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p-Kähler and balanced structures on nilmanifolds with nilpotent complex structures

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Abstract
Let (X, J) be a nilmanifold with an invariant nilpotent complex structure. We study the existence of p-Kähler structures (which include Kähler and balanced metrics) on X. More precisely, we determine an optimal p such that there are no p-Kähler structures on X. Finally, we show that, contrarily to the Kähler case, on compact complex manifolds there is no relation between the existence of balanced metrics and the degeneracy step of the Frölicher spectral sequence. More precisely, on balanced manifolds the degeneracy step can be arbitrarily large.

Keywords Balanced metric · p-Kähler form · Nilmanifold · Frölicher spectral sequence.

Mathematics Subject Classification 53C15 · 53C25 · 53C55 · 22E25

1 Introduction

Let (X, J) be a compact complex manifold of complex dimension n and let g be an Hermitian metric on (X, J) with fundamental form ω. Then, it is well known that if ω is a Kähler metric, i.e., $d\omega = 0$ then the Frölicher spectral sequence degenerates at the first step. However, if there are no Kähler metrics on (X, J) then the degeneracy step might be higher than one, as first shown in [14]. For examples of higher degeneration steps, one can refer to [6]. It is therefore natural to ask whether there are weaker metric conditions that impose restrictions on the degeneration of the Frölicher spectral sequence. In fact, in [18, Conjecture 1.3] it is conjectured that if there exists an SKT metric on (X, J), namely an Hermitian metric ω such that $\partial\overline{\partial}\omega = 0$, the degeneracy step is 2. Another important class of metrics in non-

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Kähler geometry is provided by *balanced metrics*, that are Hermitian metrics satisfying $d\omega^{n-1} = 0$. Clearly, on compact complex surfaces they coincide with Kähler metrics, but in higher dimension they are a larger class of non necessarily Kähler metrics. In [17] it was shown that the existence of a balanced metric does not imply the degeneration at the first step, see also [7]. Here, we show that in fact on balanced manifolds the degeneracy step can be arbitrarily large. More precisely, we prove the following (see Theorem 3.3 and Corollary 3.4)

Theorem *For every $n \geq 2$, there exists a nilmanifold of complex dimension $4n - 2$ that admits balanced metrics and such that the Frölicher spectral sequence does not degenerate at step n.*

Such nilmanifolds were first constructed in [5], where the degeneration of the Frölicher spectral sequence was studied. Here we investigate the existence of special structures on such manifolds. In particular, we show that, other than being balanced, they do not admit any SKT and locally conformally Kähler metrics. Moreover, in Theorem 3.2 we show that they do not admit any p-Kähler structure for $1 \leq p < 4n - 3$.
More generally, we study in Sect. 2 the existence of p-Kähler forms on nilmanifolds with nilpotent complex structures. We recall that, for $1 \leq p \leq n$, a *p-Kähler* form Ω on a complex manifold X of dimension n is a d-closed real transverse (p, p)-form. In particular, 1-Kähler forms coincide with Kähler metrics and $(n-1)$-Kähler forms coincide with balanced metrics. They were introduced in [1]. Here, we determine an optimal p such that there are no p-Kähler structures on nilmanifolds with nilpotent complex structure. More precisely, we prove the following (see Theorem 2.3)

Theorem *Let $X = \Gamma\backslash G$ be a nilmanifold of complex dimension n endowed with an invariant nilpotent complex structure J. Let $\left\{\varphi^i\right\}_{i=1,\cdots,n}$ be a co-frame of invariant $(1, 0)$-forms satisfying, for $i = 1, \cdots, n$*

$$d\varphi^i \in \Lambda^2\left\langle\varphi^1, \cdots, \varphi^{i-1}, \bar{\varphi}^1, \cdots, \bar{\varphi}^{i-1}\right\rangle,$$

and let k be the index such that

$$d\varphi^i = 0 \;\; \textit{for} \;\; i = 1, \cdots, k \;\; \textit{and} \;\; d\varphi^i \neq 0 \;\; \textit{for} \;\; i = k+1, \cdots, n.$$

Then, there are no $(n-k)$-Kähler forms on X.

In particular, for $k = 1$ we obtain the conclusion for balanced metrics. Notice that, for complex dimension 3, this result is compatible with the classification of balanced structures on 6-dimensional nilmanifolds endowed with nilpotent complex structures proved in [21].
We recall that nilpotent complex structures include abelian and bi-invariant complex structures. We further observe that with this degree of generality the index $p = n-k$ in Theorem 2.3 is optimal. Indeed, we construct in examples 2.6 and 2.7 two 2-step nilmanifolds with invariant abelian complex structures that admit a $(n-k-1)$-Kähler form and a $(n-k+1)$-Kähler form.

In Remark 2.5 we also notice that the hypothesis of nilpotency of the complex structure is crucial.

2 p-Kähler structures on nilmanifolds with nilpotent complex structures

In this section we are going to discuss the existence of special structures, called p-Kähler, on nilmanifolds with nilpotent complex structures. These structures include Kähler and balanced metrics. We recall that by the results of Benson-Gordon, and Hasegawa (see [4, 12]), the only nilmanifolds which admit Kähler metrics are tori.

2.1 *p*-Kähler structures

Let us start by recalling some definitions that will be used in the following.
Let (X, J) be a complex manifold of complex dimension n, i.e., the datum of a differentiable manifold X of real dimension $2n$ and an integrable almost complex structure J. By extending J to, respectively, the complexified tangent bundle $T_{\mathbb{C}}X := TX \otimes \mathbb{C}$, and complexified cotangent bundle $T^*_{\mathbb{C}}X := T^*X \otimes \mathbb{C}$, we obtain the following direct sum decompositions in terms of the respective $\pm i$-eigenspaces, i.e.,

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X\,,$$
$$T^*_{\mathbb{C}}X = (T^{1,0}X)^* \oplus (T^{0,1}X)^*,$$

and, by considering the exterior powers of $T^*_{\mathbb{C}}X$, we obtain

$$\bigwedge\nolimits^k_{\mathbb{C}} X := \bigwedge\nolimits^k(T^*_{\mathbb{C}}X) = \bigoplus\nolimits_{p+q=k} \bigwedge\nolimits^{p,q} X,$$

where each $\bigwedge^{p,q} X := \bigwedge^p(T^{1,0}X)^* \otimes \bigwedge^q(T^{0,1}X)^*$ is the bundle of (p, q)-*forms* on X. We will denote with $A^k(X, \mathbb{C})$ and $A^{p,q}(X)$ the spaces of smooth sections of the bundles $\bigwedge^k_{\mathbb{C}} X$ and $\bigwedge^{p,q} X$ respectively.
We recall the following pointwise definitions. Fix $x \in X$. We denote with T_xX the tangent space of X at x and with T^*_xX its dual. If we consider $\{\phi^i\}_{i=1}^n$ a basis of $\bigwedge^{1,0}(T^*_xX \otimes \mathbb{C})$, then a basis of $\bigwedge^{p,q}(T^*_xX \otimes \mathbb{C})$ is given by

$$\{\phi^{i_1} \wedge \cdots \wedge \phi^{i_p} \wedge \overline{\phi}^{j_1} \wedge \cdots \wedge \overline{\phi}^{j_q} \,:\, 1 \le i_1 < \cdots < i_p \le n,\ 1 \le j_1 < \cdots < j_q \le n\}.$$

Let us denote the constant $\sigma_p := i^{p^2}2^{-p}$ and the space of *real* (p, p)-*forms* by

$$\bigwedge\nolimits^{p,p}_{\mathbb{R}}(T^*_xX) = \{\alpha \in \bigwedge\nolimits^{p,p}(T^*_xX \otimes \mathbb{C}) : \overline{\alpha} = \alpha\}.$$

Then, a basis for $\bigwedge^{p,p}_{\mathbb{R}}(T^*_xX)$ is given by

$$\{\sigma_p\phi^{i_1} \wedge \cdots \wedge \phi^{i_p} \wedge \overline{\phi}^{i_1} \wedge \cdots \wedge \overline{\phi}^{i_p} : 1 \le i_1 < \cdots < i_p \le n\}.$$

We note that the (n, n)-form

$$vol = (\frac{i}{2}\phi^1 \wedge \overline{\phi}^1) \wedge \cdots \wedge (\frac{i}{2}\phi^n \wedge \overline{\phi}^n) = \sigma_n\phi^1 \wedge \cdots \wedge \phi^n \wedge \overline{\phi}^1 \wedge \cdots \wedge \overline{\phi}^n$$

is indeed real and, thus, it defines a volume.
We say that a real (n, n)-form α is *positive*, respectively *strictly positive*, if it holds that

$$\alpha = c \cdot vol,$$

with $c \ge 0$, respectively $c > 0$. A $(p, 0)$-form α is said to be *simple* if

$$\alpha = \alpha^1 \wedge \cdots \wedge \alpha^p,$$

for $\alpha^i \in \bigwedge^{1,0}(T^*_xX \otimes \mathbb{C})$.
A real (p, p)-form $\Omega \in \bigwedge^{p,p}_{\mathbb{R}}(T^*_xX)$ is said to be *transverse* if the (n, n)-form

$$\Omega \wedge \sigma_{n-p}\alpha \wedge \overline{\alpha}$$

is strictly positive, for every non-zero simple form $\alpha \in \bigwedge^{n-p,0}(T_x^*X \otimes \mathbb{C})$).
We recall the following

Definition 2.1 Let (X, J) be a complex manifold of complex dimension n and let $1 \leq p \leq n$. A *p-Kähler* form Ω on X is a d-closed real transverse (p, p)-form, namely $d\Omega = 0$ and, at every point $x \in X$, $\Omega_x \in \bigwedge_{\mathbb{R}}^{p,p}(T_x^*X)$ is transverse.

Notice that, by definition, for $p = 1$ we obtain Kähler metrics and for $p = n - 1$ we obtain balanced metrics. Indeed, if Ω is a $(n-1)$-Kähler form then by [15] there exists an Hermitian metric ω on X such that $\omega^{n-1} = \Omega$, so in particular $d\omega^{n-1} = 0$. We also point out that for $1 < p < n - 1$, p-Kähler forms have no metric meaning (cf. [3, Proposition 2.1]). However, notice that if Ω is a transverse (p, p)-form on X, and Y is a p-dimensional complex submanifold of X,then $\Omega_{|Y}$ is a volume form on Y. Such structures have been originally introduced and studied in [1–3]. Recently, their behaviour under small deformations of the complex structure has been studied in [19]. In [13] such forms have been extended to non-integrable almost-complex manifolds and we recall here the following lemma that provides an obstruction to their existence, see [13, Proposition 3.4].

Lemma 2.2 [13] *Let (X, J) be a compact complex manifold of complex dimension n. Suppose that there exists a non-closed $(2n-2p-1)$-form η such that the $(n-p, n-p)$-component of $d\eta$ satisfies*

$$(d\eta)^{n-p,n-p} = \sum_k c_k \psi_k \wedge \bar{\psi}_k$$

where the ψ_k are simple $(n-p, 0)$-forms and the c_k have the same sign. Then, (X, J) does not admit a p-Kähler form.

We will use this lemma to prove the non-existence of a p-Kähler form (for suitable p) on nilmanifolds with nilpotent complex structures.

2.2 Nilmanifolds with nilpotent complex structure

Let $X = \Gamma \backslash G$ be a $2n$-dimensional nilmanifold, namely G is a connected, simply-connected nilpotent Lie group and Γ is a lattice in G. Let J be an invariant complex structure on X, namely J is induced by a left-invariant complex structure on G. We denote with $\mathfrak{g}$ the Lie algebra of G. Then, J is said to be *nilpotent* if the ascending series $\left\{\mathfrak{g}_i^J\right\}_{i \geq 0}$ defined by

$$\mathfrak{g}_0^J = 0, \qquad \mathfrak{g}_i^J = \left\{X \in \mathfrak{g} \,|\, [X, \mathfrak{g}] \subseteq \mathfrak{g}_{i-1}^J,\ [JX, \mathfrak{g}] \subseteq \mathfrak{g}_{i-1}^J\right\}$$

satisfies $\mathfrak{g}_k^J = \mathfrak{g}$ for some $k > 0$.

Then, by [8, Theorem 2] (cf. also [9, Theorem 12]), J is nilpotent if and only if there exists a co-frame of invariant $(1, 0)$-forms $\left\{\varphi^i\right\}_{i=1,\cdots,n}$ satisfying, for $i = 1, \cdots, n$

$$d\varphi^i \in \Lambda^2 \left\langle \varphi^1, \cdots, \varphi^{i-1}, \bar{\varphi}^1, \cdots, \bar{\varphi}^{i-1} \right\rangle.$$

From now on we will abbreviate e.g., $\varphi^{ij\bar{k}} := \varphi^i \wedge \varphi^j \wedge \bar{\varphi}^k$ and so on.

Now we prove the following

Theorem 2.3 *Let $X = \Gamma\backslash G$ be a nilmanifold of complex dimension n endowed with a invariant nilpotent complex structure J. With the above notations, let k be the index such that*

$$d\varphi^i = 0 \ \ for \ \ i = 1, \cdots, k \ \ and \ \ d\varphi^i \neq 0 \ \ for \ \ i = k+1, \cdots, n. \tag{1}$$

Then, there are no $(n-k)$-Kähler forms on X.

Proof In order to prove the result we will exhibit a $(2k-1)$-form η satisfying the hypothesis of Lemma 2.2. Since $d\varphi^{k+1} \neq 0$ then at least one between $\partial\varphi^{k+1}$ and $\overline{\partial}\varphi^{k+1}$ is different from 0. Suppose now that $\overline{\partial}\varphi^{k+1} \neq 0$. We will deal later with the other case. Since J is nilpotent,

$$\overline{\partial}\varphi^{k+1} = \sum_{l,m=1}^{k} C_{l\bar{m}}\, \varphi^{l\bar{m}} \neq 0$$

for some constants $C_{l\bar{m}}$. Hence, we fix two indices $i, j \leq k$ such that $C_{i\bar{j}} \neq 0$.
We define the following $(2k-1)$-form

$$\eta = \varphi^{1\cdots\hat{i}\cdots k+1\,\bar{1}\cdots\hat{\bar{j}}\cdots\bar{k}},$$

where $\hat{\phi}^i$ and $\hat{\overline{\phi}}^j$ mean that we are removing the forms ϕ^i and $\overline{\phi}^j$ from η.
By the structure equations, since $d\varphi^i = 0$ for $i = 1, \cdots, k$ and J is nilpotent,

$$d\eta = \pm C_{i\bar{j}}\, \varphi^{1\cdots k\,\bar{1}\cdots\bar{k}}$$

hence η satisfies the hypothesis of Lemma 2.2 and so there is no $(n-k)$-Kähler structure on X.
On the other side, suppose that $\overline{\partial}\varphi^{k+1} = 0$ and $\partial\varphi^{k+1} \neq 0$.
Since J is nilpotent,

$$\partial\varphi^{k+1} = \sum_{l,m=1,l<m}^{k} A_{l\,m}\, \varphi^{l\,m} \neq 0$$

for some constants $A_{l\,m}$. Hence, we fix two indices $i < j \leq k$ such that $A_{i\,j} \neq 0$.
We define the following $(2k-1)$-form

$$\eta = \varphi^{1\cdots\hat{i}\cdots\hat{j}\cdots k+1\,\bar{1}\cdots\bar{k}}.$$

By the structure equations, since $d\varphi^i = 0$ for $i = 1, \cdots, k$ and J is nilpotent,

$$d\eta = \pm A_{i\,j}\varphi^{1\cdots k\,\bar{1}\cdots\bar{k}}$$

hence η satisfies the hypothesis of Lemma 2.2 and so there is no $(n-k)$-Kähler structure on X. □

As a Corollary for $k = 1$ one gets immediately

Corollary 2.4 *Let $X = \Gamma\backslash G$ be a nilmanifold of complex dimension n endowed with an invariant nilpotent complex structure J, with co-frame of $(1,0)$-forms $\{\varphi^i\}_{i=1,\cdots,n}$ satisfying the following structure equations,*

$$d\varphi^1 = 0 \ \ and \ \ d\varphi^i \neq 0 \ \ for \ \ i = 2, \cdots, n.$$

Then, there are no balanced metrics on X.

We notice that there are large classes of complex nilmanifolds where Theorem 2.3 can be applied.

For instance, if J is *abelian*, namely $[Jx, Jy] = [x, y]$ for every $x, y \in \mathfrak{g}$, or *bi-invariant*, namely $J[x, y] = [Jx, y]$ for every $x, y \in \mathfrak{g}$, then it is nilpotent (cf. [20]). Moreover, by [16] if (X, J) is a 2-step nilmanifold with invariant complex structure and J-invariant center, then J is nilpotent.

Remark 2.5 We notice that the nilpotency of the complex structure of the nilmanifold is crucial in the previous results. Indeed, when the hypotesis of nilpotency on the complex structure of the nilmanifold is dropped, Theorem 2.3 and Corollary 2.4 are not valid in general. More precisely, in [7] the authors consider the real 6-dimensional nilmanifold, whose associated Lie algebra is $\mathfrak{h}_{19}^- = (0, 0, 0, 12, 23, 14-35)$ and they prove that it is endowed with invariant non nilpotent complex structures (see [7, Theorem 2.1]) which satisfy condition (1) for $k = 1$ ([7, Table 2]), indeed the complex structure equations are

$$d\varphi^1 = 0, \quad d\varphi^2 = \varphi^{13} + \varphi^{1\bar{3}}, \quad d\varphi^3 = \pm i\left(\varphi^{1\bar{2}} - \varphi^{2\bar{1}}\right).$$

As shown in [7, Remark 5.4] such nilmanifolds admit invariant balanced metrics, i.e., 2-Kähler forms.

We now show that $p = n - k$ in Theorem 2.3 is optimal. Indeed, we will show now two examples of 2-step nilmanifolds with invariant abelian complex structures that admit a $(n - k - 1)$-Kähler form and a $(n - k + 1)$-Kähler form.

Example 2.6 Let X be the 6-dimensional 2-step nilmanifold with abelian complex structure defined by the following structure equations

$$d\varphi^1 = d\varphi^2 = 0, \qquad d\varphi^3 = \varphi^{1\bar{2}}$$

where $\left\{\varphi^i\right\}_{i=1,2,3}$ is a co-frame of $(1, 0)$-forms.
With the previous notations we have $n = 3$ and $k = 2$. So, by Theorem 2.3 there are no 1-Kähler forms on X. Of course, this was already known since on non-toral nilmanifolds there are no Kähler metrics.
Now, we show that there exists a 2-Kähler form on X, namely a $(n - k + 1)$-Kähler form. Let

$$\Omega := -\varphi^{1\bar{1}2\bar{2}} - \varphi^{1\bar{1}3\bar{3}} - \varphi^{2\bar{2}3\bar{3}}.$$

Then, Ω is a real transverse $(2, 2)$-form and by the structure equations

$$d\Omega = 0.$$

Hence, Ω is a 2-Kähler form on X. In particular, there exists a balanced metric ω on X such that $\omega^2 = \Omega$. In fact, it is easy to see that

$$\omega = i\phi^{1\bar{1}} + i\phi^{2\bar{2}} + i\phi^{3\bar{3}}.$$

Example 2.7 Let X be the 8-dimensional 2-step nilmanifold with abelian complex structure defined by the following structure equations

$$d\varphi^1 = 0, \qquad d\varphi^2 = d\varphi^3 = d\varphi^4 = \varphi^{1\bar{1}}$$

where $\left\{\varphi^i\right\}_{i=1,2,3,4}$ is a co-frame of $(1, 0)$-forms.

With the previous notations we have $n = 4$ and $k = 1$. So, by Corollary 2.4 there are no balanced metrics on X.
Now, we show that there exists a 2-Kähler form on X, namely a $(n - k - 1)$-Kähler form. Let

$$\begin{aligned}\Omega := &-\varphi^{1\bar{1}2\bar{2}} - \varphi^{1\bar{1}3\bar{3}} - \varphi^{1\bar{1}4\bar{4}} - \varphi^{2\bar{2}3\bar{3}} - \varphi^{2\bar{2}4\bar{4}} - \varphi^{3\bar{3}4\bar{4}} + \\ &+\varphi^{2\bar{2}3\bar{4}} + \varphi^{2\bar{2}4\bar{3}} + \varphi^{2\bar{4}3\bar{3}} + \varphi^{4\bar{2}3\bar{3}} + \varphi^{2\bar{3}4\bar{4}} + \varphi^{3\bar{2}4\bar{4}} .\end{aligned}$$

Then, Ω is a real transverse $(2, 2)$-form and by the structure equations one can see directly that

$$d\Omega = 0 .$$

Hence, Ω is a 2-Kähler form on X.

3 Special Hermitian metrics on the Bigalke and Rollenske's nilmanifolds

In this section we are going to discuss the existence of special Hermitian metrics and p-Kähler forms on the 2-step nilmanifolds with nilpotent complex structure constructed by Bigalke and Rollenske in [5]. In particular, for every $n \geq 2$, these $(4n - 2)$-dimensional compact complex manifolds are such that the Frölicher spectral sequence does not degenerate at the E_n term.
We start by recalling the construction. Fix $n \geq 2$ and let G_n be the real nilpotent subgroup of $GL(2n + 2, \mathbb{C})$ consisting of the matrices of the form

$$\begin{pmatrix}
1 & 0 & & & & & \cdots & & & 0 & \bar{y}_1 & w_1 \\
 & 1 & 0 & \cdots & 0 & \bar{z}_1 & -x_1 & 0 & \cdots & & 0 & w_2 \\
 & & \ddots & & & & \ddots & & & & \vdots & \vdots \\
 & & & 1 & 0\cdots & & 0 & \bar{z}_{n-1} & -x_{n-1} & & 0 & w_n \\
 & & & & 1 & 0 & & \cdots & & & 0 & y_1 \\
 & & & & & \ddots & & & & & \vdots & \vdots \\
 & & & & & & & & & & & \\
 & & & & & & & \ddots & & & \vdots & \vdots \\
 & & & & & & & & & 1 & 0 & y_n \\
 & & & & & & & & & & 1 & z_1 \\
 & & & & & & & & & & & 1
\end{pmatrix} .$$

with $x_1, \dots, x_{n-1}, y_1, \dots, y_n, z_1, \dots, z_{n-1}, w_1, \dots, w_n \in \mathbb{C}$.
Let Γ be the subgroup of G_n consisting of the matrices of the same form and entries in $\mathbb{Z}[i]$. Then, Γ is a lattice in G_n and the quotient $X^{4n-2} := \Gamma \backslash G_n$ is a compact $(4n-2)$-dimensional 2-step nilmanifold with an invariant complex structure.
A global co-frame of invariant $(1, 0)$-forms is given by

$$dx_1, \dots, dx_{n-1}, dy_1, \dots, dy_n, dz_1, \dots, dz_{n-1}, \omega_1, \dots, \omega_n$$

where

$$\omega_1 = dw_1 - \bar{y}_1 dz_1, \quad \omega_k = dw_k - \bar{z}_{k-1} dy_{k-1} + x_{k-1} dy_k \quad (k = 2, \dots, n).$$

The structure equations become

$$
\begin{aligned}
&d(dx_j) = d(dz_j) = 0, \qquad (j = 1, \dots, n-1)\\
&d(dy_j) = 0, \qquad (j = 1, \dots, n)\\
&\partial\omega_1 = 0, \quad \overline{\partial}\omega_1 = dz_1 \wedge d\bar{y}_1\\
&\partial\omega_j = dx_{j-1} \wedge dy_j, \quad \overline{\partial}\omega_j = dy_{j-1} \wedge d\bar{z}_{j-1} \qquad (j = 2, \dots, n)
\end{aligned}
$$

In [5] the authors show that that the Frölicher spectral sequence of X^{4n-2} has non-vanishing differential d_n, namely the Frölicher spectral sequence does not degenerate at the E_n term.

We now rename the forms dx_j, dy_j, dz_j, and ω_j by considering the basis of $(1,0)$-forms $\{\phi^j\}_{j=1}^{4n-2}$, defined as follows

$$
\phi^j := \begin{cases} dx_j, & 1 \le j < n\\ dy_j, & n \le j < 2n\\ dz_j, & 2n \le j < 3n-1\\ \omega_j, & 3n-1 \le j \le 4n-2. \end{cases}
$$

As a result, the structure equations become

$$
d\phi^j = \begin{cases} 0, & 1 \le j < 3n-1\\ \phi^{2n} \wedge \overline{\phi}^n, & j = 3n-1\\ \phi^{j-3n+1} \wedge \phi^{j-2n+1} + \phi^{j-2n} \wedge \overline{\phi}^{j-n}, & 3n \le j \le 4n-2, \end{cases} \tag{2}
$$

or, more precisely,

$$
\partial\phi^j = \begin{cases} 0, & 1 \le j \le 3n-1\\ \phi^{j-3n+1} \wedge \phi^{j-2n+1}, & 3n \le j \le 4n-2, \end{cases} \tag{3}
$$

and

$$
\overline{\partial}\phi^j = \begin{cases} 0, & 1 \le j < 3n-1\\ \phi^{2n} \wedge \overline{\phi}^n, & j = 3n-1\\ \phi^{j-2n} \wedge \overline{\phi}^{j-n}, & 3n \le j \le 4n-2. \end{cases} \tag{4}
$$

3.1 Special Hermitian metrics

Now we study the existence of special Hermitian metrics on Bigalke and Rollenske's nilmanifolds. In particular, one can apply Theorem 2.3 and get immediately the following

Proposition 3.1 *For every $n \ge 2$ the Bigalke and Rollenske's nilmanifold X^{4n-2} does not admit any n-Kähler form.*

In fact, we can show more, namely there are no p-Kähler forms except for balanced metrics. More precisely, we start by proving the following

Theorem 3.2 *For every $n \ge 2$ the Bigalke and Rollenske's nilmanifold X^{4n-2} does not admit any p-Kähler form for $1 \le p < 4n-3$.*

Proof We will show that on any Bigalke and Rollenske's manifold X^{4n-2}, for every fixed p, with $1 \leq p < 4n-3$, we can construct a non closed $(8n-2p-5)$-form η_p such that the $(4n-2-p, 4n-2-p)$-component of $d\eta_p$ satisfies

$$(d\eta_p)^{(4n-2-p,4n-2-p)} = \epsilon_p \psi_p \wedge \overline{\psi_p}, \tag{5}$$

with $\psi_p \in A^{4n-2-p,0}(X^{4n-2})$ a simple form and $\epsilon_p \in \{-1, 1\}$. By Lemma 2.2, this will assure that there exists no p-Kähler form on X^{4n-2}.
Let us consider separately the cases
(i) $1 \leq p < n$;
(ii) $n \leq p \leq 4n-2$.
Before doing so, we remark that, by structure equations (2), the index j such that every term of the expression of $d\phi^j$ contains forms with the highest indices, is $j = 4n-2$. Such expression is

$$d\phi^{4n-2} = \phi^{n-1} \wedge \phi^{2n-1} + \phi^{2n-2} \wedge \overline{\phi}^{3n-2},$$

whereas, in general, we have that $d\phi^j, d\overline{\phi}^j \neq 0$ if, and only if, $3n-1 \leq j \leq 4n-2$.

(i) Even though it is well-known that on non-toral nilmanifolds there are no 1-Kahler forms, since they coincide with Kähle metrics, we will consider the case $p = 1$ for the benefit of the following constructions. We must construct a non closed $(8n-7)$-form satisfying property (5). In particular, if we start from the $(8n-4)$-form

$$\phi^1 \wedge \cdots \wedge \phi^{4n-2} \wedge \overline{\phi}^1 \wedge \cdots \wedge \overline{\phi}^{4n-2},$$

we must remove three 1-forms. For this purpose, we select ϕ^{2n-2}, $\overline{\phi}^{3n-2}$, and $\overline{\phi}^{4n-2}$, therefore considering the $(4n-3, 4n-4)$-form η_1 given by

$$\eta_1 = \phi^1 \wedge \cdots \wedge \phi^{\widehat{2n-2}} \wedge \cdots \wedge \phi^{4n-2} \wedge \overline{\phi}^1 \wedge \cdots \wedge \overline{\phi}^{\widehat{3n-2}} \wedge \cdots \wedge \overline{\phi}^{4n-3}.$$

We now compute the $(4n-3, 4n-3)$-component of $d\eta_1$. By the structure equations (4), we remark that the only non trivial relevant differentials are

$$\overline{\partial}\phi^{3n-1} = \phi^{2n} \wedge \overline{\phi}^{n},$$
$$\overline{\partial}\phi^{j} = \phi^{j-2n} \wedge \overline{\phi}^{j-n}, \quad 3n \leq j \leq 4n-2.$$

In order to have a non vanishing term, we must ensure that $\overline{\partial}\phi^j = \phi^{2n-2} \wedge \overline{\phi}^{3n-2}$. However, this can happen if and only if $j = 4n-2$, resulting in

$$\begin{aligned} d\eta_1^{(4n-3,4n-3)} &= d\left(\phi^1 \wedge \cdots \wedge \phi^{\widehat{2n-2}} \wedge \cdots \wedge \phi^{4n-2} \wedge \overline{\phi}^1 \wedge \cdots \wedge \overline{\phi}^{\widehat{3n-2}} \wedge \cdots \wedge \overline{\phi}^{4n-3}\right) \\ &= \phi^1 \wedge \cdots \wedge \phi^{4n-3} \wedge \overline{\phi}^1 \wedge \cdots \wedge \overline{\phi}^{4n-3}. \end{aligned}$$

Thus, considering $\psi_1 := \phi^1 \wedge \cdots \wedge^{4n-3} \in A^{4n-3,0}(X^{4n-2})$, we can conclude by Lemma 2.2. Therefore, for the case $1 < p < n$, we can construct η_p starting from the $(8n-7)$-form η_1 and then remove the forms $\phi^{3n-1}, \phi^{3n}, \ldots \phi^{3n+p-3}, \overline{\phi}^{3n-1}, \overline{\phi}^{3n}, \ldots, \overline{\phi}^{3n+p-3}$, (which accounts to removing $2p-2$ forms), obtaining a non closed $(8n-2p-5)$-form. Then, the $(4n-2-p, 4n-2-p)$-component of $d\eta_p$ is of type

$$\psi_p \wedge \overline{\psi}_p,$$

with $\psi_p \in A^{4n-2-p,0}(X^{4n-2})$ given by

$$\psi_p = \phi^1 \wedge \cdots \wedge \phi^{3n-2} \wedge \phi^{3n+p-2} \wedge \cdots \wedge \phi^{4n-3}.$$

Again, we can conclude by Lemma 2.2.
(ii) Let us now consider the case $n \leq p \leq 4n-2$, starting from $p = n$ for the benefit of the following construction.
We must find a $(6n-5)$-form η_n such that the $(3n-2, 3n-2)$-component of $d\eta_n$ satisfies condition (5). We construct the form η_n as we have previously done, setting

$$\eta_n = \phi^1 \wedge \cdots \wedge \phi^{\hat{2n-2}} \wedge \cdots \wedge \phi^{3n-2} \wedge \phi^{4n-2} \wedge \overline{\phi}^1 \wedge \cdots \wedge \overline{\phi}^{3n-3},$$

with $\eta_n \in A^{3n-2,3n-3}(X^{4n-2})$. By structure equations, we see that we have removed all the forms with non trivial differential but $d\phi^{4n-2}$. Therefore, when computing the differential $d\eta_n$, we obtain

$$d\eta_n = -\phi^1 \wedge \cdots \wedge \phi^{3n-2} \wedge \overline{\phi}^1 \wedge \cdots \wedge \overline{\phi}^{3n-2}.$$

By setting $\psi_n := \phi^1 \wedge \cdots \wedge \phi^{3n-2}$, we conclude by Lemma 2.2.
Now, if $n+1 \leq p < 4n-3$, we construct the $(8n-2p-5)$-form η_p starting from the $(6n-5)$-form η_n and then removing the forms $\phi^1, \ldots, \phi^{p-n}, \overline{\phi}^1, \ldots, \overline{\phi}^{p-n}$. We clarify that, for $p-n \geq 2n-2$, since ϕ^{2n-2} has already been removed, we keep removing the $(1,0)$-forms with higher index starting from ϕ^{2n-1}, whereas we keep $\overline{\phi}^{2n-2}$ and remove $\overline{\phi}^{2n-1}$ and so forth, so to remove $(1,0)$-forms for a total of $p-n$ forms and $(0,1)$-forms for a total of $p-n$ forms. This procedure accounts to building $\eta_p \in A^{4n-p-2,4n-p-3}$ as

$$\eta_p = \phi^{p-n+1} \wedge \cdots \wedge \phi^{\hat{2n-2}} \wedge \cdots \wedge \phi^{3n-2} \wedge \phi^{4n-2} \wedge \overline{\phi}^{p-n+1} \wedge \cdots \wedge \overline{\phi}^{3n-3}$$

if $n+1 \leq p < 3n-3$, and

$$\eta_p = \phi^{p-n+2} \wedge \cdots \wedge \phi^{3n-2} \wedge \phi^{4n-2} \wedge \overline{\phi}^{2n-2} \wedge \overline{\phi}^{p-n+1} \wedge \cdots \wedge \overline{\phi}^{3n-3},$$

if $3n-3 \leq p \leq 4n-4$. We then compute $d\eta_p$. Since the only non trivial differential is $d\phi^{4n-2}$, we obtain

$$d\eta_p = \epsilon_p \phi^{p-n+1} \wedge \cdots \wedge \cdots \wedge \phi^{3n-2} \wedge \overline{\phi}^{p-n+2} \wedge \cdots \wedge \overline{\phi}^{3n-2}$$

if $n+1 \leq p < 3n-3$, and

$$d\eta_p = \epsilon_p \phi^{2n-2} \wedge \phi^{p-n+1} \wedge \cdots \wedge \phi^{3n-2} \wedge \overline{\phi}^{2n-2} \wedge \overline{\phi}^{p-n+2} \wedge \cdots \wedge \overline{\phi}^{3n-3},$$

if $3n-3 \leq p \leq 4n-4$. The number $\epsilon_p \in \{\pm 1\}$ is a sign term. Therefore, by setting

$$\psi_p = \phi^{p-n+1} \wedge \cdots \wedge \cdots \wedge \phi^{3n-2}$$

for $n+1 \leq p < 3n-3$ and

$$\psi_p = \phi^{2n-2} \wedge \phi^{p-n+1} \wedge \cdots \wedge \phi^{3n-2},$$

if $3n-3 \leq p \leq 4n-4$, we can finally conclude by Lemma 2.2. □

However, we show that there exist $(4n-3)$-Kähler forms. More precisely, we prove the following

Theorem 3.3 *For every $n \geq 2$ the Bigalke and Rollenske's nilmanifold X^{4n-2} admits balanced metrics.*

Proof We show that the diagonal Hermitian metric

$$\omega := \frac{i}{2}\sum_{j=1}^{4n-2}\phi^j \wedge \overline{\phi}^j$$

is balanced, i.e., $d\omega^{4n-3} = 0$. Notice that

$$\omega^{4n-3} = \left(\frac{i}{2}\right)^{4n-3}\frac{1}{(4n-3)!}\sum_{k=1}^{4n-2}\phi^1 \wedge \overline{\phi}^1 \wedge \cdots \wedge \hat{\phi}^k \wedge \hat{\overline{\phi}}^k \wedge \cdots \wedge \phi^{4n-2} \wedge \overline{\phi}^{4n-2}.$$

We denote by $\alpha_k := \phi^1 \wedge \overline{\phi}^1 \wedge \cdots \wedge \hat{\phi}^k \wedge \hat{\overline{\phi}}^k \wedge \cdots \wedge \phi^{4n-2} \wedge \overline{\phi}^{4n-2}$. From the structure equations, when we compute $d\omega^{4n-3}$ we consider separately each term

$$d\alpha_k = d(\phi^1 \wedge \overline{\phi}^1 \wedge \cdots \wedge \hat{\phi}^k \wedge \hat{\overline{\phi}}^k \wedge \cdots \wedge \phi^{4n-2} \wedge \overline{\phi}^{4n-2}).$$

By the structure equations we have that $d\alpha_k = 0$ for every $k = 1, \ldots, 4n-2$. Indeed, by Leibniz rule, the only way to have $d\alpha^k \neq 0$ would be that for some index $j = 1, \ldots, \hat{k}, \ldots, 4n-2$, $d\phi^j$ or $d\bar{\phi}^j$ contains exactly $\phi^k \wedge \bar{\phi}^k$. But this is not the case as showed by the structure equations. Hence, $d\alpha_k = 0$ for every $k = 1, \ldots, 4n-2$ and so $d\omega^{4n-3} = 0$ and so ω is balanced. □

As a consequence, combining this with [5, Theorem 1], we get that there is no relation between the existence of balanced metrics and the degeneracy step of the Frölicher spectral sequence.

Corollary 3.4 *On balanced manifolds the degeneracy step of the Frölicher spectral sequence can be arbitrarily large.*

In particular, this is in contrast with the situation in Kähler geometry where for compact Kähler manifolds the Frölicher spectral sequence degenerates at the first step and with a conjecture by Popovici stating that on compact SKT manifolds the Frölicher spectral sequence degenerates at the second step (cf. [18, Conjecture 1.3]). For completeness, we recall that an *SKT* (or *pluriclosed*) metric on a complex manifold is an Hermitian metric ω such that $\partial\overline{\partial}\omega = 0$.
In fact, in relation with this conjecture we show explicitly the following

Proposition 3.5 *For every $n \geq 2$ the Bigalke and Rollenske's nilmanifold X^{4n-2} does not admit any SKT metric.*

Proof In order to show that X^{4n-2} does not admit any SKT metric we use the characterization of [10] in terms of currents. More precisely, we will construct a non-zero positive $(1, 1)$-current which is $\partial\overline{\partial}$-exact.
Indeed, by a direct computation using the structure equations

$$\begin{aligned}\psi := {} & \phi^1 \wedge \bar{\phi}^1 \wedge \cdots \wedge \phi^{4n-3} \wedge \bar{\phi}^{4n-3} = \\ & \partial\overline{\partial}\left(\phi^1 \wedge \bar{\phi}^1 \wedge \cdots \wedge \hat{\phi}^{n-1} \wedge \hat{\bar{\phi}}^{n-1} \wedge \cdots \wedge \hat{\phi}^{2n-1} \wedge \hat{\bar{\phi}}^{2n-1} \wedge \cdots \wedge \phi^{4n-2} \wedge \bar{\phi}^{4n-2}\right).\end{aligned}$$

The $(4n-3, 4n-3)$-form ψ gives rise to a $\partial\overline{\partial}$-exact non-zero positive $(1, 1)$-current on X. □

Notice that this follows also by [11] where the authors show that on non-tori nilmanifolds balanced and SKT metrics cannot coexist.
We recall that an Hermitian metric ω on a complex manifold is called *locally conformally Kähler* if

$$d\omega = \theta \wedge \omega$$

where θ is a d-closed 1-form. We then notice also the following

Proposition 3.6 *For every $n \geq 2$ the Bigalke and Rollenske's nilmanifold X^{4n-2} does not admit any locally conformally Kähler metric.*

Proof This follows directly combining Theorem 3.3 and [16, Theorem 4.9] where it is proved that on non-tori complex nilmanifolds endowed with an invariant complex structure, locally conformally Kähler metrics and balanced metrics cannot coexist. □

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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