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Deformations of G_2-instantons on nearly G_2 manifolds

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Abstract
We study the deformation theory of G_2-instantons on nearly G_2 manifolds. There is a one-to-one correspondence between nearly parallel G_2 structures and real Killing spinors; thus, the deformation theory can be formulated in terms of spinors and Dirac operators. We prove that the space of infinitesimal deformations of an instanton is isomorphic to the kernel of an elliptic operator. Using this formulation we prove that abelian instantons are rigid. Then we apply our results to describe the deformation space of the characteristic connection on the four normal homogeneous nearly G_2 manifolds.

Keywords G_2-structures · Infinitesimal deformations · Gauge theory in higher dimensions

1 Introduction

Nearly parallel G_2 structures on a 7-manifold M are defined by a so-called positive 3-form φ. Such a 3-form induces a metric g, an orientation and a spin structure on M (see Sect. 2). We denote by ∇^g the Levi-Civita connection and its lift on the spinor bundle. The G_2-structure φ is nearly parallel if for some $\tau_0 \neq 0$

$$\mathrm{d}\varphi = \tau_0 *_\varphi \varphi,$$

or equivalently if there exists a real Killing spinor η such that

$$\nabla^g_X \eta = -\frac{\tau_0}{8} X \cdot \eta.$$

Nearly G_2 manifolds were introduced as manifolds with weak holonomy G_2 by Gray in [30]. Some examples of such manifolds are the round and squashed 7-spheres, the Aloff–Wallach spaces, and the Berger space $SO(5)/SO(3)$. The inclusion of the exceptional Lie group G_2 as a possible holonomy group for Riemannian manifolds in Berger's list [11] led mathematicians to look for examples of manifolds with holonomy G_2. In [45] Wang established the first correspondence between parallel spinors and integrable geometries. Later the classification of manifolds with real Killing spinors in [10, 25, 26, 31, 32] established a

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link between manifolds with weak holonomy and manifolds with real Killing spinors. These manifolds are Einstein with positive scalar curvature. Except for the round 7-sphere, the dimension of the space of Killing spinors on a nearly G_2 manifold is 1,2 or 3 (see[27]) giving rise to three types: proper, Sasaki–Einstein and 3-Sasakian, respectively. The cones over these manifolds have holonomy contained in Spin(7) which makes these spaces particularly important in the construction and understanding of manifolds with torsion-free Spin(7)-structures.

The correspondence between nearly parallel G_2 structures and Killing spinors has been extensively used to produce many results on nearly G_2 manifolds. The infinitesimal deformation space of nearly G_2-structures was explicitly described as an eigenspace of a Dirac operator in [9]. In the homogeneous setting, non-trivial deformations were only found for the Aloff–Wallach space which in [23] were proved to be obstructed.

The spinorial approach can also be used to study gauge theory on manifolds with weak holonomy. A connection A on M is a G_2-*instanton* if its curvature F satisfies the algebraic condition

$$F \wedge \varphi = *_\varphi F,$$

or equivalently $F \cdot \eta = 0$. In this article we describe the infinitesimal deformation space of instantons on nearly G_2 manifolds as the eigenspaces of the Dirac operators associated with the one parameter family of connections with skew-symmetric torsion

$$\nabla^t_X Y = \nabla^g_X Y + \frac{t}{3}\varphi(X, Y, \cdot),$$

described in [1–4, 7]. At $t = -1$, the connection ∇^{-1} is the *characteristic connection* which is a G_2-instanton. We explicitly describe the infinitesimal deformation space of the characteristic connections for the normal homogeneous nearly G_2 manifolds classified in [27]. In [17] an analogous description for the infinitesimal deformation space of instantons on nearly Kähler 6-manifolds is given. On an oriented manifold with real Killing spinor η the volume form vol defines a Killing spinor vol $\cdot\, \eta$. On a nearly Kähler 6-manifold $\{\eta, \mathrm{vol}\cdot\eta\}$ defines a 2-dimensional space of Killing spinors, whereas on a nearly G_2 manifold η and vol $\cdot\, \eta$ are linearly dependent. This prevents us from having a relation like in [[17], Proposition 4(iii)] which makes the computation of the infinitesimal deformation space much more convenient (See 4.3). In fact we show in Sect. 4 that such a relation does not exist in the nearly G_2 case by explicitly computing the kernel of the elliptic operator for the homogeneous nearly G_2 manifolds. In [22] the author uses the spinorial approach to describe the deformation space of instantons on asymptotically conical G_2 manifolds.

As with parallel G_2 structures ($\tau_0 = 0$), the instantons on nearly G_2 manifolds also solve the Yang–Mills equation $d^*_\nabla F = 0$ ([33]). This makes the study of these instantons important from the point of view of gauge theory in higher dimensions. However unlike G_2-instantons in the parallel case they are not necessarily minimizers of the Yang–Mills functional [14]. The first examples of G_2-instantons on parallel G_2 manifolds were constructed in [19, 42, 43]. In [14] the authors proved the existence of nearly G_2-instantons on certain Aloff–Wallach spaces and classified invariant G_2-instantons on these spaces with gauge group U(1) and SO(3). Waldron [44] obtained a smooth, complete 15-dimensional family of G_2-instantons on S^7 as the pullback of the ASD instantons on the 4-sphere via the quaternionic Hopf fibration.

In Sect. 2 we describe the 1-parameter family of connections on the spinor bundle $\$$ over nearly G_2 manifolds and the associated Dirac operators.

In Sect. 3 we describe the deformation space of a nearly G_2 instanton A as an eigenspace of a Dirac operator associated with A and the characteristic connection (Theorem 3.2). Using this description, we show that on a compact nearly G_2 manifold the G_2-instanton A is rigid if the structure group is abelian (cf. Theorem 3.7(i)) or if all the eigenvalues of a linear operator L_A are greater than $-\frac{28}{5}$ (Theorem 3.7(ii)) . The instanton A is also rigid if all the eigenvalues of L_A are less than 6, as shown in [[14], Proposition 8] where authors used a Weitzenböck formula, while the proof of Theorem 3.7(ii) uses the Schrödinger–Lichnerowicz formula for the family of Dirac operators associated with ∇^t and A.

In Sect. 4 we describe the infinitesimal deformation space of the characteristic connection on all the homogeneous nearly G_2 manifolds whose nearly G_2 metric is normal. By considering the actions of the Lie groups H and G_2 on G/H we can view the characteristic connection as an H-connection or a G_2-connection. We compute its infinitesimal deformation spaces in both of these cases. The results are recorded in Theorem 4.6. The deformations are shown to be genuine in all cases except that of the Aloff–Wallach space $\frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$. In the latter case the author is currently unaware of any known family of nearly G_2-instantons for which the infinitesimal deformations are the ones found in Theorem 4.6.

Disclaimer. No datasets were generated or analyzed during the current study.

2 Preliminaries

2.1 Nearly parallel G_2 structures

Let M be a 7-dimensional Riemannian manifold. At each point $p \in M$ we denote by $\Lambda^3_+(T^*_pM)$ the $GL(7,\mathbb{R})$ orbit of the standard G_2-structure on $\mathbb{R}^7$. The union $\Lambda^3_+(T^*M) := \cup_{p\in M}\Lambda^3_+(T^*_pM)$ is a sub-bundle of $\Lambda^3(T^*M)$. A 3-form on M is said to be *positive* if it takes values in $\Lambda^3_+(T^*_pM)$ for all $p \in M$. We denote by $\Omega^3_+(M)$ the set of positive 3-forms. Let $\varphi \in \Omega^3_(M)$, then φ defines a G_2 structure on M. It also induces an orientation and a metric on M which together define a Hodge star operator $*_\varphi$ on the space of differential forms (see [16]). The G_2 structure φ is called a nearly parallel G_2 structure on M if it satisfies the following differential equation for some nonzero $\tau_0 \in \mathbb{R}$,

$$d\varphi = \tau_0 *_\varphi \varphi. \tag{2.1}$$

We denote the 4-form $*_\varphi \varphi$ by ψ in the remainder of this article. The condition $d\varphi = \tau_0\psi$ implies $d\psi = 0$; thus, the nearly parallel G_2 structure φ is co-closed.

Every manifold with a G_2 structure is orientable and spin, and thus admits a spinor bundle $\$$. Let ∇^g be the Levi-Civita connection of the induced metric on M. A spinor $\eta \in \Gamma(\$)$ is a real Killing spinor if for some nonzero $\delta \in \mathbb{R}$,

$$\nabla^g_X\eta = \delta X\cdot\eta \qquad \forall\, X \in \Gamma(TM). \tag{2.2}$$

There is a one-to-one correspondence between nearly parallel G_2 structures and real Killing spinors on M described in [12, Chapter 4]. Given a nearly parallel G_2 structure φ that satisfies (2.1) there exists a real Killing spinor η that satisfies (2.2) with $\delta = -\frac{1}{8}\tau_0$ and vice-versa. Switching $-\frac{\tau_0}{8}$ to $\frac{\tau_0}{8}$ corresponds to changing the orientation of the cone $M \times_{r^2} \mathbb{R}^+$. See [12] and [10] for more details.

The constant τ_0 can be altered by rescaling the metric and readjusting the orientation. In this article we use $\tau_0 = 4$. With this choice of τ_0 our nearly G_2 structure φ and Killing spinor η satisfy the following equations, respectively

$$d\varphi = 4\psi,$$
$$\nabla^g_X \eta = -\frac{1}{2} X \cdot \eta. \tag{2.3}$$

Manifolds with nearly parallel G_2 structures have several nice properties which can be found in detail in [12]. In particular they are positive Einstein as shown by [30]. Let g be the metric induced by φ, then the Ricci curvature $\mathrm{Ric}_g = \frac{3}{8}\tau_0^2 g$ and the scalar curvature $\mathrm{Scal}_g = 7\mathrm{Ric}_g = \frac{21}{8}\tau_0^2$. A G_2 structure on M induces a splitting of the spaces of differential forms on M into irreducible G_2 representations. The space of 2-forms $\Lambda^2(M)$ decomposes as

$$\Lambda^2(M) = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M),$$

where Λ^2_l has pointwise dimension l. More precisely, we have the following description of the space of forms :

$$\Lambda^2_7(M) = \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} = \{\beta \in \Lambda^2(M) \mid *(\varphi \wedge \beta) = -2\beta\},$$
$$\Lambda^2_{14}(M) = \{\beta \in \Lambda^2(M) \mid \beta \wedge \psi = 0\} = \{\beta \in \Lambda^2(M) \mid *(\varphi \wedge \beta) = \beta\}.$$

Note that we are using the convention of [36] which is opposite to that of [35] and [15].

The space Λ^2_{14} is isomorphic to the Lie algebra of G_2 denoted by $\mathfrak{g}_2$. Since the group G_2 preserves the G_2 structure φ, it preserves the real Killing spinor η induced by φ. The space Λ^2_{14} can be equivalently defined as

$$\Lambda^2_{14} = \{\omega \in \Lambda^2 \mid \omega \cdot \eta = 0\}. \tag{2.4}$$

We make use of this identification when defining the instanton condition on M in Sect. 3.

2.2 The spinor bundle

For a 7-dimensional Riemannian manifold M with a nearly parallel G_2 structure φ, the spinor bundle $\$$ is a rank-8 real vector bundle over M and is isomorphic to the bundle $\underline{\mathbb{R}} \oplus TM = \Lambda^0 \oplus \Lambda^1$ (see [27, 40]). If η is the real Killing spinor on M induced by φ, then we have the isomorphism

$$\$ = (\Lambda^0 TM \cdot \eta) \oplus (\Lambda^1 TM \cdot \eta) \cong \Lambda^0 TM \oplus \Lambda^1 TM.$$

Under this isomorphism any spinor $s = (f \cdot \eta, \alpha \cdot \eta) \in \$$ can be written as $s = (f, \alpha) \in \Lambda^0 \oplus \Lambda^1$. Let $\times$ be the cross-product induced by the G_2 structure φ then as shown in [37] the Clifford multiplication of a 1-form Y and a spinor (f, Z) is given by

$$Y \cdot (f, Z) = -(-\langle Y, Z\rangle, fY + Y \times Z). \tag{2.5}$$

Note that the product defined above differs from [37] by a negative sign due to our choice of the representation of Cl_7 on $\$$ [[40], Chapter 1.8].

Throughout this article we denote by $\{e_i, i = 1..7\}$ a local orthonormal frame for both tangent vectors and 1-forms, identified using the metric. For any p-form $\beta = \beta_{i_1 \ldots i_p} e_{i_1} \wedge e_2 \wedge \cdots \wedge e_{i_p}$ the Clifford multiplication of β with a spinor is given by

$$\beta \cdot (f, X) = \beta_{i_1 \ldots i_p}(e_{i_1} \cdot (e_{i_2} \cdot \ldots \cdot (e_{i_p} \cdot (f, X)) \ldots)),$$

and the following identity holds [[40], Proposition 3.8, Ch1]

$$\sum_j e_j \cdot \beta \cdot e_j = (-1)^{p+1}(n-2p)\,\beta. \tag{2.6}$$

The vector bundle $\$$ is a G_2-representation, and since G_2 is the isotropy group of the 3-form φ, the map $\mu \mapsto \varphi \cdot \mu$ from the bundle of spinors $\$$ to itself is an isomorphism. The same argument holds for the 4-form ψ. The following formulae described in [2, 24] will prove useful in later computations.

Lemma 2.1 *Let η be the Killing spinor associated with the nearly G_2-structure and $X \in \Gamma(TM)$ then*

$$\varphi \cdot \eta = \psi \cdot \eta = 7\eta, \quad \varphi \cdot X \cdot \eta = \psi \cdot X \cdot \eta = -X \cdot \eta.$$

For an oriented Riemannian manifold (M, g), a metric connection ∇^c with skew symmetric torsion T is known as the characteristic connection if it preserves the G-structure on M. The Killing spinor η is parallel with respect to this connection and $\mathrm{Hol}(\nabla^c) \subset \mathrm{G}$. In [[5], Theorem 2.1] the authors showed that the characteristic connection (if existent) is unique if $\mathrm{G} \subsetneq \mathrm{SO(n)}$ is connected and acts irreducibly on $\mathbb{R}^n$ other than the adjoint representation. The above condition applies to many geometric situations for example, to almost hermitian structures ($\mathrm{SU}(n) \subset \mathrm{SO}(2n)$), G_2- structures and $\mathrm{Spin}(7)$ structures. If M is G-irreducible such a connection exists if and only if M is locally isometric to a non-symmetric isotropy irreducible homogeneous space or is a nearly Kähler 6-manifold or nearly G_2 manifold (see [[20], Theorem 6.3]).

For a Riemannian spin manifold (M^n, g) with a 3-form T we describe the one-parameter family of linear metric connections with totally skew-symmetric torsion as given in [3],

$$\nabla^t_X Y = \nabla^g_X Y + tT(X, Y, \cdot).$$

By [[40], Theorem 4.14] the lift of the connection ∇^t on the spinor bundle which we also denoted by ∇^t acts on $\mu \in \Gamma(\$)$ as

$$\nabla^t_X \mu = \nabla^g_X \mu + \frac{t}{2}(i_X T) \cdot \mu.$$

When M has a G_2-structure φ, setting $T = \frac{\varphi}{3}$ gives the family of connections on TM

$$\nabla^t_X Y = \nabla^g_X Y + \frac{t}{3}\varphi(X, Y, \cdot), \tag{2.7}$$

which lifts to $\$M$ as

$$\nabla^t_X \mu = \nabla^g_X \mu + \frac{t}{6}(i_X \varphi) \cdot \mu. \tag{2.8}$$

If M is nearly G_2 for a Killing spinor η and a vector field X since $X \cdot \varphi + \varphi \cdot X = -2\, i_X \varphi$ it follows from (2.3) and Lemma 2.1 that

$$\nabla^t_X \eta = -\frac{t+1}{2} X \cdot \eta.$$

Therefore η is parallel with respect to the connection ∇^{-1} and the connection ∇^{-1} is the characteristic connection ∇^c.

It is well known from [4, 18, 24] that the Ricci tensor Ric^t of the connection ∇^t is given by

$$\mathrm{Ric}^t = (6 - \frac{2t^2}{3})g. \tag{2.9}$$

2.3 Instantons on nearly G_2 manifolds

For a Lie group K let $\mathcal{P} \to M$ be a principal K-bundle. We denote by $Ad_{\mathcal{P}}$ the adjoint bundle associated with $\mathcal{P}$. Let A be a connection 1-form on $\mathcal{P}$ and $F = \mathrm{d}A + \frac{1}{2}[A \wedge A] \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{Ad}_{\mathcal{P}})$ be the curvature 2-form associated with A.

There are many ways to define the instanton condition on A. An interested reader can read further on these definitions and their relations in [33].

- If (M, g) is equipped with a G-structure such that $G \subset \mathrm{O}(n)$, the connection A is an instanton if the 2-form part of F belongs to subbundle $\mathfrak{g}(T^*M) \subset \Lambda^2 T^*M$ whose fiber is isomorphic to $\mathfrak{g} = \mathrm{Lie}(G)$,

$$F \in \Gamma(\mathfrak{g}(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \mathrm{Ad}_{\mathcal{P}}).$$

- When G is simple the quadratic Casimir is a G-invariant element of $\mathfrak{g} \otimes \mathfrak{g} \cong \Lambda^2 \otimes \Lambda^2$ and hence can be identified to a G-invariant 4-form Q by taking a wedge product. Then F is an instanton if for some $\nu \in \mathbb{R}$

$$*(*Q \wedge F) = \nu F.$$

- Furthermore if M is a spin manifold, and the spinor bundle admits one or more non-vanishing spinors η, then A is an instanton if

$$F \cdot \eta = 0,$$

  where the Clifford multiplication is between the 2-form part of F and η.

If (M, g, φ) is a manifold with a G_2 structure φ all the above definitions are equivalent. They all imply that the curvature F associated with A lies in $\Gamma(\Lambda^2_{14})$ and thus satisfies *all* of these equivalent conditions:

$$F \cdot \eta = 0, \quad F \wedge \varphi = *F, \quad F \wedge \psi = 0, \quad F \lrcorner \varphi = 0. \tag{2.10}$$

From now on in this article we use these instanton conditions interchangeably according to the context without further specification. Note that the above definitions are valid for any general G_2 structure and not only for nearly parallel ones.

On a manifold with real Killing spinors instantons solve the Yang–Mills equation [33]. For a nearly G_2 manifold M we can prove this fact by direct computations using (2.10) and the second Bianchi identity. Let K be a compact semisimple Lie group and $\mathcal{P}$ be a principal K-bundle over M then for an instanton A on $\mathcal{P}$

$$(d^A)^* F = *d^A * F = *d^A(\varphi \wedge F) = 4 * (\psi \wedge F) = 0.$$

One of the most natural examples of an instanton on the spinor bundle $\$M$ over a nearly G_2 manifold is the lift of the characteristic connection to $\$M$. In [14] the authors construct instantons on any complex line bundle and certain SO(3)-bundles over the Aloff–Wallach spaces $SU(3)/U(1)_{k,l}$.

3 Infinitesimal deformation of instantons

Let M^7 be a nearly G_2 manifold. We are interested in studying the infinitesimal deformations of nearly G_2-instantons on M. An infinitesimal deformation of a connection A represents an infinitesimal change in A and, thus, is a section of $T^*M \otimes Ad_{\mathcal{P}}$. If $\epsilon \in \Gamma(T^*M \otimes Ad_{\mathcal{P}})$ is an infinitesimal deformation of A, the corresponding change in the curvature F up to first order is given by $\mathrm{d}^A\epsilon$. A standard gauge fixing condition on this perturbation is given by $(\mathrm{d}^A)^*\epsilon = 0$. So in total the pair of equations whose solutions define an infinitesimal deformation of an instanton A is given by

$$(\mathrm{d}^A\epsilon) \cdot \eta = 0, \quad (\mathrm{d}^A)^*\epsilon = 0 \tag{3.1}$$

where η is a Killing spinor.

The 1-parameter family of connections on the spinor bundle $\$$ defined in (2.8) and the connection A on $\mathcal{P}$ can be used to construct a 1-parameter family of connections on the associated vector bundle $\$ \otimes Ad_{\mathcal{P}}$. We denote by $\nabla^{t,A}$, the connection induced by ∇^t and A for all $t \in \mathbb{R}$, respectively. We denote by $D^{t,A}$ the Dirac operator associated to $\nabla^{t,A}$. The following proposition associates the solutions to (3.1) to a particular eigenspace of $D^{t,A}$ for each t. The proposition was proved in [28] for $t = 0$.

Proposition 3.1 *Let ϵ be a section of $T^*M \otimes Ad_{\mathcal{P}}$, and let $D^{t,A}$ be the Dirac operator constructed from the connections $\nabla^{t,A}$ for $t \in \mathbb{R}$. Then ϵ solves (3.1) if and only if for any Killing spinor η*

$$D^{t,A}(\epsilon \cdot \eta) = -\frac{t+5}{2}\epsilon \cdot \eta. \tag{3.2}$$

Proof It follows from (2.8) and the identity $\sum_a e_a \cdot i_a\varphi = 3\varphi$ that

$$D^{t,A} = D^{0,A} + \frac{t}{2}\varphi \cdot .$$

Let $\{e_a, a = 1 \ldots 7\}$ be a local orthonormal frame for T^*M. Using Clifford multiplication identities we get

$$D^{0,A}(\epsilon \cdot \eta) = (\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon) \cdot \eta + e_a \cdot \epsilon \cdot \nabla^0_a \eta.$$

Applying (2.6) to the 1-form part of ϵ we get $e_a \cdot \epsilon \cdot e_a \cdot \eta = 5\epsilon \cdot \eta$. So if η is a real Killing spinor then (2.3) together with the above identity imply

$$\begin{aligned} D^{0,A}(\epsilon \cdot \eta) &= (\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon) \cdot \eta - \frac{1}{2}e_a \cdot \epsilon \cdot e_a \cdot \eta \\ &= (\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon - \frac{5}{2}\epsilon) \cdot \eta. \end{aligned}$$

Since $\epsilon \cdot \eta \in \Lambda^1 \cdot \eta$, by Lemma 2.1 we have

$$D^{t,A}(\epsilon \cdot \eta) = \left(\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon + \frac{-t-5}{2}\epsilon\right) \cdot \eta.$$

The equation $D^{t,A}(\epsilon \cdot \eta) = -\frac{t+5}{2}\epsilon \cdot \eta$ is thus equivalent to $(\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon) \cdot \eta = 0$, which in turn is equivalent to the pair of equations $(\mathrm{d}^A\epsilon) \cdot \eta = 0$, $(\mathrm{d}^A)^*\epsilon = 0$ since these two components live in complementary subspaces.

Since η is parallel with respect to ∇^{-1} we can view $D^{-1,A}$ as an operator on $\Lambda^1 \otimes Ad_P$ defined by $D^{-1,A}(\epsilon \cdot \eta) = (D^{-1,A}\epsilon) \cdot \eta$. The following theorem is an immediate consequence of the above proposition.

Theorem 3.2 *The space of infinitesimal deformations of a* G_2*-instanton* A *on a principal bundle* $\mathcal{P}$ *over a nearly* G_2 *manifold* M *is isomorphic to the kernel of the operator*

$$\left(D^{-1,A} + 2\,\mathrm{Id}\right) \colon \Gamma(\Lambda^1 \otimes Ad_{\mathcal{P}}) \to \Gamma(\Lambda^1 \otimes Ad_{\mathcal{P}}). \tag{3.3}$$

Remark 3.3 By Proposition 3.1, the $-\frac{t+5}{2}$ eigenspace of the operator $D^{t,A}$ on $\Lambda^1 \cdot \eta \otimes Ad_{\mathcal{P}}$ is isomorphic to the infinitesimal deformation space of the instanton A for all $t \in \mathbb{R}$ and all these eigenspaces are thus isomorphic to each other. In particular

$$\ker(D^{-1/3,A} + \frac{7}{3}\mathrm{id}) \cong \ker(D^{-1,A} + 2\mathrm{id}). \tag{3.4}$$

The deformation space found above can be further analyzed as an eigenspace of the square of the Dirac operator. In [3] the authors obtained a Schrödinger–Lichnerowicz-type formula relating the square of the Dirac operator with torsion T to the connection with torsion $3T$. Such a rescaling was earlier used in [29] for η-invariant homogeneous spaces and in [13] for Hermitian manifolds.

Proposition 3.4 *Let* EM *be a vector bundle associated with* $\mathcal{P}$ *and* $\mu \in \Gamma(\$ \otimes EM)$. *Let* A *be any connection on* $\mathcal{P}$. *Then for all* $t \in \mathbb{R}$,

$$(D^{t/3,A})^2\mu = (\nabla^{t,A})^*\nabla^{t,A}\mu + \frac{1}{4}\mathrm{Scal}_g\mu + \frac{t}{6}\mathrm{d}\varphi \cdot \mu - \frac{t^2}{18}\|\varphi\|^2\mu + F \cdot \mu. \tag{3.5}$$

When the connection A is an instanton on a nearly G_2 manifold the expression for $(D^{t/3,A})^2$ can be simplified further. For the G_2 structure φ, $\|\varphi\|^2 = 7$ and under our choice of convention $d\varphi = 4\psi$ and $\mathrm{Scal}_g = 42$. Thus we can calculate the action of $(D^{t/3,A})^2$ on spinors in $\Lambda^0\eta$ and $\Lambda^1 \cdot \eta$ as follows.

Let $\eta \in \Gamma(\Lambda^0 M \otimes EM)$ be a real Killing spinor, then Lemma 2.1 implies $\psi \cdot \eta = 7\eta$ and $F \cdot \eta = 0$ by (2.10). Thus by above proposition we obtain,

$$(D^{t/3,A})^2\eta = (\nabla^{t,A})^*\nabla^{t,A}\eta - \frac{7}{18}(t^2 - 12t - 27)\eta. \tag{3.6}$$

Now suppose ϵ is an infinitesimal deformation of A. Then $\epsilon \cdot \eta \in \Gamma(\Lambda^1 M \otimes EM)$. From Lemma 2.1 we know that $\psi \cdot \epsilon \cdot \eta = -\epsilon \cdot \eta$ and since $F \cdot \eta = 0$, $F \cdot \epsilon \cdot \eta = (F \cdot \epsilon + \epsilon \cdot F) \cdot \eta = -2(\epsilon \lrcorner F) \cdot \eta$. Thus by above proposition

$$(D^{t/3,A})^2(\epsilon \cdot \eta) = (\nabla^{t,A})^*\nabla^{t,A}(\epsilon \cdot \eta) - \frac{1}{18}(7t^2 + 12t - 189)\epsilon \cdot \eta - 2(\epsilon \lrcorner F) \cdot \eta. \tag{3.7}$$

In the special case when the bundle EM is equal to $Ad_{\mathcal{P}}$, the holonomy group $H \subset K$ of the connection A acts on the Lie algebra $\mathfrak{k}$ of K. Let us denote by $\mathfrak{k}_0 \subset \mathfrak{k}$ the subspace on which H acts trivially. Let $\mathfrak{k}_1$ be the orthogonal subspace of $\mathfrak{k}_0$ with respect to the Killing form of K. The corresponding splitting of the adjoint bundle is given by $Ad_{\mathcal{P}} = L_0 \oplus L_1$ and $\$ \otimes \mathrm{Ad}_{\mathcal{P}}$ decomposes as follows

$$\$ \otimes \mathrm{Ad}_{\mathcal{P}} = (\Lambda^1 M \otimes L_0) \oplus (\Lambda^1 M \otimes L_1) \oplus (\Lambda^0 M \otimes L_0) \oplus (\Lambda^0 M \otimes L_1).$$

By Proposition 3.4 the operator $(D^{-1/3,A})^2$ is self-adjoint and hence respects the above decomposition. We use the shorthand $\Lambda^i L_j$ for $\Lambda^i M \otimes L_j$ where $i, j = 0, 1$. For compact M we have the following proposition.

Proposition 3.5 *Let A be a G_2-instanton on a principal K-bundle $\mathcal{P}$ with holonomy group H where $Ad_{\mathcal{P}}$ splits as above. Then*

(i) $\ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) = \ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap (\Lambda^1 L_1 \oplus \Lambda^0 L_0)$.

(ii) $\ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap \Lambda^1 L_1 = \Big(\ker(D^{-1/3,A} + \frac{7}{3}\mathrm{id}) \oplus \ker(D^{-1/3,A} - \frac{7}{3}\mathrm{id})\Big) \cap \Lambda^1 L_1$.

Proof To prove (i) we need to show that $\ker((D^{-1/3,A})^2 - (\frac{7}{3})^2\mathrm{id}) \cap (\Lambda^0 L_1 \oplus \Lambda^1 L_0)$ is trivial.

1. Let $\mu \in \ker((D^{-1/3,A})^2 - (\frac{7}{3})^2\mathrm{id}) \cap \Lambda^0 L_1$. Thus we have by (3.6) ,

$$\begin{aligned} 0 &= \int_M (\mu, (D^{-1/3,A})^2 - (\frac{7}{3})^2)\mu) \\ &= \int_M (\mu, (\nabla^{-1,A})^* \nabla^{-1,A}\mu + (\frac{49}{9} - \left(\frac{7}{3}\right)^2)\mu) \\ &= \int_M \|\nabla^{-1,A}\mu\|^2. \end{aligned}$$

But since the action of the holonomy group of A fixes no non-trivial elements in $\mathfrak{k}_1$ and the holonomy group of ∇^{-1} acts trivially on Λ^0 we get $\mu = 0$.

2. Let $\epsilon \cdot \eta \in \ker((D^{-1/3,A})^2 - (\frac{7}{3})^2\mathrm{id}) \cap \Lambda^1 L_0$. By the definition of L_0 the curvature F acts trivially on $\epsilon \cdot \eta$ in (3.7) and we get,

$$\begin{aligned} 0 &= \int_M (\epsilon \cdot \eta, (D^{-1/3,A})^2 - (\frac{7}{3})^2)\epsilon \cdot \eta) \\ &= \int_M (\epsilon \cdot \eta, (\nabla^{-1})^* \nabla^{-1}(\epsilon \cdot \eta) + (\frac{97}{9} - \left(\frac{7}{3}\right)^2)\epsilon \cdot \eta) \\ &= \int_M \|\nabla^{-1}(\epsilon \cdot \eta)\|^2 + \frac{48}{9}\int_M \|\epsilon \cdot \eta\|^2 \end{aligned}$$

hence $\epsilon \cdot \eta = 0$.

For proving (ii) we already know that $\big(\ker((D^{-1/3,A}) + \frac{7}{3}\} \oplus \ker((D^{-1/3,A}) + \frac{7}{3}\}\big) \cap \Lambda^1 L_1 \subset \ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap \Lambda^1 L_1$. The reverse inclusion can be seen using the fact that since $D^{-1/3,A}$ and $(D^{-1/3,A})^2$ commute they have the same eigenvectors. Moreover since $D^{-1/3,A}$ is self-adjoint, $\epsilon \cdot \mu \in \ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap \Lambda^1 L_1$ implies $\|D^{-1/3,A}\epsilon \cdot \mu\| = \frac{7}{3}\|\epsilon \cdot \mu\|$; thus, the corresponding eigenvalues of $D^{-1/3,A}$ can only be $\pm\frac{7}{3}$.

Remark 3.6 Note that part (i) for the above proposition holds only for $D^{-1/3,A}$ and not for any other $D^{t,A}$ where $t \neq -1/3$ since the proof explicitly uses the fact that η is parallel with respect to ∇^{-1}. But since $D^{t,A}$ is self-adjoint for all $t \in \mathbb{R}$, for any $\lambda \in \mathbb{R}$ we have the following decomposition

$$\ker\left\{(D^{t,A})^2 - \lambda^2\mathrm{id}\right\} \cap \Lambda^1 Ad_{\mathcal{P}} = \left(\ker\left\{D^{t,A} - \lambda\mathrm{id}\right\} \oplus \ker\left\{D^{t,A} + \lambda\mathrm{id}\right\}\right) \cap \Lambda^1 Ad_{\mathcal{P}}.$$

The above proposition has the following important consequence. If the structure group K is abelian H acts as identity on the whole of $\mathfrak{k}$ which means $\mathfrak{k}_1 = 0$ and L_1 is trivial. Thus by Remark 3.3 the space of infinitesimal deformations of the G_2-instanton A which is isomorphic to $\ker(D^{-1/3,A} + \frac{7}{3}) \cap \Lambda^1 Ad_{\mathcal{P}} = \ker(D^{-1/3,A} + \frac{7}{3}) \cap \Lambda^1 L_1$ is zero dimensional.

In [[14], Proposition 24] the authors prove that the G_2-instanton A is rigid if all the eigenvalues of the operator

$$\begin{aligned} L_A \colon \Lambda^1 \otimes Ad_{\mathcal{P}} &\to \Lambda^1 \otimes Ad_{\mathcal{P}} \\ w &\mapsto -2w \lrcorner F \end{aligned}$$

are smaller than 6. We prove the lower bound for the eigenvalue as follows. Let λ be the smallest eigenvalue of L_A. If $\epsilon \in \Gamma(T^*M \otimes Ad_{\mathcal{P}})$ is an infinitesimal deformation of A, then from (3.7) and Theorem 3.2 we know that

$$(\nabla^{t,A})^* \nabla^{t,A} \epsilon \cdot \eta = \left(\frac{5t^2}{12} + \frac{3t}{2} - \frac{17}{4} \right) \epsilon \cdot \eta - L_A(\epsilon) \cdot \eta.$$

Taking the inner product with $\epsilon \cdot \eta$ on both sides we get that if $\lambda > \min \left\{ \frac{5t^2+18t-51}{12} \mid t \in \mathbb{R} \right\} = -\frac{28}{5}$ then $\epsilon = 0$ is the only solution. Thus we get the following result.

Theorem 3.7 *Any G_2-instanton A on a principal K-bundle over a compact nearly G_2 manifold M is rigid if*

(i) the structure group G is abelian, or
(ii) the eigenvalues of the operator L_A are either all greater than $-\frac{28}{5}$ or all smaller than 6.

Some immediate consequences of Theorem 3.7 are that the flat instantons are rigid. Also if all the eigenvalues of L_A are equal, then A has to be rigid.

4 Instantons on homogeneous nearly G_2 manifolds

4.1 Classification of homogeneous nearly G_2 manifolds

By the classification result in [27] there are six compact, simply connected homogeneous nearly G_2 manifolds:

$$(S^7, g_{round}) = \mathrm{Spin}(7)/G_2,\ (S^7, g_{squashed}) = \frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)},\quad \mathrm{SO}(5)/\mathrm{SO}(3),$$
$$M(3,2) = \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)},\qquad N(k,l) = \mathrm{SU}(3)/S^1_{k,l}\ k,l \in \mathbb{Z},\ Q(1,1,1) = \mathrm{SU}(2)^3/\mathrm{U}(1)^2.$$

where $S^1_{k,l} = \{\mathrm{diag}(e^{ik\theta}, e^{il\theta}, e^{-i(k+l)\theta}), \theta \in \mathbb{R}\}$ denotes the embedding of $\mathrm{U}(1)$ into $\mathrm{SU}(3)$. We describe the homogeneous structure on each of these spaces.

- In the round S^7 the embedding of G_2 in $\mathrm{Spin}(7)$ is obtained by lifting the standard embedding of G_2 into $\mathrm{SO}(7)$.
- For the squashed metric on S^7 the two copies of $\mathrm{Sp}(1)$ in $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ denoted by $\mathrm{Sp}(1)_u$ and $\mathrm{Sp}(1)_d$ [9] are

$$\mathrm{Sp}(1)_u := \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) : a \in \mathrm{Sp}(1) \right\},\quad \mathrm{Sp}(1)_d := \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, a \right) : a \in \mathrm{Sp}(1) \right\}.$$

- In the Berger space $\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}$, the Lie group $\mathrm{SO}(3)$ is embedded into $\mathrm{SO}(5)$ via the 5-dimensional irreducible representation of $\mathrm{SO}(3)$ on $\mathrm{Sym}^2_0(\mathbb{R}^3)$.

- In $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}$ the embedding of SU(2) (denoted by $\mathrm{SU}(2)_d$) and U(1) in $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is defined as [9]

$$\mathrm{SU}(2)_d := \left\{\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}, a\right) : a \in \mathrm{SU}(2)\right\}, \quad \mathrm{U}(1) := \left\{\left(\begin{pmatrix} e^{i\theta} & 0 & 0\\ 0 & e^{i\theta} & 0\\ 0 & 0 & e^{-2i\theta}\end{pmatrix}, 1\right) : \theta \in \mathbb{R}\right\}$$

- In the Aloff–Wallach spaces $N_{k,l}$ where k, l are coprime positive integers the embedding of $S^1_{k,l} = U(1)_{k,l}$ in SU(3) is described

$$S^1_{k,l} = \left\{\begin{pmatrix} e^{ik\theta} & 0 & 0\\ 0 & e^{il\theta} & 0\\ 0 & 0 & e^{-i(k+l)\theta}\end{pmatrix}, \theta \in \mathbb{R}\right\}$$

- In $Q(1,1,1)$ we denote the two copies of U(1) inside $\mathrm{SU}(2)^3$ as $\mathrm{U}(1)_u, \mathrm{U}(1)_d$ where their respective embeddings are given by

$$\mathrm{U}(1)_u = \mathrm{Span}\left\{\left(\begin{pmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}, \begin{pmatrix} e^{-i\theta} & 0\\ 0 & e^{i\theta}\end{pmatrix}, \mathrm{I}_2\right), \theta \in \mathbb{R}\right\},$$
$$\mathrm{U}(1)_d = \mathrm{Span}\left\{\left(\mathrm{I}_2, \begin{pmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}, \begin{pmatrix} e^{-i\theta} & 0\\ 0 & e^{i\theta}\end{pmatrix}\right), \theta \in \mathbb{R}\right\}.$$

Except $N_{k,l}$ and $Q(1,1,1)$ the remaining four homogeneous spaces are normal. If B denotes the Killing form of G, then the nearly G_2 metric g on G/H is given by $g = -\frac{3}{40}B$. The choice of the scalar constant $\frac{3}{40}$ is based on our convention $\tau_0 = 4$. The general formula for the constant was derived in [[9], Lemma 7.1]. For a description of the nearly G_2 metric on the non-normal cases see [46].

All the six homogeneous nearly G_2 manifolds are naturally reductive. Let $\mathfrak{m}$ be the orthogonal complement of the Lie algebra $\mathfrak{h}$ of H in $\mathfrak{g}$ with respect to g then $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ equips the principal H-bundle $G \to G/H$ with a G-invariant connection whose horizontal spaces are the left translates of $\mathfrak{m}$. This connection is known as the characteristic homogeneous connection. On homogeneous nearly G_2 manifolds the characteristic homogeneous connection has holonomy contained in G_2. If we denote by $Z_{\mathfrak{m}}$ the projection of $Z \in \mathfrak{g}$ on $\mathfrak{m}$, the torsion tensor T for any $X, Y \in \mathfrak{m}$ is given by

$$T(X,Y) = -[X,Y]_{\mathfrak{m}},$$

and is totally skew-symmetric. Thus by the uniqueness result in [20] it is the characteristic connection with respect to the nearly G_2 structure on G/H [33]. The characteristic connection is a G_2-instanton as proved in [[33], Proposition 3.1].

The adjoint representation $\mathrm{ad}\colon H \to \mathrm{GL}(\mathfrak{m})$ gives rise to the associated vector bundle $G \times_{\mathrm{ad}} \mathfrak{m}$ on G/H. Similarly since G/H has a nearly G_2 structure we have the adjoint action of G_2 on $\mathfrak{m}$ which we again denote by ad and the isotropy homomorphism $\lambda\colon H \to \mathrm{G}_2$ which we can use to construct the associated vector bundle $G \times_{\mathrm{ad}\circ\lambda} \mathfrak{m}$. The characteristic connection is a connection on both $G \times_{\mathrm{ad}} \mathfrak{m}$ and $G \times_{\mathrm{ad}\circ\lambda} \mathfrak{m}$ with structure group H and G_2, respectively. Therefore it is natural to study the infinitesimal deformation space of the characteristic connection in both these situations. Since $H \subset \mathrm{G}_2$, the deformation space as an H-connection is a subset of the deformation space as a G_2-connection.

We can completely describe the deformation space when the structure group is H, but for structure group G_2 we can only find the deformation space for the normal homogeneous nearly G_2 manifolds since our methods do not work for non-normal homogeneous metrics. However since H is abelian in both of the non-normal cases, Theorem 3.7 tells us that the characteristic connection is rigid as an H-connection. But we cannot say anything about the deformation space for the structure group G_2 in those two cases.

The remainder of this article is devoted to computing the infinitesimal deformation space of the characteristic connection with the structure group H and G_2 for the normal homogeneous spaces.

4.2 Infinitesimal deformations of the characteristic connection

Let $M = G/H$ be a homogeneous manifold. Consider the principal H-bundle $G \to M$. If (V, ρ) is an H-representation then the space of smooth sections $\Gamma(G \times_\rho V)$ of the associated vector bundle $G \times_\rho V$ is isomorphic to the space $C^\infty(G, V)_H$ of H-equivariant smooth functions $G \to V$. The space $C^\infty(G, V)_H$ carries the left regular G-representation ρ_L defined by $\rho_L(g)(f) = g.f = f \circ l_{g^{-1}}$ which is also known as the induced G-representation $\mathrm{Ind}_H^G V$.
For any connection A on G the covariant derivative associated with A on any bundle associated with A is denoted by ∇^A. Let $s \in \Gamma(G \times_\rho V)$ and $f_s : G \to V$ be the G-equivariant function given by $s(gH) = [g, f_s(g)]$. If we denote by X_h the horizontal lift of $X \in \Gamma(TM)$ via A, then ∇^A acts on s as

$$(\nabla^A_X s)(gH) = (g, X_h(f_s)(g)).$$

For the characteristic connection on $G \to M$, $X_h = X$ for every vector field. Thus the covariant derivative ∇^c is given by

$$(\nabla^c_X s)(gH) = (g, X(f_s)(g)).$$

By the Peter–Weyl theorem [[38], Theorem 1.12] the space of sections can also be formulated as follows. If we denote by G_{irr} the set of equivalence classes of irreducible H-representations then

$$\Gamma(G \times_\rho V) = \overline{\bigoplus_{W \in G_{irr}} \mathrm{Hom}(W, V)_H \otimes W}.$$

The embedding $\mathrm{Hom}(W, V)_H \otimes W$ into $C^\infty(G, V)_H = \Gamma(G \times_\rho V)$ is given by sending (ϕ, w) to the function $f_{(\phi,w)}$ defined by $f_{(\phi,w)}(g) = \phi(\tau(g^{-1})w)$. Thus (ϕ, w) defines a section $s_{(\phi,w)}(gH) = [g, f_{(\phi,w)}(g)]$ which we denote by (ϕ, w) as well.

We can compute the covariant derivative on $s_{(\phi,w)} \in \mathrm{Hom}(W, V)_H \otimes W \subset \Gamma(G \times_\rho V)$ by

$$\begin{aligned}
\nabla^c_X s_{(\phi,w)}(gH) &= X(f_{(\phi,w)})(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(e^{tX}g) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f_{(\phi,w)} \circ l_{e^{tX}})(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (e^{-tX}.f)(g) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{(\phi,\tau(e^{-tX})w)} = -f_{(\phi,\tau_*(X)w)}(gH).
\end{aligned}$$

The above can be written as

$$\nabla^c_X(\phi, w) = -(\phi, \tau_*(X)w). \tag{4.1}$$

Thus we get that for the characteristic connection the covariant derivative of a section $s \in \Gamma(G \times_\rho V)$ with respect to some $X \in \mathfrak{m}$ translates into the derivative $X(f_s)$, which is minus the differential of the left-regular representation $(\rho_L)_*(X)(f_s)$, see [41].

Let $\{a_i,\ i = 1 \dots n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $g = -\frac{3}{40}B$, then the Casimir element $\mathrm{Cas}_{\mathfrak{g}} \in \mathrm{Sym}^2(\mathfrak{g})$ is defined by $\sum_{i=1}^{\dim G} a_i \otimes a_i$. On any $\mathfrak{g}$ representation (V, μ) we can define the Casimir invariant $\mu(\mathrm{Cas}_{\mathfrak{g}}) \in \mathrm{gl}(V)$ by

$$\mu(\mathrm{Cas}_{\mathfrak{g}}) = \sum_{i=1}^{n} \mu(a_i)^2.$$

For the reductive homogeneous spaces G/H let $\{a_i, i = 1 \dots \dim(H)\}$ and $\{a_i, i = \dim(H) \dots \dim(G)\}$ be the basis of $\mathfrak{h}$ and $\mathfrak{m}$, respectively. If we define $\mathrm{Cas}_{\mathfrak{h}} = \sum_{i=1}^{\dim(H)} a_i \otimes a_i$ and $\mathrm{Cas}_{\mathfrak{m}} = \sum_{\dim(H)}^{\dim(G)} a_i \otimes a_i$ we can decompose $\mathrm{Cas}_{\mathfrak{g}}$ as

$$\mathrm{Cas}_{\mathfrak{g}} = \mathrm{Cas}_{\mathfrak{h}} + \mathrm{Cas}_{\mathfrak{m}}.$$

Note that $\mathrm{Cas}_{\mathfrak{m}}$ is just used for notational convenience as $\mathfrak{m}$ may not be a Lie algebra. Also in $\mathrm{Cas}_{\mathfrak{h}}$ the trace is taken over H.

Remark 4.1 If one uses the metric $-cB$ instead of $-B$ then the Casimir operator is divided by the scalar c.

To study the deformation space of the characteristic connection ∇^c on these homogeneous spaces we rewrite Schrödinger–Lichnerowicz formula (3.7) in terms of the Casimir operator of $\mathfrak{h}$ and $\mathfrak{g}$ and then use the Frobenius reciprocity formula to compute the deformation space of the characteristic connection in each case. Let F be the curvature associated with ∇^c, then the operator $-2\epsilon \lrcorner F$ can be reformulated in terms of $\mathrm{Cas}_{\mathfrak{h}}$ by doing similar calculations as in [[17], Lemma 4] which gives

$$-2\epsilon \lrcorner F = (\rho_{\mathfrak{m}^*}(\mathrm{Cas}_{\mathfrak{h}}) \otimes 1_E + 1_{\mathfrak{m}^*} \otimes \rho_E(\mathrm{Cas}_{\mathfrak{h}}) - \rho_{\mathfrak{m}^* \otimes E}(\mathrm{Cas}_{\mathfrak{h}}))\epsilon. \tag{4.2}$$

Let (E, ρ_E) be an H-representation. We denote the tensor product of representations on $\mathfrak{m}^*$ and E by $\rho_{m^* \otimes E}$. For every $t \in \mathbb{R}$, $D^{t,A}$ denotes the Dirac operator on $G \times_{\rho_{\mathfrak{m}^* \otimes E}} (\mathfrak{m}^* \otimes E) \otimes \$$ associated to the connection ∇^A and ∇^t on $G \times_{\rho_{\mathfrak{m}^* \otimes E}} (\mathfrak{m}^* \otimes E)$ and $\$$, respectively. From now on we use the same symbol to denote the Lie group representation and the associated Lie algebra representation wherever there is no confusion. On a naturally reductive space Kostant's formula for cubic Dirac operator relates the square of the Dirac operator to suitable Casimir operators and scalar terms (see [6, 39, 41]). We now use Proposition 3.4 to prove a similar result for $(D^{-1/3,c})^2$.

Proposition 4.2 *Let ∇^c be the characteristic connection on a homogeneous nearly G_2 manifold $M = G/H$. Let (E, ρ_E) be an H-representation and ϵ be a smooth section of $G \times_{\rho_{\mathfrak{m}^* \otimes E}} (\mathfrak{m}^* \otimes E)$. Then*

$$(D^{-1/3,c})^2 \epsilon \cdot \eta = (-\rho_L(\mathrm{Cas}_{\mathfrak{g}}) + \rho_E(\mathrm{Cas}_{\mathfrak{h}}))\epsilon + \frac{49}{9}\epsilon) \cdot \eta. \tag{4.3}$$

Proof We begin by analyzing the rough Laplacian term in the Schrödinger– Lichnerowicz formula for $(D^{-1/3,c})^2 \epsilon \cdot \eta$ from (3.7) and then substitute the F-dependent term from (4.2) in the same. We denote by ρ_L the left regular representation of G. From above calculations we know that at the center of a normal orthonormal frame $\{e_i, i = 1 \dots 7\}$ of $\mathfrak{m}$ with respect to $g = -\frac{3}{40}B$,

$$(\nabla^{-1,c})^* \nabla^{-1,c} = -\nabla_{e_i}^{-1,c} \nabla_{e_i}^{-1,c} = -\rho_L(e_i)^2 = -\rho_L(\mathrm{Cas}_{\mathfrak{m}}).$$

Since $\mathrm{Res}^H_G\,\rho_L = \mathrm{Res}^H_G\,\mathrm{Ind}^G_H(m^*\otimes E) \cong m^*\otimes E$ we have that $\rho_{\mathfrak{m}^*\otimes E}(\mathrm{Cas}_{\mathfrak{h}}) = \rho_L(\mathrm{Cas}_{\mathfrak{h}})$. Also $\rho_{\mathfrak{m}^*}(e_i)^2 = \rho_{\mathfrak{m}^*}(\mathrm{Cas}_{\mathfrak{h}})$ acts as $-\mathrm{Ric}$ of the characteristic connection on 1-forms which is equal to $-\frac{16}{3}\mathrm{id}$ from (2.9). Substituting all the terms in (3.7) for $t=-1$ we get

$$\begin{aligned}(D^{-1/3,c})^2\epsilon\cdot\eta &= (-\rho_L(\mathrm{Cas}_{\mathfrak{m}})\epsilon + \frac{97}{9}\epsilon + (\rho_{\mathfrak{m}^*}(\mathrm{Cas}_{\mathfrak{h}})\otimes 1_E \\ &\quad + 1_{\mathfrak{m}^*}\otimes\rho_E(\mathrm{Cas}_{\mathfrak{h}}) - \rho_{\mathfrak{m}^*\otimes E}(\mathrm{Cas}_{\mathfrak{h}}))\epsilon)\cdot\eta \\ &= (-(\rho_L(\mathrm{Cas}_{\mathfrak{m}}) + \rho_L(\mathrm{Cas}_{\mathfrak{h}}))\epsilon + (\frac{97}{9}-\frac{16}{3})\epsilon + \rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon)\cdot\eta \\ &= ((-\rho_L(\mathrm{Cas}_{\mathfrak{g}})\epsilon + \rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon + \frac{49}{9}\epsilon)\cdot\eta\end{aligned}$$

which completes the proof.

Since all the homogeneous spaces considered are naturally reductive and $H\subset \mathrm{G}_2$, there is an adjoint action of H on $\mathfrak{m}$, $\mathfrak{h}$ and $\mathfrak{g}_2$ and thus H-representations on $\mathfrak{m}^*\otimes\mathfrak{h}$ and $\mathfrak{m}^*\otimes\mathfrak{g}$ which we denote by $\rho_{\mathfrak{m}^*\otimes\mathfrak{h}}, \rho_{\mathfrak{m}^*\otimes\mathfrak{g}_2}$. The corresponding Lie algebra representations are denoted similarly. The infinitesimal deformation space of the instanton ∇^c is a subspace of $\Gamma(\mathfrak{m}^*\otimes E)$ where E can be either $\mathfrak{h}$ or $\mathfrak{g}_2$.

From Propositions 3.1 and 4.2 it is clear that if ϵ is an infinitesimal deformation of ∇^c on the bundle $\mathfrak{m}^*\otimes E$ over G/H then

$$\rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon = \rho_L(\mathrm{Cas}_{\mathfrak{g}})\epsilon \tag{4.4}$$

where the trace in both the Casimirs is taken over G.

Using (4.4) we can reformulate the infinitesimal deformation space of the characteristic connection. Since the Casimir operator acts as scalar multiple of the identity on irreducible representations we can solve (4.4) for irreducible subrepresentations of L. From Theorem 3.2 the deformations of the characteristic connection are the -2 eigenfunctions $\epsilon\cdot\eta$ of $D^{-1,c}$. To explicitly compute the deformation space first we need to find the solutions for (4.4) which by above proposition is identical to the space of $\frac{49}{9}$ eigenfunctions $\epsilon\cdot\eta$ of $(D^{-1/3,c})^2$. For $\alpha\in\Lambda^1\mathrm{Ad}_{\mathcal{P}}$ by Lemma 2.1 $D^{t,A}\alpha\cdot\eta = D^{0,A}\alpha\cdot\eta + \frac{t}{2}\varphi\cdot\alpha\cdot\eta = D^{0,A}\alpha\cdot\eta - \frac{t}{2}\alpha\cdot\eta$. Therefore the $\pm\frac{7}{3}$ eigenfunctions $\epsilon\cdot\eta$ of $D^{-1/3,c}$ correspond to the -2 and $\frac{8}{3}$ eigenfunction of $D^{-1,A}$, respectively. By Proposition 3.5 we have the following decomposition

$$\begin{aligned}\ker\left((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}\right)\cap\Gamma(\mathfrak{m}^*\otimes E) &= \ker(D^{-1,c}+2\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E) \\ &\quad \ker(D^{-1,c}-\frac{8}{3}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E)\end{aligned} \tag{4.5}$$

The first summand on the right-hand side is isomorphic to the space of infinitesimal deformations of ∇^c by Theorem 3.2. So in the second step we check which of the subspaces in $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id})\cap(\Gamma(\mathfrak{m}^*\otimes E)\cdot\eta)$ lie in the -2 eigenspace of $D^{-1,c}$.

The Killing spinor η is parallel with respect to ∇^{-1} therefore by the definition of the Dirac operator and Proposition we can restrict $D^{-1,c}$ and $(D^{-1/3,c})^2$ to operators from $\Gamma(\mathfrak{m}^*\otimes E)\to\Gamma(\mathfrak{m}^*\otimes E)$. On a homogeneous space we can explicitly compute the characteristic connection as we describe below.

Step 1: Calculating $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E)$:

Let $E_{\mathbb{C}} = \oplus_{i=1}^{n} V_i$ be the decomposition of $E_{\mathbb{C}}$ into complex irreducible H-representations. For each V_i we find all the complex irreducible G-representations $W_{i,j}$, $j = 1 \dots n_i$, that satisfy the equation

$$\rho_{V_i}(\mathrm{Cas}_{\mathfrak{h}}) = \rho_{W_{i,j}}(\mathrm{Cas}_{\mathfrak{g}}).$$

In order to see whether $W_{i,j} \subset \mathrm{Ind}_H^G(\mathfrak{m}^* \otimes E)_{\mathbb{C}}$ we find the multiplicity $m_{i,j}$ of $W_{i,j}$ in $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes V_i)$. Because of Schur's lemma this multiplicity is given by $\dim(\mathrm{Hom}(W_{i,j}, \mathfrak{m}^*_{\mathbb{C}} \otimes V_i)_H)$. Repeating this process for all the i, j's and summing over all irreducible G-representations $W_{i,j}$ along with their multiplicity we get,

$$\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)_{\mathbb{C}} \cong \bigoplus_{i=1}^{n} \Big(\bigoplus_{j=1}^{n_i} m_{i,j} W_{i,j} \Big). \tag{4.6}$$

Step 2: Calculating $\ker(D^{-1,c} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$:

To figure out which of the $W_{i,j}$'s found in Step 1 are in the $\ker(D^{-1,c} + 2\mathrm{id})$ we need to calculate the covariant derivative ∇^c on $\mathrm{Hom}(W_{i,j}, \mathfrak{m}^*_{\mathbb{C}} \otimes V_i)_H \otimes W_{i,j} \subseteq \Gamma(\mathfrak{m}^* \otimes E)_{\mathbb{C}}$.

If (W, τ) is an irreducible G-subrepresentation of $\mathrm{Ind}_H^G(\mathfrak{m}^* \otimes E)$ then $\mathrm{Hom}(W, \mathfrak{m}^* \otimes E)_H$ is non-trivial. By Schur's lemma the dimension of $\mathrm{Hom}(W, \mathfrak{m}^* \otimes E)_H$ is the number of common irreducible H-subrepresentations in $\mathrm{Res}_G^H W$ and $\mathfrak{m}^* \otimes E$. Let W_α be such a common irreducible H-representation. We denote by $V|_U$ the subspace of V isomorphic to U then $\mathrm{Hom}(W|_{W_\alpha}, (\mathfrak{m}^* \otimes E)|_{W_\alpha} = \mathrm{Span}\{\phi_\alpha\}$. Let τ_* be the Lie algebra $\mathfrak{g}$ representation associated with the G-representation (W, τ) then for $X \in \Gamma(TM)$ and $(\phi = \sum c_\alpha \phi_\alpha, w) \in \mathrm{Hom}(W, \mathfrak{m}^* \otimes E)_H \otimes W$, (4.1)

$$\nabla^c_X(\phi, w)(eH) = -\phi(\tau_*(X)w) \in \mathfrak{m}^* \otimes E.$$

Using this we can calculate the Dirac operator at eH by

$$D^{-1,c}(\phi_\alpha, w)(eH) = -\sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(\phi_\alpha, w)(eH) = -\sum_{i=1}^{7} e_i \cdot \phi_\alpha(\tau_*(e_i)w). \tag{4.7}$$

The above method can be extended by linearity to compute the Dirac operator on $\Gamma(\mathfrak{m}^* \otimes E)$. Note that we have omitted the Killing spinor η since it is parallel with respect to η so does not effect the eigenspace.

In the following sections we implement the above procedure on each of the four homogeneous spaces.

Remark 4.3 In a nearly Kähler 6-manifold whose structure is defined by a real Killing spinor η, the spinor $\mathrm{vol} \cdot \eta$ is another independent real Killing spinor. Any Dirac operator $\not{D}$ anti-commutes with the Clifford multiplication by vol that is $\not{D}\mathrm{vol} = -\mathrm{vol} \cdot \not{D}$,, hence for all $\lambda \in \mathbb{R}$ we have $\ker(\not{D} - \lambda\mathrm{id}) \cong \ker(\not{D} + \lambda\mathrm{id})$.. Therefore $\ker(\not{D}^2 - \lambda^2\mathrm{id}) \cong 2\ker(\not{D} \pm \lambda\mathrm{id})$ and one can compute the λ eigenspace of $\not{D}$ by computing the λ^2 eigenspace of $\not{D}^2$ as done in [[17], Proposition 4]. In the case of nearly G_2 manifolds $\not{D}$ and the 7-dimensional vol commute and thus we do not have such an isomorphism between the $\pm\lambda$ eigenspaces of the Dirac operator. In fact there is no such automatic relation between $\ker(\not{D}^2 - \lambda^2\mathrm{id})$ and $\ker(\not{D} + \lambda\mathrm{id})$ as Sect. 4.4 reveals.

Remark 4.4 The Dirac operator is always self-adjoint; therefore, the above method of finding a particular eigenspace of a Dirac operator D can be used more generally in any bundle associated with the spinor bundle over a homogeneous spin manifold. We will first compute the λ^2-eigenspace the square of the Dirac operator as it can be easily described algebraically (see Proposition 4.2). Once we know the λ^2-eigenspace of D^2 we can apply D on them to see which of them lie in the λ or $-\lambda$-eigenspace of D.

4.3 Eigenspaces of the square of the Dirac operator

In this section we follow *Step 1* of the above procedure. To see which of the irreducible representations of G satisfy (4.4), we need to compute the Casimir operator on complex irreducible representations. Given any irreducible representation ρ_λ with highest weight λ we use the Freudenthal formula to compute $\rho_\lambda(\mathrm{Cas}_{\mathfrak{g}})$. We drop the constant $\frac{40}{3}$ in our definition of Casimir operator for this section as it does not play any role in comparing the Casimir operators. Let $\mu = \frac{1}{2}$(sum of the positive roots of $\mathfrak{g}$) then the Freudenthal formula states that

$$\rho_\lambda(\mathrm{Cas}_{\mathfrak{g}}) = B(\lambda, \lambda) + 2B(\mu, \lambda). \tag{4.8}$$

We compute the deformation space of the characteristic connection for $E = \mathfrak{h}$ and $E = \mathfrak{g}_2$ as described earlier. In all the examples listed below, Case 1 is for $E = \mathfrak{h}$ and Case 2 is for $E = \mathfrak{g}_2$.

4.3.1 Spin(7)/G$_2$

For this space, $H = \mathrm{G}_2$ so there is only one case to consider.

The adjoint representation $\mathfrak{g}_2$ is the unique 14-dimensional irreducible representation of G_2. The complex irreducible representations of G_2 are identified with respect to their highest weights of the form $(p, q) \in \mathbb{Z}^2_{\geq 0}$ and are denoted by $V_{(p,q)}$. Here $V_{(1,0)}$ is the 7-dimensional standard G_2-representation and $V_{(0,1)}$ is the 14-dimensional adjoint representation. The reductive splitting of the Lie algebra is given by

$$\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}.$$

We have the following isomorphisms of G_2 representations,

$$\mathfrak{h}_{\mathbb{C}} = (\mathfrak{g}_2)_{\mathbb{C}} \cong V_{(0,1)}$$
$$\mathfrak{m}_{\mathbb{C}} \cong V_{(1,0)}.$$

The isomorphism $\mathfrak{spin}(7) \cong \mathfrak{so}(7)$ implies that the eigenvalues of their Casimir operators on irreducible representations are equal. For $\mathfrak{so}(7)$, let E_{ij} be the 7×7 skew-symmetric matrix with 1 at the (i, j)th entry and 0 elsewhere. We define $H_1 = E_{45} - E_{23}$, $H_2 = E_{67} - E_{45}$ and $H_3 = E_{45}$. A Cartan subalgebra for $\mathfrak{so}(7)$ is given by $\mathrm{Span}\{H_i, i = 1, 2, 3\}$. A set of simple roots $\{\alpha_i, i = 1, 2, 3\}$ is given by

$$\alpha_1 = \begin{bmatrix} i \\ -2i \\ i \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ i \\ -i \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}.$$

The Cartan matrix C of $\mathfrak{so}(7)$ which is given by

$$C = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$$

Then one can compute the simple co-roots F_is by $\alpha_i(F_j) = C_{ij}$ which give $F_1 = iH_2$, $F_2 = -iH_1 + 2iH_3$ and $F_3 = -2iH_2 - 2iH_3$. The set of fundamental weights is dual to the set of the simple co-roots. We denote the fundamental weights in decreasing order by λ_1, λ_2 and λ_3 which are dual to F_3, F_1, F_2, respectively. We can compute easily that

$$\begin{bmatrix} B(H_1,H_1) & B(H_1,H_2) & B(H_1,H_2) \\ B(H_2,H_1) & B(H_2,H_2) & B(H_2,H_3) \\ B(H_3,H_1) & B(H_3,H_2) & B(H_3,H_3) \end{bmatrix} = \begin{bmatrix} -20 & 10 & -10 \\ 10 & -20 & 10 \\ -10 & 10 & -10 \end{bmatrix}$$

which implies,

$$\begin{bmatrix} B(\lambda_1,\lambda_1) & B(\lambda_1,\lambda_2) & B(\lambda_1,\lambda_2) \\ B(\lambda_2,\lambda_1) & B(\lambda_2,\lambda_2) & B(\lambda_2,\lambda_3) \\ B(\lambda_3,\lambda_1) & B(\lambda_3,\lambda_2) & B(\lambda_3,\lambda_3) \end{bmatrix} = \begin{bmatrix} 3/40 & 1/10 & 1/20 \\ 1/10 & 1/5 & 1/10 \\ 1/20 & 1/10 & 1/10 \end{bmatrix}.$$

Since half the sum of positive roots is given by $\lambda_1+\lambda_2+\lambda_3$ in [[34], Section 13.3] therefore by (4.8) on an irreducible SO(7)-representation $V_{(m_1,m_2,m_3)}$ with highest weight $m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3,\ m_1, m_2, m_3 \geq 0$ we have

$$\rho_\lambda(\mathrm{Cas}_{\mathfrak{so}(7)}) = \frac{1}{40}(3m_1^2+8m_2^2+4m_3^2+8m_1m_2+4m_1m_3+8m_2m_3+18m_1+32m_2+20m_3).$$

Now we compute the eigenvalues of the Casimir operator for the irreducible representations of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. A Cartan subalgebra of $\mathfrak{g}_2$ is given by Span$\{H_1, H_2\}$. Here a pair of simple roots β_1, β_2 is given by

$$\beta_1 = \begin{bmatrix} i \\ -2i \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

and the Cartan matrix $\tilde{C}$ for $\mathfrak{g}_2$ is given by

$$\tilde{C} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

. Let μ_1, μ_2 be the fundamental weights in decreasing order then their duals with respect to B are $-iH_1 - 2iH_2, iH_2$, respectively, and one can compute

$$\begin{bmatrix} B(\mu_1,\mu_1) & B(\mu_1,\mu_2) \\ B(\mu_2,\mu_1) & B(\mu_2,\mu_2) \end{bmatrix} = \begin{bmatrix} 1/15 & 1/10 \\ 1/10 & 1/5 \end{bmatrix}.$$

Again half the sum of the positive roots is given by $\mu_1 + \mu_2$. Using these values in the Freudenthal formula for an irreducible G_2-representation $V_{(p,q)}$ with highest weight $p\mu_1 + q\mu_2$ we have

$$\rho_{(p,q)}(\mathrm{Cas}_{\mathfrak{g}_2}) = \frac{1}{15}(p^2+3q^2+3pq+5p+9q).$$

Case 1: $E = \mathfrak{g}_2$

The adjoint representation $(\mathfrak{g}_2)_{\mathbb{C}} \cong V_{(0,1)}$. From above

$$\rho_{(0,1)}(\mathrm{Cas}_{\mathfrak{g}_2}) = \frac{4}{5}.$$

Substituting the above-found values into (4.4) we get that $V_{(m_1,m_2,m_3)}$ can be an infinitesimal deformation space for the characteristic connection if

$$\frac{1}{40}(3m_1^2+8m_2^2+4m_3^2+8m_1m_2+4m_1m_3+8m_2m_3+18m_1+32m_2+20m_3) = \frac{4}{5}.$$

But since there are no positive integral solutions of this equation there are no deformations of the characteristic connection on $\mathrm{Spin}(7)/\mathrm{G}_2$.

4.3.2 SO(5)/SO(3)

The complex irreducible SO(5)-representations are characterized by highest weights $(m_1, m_2) \in \mathbb{Z}_{\geq 0}$. The complex irreducible representations of SO(3) are given by $S^k\mathbb{C}^2$ which is a $\binom{2+k-1}{k} = k+1$-dimensional space. The 3-dimensional adjoint representation $\mathfrak{so}(3)_\mathbb{C}$ and the 7-dimensional representation $\mathfrak{m}_\mathbb{C}$ are irreducible SO(3)-representations therefore

$$\mathfrak{m}_\mathbb{C} \cong S^6\mathbb{C}^2,$$
$$\mathfrak{so}(3)_\mathbb{C} \cong S^2\mathbb{C}^2.$$

A Cartan subalgebra of $\mathfrak{so}(5)$ is given by Span$\{H_1, H_2\}$ where $H_1 = E_{12}$, $H_2 = E_{34}$ where E_{ij} is the 5×5 skew-symmetric matrix with 1 at the (i, j)th position and 0 elsewhere. With respect to the Killing form B on $\mathfrak{so}(5)$, H_1 is orthogonal to H_2 with $B(H_i, H_i) = -6$ for $i = 1, 2$. Let λ_1, λ_2 be the fundamental weights whose duals are $i(H_1 - H_2)$, $2iH_2$, respectively, then half the sum of positive roots is given by $\lambda_1 + \lambda_2$. Doing similar computations as above we get

$$\begin{bmatrix} B(\lambda_1, \lambda_1) & B(\lambda_1, \lambda_2) \\ B(\lambda_2, \lambda_1) & B(\lambda_2, \lambda_2) \end{bmatrix} = \begin{bmatrix} 1/6 & 1/12 \\ 1/12 & 1/12 \end{bmatrix}.$$

Using (4.8) for the eigenvalues of the Casimir operator for irreducible representation $V_{(m_1,m_2)}$ of SO(5) with highest weight $m_1\lambda_1 + m_2\lambda_2$ for $m_1, m_2 \geq 0$ we get,

$$\rho_{(m_1,m_2)}(\mathrm{Cas}_{\mathfrak{so}(5)}) = \frac{1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2).$$

Under the embedding of $\mathfrak{so}(3)$ in $\mathfrak{so}(5)$ the Cartan subalgebra of $\mathfrak{so}(3)$ is given by Span$\{2H_1 + H_2\}$. Here the Cartan subalgebra is 1-dimensional and the fundamental weight μ_1 is dual to $4iH_1 + 2iH_2$. Using $B(H_i, H_i) = -6$ one can compute that $B(4H_1 + 2H_2, 4H_1 + 2H_2) = -120$ the eigenvalue of the Casimir operator on the irreducible representation $S^q\mathbb{C}^2$ of $\mathfrak{so}(3)$ is given by

$$\rho_q(\mathrm{Cas}_{\mathfrak{so}(3)}) = \frac{1}{120}(q^2 + 2q).$$

Case 1: $E = \mathfrak{so}(3)$
The adjoint representation of $\mathfrak{so}(3)_\mathbb{C}$ is an irreducible $\mathfrak{so}(3)$ representation with highest weight 2. Thus

$$\rho_E(\mathrm{Cas}_{\mathfrak{so}(3)}) = \rho_2(\mathrm{Cas}_{\mathfrak{so}(3)}) = \frac{1}{15}.$$

We need to find irreducible representations $V_{(m_1,m_2)}$ of $\mathfrak{so}(5)$ that satisfy (4.4) which requires

$$\frac{1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2) = \frac{1}{15}.$$

But since there are no integral solutions for the equation, the deformation space is trivial in this case.
Case 2: $E = \mathfrak{g}_2$
The adjoint representation of $(\mathfrak{g}_2)_\mathbb{C}$ splits as an $\mathfrak{so}(3)$ representation into $S^2\mathbb{C}^2 \oplus S^{10}\mathbb{C}^2$. The first component in the splitting has already been studied in case 1 and hence has no contribution to the deformation space. For the second component

$$\rho_{10}(\mathrm{Cas}_{\mathfrak{so}(3)}) = 1.$$

Thus we need to find $\mathfrak{so}(5)$ representations $V_{(m_1,m_2)}$ such that

$$\frac{1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2) = 1,$$

which has one integral solution, namely $m_1 = 0, m_2 = 2$. Thus $V_{(0,2)} \cong \mathfrak{so}(5)_{\mathbb{C}}$ is the only SO(5)-representation for which $\mathrm{Cas}_{\mathfrak{g}}$ has eigenvalue 1. As $\mathfrak{so}(3)$ representations

$$V_{(0,2)} \cong S^2\mathbb{C}^2 \oplus S^6\mathbb{C}^2,$$

$$\mathfrak{m}_{\mathbb{C}}^* \otimes S^{10}\mathbb{C}^2 \cong \bigoplus_{k=2}^{8} S^{2k}\mathbb{C}^2.$$

Thus $V_{(0,2)}$ and $\mathfrak{m}_{\mathbb{C}}^* \otimes S^{10}\mathbb{C}^2$ have 1 common irreducible $\mathfrak{so}(3)$ representation, namely $S^6\mathbb{C}^2$. Thus $V_{(0,2)}$ occurs in $\mathrm{Ind}_H^G(\mathfrak{m}_{\mathbb{C}}^* \otimes S^{10}\mathbb{C}^2)$ with multiplicity 1. Therefore in this case $(\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2))_{\mathbb{C}} \cong V_{(0,2)}$.

4.3.3 $\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$

The Lie algebra $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ decomposes as

$$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) = \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d \oplus \mathfrak{m}$$

and the embeddings $\mathfrak{sp}(1)_u, \mathfrak{sp}(1)_d$ are given by

$$\mathfrak{sp}(1)_u = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, 0\right) : a \in \mathfrak{sp}(1)\right\}, \quad \mathfrak{sp}(1)_d = \left\{\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, a\right) : a \in \mathfrak{sp}(1)\right\}$$

where we follow the notations used in [9]. Let $H_1 = (E_1, 0)$, $H_2 = (E_2, 0)$ and $H_3 = (0, E_3)$ then a Cartan subalgebra of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ is given by $\mathrm{Span}\{H_1, H_2, H_3\}$ where

$$E_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

If B denotes the Killing form of $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$, we can compute that H_is are orthogonal with respect to B and $B(H_i, H_i) = -12$ for $i = 1, 2$ and $B(H_3, H_3) = -8$. The fundamental weights $\lambda_1, \lambda_2, \lambda_3$ are dual to $i(H_1 - H_2), iH_1, iH_3$, respectively, and half the sum of positive roots is given by $\lambda_1 + \lambda_2 + \lambda_3$. By identical calculations as in other cases we get

$$\begin{bmatrix} B(\lambda_1,\lambda_1) & B(\lambda_1,\lambda_2) & B(\lambda_1,\lambda_3) \\ B(\lambda_2,\lambda_1) & B(\lambda_2,\lambda_2) & B(\lambda_2,\lambda_3) \\ B(\lambda_3,\lambda_1) & B(\lambda_3,\lambda_2) & B(\lambda_3,\lambda_3) \end{bmatrix} = \begin{bmatrix} 1/12 & 1/12 & 0 \\ 1/12 & 1/6 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}.$$

Applying Freudenthal formula (4.8) we get that the Casimir operator of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ acts on the irreducible representations $V_{(m_1,m_2,l)}$ with highest weight $m_1\lambda_1 + m_2\lambda_2 + l\lambda_3$, $m_1, m_2, l \geq 0$ with the eigenvalue

$$\rho_{(m_1,m_2,l)}(\mathrm{Cas}_{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}) = \frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l).$$

Under the embedding given above a Cartan subalgebra of $\mathfrak{sp}(1)_u, \mathfrak{sp}(1)_d$ is given by $\mathrm{Span}\{H_1\}$ and $\mathrm{Span}\{(E_2, E_3)\}$, respectively. Let P, Q be the standard 2-dimensional representation of $\mathfrak{sp}(1)_u, \mathfrak{sp}(1)_d$, respectively. Then the unique $(n+1)-$dimensional irreducible

$\mathfrak{sp}(1)_u$ (respectively, $\mathfrak{sp}(1)_d$) representation is given by S^nP (respectively, S^nQ). From previous calculations we have $B(H_1, H_1) = -12$; thus, the eigenvalue of $\mathrm{Cas}_{\mathfrak{sp}(1)_u}$ on S^nP is given by

$$\rho_n(\mathrm{Cas}_{\mathfrak{sp}(1)_u}) = \frac{1}{12}(n^2 + 2n).$$

Similarly with the help of previous work one can calculate $B((E_2, E_3), (E_2, E_3)) = -20$. Thus $\mathrm{Cas}_{\mathfrak{sp}(1)_d}$ acts on S^pQ as the scalar multiple of

$$\rho_n(\mathrm{Cas}_{\mathfrak{sp}(1)_d}) = \frac{1}{20}(n^2 + 2n).$$

The adjoint representation $\mathfrak{sp}(1)$ is an irreducible 3-dimensional $\mathfrak{sp}(1)$ representation, and hence, we have the following decompositions into $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$ representations

$$(\mathfrak{sp}(1)_u)_{\mathbb{C}} \cong S^2P, \quad (\mathfrak{sp}(1)_d)_{\mathbb{C}} \cong S^2Q, \quad \mathfrak{m}_{\mathbb{C}} \cong S^2Q \oplus PQ$$

where PQ denotes the tensor product of P and Q and we omitted the tensor product sign for clarity and will continue to do so.

Case 1: $E = \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$

We need to find the irreducible $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ representations $V_{(m_1,m_2,l)}$ that satisfy (4.4) for each irreducible component of $\mathfrak{h}_{\mathbb{C}}$ that is $(\mathfrak{sp}(1)_u)_{\mathbb{C}}$ and $(\mathfrak{sp}(1)_d)_{\mathbb{C}}$. For $\mathfrak{sp}(1)_u$ this equation takes the form

$$\frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l) = \frac{8}{12}.$$

The integral solution (m_1, m_2, l) for this equation is $(0, 1, 0)$. Thus the only irreducible $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ representations for which $\mathrm{Cas}_{\mathfrak{g}}$ has eigenvalue $\frac{2}{3}$ is $V_{(0,1,0)}$. As $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$-representations we have the following decomposition

$$V_{(0,1,0)} \cong PQ \oplus \mathbb{C},$$
$$(\mathfrak{sp}(1)_u \otimes \mathfrak{m})_{\mathbb{C}} \cong S^2PS^2Q \oplus S^3PQ \oplus PQ.$$

The irreducible $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ representation in $(\mathfrak{sp}(1)_u \otimes \mathfrak{m})_{\mathbb{C}}$ common with $V_{(0,1,0)}$ is PQ with multiplicity 1. Thus $V_{(0,1,0)}$ occurs in $\mathrm{Ind}_H^G(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u)_{\mathbb{C}}$ with multiplicity 1. Therefore the solutions to (4.4) in $\Gamma(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u)_{\mathbb{C}}$ are the 5-dimensional complex $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ representation $V_{(0,1,0)}$.

For the next irreducible $\mathfrak{h}_{\mathbb{C}}$ component $(\mathfrak{sp}(1)_d)_{\mathbb{C}}$ (4.4) for $V_{(m_1,m_2,l)}$ becomes

$$\frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l) = \frac{8}{20},$$

which has no integral solutions and thus it has no contribution to the deformation space.

Thus from Proposition 4.2 we conclude that $(\ker((D^{-1/3.c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d)_{\mathbb{C}} \cong (V_{(0,1,0)})$ when the structure group is $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$.

Case 2: $E = (\mathfrak{g}_2)_{\mathbb{C}}$

The adjoint representation of $\mathfrak{g}_2$ decomposes into irreducible $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$ as follows:

$$(\mathfrak{g}_2)_{\mathbb{C}} = S^2P \oplus S^2Q \oplus PS^3Q.$$

We have already seen the contribution of the first two irreducible components in the summation. For the third component

$$\rho_{1,3}(\mathrm{Cas}_{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d}) = 1,$$

so here we need to find the $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ representations $V_{(m_1,m_2,l)}$ such that

$$\frac{1}{12}(m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2) + \frac{1}{8}(l^2 + 2l) = 1.$$

The $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$-representations that satisfy (4.4) are $V_{(2,0,0)}$ and $V_{(0,0,2)}$, which decompose into $\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d$ representations as

$$V_{(2,0,0)} \cong \mathfrak{sp}(2)_{\mathbb{C}} \cong S^2P \oplus S^2Q \oplus PQ, \quad V_{(0,0,2)} \cong (\mathfrak{sp}(1)_d)_{\mathbb{C}} \cong S^2Q.$$

Moreover

$$PS^3Q \otimes \mathfrak{m}_{\mathbb{C}}^* \cong S^2PS^4Q \oplus S^2PS^2Q \oplus P(S^5Q \oplus S^3Q \oplus Q) \oplus S^4Q \oplus S^2Q.$$

Thus $V_{(2,0,0)}$ and $PS^3Q \otimes \mathfrak{m}_{\mathbb{C}}^*$ have two common irreducible representations PQ, S^2Q and $V_{(0,0,2)}$ and $PS^3Q \otimes \mathfrak{m}_{\mathbb{C}}^*$ have one common irreducible representation S^2Q. So by Frobenius reciprocity $V_{(2,0,0)}$ and $V_{(0,0,2)}$ lie in $\mathrm{Ind}_H^G(\mathfrak{m}_{\mathbb{C}}^* \otimes PS^3Q)$ with multiplicity 2, 1, respectively. Thus the solution of (4.4) in $\Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ is the 28-dimensional $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ complex representation $2V_{(2,0,0)} \oplus V_{(0,1,0)} \oplus V_{(0,0,2)}$. So again by Proposition 4.2 we conclude that $\ker((D^{-1/3.c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}} \cong 2V_{(2,0,0)} \oplus V_{(0,1,0)} \oplus V_{(0,0,2)}$ when the structure group is G_2.

4.3.4 $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$

The embeddings of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ in $\mathfrak{su}(3) \times \mathfrak{su}(2)$ which we denote by $\mathfrak{su}(2)_d$ and $\mathfrak{u}(1)$ following [9] in $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ are given by

$$\mathfrak{su}(2)_d = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a\right) : a \in \mathfrak{su}(2)\right\}, \quad \mathfrak{u}(1) = \mathrm{span}\left\{\left(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0\right)\right\}.$$

A Cartan subalgebra of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ is given by $\mathrm{span}\{H_1 = (E_1, 0), H_2 = (E_2, 0), H_3 = (0, E_3)\}$ where

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We can check that the H_is are orthogonal with respect to the Killing form B on $\mathrm{SU}(3)\times\mathrm{SU}(2)$. As earlier we denote by $\lambda_1, \lambda_2, \lambda_3$ the fundamental weights which are dual to $\frac{i}{2}(H_1 - H_2)$, $\frac{i}{2}(H_1 + H_2)$, iH_3, respectively. By direct computations we get

$$\begin{bmatrix} B(H_1, H_1) & B(H_1, H_2) & B(H_1, H_3) \\ B(H_2, H_1) & B(H_2, H_2) & B(H_2, H_3) \\ B(H_3, H_1) & B(H_3, H_2) & B(H_3, H_3) \end{bmatrix} = \begin{bmatrix} -12 & 0 & 0 \\ 0 & -36 & 0 \\ 0 & 0 & -8 \end{bmatrix},$$

therefore

$$\begin{bmatrix} B(\lambda_1, \lambda_1) & B(\lambda_1, \lambda_2) & B(\lambda_1, \lambda_3) \\ B(\lambda_2, \lambda_1) & B(\lambda_2, \lambda_2) & B(\lambda_2, \lambda_3) \\ B(\lambda_3, \lambda_1) & B(\lambda_3, \lambda_2) & B(\lambda_3, \lambda_3) \end{bmatrix} = \begin{bmatrix} 1/9 & 1/18 & 0 \\ 1/18 & 1/9 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}.$$

Half the sum of the positive roots is $\lambda_1 + \lambda_2 + \lambda_3$ and thus by Freudenthal formula (4.8) for a $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ representation $V_{(m_1,m_2,l)}$ with highest weight $m_1\lambda_1 + m_2\lambda_2 + l\lambda_3$ where $m_1, m_2, l \geq 0$

$$\rho_{m_1,m_2,l}(\mathrm{Cas}_{\mathfrak{su}(3)\oplus\mathfrak{su}(2)}) = \frac{1}{9}(m_1^2 + m_2^2 + m_1m_2 + 3m_1 + 3m_2) + \frac{1}{8}(l^2 + 2l).$$

Using the embeddings of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ given above we see that Cartan subalgebras of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ in $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ are given by span$\{(E_1, E_3)\}$ and span$\{H_2\}$, respectively. By calculations completely analogous to the previous case we then get that if we represent the irreducible $(n+1)$-dimensional $\mathfrak{su}(2)_d$ representations by S^nW where W is the standard $\mathfrak{su}(2)_d$ representation and the 1-dimensional $\mathfrak{u}(1)$ representation with highest weight k by $F(k)$ we get by Freudenthal formula (4.8)

$$\rho_n(\mathrm{Cas}_{\mathfrak{su}(2)_d}) = \frac{1}{20}(n^2 + 2n),$$
$$\rho_k(\mathrm{Cas}_{\mathfrak{u}(1)}) = \frac{1}{36}k^2.$$

As $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$ representations the 7-dimensional space $\mathfrak{m}_{\mathbb{C}}$ decomposes as

$$\mathfrak{m}_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3),$$

whereas the 3-dimensional adjoint representation of $(\mathfrak{su}(2)_d)_{\mathbb{C}}$ is irreducible and hence is isomorphic to S^2W.
Case 1: $E = \mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$
The adjoint representation $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$ splits as irreducible $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$ representations as follows:

$$(\mathfrak{su}(2)_d \oplus \mathfrak{u}(1))_{\mathbb{C}} \cong S^2W \oplus \mathbb{C}.$$

Since $U(1)$ is abelian we know by Theorem 3.7 that the component $\mathfrak{u}(1)$ is abelian and thus gives rise to no deformations of the characteristic connection. Therefore we only need to check for deformations corresponding to S^2W. For that we need to look for representations $V_{(m_1,m_2,l)}$ such that

$$\frac{1}{9}(m_1^2 + m_2^2 + m_1m_2 + 3m_1 + 3m_2) + \frac{1}{8}(l^2 + 2l) = \frac{8}{20},$$

which as seen before has no integral solutions.

Hence the characteristic connection admits no deformations in this case.
Case 2: $E = \mathfrak{g}_2$
The adjoint representation $(\mathfrak{g}_2)_{\mathbb{C}}$ splits as $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$ representation as follows:

$$(\mathfrak{g}_2)_{\mathbb{C}} = S^3WF(3) \oplus S^3WF(-3) \oplus S^2W \oplus F(6) \oplus F(-6) \oplus \mathbb{C}.$$

We need to follow the same procedure as above for each of the components. For each component we need to find the $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ representation $V_{(m_1,m_2,l)}$ that satisfies (4.4). We have already solved this for $S^2W \oplus \mathbb{C}$, so we just need to compute it for the rest.
From above calculations $\rho_{S^3WF(3)}(\mathrm{Cas}_{\mathfrak{h}}) = 1$ therefore $V_{(m_1,m_2,l)}$ should satisfy

$$\frac{1}{9}(m_1^2 + m_2^2 + m_1m_2 + 3m_1 + 3m_2) + \frac{1}{8}(l^2 + 2l) = 1.$$

Table 1 $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$

Homogeneous space	$\mathfrak{h}$	$\mathfrak{g}_2$
Spin(7)/G$_2$	0	0
SO(5)/SO(3)	0	$\mathfrak{so}(5)$
$\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$2\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus V_{\mathbb{R}}^{(0,1)}$
$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	0	$2\mathfrak{su}(2) \oplus 6\mathfrak{su}(3)$
$\mathrm{N}_{k,l}$	0	Unknown
$\mathrm{SU}(2)^3/\mathrm{U}(1)^2$	0	Unknown

The only possible solutions are $V_{(0,0,2)}, V_{(1,1,0)}$. As $\mathfrak{su}(2) \otimes \mathfrak{u}(1)$ representations $V_{(0,0,2)} \cong S^2W$ and $V_{(1,1,0)} \cong \mathfrak{su}(3)_{\mathbb{C}}$. Further one can compute

$$V_{(0,0,2)} \cong \mathfrak{su}(2)_{\mathbb{C}} \cong S^2W,$$
$$V_{(1,1,0)} \cong \mathfrak{su}(3)_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3) \oplus \mathbb{C},$$
$$S^3WF(3) \otimes \mathfrak{m}^*_{\mathbb{C}} \cong (S^5W \oplus S^3W \oplus W)F(3) \oplus (S^4W \oplus S^2W)F(6) \oplus S^4W \oplus S^2W.$$

Thus $V_{(0,0,2)}$ and $S^3WF(3) \otimes \mathfrak{m}^*_{\mathbb{C}}$ has one common component S^2W with multiplicity 1 and $V_{(1,1,0)}$ and $S^3WF(3) \otimes \mathfrak{m}^*_{\mathbb{C}}$ has two common components $S^2W, WF(3)$ both with multiplicity 1 each. So by Frobenius reciprocity $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))$ contains a copy of $V_{(0,0,2)} \oplus 2V_{(1,1,0)}$.

The representation $S^3WF(-3)$ is the dual of the representation $S^3WF(3)$, and since $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ representations are isomorphic to their duals, the result for this case is same as the above and $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))$ also contains a copy of $V_{(0,0,2)} \oplus 2V_{(1,1,0)}$.

For the $\mathfrak{u}(1)$ representation $F(6)$, $\rho_6(\mathrm{Cas}_{\mathfrak{u}(1)}) = 1$. Thus again the only solutions are $V_{(0,0,2)}, V_{(1,1,0)}$ by the previous case. The $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ representation $F(6) \otimes \mathfrak{m}^*_{\mathbb{C}}$ has the following decomposition

$$F(6) \otimes \mathfrak{m}^*_{\mathbb{C}} \cong S^2WF(6) \oplus WF(9) \oplus WF(3),$$

thus $V_{(0,0,2)}$ is not contained in $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes F(6))$, but $V_{(1,1,0)}$ is with multiplicity 1. Since $F(-6) \cong F(6)^*$ this case is similar to the above case.
Summing up all the parts together we get that $\ker((D^{-1/3,c})^2 - \frac{49}{9}\} \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}} \cong 2(V_{(0,0,2)} \oplus 3V_{(1,1,0)})$ when the structure group is G$_2$.

Table 1 lists the $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$ when $E = \mathfrak{h}$ and $E = \mathfrak{g}_2$ for all the normal homogeneous spaces. Note that for the remaining two homogeneous spaces $N_{k,l}, k \neq l$ and $\mathrm{SU}(2)^3/\mathrm{U}(1)^2$ our methods does not apply when $E = \mathfrak{g}_2$ although since H is abelian for both of them, there are no deformations for the $E = \mathfrak{h}$ case. The space $V^{(0,1)}$ listed in Table 1 denotes the unique irreducible 5-dimensional complex representation of $\mathfrak{sp}(2)$.

4.4 Eigenspaces of the Dirac operator

All the G-representations listed in Table 1 lie in $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$ which by (4.5) is equal to $(\ker(D^{-1,c} + 2\mathrm{id}) \oplus \ker(D^{-1,c} - \frac{8}{3}\mathrm{id})) \cap \cap\Gamma(\mathfrak{m}^* \otimes E)$. Since the characteristic connection is translation invariant, it takes an irreducible G-representation to itself.

Hence the irreducible subspaces found in Table 1 lie in either $\ker(D^{-1,c} - \frac{8}{3}\mathrm{id})$ or $\ker(D^{-1,c} + 2\mathrm{id})$ where the subspaces in the latter space constitute the infinitesimal deformations of the characteristic connection by Theorem 3.2. Thus now it remains to identify which of the subspaces in Table 1 lies in $\ker(D^{-1,c} + 2\mathrm{id})$ for each of the homogeneous spaces. For all the normal homogeneous spaces G/H the metric corresponding to the nearly G_2 structure φ is given by $-\frac{3}{40}B$ where B is the Killing form of G. For 1-forms X, Y the Clifford product between X and $Y \cdot \eta$ is given by

$$X \cdot Y \cdot \eta = \langle X, Y\rangle \eta - \varphi(X, Y, .) \cdot \eta. \tag{4.9}$$

Thus we have all the ingredients in (4.7) to calculate the action of the Dirac operator $D^{-1,c}$ on each irreducible subspace in Table 1.

4.4.1 SO(5)/SO(3)

From the previous section we know that there is no deformation of the characteristic connection when the structure group is SO(3). For the structure group G_2 we calculated that the smooth sections of $G \times_{\rho_{\mathfrak{m}^*\otimes\mathfrak{g}_2}} (\mathfrak{m}^* \otimes \mathfrak{g}_2)$ in $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cong V_{(0,2)} \cong \mathfrak{so}(5)_{\mathbb{C}}$. If we denote by E_{ij} the skew-symmetric matrix with 1 at (i, j), -1 at (j, i) and 0 elsewhere and define

$$e_1 := \frac{2}{3}(E_{12} - 2E_{34}), \quad e_2 := \frac{2}{3}(\sqrt{2}E_{45} - \frac{\sqrt{3}}{\sqrt{2}}(E_{23} - E_{14})),$$
$$e_3 := \frac{2\sqrt{5}}{3}E_{25}, \quad e_4 := \frac{2}{3}(\sqrt{2}E_{35} - \frac{\sqrt{3}}{\sqrt{2}}(E_{13} + E_{24})),$$
$$e_5 := \frac{\sqrt{10}}{3}(E_{24} - E_{13}), \quad e_6 := -\frac{\sqrt{10}}{3}(E_{23} + E_{14}), \quad e_7 := \frac{2\sqrt{5}}{3}E_{15},$$

then $\{e_i, i = 1\dots7\}$ defines a basis of $\mathfrak{m}^*$ which is orthonormal with respect to the metric $-\frac{3}{40}B$. With respect to this basis the nearly G_2 structure φ is given by

$$\varphi = e_{124} + e_{137} + e_{156} + e_{235} + e_{267} + e_{346} + e_{457}.$$

We have seen that for SO(5)/SO(3) the characteristic connection has no deformation as an SO(3) connection. Now we need to check whether the SO(5)-representation $V_{(0,2)}$ lies in the $\ker(D^{-1,c} - \frac{8}{3}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ or $\ker(D^{-1,c} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$. As seen before the common irreducible $\mathfrak{so}(3)$ representation in $V_{(0,2)}|_{\mathfrak{so}(3)}$ and $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ is $S^6\mathbb{C}^2 \cong \mathfrak{m}^*_{\mathbb{C}}$. We denote the 1-dimensional space $\mathrm{Hom}(V_{(0,2)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}) = \mathrm{Span}(\alpha)$. Let $\mu_i, i = 1\dots11$ be a basis of the 11-dimensional subspace of $(\mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to the $\mathfrak{so}(3)$ representation $S^{10}\mathbb{C}^2$. Then the subspace of $\mathfrak{m}^*_{\mathbb{C}} \otimes S^{10}\mathbb{C}^2 \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $S^6\mathbb{C}^2$ is given by $\mathrm{Span}\{v_i, i = 1\dots7\}$ where

$$\begin{aligned} v_1 =& -\frac{e_2}{14} \otimes (5(\mu_1 - \mu_7) + 3\sqrt{15}\mu_9) + e_3 \otimes (\mu_5 + \mu_{11}) - \frac{e_4}{14} \otimes (5\mu_2 + 3\sqrt{15}(\mu_3 + \mu_4)) \\ &+ e_5 \otimes (\mu_3 - \mu_4) + e_6 \otimes \mu_9 + e_7 \otimes (\mu_6 - \mu_{10}), \\ v_2 =& e_1 \otimes \mu_9 + e_2 \otimes (-2\mu_5 + \mu_4) - \frac{e_3}{28} \otimes (47\mu_1 + 37\mu_7 + 3\sqrt{5}\mu_9) - e_4 \otimes (\mu_6 + 2\mu_{10}) \\ &- \frac{e_5}{14} \otimes \mu_8 + \frac{e_7}{28} \otimes (-37\mu_2 + 3\sqrt{15}(\mu_3 + \mu_4)), \\ v_3 =& -\frac{e_1}{2} \otimes (\mu_3 - \mu_4) + \frac{e_2}{2} \otimes (2\mu_6 + \mu_{10}) + \frac{e_3}{56} \otimes (47\mu_2 + 3\sqrt{5}(\mu_3 + \mu_4)) \end{aligned}$$

$$
\begin{aligned}
&-\frac{e_4}{2}\otimes(\mu_5-2\mu_{11})-\frac{e_6}{28}\otimes\mu_8+\frac{e_7}{56}(-37\mu_1+6\sqrt{15}\mu_9),\\
v_4=&-\frac{e_1}{28}\otimes(5\mu_2+3\sqrt{15}(\mu_3+\mu_4))+\frac{5e_2}{28}\otimes\mu_8-\frac{e_3}{56}\otimes(3\sqrt{15}\mu_2+41\mu_3+13\mu_4)\\
&-\frac{e_5}{2}\otimes(\mu_5-2\mu_{11})+\frac{e_6}{2}\otimes(\mu_6+2\mu_{10})+\frac{e_7}{56}\otimes(3\sqrt{15}(\mu_1-\mu_7)+41\mu_9),\\
v_5=&e_1\otimes(\mu_5+\mu_{11})-\frac{e_2}{28}\otimes(3\sqrt{15}(\mu_1-\mu_7)+13\mu_9)+\frac{e_4}{28}\otimes(3\sqrt{15}\mu_2+41\mu_3+13\mu_4)\\
&+\frac{e_5}{28}\otimes(47\mu_2+3\sqrt{15}(\mu_3+\mu_4))+\frac{e_6}{28}\otimes(47\mu_1+37\mu_7+3\sqrt{15}\mu_9)+\frac{2e_7}{28}\otimes\mu_8,\\
v_6=&e_1\otimes(-\mu_6+\mu_{10})+\frac{e_2}{28}\otimes(3\sqrt{15}\mu_2+13\mu_3+41\mu_4)+\frac{2e_3}{7}\otimes\mu_8\\
&+\frac{e_4}{28}\otimes(3\sqrt{15}(\mu_1-\mu_7)+41\mu_9)+\frac{e_5}{28}\otimes(37\mu_1+47\mu_7-3\sqrt{15}\mu_9)\\
&+\frac{e_6}{28}\otimes(-37\mu_2+3\sqrt{15}(\mu_3+\mu_4)),\\
v_7=&\frac{e_1}{14}\otimes(5(\mu_1-\mu_7)+3\sqrt{15}\mu_9)-\frac{e_3}{28}\otimes(3\sqrt{15}(\mu_1-\mu_7)+13\mu_9)\\
&+\frac{5e_4}{14}\otimes\mu_8-2e_5\otimes(\mu_6+\mu_{10})\\
&+e_6\otimes(-2\mu_5+\mu_{11})-\frac{e_7}{28}\otimes(3\sqrt{15}\mu_2+13\mu_3+41\mu_4).
\end{aligned}
$$

The subspace of $V_{(0,2)}$ isomorphic to $S^6\mathbb{C}^2$ is $\mathrm{Span}_{\mathbb{C}}\{e_i, i=1\ldots7\}$, and the SO(3) equivariant homomorphism α between $V_{(0,2)}$ and $(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$ is given by

$$
\begin{aligned}
&\alpha(e_1)=v_1,\quad \alpha(e_2)=v_7,\quad \alpha(e_3)=-v_5,\\
&\alpha(e_4)=-2v_4,\quad \alpha(e_5)=2v_3,\quad \alpha(e_6)=-v_2,\quad \alpha(e_7)=v_6.
\end{aligned}
$$

Any section of the bundle associated with $\mathfrak{m}^*\otimes\mathfrak{g}_2$ in $\ker((D^{-1/3,c})^2-\frac{49}{9}\mathrm{id})$ can be represented by (α, v) for some $v\in V_{(0,2)}|_{S^6\mathbb{C}^2}\cong\mathfrak{m}^*_{\mathbb{C}}$. The action of the characteristic connection on such a section is then given by $\nabla^{-1,c}_X(\alpha,v)(eH)=-\alpha([X,v])$ where the Lie bracket is in $\mathfrak{so}(5)$. We can now calculate the action of the Dirac operator, $D^{-1,c}$ on $(\alpha,e_1)\cdot\eta$ at the point eH as follows. We omit the $\cdot\eta$ from the computations to reduce notational clutter and will continue to do so in every case.

$$
\begin{aligned}
D^{-1,c}(\alpha,e_1)(eH)&=\sum_{i=1}^{7}e_i\cdot\nabla^{-1,c}_{e_i}(\alpha,e_1)(eH)\\
&=\frac{-2}{3}(e_2\cdot\alpha(e_4)+e_3\cdot\alpha(e_7)+e_4\cdot\alpha(-e_2)+e_5\cdot\alpha(e_6)\\
&\quad+e_6\cdot\alpha(-e_5)+e_7\cdot\alpha(-e_3))\\
&=\frac{2}{3}(2e_2\cdot v_4-e_3\cdot v_6+e_4\cdot v_7+e_5\cdot v_2+2e_6\cdot v_3-e_7\cdot v_5)\\
&=\frac{2}{3}(-3v_1)\cdot\eta=-2\alpha(e_1).
\end{aligned}
$$

Thus by the translation invariance of the characteristic connection $V_{(0,2)}\subseteq\ker(D^{-1,c}+2\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$.

4.4.2 $\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$

From the previous section we know that for $E = \mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$ the $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id})\cap \Gamma(\mathfrak{m}^*\otimes E)_{\mathbb{C}} \cong V_{(0,1,0)}$. Let $\{e_i, i = 1\dots 7\}$ be an orthonormal basis of $\mathfrak{m}^*$ with respect to the metric $-\frac{3}{40}B$ given by

$$e_1 := \frac{1}{3}\left(\begin{pmatrix}0 & 0\\ 0 & 2i\end{pmatrix}, -3i\right),\quad e_2 := \frac{1}{3}\left(\begin{pmatrix}0 & 0\\ 0 & 2j\end{pmatrix}, -3j\right),\quad e_3 := \frac{1}{3}\left(\begin{pmatrix}0 & 0\\ 0 & 2k\end{pmatrix}, -3k\right),$$
$$e_4 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, 0\right),\ e_5 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, 0\right),\ e_6 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix}0 & j\\ j & 0\end{pmatrix}, 0\right),$$
$$e_7 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix}0 & k\\ k & 0\end{pmatrix}, 0\right).$$

With respect to this basis the nearly G_2 form is given by

$$\varphi = e_{123} - e_{145} - e_{167} - e_{246} + e_{257} - e_{347} - e_{356},$$

From Table 1 we know that as an $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ connection the deformation space of the characteristic connection is an irreducible subrepresentation of $V_{(0,1,0)}$ and is thus trivial or $(V_{(0,1,0)})_{\mathbb{R}}$. We need to check whether this space lies in the -2 eigenspace of $D^{-1,A}$
The $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$-representation $V_{(0,1,0)}$ is 5 dimensional. We need to find the space $\mathrm{Hom}(V_{(0,1,0)}, (\mathfrak{m}^*\otimes(\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d))_{\mathbb{C}})_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$. The common irreducible $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ representations in $V_{(0,1,0)}$ and $(\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}$ is PQ. Let $S^2P = Span\{I, J, K\}$ then the subspace of $(\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}$ isomorphic to the space PQ is given by $\mathrm{Span}_{\mathbb{C}}\{v_1, v_2, v_3, v_4\}$ where

$$v_1 = e_5\otimes I + e_6\otimes J + e_7\otimes K,\quad v_2 = -e_4\otimes I + e_7\otimes J - e_6\otimes K,$$
$$v_3 = -e_7\otimes I - e_4\otimes J + e_5\otimes K,\quad v_4 = e_6\otimes I - e_5\otimes J - e_4\otimes K.$$

Let the subspace of $V_{(0,1,0)}$ isomorphic to PQ be given by $\mathrm{Span}\{w_1, w_2, w_3, w_4\}$ and the homomorphism space $\mathrm{Hom}(V_{(0,1,0)}, (\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}) = \mathrm{Span}(\beta)$ where β is defined by

$$w_1 \mapsto v_3 + iv_4,\quad w_2 \mapsto v_1 - iv_2,$$
$$w_3 \mapsto v_1 + iv_2,\quad w_4 \mapsto v_3 - iv_4.$$

Using this isomorphism one can compute that the only non-trivial $\mathfrak{gl}(V_{(0,1,0)}|_{PQ})$ elements with respect to the basis $\{w_1, w_2, w_3, w_4\}$ are

$$\tau_*(e_1) = \frac{2}{3}\begin{bmatrix} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & -i\end{bmatrix},\quad \tau_*(e_2) = \frac{2}{3}\begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix},\quad \tau_*(e_3) = \frac{2}{3}\begin{bmatrix} 0 & i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\end{bmatrix}.$$

Also by the definition of the characteristic connection, $\nabla^{-1,c}_X(\beta, w)(eH) = -\beta(\tau_*(X)w)$. Thus we can calculate

$$\begin{aligned}(D^{-1,c}(\beta, w_1))(eH) &= \sum_{i=1}^{7} e_i\cdot\nabla^{-1,c}_{e_i}(\beta, w_1)(eH) = -\sum_{i=1}^{7} e_i\cdot\beta((\tau_*(e_i)w_1)|_{PQ})\\ &= -(e_1\cdot\beta(\frac{2}{3}iw_1) + e_2\cdot\beta(\frac{2}{3}w_2) + e_3\cdot\beta(\frac{2}{3}iw_2))\end{aligned}$$

$$= -\frac{2}{3}(ie_1 \cdot (v_3 + iv_4) + e_2 \cdot (v_1 - iv_2) + ie_3 \cdot (v_1 - iv_2))$$
$$= -\frac{2}{3}(3(v_3 + iv_4)) = -2\beta(w_1).$$

Thus we have shown that $V_{(0,1,0)}$ lies in the $\ker(D^{-1,c} + 2\mathrm{id})$.

For $E = \mathfrak{g}_2$ the subspace of $\Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)$ in $\ker((D^{-1/3.c})^2 - \frac{49}{9}\mathrm{id})$ is isomorphic to the $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ representation $2V_{(2,0,0)} \oplus V_{(0,1,0)} \oplus V_{(0,0,2)}$. We have already dealt with the space $V_{(0,1,0)}$. The remaining spaces are $2V_{(2,0,0)} \cong 2\mathfrak{sp}(2)$ and $V_{(0,0,2)} \cong \mathfrak{sp}(1)$. The two copies of $V_{(2,0,0)}$ arise from $\mathrm{Hom}(V_{(2,0,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$, and the one copy of $V_{(0,0,2)}$ arises from $\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$. Thus we have two cases:

Case: 1-$\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \otimes V_{(0,0,2)}$

Let $\{w_1, w_2, w_3\}$ be the standard basis of $V_{(0,0,2)} \cong \mathfrak{sp}(1)_{\mathbb{C}}$ then the non-trivial actions of $\mathfrak{m}$ on $\mathfrak{sp}(1)_{\mathbb{C}}$ are given by

$$[e_1, .] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad [e_2, .] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \quad [e_3, .] = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\{\mu_i, i = 1 \ldots 8\}$ be a basis of the $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$ subrepresentation of $(\mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to PS^3Q. The 1-dimensional space $\mathrm{Hom}(V_{(0,0,2)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}) = \mathrm{Span}\{\phi\}$ where ϕ maps

$$w_1 \mapsto e_4 \otimes (\mu_5 - \mu_2) + e_5 \otimes (\mu_1 + \mu_6) + e_6 \otimes (\mu_4 - \mu_7) - e_7 \otimes (\mu_3 + \mu_8),$$
$$w_2 \mapsto e_4 \otimes (\mu_3 - 2\mu_8) - e_5 \otimes (\mu_4 + 2\mu_7) + e_6 \otimes (\mu_1 - 2\mu_6) - e_7 \otimes (\mu_2 + 2\mu_5),$$
$$w_3 \mapsto -e_4 \otimes (2\mu_4 + \mu_7) + e_5 \otimes (\mu_8 - 2\mu_3) - e_6 \otimes (2\mu_2 + \mu_5) + e_7 \otimes (\mu_6 - 2\mu_1).$$

The connection $\nabla^{-1,c}_X(\phi, w) = -\phi([X, w])$ for $w \in \mathfrak{sp}(1)$ where the Lie bracket is in the Lie algebra $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$. Thus we can calculate

$$D^{-1,c}(\phi, w_1)(eH) = \sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(\phi, w_1)(eH) = -\sum_{i=1}^{7} e_i \cdot \phi([e_i, w_1])$$
$$= -(e_2 \cdot \phi(-2w_3) + e_3 \cdot \phi(2w_2))$$
$$= -2(e_4 \otimes (\mu_5 - \mu_2) + e_5 \otimes (\mu_1 + \mu_6) + e_6 \otimes (\mu_4 - \mu_7) - e_7 \otimes (\mu_3 + \mu_8))$$
$$= -2\phi(w_1).$$

Hence again by translation invariance of $\nabla^{-1,c}$, $V_{(0,0,2)} \subseteq \ker(D^{-1,c} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$.

Case: 2-$\mathrm{Hom}(V_{(2,0,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \otimes V_{(2,0,0)}$

The $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-representation $V_{(2,0,0)} \cong \mathfrak{sp}(2)_{\mathbb{C}} \cong S^2P \oplus S^2Q \oplus PQ$. The subspace of $(\mathfrak{sp}(2))_{\mathbb{C}}$ isomorphic to S^2Q, PQ is given by $\mathrm{Span}_{\mathbb{C}}\{e_1, e_2, e_3\}$, $\mathrm{Span}_{\mathbb{C}}\{e_4, e_5, e_6, e_7\}$, respectively. As before the basis of $PS^3Q \subset (\mathfrak{g}_2)_{\mathbb{C}}$ is denoted by $\{\mu_1, \mu_2, \ldots, \mu_8\}$ and the subspace of $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to S^2Q is given by $\mathrm{Span}\{\phi(w_1), \phi(w_2), \phi(w_3)\}$ defined above. The subspace of $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to PQ is given by $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$ where

$$v_1 = e_1 \otimes (\mu_1 + \mu_6) - e_2 \otimes (\mu_4 + 2\mu_7) - e_3 \otimes (2\mu_3 - \mu_8),$$
$$v_2 = e_1 \otimes (\mu_2 - \mu_5) - e_2 \otimes (\mu_3 - 2\mu_8) + e_3 \otimes (2\mu_4 + \mu_7),$$
$$v_3 = -e_1 \otimes (\mu_3 + \mu_8) - e_2 \otimes (\mu_2 + 2\mu_5) - e_3 \otimes (2\mu_1 - \mu_6),$$
$$v_4 = -e_1 \otimes (\mu_4 - \mu_7) - e_2 \otimes (\mu_1 - 2\mu_6) + e_3 \otimes (2\mu_2 + \mu_5).$$

Let $\{A_1, A_2\}$ be a basis of the 2-dimensional space $\mathrm{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d}$, and let $A = c_1 A_1 + c_2 A_2$ for some real constants c_1, c_2, then we have that

$$A(e_1) = c_1 w_1, \quad A(e_2) = c_1 w_2, \quad A(e_3) = c_1 w_3$$
$$A(e_4) = -c_2 v_2, \quad A(e_5) = c_2 v_1, \quad A(e_6) = -c_2 v_4, \quad A(e_7) = c_2 v_3$$

and A_1, A_2 acts trivially on $S^2 P$.

Let $s_{(A,w)} \in \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ be the section corresponding to $(A, w) \in \mathrm{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{Sp}(1) \times \mathrm{Sp}(1)} \otimes \mathfrak{sp}(2)$ then $\nabla^{-1,c}_X(A, w) = -A(\mathrm{ad}(X)w) = A([X, w]|)$ where the Lie bracket is in the Lie algebra $\mathfrak{sp}(2)$. Using this action of $\nabla^{-1,c}$ we can calculate

$$\begin{aligned}(D^{-1,c}(A, e_1))(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(A, e_1)(eH) = -\sum_{i=1}^{7} e_i \cdot A([e_i, e_1]|) \\ &= -\frac{2}{3}(-e_2 \cdot A(e_3) + e_3 \cdot A(e_2) + e_4 \cdot A(e_5) - e_5 \cdot A(e_4) \\ &\quad + e_6 \cdot A(e_7) - e_7 \cdot A(6)) \\ &= -\frac{2}{3}(c_1(-e_2 \cdot w_3 + e_3 \cdot w_2) + c_2(e_4 \cdot v_1 - e_5 \cdot (-v_2) \\ &\quad + e_6 \cdot v_3 - e_7 \cdot (-v_4))) \\ &= \frac{4c_1 - 6c_2}{3} w_1 = \frac{4c_1 - 6c_2}{3} A_1(e_1).\end{aligned}$$

By doing similar computations we get that

$$\begin{aligned}(D^{-1,c}(A, f_i))(eH) &= 0, \quad i = 1, 2, 3, \\ (D^{-1,c}(A, e_i))(eH) &= \frac{4c_1 - 6c_2}{3} A_1(e_i), \quad i = 1, 2, 3, \\ (D^{-1,c}(A, e_i))(eH) &= -\frac{20c_1 + 6c_2}{9} A_2(e_i), \quad i = 4, 5, 6, 7.\end{aligned}$$

Therefore the subspace of $\mathrm{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}$ in the $\ker(D^{-1,c} + 2\mathrm{id})$ is given by the condition $c_2 = \frac{5}{3} c_1$ and is thus 1-dimensional. Therefore $V_{(2,0,0)}$ occurs in the $\ker(D^{-1,c} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ with multiplicity 1.

Remark 4.5 We can immediately see from above that the only other possible eigenvalue for which $\mathfrak{sp}(2)$ is an eigenspace of $D^{-1,c}$ is $-\frac{8}{3}$ for $c_2 = -\frac{2}{3} c_1$. This shows that not all spaces in $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id})$ are in $\ker(D^{-1,c} + 2\mathrm{id})$.

4.4.3 $\frac{\mathrm{SU}(3) \times \mathrm{SU}(2)}{\mathrm{SU}(2) \times \mathrm{U}(1)}$

As before let $\{e_i, i = 1 \dots 7\}$ be an orthonormal basis of $\mathfrak{m}^*$ with respect to g. If we define $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, we have

$$e_1 := \frac{1}{3}\left(\begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}, -3I\right), \quad e_2 := \frac{1}{3}\left(\begin{pmatrix} 2J & 0 \\ 0 & 0 \end{pmatrix}, -3J\right), \quad e_3 := \frac{1}{3}\left(\begin{pmatrix} 2K & 0 \\ 0 & 0 \end{pmatrix}, -3K\right),$$
$$e_4 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}, 0\right), \quad e_5 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & \sqrt{2}i \\ 0 & 0 & 0 \\ \sqrt{2}i & 0 & 0 \end{pmatrix}, 0\right),$$

$$e_6 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix}0&0&0\\0&0&\sqrt{2}\\0&-\sqrt{2}&0\end{pmatrix},0\right),\quad e_7 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix}0&0&0\\0&0&\sqrt{2}i\\0&\sqrt{2}i&0\end{pmatrix},0\right).$$

With respect to this basis the nearly G_2 structure φ is given by

$$\varphi = e_{123} + e_{145} - e_{167} + e_{246} + e_{257} + e_{347} - e_{356}.$$

As an $SU(2)\times U(1)$ representation, $\mathfrak{m}^*_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3)$ where

$$S^2W = \mathrm{Span}\{e_1, e_2, e_3\},\quad WF(3) = \mathrm{Span}\{e_4 - ie_5, e_6 - ie_7\},$$
$$WF(-3) = \mathrm{Span}\{e_4 + ie_5, e_6 + ie_7\}.$$

From our previous work we know that the characteristic connection has no deformations as an $SU(2)\times U(1)$ connection, so we only have to consider the case $E = \mathfrak{g}_2$.

As an $SU(2)\times U(1)$ representation, $(\mathfrak{g}_2)_{\mathbb{C}} \cong S^3W(F(3)\oplus F(-3))\oplus S^2W \oplus F(6)\oplus F(-6)$. We have already seen that S^2W gives rise to no deformations. From previous calculations we know that $\ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(\pm 3)) \cong V_{(0,0,2)} \oplus 2V_{(1,1,0)} \cong (\mathfrak{su}(2))_{\mathbb{C}} \oplus 2(\mathfrak{su}(3))_{\mathbb{C}}$ and $\Gamma(\mathfrak{m}^*_{\mathbb{C}} \otimes F(\pm 6)) \cap \ker((D^{-1/3,c})^2 - \frac{49}{9}\mathrm{id}) \cong V_{(1,1,0)}$, respectively. Therefore there are 6 subspaces of $\Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)$ to consider here.

Case: 1-$\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))_{SU(2)\times U(1)} \otimes V_{(0,0,2)}$

We denote by $\{\mu_i, i = 1\ldots 4\}$ a basis of $S^3WF(3)$. Let $f_i, i = 1\ldots 3$ be the standard basis of $\mathfrak{su}(2)$ such that $[f_1, f_2] = -2f_3$, $[f_1, f_3] = 2f_2$, $[f_2, f_3] = -2f_1$. Then the subspace of $WF(-3) \otimes S^3WF(3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $(\mathfrak{su}(2))_{\mathbb{C}}$ is given by $\mathrm{Span}\{v_1, v_2, v_3\}$ where

$$v_1 = \frac{3i}{4}(e_4 + ie_5)\otimes\mu_1 + (e_6 + ie_7)\otimes(\frac{5i}{4}\mu_2 + \mu_4),$$
$$v_2 = (e_4 + ie_5)\otimes(-i\mu_2 + \mu_4) + (e_6 + ie_7)\otimes(-i\mu_1 - \mu_3),$$
$$v_3 = (e_4 + ie_5)\otimes(-\frac{5i}{4}\mu_1 + \mu_3) - \frac{3i}{4}(e_6 + ie_7)\otimes\mu_2$$

and the space $\mathrm{Hom}(V_{(0,0,2)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}) = \mathrm{Span}\{\gamma^A\}$ where γ^A is defined by

$$\gamma^A(f_1) = v_2,\quad \gamma^A(f_2) = i(v_1 - v_3),\quad \gamma^A(f_3) = -2(v_1 + v_3).$$

For $i = 1, 2, 3$, since $e_i = (\frac{2}{3}f_i, -f_i)$ we have $[e_i, v] = -[f_i, v]$ for all $v \in \mathfrak{su}(2)$. The action is trivial for $i = 4\ldots 7$ since $[e_i, f_j] \notin \mathrm{Span}\{f_1, f_2, f_3\}$. We can thus calculate

$$\begin{aligned} D^{-1,c}(\gamma^A, f_1)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(\gamma^A, f_1)(eH)\\ &= e_2\cdot\gamma^A(2f_3) - e_3\cdot\gamma^A(2f_2)\\ &= -(4e_2\cdot(v_1 + v_3) + 2ie_3\cdot(v_1 - v_3))\\ &= -2v_2 = -2\,\gamma^A(f_1).\end{aligned}$$

Hence $\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))|_{Sp(1)\times Sp(1)} \otimes V_{(0,0,2)} \subseteq \ker(D^{-1,c} + 2\mathrm{id})$.

Case: 2-$\mathrm{Hom}(V_{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))_{SU(2)\times U(1)} \otimes V_{(1,1,0)}$

Let a basis of the subspace of $V_{(1,1,0)} \cong (\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $S^2W \cong (\mathfrak{su}(2))_{\mathbb{C}}$ be given by

$$p_1 := \begin{pmatrix}I&0\\0&0\end{pmatrix},\quad p_2 := \begin{pmatrix}J&0\\0&0\end{pmatrix},\quad p_3 := \begin{pmatrix}K&0\\0&0\end{pmatrix}.$$

where I, J, K are defined previously. Then $[p_1, p_2] = -2p_3, [p_1, p_3] = 2p_2, [p_2, p_3] = -2p_1$. The basis of $\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3) \subset \mathfrak{m}^*_{\mathbb{C}} \otimes \mathfrak{g}_2$ isomorphic to S^2W is given by Span$\{w_1, w_2, w_3\}$ where

$$w_1 = (e_4 + ie_5) \otimes \frac{\mu_2 + i\mu_3}{2} + (e_6 + ie_7) \otimes \frac{\mu_1 - i\mu_4}{2},$$
$$w_2 = (e_4 + ie_5) \otimes \frac{\mu_4 - 2i\mu_1}{2} + (e_6 + ie_7) \otimes \frac{\mu_3 - 2i\mu_2}{2},$$
$$w_3 = -(e_4 + ie_5) \otimes \frac{\mu_1 + 2i\mu_4}{2} + (e_6 + ie_7) \otimes \frac{\mu_2 - 2i\mu_3}{2}.$$

Since $(\mathfrak{su}(3))_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathbb{C}$, the subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(3)$ is given by $\text{Span}_{\mathbb{C}}\{e_4 - ie_5, e_6 - ie_7\}$. The subspace of $S^2W \otimes S^3WF(3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $WF(3)$ is given by Span$\{u_1, u_2\}$ where

$$u_1 = ie_1 \otimes \frac{\mu_2 + i\mu_3}{2} + e_2 \otimes \frac{2\mu_1 + i\mu_4}{2} - ie_3 \otimes \frac{\mu_1 + 2i\mu_4}{2},$$
$$u_2 = ie_1 \otimes \frac{\mu_1 - i\mu_4}{2} + e_2 \otimes \frac{2\mu_2 - i\mu_3}{2} + ie_3 \otimes \frac{\mu_2 - 2i\mu_3}{2}$$

If we denote the space $\text{Hom}(V^{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))$ and $\text{Hom}(V^{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))$ by Span$\{A_1\}$, Span$\{A_2\}$, respectively, then

$$A_1(p_i) = w_i, \quad i = 1, 2, 3,$$
$$A_2(e_4 - ie_5) = u_1, \quad A_2(e_6 - ie_7) = u_2.$$

Define $A = c_1A_1 + c_2A_2$ for some constants c_1, c_2. We need to find the conditions on c_1, c_2 such that $(A, w) \in \Gamma(\mathfrak{m}^* \otimes S^3WF(3)) \cap \ker(D^{-1,c} + 2\text{id})$ for all $w \in \mathfrak{su}(3)$.

Let $s_{(A,w)}$ be the section corresponding to (A, w). Then for any vector field X, $\nabla^{-1,c}_X(A, w) = -A(\text{ad}(X)w) = A([X, w]|)$ where the Lie bracket is in the Lie algebra $\mathfrak{su}(3)$. Using this action of $\nabla^{-1,c}$ we can calculate

$$\begin{aligned}
D^{-1,c}(A, p_1)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(A, p_1)(eH) \\
&= -(\frac{2}{3}(-e_2 \cdot A(2p_3) + e_3 \cdot A(2p_2))e_4 \cdot A(-e_5) + e_5 \cdot A(e_4) \\
&\quad + e_6 \cdot A(e_7) + e_7 \cdot A(e_6)) \\
&= -\frac{2c_1}{3}(-e_2 \cdot w_1 + e_3 \cdot w_2) - c_2(-e_4 \cdot i\frac{u_1}{2} + e_5 \cdot \frac{u_1}{2} + e_6 \cdot i\frac{u_2}{2} \\
&\quad - e_7 \cdot \frac{u_2}{2}) \\
&= \frac{4c_1 + 3ic_2}{3} w_1 = \frac{4c_1 + 3ic_2}{3} A_1(e_1).
\end{aligned}$$

The operator $D^{-1,c}$ acts trivially on the subspaces of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $\mathbb{C}$ and $WF(-3)$. On the remaining subspaces we can compute the action of the Dirac operator as

$$D^{-1,c}(A, p_1)(eH) = \frac{4c_1 + 3ic_2}{3} A_1(e_i), \quad i = 1, 2, 3,$$
$$D^{-1,c}(A, e_4 - ie_5)(eH) = \frac{20c_1 - 3ic_2}{9} A_2(e_4 - ie_5),$$

$$D^{-1,c}(A, e_6 - ie_7)(eH) = \frac{20c_1 - 3ic_2}{9} A_2(e_6 - ie_7).$$

Thus for any $w \in (\mathfrak{su}(3))_{\mathbb{C}}$, $(A, w) \in \ker(D^{-1,c} + 2\mathrm{id})$ if and only if $c_2 = \frac{10i}{3}c_1$. Thus only one copy of $\mathfrak{su}(3)$ lies in $\ker(D^{-1,c} + 2\mathrm{id})$.

Note that similarly to Remark 4.5 here also for $c_2 = -\frac{4i}{3}c_1$, $(A, w) \in \ker(D^{-1,c} - \frac{8}{3}\mathrm{id})$.

Case: 3-$\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \otimes V_{(0,0,2)}$

Let $f_i, i = 1\ldots3$ be as before and denote by $\{\nu_i, i = 1\ldots4\}$ a basis of $S^3WF(-3)$. Then the subspace of $WF(3) \otimes S^3WF(-3)$ isomorphic to S^2W is given by $\mathrm{Span}\{w_1, w_2, w_3\}$ where

$$\begin{aligned}
w_1 &= (e_4 - ie_5) \otimes (\frac{-3i}{4}\nu_1) + (e_6 - ie_7) \otimes (\frac{-5i}{4}\nu_2 + \nu_4),\\
w_2 &= (e_4 - ie_5) \otimes (i\nu_2 + \nu_4) + (e_6 - ie_7) \otimes (i\nu_1 - \nu_3),\\
w_3 &= (e_4 - ie_5) \otimes (\frac{5i}{4}\nu_1 + \nu_3) + (e_6 - ie_7) \otimes (\frac{3i}{4}\nu_2)
\end{aligned}$$

and the space $\mathrm{Hom}(V_{(0,0,2)}, (\mathfrak{m}^*_C \otimes S^3WF(-3))) = \mathrm{Span}\{\gamma^B\}$ where γ^B is defined by

$$\gamma^B(f_1) = \frac{i}{2}w_2, \quad \gamma^B(f_2) = \frac{1}{2}(w_1 - w_3), \quad \gamma^B(f_3) = -i(w_1 + w_3).$$

The action of $e_i, i = 1\ldots7$ on $f_j, j = 1\ldots3$ is the same as Case 1, and thus, we can calculate $D^{-1,c}(\gamma^B, f_1)$ as

$$\begin{aligned}
D^{-1,c}(\gamma^B, f_1)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(\gamma^B, f_1)(eH)\\
&= e_2 \cdot \gamma^B(2f_3) - e_3 \cdot \gamma^B(2f_2)\\
&= -2ie_2 \cdot (w_1 + w_3) - e_3 \cdot (w_1 - w_3)\\
&= -iw_2 = -2\,\gamma^B(f_1).
\end{aligned}$$

This implies $V_{(0,0,2)} \subseteq \ker(D^{-1,c} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$.

Case: 4-$\mathrm{Hom}(V_{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))_{\mathrm{SU}(2)\times\mathrm{U}(1)} \otimes V_{(1,1,0)}$

As above in Case 2, let a basis of the subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $S^2W \cong \mathfrak{su}(2)$ be given by $\mathrm{Span}\{p_1, p_2, p_3\}$. The basis of $\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to S^2W is given by $\mathrm{Span}\{w_1, w_2, w_3\}$ where

$$\begin{aligned}
w_1 &= (e_4 - ie_5) \otimes \frac{\nu_2 - i\nu_3}{2} + (e_6 - ie_7) \otimes \frac{\nu_1 + i\nu_4}{2},\\
w_2 &= (e_4 - ie_5) \otimes \frac{\nu_4 + 2i\nu_1}{2} + (e_6 - ie_7) \otimes \frac{\nu_3 + 2i\nu_2}{2},\\
w_3 &= -(e_4 - ie_5) \otimes \frac{\nu_1 - 2i\nu_4}{2} + (e_6 - ie_7) \otimes \frac{\nu_2 + 2i\nu_3}{2}.
\end{aligned}$$

The subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}\{e_4 + ie_5, e_6 + ie_7\}$. The subspace of $S^2W \otimes S^3WF(-3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}_{\mathbb{C}}\{u_1, u_2\}$ where

$$\begin{aligned}
u_1 &= -ie_1 \otimes \frac{\nu_2 - i\nu_3}{2} + e_2 \otimes \frac{2\nu_1 - i\nu_4}{2} + ie_3 \otimes \frac{\nu_1 - 2i\nu_4}{2},\\
u_2 &= -ie_1 \otimes \frac{\nu_1 + i\nu_4}{2} + e_2 \otimes \frac{2\nu_2 + i\nu_3}{2} - ie_3 \otimes \frac{\nu_2 + 2i\nu_3}{2}.
\end{aligned}$$

Again if we denote the spaces $\mathrm{Hom}(V(1,1,0), \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))$ and $\mathrm{Hom}(V^{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))$ by $\mathrm{Span}\{B_1\}$, $\mathrm{Span}\{B_2\}$, respectively, then

$$B_1(p_i) = w_i, \quad i = 1, 2, 3,$$
$$B_2(e_4 + ie_5) = u_1, \quad B_2(e_6 + ie_7) = u_2.$$

Again as before we need to find the conditions on c_1, c_2 such that $(B = c_1B_1 + c_2B_2, w) \in \ker(D^{-1,c} + 2\mathrm{id})$ for all $w \in (\mathfrak{su}(3))_{\mathbb{C}}$. By similar computations as Case 2, we can calculate,

$$\begin{aligned}
D^{-1,c}(B, p_1)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(B, p_1)(eH) \\
&= -(\frac{2}{3}(-e_2 \cdot B(2p_3) + e_3 \cdot B(2p_2)) + e_4 \cdot B(-e_5) + e_5 \cdot B(e_4) \\
&\quad + e_6 \cdot B(e_7) + e_7 \cdot B(e_6)) \\
&= -\frac{2c_1}{3}(-e_2 \cdot w_1 + e_3 \cdot w_2) - c_2(-e_4 \cdot i\frac{u_1}{2} + e_5 \cdot \frac{u_1}{2} \\
&\quad + e_6 \cdot i\frac{u_2}{2} - e_7 \cdot \frac{u_2}{2}) \\
&= \frac{4c_1 - 3ic_2}{3} w_1 = \frac{4c_1 - 3ic_2}{3} B_1(e_1).
\end{aligned}$$

Once can check that $D^{-1,c}$ acts trivially on the subspaces of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $\mathbb{C}$, $WF(3)$ and

$$\begin{aligned}
D^{-1,c}(A, p_1)(eH) &= \frac{4c_1 - 3ic_2}{3} B_1(e_i), \quad i = 1, 2, 3, \\
D^{-1,c}(A, e_4 + ie_5)(eH) &= \frac{20c_1 + 3ic_2}{9} B_2(e_4 + ie_5), \\
D^{-1,c}(A, e_6 + ie_7)(eH) &= \frac{20c_1 + 3ic_2}{9} B_2(e_6 + ie_7).
\end{aligned}$$

Thus for all $w \in (\mathfrak{su}(3))_{\mathbb{C}}$, $(B, w) \in \ker(D^{-1,c} + 2\mathrm{id})$ if and only if $c_2 = -\frac{10i}{3}c_1$ which proves that only one copy of $\mathfrak{su}(3)$ lies in $\ker(D^{-1,c} + 2\mathrm{id})$ in this case as well. It immediately follows from the given action that for $c_2 = \frac{4i}{3}c_1$, $(B, w) \in \ker(D^{-1,c} - \frac{8}{3}\mathrm{id})$.

<u>Case: 5</u>-$\mathrm{Hom}(V_{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes F(6))_{SU(2)\times U(1)} \otimes V_{(1,1,0)}$

From before we know that the subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(3)$ is given by $\mathrm{Span}\{e_4 - ie_5, e_6 - ie_7\}$. if we denote by μ a basis vector for the 1-dimensional representation $F(6)$, the subspace of $\mathfrak{m}^*_{\mathbb{C}} \otimes F(6)$ isomorphic to $WF(3)$ is given by $\mathrm{Span}_{\mathbb{C}}\{(e_4 + ie_5) \otimes \mu, (e_6 + ie_7) \otimes \mu\}$. Let $\mathrm{Hom}(V_{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes F(6)) = \mathrm{Span}\{\alpha\}$. We can define α as follows,

$$\alpha(e_4 - ie_5) = (e_6 + ie_7) \otimes \mu, \quad \alpha(e_6 - ie_7) = -(e_4 + ie_5) \otimes \mu.$$

Since $V_{(1,1,0)}$ is isomorphic to the adjoint representation $(\mathfrak{su}(3))_{\mathbb{C}}$, $\nabla^{-1,c}_X(\alpha, v)(eH) = -\alpha([X, v])$ where $X \in \mathfrak{m}$, $v \in WF(3) \subset \mathfrak{su}(3)$. Thus we can compute

$$\begin{aligned}
D^{-1,c}(\alpha, e_4 - ie_5)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,c}_{e_i}(\alpha, e_4 - ie_5)(eH) \\
&= -(e_1 \cdot \alpha(\frac{2i}{3}(e_4 - ie_5)) + e_2 \cdot \alpha(\frac{2}{3}(e_6 - ie_7))
\end{aligned}$$

Table 2 .

G/H	Structure group	
	H	G_2
$\mathrm{Spin}(7)/\mathrm{G}_2$	0	0
$\mathrm{SO}(5)/\mathrm{SO}(3)$	0	$\mathfrak{so}(5)$
$\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus V_{\mathbb{R}}^{(0,1)}$
$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	0	$2\mathfrak{su}(2)\oplus 4\mathfrak{su}(3)$

$$
\begin{aligned}
&+ e_3\cdot\alpha(\frac{2i}{3}(e_6-ie_7)))\\
&= -\frac{2}{3}(ie_1\cdot(e_6+ie_7)\otimes\mu - e_2\cdot(e_4+ie_5)\otimes\mu - ie_3\cdot(e_4+ie_5)\otimes\mu)\\
&= -2(e_6+ie_7)\otimes\mu = -2\alpha(e_4-ie_5).
\end{aligned}
$$

Therefore $\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(6))_{\mathrm{SU}(2)\times\mathrm{U}(1)}\otimes V_{(1,1,0)}\subset\ker(D^{-1,c}+2\mathrm{id})$ and thus lies in the deformation space.

Case: 6-$\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(-6))_{\mathrm{SU}(2)\times\mathrm{U}(1)}\otimes V_{(1,1,0)}$

The subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}_{\mathbb{C}}\{e_4+ie_5, e_6+ie_7\}$. We denote $F(-6)=\mathrm{Span}\{\nu\}$. Then $\mathfrak{m}^*_{\mathbb{C}}\otimes F(-6)$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}\{(e_4-ie_5)\otimes\nu, (e_6-ie_7)\otimes\nu\}$. Let $\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(-6))=\mathrm{Span}\{\beta\}$ then

$$
\beta(e_4+ie_5) = -(e_6-ie_7)\otimes\nu,\quad \beta(e_6+ie_7)=(e_4-ie_5)\otimes\nu.
$$

Since $V_{(1,1,0)}\cong(\mathfrak{su}(3))_{\mathbb{C}}$, $\nabla^{-1,c}_X(\beta,v)(eH)=-\beta([X,v])$ where $X\in\mathfrak{m}$, $v\in WF(-3)\subset(\mathfrak{su}(3))_{\mathbb{C}}$. Thus we can compute

$$
\begin{aligned}
D^{-1,c}(\beta,e_4+ie_5)(eH) &= \sum_{i=1}^{7} e_i\cdot\nabla^{-1,c}_{e_i}(\beta,e_4+ie_5)(eH)\\
&= -(e_1\cdot\beta(\frac{-2i}{3}(e_4+ie_5)) + e_2\cdot\beta(\frac{2}{3}(e_6+ie_7))\\
&\quad + e_3\cdot\beta(\frac{-2i}{3}(e_6+ie_7)))\\
&= -\frac{2}{3}(ie_1\cdot(e_6-ie_7)\otimes\nu + e_2\cdot(e_4-ie_5)\otimes\nu - ie_3\cdot(e_4-ie_5)\otimes\nu)\\
&= 2(e_6-ie_7)\otimes\nu = -2\beta(e_4+ie_5),
\end{aligned}
$$

which by translation invariance of $D^{-1,c}$ shows that $\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(-6))_{\mathrm{SU}(2)\times\mathrm{U}(1)}\otimes V_{(1,1,0)}\subset\ker(D^{-1,c}+2\mathrm{id})$.

As an H-connection the characteristic connection is rigid for three out of the four considered normal homogeneous spaces. As a G_2-connection the deformation space is always non-trivial infinitesimal except for the round S^7. Summing up all the results found above we get the following theorem.

Theorem 4.6 *The infinitesimal deformation space for the characteristic connection on the four normal homogeneous nearly* G_2 *spaces* G/H *when the structure group is* H *or* G_2 *is isomorphic to*

where $V^{(0,1)}$ *is the unique* 5*-dimensional complex irreducible* $\mathrm{Sp}(2)$*-representation.*

4.5 Integrability of the deformation spaces

We now describe the deformation spaces obtained in Theorem 4.6.

Let M be a nearly G_2 manifold. We first observe that for the structure group G_2 the space of non-trivial deformations in Theorem 4.6 is either isomorphic to or contains as a subrepresentation one or multiple copies of the Lie algebra $\mathfrak{g}$ of the automorphism group G. A vector field X on M preserves the G_2-structure φ if $\mathcal{L}_X\varphi = 0$. We denote by $\mathcal{X}$ the space of vector fields on M preserving the G_2-structure. Since the G_2-structure on G/H is G invariant, the space $\mathfrak{g}$ is contained in $\mathcal{X}$. Note that if $X \in \mathcal{X}$ then $\mathcal{L}_X\psi = \mathcal{L}_X g = 0$.

The next proposition asserts that if we fix a section $\xi \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \mathrm{Ad}_{\mathcal{P}})$, then for any vector field $X \in \mathcal{X}$ on M the $\mathrm{Ad}_{\mathcal{P}}$ valued 1-form $\epsilon_X = i_X\xi \in \Gamma(T^*M \otimes \mathrm{Ad}_{\mathcal{P}})$ defines an infinitesimal deformation of the nearly G_2 instanton A in the sense of (3.1). The proof of the proposition follows verbatim from the proof of [17, Proposition 9] and is hence omitted.

Proposition 4.7 *Let A be an instanton on a principal G-bundle $\mathcal{P}$ over a nearly G_2 manifold M. Let $\xi \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \mathrm{Ad}_{\mathcal{P}})$ such that $\nabla^{-1,A}\xi = 0$. Then for any $X \in \mathcal{X}$, $\epsilon_X = i_X\xi \in \Gamma(T^*M \otimes \mathrm{Ad}_{\mathcal{P}})$ satisfies the linearized instanton condition*

$$d^A\epsilon_X \cdot \eta = 0.$$

The above proposition implies that for each $\xi \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}})$ such that $\nabla^{-1,A}\xi = 0$, there is a copy of $\mathfrak{g}$ in the deformation space of A. Thus the multiplicity of $\mathfrak{g}$ in the deformation space is the number of parallel sections of $\mathfrak{g}_2(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}}$. On G/H, when we see $\mathcal{P}$ as a G_2-bundle, every parallel section of $\mathfrak{g}_2(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}}$ corresponds to an H-invariant element of the H-representation $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ (since $\mathrm{Ad}_{\mathcal{P}} \cong \mathfrak{g}_2$) and vice-versa. The number of linearly independent H-invariant elements of $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ is equal to the multiplicity of the trivial H-representation in $\mathfrak{g}_2 \otimes \mathfrak{g}_2$.

For the characteristic connection ∇^c on G/H, the curvature F satisfies $\nabla^{-1,c}F = 0$ since $\mathrm{Hol}(\nabla^c) \subseteq G_2$ and $F \in \Gamma(\mathfrak{g}_2(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}})$ since ∇^c is a G_2 instanton. Hence by Proposition 4.7 for every $X \in \mathcal{X}$, $\epsilon_X = i_X F$ defines an infinitesimal deformation of ∇^c. Using the Bianchi identity and the definition of ϵ_X we have that

$$\begin{aligned} d^A\epsilon_X &= d\epsilon_X + [A, \epsilon_X] \\ &= \mathcal{L}_X F - i_X dF + [A, \epsilon_X] \\ &= \mathcal{L}_X F + i_X[A, F] + [A, \epsilon_X] \\ &= \mathcal{L}_X F + [i_X A, F]. \end{aligned}$$

Since under the action of a gauge transformation ϕ, the curvature F transforms by $\phi F \phi^{-1}$, for all $X \in \mathcal{X}$ there exists an infinitesimal gauge transformation ϕ_X such that

$$\mathcal{L}_X F = [\phi_X, F].$$

Also $i_X A$ defines an infinitesimal gauge transformation; hence, $[\phi_X + i_X A, F]$ is an action of an infinitesimal gauge transformation on F. Thus for all $X \in \mathcal{X}$ the deformations $i_X F$ arise from gauge transformations and hence do not descend to the moduli space.

Thus the multiplicity of $\mathfrak{g}$ in the deformation space (modulo gauge transformations) of the characteristic connection on G/H is the number of trivial sub-representations of H in $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ apart from the one that corresponds to F. In all the cases we consider, the trivial H-representation occurs with multiplicity one in the subrepresentation $\mathfrak{g}_2 \otimes \mathfrak{h}$ of $\mathfrak{g}_2 \otimes \mathfrak{g}_2$. The trivial representation coming from $\mathfrak{g}_2 \otimes \mathfrak{h}$ corresponds to the H-invariant element F. We deal with the four normal homogeneous spaces one by one. The notation for the irreducible H-representations in all the cases is the same as used in Sect. 4.3.

- $\mathrm{Spin}(7)/\mathrm{G}_2$
  Since $H = \mathrm{G}_2$, in this case $\mathfrak{g}_2$ is the irreducible adjoint representation. There is only one trivial $\mathfrak{g}_2$-subrepresentation of $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ which corresponds to F. Hence $\mathfrak{g} = \mathfrak{spin}(7)$ does not occur in the deformation space as proved in Theorem 4.6.
- $\mathrm{SO}(5)/\mathrm{SO}(3)$
  In this case, as an $\mathfrak{so}(3)$ representation, $\mathfrak{g}_2$ decomposes into two irreducible $\mathfrak{so}(3)$-representations, the adjoint representation $S^2\mathbb{C}^2$, and the 11-dimensional representation $S^{10}\mathbb{C}^2$. Thus as $\mathfrak{so}(3)$-representation

$$\mathfrak{g}_2 \otimes \mathfrak{g}_2 = (S^2\mathbb{C}^2 \otimes S^2\mathbb{C}^2) \oplus 2(S^2\mathbb{C}^2 \otimes S^{10}\mathbb{C}^2) \oplus (S^{10}\mathbb{C}^2 \otimes S^{10}\mathbb{C}^2).$$

  There are two trivial components occurring in the above decomposition from $S^2\mathbb{C}^2 \otimes S^2\mathbb{C}^2$ and $S^{10}\mathbb{C}^2 \otimes S^{10}\mathbb{C}^2$, respectively, but since the component coming from $S^2\mathbb{C}^2 \otimes S^2\mathbb{C}^2$ corresponds to F, up to gauge transformations the deformation space of the characteristic connection on $\mathrm{SO}(5)/\mathrm{SO}(3)$ contains only one copy of $\mathfrak{g} = \mathfrak{so}(5)$ as shown in Theorem 4.6.
- $\mathrm{Sp}(2) \times \mathrm{Sp}(1)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$
  As an $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$-representation,

$$\mathfrak{g}_2 = S^2P \oplus S^2Q \oplus PS^3Q.$$

  The trivial $\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$ components of $\mathfrak{g}_2\otimes\mathfrak{g}_2$ coming from $S^2P\otimes S^2P$ and $S^2Q\otimes S^2Q$ correspond to F and thus can be ignored. The only trivial component that corresponds to an infinitesimal deformation modulo gauge transformations comes from $PS^3Q\otimes PS^3Q$; hence, again $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ appears with multiplicity 1 as in Theorem 4.6.
- $\mathrm{SU}(3) \times \mathrm{SU}(2)/\mathrm{SU}(2) \times \mathrm{U}(1)$
  The decomposition of $\mathfrak{g}_2$ as an $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$-representation is given by

$$\mathfrak{g}_2 = S^2W \oplus \mathbb{C} \oplus S^3WF(3) \oplus S^3WF(-3) \oplus F(6) \oplus F(-6).$$

  The first two components in the above decomposition correspond to $\mathfrak{h}$; hence, the only trivial $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$-subrepresentations of $\mathfrak{g}_2 \otimes \mathfrak{g}_2$ that correspond to non-trivial deformations come from the spaces $S^2WF(3) \otimes S^2WF(-3)$ and $F(6) \otimes F(-6)$. Hence as proved in Theorem 4.6 the space $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ occurs in the deformation space with multiplicity 2.
  The only deformation spaces left to be considered in Table 2 are the $\mathrm{Sp}(2)$-representation $V^{\mathbb{R}}_{(0,1)}$ for the squashed 7-sphere and 2 copies of the $\mathrm{SU}(3)$-representation $\mathfrak{su}(3)$ on the Aloff–Wallach space $\mathrm{SU}(3) \times \mathrm{SU}(2)/\mathrm{SU}(2) \times \mathrm{U}(1)$.

On the squashed 7-sphere the characteristic connection splits into two connections with $\mathrm{Hol} = \mathrm{Sp}(1)_{\mathrm{u}}$ and $\mathrm{Sp}(1)_{\mathrm{d}}$, respectively. From Sect. 4.3 the deformations only come from the $\mathrm{Sp}(1)_{\mathrm{u}}$ part which is the pullback of the standard instanton on S^4. If we view S^4 as the symmetric homogeneous space $\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)_a\times\mathrm{Sp}(1)_b\times\mathrm{Sp}(1)_c}$ and denote by $P, Q, R \cong \mathbb{C}^2$ the irreducible representation of the three $\mathrm{Sp}(1)$ factors, respectively, we have the orthogonal decomposition

$$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) = \mathfrak{sp}(1)_a \oplus \mathfrak{sp}(1)_b \oplus \mathfrak{sp}(1)_c \oplus \mathfrak{n}.$$

As an $\mathfrak{sp}(1)_a \oplus \mathfrak{sp}(1)_b \oplus \mathfrak{sp}(1)_c$-representation

$$\mathfrak{n} \cong PQ.$$

The squashed sphere becomes a bundle over S^4 by reducing to the subgroup $\mathrm{Sp}(1)^2$ corresponding to the identification $Q = R$, so the factor $\mathrm{Sp}(1)_d$ acts diagonally. The complexified tangent space of $\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)_u\times\mathrm{Sp}(1)_d}$ is then

$$m \cong S^2Q + PQ.$$

The standard instanton on S^4 is the unique Sp(2)-invariant ASD connection on S^4 with charge 1. As a bundle over S^4, the Levi-Civita connection induces the standard instanton on P. It is also the homogeneous connection on the $\mathrm{Spin}(4) = \mathrm{Sp}(1)^2$ bundle over S^4 obtained by left-translating the subspace $\mathfrak{n}$ in $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) = 3\mathfrak{sp}(1) \oplus \mathfrak{n}$ by $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$. Thus the horizontal distribution corresponding to the standard instanton is $\mathfrak{n}$.

On the other hand the characteristic connection on the squashed 7-sphere is the characteristic homogeneous connection defined by the horizontal distribution $\mathfrak{m}$ in the decomposition $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) = 2\mathfrak{sp}(1) \oplus \mathfrak{m} = 2\mathfrak{sp}(1) \oplus (S^2Q \oplus \mathfrak{n})$. The characteristic connection on squashed 7-sphere reduces to $\mathrm{Sp}(1)^2$ and preserves the horizontal distribution D defined by $\mathfrak{n}$ which is stable under both $\mathrm{Ad}(\mathrm{Sp}(1)^3)$ and $\mathrm{Ad}(\mathrm{Sp}(1)^2)$.

If we consider the map

$$p : S^7 = \frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d} \to S^4 = \frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1)_a \times \mathrm{Sp}(1)_b \times \mathrm{Sp}(1)_c}$$

then the connection induced on D is the pullback of the homogeneous connection defined by n on $T\left(S^4\right)$ via p.

Let $\mathcal{M}$ be the moduli space of charge-1 instantons on S^4 with structure group SU(2). Then, there is a diffeomorphism from $\mathcal{M}$ to $B^5 \subset \mathbb{R}^5$ which to an instanton associates its center. The standard instanton on S^4 is the charge-1 instanton that corresponds to the center of the ball, that is to $0 \in B^5$, and is the unique homogeneous charge-1 instanton. As the name suggests, the homogeneous charge-1 instanton is invariant with respect to the Sp(2)-action. The pullback of the homogeneous charge-1 instanton to the squashed S^7 is a G_2-instanton (see [14, 19]). As shown in [8] the moduli space of the standard instanton on S^4 can be identified as a topological space and as a differentiable manifold with $\mathbb{R}^+ \times \mathbb{H}$ (see [21, sec 4.1]). As shown above the Sp(1) part of the characteristic connection on the squashed 7-sphere is the pullback of the standard instanton; hence, the deformation space of the characteristic connection on the squashed 7-sphere must contain the deformation space of the standard ASD instanton on S^4 and thus be at least 5-dimensional. From Table 2, we know that the moduli space of the deformations of the characteristic connection on the squashed 7-sphere is exactly 5-dimensional and hence we get the following Corollary.

Theorem 4.8 *The deformations of the characteristic connection on the squashed 7-sphere are lifts of the deformations of the standard ASD connection on S^4 and are thus integrable.*

As of the deformation subspace isomorphic to $2\mathfrak{su}(3)$ of the characteristic connection on $\mathrm{SU}(3) \times \mathrm{SU}(2)/\mathrm{SU}(2) \times \mathrm{U}(1)$ with structure group G_2, the author is unaware of any such explicit description. It would be interesting to see whether these deformations are genuine.

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