A horizontal Chern–Gauss–Bonnet formula on totally geodesic foliations

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Abstract

Under suitable conditions, we show that the Euler characteristic of a foliated Riemannian manifold can be computed only from curvature invariants which are transverse to the leaves. Our proof uses the hypoelliptic sub-Laplacian on forms recently introduced by two of the authors in Baudoin and Grong (Ann Glob Anal Geom 56(2):403–428, 2019).

1 Introduction

The goal of the paper is to prove the following result:

Theorem 1.1 *Let $\mathbb{M}$ be a smooth, connected, oriented and $n + m$ dimensional compact manifold. We assume that $\mathbb{M}$ is equipped with a Riemannian foliation $\mathcal{F}$ with bundle-like metric g and totally geodesic m -dimensional leaves. We also assume that the horizontal distribution $\mathcal{H} = \mathcal{F}^\perp$ is bracket-generating and that there exists $\varepsilon > 0$ such that*

$$(\nabla_v J)_w = -\frac{1}{2\varepsilon} [J_v, J_w] \quad (1.1)$$

for any $v, w \in T_x \mathbb{M}$, $x \in \mathbb{M}$, where ∇ is the Bott connection of the foliation and J is the tensor defined in (2.2). Denoting $\chi(\mathbb{M})$ the Euler characteristic of $\mathbb{M}$:

- *If n or m is odd, then $\chi(\mathbb{M}) = 0$;*

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- If n and m are both even, then

$$\chi(\mathbb{M}) = \int_{\mathbb{M}} \hat{\omega}_{\mathcal{H}}^{\varepsilon} \wedge \left[\det \left(\frac{\mathcal{T}}{\sinh(\mathcal{T})} \right)^{1/2} \right]_m.$$

Notations are further explained in Sect. 4, but we point out that a remarkable feature of that result is that the density $\hat{\omega}_{\mathcal{H}}^{\varepsilon} \wedge \left[\det \left(\frac{\mathcal{T}}{\sinh(\mathcal{T})} \right)^{1/2} \right]_m$ essentially only depends on *horizontal curvature quantities*. Therefore, the theorem illustrates further the fact already observed in [4] that topological properties of $\mathbb{M}$ might be obtained from horizontal curvature invariants only provided that the bracket-generating condition of the horizontal distribution is satisfied; thus, in essence, the theorem is a sub-Riemannian result. We also note that the condition (1.1) is satisfied in a large class of examples including the H-type foliations introduced in [5], see Example 2.4.

The proof of Theorem 1.1 is based on the study of the heat semigroup generated by the hypoelliptic sub-Laplacian on forms recently introduced in [4]. The heat equation approach to Chern–Gauss–Bonnet type formulas (or index formulas) that we are using is of course not new: It was suggested by Atiyah–Bott [1] and McKean–Singer [16] and first carried out by Patodi [18] and Gilkey [12] and is by now classical, see the book [9]. However, a difficulty in our setting is that the sub-Laplacian on forms we consider is only hypoelliptic but not elliptic. To carry out the required small-time asymptotics analysis to obtain the horizontal Chern–Gauss–Bonnet formula, we will make use of the probabilistic Brownian Chen series parametrix method first introduced in [3] and which is easy to adapt to hypoelliptic situations, see [2].

The paper is organized as follows. In Sect. 2, we introduce the horizontal Laplacian on forms $\Delta_{\mathcal{H},\varepsilon}$ and prove that it is a self-adjoint operator if and only if the condition (1.1) is satisfied. In Sect. 3, we prove a McKean–Singer type formula for $\Delta_{\mathcal{H},\varepsilon}$, namely that for every $t > 0$,

$$\mathbf{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}}) = \chi(\mathbb{M}).$$

Finally, in Sect. 4 we study the small-time asymptotics of $\mathbf{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}})$ and conclude the proof of Theorem 1.1.

2 Preliminaries

In this section, we first recall the framework and notations of Baudoin and Grong [4] and the references therein to which we refer for further details. We then prove a necessary and sufficient condition for the form horizontal Laplacian of a totally geodesic foliation to be a symmetric operator.

2.1 Totally geodesic foliations

Let $(\mathbb{M}, g)$ be a smooth, oriented, connected, compact Riemannian manifold with dimension $n + m$. We assume that $\mathbb{M}$ is equipped with a foliation $\mathcal{F}$ with m -dimensional leaves. The distribution $\mathcal{V}$ formed by vectors tangent to the leaves is referred to as the set of *vertical directions* (or *vertical subbundle*). Define the *horizontal subbundle* $\mathcal{H} = \mathcal{V}^{\perp}$ as its orthogonal complement. We will always assume in this paper that the horizontal distribution $\mathcal{H}$ is

everywhere bracket-generating. The foliation is called *Riemannian* and *totally geodesic* if for any $X \in \Gamma(\mathcal{H})$, $Z \in \Gamma(\mathcal{V})$, the respective conditions are satisfied,

$$(\mathcal{L}_Z g)(X, X) = 0, \quad (\mathcal{L}_X g)(Z, Z) = 0.$$

Equivalently, we can describe these conditions using *the Bott connection*. Write $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{V}}$ for the respective orthogonal projections to $\mathcal{H}$ and $\mathcal{V}$. Let ∇^g be the Levi–Civita connection of g . Introduce a new connection ∇ on $T\mathbb{M}$ according to the rules,

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla_X^g Y) & \text{for any } X, Y \in \Gamma(\mathcal{H}), \\ \pi_{\mathcal{H}}([X, Y]) & \text{for any } X \in \Gamma(\mathcal{V}), Y \in \Gamma(\mathcal{H}), \\ \pi_{\mathcal{V}}([X, Y]) & \text{for any } X \in \Gamma(\mathcal{H}), Y \in \Gamma(\mathcal{V}), \\ \pi_{\mathcal{V}}(\nabla_X^g Y) & \text{for any } X, Y \in \Gamma(\mathcal{V}). \end{cases} \quad (2.1)$$

We observe that ∇ preserves $\mathcal{H}$ and $\mathcal{V}$ under parallel transport. The foliation $\mathcal{F}$ is then both Riemannian and totally geodesic if and only if $\nabla g = 0$. For the rest of the paper, we will assume that ∇ is indeed compatible with the metric g . The torsion T of ∇ is given by

$$T(X, Y) = -\pi_{\mathcal{V}}[\pi_{\mathcal{H}}X, \pi_{\mathcal{H}}Y].$$

Define a corresponding endomorphism valued one-form $Z \mapsto J_Z$ by

$$\langle J_Z X, Y \rangle_g = \langle Z, T(X, Y) \rangle_g, \quad X, Y, Z \in \Gamma(T\mathbb{M}). \quad (2.2)$$

Let $g_{\mathcal{H}}$ and $g_{\mathcal{V}}$ be the respective restrictions of g to $\mathcal{H}$ and $\mathcal{V}$. We then define *the canonical variation* g by $g_{\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$, $\varepsilon > 0$, and make the following observations:

- (i) If $(\mathbb{M}, \mathcal{F}, g)$ is a Riemannian, totally geodesic foliation, then so is $(\mathbb{M}, \mathcal{F}, g_{\varepsilon})$.
- (ii) Although the Levi-Civita connection $\nabla^{g_{\varepsilon}}$ of g_{ε} is different from the connection ∇^g of g , replacing ∇^g with $\nabla^{g_{\varepsilon}}$ in formula (2.1) will lead to exactly the same connection. In other words, when defining the Bott connection ∇ , we obtain the same connection for any metric g_{ε} in the family of canonical variations.
- (iii) For any fixed $\varepsilon > 0$, define a connection

$$\hat{\nabla}_X^{\varepsilon} Y = \nabla_X Y + \frac{1}{\varepsilon} J_X Y. \quad (2.3)$$

This connection preserves $\mathcal{H}$ and $\mathcal{V}$ under parallel transport and is compatible with $g_{\varepsilon'}$ for any $\varepsilon' > 0$. Furthermore, its torsion

$$\hat{T}^{\varepsilon}(X, Y) = T(X, Y) + \frac{1}{\varepsilon} J_X Y - \frac{1}{\varepsilon} J_Y X,$$

is skew-symmetric with respect to g_{ε} . Hence, if we consider its adjoint connection

$$\nabla_X^{\varepsilon} Y = \hat{\nabla}_X^{\varepsilon} Y - \hat{T}^{\varepsilon}(X, Y) = \nabla_X Y - T(X, Y) + \frac{1}{\varepsilon} J_Y X, \quad (2.4)$$

it will also be compatible with g_{ε} . However, $\mathcal{H}$ and $\mathcal{V}$ are not parallel with respect to ∇^{ε} .

2.2 Horizontal Laplacian on forms

For the totally geodesic Riemannian foliation $(\mathbb{M}, \mathcal{F}, g)$, define its horizontal Laplacian on functions $f \in C^{\infty}(\mathbb{M})$ by

$$\Delta_{\mathcal{H}} f = \text{tr}_{\mathcal{H}} \nabla_{\times} df(\times). \quad (2.5)$$

We note that since $\mathcal{H}$ is assumed to be bracket-generating, from Hörmander's theorem, $\Delta_{\mathcal{H}}$ is a subelliptic operator. We also note that since $g_{\mathcal{H}}$ and the Bott connection are independent of $\varepsilon > 0$, the horizontal Laplacian is as well; that is, the choice of any metric g_{ε} in the canonical variation family will not change $g_{\mathcal{H}}$, the Bott connection, or the horizontal Laplacian.

Consider now the totally geodesic Riemannian foliation $(\mathbb{M}, \mathcal{F}, g_{\varepsilon})$ for some fixed $\varepsilon > 0$. We want to extend the horizontal Laplacian on functions (2.5) to a differential operator on forms $\Delta_{\mathcal{H},\varepsilon}$ satisfying the following requirements:

- (I) $\Delta_{\mathcal{H},\varepsilon} f = \Delta_{\mathcal{H}} f$ for any smooth function f ;
- (II) The operator $\Delta_{\mathcal{H},\varepsilon}$ is of Weitzenböck type, i.e., $\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} + \mathcal{R}_{\varepsilon}$ where $\mathcal{R}_{\varepsilon}$ is a zero-order differential operator and

$$L_{\mathcal{H},\varepsilon} = \text{tr}_{\mathcal{H}} \tilde{\nabla}_{\times,\times}^2, \quad (2.6)$$

is the connection horizontal Laplacian of some connection $\tilde{\nabla}$ compatible with g_{ε} ;

- (III) If d is the exterior differential, then

$$[\Delta_{\mathcal{H},\varepsilon}, d] = 0.$$

Given these requirements, there is an essentially unique extension of $\Delta_{\mathcal{H}}$ to forms, see [4, 15] for details. We call $\Delta_{\mathcal{H},\varepsilon}$ the ε -horizontal Laplacian on forms. This operator can be described as follows.

Proposition 2.1 (Horizontal Laplacian on forms, see [4]) *Consider the ε -horizontal divergence operator defined by*

$$\delta_{\mathcal{H},\varepsilon} \eta = -\text{tr}_{\mathcal{H}}(\nabla_{\times}^{\varepsilon} \eta)(\times, \cdot).$$

The operator

$$\Delta_{\mathcal{H},\varepsilon} = -\delta_{\mathcal{H},\varepsilon} d - d \delta_{\mathcal{H},\varepsilon}$$

is called the ε -horizontal Laplacian on forms, and it satisfies the requirements (I), (II), (III). In particular, this operator has Weitzenböck decomposition $\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} + \mathcal{R}_{\varepsilon}$ where $L_{\mathcal{H},\varepsilon}$ is defined as in (2.6) relative to ∇^{ε} .

We can describe the zero order operator $\mathcal{R}_{\varepsilon}$ can be made explicit, see [4]. For later use, we will prefer to write the operators using Fermion calculus, see Appendix A.1. Let $X_1, \dots, X_n$ and $Z_1, \dots, Z_m$ be local orthonormal bases of, respectively, $\mathcal{H}$ and $\mathcal{V}$. Define $a_i = \iota_{X_i}$ and $b_r = \iota_{Z_r}$ for the corresponding annihilation operators, with the dual operators $a_i^* = X_i^* \wedge$ and $b_r^* = Z_r^* \wedge$ acting by wedge products. The dual are here relative to the L^2 inner product with respect to the fixed metric g . Relative to the curvature tensor $\hat{R}^{\varepsilon}$ of $\hat{\nabla}^{\varepsilon}$, write

$$\hat{R}_{ijk}^{\varepsilon,l} = \langle \hat{R}^{\varepsilon}(X_i, X_j)X_k, X_l \rangle_g, \quad (2.7)$$

and use similar notation for other tensors with indices i, j, k, l denoting evaluations with respect to the basis of $\mathcal{H}$, indices r, s with respect to the basis of $\mathcal{V}$. We emphasize that these indices are always defined relative to the fixed metric g . Then, $\mathcal{R}_{\varepsilon}$ is given by

$$\begin{aligned} \mathcal{R}_{\varepsilon} = & \sum_{i,j,k=1}^n \hat{R}_{ijk}^{\varepsilon,i} a_k^* a_j + \sum_{i,k=1}^n \sum_{r=1}^m \hat{R}_{irk}^{\varepsilon,i} a_k^* b_r + \frac{1}{2} \sum_{i,j,k,l=1}^n \hat{R}_{ijk}^{\varepsilon,l} a_k^* a_l^* a_j a_i \\ & + \sum_{i,j,k=1}^n \sum_{r=1}^m \hat{R}_{irk}^{\varepsilon,l} a_k^* a_l^* b_r a_i + \frac{1}{2} \sum_{i,j=1}^n \sum_{r,s=1}^m \hat{R}_{rsi}^{\varepsilon,j} a_i^* a_j^* b_r b_s. \end{aligned} \quad (2.8)$$

We want to give a formula for this operator that shows the dependence of ε explicitly. Let T and R be the curvature of the Bott connection ∇ and use indices after semi-colons to denote covariant derivatives with respect to this connection. Using Lemma A.2, Appendix, we can write

$$\begin{aligned} \mathcal{R}_\varepsilon = & \sum_{i,j,k=1}^n \left(R_{kji}^k + \frac{1}{\varepsilon} \sum_{r=1}^m T_{ik}^r T_{jk}^r \right) a_i^* a_j - \sum_{i,j=1}^n \sum_{r=1}^m T_{ij;i}^r a_j^* b_r \\ & + \frac{1}{2} \sum_{i,j,k,l=1}^n \left(R_{kli}^j + \frac{1}{\varepsilon} \sum_{r=1}^m T_{kl}^r T_{ij}^r \right) a_i^* a_j^* a_l a_k + \sum_{i,j,k=1}^n \sum_{r=1}^m \frac{1}{\varepsilon} T_{ij;k}^r a_i^* a_j^* b_r a_k \\ & + \frac{1}{2} \sum_{i,j=1}^n \sum_{r,s=1}^m \left(\frac{2}{\varepsilon} T_{ij;r}^s + \frac{1}{\varepsilon^2} \sum_{k=1}^n (T_{kj}^r T_{ik}^s - T_{kj}^s T_{ik}^r) \right) a_i^* a_j^* b_s b_r. \end{aligned} \quad (2.9)$$

2.3 Symmetry of the horizontal Laplacian

Consider the exterior algebra

$$\Omega = \Omega(\mathbb{M}) = \bigoplus_{k=0}^{\dim \mathbb{M}} \Omega^k,$$

with the L^2 -inner product from g_ε . When restricted to elements in $\Omega^0 \oplus \Omega^1$, the operator $\Delta_{\mathcal{H},\varepsilon}$ is symmetric if and only if $\mathcal{H}$ satisfies the Yang–Mills condition, i.e., if

$$\sum_{i=1}^n T_{ij;i}^r = 0, \quad \text{for any } j = 1, \dots, n, r = 1, \dots, m.$$

see [6]. Considering all forms, we have the following result.

Proposition 2.2 *The operator $\Delta_{\mathcal{H},\varepsilon}$ is symmetric with respect to the L^2 -inner product of g_ε if and only if*

$$(\nabla_v J)_w = -\frac{1}{2\varepsilon} [J_v, J_w], \quad (2.10)$$

for any $v, w \in T_x M$, $x \in M$. In particular, $\nabla_v J = 0$ for any $v \in \mathcal{H}$.

We note that under the above condition, the expression of $\mathcal{R}_\varepsilon$ reduces to

$$\mathcal{R}_\varepsilon = \sum_{i,j,k=1}^n \left(R_{kji}^k + \frac{1}{\varepsilon} \sum_{r=1}^m T_{ik}^r T_{jk}^r \right) a_i^* a_j + \frac{1}{2} \sum_{i,j,k,l=1}^n \left(R_{kli}^j + \frac{1}{\varepsilon} \sum_{r=1}^m T_{kl}^r T_{ij}^r \right) a_i^* a_j^* a_l a_k. \quad (2.11)$$

Proof $L_{\mathcal{H},\varepsilon}$ is symmetric by Grong and Thalmaier [15, Lemma A.1], so we only need to determine when $\mathcal{R}_\varepsilon$ is symmetric. We choose a local bases $X_1, \dots, X_n$ and $Z_1, \dots, Z_m$ of, respectively, $\mathcal{H}$ and $\mathcal{V}$. We consider the representation of $\mathcal{R}_\varepsilon$ as in (2.9). Then, for $\mathcal{R}_\varepsilon$ to be

symmetric, we must have

$$\begin{aligned} 0 &= \langle \mathcal{R}_\varepsilon X_k^* \wedge Z_r^*, X_i^* \wedge X^j \rangle_\varepsilon - \langle \mathcal{R}_\varepsilon X_i^* \wedge X_j, X_k^* \wedge Z_r^* \rangle_\varepsilon = \frac{1}{\varepsilon} T_{ij;k}^r, \\ 0 &= \langle \mathcal{R}_\varepsilon Z_r^* \wedge Z_s^*, X_i^* \wedge X^j \rangle_\varepsilon - \langle \mathcal{R}_\varepsilon X_i^* \wedge X_j, Z_r^* \wedge Z_s^* \rangle_\varepsilon \\ &= \frac{2}{\varepsilon} T_{ij;r}^s + \frac{1}{\varepsilon^2} \sum_{k=1}^n (T_{kj}^r T_{ik}^s - T_{kj}^s T_{ik}^r). \end{aligned}$$

These equations are clearly equivalent to (2.10). If these hold, then $\mathcal{R}_\varepsilon$ reduces to the expression (2.11), which is symmetric by Lemma A.3 (i). $\square$

Remark 2.3 If we assume that $m = 1$ (i.e., the leaves are one-dimensional), then it is immediate from the previous result that the following are equivalent:

- (i) $\Delta_{\mathcal{H},\varepsilon}$ is symmetric for some $\varepsilon > 0$.
- (ii) $\Delta_{\mathcal{H},\varepsilon}$ is symmetric for all $\varepsilon > 0$.
- (iii) $\nabla J = 0$.

Recall that the statement $\nabla J = 0$ is equivalent to $\nabla T = 0$. For $m > 1$, the above statement remains true if we replace (i) by the following assumption

- (i') $\Delta_{\mathcal{H},\varepsilon}$ is symmetric at least two values $\varepsilon > 0$ and $\varepsilon' > 0$.

Example 2.4 (*H-type foliations*) Following definitions given in [5], we say that a foliated Riemannian manifold $(\mathbb{M}, \mathcal{F}, g)$ is of *H-type* if for every $Z \in \Gamma(\mathcal{V})$, we have $J_Z^2 = -\|Z\|_{\mathcal{V}}^2 \pi_{\mathcal{H}}$. Expand the definition of J from taking values from $\mathcal{V}$ to its Clifford algebra $\mathbf{Cl}(\mathcal{V})$ by the rule $J_1 = \pi_{\mathcal{H}}$ and iteratively $J_{u \cdot v} = J_u J_v$, $u, v \in \mathbf{Cl}(\mathcal{V})$. We then further say that the foliation is of horizontally parallel Clifford type if $\nabla_X J = 0$ for any horizontal vector fields $X \in \Gamma(\mathcal{H})$ and while for $u, v \in \mathcal{V}$.

$$(\nabla_u J)_v \in J_{\mathbf{Cl}(\mathcal{V})}.$$

It then turns out that for some $\kappa \in \mathbb{R}$,

$$(\nabla_u J)_v = -\kappa J_{u \cdot v + \langle u, v \rangle} = -\frac{\kappa}{2} [J_u, J_v].$$

The number κ determines the Ricci curvature of ∇ , see [5, Theorem 3.16]. We see that if we have an H-type Riemannian foliation $(\mathbb{M}, \mathcal{F}, g)$ of horizontally parallel Clifford type, then $\Delta_{\mathcal{H},\varepsilon}$ is symmetric with respect to g_ε for $\varepsilon = \frac{1}{\kappa}$.

Finally, to conclude the section we point out the following result. For the definition of the Carnot–Carathéodory metric d_{cc} of the sub-Riemannian manifold $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ and the tangent cone of a metric space, see, e.g., [13].

Corollary 2.5 *Assume that $\Delta_{\mathcal{H},\varepsilon}$ is symmetric on forms for some fixed $\varepsilon > 0$. Then, the following holds:*

- (a) *The horizontal bundle $\mathcal{H}$ has step 2, that is $\mathcal{H} + [\mathcal{H}, \mathcal{H}] = T\mathbb{M}$. In particular, the torsion T of the Bott connection ∇ will be surjective on $\mathcal{V}$.*
- (b) *The tangent cones of the metric space $(\mathbb{M}, d_{cc})$ at any pair of points $x, y \in \mathbb{M}$ are isometric.*

Proof (a) Recall that if $\Delta_{\mathcal{H},\varepsilon}$ is symmetric on forms for some $\varepsilon > 0$, then in particular $\nabla_v J = 0$ for any $v \in \mathcal{H}$. We can rewrite it as $\nabla_v T = 0$ for any $v \in \mathcal{H}$ since ∇ is compatible with g . Define $\mathcal{H}^2 = \mathcal{H} + [\mathcal{H}, \mathcal{H}]$ and let $X_1, X_2, X_3 \in \Gamma(\mathcal{H})$ be arbitrary. We first see that

$$T(X_2, X_3) = \nabla_{X_2} X_3 - \nabla_{X_3} X_2 - [X_2, X_3] = 0 \mod \mathcal{H}^2,$$

since ∇ preserves $\mathcal{H}$. Furthermore, by the definition of the Bott connection

$$\begin{aligned} [X_1, [X_2, X_3]] &= -[X_1, T(X_2, X_3)] \mod \mathcal{H}^2 = -\nabla_{X_1} T(X_2, X_3) \mod \mathcal{H}^2 \\ &= -T(\nabla_{X_1} X_2, X_3) - T(X_2, \nabla_{X_1} X_3) \mod \mathcal{H}^2 = 0 \mod \mathcal{H}^2. \end{aligned}$$

It follows that $\mathcal{H}$ only generates $\mathcal{H}^2$. As we assumed that $\mathcal{H}$ is bracket generating, we have $\mathcal{H}^2 = TM$.

- (b) Since both $\mathcal{H}$ and $\mathcal{H}^2 = \mathcal{H} + [\mathcal{H}, \mathcal{H}] = TM$ have constant rank, it follows by Mitchell [17] and Bellaïche [8] that the tangent cone at a point x is a Carnot group G_x . Its Lie algebra $\mathfrak{g}_x$ is given by

$$\mathfrak{g}_x = \mathfrak{g}_{x,1} \oplus \mathfrak{g}_{x,2} = \mathcal{H}_x \oplus T_x M / \mathcal{H}_x,$$

where $T_x M / \mathcal{H}_x$ is the center, and for $X_x, Y_x \in \mathcal{H}_x = \mathfrak{g}_{x,1}$ the Lie bracket is defined as

$$[[X_x, Y_x]] = [X, Y]|_x \mod \mathcal{H}_x.$$

where X, Y are any pair of vector fields extending this vectors. The Carnot group G_x is then the corresponding simply connected Lie group of $\mathfrak{g}_x$ with the sub-Riemannian structure given by left translation of $\mathfrak{g}_x = \mathcal{H}_x$ and its inner product.

If identify $\mathfrak{g}_x = \mathcal{H}_x \oplus T_x M / \mathcal{H}_x$ with $T_x M = \mathcal{H}_x \oplus \mathcal{V}_x$ through the map $v \mod \mathcal{H}_x \mapsto \pi_{\mathcal{V}_x}(v)$, $v \in T_x M$, then the Lie bracket becomes,

$$[[v, w]] = -T(v, w), \quad v, w \in T_x M.$$

Let now y be any other point and let $\gamma : [0, 1] \rightarrow \mathbb{M}$ be any horizontal curve from x to y , which exists from our assumption that $\mathcal{H}$ satisfies the bracket-generating condition. Then, $\nabla_{\dot{\gamma}(t)} T = 0$ for any $t \in [0, 1]$, so if we write

$$I_{\gamma,t} = I_t : T_x \mathbb{M} \rightarrow T_{\gamma(t)} \mathbb{M},$$

for the parallel transport map along γ , then this satisfies

$$I_t T(u, v) = T(I_t u, I_t v), \quad u, v \in T_x \mathbb{M}.$$

As a consequence, $I_1 : \mathfrak{g}_x = T_x \mathbb{M} \rightarrow \mathfrak{g}_y = T_y \mathbb{M}$ is a Lie algebra isomorphism, which can be integrated to a Lie group isomorphism from G_x to G_y . Since the parallel transport I_1 also maps $\mathcal{H}_x$ onto $\mathcal{H}_y$ isometrically, the induced map on Carnot groups is in fact a sub-Riemannian isometry.

□

3 Horizontal McKean–Singer theorem

We work on a totally geodesic foliation $(\mathbb{M}, \mathcal{F}, g)$ and assume that there is some $0 < \varepsilon < +\infty$ such that horizontal Laplacian $\Delta_{\mathcal{H},\varepsilon}$ is symmetric. From Proposition 2.2, this assumption is

equivalent to the fact that

$$(\nabla_v J)_w = -\frac{1}{2\varepsilon}[J_v, J_w].$$

Since $\Delta_{\mathcal{H},\varepsilon}$ commutes with d on smooth forms and is symmetric, it also commutes on smooth forms with the coderivative δ_ε , and thus, it also commutes with the Hodge–de Rham operator $\Delta_\varepsilon := -d\delta_\varepsilon - \delta_\varepsilon d$ on smooth forms. From Hodge theorem, the operator Δ_ε is elliptic with a compact resolvent and the space of L^2 -forms can be decomposed as $\bigoplus_{k=0}^{+\infty} E_{\lambda_k}$ where the E_{λ_k} 's are the eigenspaces of Δ_ε . Those eigenspaces only contain smooth forms, therefore $\Delta_{\mathcal{H},\varepsilon}(E_{\lambda_k}) \subset E_{\lambda_k}$. This implies that $\Delta_{\mathcal{H},\varepsilon}$ is essentially self-adjoint and generates the semigroup:

$$e^{t\Delta_{\mathcal{H},\varepsilon}} = \bigoplus_{k=0}^{+\infty} e^{t\Delta_{\mathcal{H},\varepsilon}|_{E_{\lambda_k}}} \quad (3.1)$$

By hypoellipticity (see [4, Lemma 4.9]), this semigroup has a smooth kernel $p_{\mathcal{H},\varepsilon}(t, x, y)$ and is a bounded trace class operator in $L^2_\mu(\wedge^* \mathbb{M}, g_\varepsilon)$. Let us denote by $E_0^+(\Delta_{\mathcal{H},\varepsilon})$ (resp. $E_0^-(\Delta_{\mathcal{H},\varepsilon})$) the space of harmonic even forms for $\Delta_{\mathcal{H},\varepsilon}$ (resp. the space of harmonic odd forms for $\Delta_{\mathcal{H},\varepsilon}$).

The goal of the section is to prove the following theorem, which is an analogue for our horizontal Laplacian of the classical McKean–Singer formula found in [16] :

Theorem 3.1 (Horizontal McKean–Singer formula) *For every $t > 0$,*

$$\begin{aligned} \text{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}}) &:= \int_{\mathbb{M}} \text{Tr}(p_{\mathcal{H},\varepsilon}^+(t, x, x)) d\mu(x) - \int_{\mathbb{M}} \text{Tr}(p_{\mathcal{H},\varepsilon}^-(t, x, x)) d\mu(x) \\ &= \dim E_0^+(\Delta_{\mathcal{H},\varepsilon}) - \dim E_0^-(\Delta_{\mathcal{H},\varepsilon}) \\ &= \chi(\mathbb{M}) \end{aligned}$$

where $\chi(\mathbb{M})$ is the Euler characteristic of $\mathbb{M}$.

We turn to the proof of Theorem 3.1. We denote by

$$\mathbf{D}_\varepsilon = d + \delta_\varepsilon$$

the Dirac operator of the metric g_ε . Observe that $\mathbf{D}_\varepsilon$ commutes with $\Delta_{\mathcal{H},\varepsilon}$ since both d and δ_ε commute with it. The main idea to prove Theorem 3.1 is to introduce a deformation of $\Delta_{\mathcal{H},\varepsilon}$ as follows:

$$\square_{\varepsilon,\theta} = (1 - \theta)\Delta_{\mathcal{H},\varepsilon} - \theta \mathbf{D}_\varepsilon^2, \quad \theta \in [0, 1].$$

A first lemma is the following:

Lemma 3.2 *Let λ be a nonzero eigenvalue of $\square_{\varepsilon,\theta}$. Then, $\mathbf{D}_\varepsilon : E_\lambda^+(\square_{\varepsilon,\theta}) \rightarrow E_\lambda^-(\square_{\varepsilon,\theta})$ is an isomorphism. Therefore, $\dim E_\lambda^+(\square_{\varepsilon,\theta}) = \dim E_\lambda^-(\square_{\varepsilon,\theta})$.*

Proof Let λ be a nonzero eigenvalue of $\square_{\varepsilon,\theta}$. The corresponding eigenspace $E_\lambda(\square_{\varepsilon,\theta})$ is finite-dimensional since $e^{t\square_{\varepsilon,\theta}}$ is a compact operator for $t > 0$. Moreover, since $\mathbf{D}_\varepsilon$ commutes with $\square_{\varepsilon,\theta}$, $\mathbf{D}_\varepsilon : E_\lambda^+(\square_{\varepsilon,\theta}) \rightarrow E_\lambda^-(\square_{\varepsilon,\theta})$ is well defined. Let now $\alpha \in E_\lambda^+(\square_{\varepsilon,\theta})$ such that $\mathbf{D}_\varepsilon \alpha = 0$. One has then

$$d\alpha = -\delta_\varepsilon \alpha.$$

This implies that

$$\|d\alpha\|_{L^2(\wedge^* \mathbb{M}, g_\varepsilon)}^2 = -\langle d\alpha, \delta_\varepsilon \alpha \rangle_{L^2(\wedge^* \mathbb{M}, g_\varepsilon)} = 0,$$

so $d\alpha = 0$. Similarly, one has $\|\delta_\varepsilon \alpha\|_{L^2(\wedge^1 \mathbb{M}, g_\varepsilon)}^2 = 0$, so $\delta_\varepsilon \alpha = 0$. Therefore,

$$\alpha = \frac{1-\theta}{\lambda} \Delta_{\mathcal{H}, \varepsilon} \alpha = -\frac{1-\theta}{\lambda} (d\delta_{\mathcal{H}, \varepsilon} + \delta_{\mathcal{H}, \varepsilon} d)\alpha = -\frac{1-\theta}{\lambda} d\delta_{\mathcal{H}, \varepsilon} \alpha.$$

One deduces

$$\|\alpha\|_{L^2(\wedge^1 \mathbb{M}, g_\varepsilon)}^2 = -\frac{1-\theta}{\lambda} \langle \alpha, d\delta_{\mathcal{H}, \varepsilon} \alpha \rangle_{L^2(\wedge^1 \mathbb{M}, g_\varepsilon)} = -\frac{1-\theta}{\lambda} \langle \delta_\varepsilon \alpha, \delta_{\mathcal{H}, \varepsilon} \alpha \rangle_{L^2(\wedge^1 \mathbb{M}, g_\varepsilon)} = 0.$$

As a consequence, $\mathbf{D}_\varepsilon : E_\lambda^+(\square_{\varepsilon, \theta}) \rightarrow E_\lambda^-(\square_{\varepsilon, \theta})$ is injective. Let us now prove that it is surjective. Let $\alpha \in E_\lambda^-(\square_{\varepsilon, \theta})$ which is orthogonal to the space $\mathbf{D}_\varepsilon E_\lambda^+(\square_{\varepsilon, \theta})$. For every $\omega \in E_\lambda^+(\square_{\varepsilon, \theta})$, one has

$$0 = \langle \alpha, \mathbf{D}_\varepsilon \omega \rangle_{L^2(\wedge^1 \mathbb{M}, g_\varepsilon)} = \langle \mathbf{D}_\varepsilon \alpha, \omega \rangle_{L^2(\wedge^1 \mathbb{M}, g_\varepsilon)}.$$

Thus, $\mathbf{D}_\varepsilon \alpha = 0$ and from the first part of the proof, we deduce that $\alpha = 0$. We conclude that $\mathbf{D}_\varepsilon : E_\lambda^+(\square_{\varepsilon, \theta}) \rightarrow E_\lambda^-(\square_{\varepsilon, \theta})$ is indeed an isomorphism. $\square$

A second lemma is the following:

Lemma 3.3 *For every $t > 0$, the map $\theta \rightarrow \mathbf{Str}(e^{t\square_{\varepsilon, \theta}})$ is continuous on $[0, 1]$.*

Proof Let $q_{\varepsilon, \theta}(t, x, y)$ be the heat kernel of $\square_{\varepsilon, \theta} = (1-\theta)\Delta_{\mathcal{H}, \varepsilon} - \theta\mathbf{D}_\varepsilon^2$, $p_{\mathcal{H}, \varepsilon}(t, x, y)$ be the heat kernel of $\Delta_{\mathcal{H}, \varepsilon}$ and $p_\varepsilon(t, x, y)$ be the heat kernel of $-\mathbf{D}_\varepsilon^2$. Since $-\mathbf{D}_\varepsilon^2$ and $\Delta_{\mathcal{H}, \varepsilon}$ commute, we have

$$e^{t\square_{\varepsilon, \theta}} = e^{t(1-\theta)\Delta_{\mathcal{H}, \varepsilon}} e^{-t\theta\mathbf{D}_\varepsilon^2}.$$

Therefore:

$$q_{\varepsilon, \theta}(t, x, y) = \int_{\mathbb{M}} p_{\mathcal{H}, \varepsilon}(t(1-\theta), x, z) p_\varepsilon(t\theta, z, y) dz$$

and the result easily follows since

$$\mathbf{Str}(e^{t\square_{\varepsilon, \theta}}) = \int_{\mathbb{M}} q_{\varepsilon, \theta}(t, x, x) dx.$$

$\square$

We are now ready for the proof of Theorem 3.1.

Proof From the first lemma:

$$\begin{aligned} \mathbf{Str}(e^{t\square_{\varepsilon, \theta}}) &= \dim E_0^+(\square_{\varepsilon, \theta}) - \dim E_0^-(\square_{\varepsilon, \theta}) + \sum_{\lambda \neq 0} (\dim E_\lambda^+(\square_{\varepsilon, \theta}) - \dim E_\lambda^-(\square_{\varepsilon, \theta})) e^{\lambda t} \\ &= \dim E_0^+(\square_{\varepsilon, \theta}) - \dim E_0^-(\square_{\varepsilon, \theta}). \end{aligned}$$

Therefore, $\mathbf{Str}(e^{t\square_{\varepsilon, \theta}}) \in \mathbb{Z}$. From the second lemma, $\theta \rightarrow \mathbf{Str}(e^{t\square_{\varepsilon, \theta}})$ is continuous, thus constant. We deduce

$$\mathbf{Str}(e^{t\square_{\varepsilon, 0}}) = \mathbf{Str}(e^{t\square_{\varepsilon, 1}}).$$

Since $\square_{\varepsilon^*, 1} = -\mathbf{D}_\varepsilon^2$ is the Hodge–de Rham Laplacian of the Riemannian manifold $(\mathbb{M}, g_\varepsilon)$, from the usual Riemannian Hodge theory (see [16]), we have

$$\mathbf{Str}(e^{t\square_{\varepsilon, 1}}) = \chi(\mathbb{M}),$$

4 which concludes the proof. $\square$

Remark 3.4 (*Dependence on the symmetry condition*) It would obviously be beneficial to prove the above statement without the assumption of symmetry on $\Delta_{\mathcal{H},\varepsilon}$. A semigroup approach to non-symmetric horizontal Laplacians has been used, see [15, Appendix A]. In the above proof, however, we really rely on the fact that $\Delta_{\mathcal{H},\varepsilon}$ commutes with the codifferential δ_ε , and with the Laplace–Beltrami operator $-\mathbf{D}_\varepsilon^2$. We can no longer use these properties if we remove the symmetry assumption.

4 Horizontal Chern–Gauss–Bonnet formula

As before, we consider the horizontal Laplacian

$$\Delta_{\mathcal{H},\varepsilon} = -d\delta_{\mathcal{H},\varepsilon} - \delta_{\mathcal{H},\varepsilon}d,$$

and assume that it is symmetric for a fixed ε . As seen earlier, $\Delta_{\mathcal{H},\varepsilon}$ satisfies the Weitzenböck identity

$$\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} - \mathcal{R}_\varepsilon = -(\nabla_{\mathcal{H}}^\varepsilon)^* \nabla_{\mathcal{H}}^\varepsilon - \mathcal{R}_\varepsilon. \quad (4.1)$$

where the later equality follows from [15, Lemma 2.1]. The goal of the section is to compute the pointwise limit

$$\lim_{t \rightarrow 0} \mathbf{Str} (p_{\mathcal{H},\varepsilon}(t, x, x))$$

and deduce from it our horizontal Chern–Gauss–Bonnet formula. The computation of that limit will be based on the probabilist method of Brownian Chen series (see [3,7]) which has the advantage of being easily adapted to subelliptic operators like $\Delta_{\mathcal{H},\varepsilon}$, see [2]. For convenience and to introduce notation, we include in Appendix A.2 the main elements of that theory.

A first step to implement the method in [2] is to study the small-time heat kernel asymptotics of a diffusion tangent to the scalar horizontal Laplacian $\Delta_{\mathcal{H}}$. Since we assume that $\Delta_{\mathcal{H},\varepsilon}$ is symmetric, from Corollary 2.5 one has $T\mathbb{M} = \mathcal{H} + [\mathcal{H}, \mathcal{H}]$, and thus the tangent diffusion will take its values in a two-step Carnot group [the so-called tangent cone, see Corollary 2.5(b)] for which an explicit formula for the heat kernel is known (see [10,11]). In a local horizontal frame $\{X_1, \dots, X_n\}$ around x_0 write

$$V_t(x_0) = \sum_{i=1}^n \sqrt{2} X_i(x_0) B_t^i + \sum_{1 \leq i < j \leq n} \pi_{\mathcal{V}}([X_i, X_j](x_0)) \int_0^t B_s^i dB_s^j - B_s^j dB_s^i,$$

where $(B_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}^n$. We note that $V_t(x_0)$ can be written in a basis free way as

$$\sqrt{2} B_t(x_0) - \int_0^t T(B_s(x_0), dB_s(x_0))$$

where $B_t(x_0) = \sum_{i=1}^n X_i(x_0) B_t^i$ is a standard Brownian motion in $\mathcal{H}_{x_0}$.

Lemma 4.1 *Let $x_0 \in \mathbb{M}$. For $t > 0$, let $d_t(x_0)$ be the density at 0 of the $T_{x_0}\mathbb{M}$ valued random variable $V_t(x_0)$. Then, when $t \rightarrow 0$,*

$$d_t(x_0) \sim \frac{2^m}{(4\pi t)^{\frac{n}{2}+m}} \int_{\mathcal{V}_{x_0}} \det \left(\frac{\sqrt{J_z^* J_z}}{\sinh \sqrt{J_z^* J_z}} \right)^{1/2} dz.$$

Proof The process $(V_t(x_0))_{t \geq 0}$ is the horizontal Brownian motion in the tangent cone G_{x_0} which is a 2-step Carnot group when it is identified with $T_{x_0}\mathbb{M}$ using the group exponential map. The heat kernel of the horizontal Laplacian is known explicitly in 2-step Carnot groups (see [10, 11]) which yields the small-time asymptotics. $\square$

Remark 4.2 We note that $d_t(x_0)$ is independent of x_0 because of Corollary 2.5(b).

In the sequel, we will use the notation $\mathcal{F}_I$ (defined with respect to the connection $D = \nabla^\varepsilon$) and $\Lambda_I(B)_t$, as introduced and discussed in Appendix A.2.

Corollary 4.3 *It will hold that as $t \rightarrow 0$*

$$\mathbf{Str}(p_{\mathcal{H}, \varepsilon}(t, x_0, x_0)) \sim d_t(x_0) \mathbb{E} \left(\mathbf{Str} \left(\exp \left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t \mathcal{F}_I \right) (x_0) \right) \middle| B_1 = 0 \right)$$

where $d_t(x_0)$ is the density at 0 of $V_t(x)$, as in Lemma 4.1.

Proof Since $\mathcal{H}$ is two-step bracket generating, the homogeneous dimension is $Q = \dim \mathcal{H} + 2 \dim \mathcal{V} = n + 2m$. Taking $N = n + 2m$ in Theorem A.1, and applying similar arguments as in the proof of Proposition 4.2 in [3], the corollary follows by recognizing that for $|I| > 2$, X_I is a linear combination of X_i , $[X_j, X_k]$ so that when $t \rightarrow 0$ the density at 0 of

$$\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t X_I$$

is equivalent to $d_t(x_0)$ from the previous lemma. $\square$

Applying the previous results, we are now able to compute $\lim_{t \rightarrow 0} \mathbf{Str}(p_{\mathcal{H}, \varepsilon}(t, x_0, x_0))$. Choose local orthonormal bases $X_1, \dots, X_n$ and $Z_1, \dots, Z_m$ of, respectively, $\mathcal{H}$ and $\mathcal{V}$.

Lemma 4.4 *The integral*

$$\mathcal{J} = \mathcal{J}(x_0) = \frac{2^m}{(2\pi)^{\frac{n}{2}+m}} \int_{\mathcal{V}_{x_0}} \det \left(\frac{\sqrt{J_z^* J_z}}{\sinh \sqrt{J_z^* J_z}} \right)^{1/2} dz,$$

is a constant, so independent of the point $x_0 \in \mathbb{M}$ chosen. Furthermore, it holds that

$$\lim_{t \rightarrow 0} \mathbf{Str}(p_{\mathcal{H}, \varepsilon}(t, x_0, x_0)) = \begin{cases} \frac{\mathcal{J}}{(\frac{n}{2}+m)!} \mathbb{E} \left(\mathbf{Str} \left[A_{x_0}^{\frac{n}{2}+m} \right] \middle| B_1 = 0 \right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

where the random variable A_{x_0} is given by

$$\begin{aligned} A_{x_0} = & -\frac{1}{2} \sum_{i,j,k,l=1}^n \left(R_{kli}^j + \frac{1}{\varepsilon} \sum_{r=1}^m T_{kl}^r T_{ij}^r \right) a_i^* a_j^* a_l a_k \\ & , \sum_{1 \leq i < j \leq n} \sum_{r,s=1}^m T_{ij;r}^s b_r^* b_s \int_0^1 B_t^i d B_t^j - B_t^j d B_t^i. \end{aligned} \quad (4.2)$$

Proof First, observe that

$$\mathcal{J}(x_0) = (2t)^{\frac{n}{2}+m} d_t(x_0),$$

and so the independence of $\mathcal{J}(x_0)$ from x_0 follows from Corollary 2.5(b) as in Remark 4.2.

Consider the expansion

$$\mathbf{Str} \left[\exp \left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t \mathcal{F}_I \right) (x_0) \right] = \sum_{k \geq 0} \frac{1}{k!} \mathbf{Str} \left[\left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t \mathcal{F}_I \right)^k (x_0) \right].$$

From the Weitzenböck identity (4.1), we have for $i, j \in \{1, \dots, n+m\}$ that

$$\mathcal{F}_0 = -\mathcal{R}_\varepsilon, \quad \mathcal{F}_i = 0, \quad \mathcal{F}_{(i,j)} = \hat{R}^\varepsilon(Y_i, Y_j)$$

where $\{Y_1, \dots, Y_{n+m}\}$ form a local orthonormal frame and the $\{c_i, c_i^*\}_{i=1}^{n+m}$ form the associated Fermion calculus of $T\mathbb{M}$. Equation (2.11) allows us to write

$$\mathcal{R}_\varepsilon = \sum_{i,j,k=1}^n \langle \hat{R}^\varepsilon(X_i, X_k) X_j, X_i \rangle_g a_k^* a_i + \sum_{i,j,k,l} \langle \hat{R}^\varepsilon(X_i, X_j) X_k, X_l \rangle_g a_i^* a_j^* a_l a_k$$

where $\{a_i, a_i^*\}$ form the Fermion calculus for $\mathcal{H}$.

Recalling equation (A.1) in the appendix, we see that the supertrace will vanish for any term that is not of full degree; from our expressions for $\mathcal{F}_I$, it is thus clear that for $k < \frac{n}{2} + m$

$$\mathbf{Str} \left[\left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t \mathcal{F}_I \right)^k (x_0) \right] = 0.$$

Let us assume that n is even. Applying the scaling property of Brownian motion, when $t \rightarrow 0$ the term $k = \frac{n}{2} + m$ will be dominant. More precisely,

$$\begin{aligned} & \mathbb{E} \left(\mathbf{Str} \left[\exp \left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t \mathcal{F}_I \right) (x_0) \right] \middle| B_1 = 0 \right) \\ &= \frac{1}{(\frac{n}{2}+m)!} \mathbb{E} \left(\mathbf{Str} \left[\left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t \mathcal{F}_I \right)^{\frac{n}{2}+m} (x_0) \right] \middle| B_1 = 0 \right) + O \left(t^{\frac{n}{2}+m+\frac{1}{2}} \right). \end{aligned} \quad (4.3)$$

Then, we have,

$$\begin{aligned} & \mathbb{E} \left(\mathbf{Str} \left[\left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_t \mathcal{F}_I \right)^{\frac{n}{2}+m} (x_0) \right] \middle| B_1 = 0 \right) \\ &= \mathbb{E} \left(\mathbf{Str} \left[\left(-t \mathcal{R}_\varepsilon(x_0) + \sum_{1 \leq i < j \leq n} \sum_{r,s=1}^r \hat{R}_{iir}^{\varepsilon,s} b_r^* b_s \int_0^t B_u^i d B_u^j - B_u^j d B_u^i \right)^{\frac{n}{2}+m} \right] \middle| B_1 = 0 \right) + O \left(t^{\frac{n}{2}+m+\frac{1}{2}} \right). \end{aligned} \quad (4.4)$$

We can further simplify this expression using that by Lemma A.2, Appendix, we know that $\hat{R}_{ijr}^{\varepsilon,s} = R_{ijr}^s = T_{ij;r}^s$. We also use (2.11) and the fact that only the last term in $\mathcal{R}_\varepsilon$ contributes to the supertrace. Combining Lemma 4.1, Corollary 4.3, and Eqs. (4.3) and (4.4), we apply the scaling property of Brownian motion again to find

$$\begin{aligned} \mathbf{Str}(p_{\mathcal{H},\varepsilon}(t, x_0, x_0)) &= \frac{\mathcal{J}}{(\frac{n}{2}+m)!} \mathbb{E} \left(\mathbf{Str} \left[A_{x_0}^{\frac{n}{2}+m} \right] \middle| \right. \\ & \left. B_1 = 0 \right) + O \left(t^{\frac{1}{2}} \right). \end{aligned}$$

If n is odd, we get by similar arguments that

$$\mathbf{Str}(p_{\mathcal{H},\varepsilon}(t, x_0, x_0)) = O \left(t^{\frac{1}{2}} \right).$$

completing the proof. $\square$

In what follows, we will introduce the tensor $\mathcal{T}$ by

$$\mathcal{T}(Y_1, Y_2) = \hat{R}^\varepsilon(\pi_{\mathcal{H}}Y_1, Y_2)\pi_{\mathcal{V}} = \pi_{\mathcal{V}}\hat{R}^\varepsilon(\pi_{\mathcal{H}}Y_1, Y_2).$$

We observe that for any $X_1, X_2 \in \Gamma(\mathcal{H})$ and $Z \in \mathcal{V}$,

$$\mathcal{T}(X_1, X_2)Z = (\nabla_Z T)(X_1, X_2) = \frac{1}{2\varepsilon} (T(J_Z X_1, X_2) + T(X_1, J_Z X_2)),$$

where the latter equality follows from the symmetry condition of $\Delta_{\mathcal{H},\varepsilon}$.

Example 4.5 (H-type foliation) We again consider the case of the H-type foliations as in Example 2.4. We recall that in this case, we have that $\Delta_{\mathcal{H},\varepsilon}$ for $\varepsilon = \frac{1}{\kappa}$. Let $x \in \mathbb{M}$ be a fixed point and let $\mathbf{Cl}(\mathcal{V}_x)$ be the Clifford algebra of the vertical space. We remark that in this case, for any $u, v \in \mathcal{H}_x$ with $v \in (\text{span}_{\zeta \in \mathbf{Cl}(\mathcal{V}_x)} J_\zeta u)^\perp$, we have $\mathcal{T}(u, v) = 0$. On the other hand, if $v = J_\zeta u$, then for any $z \in \mathcal{V}_x$,

$$\mathcal{T}(u, J_\zeta u)z = \kappa \pi_{\mathcal{V}_x}(z \cdot \zeta^{\text{odd}}),$$

where ζ^{odd} is the odd part of ζ and $\pi_{\mathcal{V}_x} \text{Cl}(\mathcal{V}_x) \rightarrow \mathcal{V}_x$ is the projection to the first-order part.

We can use the above definition and the previous lemma to prove the following.

Proposition 4.6 Assume that n or m is odd, then

$$\lim_{t \rightarrow 0} \mathbf{Str} (p_{\mathcal{H},\varepsilon}(t, x, x)) dx = 0$$

Assume that both n and m are even, then

$$\lim_{t \rightarrow 0} \mathbf{Str} (p_{\mathcal{H},\varepsilon}(t, x, x)) dx = \hat{\omega}_{\mathcal{H}}^\varepsilon \wedge \left[\det \left(\frac{\mathcal{T}}{\sinh(\mathcal{T})} \right)^{1/2} \right]_m$$

where $[\cdot]_m$ denotes the m -form part and $\hat{\omega}_{\mathcal{H}}^\varepsilon$ is the horizontal Euler form, locally defined as

$$\hat{\omega}_{\mathcal{H}}^\varepsilon = \frac{(-1)^{n/2} m!}{2^{n/2} \left(\frac{n}{2} + m \right)!} \mathcal{T} \sum_{\sigma, \tau \in \mathfrak{S}_n} \epsilon(\sigma) \epsilon(\tau) \prod_{i=1}^{n-1} \hat{R}_{\sigma(i)\sigma(i+1)\tau(i)}^{\varepsilon, \tau(i+1)} dx_{\mathcal{H}},$$

In the above formula, $\mathfrak{S}_n$ is the set of the permutations of the indices $\{1, \dots, n\}$, ϵ the signature of a permutation, $\hat{R}_{ijk}^{\varepsilon, l}$ is as in (2.7) and $dx_{\mathcal{H}}$ the n -form $X_1^* \wedge \dots \wedge X_n^*$.

Proof We first assume that both n and m are even. It remains to compute $\mathbb{E} \left(\mathbf{Str} \left[A_{x_0}^{\frac{n}{2}+m} \right] \middle| B_1 = 0 \right)$.

Looking at (4.2), we have

$$\begin{aligned} & \mathbb{E} \left(\mathbf{Str} \left[A_{x_0}^{\frac{n}{2}+m} \right] \middle| B_1 = 0 \right) \\ &= \mathbf{Str} \left[\left(-\sum_{i,j,k,l} \hat{R}^\varepsilon(X_i, X_j) X_k, X_l \rangle_g a_i^* a_j^* a_l a_k \right)^{n/2} \mathbb{E} \left[\left(\sum_{1 \leq i < j \leq n} \mathcal{T}(X_i, X_j)(x_0) \int_0^1 B_s^i d B_s^j - B_s^j d B_s^i \right)^m \middle| B_1 = 0 \right] \right] \end{aligned}$$

The term $\left(\sum_{i,j,k,l} \hat{R}^\varepsilon(X_i, X_j) X_k, X_l \rangle_g a_i^* a_j^* a_l a_k \right)^{n/2}$ is then analyzed as in the proof of Proposition 5.6 in [7] (see also Lemma 2.35 in [19]) and up to constant yields the horizontal Euler form $\hat{\omega}_{\mathcal{H}}^\varepsilon$. On the other hand, using again the formula for the supertrace, the term

$$\mathbb{E} \left[\left(\sum_{1 \leq i < j \leq n} \mathcal{T}(X_i, X_j)(x_0) \int_0^1 B_s^i d B_s^j - B_s^j d B_s^i \right)^m \middle| B_1 = 0 \right]$$

can be replaced with

$$m! \mathbb{E} \left[\exp \left(\sum_{1 \leq i < j \leq n} \mathcal{T}(X_i, X_j)(x_0) \int_0^1 B_s^i d B_s^j - B_s^j d B_s^i \right) \middle| B_1 = 0 \right]$$

and is analyzed using the Lévy area formula as in the proof of Theorem 4.3 in [3]; it yields the top degree Fermionic piece of $\det \left(\frac{\mathcal{T}}{\sinh(\mathcal{T})} \right)^{1/2} (x_0) \in \mathbf{End}(\wedge \mathcal{V}_{x_0}^*)$ (Fermionic calculus is done here on $\mathcal{V}_{x_0}$).

If n is even and m is odd, a similar analysis shows that

$$\mathbb{E} \left(\mathbf{Str} \left[A_{x_0}^{\frac{n}{2}+m} \right] \middle| B_1 = 0 \right) = 0.$$

□

Combining Theorem 3.1 and Proposition 4.6 finally yields our main theorem:

Theorem 4.7 *Assume that both n and m are even, then*

$$\chi(\mathbb{M}) = \int_{\mathbb{M}} \hat{\omega}_{\mathcal{H}}^{\varepsilon} \wedge \left[\det \left(\frac{\mathcal{T}}{\sinh \mathcal{T}} \right)^{1/2} \right]_m.$$

Assume that n or m is odd, then $\chi(\mathbb{M}) = 0$.

As a corollary, since $\nabla J = 0$ implies $\mathcal{T} = 0$, we obtain the following result:

Corollary 4.8 *Assume that $\nabla J = 0$, then $\chi(\mathbb{M}) = 0$.*

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A Appendices

A.1 Fermion calculus and supertraces

In this section, we recall some basic elements of Fermion calculus, see section 2.2.2 in [19] for more details. Let V be a d -dimensional Euclidean vector space. We denote V^* its dual and $\wedge V^* = \bigoplus_{k \geq 0} \wedge^k V^*$, its exterior algebra. If $u \in V^*$, we denote a_u^* the map $\wedge V^* \rightarrow \wedge V^*$, such that $a_u^*(\omega) = u \wedge \omega$. The dual map is denoted a_u . Let now $\theta_1, \dots, \theta_d$ be an orthonormal basis of V^* . We denote $a_i = a_{\theta_i}$. If I and J are two words with $1 \leq i_1 < \dots < i_k \leq d$ and $1 \leq j_1 < \dots < j_l \leq d$, we denote

$$A_{IJ} = a_{i_1}^* \cdots a_{i_k}^* a_{j_1} \cdots a_{j_l}.$$

The family of all the possible A_{IJ} forms a basis of the 2^{2d} -dimensional vector space $\mathbf{End}(\wedge V^*)$.

If $A \in \mathbf{End}(\wedge V^*)$, the supertrace $\mathbf{Str}(A)$ is the difference of the trace of A on even forms minus the trace of A on odd forms. If $A = \sum_{I,J} c_{IJ} A_{IJ}$, then we have

$$\mathbf{Str}(A) = (-1)^{\frac{d(d-1)}{2}} c_{\{1,\dots,d\}\{1,\dots,d\}}. \quad (\text{A.1})$$

In this paper, $c_{\{1,\dots,d\}\{1,\dots,d\}}$ will be called the top degree Fermionic piece of A and

$$[A]_d := (-1)^{\frac{d(d-1)}{2}} c_{\{1,\dots,d\}\{1,\dots,d\}} \theta_1 \wedge \dots \wedge \theta_d$$

the d -form part of A .

A.2 The Brownian Chen series parametrix method

For the sake of completeness and to introduce some notations used in the paper, we reproduce here the essential ideas from [2,3,7] to which we refer for further details. Let $\mathcal{E}$ be a finite-dimensional vector bundle over a compact manifold $\mathbb{M}$ equipped with a connection D and consider a second-order differential operator $\mathcal{L} = D_0 + \sum_{i=1}^d D_i^2$ with $D_i = \mathcal{F}_i + D_{X_i}$ for some smooth vector fields X_i and potentials $\mathcal{F}_i$ on $\mathcal{E}$. It is known that the differential equation

$$\frac{\partial \Phi}{\partial t} = \mathcal{L}\Phi, \quad \Phi(0, x) = f(x)$$

has solution

$$\Phi(t, x) = (e^{t\mathcal{L}} f)(x) = P_t f(x).$$

At strongly regular points $x_0 \in \mathbb{M}$, it is furthermore true that P_t admits a smooth heat kernel

$$\begin{aligned} p_t(x_0, \cdot) : \mathbb{R}_{>0} &\rightarrow \Gamma(\mathbb{M}, \text{Hom}(\mathcal{E})) \\ t &\mapsto p_t(x_0, \cdot) \end{aligned}$$

which is to say

$$(P_t f)(x_0) := (e^{t\mathcal{L}} f)(x_0) = \int_{\mathbb{M}} p_t(x_0, y) f(y) dy.$$

We have a method of approximation for the heat kernel in this setting.

Theorem A.1 *Let $N \geq 1$ and define $(P_t^N f)(x) = \mathbb{E}(\Psi(1, x))$ where $\Psi(\tau, x)$ solves the random differential equation*

$$\frac{\partial \Psi}{\partial \tau} = \sum_{I: d(I) \leq N} \Lambda_I(B)_t (D_I \Psi)(\tau, x), \quad \Psi(0, x) = f(x). \quad (\text{A.2})$$

where $I = (i_1, \dots, i_k) \in \{0, \dots, d\}^k$ is a word, $D_I = [D_{i_1}, [\dots, [D_{i_{k-1}}, D_{i_k}] \dots]]$, $d(I) = n(I) + k$ with $n(I)$ the number of 0's in I , and the random coefficients are defined by

$$\Lambda_I(B)_t = 2^{d(I)/2} \sum_{\sigma \in \mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{\Delta^k[0,t]} \circ d B^{\sigma^{-1}(I)}$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$. Then,

- For $k \geq 0$, define the norm

$$\|f\|_k = \sup_{0 \leq l \leq k} \sup_{0 \leq i_1, \dots, i_k} \sup_{x \in \mathbb{M}} \|D_{i_1} \cdots D_{i_k} f(x)\|.$$

It will hold that for any $k \geq 0$

$$\|P_t f - P_t^N f\|_k = O\left(t^{\frac{N+1}{2}}\right), \quad t \rightarrow 0$$

- P_t^N admits a smooth kernel p_t^N such that for $N \geq 2$

$$p_t(x_0, x_0) = p_t^N(x_0, x_0) + O\left(t^{\frac{N+1-Q}{2}}\right), \quad t \rightarrow 0$$

where Q is the homogeneous dimension at x_0 .

- Write $\mathcal{F}_I = D_I - D_{X_I}$. For $N \geq 2$, it holds as $t \rightarrow 0$ that

$$\begin{aligned} p_t^N(x_0, x_0) &= d_t^N(x_0) \mathbb{E} \left(\exp \left(\sum_{I, d(I) \leq N} \Lambda_I(B)_t \mathcal{F}_I \right) (x_0) \middle| \sum_{I, d(I) \leq N} \Lambda_I(B)_t X_I(x_0) = 0 \right) + O\left(t^{\frac{N+1-Q}{2}}\right) \end{aligned}$$

where $d_t^N(x)$ is the density at 0 of the random variable $\sum_{I, d(I) \leq N} \Lambda_I(B)_t X_I(x)$.

We refer to Baudoin [2] and Baudoin [7, Section 5.1] for the proofs and further details, but we remark that roughly the theorem says that in small time we can approximate the heat kernel of $\mathcal{L}$ by the kernel associated with solutions of Eq. (A.2), for which we will be able to say much more.

A.3 Curvature of the connection $\hat{\nabla}^\varepsilon$

We want to give details on writing the curvatures of $\hat{\nabla}^\varepsilon$ in terms of the Bott connection ∇ .

Lemma A.2 *Relative to the notation of (2.7) we have the following identities. Recall that i, j, k, l denotes vector fields from a basis of $\mathcal{H}$, while indices r, s denotes such elements from a basis of $\mathcal{V}$*

- (i) $R_{ijk}^l = R_{kli}^j$, $R_{r_1 s_1 r_1}^{s_2} = R_{r_2 s_2 r_1}^{s_1}$,
- (ii) $R_{ijr}^s = T_{ij;r}^s$, $R_{irk}^l = 0$, $R_{is_1 r_2}^{s_2} = 0$,
- (iii) $T_{ij;r}^l = 0$. Equivalently $(\nabla_Z J)_Z = 0$ for any vector field Z with values in $\mathcal{V}$.
- (iv) $\hat{R}_{ijk}^{\varepsilon, l} = R_{ijk}^l + \frac{1}{\varepsilon} \sum_{s=1}^m T_{ij}^s T_{kl}^s$.
- (v) $\hat{R}_{irk}^{\varepsilon, l} = \frac{1}{\varepsilon} T_{kl;i}^s$.
- (vi) $\hat{R}_{rsk}^{\varepsilon, l} = \frac{2}{\varepsilon} T_{kl;r}^s + \frac{1}{\varepsilon^2} \sum_{i=1}^n (T_{il}^r T_{ki}^s - T_{il}^s T_{ki}^r)$

Proof From (2.3), we observe that

$$\begin{aligned} \hat{R}^\varepsilon(X, Y)Z &= R(X, Y)Z + \frac{1}{\varepsilon} (\nabla_X J)_Y Z - \frac{1}{\varepsilon} (\nabla_Y J)_X Z \\ &\quad + \frac{1}{\varepsilon} J_{T(X, Y)} Z + \frac{1}{\varepsilon^2} [J_X, J_Y]Z. \end{aligned} \quad (\text{A.3})$$

We will also use the first Bianchi identity for connections with torsion

$$\circlearrowleft R(X, Y)Z = \circlearrowleft (\nabla_X T)(X, Y) + \circlearrowleft T(T(X, Y), Z),$$

where $\circlearrowleft$ denotes the cyclic sum. We furthermore observe the following identities.

- (i) Since $\langle T(Y_1, Y_2), Y_3 \rangle$ and $T(T(Y_1, Y_2), Y_3)$ vanishes if Y_1, Y_2, Y_3 are either all vertical or all horizontal,

$$\begin{aligned}\langle R(X_1, X_2)X_3, X_4 \rangle_g &= \langle R(X_3, X_4)X_1, X_2 \rangle_g, \\ \langle R(Z_1, Z_2)Z_3, Z_4 \rangle_g &= \langle R(Z_3, Z_4)Z_1, Z_2 \rangle_g,\end{aligned}$$

for any $X_i \in \Gamma(\mathcal{H})$, $Z_i \in \Gamma(\mathcal{V})$, $i = 1, 2, 3, 4$.

- (ii) From Grong [14, Appendix A], we know that for $X_1, X_2 \in \Gamma(\mathcal{H})$, $Z_1, Z_2 \in \Gamma(\mathcal{V})$,

$$R(X_1, X_2)Z_1 = (\nabla_{Z_1}T)(X_1, X_2), \quad R(X_1, Z_1)X_2 = 0 \quad R(X_1, Z_1)Z_2 = 0.$$

- (iii) Since ∇ is compatible with the metric then $(\nabla_Z J)_Z = 0$ for any $Z \in \Gamma(\mathcal{V})$, as for any $X_1, X_2 \in \Gamma(\mathcal{H})$,

$$\begin{aligned}0 &= \langle Z, R(X_1, X_2)Z \rangle_g = \langle Z, \circlearrowleft R(X_1, X_2)Z \rangle_g \\ &= \langle Z, (\nabla_Z T)(X_1, X_2) \rangle_g = \langle X_2, (\nabla_Z J)_Z X_1 \rangle_g.\end{aligned}$$

- (iv) We observe first that from (A.3), for any $X_1, X_2, X_3, X_4 \in \Gamma(\mathcal{H})$

$$\begin{aligned}\langle \hat{R}^\varepsilon(X_1, X_2)X_3, X_4 \rangle_g &= \langle R(X_1, X_2)X_3, X_4 \rangle_g + \frac{1}{\varepsilon} \langle J_{T(X_1, X_2)}X_3, X_4 \rangle_g \\ &\stackrel{(i)}{=} \langle R(X_3, X_4)X_1, X_2 \rangle_g + \frac{1}{\varepsilon} \langle T(X_1, X_2), T(X_3, X_4) \rangle_g.\end{aligned}$$

- (v) Next, for any $X_1, X_2 \in \Gamma(\mathcal{H})$, $Z \in \Gamma(\mathcal{V})$,

$$\hat{R}^\varepsilon(X_1, Z)X_2 \stackrel{(ii)}{=} \frac{1}{\varepsilon} (\nabla_{X_1} J)_Z X_2.$$

- (vi) For the final property observe that

$$R(Z_1, Z_2)X_1 \stackrel{(iii)}{=} \circlearrowleft R(Z_1, Z_2)X_1 = 0.$$

Hence,

$$\begin{aligned}\hat{R}^\varepsilon(Z_1, Z_2)X_1 &= \frac{1}{\varepsilon} (\nabla_{Z_1} J)_{Z_2} X_1 - \frac{1}{\varepsilon} (\nabla_{Z_2} J)_{Z_1} X_1 + \frac{1}{\varepsilon^2} [J_{Z_1}, J_{Z_2}]X_1 \\ &\stackrel{(iii)}{=} \frac{2}{\varepsilon} (\nabla_{Z_1} J)_{Z_2} X_1 + \frac{1}{\varepsilon^2} [J_{Z_1}, J_{Z_2}]X_1.\end{aligned}$$

□

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