Annals of Global Analysis and Geometry (2023) 63:5
https://doi.org/10.1007/s10455-022-09881-x

Complexes, residues and obstructions for log-symplectic manifolds

Ziv Ran[1]

Received: 7 April 2021 / Accepted: 29 September 2022 / Published online: 7 November 2022

Abstract

We consider compact Kählerian manifolds X of even dimension 4 or more, endowed with a log-symplectic structure Φ, a generically nondegenerate closed 2-form with simple poles on a divisor D with local normal crossings. A simple linear inequality involving the iterated Poincaré residues of Φ at components of the double locus of D ensures that the pair (X, Φ) has unobstructed deformations and that D deforms locally trivially.

Keywords Poisson structure · Log-symplectic manifold · Deformation theory · Log complex · Mixed Hodge theory

Mathematics Subject Classification 14J40 · 32G07 · 32J27 · 53D17

Introduction

A log-symplectic manifold is a pair consisting of a complex manifold X, usually compact and Kählerian, together with a log-symplectic structure. A log-symplectic structure can be defined either as a generically nondegenerate meromorphic closed 2-form Φ with normal-crossing (anticanonical) polar divisor D, or equivalently as a generically nondegenerate holomorphic tangential 2-vector Π such that $[\Pi, \Pi] = 0$ with normal-crossing degeneracy divisor D. The two structures are related via $\Pi = \Phi^{-1}$. See [3] or [11] or [2] or [12] for basic facts on Poisson and log-symplectic manifolds and [4] (especially the appendix), [5, 7, 8] or [10], and references therein, for deformations.

Understanding log-symplectic manifolds unavoidably involves understanding their deformations. In the very special case of *symplectic* manifolds, where $D = 0$, the classical theorem of Bogomolov [1] shows that the pair (X, Φ) has unobstructed deformations. In [13] we obtained a generalization of this result which holds when Φ satisfied a certain 'very general position' condition with respect to D (the original statement is corrected in the subsequent erratum/corrigendum). Namely, we showed in this case that (X, Φ) has 'strongly unob-

---

✉ Ziv Ran
ziv.ran@ucr.edu
http://math.ucr.edu/∼ziv/

[1] UC Mathematics Department, Big Springs Road Surge Facility, Riverside, CA 92521, USA

structed' deformations, in the sense that it has unobstructed deformations and D deforms locally trivially.

Further results on unobstructed deformations (in the sense of Hitchin's generalized geometry [6]) and Torelli theorems in the case where D has global normal crossings were obtained by Matviichuk, Pym and Schedler [9], based on their notion of holonomicity.

Our purpose here is to prove a more precise strong unobstructedness result compared to [13], nailing down the generality required: we will show in Theorem 6 that strong unobstructedness can fail only when the log-symplectic structure Φ, more precisely its (iterated Poincaré) residues at codimension-2 strata of the polar divisor D (which are essentially the (locally constant) coefficients of Φ with respect to a suitable basis of the log forms adapted to D) satisfy certain special linear relations with integer coefficients. Explicitly, at a triple point of D with branches labeled 1,2,3 and associated residues c_{12}, c_{23}, c_{31}, the condition is

$$c_{23} + c_{31} \in \mathbb{N}c_{12}.$$

Essentially, if this never happens over the entire triple locus then (X, Φ) has strongly unobstructed deformations.

The strategy of the proof as in [13] is to study the inclusion of complexes

$$(T^\bullet_X\langle -\log D\rangle, [\ .\ , \Pi]) \to (T^\bullet_X, [\ .\ , \Pi])\ ,$$

albeit from a more global viewpoint. In fact as in [13] it turns out to be more convenient to transport the situation over to the De Rham side where it becomes an inclusion

$$(\Omega^\bullet_X\langle\log D\rangle, d) \to (\Omega^\bullet_X\langle\log^+ D\rangle, d)$$

where the latter 'log-plus' complex is a certain complex of meromorphic forms with poles on D. We study a filtration, introduced in [13], interpolating between the two complexes, especially its first two graded pieces. As we show, the first piece is automatically exact, while 0-acyclicity for the second piece leads to the above cocycle condition. See Sect. 3 for details.

We begin the paper with a couple of auxiliary, independent sections. In Sect. 1 we construct a 'principal parts complex' associated to an invertible sheaf L on a smooth variety, extending the principal parts sheaf $P(L)$ together with the universal derivation $L \to P(L)$. We show this complex is always exact. In Sect. 2 we show that, for any normal-crossing divisor $D \subset X$ on any smooth variety, the log complex $\Omega^\bullet_X\langle\log D\rangle$—unlike $\Omega^\bullet_X$ itself—can be pulled back to a complex of vector bundles on the normalization of D. These complexes play a role in our analysis of the aforementioned inclusion map.

I am grateful to Brent Pym for helpful communications, in particular for communicating Example 8.

1 Principal parts complex

In this section X denotes an arbitrary n-dimensional smooth complex variety and L denotes an invertible sheaf on X.

1.1 Principal parts

The Grothendieck principal parts sheaf $P(L)$ (see EGA) is a rank-$(n+1)$ bundle on X defined as

$$P(L) = p_{1*}(p_2^*L \otimes (\mathcal{O}_{X\times X}/\mathcal{I}_\Delta^2))$$

where $\Delta \subset X \times X$ is the diagonal and $p_1, p_2 : X \times X \to X$ are the projections. We have a short exact sequence

$$0 \to \Omega^1_X \otimes L \to P(L) \to L \to 0$$

whose corresponding extension class in $\mathrm{Ext}^1(L, \Omega^1_X \otimes L) = H^1(X, \Omega^1_X)$ coincides with $c_1(L)$. The sheaf

$$P_0(L) = P(L) \otimes L^{-1},$$

which likewise has extension class $c_1(L)$, is called the *normalized* principal parts sheaf. The map $P(L) \to L$ admits a splitting $d_L : L \to P(L)$ that is a derivation, i.e.,

$$d_L(fu) = f d_L u + df \otimes u.$$

In fact, d_L the universal derivation on L. Moreover, $P(L)$ is generated over $\mathcal{O}_X$ by the image of d_L. Likewise, $P_0(L)$ is generated by elements of the form $\mathrm{dlog}(u) := d_L u \otimes u^{-1}$ where u is a local generator of L.

1.2 Complex

It is well known that $P(L^{m+1}) \simeq P(L) \otimes L^m, m \geq 0$ which in particular yields a derivation $L^{n+1} \to P(L) \otimes L^n, n \geq 0$. In fact, this map extends to a complex that we denote by $P^\bullet_{n+1}(L)$ or just $P^\bullet(L)$ and call the ($(n+1)$st) *principal parts complex* of L:

$$P^\bullet(L) : L^{n+1} \to P(L)L^n \to \wedge^2 P(L)L^{n-1} \to \ldots \wedge^{n+1} P(L) = \Omega^n_X \otimes L^{n+1}. \qquad (1)$$

The differential is given, in terms of local $\mathcal{O}_X$-generators $u_1, \ldots, u_k, v_1, \ldots, v_\ell$ of L, by

$$d(u_1 \ldots u_k d_L(v_1) \wedge ...d_L(v_\ell) = \sum u_1 ... \hat{u_i} \ldots u_k d_L(u_i) \wedge d_L(v_1) \wedge \ldots \wedge d_L(v_\ell)$$

and extending using additivity and the derivation property. There are also similar shorter complexes

$$L^m \to P(L)L^{m-1} \to \ldots \to \wedge^m P(L).$$

Note the exact sequences

$$0 \to \Omega^m_X L^m \to \wedge^m P(L) \to \Omega^{m-1}_X L^m \to 0.$$

These sequences splits locally and also split globally whenever L is a flat line bundle. In such cases, we get a short exact sequence

$$0 \to \Omega^\bullet_X L^{n+1}[-1] \to P^\bullet(L) \to \Omega^\bullet_X L^{n+1} \to 0 \qquad (2)$$

The principal parts complex $P^\bullet(L)$ may be tensored with L^{j-n-1}, for any $j > 0$, yielding the j-th principal parts complex:

$$P^\bullet_j(L) : L^j \to P_0(L)L^j \to \wedge^2 P_0(L)L^j \to \ldots \to \wedge^{n+1} P_0(L)L^j \qquad (3)$$

The differential is defined by setting

$$d(\mathrm{dlog}(u_1) \wedge ... \wedge \mathrm{dlog}(u_i)v^j) = j\, \mathrm{dlog}(u_1) \wedge ... \wedge \mathrm{dlog}(u_i)\, \mathrm{dlog}(v) v^j$$

where $u_1, ..., u_i, v$ are local generators for L, and extending by additivity and the derivation property. Thus, $P^\bullet(L) = P^\bullet_{n+1}(L)$.

An important property of principal parts complexes is the following:

Proposition 1 *For any local system S, the complexes $P^\bullet_j(L) \otimes S$ are null-homotopic and exact for all $j > 0$.*

Proof The assertion being local, we may assume L is trivial and $S = \mathbb{C}$, so the i-th term of $P^\bullet_j(L) \otimes S$ is just $\Omega^{i-1}_X \oplus \Omega^i_X$ and the differential is $\begin{pmatrix} d & \mathrm{id} \\ 0 & d \end{pmatrix}$. Then, a homotopy is given by $\begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}$. Thus, $P^\bullet_j(L)$ is null-homotopic, hence exact. □

1.3 Log version

The above constructions have an obvious extension to the log situation. Thus, let D be a divisor with normal crossings on X. We define $P(L)\langle \log D\rangle$ as the image of $P(L)$ under the inclusion $\Omega_X \to \Omega_X\langle \log D\rangle$, and likewise for $P_0(L)\langle \log D\rangle$. Then, as above we get complexes

$$P^\bullet_j(L)\langle \log D\rangle : L^j \to P_0(L)\langle \log D\rangle L^j \to \ldots \to \wedge^{n+1} P_0(L)\langle \log D\rangle L^j. \tag{4}$$

1.4 Foliated version

Let $F \subset \Omega_X\langle \log D\rangle$ be an integrable subbundle of rank m. Then, F gives rise to a foliated De Rham complex $\wedge^\bullet(\Omega_X\langle \log D\rangle/F)$, we well as a foliated principal parts sheaf $P^1_F(L)\langle \log D\rangle = P^1(L)\langle \log D\rangle/F \otimes L$. Putting these together, we obtain the foliated principal parts complexes (where $P_{0,F}(L)\langle \log D\rangle := P_0(L)\langle \log D\rangle/F$):

$$P^\bullet_{j,F}(L)\langle \log D\rangle : L^j \to P_{0,F}(L)\langle \log D\rangle L^j \to \ldots \to \wedge^{n-m+1} P_{0,F}(L)\langle \log D\rangle \tag{5}$$

Note that the proof of Proposition 1 made no use of the acyclicity of the De Rham complex. Hence, the same proof applies verbatim to yield

Proposition 2 *For any local system S, the complexes $P^\bullet_{j,F}(L)\langle \log D\rangle \otimes S$ are null-homotopic and exact for all $j > 0$.*

2 Calculus on normal crossing divisors

In this section X denotes a smooth variety or complex manifold and D denotes a locally normal-crossing divisor on X. Our aim is to show that the log complex on X, unlike its De Rham analogue, can be pulled back to the normalization of D.

2.1 Branch normal

Let $f_i : X_i \to X$ be the normalization of the i-fold locus of D. A point on X_i consists of a point on D together with a choice of i distinct local branches of D at it. There is a canonical induced normal-crossing divisor D_i on X_i: at a point where $x_1 \ldots x_m$ is an equation for D and $x_1, \ldots, x_i$ are the chosen branches, the equation of D_i is $x_{i+1} \ldots x_m$. Note the exact sequence

$$0 \to T_X\langle -\log D\rangle \to T_X \to f_{1*}N_{f_1} \to 0 \tag{6}$$

where N_{f_1} is the normal bundle to f_1 which fits in an exact sequence

$$0 \to T_{X_1} \to f_1^* T_X \to N_{f_1} \to 0.$$

Locally, N_{f_1} coincides with $x_1^{-1}\mathcal{O}_X/\mathcal{O}_X$ where x_1 is a 'branch equation': to be precise, if K denotes the kernel of the natural surjection $f_1^{-1}\mathcal{O}_X \to \mathcal{O}_{X_1}$, then $J = K/K^2 = K \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X_1}$ is an invertible $\mathcal{O}_{X_1}$-module locally generated by x_1 and $N_{f_1} = J^{-1}$. Note that

$$N_{f_1} \otimes \mathcal{O}_{X_1}(D_1) = f_1^*(\mathcal{O}_X(D)).$$

2.2 Pulling back log complexes

Interestingly, even though the differential on the pullback De Rham complex $f_1^{-1}\Omega_X^\bullet$ does not extend to $f^{-1}\Omega_X^\bullet \otimes \mathcal{O}_{X_1}$, the analogous assertion for the log complex does hold: the differential on $f_1^{-1}\Omega_X^\bullet\langle\log D\rangle$ extends to what might be called the restricted log complex:

$$f_1^*\Omega_X^\bullet\langle\log D\rangle = f_1^{-1}\Omega_X^\bullet\langle\log D\rangle \otimes \mathcal{O}_{X_1}.$$

This is due to the identity (where x_1 denotes a branch equation)

$$dx_1 = x_1\,\mathrm{dlog}(x_1).$$

Note that the residue map yields an exact sequence

$$0 \to \Omega^1_{X_1}\langle\log D_1\rangle \overset{j}{\to} f_1^*\Omega^1_X\langle\log D\rangle \overset{\mathrm{Res}}{\to} \mathcal{O}_{X_1} \to 0. \tag{7}$$

Note that the residue map commutes with exterior derivative. Therefore, this sequence induces a short exact sequence of complexes

$$0 \to \Omega^\bullet_{X_1}\langle\log D_1\rangle \to f_1^*\Omega^\bullet_X\langle\log D\rangle \to \Omega^\bullet_{X_1}\langle\log D_1\rangle[-1] \to 0. \tag{8}$$

Furthermore, a twisted form of the restricted log complex, called the normal log complex, also exists:

$$N_{f_1} \otimes f_1^*\Omega^\bullet_X\langle\log D\rangle : N_{f_1} \to N_{f_1} \otimes f_1^*\Omega^1_X\langle\log D\rangle \to \dots \tag{9}$$

this is thanks to the identity, where ω is any log form,

$$d(\omega/x_1) = (d\omega)/x_1 - \mathrm{dlog}(x_1) \wedge \omega/x_1.$$

Now recall the exact sequence coming from the residue map

$$0 \to \Omega_{X_1}\langle\log D_1\rangle \to f_1^*\Omega_X\langle\log D\rangle \to \mathcal{O}_{X_1} \to 0$$

In fact, it is easy to check that this exact sequence has extension class $c_1(N_{f_1})$ hence identifies $f_1^*\Omega_X\langle\log D\rangle$ with $P_0(N_{f_1})$ so that the normal log complex (9) may be identified with the principal parts complex $P^\bullet(N_{f_1})$:

Lemma 3 *The normal log complex $N_{f_1} \otimes f_1^*\Omega_X\langle\log D\rangle$ is isomorphic to $P^\bullet(N_{f_1})$, hence is exact.*

Similarly, a pull back log complex $f_k^*\Omega_X^\bullet\langle\log D\rangle = f_k^{-1}\Omega_X^\bullet\langle\log D\rangle \otimes \mathcal{O}_{X_k}$ exists for all $k \geq 1$. A similar twisted log complex also exists the determinant of the normal bundle N_{f_k}:

$$\det N_{f_k} \otimes f_k^*\Omega_X^\bullet\langle\log D\rangle : \det N_{f_k} \to \det N_{f_k} \otimes \Omega_X^1\langle\log D\rangle \to \dots \tag{10}$$

This comes from (where $x_1, \dots, x_k$ are the branch equations at a given point of X_k):

$$d(\omega/x_1 \dots x_k) = d\omega/x_1 \dots x_k - \mathrm{dlog}(x_1 \dots x_k)\omega/x_1 \dots x_k).$$

2.3 Iterated residue

We have a short exact sequence of vector bundles on X_k:

$$0 \to \Omega_{X_k}\langle\log D_k\rangle \to f_k^*\Omega_X\langle\log D\rangle \to \nu_k \otimes \mathcal{O}_{X_k} \to 0 \tag{11}$$

where ν_k is the local system of branches of D along X_k and the right map is multiple residue. Taking exterior powers, we get various exact Eagon–Northcott complexes. In particular, we get surjections, called iterated Poincaré residue:

$$f_k^*\Omega_X^i\langle\log D\rangle \to \Omega_{X_k}^{i-k}\langle\log D_k\rangle \otimes \det{}_{\mathbb{C}}(\nu_k),\ i \geq k, \tag{12}$$

$$f_k^*\Omega_X^i\langle\log D\rangle \to \wedge_{\mathbb{C}}^i \nu_k \otimes \mathcal{O}_{X_k},\ i \leq k. \tag{13}$$

$\det_{\mathbb{C}}(\nu_k)$ is a rank-1 local system on X_k which may be called the 'normal orientation sheaf.' The maps for $i \geq k$ together yield a surjection

$$f_k^*\Omega_X^\bullet\langle\log D\rangle \to \Omega_{X_k}^\bullet\langle\log D_k\rangle[-k] \otimes \det(\nu_k). \tag{14}$$

3 Comparing log and log plus complexes

In this section X denotes a log-symplectic smooth variety with log-symplectic form Φ and corresponding Poisson vector $\Pi = \Phi^{-1}$, and D denotes the degeneracy divisor of Π or polar divisor of Φ. Our aim is to prove Theorem 6 which shows that deformations of (X, Φ) coincide with locally trivial deformations of (X, Φ, D) and are unobstructed.

3.1 Setting up

We will use $\Omega_X^{+\bullet}$ to denote $\bigoplus\limits_{i>0} \Omega_X^i$ and similarly for the log versions. This is to match with the Lichnerowicz–Poisson complex $T_X^\bullet$ and $T_X^\bullet\langle-\log D\rangle$. Thus, interior multiplication by Φ induces and isomorphism $T_X^\bullet\langle-\log D\rangle \to \Omega_X^\bullet\langle\log D\rangle$. Equivalently, Φ itself is a form in $\Omega_X^2\langle\log D\rangle$ inducing a nondegenerate pairing on $T_X\langle-\log D\rangle$. In terms of local coordinates, at a point of multiplicity m on D, we have a basis for $\Omega_X\langle\log D\rangle$ of the form

$$\eta_1 = \mathrm{dlog}(x_1), \dots, \eta_m = \mathrm{dlog}(x_m), \eta_{m+1} = \mathrm{dlog}(x_{m+1}), \dots$$

and then

$$\Phi = \sum b_{ij}\eta_i \wedge \eta_j.$$

We have an inclusion of complexes

$$T_X^\bullet\langle-\log D\rangle \to T_X^\bullet$$

where, for X compact Kähler, the first complex controls 'locally trivial' deformations of (X, Π), i.e., deformations of (X, Π) inducing a locally trivial deformation of $D = [\Pi^n]$, and the second complex controls all deformations of (X, Π). It is known (see, e.g., [13]) that locally trivial deformations of (X, Π) are always unobstructed and have an essentially Hodge-theoretic (hence topological) character, so one is interested in conditions to ensure that the above inclusion induces an isomorphism on deformation spaces; as is well known, the latter would follow if one can show that the cokernel of this inclusion has vanishing $\mathbb{H}^1$.

Our approach to this question starts with the above 'multiplication by Φ' isomorphism

$$(T^\bullet_X\langle-\log D\rangle, [\,.\,, \Pi]) \to (\Omega^{+\bullet}_X\langle\log D\rangle, d).$$

This isomorphism extends to an isomorphism to $T^\bullet_X$ with a certain subcomplex of $\Omega^{+\bullet}_X(*D)$, the meromorphic forms regular off D, that we call the log plus complex and denote by $\Omega^{+\bullet}_X\langle\log{}^+D\rangle$.

Our goal then becomes that of comparing the log and log-plus complexes. To this end we introduce a filtration on $\Omega^{+\bullet}_X\langle\log{}^+D\rangle$, essentially the filtration induced by the exact sequence

$$0 \to T_X\langle-\log D\rangle \to T_X \to f_{1*}N_{f_1} \to 0$$

and its isomorphic copy

$$0 \to \Omega_X\langle\log D\rangle \to \Omega_X\langle\log{}^+D\rangle \to f_*N_{f_1} \to 0$$

where $f_1 : X_1 \to D \subset X$ is the normalization of D and N_{f_1} is the associated normal bundle ('branch normal bundle'). We will show that the first graded piece is always an exact complex. The second graded piece is much more subtle. We will show that it is locally exact in degree 0 unless the log-symplectic form Φ, i.e., the matrix (b_{ij}) above satisfies some special relations with integer coefficients.

The computations of this section are all local in character, though the applications are global.

3.2 Residues and duality

Let $f_i : X_i \to X$ be the normalization of the i-fold locus of D, D_i the induced normal-crossing divisor on X_i. Thus, a point of X_i consists of a point p of D together with a choice of an unordered set S of i branches of D through p and D_i is the union of the branches of D not in S. We consider first the codimension-1 situation. As above, we have a residue exact sequence

$$0 \to \Omega^1_{X_1}\langle\log D_1\rangle \overset{j}{\to} f_1^*\Omega^1_X\langle\log D\rangle \overset{\mathrm{Res}}{\to} \mathcal{O}_{X_1} \to 0 \tag{15}$$

(the right-hand map given by residue is locally evaluation on $x_1 @_{x_1}$ where x_1 is a local equation for the branch of D through the given point of X_1). Note that if η comes from a closed form on X near D, then $\mathrm{Res}(\eta)$ is a constant.

Dualizing (15), we get

$$0 \to \mathcal{O}_{X_1} \overset{\check{R}_1}{\to} f_1^*T_X\langle-\log D\rangle \overset{\check{j}}{\to} T_{X_1}\langle-\log D_1\rangle \to 0, \tag{16}$$

where the left-hand map, the 'co-residue,' is locally multiplication by $x_1 @_{x_1}$ where x_1 is a branch equation). Set

$$v_1 = x_1 @_{x_1}.$$

Then, v_1 is canonical as section of $f_1^* T_X\langle -\log D\rangle$, independent of the choice of local equation x_1. By contrast, $@_{x_1}$ as section of $f_1^* T_X$ is canonical only up to a tangential field to X_1, and generates $f_1^* T_X$ modulo $T_{X_1}\langle -\log D\rangle$.

Now $f_1^* \Omega_X^1\langle \log D\rangle$ and $f_1^* T_X\langle -\log D\rangle$ admit mutually inverse isomorphisms

$$i_{X_1}\Pi := \langle \Pi, .\rangle_{X_1} = f_1^*\langle \Pi, .\rangle, i_{X_1}\Phi := \langle \Phi, .\rangle_{X_1} = f_1^*\langle \Phi, .\rangle.$$

The composite

$$\check{j} \circ i_{X_1}\Pi \circ j : \Omega^1_{X_1}\langle \log D_1\rangle \to T_{X_1}\langle -\log D_1\rangle$$

has a rank-1 kernel that is the kernel of the Poisson vector on X_1 induced by Π, aka the conormal to the symplectic foliation on X_1. Now set

$$\psi_1 = i_{X_1}(\Phi)(v_1) = \langle \Phi, v_1\rangle_{X_1}.$$

Then, ψ_1 is locally the form in $\Omega_{X_1}\langle \log D_1\rangle$ denoted by $x_1\phi_1$ in [13]. Again ψ_1 is canonically defined, independent of choices and corresponds to the first column of the $B = (b_{ij})$ matrix for a local coordinate system $x_1, x_2, \ldots$ compatible with the normal-crossing divisor D. By contrast, ϕ_1, which depends on the choice of local equation x_1, is canonical up to a log form in $\Omega_{X_1}\langle \log D_1\rangle$ and generates $\Omega_{X_1}\langle \log^+ D_1\rangle$ modulo the latter.

In $X_1 \setminus D_1$, Φ is locally of the form $\mathrm{dlog}(x_1) \wedge dx_2$+(symplectic), so there $\psi_1 = dx_2$. Note that by skew-symmetry we have

$$\mathrm{Res} \circ i_{X_1}(\Phi) \circ \check{R}_1 = 0.$$

Thus, locally $\psi_1 \in \Omega_{X_1}\langle \log D_1\rangle$. In terms of the matrix B above, $\psi_1 = \sum_{j>1} b_{1j}\, \mathrm{dlog}(x_j)$. Note that ψ_1 which corresponds to the Hamiltonian vector field v_1, is a closed form. Consequently, ψ_1 defines a foliation on X_1. Let $Q_1^\bullet = \psi_1\Omega^\bullet_{X_1}$ be the associated foliated De Rham complex $\psi_1\Omega^\bullet_{X_1}$:

$$Q_1^0 = \mathcal{O}_{X_1}\phi_1 \to Q_1^1 = \psi_1\Omega^1_{X_1} \simeq \Omega^1_{X_1}/\mathcal{O}_{X_1}\psi_1 \to \ldots \to Q_1^i = \wedge^i Q_1^1 \to \ldots$$

endowed with the foliated differential.

Note that the residue exact sequence (15) induces the Poincaré residue sequence

$$0 \to \Omega^\bullet_{X_1}\langle \log D_1\rangle \to f_1^*\Omega^\bullet_X\langle \log D\rangle \to \Omega^\bullet_{X_1}\langle \log D_1\rangle[-1] \to 0.$$

Again the Poincaré residue of a closed form is closed. Now the exact sequence

$$0 \to T_X\langle -\log D\rangle \to T_X \to f_{1*}N_{f_1} \to 0$$

yields

$$0 \to \Omega_X\langle \log D\rangle \to \Omega_X\langle \log^+ D\rangle \to f_{1*}N_{f_1} \to 0. \tag{17}$$

and this sequence induces the $\mathcal{F}_\bullet$ filtration on the log-plus complex $\Omega^\bullet_X\langle \log^+ D\rangle$.

3.3 First graded piece

Now consider first the first graded $\mathcal{G}_1^\bullet = (\mathcal{F}_1^\bullet/\mathcal{F}_0^\bullet)[1]$ which is supported in codimension 1. (the shift is so that $\mathcal{G}^\bullet$ starts in degree 0). Then, $\mathcal{G}_1^\bullet$ is a (finite) direct image of a complex of X_1 modules:

$$\mathcal{E}_1 : N_{f_1} \to N_{f_1} \otimes Q_1 \to N_{f_1} \otimes Q_1^2 \to \ldots$$

Using Lemma 3, we can easily show:

Proposition 4 *$\mathcal{E}_1$ is isomorphic to $P^{\bullet}_{R_1'}(N_{f_1})$, hence is null-homotopic and exact, hence $\mathcal{G}_1^{\bullet}$ is exact.*

3.4 Second graded piece

Next we study $\mathcal{G}_2$, which is supported on X_2. We consider a connected, nonempty open subset $W \subset X_2$, for example an entire component, over which the 'normal orientation sheaf' $\nu_2 : X_{2,1} \to X_2$, i.e., the local $\mathbb{Z}_2$-system of branches of X_1 along X_2, is trivial (we can take $W = X_2$ if, e.g., D has global normal crossings). Such a subset W of X_2 is said to be a *normally split* subset of X_2, and a *normal splitting* of W is an ordering of the branches is specified. Obviously X_2 is covered by such subsets W. Likewise, for a subset $Z \subset X_k$.

3.4.1 Iterated residue

Over a normally split subset W, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}_W \arrow[r, "\check{R}_2"] & f_2^* T_X\langle -\log D\rangle|_W \arrow[r] \arrow[d] & T_{X_2}\langle -\log D_2\rangle|_W \arrow[r] & 0 \\
0 \arrow[r] & \Omega_W\langle \log D_2\rangle \arrow[r] & f_2^*\Omega_X\langle \log D\rangle|_W \arrow[r, "R_2"] & 2\mathcal{O}_W \arrow[r] & 0
\end{tikzcd} \tag{18}$$

where $\check{R}_2$ is the map induced by $\check{R}_1$. The composite map $R_2\check{R}_2 : 2\mathcal{O}_W \to 2\mathcal{O}_W$ is just the alternating form induced by Φ and has the form $c_W H_2$ where H_2 is the hyperbolic plane $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In terms of a local frame for $\Omega_X\langle \log D\rangle$ containing $\mathrm{dlog}(x_1)$, $\mathrm{dlog}(x_2)$, c_W is the coefficient of $\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2)$ in Φ. Note c_W must be constant because Φ is closed. In fact, we have

$$c_W = \mathrm{Res}_1\mathrm{Res}_2(\Phi)$$

where Res_i denotes the (Poincaré) residues along the branches of X_1 over X_2. Set

$$\mathrm{Res}_W(\Phi) := c_W.$$

This is essentially what is called the biresidue by Matviichuk et al., see [9]. Thus, when $c_W \neq 0$, we have a basis for the log forms

$$\eta_1 = \mathrm{dlog}(x_1), \ldots, \eta_m = \mathrm{dlog}(x_m), \eta_{m+1} = dx_{m+1}, \ldots, \eta_2 n = dx_{2n}$$

$m =$ multiplicity of D, $m \geq 2$, and then

$$\Phi = \sum b_{ij}\eta_i \wedge \eta_j$$

where

$$b_{12} = -b_{21} = c_W.$$

If W may be not be normally orientable (e.g., an entire component of X_2), then c_W is defined only up to sign; if $c_W = 0$, we say that W is nonresidual; otherwise, it is residual.

3.4.2 Nonresidual case

Here we consider the case $c_W = 0$.

Note that in that case we may express Φ along W in the form

$$\Phi = \mathrm{dlog}(x_1)\gamma_3 + \mathrm{dlog}(x_2)\gamma_4 + \gamma_5$$

where the gammas are closed log forms in the coordinates on W, i.e., $x_3, \dots, x_{2n}$. Moreover, $\gamma_3 \wedge \gamma_4 \neq 0$ because Φ^n is divisible by $\mathrm{dlog}(x_1)\,\mathrm{dlog}(x_2)$. Also, unless γ_3, γ_4 are both holomorphic (pole-free), there is another component W' of X_2 such that $c_{W'} \neq 0$ (in particular, $W \cap D_2 \neq \emptyset$). Hence, if no such W' exists, we may by suitably modifying coordinates, assume locally that $\gamma_3 = dx_3, \gamma_4 = dx_4$. A similar argument, or induction, applies to γ_5. This means we are essentially in the P-normal case considered in [14]. This we conclude:

Lemma 5 *Unless Π is P-normal, there exists a nonempty residual open subset W of X_2.*

3.4.3 Residual case: identifying $\mathcal{G}_2$

Next we analyze a residual normally oriented open subset $W \subset X_2$. As above, we get a composite map of $R_2' : 2\mathcal{O}_W \to f_2^*\Omega_X\langle\log D\rangle|_W$, whose image we denote by M_{2W}. It has a local basis $(\psi_{11} = x_1\phi_1, \psi_{12} = x_2\phi_2)$ corresponding to the basis (e_1, e_2) of $2\mathcal{O}_W$. In terms of the B-matrix, we have

$$\psi_{11} = \sum b_{1j}\eta_j = -\sum b_{j1}\eta_j, \psi_{12} = -\sum b_{2j}\eta_j = \sum b_{j2}\eta_j.$$

As ψ_{11}, ψ_{12} are closed, M_2 is integrable. Let $\bar\Omega$ denote the quotient $f_2^*\Omega_X\langle\log D\rangle|_W/M_{2W}$. Then, we have an isomorphism

$$\bar\Omega \to \Omega_W\langle\log D_2\rangle \tag{19}$$

given explicitly by

$$\bar\omega \mapsto \omega - \mathrm{Res}_1(\omega)\psi_{12}/c_W - \mathrm{Res}_2(\omega)\psi_{11}/c_W$$

(because $\mathrm{Res}_2(\psi_{11}) = \mathrm{Res}_1(\psi_{12}) = c_W$, residues with respect to the two branches of D). Now set $N_2 = \det N_{f_2}$, an invertible sheaf on X_2. Then, $\mathcal{G}_2^\bullet = (\mathcal{F}_2^\bullet/\mathcal{F}_1^\bullet)[2]$ is the direct image of a complex on X_2:

$$\mathcal{E}_2^\bullet : N_2 \to N_2 \otimes \bar\Omega \to N_2 \otimes \wedge^2\bar\Omega \to \dots \tag{20}$$

where a local generator of N_2 has the form $1/x_1x_2$ and the differential has the form

$$\bar\omega/x_1x_2 \mapsto d\bar\omega/x_1x_1 \pm (\bar\omega/x_1x_2)\,\mathrm{dlog}(x_1x_2).$$

3.4.4 Zeroth differential

Using the identification (19), the zeroth differential has the form

$$\tilde d(g/x_1x_2) = \frac{1}{x_1x_2}(dg + g(\mathrm{dlog}(x_1x_2) - (\psi_{11} + \psi_{12})/c_W)), g \in \mathcal{O}_{X_2}. \tag{21}$$

The form $\psi_2 = -\mathrm{dlog}(x_1x_2) + (\psi_{11} + \psi_{12})/c_W$ has zero residues with respect to x_1, x_2, hence yields a form in $\Omega_{X_2}\langle\log D_2\rangle$. Changing the local equations x_1, x_2 changes ψ by adding a holomorphic (pole-free) form on X_2.

For g nonzero (21) can be rewritten

$$\tilde{d}(g/x_1x_2) = \frac{g}{x_2x_2}(\mathrm{dlog}(g) - \psi_2) \tag{22}$$

When does this operator have a nontrivial kernel? First, if g is constant, then $\psi_2 = 0$ on W which is impossible if W meets D_2. Next, locally at a point $x \in W \setminus D_2 \cap W$, clearly g/x_1x_2 holomorphic and nonzero in the kernel exists locally since ψ_2 is closed and holomorphic so $\psi_2 = dh$ for a holomorphic function h and we can take $g = e^h$. Moreover, nonzero solutions to $d(g/x_1x_2) = 0$ differ by a multiplicative constant. The condition that the local solutions patch is clearly that $\frac{1}{2\pi i}\int_\gamma \psi_2$ be an integer for any loop γ in $W \setminus D_2 \cap W$. Now ψ_2 is defined only modulo a holomorphic form on X_2 while $H_1(W \setminus D_2 \cap W)$ is generated modulo $H_1(W)$ by small loops normal to components of D_2, so the relevant condition is just integrality over such loops γ.

At a simple point of $D_2 \cap W$, the condition that g exist locally as a holomorphic function with no pole on D_2 is clearly that for γ as above, oriented positively, the integer $\frac{1}{2\pi i}\int_\gamma \psi_2$ is nonnegative, so that g has no pole on D_2. In other words, that the sum of the first 2 columns of the B matrix, normalized so that $b_{12} = -b_{21} = 1$, should be a nonnegative integer vector. Finally by Hartogs, if g is holomorphic off the singular locus of $D_2 \cap W$, it extends holomorphically to W.

3.4.5 Special components

Now let Z be a component of $D_2 \cap W$ and assume W and Z are both normally split so that the branches of D along W may be labeled 12, while those along Z may be labeled 123. Thus, branches of X_2 over Z are labeled 12, 23, 31 and the preceding discussion shows that the zeroth differential has nontrivial kernel along Z only if the iterated residues of Φ along these branches, denoted c_{21}, c_{23}, c_{31}, assuming $c_{12} \neq 0$, satisfy

$$c_{23} + c_{31} = kc_{21}, k \in \mathbb{N}. \tag{23}$$

We call such a component Z *special*; then, W is said to be special if every (normally split) component of $D_2 \cap W$ is special.

What about the normally split hypothesis? Suppose first W is contained in a connected open set W' which is not normally split. Then, as c_{12} is locally constant in W', it follows that $c_{12} = 0$, i.e., W is not residual. Now suppose Z is contained in Z' open connected and not normally split. Then, monodromy acts on the branches of X_2 along Z' cyclically and consequently the c_{ij} above are all equal. Then, (23) holds automatically with $k = 2$, so Z is special.

3.4.6 Conclusion

What we have so far proven is the following: if W is a normally oriented residual open subset of X_2, then the stalk of the zeroth cohomology $\mathcal{H}^0(\mathcal{G}_2^\bullet)$ vanishes somewhere on W *unless* either

(i) $W \cap D_2 = \emptyset$, or

(ii) W is special.

Note that if the stalk of $\mathcal{H}^0(\mathcal{G}_2^\bullet)$ vanishes somewhere in W, then because $\mathcal{G}_2^0$ is coherent and torsion-free, it follows that $H^0(\mathcal{G}_2^\bullet)|_W = 0$, hence a similar vanishing holds for the entire component of X_2 containing W. Now recall that, minding the index shift, if $H^0(\mathcal{G}_2^\bullet) = 0$, then the cokernel of the inclusion $\Omega_X^{+\bullet}\langle \log D\rangle \to \Omega_X^{+\bullet}\langle \log^+ D\rangle$ has vanishing $\mathbb{H}^1$ (and $\mathbb{H}^0$). On the other hand, it is well known (see, e.g., [13]) that $\Omega_X^{+\bullet}\langle \log D\rangle \simeq T_X^\bullet\langle -\log D\rangle$ controls deformations of (X, Φ) or (X, Π) where D deforms locally trivially, and those deformations are unobstructed thanks to Hodge theory.

Summarizing this discussion, we conclude:

Theorem 6 *Let (X, Φ) be a log-symplectic manifold with polar divisor D. With notations as above, let*

$$\Omega_X^{+\bullet}\langle \log D\rangle = \bigoplus_{i>0} \Omega_X^i\langle \log D\rangle,\ \Omega_X^{+\bullet}\langle \log^+ D\rangle = \bigoplus_{i>0} \Omega_X^i\langle \log^+ D\rangle.$$

Then, the inclusions

$$\Omega_X^{+\bullet}\langle \log D\rangle \to \Omega_X^{+\bullet}\langle \log^+ D\rangle,$$
$$T_X^\bullet\langle -\log D\rangle \to T_X^\bullet$$

induce isomorphisms on $\mathcal{H}^2$ and injections on $\mathcal{H}^3$; hence, isomorphisms on $\mathbb{H}^1$ and injections on $\mathbb{H}^2$, unless either

(i) X_2 has a nonresidual component; or

(ii) X_2 has a special component.

As noted above, any component of X_2 that is disjoint from D_2, i.e., contains no triple points of D, is automatically nonresidual.

Corollary 7 *Notations as above, if X is compact and Kählerian and conditions (i), (ii) both fail, then the pair (X, Φ) has unobstructed deformations and the polar divisor of Φ deforms locally trivially.*

In the case where D has global normal crossings, i.e., is a union of smooth divisors, this result also follows from results in [9], which also states a partial converse: when $T_X^\bullet\langle -\log D\rangle \to T_X^\bullet$ is not a quasi-isomorphism, (X, Φ) has obstructed deformations and admits deformations where D either smooths or deforms locally trivially.

Example 8 (Due to M. Matviichuk, B. Pym, T. Schedler, see [9], communicated by B. Pym) Consider the matrix

$$B = (b_{ij}) = \begin{pmatrix} 0 & 1 & 2 & 4 \\ -1 & 0 & 3 & 5 \\ -2 & -3 & 0 & 6 \\ -4 & -5 & 1 & 0 \end{pmatrix} \tag{24}$$

and the corresponding log-symplectic form on $\mathbb{C}^4$, $\Phi = \sum\limits_{i<j} b_{ij} \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$ and corresponding Poisson structure $\Pi = \Phi^{-1}$, both of which extend to $\mathbb{P}^4$ with Pfaffian divisor $D = (z_0z_1z_2z_3z_4)$, $z_0 =$ hyperplane at infinity. Then Π admits the 1st order Poisson deformation with bivector $z_3z_4\ @_{z_1}\ @_{z_2}$, which in fact extends to a Poisson deformation of $(\mathbb{P}^4, \Pi)$ over the affine line $\mathbb{C}$, and the Pfaffian divisor deforms as $(z_3z_4z_0(z_1z_2 - tz_3z_4))$, hence non locally-trivially. Correspondingly, the log-plus form $z_3z_4\phi_1\phi_2$ is closed (and not exact). That $d(z_3z_4\phi_1\phi_2) = 0$ corresponds to the integral column relation

$$k_1 - k_2 + (e_1 + e_2) - (e_3 + e_4) = 0$$

where the k_i and e_j are the columns of the B matrix and the identity, respectively, showing that $(z_1z_2z_3)$ and $(z_1z_2z_4)$ are residual triples of type II and (12), i.e., $(x_1) \cap (x_2)$ is a special component of X_2.

Remark 9 As we saw above, the presence of monodromy on the branches of D is related to nonresidual or special components. This suggests that log-symplectic manifolds with irreducible polar divisor may often be obstructed. However, we do not have specific examples.

Data availability statement There is no data set associated with this paper.

References

1. Bogomolov, F.A.: Hamiltonian Kählerian manifolds. Dokl. Akad. Nauk SSSR **243**(5), 1101–1104 (1978)
2. Ciccoli, N.: From Poisson to quantum geometry, Notes taken by P. Witkowski. Available on http://toknotes.mimuw.edu.pl/sem4/files/Ciccoli_fpqg.pdf
3. Dufour, J.-P., Zung, N.T.: Poisson Structures and Their Normal Forms. Progress in Mathematics, vol. 242. Birkhauser, Basel (2005)
4. Ginzburg, V., Kaledin, D.: Poisson deformations of symplectic quotient singularities. Adv. Math. **186**, 1–57 (2004). arXiv:math/0212279v5
5. Goto, R., Rozanski-Witten invariants of log-symplectic manifolds. In: Integrable Systems, Topology, and Physics 2000. Contemporary Mathematics, Tokyo, vol. 309 (2002)
6. Hitchin, N.: Generalized Calabi–Yau manifolds. Q. J. Math. **54**(3), 281–308 (2003)
7. Lima, R., Pereira, J.V.: A characterization of diagonal Poisson structures. Bull. Lond. Math. Soc. **46**, 1203–1217 (2014)
8. Mărcut, I., Torres, B.O.: Deformations of log symplectic structures. J. Lond. Math. Soc. (2014)
9. Matviichuk, M., Pym, B., Schedler, T.: A local torelli for log symplectic manifolds. arXiv:2010.08692 (2020)
10. Namikawa, Y.: Flops and Poisson deformations of symplectic varieties. Publ. RIMS Kyoto Univ. **44**, 259–314 (2008)
11. Polishchuk, A.: Algebraic geometry of Poisson brackets. J. Math. Sci. **84**, 1413–1444 (1997)
12. Pym, B.: Constructions and classifications of projective Poisson varieties. arxiv:1701.08852v1 (2017)
13. Ran, Z.: A Bogomolov unobstructedness theorem for log-symplectic manifolds in general position., J. Inst. Math. Jussieu (2018). Erratum/corrigendum (2021); arXiv:1705.08366
14. Ran, Z.: Deformations of holomorphic pseudo-symplectic Poisson manifolds. Adv. Math. **304**, 1156–1175 (2017). arXiv:1308.2442