_P \rangle)^2 \\
&\le 2 \int_{F_y} \tilde{f}^2 (\Delta \chi_P)^2 + 8 \int_{F_y} \|\operatorname{grad} \tilde{f}\|^2 \|\operatorname{grad} \chi_P\|^2 \\
&\le 2 C_2^2 \|f\|_{L^2(M_1)}^2 + 8 C_2^2 \int_{M_1} \|\operatorname{grad} f\|^2,
\end{aligned}
$$

where C_2 is the constant from (5). Moreover, we deduce that

$$\int_{\partial M_2 \cap F_y} (\nu_2(f_P) - \lambda f_P)^2 = \int_{\partial M_2 \cap F_y} (\nu_2(\chi_P)\tilde{f})^2 \le C_2^2$$

where we used that $\nu_2(\tilde{f}) = \lambda \tilde{f}$ on ∂M_2, $\|f\|_{L^2(\partial M_1)} = 1$ and (2).

Furthermore, we derive from (2) and (3) that

$$\|f_P\|_{L^2(\partial M_2)}^2 \ge \sum_{y \in Q_+} \int_{\partial M_2 \cap F_y} f_P^2 = |Q_+|.$$

Combining the above estimates, together with (3) and Proposition 3.6, yields that

$$
\begin{aligned}
\|\nu(\mathcal{H}f_P) - \lambda f_P\|_{L^2(\partial M_2)}^2 &\leq 2\|\nu(f_P) - \lambda f_P\|_{L^2(\partial M_2)}^2 + 2C^2\|\Delta f_P\|_{L^2(M_2)}^2 \\
&= 2\sum_{y \in Q_-}\left(\int_{\partial M_2 \cap F_y}(\nu(f_P) - \lambda f_P)^2 + C^2\int_{F_y}(\Delta f_P)^2\right) \\
&\leq 2C_2^2|Q_-|\left(1 + 2C^2\|f\|_{L^2(M_1)}^2 + 8C^2\int_{M_1}\|\operatorname{grad} f\|^2\right).
\end{aligned}
$$

Since the constants involved in these estimates are independent from P, it follows from Proposition 4.3 that for any $\varepsilon > 0$ there exists a finite subset P of $p^{-1}(x)$ such that

$$
\|\nu(\mathcal{H}f_P) - \lambda f_P\|_{L^2(\partial M_2)}^2 < \varepsilon\|f_P\|_{L^2(\partial M_2)}^2.
$$

We conclude from Proposition 2.1 that $\lambda \in \sigma(M_2)$, $\varepsilon > 0$ being arbitrary.

It is clear from Proposition 3.3 that $\sigma_0(M_2) \geq 0$. Therefore, if $\sigma(M_1) \subset \sigma(M_2)$, then $\sigma_0(M_2) = 0$, because $\sigma_0(M_1) = 0$. Suppose now that $\sigma_0(M_2) = 0$. Then Theorem 3.5 states that $\lambda_0^N(M_2) = 0$, and thus, the covering is amenable, in view of Theorem 2.12. □

5 Steklov spectrum under Riemannian submersions

The aim of this section is to establish Theorem 1.3. Throughout, we consider a Riemannian submersion $p\colon M_2 \to M_1$ arising from the action of a connected Lie group G, where M_1 is compact with boundary, and denote by ν_i the outward pointing unit normal to ∂M_i, $i = 1, 2$. For more details on Riemannian submersions, see [9, 12]. We begin by showing that M_2 has bounded geometry and $\lambda_0^D(M_2) > 0$.

Proposition 5.1 *In the aforementioned setting, M_2 has bounded geometry.*

Proof Bearing in mind that G acts on M_2 via isometries, it is easy to verify properties (ii)-(iv) of Definition 2.6, after noticing that $B(\partial M_2, r) = p^{-1}(B(\partial M_1, r))$ for any $r > 0$. To check the validity of (i), observe that there exists $r > 0$ such that the map

$$
\exp\colon \partial M_1 \times [0, r) \to M_1, \ (y, t) \mapsto \exp_y(-t\nu_1)
$$

is a diffeomorphism onto its image, M_1 being compact. Choose a precompact, open $U \subset \partial M_2$ such that $GU = \partial M_2$, and consider $r' \leq r$ such that

$$
\exp\colon U \times [0, r') \to M_2, \ (z, t) \mapsto \exp_z(-t\nu_2)
$$

is a diffeomorphism onto its image. Since G acts on M_2 via isometries, we derive that

$$
\exp\colon \partial M_2 \times [0, r') \to M_2, \ (z, t) \mapsto \exp_z(-t\nu_2)
$$

is a local diffeomorphism onto its image. Hence, it remains to show that this map is injective. To this end, let $z_1, z_2 \in \partial M_2$ and $t_1, t_2 \in [0, r')$ with $\exp_{z_1}(-t_1\nu_2) = \exp_{z_2}(-t_2\nu_2) = z$. From the fact that $\gamma_i(t) = \exp_{z_i}(-t\nu_2)$ is a horizontal geodesic, we readily see that $(p \circ \gamma_i)(t) = \exp_{p(z_i)}(-t\nu_1)$, $i = 1, 2$. Since $r' \leq r$, this yields that $p(z_1) = p(z_2)$ and $t_1 = t_2$. In particular, the geodesics $p \circ \gamma_i$ coincide, and hence, so do their horizontal lifts γ_i with endpoint z. We conclude that $z_1 = z_2$ and $t_1 = t_2$, as we wished. □

Consider the Schrödinger operator

$$S = \Delta + \frac{1}{4}\|p_*H\|^2 - \frac{1}{2}\operatorname{div} p_*H \tag{6}$$

on M_1. We know from [23, Theorem 1.1] that $\lambda_0^D(M_2) \geq \lambda_0^D(S)$. It is worth to mention that [23, Theorem 1.1] is formulated for manifolds without boundary, which however do not have to be complete. Hence, the assertion is readily extended to manifolds with boundary.

Proposition 5.2 *In the aforementioned setting, we have that $\lambda_0^D(S) > 0$.*

Proof Given a nonzero $f \in C_c^\infty(M_1)$ with $f = 0$ on ∂M_1, we compute

$$
\begin{aligned}
\langle Sf, f \rangle_{L^2(M_1)} &= \int_{M_1} (\|\operatorname{grad} f\|^2 + \frac{1}{4}\|p_*H\|f^2 - \frac{1}{2}f^2 \operatorname{div} p_*H) \\
&= \int_{M_1} (\|\operatorname{grad} f\|^2 + \frac{1}{4}\|p_*H\|f^2 + \frac{1}{2}\langle \operatorname{grad} f^2, p_*H\rangle) \\
&= \int_{M_1} \|\operatorname{grad} f + \frac{f}{2}p_*H\|^2,
\end{aligned}
$$

where we used the divergence formula. In particular, we deduce that $\mathcal{R}_S(f) \geq 0$ for any such f, which means that $\lambda_0^D(S) \geq 0$, in view of Proposition 2.4.

Assume to the contrary that $\lambda_0^D(S) = 0$. Since M_1 is compact, the Dirichlet spectrum of S is discrete. This yields that there exists $f \in C^\infty(M_1)$ positive in M_1° and vanishing on ∂M_1 such that $Sf = 0$, which implies that $\mathcal{R}_S(f) = 0$. From the preceding computation, we conclude that

$$p_*H = -2\operatorname{grad} \ln f$$

in M_1°. This is a contradiction, since p_*H is smooth on M_1, while f vanishes on ∂M_1. □

From the above, it follows that M_2 has bounded geometry and $\lambda_0^D(M_2) > 0$. Therefore, we may define the Steklov spectrum of M_2 as in Sect. 3.

The proof of Theorem 1.3 relies on the methods of [23]. For convenience of the reader, we briefly discuss what will be used in the sequel. Given a section $s\colon U \subset M_1 \to M_2$, the map $\Phi\colon G \times U \to p^{-1}(U)$ defined by $\Phi(x, y) = xs(y)$ is a diffeomorphism. We denote by $g_{s(y)}$ the pullback of the Riemannian metric of the fiber $F_y = p^{-1}(y)$ via $\Phi(\cdot, y)$. Then $g_{s(y)}$ is a left-invariant metric depending smoothly on $y \in U$, according to [23, Proposition 4.1]. The behavior of the volume elements of these metrics is illustrated in the following.

Proposition 5.3 *([23, Corollaries 4.2 and 4.3]) Let g be a fixed left-invariant metric on G. Given a section $s\colon U \subset M_1 \to M_2$, there exists $V_s \in C^\infty(U)$ such that the volume elements are related by*

$$d\mathrm{vol}_{g_{s(y)}} = V_s(y)d\mathrm{vol}_g.$$

If, in addition, G is unimodular, then there exists $V \in C^\infty(M)$ such that

$$d\mathrm{vol}_{g_{s(y)}} = V(y)d\mathrm{vol}_g$$

for any section $s\colon U \subset M_1 \to M_2$ and $y \in U$. Moreover, the gradient of V is given by $\operatorname{grad} V = -Vp_*H$.

In the case where G is unimodular, it follows that the Schrödinger operator defined in (6) is written as

$$S = \Delta - \frac{\Delta\sqrt{V}}{\sqrt{V}},$$

and, therefore, corresponds to the symmetric diffusion operator

$$L = m_\varphi^{-1} \circ S \circ m_\varphi = \Delta - 2\,\mathrm{grad}\,\ln\sqrt{V} = \Delta + p_* H.$$

It is noteworthy that

$$\Delta(f \circ p) = (Lf) \circ p \tag{7}$$

for any $f \in C^\infty(M_1)$ (cf. for instance [23, Lemma 2.6]). Moreover, since the Dirichlet spectra of S and L coincide (m_φ being an isometric isomorphism), we derive from Proposition 5.2 that $\lambda_0^D(L) > 0$. Hence, any $f \in C^\infty(M_1)$ admits a unique L-harmonic extension $\mathcal{H}_L f \in C^\infty(M_1)$. This gives rise to the Dirichlet-to-Neumann map

$$\Lambda_L \colon C^\infty(\partial M_1) \subset L^2_{\sqrt{V}}(\partial M_1) \to L^2_{\sqrt{V}}(\partial M_1), \ f \mapsto \nu(\mathcal{H}_L f).$$

It is standard that the spectrum $\sigma_L(M_1)$ of the Friedrichs extension of this map is discrete and the corresponding eigenfunctions are smooth.

Given a section $s \colon U \subset M_1 \to M_2$ and $f \colon G \to \mathbb{R}$, we denote by $f_s \colon p^{-1}(U) \to \mathbb{R}$ the function satisfying

$$f_s(\Phi(x, y)) = f(x)$$

for any $x \in G$ and $y \in U$.

Proposition 5.4 *([23, Lemma 4.6 and Proposition 4.7]) Fix a left-invariant metric on G. Then for any $r > 0$ and any bounded, open $W \subset G$, there exists $\chi \in C_c^\infty(G)$ with $\chi = 1$ in $W \smallsetminus B(\partial W, r)$, $\mathrm{supp}\,\chi \subset B(W, r/2)$, such that for any extensible section $s \colon U \to M_2$, there exists $C > 0$ independent from W, satisfying*

$$|\Delta(\chi_s)(z)| \le C \ \text{and} \ \|\,\mathrm{grad}(\chi_s)(z)\| \le C$$

for any $z \in p^{-1}(U)$.

The point of this proposition is that the constant depends only on the section, and not on the corresponding W. These functions are obtained from a partition of unity which is constructed by translates of a fixed function (conceptually related to the partition of unity of the previous section). The importance of this construction becomes more clear in the following consequence of Theorem 2.13, together with the Cheeger and Buser inequalities (more precisely, the main ingredient in the proof of the latter one).

Proposition 5.5 *([23, Corollary 2.11]) Suppose that G is non-compact, unimodular and amenable, and choose a left-invariant metric on it. Then for any $\varepsilon > 0$ and $r > 0$, there exists a bounded, open $W \subset G$ such that*

$$|B(\partial W, r)| < \varepsilon|W \smallsetminus B(\partial W, r)|.$$

The most technical part of the proof of Theorem 1.3 is contained in the following.

Proposition 5.6 *If G is unimodular and amenable, then for any $\lambda \in \sigma_L(M_1)$ and $\varepsilon > 0$, there exists $h \in C_c^\infty(M_2)$ such that $\|(\Lambda - \lambda)h\|_{L^2(\partial M_2)} < \varepsilon\|h\|_{L^2(\partial M_2)}$.*

Proof Since M_1 is compact, for any $\lambda \in \sigma_L(M_1)$, there exists $f \in C^\infty(M_1)$ with $\|f\|_{L^2_{\sqrt{V}}(\partial M_1)} = 1, Lf = 0$ in M_1 and $\nu_1(f) = \lambda f$ on ∂M_1. If G is compact, it is immediate to verify that its lift $\tilde{f} = f \circ p$ is harmonic and satisfies $\nu_2(\tilde{f}) = \lambda \tilde{f}$ on ∂M_2, by virtue of (7). Hence, it remains to prove the assertion in the case where G is non-compact.

To this end, cover M_1 with finitely many open domains U_i that admit extensible sections $s_i \colon U_i \subset M_1 \to M_2, i = 1, \ldots, k$, and consider a smooth partition of unity $\{\varphi_i\}_{1 \le i \le k}$ subordinate to $\{U_i\}_{1 \le i \le k}$. Denote by $x_{ij} \colon U_i \cap U_j \to G$ the transition maps, which are defined by $s_j(y) = x_{ij}(y)s_i(y)$ for all $y \in U_i \cap U_j$, and by $\Phi_i \colon G \times U_i \to p^{-1}(U_i)$ the diffeomorphisms defined by $\Phi_i(x, y) = xs_i(y), i, j = 1, \ldots, k$.

Choose a left-invariant metric g on G. Using that U_i is precompact and s_i is extensible, we readily see that there exists $r > 0$ such that $x_{ij}(U_i \cap U_j) \subset B_g(e, r)$ for any $i, j = 1, \ldots, k$, where e is the neutral element of G. Given a bounded, open $W \subset G$, denote by χ the corresponding function, according to Proposition 5.4, for r as above, where we regard G endowed with the fixed Riemannian metric g. Consider the compactly supported, smooth function

$$h_i := \chi_{s_i} \tilde{\varphi}_i \tilde{f}$$

in $p^{-1}(U_i), i = 1, \ldots, k$, where $\tilde{\varphi}_i = \varphi_i \circ p$ and $\tilde{f} = f \circ p$. For $h = \sum_{i=1}^k h_i$, we obtain from Proposition 5.4, that there exists $C_f > 0$ independent from W, such that $|h(z)| \le C_f$, $\|\operatorname{grad} h(z)\| \le C_f$ and $|\Delta h(z)| \le C_f$ for any $z \in M_2$. It follows from Proposition 5.5 that there exists a bounded, open $W \subset G$ such that

$$\frac{|W_0'|_g}{|W_0|_g} < \min\left\{ \frac{\varepsilon^2}{8(\lambda^2 + 1)C_f^2 \int_{\partial M_1} V}, \frac{\varepsilon^2}{4C^2 C_f^2 \int_{M_1} V} \right\}, \tag{8}$$

where $W_0' = B(\partial W, 3r), W_0 = W \smallsetminus W_0'$, and C is the constant from Proposition 3.6 on M_2. To simplify the notation, set $D_0 = W \smallsetminus C(\partial W, 2r), D_0' = C(W, 2r)$, and given $y \in U_i$, let $W_i(y) = \Phi_i(W_0, y), W_i'(y) = \Phi_i(W_0', y), D_i(y) = \Phi_i(D_0, y)$ and $D_i'(y) = \Phi_i(D_0', y)$, $i = 1, \ldots, k$. Here, $B(\cdot, \cdot)$ and $C(\cdot, \cdot)$ stand for open and closed tubular neighborhoods with respect to the fixed Riemannian metric g, respectively. Using that

$$\Phi_i(x, y) = \Phi_j(xx_{ji}(y), y)$$

for any $y \in U_i \cap U_j$ and $x \in G$, it is immediate to verify that $h(z) = \tilde{f}(z)$ for any $z \in D_i(y) \supset W_i(y)$ and that $\operatorname{supp} h \cap F_y \subset D_i(y) \cup D_i'(y) \subset W_i(y) \cup W_i'(y)$ for any $y \in U_i, i = 1, \ldots, k$.

By virtue of Proposition 5.3, we compute

$$\|h\|_{L^2(\partial M_2)}^2 = \sum_{i=1}^k \int_{\partial M_2} \tilde{\varphi}_i h^2 \ge \sum_{i=1}^k \int_{\partial M_1 \cap U_i} \int_{W_i(y)} \tilde{\varphi}_i h^2 \, \mathrm{d}y$$

$$= \sum_{i=1}^k \int_{\partial M_1 \cap U_i} \varphi_i(y) f^2(y) |W_0|_{g_{s_i(y)}} \, \mathrm{d}y$$

$$= |W_0|_g \sum_{i=1}^k \int_{\partial M_1 \cap U_i} \varphi_i f^2 V = |W_0|_g,$$

where we used that $\|f\|_{L^2_{\sqrt{V}}(\partial M_1)} = 1$. Moreover, it is evident that

$$\|v_2(h) - \lambda h\|^2_{L^2(\partial M_2)} = \sum_{i=1}^{k} \int_{\partial M_2} \tilde{\varphi}_i (v_2(h) - \lambda h)^2 = Q_1 + Q_2,$$

where

$$Q_1 = \sum_{i=1}^{k} \int_{\partial M_1 \cap U_i} \int_{W_i(y)} \tilde{\varphi}_i (v_2(h) - \lambda h)^2 \, \mathrm{dy}, \quad Q_2 = \sum_{i=1}^{k} \int_{\partial M_1 \cap U_i} \int_{W_i'(y)} \tilde{\varphi}_i (v_2(h) - \lambda h)^2 \, \mathrm{dy}.$$

To estimate these quantities, keeping in mind that D_0 is an open subset of G, it is immediate to verify that so is $\Phi_i(D_0 \times U_i) \subset M_2$, which is a neighborhood of $W_i(y)$ for any $y \in U_i \cap \partial M_1$, $i = 1, \ldots, k$. Since $h = \tilde{f}$ in $\Phi_i(D_0 \times U_i)$, in $W_i(y)$ we have that

$$v_2(h) = v_2(\tilde{f}) = \langle v_2, \operatorname{grad} \tilde{f} \rangle = \langle v_1, \operatorname{grad} f \rangle \circ p = \lambda \tilde{f} = \lambda h,$$

and thus, $Q_1 = 0$. In $W_i'(y)$, using Proposition 5.3 and that $(v_2(h) - \lambda h)^2 \leq 2C_f^2(\lambda^2 + 1)$, we readily see that

$$Q_2 \leq 2C_f^2(\lambda^2 + 1) \sum_{i=1}^{k} \int_{\partial M_1 \cap U_i} \varphi_i(y) |W_0'|_{g_{s_i(y)}} \, \mathrm{dy} = 2C_f^2(\lambda^2 + 1)|W_0'|_g \int_{\partial M_1} V.$$

Furthermore, it is apparent that

$$\|\Delta h\|^2_{L^2(M_2)} = \sum_{i=1}^{k} \int_{M_2} \tilde{\varphi}_i (\Delta h)^2 = Q_3 + Q_4,$$

where

$$Q_3 = \sum_{i=1}^{k} \int_{U_i} \int_{W_i(y)} \tilde{\varphi}_i (\Delta h)^2 \, \mathrm{dy}, \quad Q_4 = \sum_{i=1}^{k} \int_{U_i} \int_{W_i'(y)} \tilde{\varphi}_i (\Delta h)^2 \, \mathrm{dy}.$$

Using again that $h = \tilde{f}$ in $\Phi_i(D_0 \times U_i) \supset W_i(y)$, we obtain from (7) that $\Delta h = 0$ in $W_i(y)$, and hence, $Q_3 = 0$. Finally, Proposition 5.3 implies that

$$Q_4 \leq C_f^2 \sum_{i=1}^{k} \int_{U_i} \varphi_i(y) |W_0'|_{g_{s_i(y)}} \mathrm{dy} = C_f^2 |W_0'|_g \int_{M_1} V.$$

From the above estimates and Proposition 3.6, we conclude that

$$\frac{\|(\Lambda - \lambda)h\|^2_{L^2(\partial M_2)}}{\|h\|^2_{L^2(\partial M_2)}} \leq 2 \frac{\|v(h) - \lambda h\|^2_{L^2(\partial M_2)}}{\|h\|^2_{L^2(\partial M_2)}} + 2C^2 \frac{\|\Delta h\|^2_{L^2(M_2)}}{\|h\|^2_{L^2(\partial M_2)}} < \varepsilon^2,$$

by virtue of (8). $\square$

Another important ingredient in the proof of Theorem 1.3 involves the behavior of the Neumann spectrum under Riemannian submersions. One can establish the following by arguing as in [23]. However, in our setting, where the base manifold is compact, we can prove it in a simpler way (which also establishes the analogous assertion if the base manifold is closed).

Theorem 5.7 *If $\lambda_0^N(M_2) = 0$, then G is unimodular and amenable.*

Proof Cover M_1 with finitely many open domains U_i admitting extensible sections $s_i : U_i \subset M_1 \to M_2$, and denote by $\Phi_{i,y} : G \to F_y$ the diffeomorphism $\Phi_{i,y}(x) = x s_i(y)$ with $y \in U_i$, $i = 1, \ldots, k$. Let $\{\varphi_i\}_{1 \le i \le k}$ be a smooth partition of unity subordinate to $\{U_i\}_{1 \le i \le k}$. Fix a left-invariant Riemannian metric g on G and consider the functions $V_{s_i} \in C^\infty(U_i)$ from Proposition 5.3. Since s_i is extensible, it follows that there exists $c > 0$ such that

$$\|\mathrm{grad}_{g_{s_i(y)}} f\|_{g_{s_i(y)}} \ge c \|\mathrm{grad}_g f\|_g$$

for any $f \in C^\infty(G)$ and $y \in U_i$, $i = 1, \ldots, k$.

Since $\lambda_0^N(M_2) = 0$, we obtain from Proposition 2.5 that for any $\varepsilon > 0$ there exists a nonzero $f \in C_c^\infty(M_2)$ such that

$$
\begin{aligned}
\varepsilon &> \frac{\int_{M_2} \|\mathrm{grad}\, f\|^2}{\int_{M_2} f^2} \ge \frac{\int_{M_2} \|(\mathrm{grad}\, f)^v\|^2}{\int_{M_2} f^2} = \frac{\int_{M_1} \int_{F_y} \|\mathrm{grad}(f|_{F_y})\|^2 \, \mathrm{d}y}{\int_{M_1} \int_{F_y} f^2 \, \mathrm{d}y} \\
&= \frac{\sum_{i=1}^k \int_{U_i} \varphi_i(y) \int_{F_y} \|\mathrm{grad}(f|_{F_y})\|^2 \, \mathrm{d}y}{\sum_{i=1}^k \int_{U_i} \varphi_i(y) \int_{F_y} f^2 \, \mathrm{d}y} \\
&= \frac{\sum_{i=1}^k \int_{U_i} \varphi_i(y) \int_G \|\mathrm{grad}_{g_{s_i(y)}}(f \circ \Phi_{i,y})\|_{g_{s_i(y)}}^2 V_{s_i}(y) \, \mathrm{d}y}{\sum_{i=1}^k \int_{U_i} \varphi_i(y) \int_G (f \circ \Phi_{i,y})^2 V_{s_i}(y) \, \mathrm{d}y}
\end{aligned}
$$

where $(\mathrm{grad}\, f)^v$ stands for the vertical component of $\mathrm{grad}\, f$, and the integrals over G are with respect to the fixed Riemannian metric g. It is now clear that there exists $1 \le i \le k$ and $y \in U_i$ such that $\varphi_i(y) > 0$, f is not identically zero on F_y, and we have that

$$
\begin{aligned}
\varepsilon &> \frac{\varphi_i(y) \int_G \|\mathrm{grad}_{g_{s_i(y)}}(f \circ \Phi_{i,y})\|_{g_{s_i(y)}}^2 V_{s_i}(y)}{\varphi_i(y) \int_G (f \circ \Phi_{i,y})^2 V_{s_i}(y)} \ge c^2 \frac{\int_G \|\mathrm{grad}_g(f \circ \Phi_{i,y})\|_g^2}{\int_G (f \circ \Phi_{i,y})^2} \\
&= \mathcal{R}_g(f \circ \Phi_{i,y}).
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we conclude from Proposition 2.3 that $\lambda_0(G, g) = 0$, and therefore, G is unimodular and amenable, by Theorem 2.13. □

Proof of Theorem 1.3 If G is unimodular and amenable, then $\sigma_L(M_1) \subset \sigma(M_2)$ by virtue of Propositions 5.6 and 2.1. Taking into account that $0 \in \sigma_L(M_1)$, it evident that the second statement implies the third. Finally, if $\sigma_0(M_2) = 0$, then we derive from Theorem 3.5 that $\lambda_0^N(M_2) = 0$, and hence G is unimodular and amenable, in view of Theorem 5.7. □

References

1. Ballmann, W., Matthiesen, H., Polymerakis, P.: On the bottom of spectra under coverings. Math. Z. **288**(3–4), 1029–1036 (2018)
2. Ballmann, W., Polymerakis, P.: Bottom of spectra and coverings. Surv. Differ. Geom. **23**, 1–13 (2020)
3. Bessa, G.P., Montenegro, J.F., Piccione, P.: Riemannian submersions with discrete spectrum. J. Geom. Anal. **22**(2), 603–620 (2012)
4. Bordoni M.: Spectra of submersions, Contemporary geometry and related topics, Univ. Belgrade Fac. Math., Belgrade, pp. 51–62, available at https://emis.de/proceedings/CGRT2005/ (2006)
5. Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. **56**(4), 581–598 (1981)
6. Cavalcante, M.P., Manfio, F.: On the fundamental tone of immersions and submersions. Proc. Amer. Math. Soc. **146**(7), 2963–2971 (2018)
7. Dunn, C., Gilkey, P., Park, J.: The spectral geometry of the canonical Riemannian submersion of a compact Lie group. J. Geom. Phys. **57**(10), 2065–2076 (2007)
8. Eschenburg, J.-H., Heintze, E.: Comparison theory for Riccati equations. Manuscripta Math. **68**(2), 209–214 (1990)
9. Falcitelli, M., Ianus, S., Pastore, A.M.: Riemannian Submersions and Related Topics. World Scientific Publishing, River Edge, NJ (2004)
10. Følner, E.: On groups with full Banach mean value. Math. Scand. **3**, 243–254 (1955)
11. Fraser, A., Schoen, R.: The first Steklov eigenvalue, conformal geometry, and minimal surfaces. Adv. Math. **226**(5), 4011–4030 (2011)
12. Gilkey, P., Leahy, J., Park, J.: Spectral Geometry, Riemannian Submersions, and the Gromov- Lawson Conjecture, Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL (1999)
13. Girouard, A., Polterovich, I.: Shape optimization for low Neumann and Steklov eigenvalues. Math. Methods Appl. Sci. **33**(4), 501–516 (2010)
14. Greenleaf F.P.: Invariant means on topological groups and their applications, Van Nostrand Math- ematical Studies, No. 16, Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, (1969)
15. Grosse N., Nistor V.: Neumann and mixed problems on manifolds with boundary and bounded geometry hal-01493070 (2017)
16. Hislop, P.D., Sigal, I.M.: Introduction to Spectral Theory: With Applications to Schrödinger Operators, Applied Mathematical Sciences, vol. 113. Springer-Verlag, New York (1996)
17. Hoke, H.F., III.: Lie groups that are closed at infinity. Trans. Amer. Math. Soc. **313**(2), 721–735 (1989)
18. Kato, T.: Perturbation Theory for Linear Operators, Classics in Mathematics. Springer Verlag, Berlin (1995)
19. Milnor, J.: On fundamental groups of complete affinely flat manifolds. Adv. in Math. **25**(2), 178–187 (1977)
20. Polymerakis, P.: Coverings preserving the bottom of the spectrum. J. Spectr. Theory (2018)
21. Polymerakis, P.: On the spectrum of differential operators under Riemannian coverings. J. Geom. Anal. **30**(3), 3331–3370 (2020)
22. Polymerakis, P.: Spectral estimates and discreteness of spectra under Riemannian submersions. Ann. Global Anal. Geom. **57**(2), 349–363 (2020)
23. Polymerakis, P.: Spectral estimates for Riemannian submersions with fibers of basic mean curvature. J. Geom. Anal. **31**(10), 9951–9980 (2021)
24. Rosenblatt, J.M.: A generalization of Følner's condition. Math. Scand. **33**, 153–170 (1973)
25. Schick, T.: Manifolds with boundary and of bounded geometry. Math. Nachr. **223**, 103–120 (2001)