The spinor and tensor fields with higher spin on spaces of constant curvature

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Abstract

In this article, we give all the Weitzenböck-type formulas among the geometric first-order differential operators on the spinor fields with spin $j + 1/2$ over Riemannian spin manifolds of constant curvature. Then, we find an explicit factorization formula of the Laplace operator raised to the power $j + 1$ and understand how the spinor fields with spin $j + 1/2$ are related to the spinors with lower spin. As an application, we calculate the spectra of the operators on the standard sphere and clarify the relation among the spinors from the viewpoint of representation theory. Next we study the case of trace-free symmetric tensor fields with an application to Killing tensor fields. Lastly we discuss the spinor fields coupled with differential forms and give a kind of Hodge–de Rham decomposition on spaces of constant curvature.

Keywords The Lichnerowicz Laplacian · The (higher spin) Dirac · Weitzenböck formulas · Generalized gradients · Space of constant curvature · Harmonic analysis on spheres

Mathematics Subject Classification 53C27 · 58C40 · 58J50

1 Introduction

Recently, many researchers in mathematics and physics have tried to understand geometry and analysis for the spinor fields with higher spin and the symmetric tensor fields. In fact, there are a lot of articles for the Rarita–Schwinger fields with spin $3/2$ and Killing tensor fields, which are special fields satisfying geometric differential equations. One of the methods to define such meaningful geometric differential equations is to use the generalized gradients or the Stein–Weiss operators. The operators are first-order differential operators

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naturally defined on spinor and tensor fields. In [12], Fegan classified them and showed they are conformally covariant. In [4], Branson studied the ellipticity and Weitzenböck formulas for generalized gradients using spectra of them on the standard sphere. He also studied another type of Weitzenböck formula in [5]. After a decade, the first author of this paper showed an explicit method to construct the Weitzenböck formulas for the generalized gradients in [16], which produce a lot of applications, vanishing theorem, eigenvalue estimates and so on. There are also many articles to study the generalized gradients and their applications to mathematics and physics ([15, 17, 18, 22], etc.). Moving on to analysis, we know that one of the main topics in Clifford analysis is to generalize spherical harmonic analysis on Euclidean space to such spinor and tensor fields. Polynomial solutions called monogenic functions, fundamental solutions and Fueter theorem for the higher spin Dirac operators have been constructed in [9–11], etc. The key to give the fundamental solution is a so-called factorization formula. The $j + 1$ -st power of the Laplace operator Δ on $\mathbb{R}^n$ is factorized as $\Delta^{j+1} = D_j \circ A_{2j+1}$, where D_j is a generalized gradient (the higher spin Dirac operator) and A_{2j+1} is a differential operator with order $2j + 1$. To consider such a formula on curved manifold would be important to analyze higher spin fields and give a variety of applications to geometry.

In this article, we combine Clifford analysis and differential geometry. We study the generalized gradients on spinor fields with spin $j + 1/2$ over a Riemannian spin manifold (M, g) and try to give a factorization formula for the exponentiation of the Laplace operator Δ . Since Weitzenböck formulas are too complicated for $j \geq 2$ to handle them on curved manifolds, we assume that (M, g) has a constant sectional curvature. Then, we see any operators commute with each other in a way and construct a factorization formula explicitly in Theorem 2.9. To clarify the meaning of the factorization, we show how the spinor fields with spin $j + 1/2$ are influenced by the spinor fields with lower spin in Theorem 2.11. Remark that, in the case of $j = 1$, we need only the assumption of Einstein manifold and can develop fruitful geometry and analysis in [1, 18] and [19]. Next, we study harmonic analysis on spinor fields with spin $j + 1/2$ on the standard sphere as a model case for spinor analysis on a curved space in Sect. 3. Interestingly the factorization formula and Weitzenböck formulas yield all the eigenvalues of the generalized gradients from the eigenvalues of the Laplacian, which are calculated by Freudenthal's formula for Casimir operator (cf. [3]). This method would be easier to understand than in [6] or originally in [7]. We also show how the spaces of the spinor fields as Spin-modules relate to each other through the generalized gradients. In Sect. 4, we discuss the trace-free symmetric tensor fields on a space of constant curvature and give a factorization formula in Theorem 4.5. We study harmonic analysis on such fields over the standard sphere in Sect. 5. As an application, we give a decomposition of the Killing tensor fields on the sphere from the viewpoint of representation theory. In Sects. 6 and 7, we discuss the spinor fields coupled with differential forms on a space of constant curvature and harmonic analysis on the sphere. In particular, we give a kind of Hodge–de Rham decomposition for spinor fields with differential forms. In Appendix A, we show how to calculate the Weitzenböck formulas needed in Sect. 2.

2 Higher spin Dirac operators on spinor fields with spin $j + 1/2$

Let (M, g) be an n -dimensional Riemannian spin manifold with a spin structure $\text{Spin}(M)$, which is a principal $\text{Spin}(n)$ -bundle over M and doubly covering the orthonormal frame bundle $\text{SO}(M)$. Throughout this paper, we assume $n \geq 3$. The case of dimension two is

left as an exercise to the reader. We consider the spin $j + 1/2$ (unitary) representation π_j on W_j of $\text{Spin}(n)$ for $j = 0, 1, \dots$. For example, (π_0, W_0) is a usual spinor representation. For $n = 2m$, the space W_j is decomposed into the direct sum of W_j^+ and W_j^- . Each $W_j^\pm$ is an irreducible $\text{Spin}(n)$ -module whose highest weight is

$$(j + 1/2, \underbrace{1/2, \dots, 1/2}_{m-2}, \pm 1/2) = (j + 1/2, (1/2)_{m-2}, \pm 1/2).$$

Here, $(1/2)_k$ denotes a sequence $1/2, \dots, 1/2$ with length k as an abbreviation. We write the direct sum representation $\pi_j^+ \oplus \pi_j^-$ by π_j . For $n = 2m - 1$, the spin $j + 1/2$ spinor space W_j is an irreducible $\text{Spin}(n)$ -module whose highest weight is $(j + 1/2, (1/2)_{m-2})$. It follows from Weyl's dimension formula that the dimension of W_j is

$$\dim W_j = 2^{\lfloor n/2 \rfloor} \binom{n+j-2}{j}, \quad \dim W_j^\pm = 2^{\lfloor n/2 \rfloor - 1} \binom{n+j-2}{j}.$$

The representation π_j (resp. $\pi_j^\pm$) induces a vector bundle S_j (resp. $S_j^\pm$) associated with the principal bundle $\text{Spin}(M)$. Indeed, we consider the action of $\text{Spin}(n)$ on $\text{Spin}(M) \times W_j$ by

$$\text{Spin}(n) \times (\text{Spin}(M) \times W_j) \ni (g, (p, v)) \mapsto (pg^{-1}, \pi_j(g)v) \in \text{Spin}(M) \times W_j.$$

Then, we have a Hermitian vector bundle whose fiber is W_j ,

$$S_j := \text{Spin}(M) \times_{\text{Spin}(n)} W_j = (\text{Spin}(M) \times W_j) / \text{Spin}(n).$$

For example, S_0 is the spinor bundle of M . We call a section of S_j a *spin $j + 1/2$ field* or a *spinor field with spin $j + 1/2$* .

From now on, we study some basic properties for the first-order differential operators naturally defined on the space of the spin $j + 1/2$ fields $\Gamma(S_j)$. They are called *generalized gradients* or *Stein–Weiss operators*, which are defined by composing the orthogonal bundle projections and the covariant derivative ∇ induced by the Levi-Civita connection, [4, 12]. The covariant derivative on $\Gamma(S_j)$ is

$$\nabla : \Gamma(S_j) \ni \phi \mapsto \nabla \phi = \sum \nabla_{e_i} \phi \otimes e_i \in \Gamma(S_j \otimes TM^c),$$

where $\{e_i\}_i$ is a local orthonormal frame and TM^c is $TM \otimes \mathbb{C} \cong T^*M \otimes \mathbb{C}$ by Riemannian metric g . We split the fiber $W_j \otimes \mathbb{C}^n$ into the sum of $\text{Spin}(n)$ -modules,

$$W_j \otimes \mathbb{C}^n = W_{j+1} \oplus W_{j,1} \oplus W_j \oplus W_{j-1},$$

where $W_{j,1}$ is a $\text{Spin}(n)$ -module with the highest weight

$$\begin{cases} (j + 1/2, 3/2, (1/2)_{m-2}) \oplus (j + 1/2, 3/2, (1/2)_{m-3}, -1/2) & \text{for } n = 2m, \\ (j + 1/2, 3/2, (1/2)_{m-3}) & \text{for } n = 2m - 1. \end{cases}$$

We note that $W_{j,1}$ does not appear for $n = 3$ or $j = 0$ and W_{j-1} does not appear for $j = 0$.

The space W_j has a $\text{Spin}(n)$ -invariant Hermitian inner product (unique up to a constant factor), so that the above decomposition is orthogonal. Since the fiber metric on S_j is induced from the inner product, we have the orthogonal bundle projection onto each irreducible summand. For example, composing ∇ and the projection

$$\Pi_j : S_j \otimes TM^c \rightarrow S_j,$$

we have so-called *the higher spin Dirac operator*,

$$\tilde{D}_j := \Pi_j \circ \nabla, \quad \Gamma(S_j) \xrightarrow{\nabla} \Gamma(S_j \otimes TM^c) \xrightarrow{\Pi_j} \Gamma(S_j). \quad (1)$$

In this manner, we construct four generalized gradients on $\Gamma(S_j)$ and name them as follows;

$$\begin{aligned} \tilde{T}_j^+ : \Gamma(S_j) &\rightarrow \Gamma(S_{j+1}) && \text{the (first) twistor operator,} \\ U_j : \Gamma(S_j) &\rightarrow \Gamma(S_{j,1}) && \text{the (second) twistor operator,} \\ \tilde{D}_j : \Gamma(S_j) &\rightarrow \Gamma(S_j) && \text{the higher spin Dirac operator,} \\ \tilde{T}_j^- : \Gamma(S_j) &\rightarrow \Gamma(S_{j-1}) && \text{the co-twistor operator.} \end{aligned}$$

Here, we set $U_j = 0$ for $n = 3$, $U_0 = 0$ and $\tilde{T}_0^- = 0$. When $n = 2m$, each operator has the form of 2×2 matrix as

$$\tilde{D}_j = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad \tilde{T}_j^+, U_j, \tilde{T}_j^- = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

along the decomposition $\Gamma(S_j) = \Gamma(S_j^+) \oplus \Gamma(S_j^-)$. We introduce an L^2 -inner product on $\Gamma(S_j)$ by

$$(\phi_1, \phi_2) := \int_M \langle \phi_1, \phi_2 \rangle \text{vol}_g, \quad (\phi_1, \phi_2 \in \Gamma(S_j)).$$

Then, the co-twistor operator $\tilde{T}_j^-$ is a nonzero constant multiple of the formal adjoint of the twistor operator $\tilde{T}_{j-1}^+$ with spin $j - 1/2$,

$$(\tilde{T}_{j-1}^+)^* = c\tilde{T}_j^- : \Gamma(S_j) \rightarrow \Gamma(S_{j-1}).$$

In the case of $j = 0$, the operator $\tilde{D}_0$ seems to coincide with the Dirac operator. However we know $\tilde{D}_0 = 1/\sqrt{n}D$. Therefore, we need a normalization of $\tilde{D}_j$ to get the (usual) higher spin Dirac operator D_j in other articles ([10] etc.). An advantage of our definition with projection is that we can apply Weitzenböck formulas in [16] and [17]. Furthermore such formulas allow us to give a normalization of $\tilde{D}_j$ explicitly. Before normalizing the operators, we show some known results for analytic properties of the generalized gradients.

Proposition 2.1

1. $\tilde{D}_j, \tilde{T}_j^\pm, U_j$ are conformally covariant.
2. $\tilde{D}_j$ is a first-order, elliptic and (formally) self-adjoint differential operator.
3. $\tilde{T}_j^+$ is an overdetermined elliptic operator in the sense that the principal symbol $\sigma_{\xi, x}(\tilde{T}_j^+)$ is injective for every x in M and nonzero covector ξ in T_x^*M . Then, there is an orthogonal decomposition for the space of sections on a compact Riemannian spin manifold M ,

$$\Gamma(S_j) = \ker(\tilde{T}_{j-1}^+)^* \oplus \tilde{T}_{j-1}^+(\Gamma(S_{j-1})) = \ker \tilde{T}_j^- \oplus \tilde{T}_{j-1}^+(\Gamma(S_{j-1})).$$

Moreover, $\tilde{T}_j^+$ is a differential operator of finite type in the sense that the kernel of $\tilde{T}_j^+$ is finite dimensional (even if M is non-compact).

Proof The first claim follows from the fact that generalized gradient is conformally covariant [12, 16]. As for the second claim, a general result for the ellipticity is known in [4, 22]. We can also get the ellipticity for D_j from Remark 2.13. For the third one, we require a discussion. Let U and V be irreducible $\text{Spin}(n)$ -modules and W be the Cartan summand of $U \otimes V$, the irreducible summand whose highest weight is the sum of those of U and V . We consider the orthogonal projection Π onto an irreducible summand of $U \otimes V$. Then, it is shown in [21] that the following two statements are equivalent:

- for $u \otimes v$ in $U \otimes V$, $u \otimes v \neq 0 \Rightarrow \Pi(u \otimes v) \neq 0$.
- Π is the projection onto W .

Applying this to $U = W_j$, $V = \mathbb{C}^n = T_x^* M^c$ and $W = W_{j+1}$, we know $\Pi(u \otimes \xi) \neq 0$ for nonzero $u \otimes \xi$ in $W_j \otimes \mathbb{C}^n$. On the other hand, there is a result in [23] for finite type of differential equations such that this condition holds if and only if the generalized gradient is finite type. Thus, we have proved that $\tilde{T}_j^+$ is finite type and $\sigma_{\xi, x}(\tilde{T}_j^+)$ is injective. The decomposition of the space of sections with respect to an overdetermined elliptic operator is well known in [2]. Then, we get the decomposition of $\Gamma(S_j)$. $\square$

Remark 2.2 Let S_ρ be an irreducible vector bundle with the highest weight $\rho = (\rho^1, \dots, \rho^m)$. Then, we have the generalized gradient $D_{\rho+\mathbf{e}_1}^\rho$ from S_ρ to the Cartan summand $S_{\rho+\mathbf{e}_1}$ in $S_\rho \otimes TM^c$ where $\rho + \mathbf{e}_1 = (\rho^1 + 1, \rho^2, \dots, \rho^m)$. In the same manner, the operator $D_{\rho+\mathbf{e}_1}^\rho$ is the only generalized gradient of finite type on S_ρ .

We shall give another definition of the higher spin Dirac operator by “twisting” the spinor bundle S_0 with the bundle of symmetric tensors in [10]. Let $\text{Sym}^j = \text{Sym}^j(TM^c)$ be the j -th symmetric tensor product bundle of TM^c over (M, g) and Sym_0^j be its primitive irreducible component whose fiber is an irreducible $\text{Spin}(n)$ -module with the highest weight $(j, 0_{m-1})$. The bundle Sym^j is the direct sum of Sym_0^j and lower-order summands,

$$\text{Sym}^j = \text{Sym}_0^j \oplus g \cdot \text{Sym}_0^{j-2} \oplus g^2 \cdot \text{Sym}_0^{j-4} \oplus \dots \oplus g^{[j/2]} \cdot \begin{cases} \text{Sym}^1 & j \text{ odd} \\ \text{Sym}^0 & j \text{ even}, \end{cases} \quad (2)$$

where $\text{Sym}^0 = \text{Sym}_0^0$ is the trivial line bundle, $\text{Sym}^1 = \text{Sym}_0^1$ is the tangent bundle TM^c , and $g^k \cdot$ is the symmetric tensor product of $g^k = g \cdots g$. We consider the tensor bundle $S_0 \otimes \text{Sym}_0^j$ and the twisted Dirac operator defined by

$$D(j) = \sum_{k=1}^n (e_k \cdot \otimes \text{id}_{\text{Sym}_0^j}) \circ \nabla_{e_k}.$$

Here, ∇ is the covariant derivative on $S_0 \otimes \text{Sym}_0^j$, and $e_k \cdot$ is the Clifford multiplication by e_k . Along the decomposition of the bundle $S_0 \otimes \text{Sym}_0^j = S_j \oplus S_{j-1}$, we can realize $D(j)$ as a 2×2 matrix. As stated in [17], each component is a generalized gradient up to a nonzero multiplicative constant,

$$D(j) = \begin{pmatrix} D_j & T_{j-1}^+ \\ T_j^- & D_{j-1}' \end{pmatrix}, \quad \Gamma(S_j \oplus S_{j-1}) \rightarrow \Gamma(S_j \oplus S_{j-1}). \quad (3)$$

Since $D(j)$ is a formally self-adjoint operator, we get

$$D_j^* = D_j, \quad (D'_{j-1})^* = D'_{j-1}, \quad (T_{j-1}^+)^* = T_j^-.$$

We will see that the square of $D(j)$ gives Weitzenböck formulas. From the twisted Lichnerowicz formula,

$$D(j)^2 = \Delta_{S_0 \otimes \text{Sym}_0^j} + \frac{\text{Scal}}{8} - \frac{1}{2} \text{id}_{S_0} \otimes R_{\text{Sym}_0^j}. \quad (4)$$

We explain the first term Δ and the third term R on the right side. First, we see the third term, the curvature action $R_{\text{Sym}_0^j}$ on $\text{Sym}_0^j(TM^c)$. Let $\{e_i = [p, \mathbf{e}_i]\}_{i=1}^n$ be a local orthonormal tangent frame of $TM = \text{SO}(M) \times_{\text{SO}(n)} \mathbb{R}^n$ and $\{e_{ij} = e_i \wedge e_j\}_{1 \leq i < j \leq n}$ be a local frame of

$$\Lambda^2(T^*M) \simeq \mathfrak{o}(TM) = \mathfrak{so}(TM) = \text{SO}(M) \times_{\text{SO}(n)} \mathfrak{so}(n).$$

We have an action of the local section e_{ij} of $\mathfrak{so}(TM)$ on an associated vector bundle $S_\rho := \text{Spin}(M) \times_{\text{Spin}(n)} W_\rho$ with respect to a (not necessarily irreducible) representation (π_ρ, W_ρ) :

$$\mathfrak{so}(TM) \times S_\rho \ni (e_{ij} = [p, \mathbf{e}_i \wedge \mathbf{e}_j], \phi = [p, v]) \mapsto \pi_\rho(e_{ij})\phi := [p, \pi_\rho(\mathbf{e}_i \wedge \mathbf{e}_j)v] \in S_\rho.$$

Then, we define the curvature action R_ρ on S_ρ by

$$\begin{aligned} R_\rho &= \frac{1}{2} \sum_{1 \leq i,j,k,l \leq n} R_{ijkl} \pi_\rho(e_{ij} e_{kl}) \\ &= \frac{1}{2} \sum_{1 \leq i,j,k,l \leq n} W_{ijkl} \pi_\rho(e_{ij} e_{kl}) + 2 \sum_{1 \leq i,k,j \leq n} E_{ij} \pi_\rho(e_{ik} e_{jk}) + \frac{\pi_\rho(c_2)}{n(n-1)} \text{Scal} \\ &=: \text{Weyl}_\rho + \text{Ein}_\rho + \text{Scal}_\rho, \end{aligned}$$

where $R_{ijkl} = g(R(e_i, e_j)e_k, e_l)$ is the Riemannian curvature, $\}\}$ Weyl the conformal Weyl tensor, $\}\}$ Ein the traceless Ricci tensor and $\}\}$ Scale the scalar curvature. The coefficient $\pi_\rho(c_2)$ of Scal is an action of the second Casimir element $c_2 = \sum_{1 \leq i,j \leq n} e_{ij} e_{ji}$ on W_ρ , which acts on each irreducible bundle by a constant. Next we explain the first term of (4), the standard Laplacian or the Lichnerowicz Laplacian. On the bundle S_ρ , we define the second-order Laplace type operator by

$$\Delta_\rho := \nabla^* \nabla + \frac{1}{2} R_\rho. \quad (5)$$

This operator coincides with the Hodge Laplacian $\Delta = dd^* + d^*d$ on $S_\rho = \Lambda^k(T^*M)$ and $\Delta = D^2 - \text{Scal}/8$ on the spinor bundle. On an irreducible compact symmetric space G/K , it corresponds to the second Casimir operator (nonnegative operator) for G . Remark that Δ_ρ is not necessarily nonnegative operator on a compact Riemannian manifold in general. One of important observation for the standard Laplacian is that, when we have a decomposition $S_\rho = S_{\lambda_1} \oplus \cdots \oplus S_{\lambda_N}$ as an associated vector bundle to $\text{SO}(M)$ or $\text{Spin}(M)$, the standard Laplacian Δ_ρ acts on each bundle diagonally (for more detail, [17, 23]).

We return to the case of $D(j)$. By the above observation, we have

$$D(j)^2 = \begin{pmatrix} \Delta_j + \frac{\text{Scal}}{8} & 0 \\ 0 & \Delta_{j-1} + \frac{\text{Scal}}{8} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} (\text{id} \otimes R_{\text{Sym}_0^j})_j^j & (\text{id} \otimes R_{\text{Sym}_0^j})_j^{j-1} \\ (\text{id} \otimes R_{\text{Sym}_0^j})_{j-1}^j & (\text{id} \otimes R_{\text{Sym}_0^j})_{j-1}^{j-1} \end{pmatrix}$$

along the decomposition $S_j \oplus S_{j-1}$. On the other hand, we take the square of the block matrix realization for $D(j)$,

$$D(j)^2 = \begin{pmatrix} D_j^2 + T_{j-1}^+ T_j^- & D_j T_{j-1}^+ + T_{j-1}^+ D'_{j-1} \\ T_j^- D_j + D'_{j-1} T_j^- & T_j^- T_{j-1}^+ + (D'_{j-1})^2 \end{pmatrix}.$$

Comparing the above two equations, we show

$$\begin{aligned} D_j^2 + T_{j-1}^+ (T_{j-1}^+)^* &= \Delta_j + \frac{\text{Scal}}{8} - \frac{1}{2} (\text{id} \otimes R_{\text{Sym}_0^j})_j^j, \\ (T_{j-1}^+)^* T_{j-1}^+ + (D'_{j-1})^2 &= \Delta_{j-1} + \frac{\text{Scal}}{8} - \frac{1}{2} (\text{id} \otimes R_{\text{Sym}_0^j})_{j-1}^{j-1}, \\ (T_{j-1}^+)^* D_j + D'_{j-1} (T_{j-1}^+)^* &= -\frac{1}{2} (\text{id} \otimes R_{\text{Sym}_0^j})_{j-1}^j, \\ D_j T_{j-1}^+ + T_{j-1}^+ D'_{j-1} &= -\frac{1}{2} (\text{id} \otimes R_{\text{Sym}_0^j})_j^{j-1}. \end{aligned}$$

The curvature term $(\text{id} \otimes R_{\text{Sym}_0^j})_{j-1}^j$ is a bundle homomorphism from S_j to S_{j-1} depending only on the conformal Weyl tensor part $\text{Weyl}_{\text{Sym}_0^j}$ and the traceless Ricci tensor part $\text{Ein}_{\text{Sym}_0^j}$ ([17]). Remark that, in the case of $j = 1$, it depends only on the traceless Ricci tensor, so that there are fruitful result of the Rarita–Schwinger operator on an Einstein manifold in [18]. On the other hand, the curvature term $(\text{id} \otimes R_{\text{Sym}_0^j})_j^j$ on S_j depends not only on $\text{Weyl}_{\text{Sym}_0^j}$ and $\text{Ein}_{\text{Sym}_0^j}$ but also on the scalar curvature part. Therefore, we can calculate the curvature terms explicitly on a Riemannian manifold of constant sectional curvature $K = c$ and obtain the following proposition.

Proposition 2.3 *If (M, g) is a Riemannian manifold of constant sectional curvature $K = c$ with a spin structure, then*

$$D_j^2 + T_{j-1}^+ (T_{j-1}^+)^* = \Delta_j - (j(n+j-2) - \frac{n(n-1)}{8})c, \quad (6)$$

$$(T_{j-1}^+)^* T_{j-1}^+ + (D'_{j-1})^2 = \Delta_{j-1} - (j(n+j-2) - \frac{n(n-1)}{8})c, \quad (7)$$

$$(T_{j-1}^+)^* D_j + D'_{j-1} (T_{j-1}^+)^* = 0, \quad D_j T_{j-1}^+ + T_{j-1}^+ D'_{j-1} = 0. \quad (8)$$

Proof The conformal Weyl tensor and the traceless Ricci tensor are zero, and the scalar curvature is $\text{Scal} = n(n-1)c$ on (M, g) . The action of $-\frac{1}{2}\text{id} \otimes R_{\text{Sym}_0^j}$ is constant by

$$-\frac{\pi_{\text{Sym}_0^j}(c_2)}{2n(n-1)} \text{Scal} = -j(n+j-2)c.$$

Then, we have

$$D_j^2 + T_{j-1}^+ T_j^- = \Delta_j - (j(n+j-2) - \frac{n(n-1)}{8})c.$$

Similarly we prove (7) and (8). $\square$

Furthermore, because of $D(j)^2 D(j) = D(j) D(j)^2$, we have

$$\begin{pmatrix} \Delta_j + c' & 0 \\ 0 & \Delta_{j-1} + c' \end{pmatrix} \begin{pmatrix} D_j & T_{j-1}^+ \\ T_j^- & D_{j-1}' \end{pmatrix} = \begin{pmatrix} D_j & T_{j-1}^+ \\ T_j^- & D_{j-1}' \end{pmatrix} \begin{pmatrix} \Delta_j + c' & 0 \\ 0 & \Delta_{j-1} + c' \end{pmatrix},$$

where $c' = -(j(n+j-2) - \frac{n(n-1)}{8})c$. Then, we conclude that

$$\Delta_j D_j = D_j \Delta_j, \quad T_{j-1}^+ \Delta_{j-1} = \Delta_j T_{j-1}^+, \quad T_j^- \Delta_j = \Delta_{j-1} T_j^-$$

on a space of constant curvature. Note that these commutation relations hold on a locally symmetric space (M, g) because its curvature R satisfies $\nabla R = 0$ (see Proposition 8 in [17]).

Now we calculate a normalizing constant c_j in the equation $D_j = c_j \tilde{D}_j$. Thanks to Weitzenböck formulas in [16] (see Appendix A for more detail), the generalized gradients $\{\tilde{D}_j, \tilde{T}_j^\pm\}_j$ satisfy

$$\frac{(n+2j)(n-2)}{n+2j-2} \tilde{D}_j^2 + \frac{4(n+j-2)}{n+2j-2} (\tilde{T}_j^-)^* \tilde{T}_j^- = \Delta_j + \text{curv}, \quad (9)$$

$$\frac{4j}{n+2j-2} (\tilde{T}_{j-1}^+)^* \tilde{T}_{j-1}^+ + \frac{(n+2j-4)(n-2)}{n+2j-2} (\tilde{D}_{j-1})^2 = \Delta_{j-1} + \text{curv}, \quad (10)$$

where “curv” means a bundle endomorphism depending on the Riemannian curvature as before. We also know that D_j^2 and $T_{j-1}^+ T_j^-$ are independent in the sense that there is no relation of the form

$$aD_j^2 + bT_{j-1}^+ T_j^- = \text{curv}, \quad (a, b) \neq (0, 0)$$

and so are D_j^2 and $T_{j+1}^- T_j^+$. Then, comparing the equations (6), (7), (9) and (10), we obtain the normalizing constant for D_j with respect to $\tilde{D}_j = \Pi_j \circ \nabla$ in the next proposition.

Proposition 2.4 *We consider a Riemannian spin manifold (M, g) and the vector bundle $S_j = \text{Spin}(M) \times_{\text{Spin}(n)} W_j$ with spin $j + 1/2$. Let $\tilde{D}_j = \Pi_j \circ \nabla$ be the generalized gradient on S_j defined by (1), and D_j (resp. D_{j-1}') be the operator defined by restricting the twisted Dirac operator $D(j)$ on S_j (resp. on S_{j-1}) in (3). Then,*

$$D_j = \sqrt{\frac{(n+2j)(n-2)}{n+2j-2}} \tilde{D}_j, \quad D_{j-1}' = -\sqrt{\frac{(n+2(j-1)-2)(n-2)}{n+2(j-1)}} \tilde{D}_{j-1},$$

and hence,

$$D_j^2 = \frac{(n+2j)(n-2)}{n+2j-2} \tilde{D}_j^2, \quad (D_{j-1}')^2 = \frac{(n+2(j-1)-2)(n-2)}{n+2(j-1)} \tilde{D}_{j-1}^2.$$

As a result, we have

$$D'_j = -\frac{n+2j-2}{n+2j}D_j.$$

We also have normalizing constants for the twistor operators $T_j^\pm$ on S_j ,

$$T_j^- = 2\sqrt{\frac{n+j-2}{n+2j-2}}\tilde{T}_j^-, \quad T_j^+ = 2\sqrt{\frac{j+1}{n+2j}}\tilde{T}_j^+.$$

Hence, we have

$$\begin{aligned} \nabla^*\nabla &= (\tilde{T}_j^+)^*\tilde{T}_j^+ + U_j^*U_j + \tilde{D}_j^2 + (\tilde{T}_j^-)^*\tilde{T}_j^-, \\ &= \frac{n+2j}{4(j+1)}(T_j^+)^*T_j^+ + U_j^*U_j + \frac{n+2j-2}{(n+2j)(n-2)}D_j^2 + \frac{n+2j-2}{4(n+j-2)}(T_j^-)^*T_j^-. \end{aligned}$$

Remark 2.5 The generalized gradient is defined up to constant multiple of a complex number u with norm $|u| = 1$, so that the above normalizing constant is unique up to such u . In the case of D_j (resp. D'_j), taking into account the self-adjointness, we put $u = 1$ (resp. $u = -1$).

Remark 2.6 By the homomorphism type Weitzenböck formula given in [17], we can give an alternative proof of the calculation of the normalizing constant for $D'_j = cD_j$. In fact, we have

$$\frac{1}{\sqrt{n/2+j-2}}\tilde{D}_j\tilde{T}_{j-1}^+ = \frac{1}{\sqrt{n/2+j}}\tilde{T}_{j-1}^+\tilde{D}_{j-1}.$$

Therefore,

$$D_jT_{j-1}^+ = \frac{n+2j-4}{n+2j-2}T_{j-1}^+D_{j-1}.$$

Then, we have concluded $D'_j = -\frac{n+2j-2}{n+2j}D_j$.

Example 2.7 The operator D_0 is just the Dirac operator D . The operator D_1 is the Rarita–Schwinger operator Q in [18].

As a corollary of Proposition 2.3 and 2.4, we rewrite the Weitzenböck formula in terms of D_j and $T_j^\pm$, which will be useful in later calculations.

Corollary 2.8 Let (M, g) be a space of constant curvature $K = c$ with a spin structure. Then, the following identities for operators from $\Gamma(S_j)$ to $\Gamma(S_j)$ hold:

$$\begin{aligned} \Delta_j &= \nabla^*\nabla + \frac{1}{2}R_j = \nabla^*\nabla + (j(n+j-1) + \frac{n(n-1)}{8})c \\ &= D_j^2 + (T_j^-)^*T_j^- + (j(n+j-2) - \frac{n(n-1)}{8})c \\ &= (T_j^+)^*T_j^+ + \frac{(n+2j-2)^2}{(n+2j)^2}D_j^2 + ((j+1)(n+j-1) - \frac{n(n-1)}{8})c. \end{aligned} \quad (11)$$

The generalized gradients among $\Gamma(S_{j-1})$ and $\Gamma(S_j)$ are related as follows:

$$T_j^- D_j = \frac{n+2j-4}{n+2j-2} D_{j-1} T_j^-, \quad D_j T_{j-1}^+ = \frac{n+2j-4}{n+2j-2} T_{j-1}^+ D_{j-1}, \quad T_{j-1}^+ = (T_j^-)^*. \quad (12)$$

The main theorem in this section is a factorization formula for the power of the Laplacian on a space of constant curvature. On the flat space $\mathbb{R}^n$, it is known in [9, 10] and [11] that D_j factors through $(\Delta_j)^{j+1}$. In other words, there is a differential operator A_{2j+1} with order $2j+1$ such that $(\Delta_j)^{j+1} = D_j \circ A_{2j+1}$. It follows that the known fundamental solution G for $(\Delta_j)^{j+1}$ gives a fundamental solution $A_{2j+1}(G)$ for D_j . We can generalize this factorization on $\mathbb{R}^n$ to a space of constant curvature explicitly.

Theorem 2.9 (Factorization formula) *Let (M, g) be a space of constant curvature $K = c$ with a spin structure. For the higher spin Dirac operator D_j and the standard Laplacian Δ_j on S_j , we define a second-order operator on $\Gamma(S_j)$,*

$$B(s;j) := D_j^2 - \frac{(n+2s-2)^2}{(n+2j-2)^2} \left(\Delta_j - \left(s(n+s-2) - \frac{n(n-1)}{8} \right) c \right)$$

for $s = 0, 1, \dots, j+1$. Then, we have

$$\prod_{s=0}^j B(s;j) = 0. \quad (13)$$

Proof We prove the theorem by induction for j . We start from the equation for $j = 0$,

$$\prod_{s=0}^0 B(s;0) = B(0;0) = D_0^2 - (\Delta_0 + \frac{n(n-1)}{8} c) = 0.$$

This is just the Lichnerowicz formula for the Dirac operator. We assume the equation (13) holds for j . From (12), we easily show the key relations

$$\begin{aligned} T_j^+ B(s;j) &= \frac{(n+2j)^2}{(n+2j-2)^2} B(s;j+1) T_j^+, \\ T_{j+1}^- B(s;j+1) &= \frac{(n+2j-2)^2}{(n+2j)^2} B(s;j) T_{j+1}^-, \\ (T_j^-)^* T_j^- &= -B(j;j), \quad (T_j^+) T_j^+ = -\frac{(n+2j-2)^2}{(n+2j)^2} B(j+1;j). \end{aligned} \quad (14)$$

Sandwich (13) by T_j^+ and $(T_j^+)^*$, and we obtain

$$\begin{aligned} 0 &= T_j^+ \left(\prod_{s=0}^j B(s;j) \right) (T_j^+)^* = \left(\prod_{s=0}^j \frac{(n+2j)^2}{(n+2j-2)^2} B(s;j+1) \right) T_j^+ (T_j^+)^*, \\ &= -\frac{(n+2j)^{2(j+1)}}{(n+2j-2)^{2(j+1)}} \left(\prod_{s=0}^j B(s;j+1) \right) B(j+1;j+1). \end{aligned}$$

Thus, we have proved (13) holds for $j+1$. $\square$

To clarify the meaning of the above factorization formula, we introduce a filtration on $\Gamma(S_j)$. Put

$$F_j := \ker T_j^-, \quad F_{j-1} := \ker T_{j-1}^- T_j^-, \quad \dots \quad F_1 := \ker T_1^- \cdots T_j^-, \quad F_0 := \Gamma(S_j),$$

and we have a filtration on $\Gamma(S_j)$,

$$F_j \subset F_{j-1} \subset \dots \subset F_1 \subset F_0 = \Gamma(S_j).$$

Proposition 2.10 *On a compact space (M, g) of constant curvature $K = c$ with a spin structure,*

$$F_j = \ker B(j;j), \quad \dots, \quad F_k = \ker \prod_{s=k}^j B(s;j), \quad \dots, \quad F_0 = \Gamma(S_j),$$

for $0 \leq k \leq j$.

Proof If ϕ is in F_k , then $(T_j^-)^* \cdots (T_k^-)^* T_k^- \cdots T_j^- \phi = 0$. Conversely, if

$$(T_j^-)^* \cdots (T_k^-)^* T_k^- \cdots T_j^- \phi = 0,$$

then

$$\begin{aligned} 0 &= \int_M \langle (T_j^-)^* \cdots (T_k^-)^* T_k^- \cdots T_j^- \phi, \phi \rangle \text{vol}_g \\ &= \int_M |T_k^- \cdots T_j^- \phi|^2 \text{vol}_g = \|T_k^- \cdots T_j^- \phi\|^2. \end{aligned}$$

Thus, ϕ is in F_k if and only if $(T_j^-)^* \cdots (T_k^-)^* T_k^- \cdots T_j^- \phi = 0$. From (14),

$$\begin{aligned} (T_j^-)^* \cdots (T_k^-)^* T_k^- \cdots T_j^- &= - (T_j^-)^* \cdots (T_{k+1}^-)^* B(k;k) T_{k+1}^- \cdots T_j^- \\ &= c B(k;j) (T_j^-)^* \cdots (T_{k+1}^-)^* T_{k+1}^- \cdots T_j^- \quad (\exists c \neq 0) \\ &= -c B(k;j) (T_j^-)^* \cdots (T_{k+2}^-)^* B(k+1;k+1) T_{k+2}^- \cdots T_j^- \\ &\quad \dots \\ &= c' B(k;j) \cdots B(j;j), \quad (\exists c' \neq 0). \end{aligned}$$

Then, we can prove the proposition. $\square$

The above natural filtration allows us to get an associated grading on $\Gamma(S_j)$. In other words, there are subspaces W_s for $s = 0, \dots, j$ such that $F_s = W_s \oplus F_{s+1}$ and $\Gamma(S_j) = \bigoplus_{0 \leq s \leq j} W_s$.

Theorem 2.11 *Let S_j be the bundle of spinor fields with spin $j + 1/2$ on a compact space of constant curvature $K = c$ with a spin structure. We put*

$$W_s := \begin{cases} T_{j-1}^+ \cdots T_0^+ (\Gamma(S_0)), & s = 0, \\ T_{j-1}^+ \cdots T_s^+ (\ker T_s^-), & 1 \leq s \leq j-1, \\ \ker T_j^-, & s = j. \end{cases}$$

Then, the space $\Gamma(S_j)$ has an orthogonal decomposition associated with the filtration $\{F_j\}_j$,

$$\Gamma(S_j) = \bigoplus_{0 \leq s \leq j} W_s, \quad F_s = W_s \oplus F_{s+1} \quad (0 \leq s \leq j).$$

Furthermore, $W_s \subset \ker B(s;j)$ for $s = 0, \dots, j$.

Proof From Proposition 2.1, there are the orthogonal decompositions

$$\Gamma(S_s) = \ker T_s^- \oplus T_{s-1}^+(\Gamma(S_{s-1}))$$

for $s = 0, 1, \dots$. When we let T_s^+ act on $\Gamma(S_s)$, we can prove that the decomposition is preserved,

$$T_s^+(\Gamma(S_s)) = T_s^+(\ker T_s^-) \oplus T_s^+ T_{s-1}^+(\Gamma(S_{s-1})).$$

Since $T_s^+(\Gamma(S_s)) = T_s^+(\ker T_s^-) + T_s^+ T_{s-1}^+(\Gamma(S_{s-1}))$, all we have to do is to prove that the components intersect orthogonally. Take ϕ_s in $\ker T_s^-$, and we show

$$\begin{aligned} T_s^- T_{s+1}^- T_s^+(\phi_s) &= T_s^-(T_s^+)^* T_s^+ \phi_s = -\frac{(n+2s-2)^2}{(n+2s)^2} T_s^- B(s+1;s) \phi_s \\ &= -\frac{(n+2s-4)^2}{(n+2s)^2} B(s+1;s-1) T_s^- \phi_s = 0. \end{aligned}$$

Then,

$$(T_s^+ \phi_s, T_s^+ T_{s-1}^+ \psi_{s-1}) = (T_s^- T_{s+1}^- T_s^+ \phi_s, \psi_{s-1}) = 0 \quad \text{for } \psi_{s-1} \text{ in } \Gamma(S_{s-1}).$$

Thus, we have obtained an orthogonal decomposition $T_s^+(\Gamma(S_s)) = T_s^+(\ker T_s^-) \oplus T_s^+ T_{s-1}^+(\Gamma(S_{s-1}))$. As a consequence, we have

$$\begin{aligned} \Gamma(S_{s+1}) &= \ker T_{s+1}^- \oplus T_s^+(\ker T_s^-) \oplus T_{s+1}^+ T_s^+(\Gamma(S_{s-1})), \\ F_{s+1} &= \ker T_{s+1}^-, \quad F_s = \ker T_{s+1}^- \oplus T_s^+(\ker T_s^-). \end{aligned}$$

Repeating this, we have concluded $\Gamma(S_j) = \bigoplus_s W_s$.

Next we prove $W_s \subset \ker B(s;j)$. For ϕ_s in $\ker T_s^-$,

$$\begin{aligned} B(s;j) T_{j-1}^+ \cdots T_s^+ \phi_s &= \frac{(n+2j-4)^2}{(n+2j-2)^2} T_{j-1}^+ B(s;j-1) T_{j-2}^+ \cdots T_s^+ \phi_s \\ &= \frac{(n+2s-2)^2}{(n+2j-2)^2} T_{j-1}^+ \cdots T_{s+1}^+ T_s^+ B(s;s) \phi_s \\ &= -\frac{(n+2s-2)^2}{(n+2j-2)^2} T_{j-1}^+ \cdots T_{s+1}^+ T_s^+ (T_s^-)^* T_s^- \phi_s = 0. \end{aligned}$$

Thus, we have concluded that $W_s \subset \ker B(s;j)$. Remark that the reverse inclusion does not hold in general (for example, the case of $K = 0$). $\square$

Since $D_j \Delta_j = \Delta_j D_j$, the operator D_j and Δ_j have simultaneous eigenstates on a compact space (M, g) of constant curvature and their eigenvalues relate to each other. Let ϕ is a simultaneous eigenspinor in W_k with eigenvalue μ for Δ_j and λ^2 for D_j^2 . Then,

$$\lambda^2 = \frac{(n+2k-2)^2}{(n+2j-2)^2} \left(\mu - \left(k(n+k-2) - \frac{n(n-1)}{8} \right) c \right).$$

For the non-compact case, the situation is more difficult. In fact, it has already been known that $\Delta_j \phi = 0$ does not follow $D_j \phi = 0$ on $\mathbb{R}^n$, and vice versa ([10]).

At the end of this section, we state some remarks.

Remark 2.12 It was shown in [18] that the Rarita–Schwinger operator D_1 satisfies

$$\left(D_1^2 - \left(\Delta_1 + \frac{n-8}{8n} \text{Scal} \right) \right) \left(D_1^2 - \frac{(n-2)^2}{n^2} \left(\Delta_1 + \frac{\text{Scal}}{8} \right) \right) = 0$$

on an Einstein manifold (M, g) . To get a factorization formula for higher spin cases of $j > 1$, we have to assume the sectional curvature is constant because the curvature term and their derivatives remain in Weitzenböck formulas and the formula is too complicated to get a factorization. To calculate such error terms is a future problem.

Remark 2.13 The above theorem shows that $\sigma_\xi(D_j)$ is a factor of $\sigma_\xi(\Delta_j^{j+1}) = (-\|\xi\|^2)^{j+1}$ on the principal symbol level, and so that D_j is elliptic.

Remark 2.14 We can rewrite the equation (13) for T_j^+ as

$$\prod_{s=0}^j \left((T_j^+)^* T_j^+ - a'(s;j)(\Delta_j - b'(s;j)c) \right) = 0,$$

where

$$a'(s;j) = \frac{4(j-s+1)(n+j+s-1)}{(n+2j)^2}, \quad b'(s;j) = j(n+j) + s(n+s-2) + \frac{n(n+1)}{8}.$$

Then, the eigenvalue of the standard Laplacian Δ_j restricted to $\ker T_j^+ \cap W_k$ on a compact space of constant curvature is just

$$(j(n+j) + k(n+k-2) + \frac{n(n+1)}{8})c.$$

3 Harmonic analysis for spinor fields with higher spin on S^n

In this section, we study harmonic analysis for spin $j+1/2$ fields on the sphere. We calculate the eigenvalues of D_j^2 and other related second-order operators and show how the space of spinor fields are related to each other as $\text{Spin}(n+1)$ -modules.

Let S^n be the n -dimensional standard sphere. We realize it as a $\text{Spin}(n+1)$ -orbit of $e_{n+1} = (0, \dots, 0, 1)$ in $\mathbb{R}^{n+1}$. The isotropy group is

$$\text{Spin}(n) = \{g \in \text{Spin}(n+1) | g \cdot e_{n+1} = e_{n+1}\},$$

and the sphere is a symmetric space $\text{Spin}(n+1)/\text{Spin}(n)$. The (unique) spin structure of S^n is the principal $\text{Spin}(n)$ -bundle $\text{Spin}(n+1)$ where $\text{Spin}(n)$ acts naturally from the right side. The bundle of higher spinor fields is given by

$$S_j = \text{Spin}(n+1) \times_{\text{Spin}(n)} W_j$$

with a natural fiber metric induced from a $\text{Spin}(n)$ -invariant inner product on W_j . The space $L^2(S^n, S_j)$ of L^2 -integrable sections of S_j is a unitary representation space of $\text{Spin}(n+1)$ with respect to L^2 -inner product $\langle \phi, \psi \rangle = \int_{S^n} \langle \phi, \psi \rangle \text{vol}_g$. Frobenius reciprocity provides its irreducible decomposition,

$$L^2(S^n, S_j) \cong \bigoplus_{\rho \in \widehat{\text{Spin}(n+1)}} V_\rho \otimes \text{Hom}_{\text{Spin}(n)}(V_\rho, W_j),$$

where $\widehat{\text{Spin}(n+1)}$ means the equivalent classes of irreducible unitary finite dimensional representations of $\text{Spin}(n+1)$. Such a class is parametrized by its highest weight $\rho = (\rho^1, \dots, \rho^m)$ for $m = \lfloor \frac{n+1}{2} \rfloor$ satisfying ρ in $\mathbb{Z}^m \cup (\mathbb{Z} + 1/2)^m$ and the dominant condition

$$\begin{aligned} \rho^1 &\geq \rho^2 \geq \dots \geq \rho^m \geq 0, & (\text{for } n+1 = 2m+1), \\ \rho^1 &\geq \rho^2 \geq \dots \geq \rho^{m-1} \geq |\rho^m|, & (\text{for } n+1 = 2m). \end{aligned}$$

We state the branching rule to describe the restriction of an irreducible representation of $\text{Spin}(n+1)$ to $\text{Spin}(n)$ (cf. [27]). Fix an irreducible $\text{Spin}(n+1)$ -module V_ρ with the highest weight ρ . When V_ρ is considered as a $\text{Spin}(n)$ -module, the following $\text{Spin}(n)$ -modules appear each with multiplicity 1 as irreducible summands:

- In the case of $n+1 = 2m+1$, $\lambda = (\lambda^1, \dots, \lambda^m)$ in $\widehat{\text{Spin}(2m)}$ satisfying

$$\rho^1 \geq \lambda^1 \geq \rho^2 \geq \lambda^2 \geq \dots \geq \rho^m \geq \lambda^m \geq -\rho^m, \quad \rho^1 - \lambda^1 \in \mathbb{Z}.$$

- In the case of $n+1 = 2m$, $\lambda = (\lambda^1, \dots, \lambda^{m-1})$ in $\widehat{\text{Spin}(2m-1)}$ satisfying

$$\rho^1 \geq \lambda^1 \geq \rho^2 \geq \lambda^2 \geq \dots \geq \rho^{m-1} \geq \lambda^{m-1} \geq |\rho^m|, \quad \rho^1 - \lambda^1 \in \mathbb{Z}.$$

We denote by $V_j(k, s)$ an irreducible $\text{SO}(n+1)$ -module with the highest weight

$$(k+j, s, 0_{m-2})$$

for k in $\mathbb{Z}_{\geq 0}$ and $s = 0, \dots, j$. We also denote by $V_j(k, s)'$ an irreducible $\text{Spin}(2m+1)$ -module with the highest weight

$$(k+j+1/2, s+1/2, (1/2)_{m-2}),$$

and by $V_j^\pm(k, s)'$ an irreducible $\text{Spin}(2m)$ -module with the highest weight

$$(k+j+1/2, s+1/2, (1/2)_{m-3}, \pm 1/2)$$

for k in $\mathbb{Z}_{\geq 0}$ and $s = 0, \dots, j$.

Remark 3.1 In the case of $n+1 = 4$, the space $V_j(k, s)$ splits into the sum of $\text{SO}(4)$ -modules with the highest weight $(k+j, s)$ and $(k+j, -s)$. The space $V_j^\pm(k, s)'$ means an irreducible $\text{Spin}(4)$ -module with the highest weight $(k+j+1/2, \pm(s+1/2))$.

Applying Frobenius reciprocity and the branching rule to our case $\lambda = (j+1/2, 1/2, \dots, 1/2)$, we have the proposition.

Proposition 3.2

1. For $n = 2m$,

$$L^2(S^{2m}, S_j^\pm) \cong \bigoplus_{0 \leq s \leq j} \bigoplus_{k \geq 0} V_j(k, s)', \quad L^2(S^{2m}, S_j) \cong \bigoplus_{0 \leq s \leq j} \bigoplus_{k \geq 0} 2V_j(k, s)'.$$

2. For $n = 2m - 1$,

$$L^2(S^{2m-1}, S_j) \cong \bigoplus_{0 \leq s \leq j} \bigoplus_{k \geq 0} V_j^+(k, s)' \oplus V_j^-(k, s)'.$$

From now on, we put $V_j(k, s)' = V_j^+(k, s)' \oplus V_j^-(k, s)'$ for $n = 2m - 1$. By Weyl's dimension formula, we have

$$\dim V_j(k, s)' = 2^{\lfloor \frac{n+1}{2} \rfloor} \frac{(l+n-1+s)(l+1-s)}{(n-1)(n-2)} \binom{l+n-2}{l+1} \binom{s+n-3}{s},$$

where we set $l = k + j$. To calculate the eigenvalues of the square of the higher spin Dirac operator D_j^2 , we need the following lemma.

Lemma 3.3 On $V_j(k, s)'$, it holds that

$$\left(\Delta_j - (s(s+n-2) - \frac{n(n-1)}{8}) \right) \Big|_{V_j(k, s)'} = \left(j + k + \frac{n}{2} \right)^2.$$

Proof The standard Laplacian Δ_j on S_j coincides with the Casimir operator $c_2/2$ for $\text{Spin}(n+1)$. By Freudenthal's formula, the eigenvalue of the $c_2/2$ is $\pi_\rho(c_2/2) = \langle \rho, \rho \rangle + 2\langle \rho, \delta_{\text{Spin}(n+1)} \rangle$. The inner product is given by $\langle \mu, \nu \rangle = \sum_{1 \leq i \leq m} \mu^i \nu^i$, and $\delta_{\text{Spin}(n+1)}$ is half the sum of the positive roots,

$$\delta_{\text{Spin}(n+1)} = \begin{cases} (m-1/2, m-3/2, \dots, 3/2, 1/2) & \text{for } n+1 = 2m+1, \\ (m-1, m-2, \dots, 1, 0) & \text{for } n+1 = 2m. \end{cases}$$

Then,

$$\begin{aligned} & \pi_\rho(c_2/2) - (s(s+n-2) - \frac{n(n-1)}{8}) \\ &= (k+j+1/2)(k+j+n-1/2) + (s+1/2)(s+n-5/2) \\ & \quad + (n-4)(n-3)/8 - (s(s+n-2) - \frac{n(n-1)}{8}) \\ &= (j+k+n/2)^2. \end{aligned}$$

□

From the factorization formula in Theorem 2.9, we can guess the eigenvalue λ^2 of D_j^2 on $V_j(k, s)'$ is $\frac{(n+2s-2)^2}{(n+2j-2)^2} \left(j + k + \frac{n}{2} \right)^2$. In fact, we have

Theorem 3.4 *On the n -dimensional standard sphere S^n , the eigenvalue of D_j^2 on $V_j(k, s)'$ is*

$$\frac{(n+2s-2)^2}{(n+2j-2)^2} \left(j+k+\frac{n}{2} \right)^2$$

with multiplicity $\dim V_j(k, s)'$.

Proof We shall prove the case of $n = 2m - 1$ by induction for $j \geq 0$. First, we consider the case of $j = 0$. It is well known that the spectrum of the Dirac operator D_0^2 on the sphere is $(k + n/2)^2$ on $V_0(k, 0)'$ for $k = 0, 1, 2, \dots$. Next, assuming the statement in the theorem for j holds, we prove the one for $j + 1$. We take ϕ in $V_j(k, s)'$. It follows from our assumption, Lemma 3.3 and (11) that

$$(T_j^+)^* T_j^+ \phi = \frac{k(j-s+1)(2j+k+n)(j+s+n-1)}{(j+n/2)^2} \phi.$$

Then,

$$\ker T_j^+ = \bigoplus_{0 \leq s \leq j} V_j(0, s)',$$

and $T_j^+ : V_j(k, s)' \rightarrow V_{j+1}(k-1, s)'$ for any $k \geq 1$ is an isomorphism as $\text{Spin}(n+1)$ -module. Here, we use Schur's lemma with respect to the $\text{Spin}(n+1)$ -invariant operator T_j^+ . Taking such a $T_j^+ \phi$ in $V_{j+1}(k-1, s)'$, we show

$$D_{j+1}^2 T_j^+ \phi = \frac{(n+2j-2)^2}{(n+2j)^2} T_j^+ D_j^2 \phi = \frac{(n+2s-2)^2}{(n+2j)^2} \left(j+1+(k-1)+\frac{n}{2} \right)^2 T_j^+ \phi.$$

The case of $s = j + 1$ remains to be proved. According to Proposition 2.1, $L^2(S_{j+1})$ decomposes as $\ker T_{j+1}^- \oplus \text{Image } T_j^+$. Then,

$$\ker T_{j+1}^- = \bigoplus_{k \geq 0} V_{j+1}(k, j+1)'.$$

When ϕ is in $V_{j+1}(k, j+1)'$, Weitzenböck formula (11) gives

$$D_{j+1}^2 \phi = \Delta_{j+1} \phi - \left((j+1)(n+j-1) - \frac{n(n-1)}{8} \right) \phi = \left((j+1)+k+\frac{n}{2} \right)^2 \phi.$$

Thus, we have proved the statement for $j + 1$. We can prove the case of $n = 2m$ when we double each component $V_j(k, s)'$ in the above proof. $\square$

By using Corollary 2.8, we calculate all the eigenvalues of the other generalized gradients on S_j .

Corollary 3.5 *The eigenvalues of $(T_j^+)^* T_j^+$, $(T_j^-)^* T_j^-$ and $U_j^* U_j$ on $V_j(k, s)'$ are given by*

$$\frac{k(j-s+1)(2j+k+n)(j+s+n-1)}{(j+n/2)^2}, \quad (15)$$

$$\frac{(k+1)(j-s)(2j+k+n-1)(j+s+n-2)}{(j+n/2-1)^2}, \quad (16)$$

$$\text{and } \frac{(n-3)s(j+k+1)(n+s-2)(j+k+n-1)}{(n-2)(j+1)(j+n-2)}, \quad (17)$$

respectively.

We investigate how the eigenspaces $\{V_j(k, s')\}_{k,j,s}$ relate to each other through the generalized gradients. Put

$$\mathbf{V}_j(s)' := \bigoplus_{k \geq 0} V_j(k, s)'$$

for each j and $s = 0, \dots, j$, and we show that $\mathbf{V}_j(s)'$ is just W_s in Theorem 2.11. Remark that, in the case of the sphere, W_s coincides with the kernel of $B(s; j)$. The following diagram is useful to understand relations between the spaces:

$$\begin{array}{ccccccc} L^2(S_0) & = & \mathbf{V}_0(0)' & & & & \\ & & \downarrow T_0^+ & & & & \\ & & \vdots & & & & \\ L^2(S_{j-1}) & = & \mathbf{V}_{j-1}(0)' \oplus \cdots \oplus \mathbf{V}_{j-1}(j-1)' \oplus \{0\} & & & & \\ & & \begin{array}{c} T_j^- \downarrow \\ T_{j-1}^+ \end{array} & & \begin{array}{c} T_j^- \downarrow \\ T_{j-1}^+ \end{array} & & \begin{array}{c} T_j^+ \uparrow \end{array} \\ L^2(S_j) & = & \mathbf{V}_j(0)' \oplus \cdots \oplus \mathbf{V}_j(j-1)' \oplus \mathbf{V}_j(j)' \oplus \{0\} & & & & \\ & & \begin{array}{c} T_{j+1}^- \downarrow \\ T_j^+ \end{array} & & \begin{array}{c} T_{j+1}^- \downarrow \\ T_j^+ \end{array} & & \begin{array}{c} T_{j+1}^- \downarrow \\ T_j^+ \end{array} & & \begin{array}{c} T_{j+1}^- \uparrow \end{array} \\ L^2(S_{j+1}) & = & \mathbf{V}_{j+1}(0)' \oplus \cdots \oplus \mathbf{V}_{j+1}(j-1)' \oplus \mathbf{V}_{j+1}(j)' \oplus \mathbf{V}_{j+1}(j+1)' & & & & \\ & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

Here, we change each $\mathbf{V}_j(s)'$ by $2\mathbf{V}_j(s)'$ when n is $2m$. Then, we have the proposition.

Proposition 3.6 *The kernels and images of the generalized gradients are realized as $\text{Spin}(n+1)$ -module in $L^2(S^n, S_j)$ for $j = 0, 1, \dots$ as follows.*

For $n = 2m - 1$,

$$\begin{aligned} \ker T_j^+ &\cong \bigoplus_{0 \leq s \leq j} V_j(0, s)', \quad \ker T_j^- \cong V_j(j)', \quad \ker U_j \cong V_j(0)', \\ \text{Image } T_{j-1}^+ &\cong \bigoplus_{0 \leq s \leq j-1} V_j(s)', \quad \text{Image } T_{j+1}^- \cong L^2(S^n, S_j) \ominus \ker T_j^+. \end{aligned}$$

In particular, we have

$$\ker T_j^+ \cap \ker T_j^- \cong V_j(0, j)', \quad \ker T_j^+ \cap \ker U_j \cong V_j(0, 0)'.$$

For $n = 2m$, we double each component $V_j(k, s)'$ and $\mathbf{V}_j(s)'$ on the above equations.

4 Generalized gradients on j -th symmetric tensor fields

In this section, we study the generalized gradients on the bundle of trace-free symmetric tensors $\text{Sym}_0^j = \text{Sym}_0^j(TM^c)$. Let (M, g) be an oriented Riemannian manifold. The bundle $\text{Sym}^j(TM^c)$ for j -th symmetric tensor fields splits as (2) and the primitive component Sym_0^j is a vector bundle associated with the (oriented) orthonormal frame bundle $\text{SO}(M)$ with the highest weight $(j, 0_{m-1})$. We compose the covariant derivative ∇ and the orthogonal projection along the decomposition

$$\text{Sym}_0^j \otimes TM^c = \text{Sym}_0^{j,1} \oplus \text{Sym}_0^{j+1} \oplus \text{Sym}_0^{j-1},$$

where $\text{Sym}_0^{j,1}$ is an irreducible vector bundle with the highest weight $(j, 1, 0_{m-2})$. Then, we have three generalized gradients

$$\begin{aligned} T_j^+ : \Gamma(\text{Sym}_0^j) &\rightarrow \Gamma(\text{Sym}_0^{j+1}), & U_j : \Gamma(\text{Sym}_0^j) &\rightarrow \Gamma(\text{Sym}_0^{j,1}), \\ T_j^- : \Gamma(\text{Sym}_0^j) &\rightarrow \Gamma(\text{Sym}_0^{j-1}), \end{aligned}$$

where we set $U_0 = 0$ and $T_0^- = 0$.

Remark 4.1 For $n = 4$, $\text{Sym}_0^{j,1}$ splits into the sum of $\text{SO}(4)$ -modules with the highest weight $(j, 1)$ and $(j, -1)$. For $n = 3$ and $j \geq 1$, Sym_0^j appears again instead of $\text{Sym}_0^{j,1}$. We denote by U_j the self-gradient from $\Gamma(\text{Sym}_0^j)$ to $\Gamma(\text{Sym}_0^j)$ on a 3-dim Riemannian manifold, which is not an elliptic operator.

It follows from Weitzenböck formulas in [16] that there are two identities among them

$$\begin{aligned} \nabla^* \nabla &= (T_j^+)^* T_j^+ + U_j^* U_j + (T_j^-)^* T_j^-, \\ \frac{1}{2} R_{\text{Sym}_0^j} &= -j(T_j^+)^* T_j^+ + U_j^* U_j + (n+j-2)(T_j^-)^* T_j^-. \end{aligned}$$

Eliminating U_j from the above equations, we find

$$\begin{aligned} \Delta_j &:= \nabla^* \nabla + \frac{1}{2} R_{\text{Sym}_0^j} = \nabla^* \nabla - \frac{1}{2} R_{\text{Sym}_0^j} + R_{\text{Sym}_0^j} \\ &= (j+1)(T_j^+)^* T_j^+ - (n+j-3)(T_j^-)^* T_j^- + R_{\text{Sym}_0^j}. \end{aligned}$$

As in the case of the higher spin fields, we have to compare T_{j+1}^- and $(T_j^+)^*$ from $\Gamma(\text{Sym}_0^{j+1})$ to $\Gamma(\text{Sym}_0^j)$. The next lemma follows from [16].

Lemma 4.2 Let $D_\lambda^\rho = \Pi_\lambda \circ \nabla : \Gamma(S_\rho) \rightarrow \Gamma(S_\lambda)$ be a generalized gradient on an irreducible vector bundle $S_\rho = \text{SO}(M) \times_{\text{SO}(n)} W_\rho$ over a Riemannian manifold (M, g) . Here, Π_λ is the orthogonal projection onto S_λ from $S_\rho \otimes TM^c$. Then,

$$(D_\lambda^\rho)^* D_\lambda^\rho = \frac{\dim W_\lambda}{\dim W_\rho} D_\rho^\lambda (D_\rho^\lambda)^*. \quad (18)$$

Proof The principal symbol of D_λ^ρ is given by the linear map $p_\lambda^\rho(\xi)$ from W_ρ to W_λ defined by

$$p_\lambda^\rho(\xi)\phi := \Pi_\lambda(\phi \otimes \xi), \quad \text{for } \phi \text{ in } W_\rho \text{ and } \xi \text{ in } \mathbb{C}^n = T_x^*M^c.$$

In other words, D_λ^ρ is $\sum_i p_\lambda^\rho(e_i)\nabla_{e_i}$ as the Dirac operator D is given by $\sum_i e_i \cdot \nabla_{e_i}$. We denote by $p_\lambda^\rho(\xi)^*$ its adjoint map from W_λ to W_ρ . Conversely we know that the orthogonal projection Π_λ is realized by using the principal symbol (see Lemma 4.13 in [16]),

$$\Pi_\lambda(\phi \otimes \xi) = \sum_{1 \leq i \leq n} (p_\lambda^\rho(e_i)^* p_\lambda^\rho(\xi)\phi) \otimes e_i,$$

where $\{e_i\}_i$ is an orthonormal basis for $\mathbb{C}^n$. We shall calculate the following two $\text{SO}(n)$ -invariant maps from W_ρ to W_ρ , which are constants by Schur's lemma,

$$\sum_i p_\lambda^\rho(e_i)^* p_\lambda^\rho(e_i), \quad \sum_i p_\rho^\lambda(e_i) p_\rho^\lambda(e_i)^*.$$

Let $\{\phi_\alpha\}_\alpha$ be a basis of W_ρ . We consider the trace of $\Pi_\lambda : W_\rho \otimes \mathbb{C}^n \rightarrow W_\lambda$.

$$\begin{aligned} \dim W_\lambda &= \sum_{\alpha,i} \langle \Pi_\lambda(\phi_\alpha \otimes e_i), \phi_\alpha \otimes e_i \rangle = \sum_{\alpha,i,j} \langle (p_\lambda^\rho(e_j)^* p_\lambda^\rho(e_i)\phi_\alpha) \otimes e_j, \phi_\alpha \otimes e_i \rangle \\ &= \sum_{\alpha,i} \langle p_\lambda^\rho(e_i)^* p_\lambda^\rho(e_i)\phi_\alpha, \phi_\alpha \rangle = \sum_i p_\lambda^\rho(e_i)^* p_\lambda^\rho(e_i) \sum_\alpha \langle \phi_\alpha, \phi_\alpha \rangle \\ &= \dim W_\rho \sum_i p_\lambda^\rho(e_i)^* p_\lambda^\rho(e_i). \end{aligned}$$

Then, we have $\sum_i p_\lambda^\rho(e_i)^* p_\lambda^\rho(e_i) = \frac{\dim W_\lambda}{\dim W_\rho} \text{id}_{W_\rho}$. On the other hand, we have

$$\begin{aligned} \langle p_\rho^\lambda(\xi)\phi, p_\rho^\lambda(\eta)\psi \rangle &= \langle \Pi_\rho(\phi \otimes \xi), \Pi_\rho(\psi \otimes \eta) \rangle \\ &= \sum_{i,j} \langle p_\rho^\lambda(e_i)^* p_\rho^\lambda(\xi)\phi \otimes e_i, p_\rho^\lambda(e_j)^* p_\rho^\lambda(\eta)\psi \otimes e_j \rangle \\ &= \sum_i \langle p_\rho^\lambda(e_i)^* p_\rho^\lambda(\xi)\phi, p_\rho^\lambda(e_i)^* p_\rho^\lambda(\eta)\psi \rangle \\ &= \sum_i \langle p_\rho^\lambda(e_i) p_\rho^\lambda(e_i)^* p_\rho^\lambda(\xi)\phi, p_\rho^\lambda(\eta)\psi \rangle \\ &= \left(\sum_i p_\rho^\lambda(e_i) p_\rho^\lambda(e_i)^* \right) \langle p_\rho^\lambda(\xi)\phi, p_\rho^\lambda(\eta)\psi \rangle. \end{aligned}$$

Then, we have $\sum_i p_\rho^\lambda(e_i) p_\rho^\lambda(e_i)^* = \text{id}_{W_\rho}$. Since there is a nonzero constant a such that $p_\lambda^\rho(\xi) = ap_\rho^\lambda(\xi)^*$, we have concluded that $|a|^2 = \dim W_\lambda / \dim W_\rho$ and hence $(D_\lambda^\rho)^* D_\lambda^\rho = \frac{\dim W_\lambda}{\dim W_\rho} D_\rho^\lambda (D_\rho^\lambda)^*$. $\square$

Thus, we have given relations among the generalized gradients $\{T_j^\pm\}_j$ to construct a factorization formula on a space of constant curvature.

Proposition 4.3 *On a space (M, g) of constant curvature $K = c$, the operators T_j^+ , T_j^- and Δ_j for $j = 0, 1, 2, \dots$ satisfy*

$$\begin{aligned}
\Delta_j &= (j+1)(T_j^+)^* T_j^+ - (n+j-3)(T_j^-)^* T_j^- + 2j(n+j-2)c, \\
\frac{1}{2} R_{\text{Sym}_0^j} &= j(n+j-2)c = -j(T_j^+)^* T_j^+ + U_j^* U_j^+ + (n+j-2)(T_j^-)^* T_j^-, \\
(T_{j+1}^-)^* T_{j+1}^- &= \frac{(j+1)(n+2j-2)}{(n+j-2)(n+2j)} T_j^+ (T_j^+)^*.
\end{aligned} \tag{19}$$

Remark 4.4 We do not normalize constant multiples of the operators $\{T_j^\pm\}_j$ as the previous section. When we use a notation in [14], we find

$$(T_j^+)^* T_j^+ = \frac{1}{j+1} d_0^* d_0, \quad (T_j^-)^* T_j^- = \frac{n+2j-4}{(n+2j-2)(n+j-3)} \delta^* \delta.$$

We show the factorization formula for the $j+1$ -st power of the Laplacian on j -th symmetric tensor fields, which will be useful to calculate the eigenvalues on the sphere in the next section.

Theorem 4.5 (Factorization formula) *Let (M, g) be a space of constant curvature $K = c$, and Sym_0^j the vector bundle of trace-free symmetric tensor fields on (M, g) . The operator $T_j^+ : \Gamma(\text{Sym}_0^j) \rightarrow \Gamma(\text{Sym}_0^{j+1})$ and the Lichnerowicz Laplacian $\Delta_j : \Gamma(\text{Sym}_0^j) \rightarrow \Gamma(\text{Sym}_0^j)$ satisfy*

$$\begin{aligned}
\prod_{s=0}^j \left((T_j^+)^* T_j^+ - a(s;j) (\Delta_j - b(s;j)c) \right) &= 0, \\
\text{where } a(s;j) &= \frac{(j-s+1)(n+j+s-2)}{(j+1)(n+2j-2)}, \quad b(s;j) = j(n+j-1) + s(n+s-3).
\end{aligned} \tag{20}$$

Proof We prove the theorem by induction for j . The case of $j = 0$ follows from the definition of Δ_0 , that is, $\Delta_0 = (T_0^+)^* T_0^+$. We assume that the equation (20) holds for j . By (19),

$$\begin{aligned}
& T_j^+ \left((T_j^+)^* T_j^+ - a(s;j) (\Delta_j - b(s;j)c) \right) \\
&= \left(T_j^+ (T_j^+)^* - a(s;j) (\Delta_{j+1} - b(s;j)c) \right) T_j^+ \\
&= \underbrace{\frac{(j+2)(n+2j)}{(j+1)(n+2j-2)}}_{=:k(j)} \left((T_{j+1}^+)^* T_{j+1}^+ - a(s;j+1) (\Delta_{j+1} - b(s;j+1)c) \right) T_j^+.
\end{aligned}$$

Then,

$$\begin{aligned}
0 &= T_j^+ \left\{ \prod_{s=0}^j \left((T_j^+)^* T_j^+ - a(s;j) (\Delta_j - b(s;j)c) \right) \right\} (T_j^+)^* \\
&= k(j)^j \left\{ \prod_{s=0}^j \left((T_{j+1}^+)^* T_{j+1}^+ - a(s;j+1) (\Delta_{j+1} - b(s;j+1)c) \right) \right\} T_j^+ (T_j^+)^* \\
&= k(j)^{j+1} \left\{ \prod_{s=0}^j \left((T_{j+1}^+)^* T_{j+1}^+ - a(s;j+1) (\Delta_{j+1} - b(s;j+1)c) \right) \right\} \\
&\quad \times \left((T_{j+1}^+)^* T_{j+1}^+ - a(j+1;j+1) (\Delta_{j+1} - b(j+1;j+1)c) \right).
\end{aligned}$$

Finally we reach (20) for $j+1$. $\square$

If a tensor field ϕ in $\Gamma(\text{Sym}_0^j)$ satisfies $T_j^+ \phi = 0$, then we call it *trace-free conformal Killing tensor*, which has been investigated not only in differential geometry but also in relativity theory. As a corollary, we have an information of the eigenvalues of the Laplacian on trace-free conformal Killing tensors on a compact space of constant curvature ([25, 26] for the sphere case). We introduce a filtration on $\Gamma(\text{Sym}_0^j)$ as

$$F_j \subset F_{j+1} \subset \cdots \subset F_0 = \Gamma(\text{Sym}_0^j), \quad F_k = \ker T_k^- \cdots T_j^-.$$

and an associated grading

$$\Gamma(\text{Sym}_0^j) = \bigoplus_{0 \leq s \leq j} W_s, \quad W_s := T_{j-1}^+ \cdots T_s^+ (\ker T_s^-).$$

Since the inclusion

$$W_s \subset \ker \left((T_j^+)^* T_j^+ - a(s;j) (\Delta_j - b(s;j)c) \right)$$

follows from the same as in Theorem 2.11, we obtain the following corollary.

Corollary 4.6 *On a compact space of constant curvature $K = c$, the eigenvalue of Δ_j on $\ker T_j^+ \cap W_s$ is $b(s;j) = (j(n+j-1) + s(n+s-3))c$.*

5 Harmonic analysis for trace-free symmetric tensor fields on S^n

The method to calculate the eigenvalues of the operators $\{(T_j^\pm)^* T_j^\pm, U_j^* U_j, \Delta_j\}_j$ on the n -dimensional sphere $S^n = \text{SO}(n+1)/\text{SO}(n)$ is the same as in the spinor case. From Frobenius reciprocity and the branching rule, the space of the trace-free symmetric tensor fields are decomposed with respect to $\text{SO}(n+1)$,

$$L^2(S^n, \text{Sym}_0^j) \cong \bigoplus_{0 \leq s \leq j} V_j(s) = \bigoplus_{0 \leq s \leq j} \left(\bigoplus_{k \geq 0} V_j(k, s) \right).$$

As mentioned in Sect. 3, the space $V_j(k, s)$ stands for an irreducible $\text{SO}(n+1)$ -module with the highest weight $(k+j, s, 0_{m-2})$ and its dimension is

$$\frac{(2l+n-1)(2s+n-3)(l+n-2+s)(l+1-s)}{(n-1)(n-2)(s+n-3)(l+n-2)} \binom{l+n-2}{l+1} \binom{s+n-3}{s}$$

by setting $l = k + j$. Note that, for $n = 3$, $V_j(k, s)$ is the sum of $\mathrm{SO}(4)$ -modules with the highest weight $(k + j, s)$ and $(k + j, -s)$. We also put $\mathbf{V}_j(s) := \bigoplus_{k \geq 0} V_j(k, s)$. As in [3] Freudenthal's formula gives the eigenvalue of Δ_j on $V_j(k, s)$,

$$\Delta_j \Big|_{V_j(k,s)} = (j+k)(n+k+j-1) + s(s+n-3),$$

so that we have

$$(\Delta_j - b(s; j)) \Big|_{V_j(k,s)} = k(n+k+2j-1).$$

The factorization formula (20) allows us to guess that the eigenvalue of $(T_j^+)^* T_j^+$ on $V_j(k, s)$ is

$$a(s; j)k(n+k+2j-1) = \frac{k(n+k+2j-1)(j-s+1)(n+j+s-2)}{(j+1)(n+2j-2)}. \quad (21)$$

We shall prove the claim that the eigenvalue of $(T_j^+)^* T_j^+$ on $V_j(k, s)$ coincides with (21) by induction for j . When $j = 0$, the claim is true because of $\Delta_0 = (T_0^+)^* T_0^+$. We suppose the claim holds for j . Then,

$$\ker T_j^+ \cong \bigoplus_{0 \leq s \leq j} V_j(0, s)$$

and $T_j^+ : V_j(k, s) \rightarrow V_{j+1}(k-1, s)$ is an isomorphism for any $k \geq 1$ and $s = 0, \dots, j$. For ϕ in $V_j(k, s)$,

$$\begin{aligned} & (j+2)(T_{j+1}^+)^* T_{j+1}^+ (T_j^+ \phi) \\ &= \left(\Delta_{j+1} + (n+j-2)(T_{j+1}^-)^* T_{j+1}^- - 2((j+1)(n+(j+1)-2)) \right) (T_j^+ \phi) \\ &= \left(\Delta_{j+1} + \frac{(j+1)(n+2j-2)}{n+2j} T_j^+ (T_j^+)^* - 2((j+1)(n+j-1)) \right) (T_j^+ \phi) \\ &= T_j^+ \Delta_j \phi + \frac{k(n+k+2j-1)(j-s+1)(n+j+s-2)}{n+2j} T_j^+ \phi \\ & \quad - 2((j+1)(n+j-1))(T_j^+ \phi) \\ &= \frac{(k-1)(n+k+2j)(j-s+2)(n+j+s-1)}{n+2j} T_j^+ \phi. \end{aligned}$$

Thus, the eigenvalues of $(T_{j+1}^+)^* T_{j+1}^+$ on $V_{j+1}(k-1, s)$ for any $k \geq 1$ and $s = 0, \dots, j$ coincide with (21). The cases of $V_{j+1}(k, j+1)$ for any $k \geq 0$ remain to be proved. Since T_j^+ is an overdetermined elliptic operator, the space of the sections $\Gamma(\mathrm{Sym}_0^{j+1})$ decomposes into the orthogonal direct sum of $\ker T_j^+$ and $\mathrm{Image} T_j^+$. From the above discussion, it holds that $\mathrm{Image} T_j^+$ is given by $\bigoplus_{0 \leq s \leq j} \mathbf{V}_{j+1}(s)$. Then, its orthogonal complement $\ker T_j^+$ is $\mathbf{V}_{j+1}(j+1) = \bigoplus_{k \geq 0} V_{j+1}(k, j+1)$. For ϕ in $V_{j+1}(k, j+1)$, by Weitzenböck formula, we have

$$(T_{j+1}^+)^* T_{j+1}^+ \phi = \frac{1}{j+2} (\Delta_{j+1} \phi - 2((j+1)(n+(j+1)-2))\phi) = \frac{k(n+k+2j+1)}{j+2} \phi,$$

and check that this gives (21) on $V_{j+1}(k, j+1)$. Thus, we have proved that (21) gives the eigenvalue of $(T_j^+)^* T_j^+$ on $V_j(k, s)$ for any j . We can also have the eigenvalue of the other operators by using Weitzenböck formulas in Proposition 4.3.

Theorem 5.1 *On the n -dimensional standard sphere S^n , the eigenvalues of $(T_j^+)^* T_j^+$, $(T_j^-)^* T_j^-$, $U_j^* U_j$ and Δ_j on $V_j(k, s)$ are given by*

$$\frac{k(n+k+2j-1)(j-s+1)(n+j+s-2)}{(j+1)(n+2j-2)}, \quad (22)$$

$$\frac{(j-s)(k+1)(n+k+2j-2)(n+j+s-3)}{(n+j-3)(n+2j-2)}, \quad (23)$$

$$\frac{s(k+j+1)(n+s-3)(n+k+j-2)}{(j+1)(n+j-3)}, \quad (24)$$

$$\text{and } (k+j)(k+j+n-1)+s(n+s-3), \quad (25)$$

respectively. The kernels and images of the operators are realized as $\text{SO}(n+1)$ -modules in $L^2(S^n, \text{Sym}_0^j)$ as follows:

$$\begin{aligned} \ker T_j^+ &\cong \bigoplus_{0 \leq s \leq j} V_j(0, s), & \ker T_j^- &\cong \mathbf{V}_j(j), & \ker U_j^- &\cong \mathbf{V}_j(0), \\ \text{Image } T_{j-1}^+ &\cong \bigoplus_{0 \leq s \leq j-1} \mathbf{V}_j(s), & \text{Image } T_{j+1}^- &\cong L^2(S^n, \text{Sym}_0^j) \ominus \ker T_j^+. \end{aligned}$$

We also know that

$$\ker T_j^+ \cap \ker T_j^- \cong V_j(0, j), \quad \ker T_j^+ \cap \ker U_j \cong V_j(0, 0).$$

As an interesting application to geometry, we shall discuss the space of Killing tensor fields on S^n from the viewpoint of representation theory. We consider the differential d on the symmetric tensor fields defined by

$$d : \Gamma(\text{Sym}^j) \ni K \mapsto dK := \sum_{i=1}^n e_i \cdot \nabla_{e_i} K \in \Gamma(\text{Sym}^{j+1}),$$

where $\{e_i\}_i$ is a local orthonormal frame for TM and $e_i \cdot$ denotes the symmetric tensor product by e_i . If dK is zero, then K is said to be a *Killing tensor field*. Since Killing tensor fields give the first integrals for geodesics, they play an important role in the physics literature, especially in relativity theory. We refer to [14] for general results of Killing and conformal Killing tensor fields in Riemannian geometry. Since the differential d is a derivation on $\bigoplus_j \Gamma(\text{Sym}^j)$, that is,

$$d(K \cdot K') = (dK) \cdot K' + K \cdot (dK') \quad \text{for } K, K' \text{ in } \bigoplus_j \Gamma(\text{Sym}^j),$$

the space $K(M)$ of the Killing tensor fields is a graded algebra, $K(M) = \bigoplus_j K^j(M)$, where $K^j(M)$ is the space of the Killing tensor fields with degree j . Let K be a symmetric tensor field with degree j . This K is decomposed as

$$K = K_0 + g \cdot K_1 + g^2 \cdot K_2 + \cdots + g^l \cdot K_l \quad (K_i \in \Gamma(\text{Sym}_0^{j-2i}), \quad 0 \leq i \leq l = [j/2])$$

with respect to (2), where K_0 is the trace-free part of K . Since the derivative d on $\Gamma(\text{Sym}_0^{j-2i})$ is realized as a linear combination of the generalized gradients T_{j-2i}^+ and T_{j-2i}^- , the tensor field dK_i has components only in $\Gamma(\text{Sym}_0^{j-2i+1})$ and $g \cdot \Gamma(\text{Sym}_0^{j-2i-1})$, and hence $dK_i = (dK_i)_0 + g \cdot (dK_i)_1$. Then, it is easily shown that K is a Killing tensor field if and only if $\{K_i\}_{0 \leq i \leq l}$ satisfy

$$(dK_0)_0 = 0, \quad (dK_0)_1 + g \cdot (dK_1)_0 = 0, \quad \dots, \quad (dK_{l-1})_1 + g \cdot (dK_l)_0 = 0, \quad (dK_l)_1 = 0. \quad (26)$$

Note that if K satisfies only the first condition $(dK_0)_0 = 0$, then K is said to be a *conformal Killing tensor field*. For a trace-free symmetric tensor $K = K_0$ with degree j , K is a trace-free Killing tensor (resp. trace-free conformal Killing tensor) if and only if K is in $\ker T_j^+ \cap \ker T_j^-$ (resp. in $\ker T_j^+$). An important observation is that there is a nonzero constant $c = c(j)$ such that $(dK_i)_1 + g \cdot (dK_{i+1})_0 = 0$ can be rewritten as $T_{j-2i}^-(K_i) = cT_{j-2i-2}^+(K_{i+1})$.

We consider Killing tensor fields on the standard sphere. Let $K = K_0 + g \cdot K_1 + \cdots$ be a *primitive Killing tensor field* with degree j , that is, K is in $K^j(S^n)$ and orthogonal to $g \cdot K^{j-2}(S^n)$ with respect to the natural L^2 -inner product on the space of the symmetric tensors $\Gamma(\text{Sym}^j)$:

$$K^j(S^n) = P^j(S^n) \oplus g \cdot K^{j-2}(S^n),$$

where $P^j(S^n)$ is the space of the primitive Killing tensor fields with degree j . Then, we may assume K_0 is in $V_j(0, s)$ because of $T_j^+(K_0) = 0$. When $s = j$, we know $T_j^-(K_0)$ is zero by Theorem 5.1 and hence $T_{j-2}^+(K_1)$ is zero by (26). Thus, we know K is the form of $K = K_0 + g \cdot \tilde{K}_1$ with $\tilde{K}_1$ in $K^{j-2}(S^n)$. Therefore, $K = K_0$ due to primitiveness of K . When $s = j - 1$, we know that $T_j^-(K_0) = cT_{j-2}^+(K_1)$ is not zero and K_1 has to be in a $\text{SO}(n+1)$ -module with the highest weight $(j, j-1, 0_{m-2})$ because $T_j^\pm$ is an invariant operator. Since $\Gamma(\text{Sym}_0^{j-2})$ does not include such a component, there is no Killing tensor field K with K_0 in $V_j(0, j-1)$. In the same manner, when K_0 is in $V_j(0, j-2i)$, the primitive Killing tensor K is a form of $K = K_0 + g \cdot K_1 + \cdots + g^i \cdot K_i$ with K_k in $V_{j-2k}(2k, j-2i) \cong V_j(0, j-2i)$ for $k = 0, \dots, i$, where each K_k is determined by K_0 through (26). We also know there is no Killing tensor field K with nonzero K_0 in $V_j(0, j-2i+1)$.

Proposition 5.2 ([26]) *Let $P^j(S^n)$ be the space of the primitive Killing tensors with degree j on the standard sphere. Then,*

$$P^j(S^n) \cong \bigoplus_{0 \leq i \leq [j/2]} V_j(0, j-2i) = \bigoplus_{0 \leq i \leq [j/2]} (j, j-2i, 0_{m-2}),$$

$$K^j(S^n) = \bigoplus_{0 \leq i \leq [j/2]} g^i \cdot P^{j-2i}(S^n).$$

6 Higher spin Dirac operators on spinor fields with differential forms

First we survey the analysis of generalized gradients on differential forms. Let (M, g) be an oriented Riemannian manifold and $\Lambda^j(T^*M)$ be the bundle of differential forms. Due to the star operator $*$: $\Lambda^j(T^*M) \rightarrow \Lambda^{n-j}(T^*M)$, we have to study only the case of $j \leq [n/2]$. As presented in [16, 24], there are three differential operators

$$\begin{aligned} C : \Omega^j(M) &= \Gamma(\Lambda^j(T^*M)) \rightarrow \Gamma(\Lambda^{j+1}(T^*M)), \\ d : \Omega^j(M) &\rightarrow \Omega^{j+1}(M), \quad d^* : \Omega^j(M) \rightarrow \Omega^{j-1}(M) \end{aligned}$$

satisfying

$$\Delta_j = \nabla^* \nabla + \frac{1}{2} R_{\Lambda^j} = d^* d + d d^*, \quad d d = 0, \quad d^* d^* = 0, \quad (27)$$

$$\nabla^* \nabla = C^* C + \frac{1}{j+1} d^* d + \frac{1}{n-j+1} d d^*, \quad (28)$$

where the highest weight of the bundle $\Lambda^{j+1}(T^*M)$ with respect to $SO(n)$ is

$$\begin{cases} (2, 1_{j-1}, 0_{m-1-j}) & \text{for } n = 2m - 1 \\ (2, 1_{j-1}, 0_{m-j}) & \text{for } n = 2m. \end{cases}$$

Because of the equation (27) and ellipticity of Δ_j , the Hodge–de Rham decomposition holds on a compact manifold (M, g) ,

$$\Omega^j(M) = H^j(M) \oplus d(\Omega^{j-1}(M)) \oplus d^*(\Omega^{j+1}(M)),$$

where $H^j(M)$ is the space of the harmonic j forms. Therefore, if we know the eigenvalues of the Laplacian Δ_j , then we have the eigenvalues of $d^* d$, $d d^*$. In addition, from the action of curvature $R_{\Lambda^j}/2$, we can get the eigenvalues of $C^* C$. For example, on a space of constant curvature $K = c$, the curvature $R_{\Lambda^j}/2$ acts by a constant $j(n-j)c$. We will calculate the eigenvalues of the operators on the standard sphere in the next section.

We move on to the higher spin Dirac operator on spinor fields coupled with differential forms. In other words, we shall study the case of the representation δ_j on W_{δ_j} with the highest weight

$$\begin{cases} ((3/2)_j, (1/2)_{m-j}) \oplus ((3/2)_j, (1/2)_{m-j-1}, -1/2) & \text{for } n = 2m, \\ ((3/2)_j, (1/2)_{m-1-j}) & \text{for } n = 2m - 1, \end{cases}$$

and the dimension $2^{[n/2]} \frac{n-2j+1}{n-j+1} \binom{n}{j}$, where δ_j splits into the sum of irreducible representation $\delta_j^\pm$ for $n = 2m$. These spinor fields are realized as sections of the tensor bundle $S_0 \otimes \Lambda^j(T^*M^c)$, that is, *spinor fields coupled with differential forms* (see Remark 6.3). We consider the vector bundle with fiber W_{δ_j} ,

$$E_j := \text{Spin}(M) \times_{\text{Spin}(n)} W_{\delta_j}.$$

There are four generalized gradients on $\Gamma(E_j)$ for $0 \leq j \leq [n/2]$,

$$\begin{aligned}
U_j : \Gamma(E_j) &\rightarrow \Gamma(E_{j,1}) && \text{the (first) twistor operator,} \\
\tilde{T}_j^+ : \Gamma(E_j) &\rightarrow \Gamma(E_{j+1}) && \text{the (second) twistor operator,} \\
\tilde{D}_j : \Gamma(E_j) &\rightarrow \Gamma(E_j) && \text{the higher spin Dirac operator,} \\
\tilde{T}_j^- : \Gamma(E_j) &\rightarrow \Gamma(E_{j-1}) && \text{the co-twistor operator,}
\end{aligned}$$

where $E_{j,1}$ is the vector bundle with the highest weight

$$\begin{cases} (5/2, (3/2)_{j-1}, (1/2)_{m-j}) \oplus (5/2, (3/2)_{j-1}, (1/2)_{m-j-1}, -1/2) & \text{for } n = 2m, \\ (5/2, (3/2)_{j-1}, (1/2)_{m-1-j}) & \text{for } n = 2m - 1. \end{cases}$$

Remark 6.1 When $j = 0$, there are two generalized gradients on E_0 ,

$$\tilde{T}_0^+ : \Gamma(E_0) \rightarrow \Gamma(E_1), \quad \tilde{D}_0 : \Gamma(E_0) \rightarrow \Gamma(E_0).$$

Then, we set $U_0 = 0$ and $\tilde{T}_0^- = 0$. When $n = 2m$ and $j = m$, there are only two generalized gradients on the bundle $E_m = E_m^+ \oplus E_m^-$ with the highest weight $((3/2)_m) \oplus ((3/2)_{m-1}, -3/2)$,

$$U_m : \Gamma(E_m) \rightarrow \Gamma(E_{m,1}), \quad \tilde{T}_m^- : \Gamma(E_m) \rightarrow \Gamma(E_{m-1}).$$

Note that the higher spin Dirac operator does not exist. Then, we set $\tilde{D}_m = 0$ and $\tilde{T}_m^+ = 0$ for $n = 2m$. When $n = 2m - 1$ and $j = m - 1$, there are three generalized gradients on the bundle E_{m-1} with the highest weight $((3/2)_{m-1})$,

$$\begin{aligned}
U_{m-1} : \Gamma(E_{m-1}) &\rightarrow \Gamma(E_{m-1,1}), && \tilde{T}_{m-1}^- : \Gamma(E_{m-1}) \rightarrow \Gamma(E_{m-2}), \\
\tilde{D}_{m-1} : \Gamma(E_{m-1}) &\rightarrow \Gamma(E_{m-1}).
\end{aligned}$$

Then, we set $\tilde{T}_{m-1}^+ = 0$ for $n = 2m - 1$.

From now on, we assume that (M, g) is a space of constant curvature $K = c$ with a spin structure. We calculate the Weitzenböck formulas on $\Gamma(E_j)$ from [16],

$$\begin{aligned}
\Delta_j &= \nabla^* \nabla + \frac{1}{2} R_{\delta_j} = \nabla^* \nabla + (j(n-j+1) + \frac{n(n-1)}{8})c \\
&= \frac{(n+2)(n-2j)}{n-2j+2} \tilde{D}_j^2 + \frac{4(n-2j+1)(n-j+2)}{(n-2j+3)(n-2j+2)} (\tilde{T}_j^+)^* \tilde{T}_j^- \\
&\quad + (j(n-j) - \frac{n(n-1)}{8})c \\
&= -\frac{4(n-2j+1)(j+1)}{(n-2j-1)(n-2j)} (\tilde{T}_j^+)^* \tilde{T}_j^+ + \frac{(n-2j+2)(n+2)}{n-2j} \tilde{D}_j^2 \\
&\quad + ((j-1)(n-j+1) - \frac{n(n-1)}{8})c \\
&= \frac{(n-j+2)(n+2)}{(n-j+1)(n+1)} U_j^* U_j + \frac{(n-2j)(n-2j+1)}{(n-2j-1)(n-j+1)} (\tilde{T}_j^+)^* \tilde{T}_j^+ \\
&\quad + (j(n-j+2) + \frac{n(n+1)}{8})c.
\end{aligned} \tag{29}$$

If denominators in the above equation are zero, then we ignore the equation. For example, when $n = 2m$ and $j = m$, we do not consider the third equality. From formulas in [17] and the equation (18) in Lemma 4.2, we also have

$$\frac{1}{\sqrt{n-2j}} \tilde{D}_j \tilde{T}_{j-1}^+ = \frac{1}{\sqrt{n-2j+4}} \tilde{T}_{j-1}^+ \tilde{D}_{j-1}, \quad \tilde{T}_{j+1}^+ \tilde{T}_j^+ = 0, \quad \tilde{T}_{j-1}^- \tilde{T}_j^- = 0,$$

$$(\tilde{T}_{j-1}^+)^* \tilde{T}_{j-1}^+ = \frac{(n-2j+1)(n-j+2)}{j(n-2j+3)} \tilde{T}_j^- (\tilde{T}_j^-)^*.$$

We normalize these generalized gradients

$$D_j := \sqrt{\frac{(n+2)(n-2j)}{n-2j+2}} \tilde{D}_j,$$

$$T_j^- := 2 \sqrt{\frac{(n-2j+1)(n-j+2)}{(n-2j+3)(n-2j+2)}} \tilde{T}_j^-, \quad T_j^+ := 2 \sqrt{\frac{j+1}{n-2j}} \tilde{T}_j^+.$$

to compare with other articles as Remark 6.3. We summarize the calculations so far in the next theorem.

Theorem 6.2 *Let (M, g) be a space of constant curvature $K = c$ with a spin structure. The generalized gradients on $\Gamma(E_j)$ satisfy*

$$\begin{aligned} \Delta_j &= D_j^2 + (T_j^-)^* T_j^- + (j(n-j) - \frac{n(n-1)}{8})c \\ &= -\frac{n-2j+1}{n-2j-1} (T_j^+)^* T_j^+ + \frac{(n-2j+2)^2}{(n-2j)^2} D_j^2 + ((j-1)(n-j+1) - \frac{n(n-1)}{8})c, \\ D_j T_{j-1}^+ &= \frac{n-2j}{n-2j+2} T_{j-1}^+ D_{j-1}, \quad T_{j+1}^+ T_j^+ = 0, \quad T_{j-1}^- T_j^- = 0, \quad T_{j-1}^+ = (T_j^-)^*. \end{aligned}$$

Remark 6.3 In [8], these operators are realized as components of the twisted Dirac operator $D(j)$ on $S_0 \otimes \mathcal{A}^i(T^*M^c)$ up to a constant multiple. Indeed, the set of the highest weights of irreducible summands in $S_0 \otimes \mathcal{A}^i(T^*M^c)$ is given by

$$\{\lambda = (1_j, 0_{m-j}) + (\epsilon_1/2, \dots, \epsilon_m/2) \mid \lambda \text{ satisfies the dominant condition, } \epsilon_i \in \{\pm 1\}\}$$

because each nonzero weight $(\epsilon_1/2, \dots, \epsilon_m/2)$ of the spinor representation has multiplicity one. Then, we decompose the tensor bundle into the sum of irreducible bundles,

$$S_0 \otimes \mathcal{A}^i(T^*M^c) = E_j \oplus E_{j-1} \oplus \dots \oplus E_0.$$

Along this bundle decomposition, the twisted Dirac operator $D(j)$ is represented as

$$D(j) = \begin{pmatrix} D(j)_j & T(j)_{j-1}^+ & 0 & 0 & \dots & 0 \\ T(j)_j^- & D(j)_{j-1} & T(j)_{j-2}^+ & 0 & \dots & 0 \\ 0 & T(j)_{j-1}^- & D(j)_{j-2} & T(j)_{j-3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & D(j)_1 & T(j)_0^+ \\ 0 & 0 & 0 & \dots & T(j)_1^- & D(j)_0 \end{pmatrix}$$

where $D(j)_k$ (resp. $T(j)_k^\pm$) is a nonzero constant multiple of $\tilde{D}_k : \Gamma(E_k) \rightarrow \Gamma(E_k)$ (resp. $\tilde{T}_k^\pm : \Gamma(E_k) \rightarrow \Gamma(E_{k\pm 1})$) for $k = 0, \dots, j$. Considering the square of $D(j)$, we have Weitzenböck formulas in Theorem 6.2 and check that $D(j)_j$, $T(j)_j^-$ and $T(j)_{j-1}^+$ coincide with our D_j , T_j^- and T_{j-1}^+ , respectively.

As applications of Theorem 6.2, we obtain a factorization formula and the Hodge–de Rham decomposition with respect to the operators on $\Gamma(E_j)$.

Corollary 6.4 (Factorization formula) *We consider a space (M, g) of constant curvature $K = c$ with a spin structure. On $\Gamma(E_j)$ for $1 \leq j \leq [n/2] - 1$, we have*

$$0 = \left(D_j^2 - \frac{(n-2j)^2}{(n-2j+2)^2} (\Delta_j - ((j-1)(n-j+1) - \frac{n(n-1)}{8})c) \right) \\ \times \left(D_j^2 - (\Delta_j - (j(n-j) - \frac{n(n-1)}{8})c) \right).$$

Proof The corollary follows from $(T_j^+)^* T_j^+ (T_j^-)^* T_j^- = (T_j^+)^* T_j^+ T_{j-1}^+ T_j^- = 0$. $\square$

Eliminating D_j^2 from two equations for Δ_j in Theorem 6.2, we get

$$\Delta_j = \frac{(n-2j)^2}{4(n-2j-1)} (T_j^+)^* T_j^+ + \frac{(n-2j+2)^2}{4(n-2j+1)} (T_j^-)^* T_j^- + \frac{n(n+1)}{8} c.$$

Taking the usual Hodge–de Rham decomposition as a model, we can show the following L^2 -orthogonal decomposition.

Proposition 6.5 *On a compact spin manifold (M, g) of constant sectional curvature $K = c$, we have the Hodge–de Rham decomposition for spinor fields coupled with differential forms,*

$$\Gamma(E_j) = T_{j-1}^+(\Gamma(E_{j-1})) \oplus T_{j+1}^-(\Gamma(E_{j+1})) \oplus \ker(\Delta_j - \frac{n(n+1)}{8} c), \\ \ker T_j^+ = T_{j-1}^+(\Gamma(E_{j-1})) \oplus \ker(\Delta_j - \frac{n(n+1)}{8} c), \\ \ker T_j^- = T_{j+1}^-(\Gamma(E_{j+1})) \oplus \ker(\Delta_j - \frac{n(n+1)}{8} c).$$

Here, we exclude the case of $n = 2m - 1$ and $j = m - 1$. When $j = 0$, the first equation means

$$\Gamma(E_0) = T_1^-(\Gamma(E_1)) \oplus \ker(\Delta_0 - \frac{n(n+1)}{8} c).$$

When $n = 2m$ and $j = m$, it does

$$\Gamma(E_m) = T_{m-1}^+(\Gamma(E_{m-1})) \oplus \ker(\Delta_m - \frac{n(n+1)}{8} c).$$

Note that $\ker(\Delta_j - \frac{n(n+1)}{8} c)$ is zero on a compact spin manifold of positive constant curvature for $j \geq 1$ because we have $\Delta_j - \frac{n(n+1)}{8} c \geq j(n-j+2)c$ from the last equality in (29).

7 Harmonic analysis for spinor fields with differential forms on S^n

We review a well-known result for the eigenvalues of the operators $\{\Delta_j, dd^*, d^*d, C^*C\}$ acting on differential forms on the standard sphere, [13, 20]. Let $V_j(k)$ be an irreducible $SO(n+1)$ -module with the highest weight

$$(k+1, 1_{j-1}, 0_{m-j}) = (k, 0_{m-1}) + (1_j, 0_{m-j})$$

for $0 \leq j \leq m = \lceil \frac{n+1}{2} \rceil$. Then, Frobenius reciprocity and the branching rule give us the decomposition of $L^2(S^n, \Lambda^j(T^*M^c))$ as an $SO(n+1)$ -module.

1. For $j = 0$,

$$L^2(S^n, \Lambda^0(T^*M^c)) \cong \bigoplus_{k \geq 0} V_1(k) \oplus V_0(0), \quad \ker d = H^0(S^n) \cong V_0(0).$$

2. For $n = 2m$ and $j = m$,

$$L^2(S^{2m}, \Lambda^m(T^*M^c)) \cong \bigoplus_{k \geq 0} 2V_m(k), \quad \ker d^* \cong \bigoplus_{k \geq 0} V_m(k), \quad \ker d \cong \bigoplus_{k \geq 0} V_m(k).$$

3. For $n = 2m-1$ and $j = m-1$,

$$L^2(S^{2m-1}, \Lambda^{m-1}(T^*M^c)) \cong \bigoplus_{k \geq 0} V_m(k) \oplus \bigoplus_{k \geq 0} V_{m-1}(k),$$

$$V_m(k) := V_m^+(k) \oplus V_m^-(k), \quad \ker d^* \cong \bigoplus_{k \geq 0} V_m(k), \quad \ker d \cong \bigoplus_{k \geq 0} V_{m-1}(k),$$

where $V_m^\pm(k)$ is an irreducible $SO(2m)$ -module with the highest weight $(k+1, 1_{m-2}, \pm 1)$.

4. Otherwise,

$$L^2(S^n, \Lambda^j(T^*M^c)) \cong \bigoplus_{k \geq 0} V_{j+1}(k) \oplus \bigoplus_{k \geq 0} V_j(k),$$

$$\ker d^* \cong \bigoplus_{k \geq 0} V_{j+1}(k), \quad \ker d \cong \bigoplus_{k \geq 0} V_j(k).$$

We calculate the eigenvalue of Δ_j on $V_j(k)$ by Freudenthal's formula. Then, the next proposition follows from (27) and (28).

Proposition 7.1 ([13, 20]) *The eigenvalues of the operators $\{\Delta_j, dd^*, d^*d, C^*C\}$ on $V_{j+1}(k)$ and $V_j(k)$ are given as follows:*

	on $V_{j+1}(k) \subset \ker d^*$	on $V_j(k) \subset \ker d$
Δ_j	$(k+j+1)(n+k-j)$	$(k+j)(n-j+k+1)$
dd^*	0	$(k+j)(n-j+k+1)$
d^*d	$(k+j+1)(n+k-j)$	0
C^*C	$\frac{j}{j+1}k(n+k+1)$	$\frac{n-j}{n-j+1}k(n+k+1)$

In particular, the space of the Killing j -forms $\ker C \cap \ker d^*$ is isomorphic to $V_{j+1}(0)$, and the space of the co-Killing j -forms $\ker C \cap \ker d$ is isomorphic to $V_j(0)$.

The (co-) Killing forms have interesting geometric meaning like Killing tensor fields [24].

We shall study harmonic analysis for the spinor fields coupled with differential forms on the standard sphere. The space $L^2(S^n, E_j)$ is decomposed as $\text{Spin}(n+1)$ -module.

Proposition 7.2 1. In the case of $n = 2m$, we denote by $V_j(k)'$ an irreducible $\text{Spin}(2m+1)$ -module with the highest weight

$$(k + 3/2, (3/2)_{j-1}, (1/2)_{m-j}) = (k + 1/2, (1/2)_{m-1}) + (1_j, 0_{m-j})$$

for $k = 0, 1, 2, \dots$. Then,

1. For $j = 0$,

$$L^2(S^{2m}, E_0^\pm) \cong \bigoplus_{k \geq 0} V_1(k)' \oplus V_0(0)', \quad L^2(S^{2m}, E_0) \cong \bigoplus_{k \geq 0} 2V_1(k)' \oplus 2V_0(0)',$$

$$\ker T_0^+ \cong 2V_0(0)'.$$

2. For $1 \leq j \leq m-1$,

$$L^2(S^{2m}, E_j^\pm) \cong \bigoplus_{k \geq 0} V_{j+1}(k)' \oplus \bigoplus_{k \geq 0} V_j(k)',$$

$$L^2(S^{2m}, E_j) \cong \bigoplus_{k \geq 0} 2V_{j+1}(k)' \oplus \bigoplus_{k \geq 0} 2V_j(k)',$$

$$\ker T_j^- \cong \bigoplus_{k \geq 0} 2V_{j+1}(k)', \quad \ker T_j^+ \cong \bigoplus_{k \geq 0} 2V_j(k)'.$$

3. For $j = m$,

$$L^2(S^{2m}, E_m^\pm) \cong \bigoplus_{k \geq 0} V_m(k)', \quad L^2(S^{2m}, E_m) \cong \bigoplus_{k \geq 0} 2V_m(k)', \quad \ker T_m^- = \{0\}.$$

In the case of $n = 2m-1$, we denote by $V_j^\pm(k)'$ an irreducible $\text{Spin}(2m)$ -module with the highest weight

$$(k + 3/2, (3/2)_{j-1}, (1/2)_{m-j-1}, \pm 1/2) = (k + 1/2, (1/2)_{m-1}, \pm 1/2) + (1_j, 0_{m-j})$$

for $k = 0, 1, 2, \dots$, and put $V_j(k)' = V_j^+(k) \oplus V_j^-(k)$. Then,

1. For $j = 0$,

$$L^2(S^{2m-1}, E_0) \cong \bigoplus_{k \geq 0} V_1(k)' \oplus V_0(0)', \quad \ker T_0^+ \cong V_0(0)'.$$

2. For $1 \leq j \leq m-2$,

$$L^2(S^{2m-1}, E_j) \cong \bigoplus_{k \geq 0} V_{j+1}(k)' \oplus \bigoplus_{k \geq 0} V_j(k)',$$

$$\ker T_j^- \cong \bigoplus_{k \geq 0} V_{j+1}(k)', \quad \ker T_j^+ \cong \bigoplus_{k \geq 0} V_j(k)'.$$

3. For $j = m - 1$,

$$L^2(S^{2m-1}, E_{m-1}) \cong \bigoplus_{k \geq 0} V_m(k)' \oplus \bigoplus_{k \geq 0} V_{m-1}(k)', \quad \ker T_{m-1}^- \cong \bigoplus_{k \geq 0} V_m(k)'.$$

We know already how the sections of $\{E_j\}_j$ relate to each other through operators $\{T_j^\pm\}_j$. In fact, $\ker T_j^+ = \text{Image } T_{j-1}^+$ and $\ker T_j^- = \text{Image } T_{j+1}^-$. Then, all we have to do is to calculate the eigenvalues of the operators on $\Gamma(E_j)$. The eigenvalues of D_j^2 were calculated in [8] by using a method from parabolic geometry in [7]. We calculate them only from the eigenvalues of Δ_j and Weitzenböck formulas.

Theorem 7.3 *The eigenvalues of D_j^2 , $(T_j^-)^* T_j^-$, $(T_j^+)^* T_j^+$ and $U_j^* U_j$ on $L^2(S^n, E_j)$ are given as follows:*

	on $V_{j+1}(k)' \subset \ker T_j^-$	on $V_j(k)' \subset \ker T_j^+$
Δ_j	$(k + j + 1)(n - j + k + 1) + \frac{n(n+1)}{8}$	$(k + j)(n - j + k + 2) + \frac{n(n+1)}{8}$
D_j^2	$\left(\frac{n}{2} + k + 1\right)^2$	$\frac{(n-2j)^2}{(n-2j+2)^2} \left(\frac{n}{2} + k + 1\right)^2$
$(T_j^-)^* T_j^-$	0	$\frac{4(n-2j+1)}{(n-2j+2)^2} (k + j)(n - j + k + 2)$
$(T_j^+)^* T_j^+$	$\frac{4(n-2j-1)}{(n-2j)^2} (n - j + k + 1)(k + j + 1)$	0
$U_j^* U_j$	$\frac{(n+1)j}{(n+2)(j+1)} k(n + k + 2)$	$\frac{(n+1)(n-j+1)}{(n+2)(n-j+2)} k(n + k + 2)$

In particular, for $n = 2m - 1$ (resp. $n = 2m$), $\ker U_j \cap \ker T_j^- \cong V_{j+1}(0)'$ (resp. $2V_{j+1}(0)'$) and $\ker U_j \cap \ker T_j^+ \cong V_j(0)'$ (resp. $2V_j(0)'$).

A Appendix: calculation

In this appendix, we give the calculation of the eigenvalues of Casimir elements and Weitzenböck formulas needed in Sect. 2. For more detail, see [16]. Let (π_j, W_j) be an irreducible representation with the highest weight $(j + 1/2, (1/2)_{[n/2]-1})$. The conformal weights $\{w(\lambda)\}_\lambda$ and the shifted conformal weights $\{\widehat{w}(\lambda) = w(\lambda) + \frac{n-1}{2}\}_\lambda$ associated with the irreducible summand of $W_j \otimes \mathbb{C}^n$ are given by

$$\begin{aligned} w(\pi_{j+1}) &:= w(\pi_{j+1}; \pi_j) = j + 1/2, & \widehat{w}(\pi_{j+1}) &= j + n/2, \\ w(\pi_{j,1}) &:= w(\pi_{j,1}; \pi_j) = -1/2, & \widehat{w}(\pi_{j,1}) &= n/2 - 1, \\ w(\pi_j) &:= w(\pi_j; \pi_j) = -n/2 + 1/2, & \widehat{w}(\pi_j) &= 0, \\ w(\pi_{j-1}) &:= w(\pi_{j-1}; \pi_j) = -n - j + 3/2, & \widehat{w}(\pi_{j-1}) &= 1 - j - n/2. \end{aligned}$$

Then, we can easily calculate the relative dimensions,

$$\begin{aligned} \frac{\dim W_{j+1}}{\dim W_j} &= \frac{n + j - 1}{j + 1}, & \frac{\dim W_{j,1}}{\dim W_j} &= \frac{(n - 3)(n + j - 1)j}{(n + j - 2)(j + 1)}, \\ \frac{\dim W_j}{\dim W_j} &= 1, & \frac{\dim W_{j-1}}{\dim W_j} &= \frac{j}{n + j - 2}. \end{aligned}$$

The Casimir element c_k with order k is an $\mathrm{SO}(n)$ -invariant element in the enveloping algebra $U(\mathfrak{so}(n))$ given by

$$c_k = \sum_{1 \leq i_1, i_2, \dots, i_k \leq n} e_{i_1 i_2} e_{i_2 i_3} \dots e_{i_k i_1},$$

where $\{e_{ij} = e_i \wedge e_j\}_{i < j}$ is a standard basis for $\mathfrak{so}(n)$. Shifting e_{ij} to $\hat{e}_{ij} = e_{ij} + \frac{n-1}{2}$, we define the shifted Casimir element $\hat{c}_k = \sum \hat{e}_{i_1 i_2} \hat{e}_{i_2 i_3} \dots \hat{e}_{i_k i_1}$. The eigenvalue of c_k on W_j is computed by the conformal weights and the relative dimensions.

$$\pi_j(c_k) = \sum_{\lambda} w(\lambda)^k \frac{\dim W_{\lambda}}{\dim W_j}, \quad \pi_j(\hat{c}_k) = \sum_{\lambda} \hat{w}(\lambda)^k \frac{\dim W_{\lambda}}{\dim W_j},$$

where λ runs among π_{j+1} , $\pi_{j,1}$, π_j and π_{j-1} . Since the number of the generalized gradients on S_j is four, there are two independent Weitzenböck formulas. In our case, we have the following two independent formulas on S_j . One is

$$-\frac{1}{2}R_j^1 = w(\pi_{j+1})(\tilde{T}_j^+)^* \tilde{T}_j^+ + w(\pi_{j,1})U_j^* U_j + w(\pi_j)\tilde{D}_j^2 + w(\pi_{j-1})(\tilde{T}_j^-)^* \tilde{T}_j^-.$$

The second one is complicated,

$$\hat{R}_j^4 = a(\pi_{j+1})(\tilde{T}_j^+)^* \tilde{T}_j^+ + a(\pi_{j,1})U_j^* U_j + a(\pi_j)\tilde{D}_j^2 + a(\pi_{j-1})(\tilde{T}_j^-)^* \tilde{T}_j^-,$$

where

$$a(\lambda) = \sum_{p=0}^3 \pi_j(\hat{c}_{3-p})(-\hat{w}(\lambda))^p, \quad \text{for } \lambda = \pi_{j+1}, \pi_{j,1}, \pi_j, \pi_{j-1}.$$

Furthermore, the curvature actions R_j^1 and $\hat{R}_j^4$ on the standard sphere are

$$R_j^1 = \pi_j(c_2), \quad \hat{R}_j^4 = \pi_j(\hat{c}_5 - \frac{n-1}{2}\hat{c}_4).$$

On the other hand, it follows from the definition of the generalized gradients that

$$\nabla^* \nabla = (\tilde{T}_j^+)^* \tilde{T}_j^+ + U_j^* U_j + \tilde{D}_j^2 + (\tilde{T}_j^-)^* \tilde{T}_j^-.$$

Deleting two generalized gradients from the above equation by using two Weitzenböck formulas, we obtain (9) and (10).

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