Relative non-pluripolar product of currents

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Abstract

Let X be a compact Kähler manifold. Let $T_1, \dots, T_m$ be closed positive currents of bi-degree $(1, 1)$ on X and T an arbitrary closed positive current on X . We introduce the *non-pluripolar product relative to T* of $T_1, \dots, T_m$. We recover the well-known non-pluripolar product of $T_1, \dots, T_m$ when T is the current of integration along X . Our main results are a monotonicity property of relative non-pluripolar products, a necessary condition for currents to be of relative full mass intersection in terms of Lelong numbers, and the convexity of weighted classes of currents of relative full mass intersection. The former two results are new even when T is the current of integration along X .

Keywords Non-pluripolar product · Quasi-continuity · Weighted class · Monge–Ampère operator · Full mass intersection

Mathematics Subject Classification 32U15 · 32U25 · 32U40 · 32U20 · 32Q15

1 Introduction

The problem of defining the intersection of closed positive currents on a complex manifold is a central question in the pluripotential theory. This question is already important for currents of bi-degree $(1, 1)$ with deep applications to complex geometry as well as complex dynamics; see [1, 11, 15, 16, 18] and references therein for an introduction to the subject.

Let X be an arbitrary complex manifold. Let $T_1, \dots, T_m$ be closed positive currents of bi-degree $(1, 1)$ on X . In [2, 4, 17], the non-pluripolar product $\langle T_1 \wedge \dots \wedge T_m \rangle$ of $T_1, \dots, T_m$ was defined by using the quasi-continuity of plurisubharmonic (psh for short) functions with respect to capacity. Since then, this notion has played an important role in complex geometry. We refer to [7–10] for recent developments.

In this paper, our first goal is to generalize the notion of non-pluripolar product to a more general natural setting. Given a closed positive (p, p) -current T on X , we introduce the *non-pluripolar product relative to T* of $T_1, \dots, T_m$ which we denote by $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$. We recover the above-mentioned non-pluripolar product when T is

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the current of integration along X (see also Remark 3.8); and if $T_1, \dots, T_m$ are of locally bounded potentials, then $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ is nothing but the classical intersection of $T_1, \dots, T_m, T$. When well defined, $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ is a closed positive $(p+m, p+m)$ -current on X . As in [4], the proof of the last fact follows from ideas in [21]. The product $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ is symmetric, homogeneous and sub-additive in $T_1, \dots, T_m$. One can consider the last product as a sort of intersection of $T_1, \dots, T_m, T$.

The main ingredient in the construction of relative non-pluripolar products is a (*uniform*) *quasi-continuity* property of *bounded* psh functions with respect to the capacity associated to closed positive currents (Theorem 2.4). This quasi-continuity property is stronger than the usual one. We also need to establish some convergence properties of mixed Monge–Ampère operators which are of independent interest.

Consider now the case where X is a compact Kähler manifold. As one can expect, the relative non-pluripolar products are always well defined in this setting. For any closed positive current R in X , we denote by $\{R\}$ the cohomology class of R . For cohomology classes α, β in $H^{p,p}(X, \mathbb{R})$, we write $\alpha \leq \beta$ if $(\beta - \alpha)$ can be represented by a closed positive current. The following result gives a monotonicity property of relative non-pluripolar products.

Theorem 1.1 (Theorem 4.4) *Let X be a compact Kähler manifold and $T_1, \dots, T_m, T$ closed positive currents on X such that T_j is of bi-degree $(1, 1)$ for $1 \leq j \leq m$. Let T'_j be closed positive $(1, 1)$ -current in the cohomology class of T_j on X such that T'_j is less singular than T_j for $1 \leq j \leq m$. Then we have*

$$\{\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle\} \leq \{\langle T'_1 \wedge \dots \wedge T'_m \wedge T \rangle\}.$$

Note that even when T is the current of integration along X , this result is new. In the last case, the above result was conjectured in [4] and proved there for T_j of potentials locally bounded outside a closed complete pluripolar set. The last condition was relaxed in [9, 22] but it was required there that $m = \dim X$ (always for T to be the current of integration along X). The proofs of these results presented there do not extend to our setting. A key ingredient in our proof is Theorem 2.6 (and Remark 2.7) giving a generalization of well-known convergence properties of Monge–Ampère operators. To prove Theorem 2.6, we will need the strong quasi-continuity of *bounded* psh functions mentioned above. When T is of bi-degree $(1, 1)$, we actually have a stronger monotonicity property, see Remark 4.5.

Theorem 1.1 allows us to define the notion of *full mass intersection relative to T* for currents $T_1, \dots, T_m$ as in [4], see Definition 4.7. Our next goal is to study currents of relative full mass intersection. We will prove that the relative non-pluripolar products of currents with full mass intersection are continuous under decreasing or increasing sequences (Theorem 4.8).

Now we concentrate on the case where the cohomology classes of $T_1, \dots, T_m$ are Kähler. In this case, $T_1, \dots, T_m$ are of full mass intersection relative to T if and only if $\{\langle \bigwedge_{j=1}^m T_j \wedge T \rangle\}$ is equal to $\{\langle \bigwedge_{j=1}^m \theta_j \wedge T \rangle\}$, where θ_j is a closed smooth $(1, 1)$ -form cohomologous to T_j for $1 \leq j \leq m$. Our next main result explains why having positive Lelong numbers is an obstruction for currents to be of full mass intersection.

Theorem 1.2 (Theorem 4.11) *Let X be a compact Kähler manifold. Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents on X such that the cohomology class of T_j is Kähler for $1 \leq j \leq m$ and T a closed positive (p, p) -current on X with $p+m \leq n$. Let V be an irreducible analytic subset in X such that the generic Lelong numbers of $T_1, \dots, T_m, T$ along V are strictly*

positive. Assume $T_1, \dots, T_m$ are of full mass intersection relative to T . Then, we have $\dim V < n - p - m$.

Recall that the generic Lelong number of T along V is the smallest value among the Lelong numbers of T at points in V . When $p + m = n$, the above theorem says that V is empty. To get motivated about Theorem 1.2, let us consider the case where $p + m = n$ and the current T_j has a potential which is locally bounded outside a point x_0 in X . In this case, $T_1 \wedge \dots \wedge T_m \wedge T$ is well defined in the sense given in [1, 11, 16] and we can see that $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle = T_1 \wedge \dots \wedge T_m \wedge T$ on $X \setminus \{x_0\}$. The latter current has strictly positive mass on x_0 by [11, Corollary 7.9]. Hence, $T_1, \dots, T_m$ cannot be of full mass intersection relative to T . By considering T_j to be suitable currents with analytic singularities (for example when X is projective) and T is the current of integration along X , one can see that the conclusion of Theorem 1.2 is optimal.

When $m = n$ and $T_1 = \dots = T_m$, Theorem 1.2 was proved in [17]. Their proof uses comparison principle and hence does not apply to our setting because $T_1, \dots, T_m$ are different in general. When the class of T_j is not Kähler, the above theorem no longer holds because currents with minimal singularities in a big and non-nef cohomology class always have positive Lelong numbers at some points in X (see [3]). The proof of Theorem 1.2, which is Theorem 4.11 below, uses Theorem 1.1.

Let $\mathcal{W}^-$ be the set of convex increasing functions $\chi : \mathbb{R} \rightarrow \mathbb{R}$ with $\chi(-\infty) = -\infty$. A function in $\mathcal{W}^-$ is called a (convex) weight. We will define the notion of having full mass intersection relative to T with weight χ and the weighted class of these currents, see Definition 5.3. When T is the current of integration along X and $T_1, \dots, T_m$ are all equal, the notion of weighted class (or pluricomplex energy) was introduced in [6] in the local context. Analogous global notions were later studied in [4, 17] and in other subsequent works. Here is our main result in this part.

Theorem 1.3 *Let $X, T_1, \dots, T_m, T$ be as in Theorem 1.2. Let $\chi \in \mathcal{W}^-$. The following three assertions hold:*

- (i) *If $T_j, \dots, T_j$ (m times T_j) are of full mass intersection relative to T for $1 \leq j \leq m$, then $T_1, \dots, T_m$ are also of full mass intersection relative to T .*
- (ii) *Let T'_j be a closed positive $(1, 1)$ -current whose cohomology class is Kähler for $1 \leq j \leq m$. Assume that T'_j is less singular than T_j and $T_1, \dots, T_m$ are of full mass intersection relative to T with weight χ . Then $T'_1, \dots, T'_m$ are also of full mass intersection relative to T with weight χ .*
- (iii) *If $T_j, \dots, T_j$ (m times T_j) are of full mass intersection relative to T with weight χ for $1 \leq j \leq m$, then $T_1, \dots, T_m$ are also of full mass intersection relative to T with weight χ .*

Theorem 1.3 is a combination of Theorems 5.1, 5.8 and 5.9 below. Property (i) – (ii) of Theorem 1.3 was proved in [17] in the case where T is the current of integration along X and $T_1, \dots, T_m$ are equal and $T'_1, \dots, T'_m$ are equal. When T is the current of integration along X , Property (iii) of Theorem 1.3 was proved in [17] for $\chi(t) = -(t)^r$ with $0 < r \leq 1$ (see also the last paper and [6] for the case where $r \geq 1$) and it was proved in [7] for every $\chi \in \mathcal{W}^-$ under an extra condition that $m = n$. The proof given in [7] used Monge–Ampère equations, hence, does not extend to our setting.

We have some comments about the proof of Theorem 1.3. Firstly, arguments similar to those in [17] are sufficient to obtain (ii) of Theorem 1.3. However, as pointed out above, to prove (iii), we need new arguments. We prove (i), (iii) using the same approach. It is possible that we can use the comparison principle to get (i) as in the case where T is the current of integration along X . But we choose to invoke a more flexible idea which is also applicable to get (iii). The idea is to combine a monotonicity property and generalizations of results in [2] about plurifine topology properties of Monge–Ampère operators. Here, the monotonicity mentioned in the last sentence is Theorem 1.1 in case of proving (i) and (ii) in case of proving (iii).

The paper is organized as follows. In Sect. 2, we establish the uniform strong quasi-continuity of bounded psh functions and derive from it some consequences concerning the convergence of Monge–Ampère operators. In Sect. 3, we define the notion of relative non-pluripolar products and prove its basic properties. In Sect. 4, we prove the monotonicity of relative non-pluripolar products and explain Lelong number obstruction for currents having full mass intersection. Section 5 is devoted to the weighted class of currents with relative full mass intersection.

Notation and convention. For a closed positive current T on a compact complex manifold X , we denote by $\{T\}$ the cohomology class of T and $v(T, x)$ denotes the Lelong number of T at x . Recall that the wedge product on closed smooth forms induces the cup-product on their de Rham cohomology classes. We use the same notation $\} \wedge \varepsilon$ to denote these two products.

Recall $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ and $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. We use $\gtrsim, \lesssim$ to denote $\geq, \leq$ modulo some multiplicative constant independent of parameters in question. For a Borel set $A \subset X$, we denote by $\mathbf{1}_A$ the characteristic function of A , that means $\mathbf{1}_A$ is equal to 1 on A and 0 elsewhere. For every current R of order 0 on X , we denote by $\|R\|_A$ the mass of R on A .

2 Quasi-continuity for bounded psh functions

In this section, we prove a quasi-continuity for bounded plurisubharmonic (psh for short) functions which is stronger than the one for general psh functions. This property is the key to define our generalization of non-pluripolar products.

Let U be an open subset of $\mathbb{C}^n$. Let K be a Borel subset of U . The *capacity* $\text{cap}(K, U)$ of K in U , which was introduced in [1], is given by

$$\text{cap}(K, U) := \sup \left\{ \int_K (dd^c u)^n : u \text{ is psh on } U \text{ and } 0 \leq u \leq 1 \right\}.$$

For a closed positive current T of bi-dimension (m, m) on U ($0 \leq m \leq n$), we define

$$\text{cap}_T(K, U) := \sup \left\{ \int_K (dd^c u)^m \wedge T : u \text{ is psh on } U \text{ and } 0 \leq u \leq 1 \right\}.$$

We say that a sequence of functions $(u_k)_{k \in \mathbb{N}}$ converges to u with respect to the capacity cap_T (relative in U) if for any constant $\varepsilon > 0$ and $K \Subset U$, we have $\text{cap}_T(\{|u_k - u| \geq \varepsilon\} \cap K, U) \rightarrow 0$ as $k \rightarrow \infty$. The notion of relative cap_T was introduced in [18, 23]. We say that a subset A in U is *locally complete pluripolar* set if locally $A = \{\psi = -\infty\}$ for some psh function ψ . We begin with the following lemma which is probably well-known.

Lemma 2.1 *Let A be a locally complete pluripolar set in U . Let T be a closed positive current of bi-dimension (m, m) on U . Assume that T has no mass on A . Then, we have $\text{cap}_T(A, U) = 0$.*

Proof The proof is standard. We present the details for readers' convenience. Since the problem is of local nature, we can assume that there is a negative psh function ψ on U such that $A = \{\psi = -\infty\}$. Let $u_1, \dots, u_m$ be bounded psh functions on U such that $0 \leq u_j \leq 1$ for $1 \leq j \leq m$. Let ω is the standard Kähler form on $\mathbb{C}^n$. Let $k \in \mathbb{N}$ and $\psi_k := k^{-1} \max\{\psi, -k\}$. We have $-1 \leq \psi_k \leq 0$. Let χ be a nonnegative smooth function with compact support in U . Let $0 \leq l \leq m$ be an integer. Put

$$I_k := \int_U \chi \psi_k \text{dd}^c u_1 \wedge \cdots \wedge \text{dd}^c u_l \wedge \omega^{m-l} \wedge T.$$

Since $\psi_k = -1$ on $\{\psi < -k\}$, in order to prove the desired assertion, it is enough to show that for every $0 \leq l \leq q$, we have

$$I_k \rightarrow 0 \quad (2.1)$$

as $k \rightarrow \infty$ uniformly in $u_1, \dots, u_l$. We will prove (2.1) by induction on l . Firstly, (2.1) is trivial if $l = 0$ because T has no mass on A . Assume that it holds for $(l-1)$. We prove it for l . Put

$$R := \text{dd}^c u_2 \wedge \cdots \wedge \text{dd}^c u_l \wedge \omega^{m-l} \wedge T.$$

By integration by parts, we have

$$I_k = \int_U u_1 \chi \text{dd}^c \psi_k \wedge R + \int_U u_1 \psi_k \text{dd}^c \chi \wedge R + 2 \int_U u_1 d\psi_k \wedge \text{d}^c \chi \wedge R.$$

Denote by $I_{k,1}, I_{k,2}, I_{k,3}$ the first, second and third term, respectively, in the right-hand side of the last equality. Since u_1 is bounded by 1, by integration by parts, we get

$$|I_{k,1}| \leq C \int_{\text{Supp} \chi} -\psi_k R \wedge \omega, \quad |I_{k,2}| \leq C \int_{\text{Supp} \chi} -\psi_k R \wedge \omega,$$

for some constant C depending only on χ . By induction hypothesis, we have

$$\lim_{k \rightarrow \infty} \int_{\text{Supp} \chi} \psi_k R \wedge \omega = 0.$$

Thus $\lim_{k \rightarrow \infty} I_{k,j} = 0$ for $j = 1, 2$. To treat $I_{k,3}$, we use the Cauchy–Schwarz inequality to get

$$|I_{k,3}| \leq \left(\int_{\text{Supp} \chi} d\psi_k \wedge \text{d}^c \psi_k \wedge R \right)^{1/2}.$$

Let $0 \leq \chi_1 \leq 1$ be a smooth cut-off function compactly supported on U such that $\chi_1 = 1$ on $\text{Supp} \chi$. Let $U_1 \Subset U$ be an open subset containing $\text{Supp} \chi_1$. Since $d\psi_k \wedge \text{d}^c \psi_k \wedge R \geq 0$, we have

$$\begin{aligned}
\int_{\text{Supp}\chi} d\psi_k \wedge dd^c \psi_k \wedge R &\leq \int_{\Omega} \chi_1 d\psi_k \wedge dd^c \psi_k \wedge R \\
&= \int_{\Omega} \chi_1 (dd^c \psi_k^2 - \psi_k dd^c \psi_k) \wedge R \\
&= \int_{\Omega} \chi_1 dd^c \psi_k^2 \wedge R - \int_{\Omega} \chi_1 \psi_k dd^c \psi_k \wedge R \\
&= \int_{\Omega} \psi_k^2 dd^c \chi_1 \wedge R - \int_{\Omega} \chi_1 \psi_k dd^c \psi_k \wedge R \\
&\lesssim \int_{\text{Supp}\chi_1} -\psi_k R \wedge \omega + \int_{\Omega} \chi_1 dd^c \psi_k \wedge R \\
&\lesssim \int_{\text{Supp}\chi_1} -\psi_k R \wedge \omega + \int_{U_1} -\psi_k R \wedge \omega.
\end{aligned}$$

because $-1 \leq \psi_k \leq 0$ and $-\omega \lesssim dd^c \chi_1 \lesssim \omega$. We infer that

$$|I_{k,3}| \lesssim \left(\int_{U_1} -\psi_k R \wedge \omega \right)^{1/2}.$$

By induction hypothesis, $\lim_{k \rightarrow \infty} \int_{U_1} \psi_k R \wedge \omega = 0$. So $\lim_{k \rightarrow \infty} I_{k,3} = 0$. In conclusion, (2.1) follows. This finishes the proof. $\square$

We now give a definition which will be important later. Let $(T_k)_k$ be a sequence of closed positive currents of bi-dimension (m, m) on U . We say that $(T_k)_k$ satisfies *Condition (*)* if $(T_k)_k$ is of uniformly bounded mass on compact subsets of U , and for every open set $U' \subset U$ and every bounded psh function u on U' and every sequence $(u_k)_k$ of psh functions on U' decreasing to u , we have

$$\lim_{k \rightarrow \infty} (u_k - u)(dd^c u)^m \wedge T_k = 0 \quad (2.2)$$

An obvious example for sequences satisfying Condition (*) is constant sequences: $T_k = T$ for every k . We can also take $(T_k)_k$ to be a sequence of suitable Monge–Ampère operators with decreasing potentials, see [1, 11, 16]. Concerning Condition (*), we will use mostly the example of constant sequences and the one provided by the following result.

Theorem 2.2 *Let S be a closed positive current on U . Let v be a psh function on U such that v is locally integrable with respect to the trace measure of S and $(v_k)_k$ a sequence of psh functions on U such that $v_k \rightarrow v$ in L^1_{loc} as $k \rightarrow \infty$ and $v_k \geq v$ for every k . Let $T := dd^c v \wedge S$ and $T_k := dd^c v_k \wedge S$. Let u_j be a bounded psh function on U for $1 \leq j \leq m$. Let $(u_{jk})_{k \in \mathbb{N}}$ be a sequence of uniformly bounded psh functions such that $u_{jk} \rightarrow u_j$ in L^1_{loc} as $k \rightarrow \infty$ and $u_{jk} \geq u_j$ for every j, k . Then we have*

$$u_{1k} dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk} \wedge T_k \rightarrow u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge T \quad (2.3)$$

as $k \rightarrow \infty$. In particular, the sequence $(T_k)_k$ satisfies Condition (*).

Proof By Hartog's lemma, v_k, u_{jk} are uniformly bounded from above in k on compact subsets of U for every j . Since the problem is local, as usual, we can assume that U is relatively compact open set with smooth boundary in $\mathbb{C}^n$, every psh function in questions is defined

on an open neighborhood of $\overline{U}$, $v_k, v \leq 0$ on U for every k and u_{jk}, u_j are all equal to a smooth psh function ψ outside some fixed compact subset of U such that $\psi = 0$ on ∂U . We claim that

$$Q_k := v_k dd^c u_{1k} \wedge \cdots \wedge dd^c u_{mk} \wedge S \rightarrow v dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S \quad (2.4)$$

as $k \rightarrow \infty$. In particular, this implies that v is locally integrable with respect to $dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge S$. We will prove (2.3) and (2.4) simultaneously by induction on m . When $m = 0$, this follows directly from [16, Proposition 3.2]. Assume that (2.3) and (2.4) hold for $(m - 1)$ in place of m . Let

$$R_{j,k} := dd^c u_{jk} \wedge \cdots \wedge dd^c u_{mk} \wedge T_k$$

for $1 \leq j \leq m$. By induction hypothesis, we have

$$R_{j,k} \rightarrow R_j := dd^c u_j \wedge \cdots \wedge dd^c u_m \wedge T$$

for $j \geq 2$. Since u_{1k} is uniformly bounded on U , the family $u_{1k} R_{2,k}$ is of uniformly bounded mass. Let R_∞ be a limit current of the last family. Without loss of generality, we can assume $R_\infty = \lim_{k \rightarrow \infty} u_{1k} R_{2,k}$ and S is of bi-degree $(n - m, n - m)$. By standard arguments, we have $R_\infty \leq u_1 R_2$ (see [16, Proposition 3.2]). Thus, in order to have $R_\infty = u_1 R_2$, we just need to check that

$$\int_U R_\infty \geq \int_U u_1 R_2 \quad (2.5)$$

(both sides are finite because of the assumption we made at the beginning of the proof). Since $\psi = 0$ on ∂U and $u_{1k} = \psi$ on outside a compact of U , we have

$$\int_U u_{1k} R_{2,k} \rightarrow \int_U R_\infty, \quad \int_U \psi R_{2,k} \rightarrow \int_U \psi R_2. \quad (2.6)$$

Let $u_{jk}^\epsilon, \psi^\epsilon$ be standard regularizations of u_{jk}, ψ , respectively. Since $u_{jk} = \psi$ outside some compact of U , we have $u_{jk}^\epsilon = \psi^\epsilon$ outside some compact K of U , for ϵ small enough and K independent of j, k, ϵ . Consequently, $u_{jk}^\epsilon - \psi^\epsilon$ is supported in $K \Subset U$. Note that since ψ is smooth, $\psi^\epsilon \rightarrow \psi$ in $\mathcal{C}^\infty$ -topology. By integration by parts and the fact that $u_{jk} \geq u_j$ for $j = 1, 2$, we have

$$\begin{aligned} \int_U (u_1 - \psi) R_2 &\leq \lim_{\epsilon \rightarrow 0} \int_U (u_{1k}^\epsilon - \psi^\epsilon) R_2 = \lim_{\epsilon \rightarrow 0} \int_U u_2 dd^c (u_{1k}^\epsilon - \psi^\epsilon) R_3 \\ &\leq \lim_{\epsilon \rightarrow 0} \int_U u_{2k}^\epsilon dd^c (u_{1k}^\epsilon - \psi^\epsilon) R_3 + \lim_{\epsilon \rightarrow 0} \int_U (u_{2k}^\epsilon - u_2) dd^c \psi^\epsilon \wedge R_3 \\ &= \lim_{\epsilon \rightarrow 0} \int_U (u_{1k}^\epsilon - \psi^\epsilon) dd^c u_{2k}^\epsilon \wedge R_3 + o_{k \rightarrow \infty}(1) \end{aligned}$$

by induction hypothesis for $(m - 1)$ of (2.3) and the fact that $\|dd^c \psi_\epsilon - dd^c \psi\|_{\mathcal{C}^0} = O(\epsilon)$. We now apply similar arguments to u_{3k} in place of u_{2k} . Precisely, as above we have

$$\begin{aligned}
\int_U (u_{1k}^\epsilon - \psi^\epsilon) dd^c u_{2k}^\epsilon \wedge R_3 &= \int_U u_3 dd^c (u_{1k}^\epsilon - \psi^\epsilon) \wedge dd^c u_{2k}^\epsilon \wedge R_4 \\
&\leq \int_U u_{3k}^\epsilon dd^c (u_{1k}^\epsilon - \psi^\epsilon) \wedge dd^c u_{2k}^\epsilon \wedge R_4 \\
&\quad + \int_U (u_{3k}^\epsilon - u_3) dd^c \psi^\epsilon \wedge dd^c u_{2k}^\epsilon \wedge R_4.
\end{aligned}$$

Letting $\epsilon \rightarrow 0$ and applying the induction hypothesis to the second term in the right-hand side of the last inequality (noticing again that $\|dd^c \psi_\epsilon - dd^c \psi\|_{\phi^0} = O(\epsilon)$), we obtain

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \int_U (u_{1k}^\epsilon - \psi^\epsilon) dd^c u_{2k}^\epsilon \wedge R_3 &\leq \lim_{\epsilon \rightarrow 0} \int_U u_{3k}^\epsilon dd^c (u_{1k}^\epsilon - \psi^\epsilon) \wedge dd^c u_{2k}^\epsilon \wedge R_4 + o_{k \rightarrow \infty}(1) \\
&\leq \lim_{\epsilon \rightarrow 0} \int_U (u_{1k}^\epsilon - \psi^\epsilon) \wedge dd^c u_{2k}^\epsilon \wedge dd^c u_{3k}^\epsilon \wedge R_4 + o_{k \rightarrow \infty}(1).
\end{aligned}$$

Put $R_{2,k}'^\epsilon := dd^c u_{2k}^\epsilon \wedge \cdots \wedge dd^c u_{mk}^\epsilon$. Repeating the above arguments for every u_{jk} ($j \geq 2$) and v, v_k gives

$$\begin{aligned}
\int_U (u_1 - \psi) R_2 &\leq \lim_{\epsilon \rightarrow 0} \int_U (u_{1k}^\epsilon - \psi^\epsilon) R_{2,k}'^\epsilon \wedge dd^c v \wedge S + o_{k \rightarrow \infty}(1) \\
&\leq \lim_{\epsilon \rightarrow 0} \int_U v dd^c (u_{1k}^\epsilon - \psi^\epsilon) \wedge R_{2,k}'^\epsilon \wedge S + o_{k \rightarrow \infty}(1) \\
&\leq \lim_{\epsilon \rightarrow 0} \int_U v_k dd^c (u_{1k}^\epsilon - \psi^\epsilon) \wedge R_{2,k}'^\epsilon \wedge S \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_U (v_k - v) dd^c \psi^\epsilon \wedge R_{2,k}'^\epsilon \wedge S + o_{k \rightarrow \infty}(1) \\
&= \lim_{\epsilon \rightarrow 0} \int_U (u_{1k}^\epsilon - \psi^\epsilon) \wedge R_{2,k}'^\epsilon \wedge dd^c v_k \wedge S + o_{k \rightarrow \infty}(1) \\
&= \int_U (u_{1k} - \psi) \wedge R_{2,k} + o_{k \rightarrow \infty}(1)
\end{aligned}$$

by (2.4) for $(m-1)$ and the usual convergence of Monge–Ampère operators. Letting $k \rightarrow \infty$ in the last inequality and using (2.6) give (2.5). Hence, (2.3) for m follows.

It remains to prove (2.4) for m . Put $R_{2,k}' := dd^c u_{2k} \wedge \cdots \wedge dd^c u_{mk}$ and $R_2' := dd^c u_2 \wedge \cdots \wedge dd^c u_m$. We check that Q_k is of uniformly bounded mass. Decompose

$$Q_k = v_k dd^c (u_{1k} - \psi) \wedge R_{2,k}' \wedge S + v_k dd^c \psi \wedge R_{2,k}' \wedge S.$$

The second term converges to $v dd^c \psi \wedge R_2' \wedge S$ as $k \rightarrow \infty$ by induction hypothesis for $(m-1)$. Denote by $Q_{k,1}$ the first term. Let v_k^ϵ be standard regularizations of v_k . By integration by parts, we have

$$\begin{aligned}
\int_U Q_{k,1}^\epsilon &:= \int_U v_k^\epsilon dd^c (u_{1k} - \psi) \wedge R_{2,k}' \wedge S \\
&= \int_U (u_{1k} - \psi) dd^c v_k^\epsilon \wedge R_{2,k}' \wedge S = (u_{1k} - \psi) R_{2,k}' \wedge dd^c v_k^\epsilon \wedge S
\end{aligned}$$

which converges to $\int_U (u_{1k} - \psi) R_{2,k}' \wedge dd^c v_k \wedge S$ as $\epsilon \rightarrow 0$ by (2.3) for m . Thus,

$$\int_U Q_{k,1} = \int_U (u_{1k} - \psi) R'_{2,k} \wedge dd^c v_k \wedge S.$$

This combined with (2.3) for m again implies that $\int_U Q_{k,1} \rightarrow \int_U (u_1 - \psi) R'_2 \wedge dd^c v \wedge S$ as $k \rightarrow \infty$. The last limit is equal to $\int_U v dd^c(u_1 - \psi) \wedge R'_2$ by integration by parts which can be performed thanks to (2.3) for m . Thus, we have proved that Q_k is of uniformly bounded mass and

$$\int_U Q_k \rightarrow \int_U v R_1$$

as $k \rightarrow \infty$. This combined with the fact that $v R_1 \geq Q_\infty$ for every limit current Q_∞ of the family $(Q_k)_k$ gives the desired assertion (2.4) for m . This finishes the proof. $\square$

Remark 2.3 We can apply Theorem 2.2 to the case where S is a constant function and v is an arbitrary psh function. By the above proof, we can check the following observations:

(i) if S is a closed positive current on U , $u_1, \dots, u_m$ are psh functions on U which are locally integrable with respect to S such that u_j is locally bounded for every $1 \leq j \leq m$ except possibly for one index, then the current $dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge S$, which is defined inductively as usual, is symmetric with respect to $u_1, \dots, u_m$ and satisfies the convergence under decreasing sequences,

(ii) let u_0 be another psh function locally integrable with respect to S such that u_0 is locally bounded if there is an index $1 \leq j \leq m$ so that u_j is not locally bounded. Then, $u_0 dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge S$ is convergent under decreasing sequences and for every compact K in U , if we have $0 \leq u_1, \dots, u_m \leq 1$, then

$$\|u_0 dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge S\|_K \leq C \|u_0 S\|_U$$

for some constant C independent of $u_0, \dots, u_m, S$.

The following result explains the reason for the use of Condition (*).

Theorem 2.4 *Let $(T_l)_l$ be a sequence of closed positive currents satisfying Condition (*). Let u be a bounded psh function on U and $(u_k)_k$ a sequence of psh functions on U decreasing to u . Then for every constant $\epsilon > 0$ and every compact K in U , we have $\text{cap}_{T_l}(\{|u_k - u| \geq \epsilon\} \cap K) \rightarrow 0$ as $k \rightarrow \infty$ uniformly in l . In particular, for every constant $\epsilon > 0$, there exists an open subset U' of U such that $\text{cap}_{T_l}(U', U) < \epsilon$ for every l and the restriction of u to $U \setminus U'$ is continuous.*

Consider the case where $T_l = T$ for every l . Then, the above theorem gives a quasi-continuity with respect to cap_T for bounded psh function which is *stronger* than the usual one for general psh functions with respect to cap (see [1]). We refer to Theorem 2.4 as a (uniform) strong quasi-continuity of bounded psh functions.

Proof We follow ideas presented in [18, Proposition 1.12]. Let $K \Subset U$. Let T_l be of bi-dimension (m, m) . We will prove that

$$\int_K (u_k - u) dd^c v_1 \wedge \dots \wedge dd^c v_m \wedge T_l \rightarrow 0 \quad (2.7)$$

uniformly in psh functions $0 \leq v_1, \dots, v_m \leq 1$ and in l . The desired assertion concerning the uniform convergence in cap_{T_l} is a direct consequence of (2.7).

By Hartog's lemma and the boundedness of u , we obtain that u_k is uniformly bounded in k in compact subsets of U . Since the problem is local and $u_k, u, v_1, \dots, v_m$ are uniformly bounded on $\overline{U}$, we can assume that $U \Subset \mathbb{C}^n$, $u_k, u, v_1, \dots, v_m$ are defined on an open neighborhood of $\overline{U}$ and there exist a smooth psh function ψ defined on an open neighborhood of $\overline{U}$ and an open neighborhood W of ∂U such that $K \subset U \setminus W$ and $u_k = u = v_j = \psi$ on W for every j, k . Let

$$T'_l := dd^c v_2 \wedge \dots \wedge dd^c v_m \wedge T_l.$$

Observe $u_k - u$ is of compact support in some open set $U_1 \Subset U$ containing K . Hence, by integration by parts, we get

$$\begin{aligned} \int_{U_1} (u_k - u) dd^c v_1 \wedge T'_l &= - \int_{U_1} d(u_k - u) \wedge d^c v_1 \wedge T'_l \\ &\leq \left(\int_{U_1} d(u_k - u) \wedge d^c(u_k - u) \wedge T'_l \right)^{1/2} \left(\int_{U_1} dv_1 \wedge d^c v_1 \wedge T'_l \right)^{1/2} \end{aligned}$$

which is $\lesssim \left(\int_{U_1} d(u_k - u) \wedge d^c(u_k - u) \wedge T'_l \right)^{1/2}$ by the Chern–Levine–Nirenberg inequality. Denote by I the integral in the last quantity. We have

$$I = - \int_{U_1} (u_k - u) \wedge dd^c(u_k - u) \wedge T' \leq \int_{U_1} (u_k - u) \wedge dd^c u \wedge T'_l.$$

Applying similar arguments to $v_2, \dots, v_m$ consecutively and the right-hand side of the last inequality, we obtain that

$$\int_K (u_k - u) dd^c v_1 \wedge \dots \wedge dd^c v_m \wedge T_l \leq C \left(\int_{U_1} (u_k - u) (dd^c u)^m \wedge T_l \right)^{2^{-m}}, \quad (2.8)$$

where C is independent of k and l (note that the mass of T_l on compact subsets of U is bounded uniformly in l). Let

$$H_{k,l} := \int_{U_1} (u_k - u) (dd^c u)^m \wedge T_l.$$

By (2.8), in order to obtain (2.7), it suffices to prove that $H_{k,l}$ converges to 0 as $k \rightarrow \infty$ uniformly in l . Suppose that this is not the case. This means that there exists a constant $\epsilon > 0$, $(k_s)_s \rightarrow \infty$ and $(l_s)_s \rightarrow \infty$ such that $H_{k_s, l_s} \geq \epsilon$ for every s . However, by Condition (*), we get $(u_{k_s} - u)(dd^c u)^m \wedge T_{l_s} \rightarrow 0$ as $s \rightarrow \infty$. This is contradiction. Hence, (2.7) follows.

We prove the second desired assertion, let $K \Subset U$ and $(u_k)_k$ a sequence of smooth psh functions defined on an open neighborhood of K decreasing to u . Let $\epsilon > 0$ be a constant. Since $u_k \rightarrow u$ in cap_{T_l} as $k \rightarrow \infty$ uniformly in l , there is a sequence $(j_k)_k$ converging to ∞ for which

$$\text{cap}_{T_l}(K \cap \{u_{j_k} > u + 1/k\}, U) \leq \epsilon 2^{-k}$$

for every $k, l \in \mathbb{N}^*$. Consequently, for $K_\epsilon := K \setminus \bigcup_{k=1}^\infty \{u_{j_k} > u + 1/k\}$, we have that $\text{cap}_{T_l}(K \setminus K_\epsilon, U) \leq \epsilon$ and u_{j_k} is convergent uniformly on K_ϵ . Hence, u is continuous on K_ϵ .

Let $(U_s)_s$ be an increasing exhaustive sequence of relatively compact open subsets of U and $K_s := \overline{U_s} \setminus U_{s-1}$ for $s \geq 1$, where $U_0 := \emptyset$. Observe that K_l is compact, $U = \bigcup_{s=1}^\infty K_s$ and

$$K_s \cap \overline{\bigcup_{s' \geq s+2} K_{s'}} = \emptyset \quad (2.9)$$

for every $s \geq 1$. By the previous paragraph, there exists a compact subset K'_s of K_s such that $\text{cap}_{T_l}(K_s \setminus K'_s, U) \leq \epsilon 2^{-s}$ and u is continuous on K'_s . Observe that $K' := \bigcup_{s=1}^\infty K'_s$ is closed in U and u is continuous on K' because of (2.9). We also have $U \setminus K' \subset \bigcup_{s=1}^\infty (K_s \setminus K'_s)$. Hence, $\text{cap}_{T_l}(U \setminus K', U) \leq \epsilon$ for every l . The proof is finished. $\square$

As one can expect, the above quasi-continuity of bounded psh functions allows us to treat, to certain extent, these functions as continuous functions with respect to closed positive currents.

Corollary 2.5 *Let $R_k := \text{dd}^c v_{1k} \wedge \cdots \wedge \text{dd}^c v_{mk} \wedge T_k$ and $R := \text{dd}^c v_1 \wedge \cdots \wedge \text{dd}^c v_m \wedge T$, where v_{jk}, v_j are uniformly bounded psh functions on U and T_k, T closed positive currents of bi-degree (p, p) . Let u be a bounded psh function on U and χ a continuous function on $\mathbb{R}$. Assume that $R_k \rightarrow R$ as $k \rightarrow \infty$ on U and $(T_k)_k$ satisfies Condition (*). Then, we have*

$$\chi(u)R_k \rightarrow \chi(u)R$$

as $k \rightarrow \infty$. In particular, the last convergence holds when $T_k = T$ for every k or $T_k = \text{dd}^c w_k \wedge S$, $T = \text{dd}^c w \wedge S$, where S is a closed positive current, w is a psh function locally integrable with respect to S , and w_k is a psh function converging to w in L^1_{loc} as $k \rightarrow \infty$ so that $w_k \geq w$ for every k .

Proof The problem is local. Hence, we can assume U is relatively compact in $\mathbb{C}^n$. Since u is bounded, using Theorem 2.4, we have that u is uniformly quasi-continuous with respect to the family cap_{T_k} with $k \in \mathbb{N}$. This means that given $\epsilon > 0$, we can find an open subset U' of U such that $\text{cap}_{T_k}(U', U) < \epsilon$ and $u|_{U \setminus U'}$ is continuous. Let $\tilde{u}$ be a bounded continuous function on U extending $u|_{U \setminus U'}$ (see [20, Theorem 20.4]). We have $\chi(\tilde{u})R_k \rightarrow \chi(\tilde{u})R$ because $\chi, \tilde{u}$ are continuous. Moreover,

$$\|(\chi(\tilde{u}) - \chi(u))R_k\| \lesssim \|R_k\|_{U \setminus U'} \leq \text{cap}_{T_k}(U \setminus U', U) < \epsilon$$

(we used here the boundedness of U) and a similar estimate also holds for $(\chi(\tilde{u}) - \chi(u))R$. The desired assertion then follows. This finishes the proof. $\square$

The following result generalizes well-known convergence properties of Monge–Ampère operators in [1].

Theorem 2.6 *Let $U \subset \mathbb{C}^n$ be an open set. Let $(T_k)_k$ be a sequence of closed positive currents satisfying Condition (*) so that T_k converges to a closed positive current T on U as $k \rightarrow \infty$. Let u_j be a locally bounded psh function on U for $1 \leq j \leq m$. Let $(u_{jk})_{k \in \mathbb{N}}$ be a sequence of locally bounded psh functions converging to u_j in L^1_{loc} as $k \rightarrow \infty$. Then, the convergence*

$$u_{1k} \text{dd}^c u_{2k} \wedge \cdots \wedge \text{dd}^c u_{mk} \wedge T_k \rightarrow u_1 \text{dd}^c u_2 \wedge \cdots \wedge \text{dd}^c u_m \wedge T \quad (2.10)$$

as $k \rightarrow \infty$ holds provided that one of the following conditions is fulfilled:

- (i) $u_{jk}(x) \nearrow u_j(x)$ for every $x \in U$ as $k \rightarrow \infty$,
- (ii) $u_{jk}(x) \nearrow u_j(x)$ for almost everywhere $x \in U$ (with respect to the Lebesgue measure) and T has no mass on pluripolar sets,
- (iii) $u_{jk} \geq u_j$ for every j, k .

Proof Given that we already have a uniform strong quasi-continuity for bounded psh functions, the desired result can be deduced without difficulty from proofs of classical results on the convergence of Monge–Ampère operators, for example, see [18].

We present here a proof of Theorem 2.6 for the readers' convenience. First of all, observe that if $u_{jk} \nearrow u_j$ almost everywhere then, we have $u_{jk} \leq u_{j(k+1)} \leq u_j$ pointwise on U and the set $\{x \in U : u_j(x) \neq \lim_{k \rightarrow \infty} u_{jk}(x)\}$ is pluripolar. By the localization principle ([18, Page 7]), we can assume that u_{jk}, u_j are all equal to some smooth psh function ψ outside some set $K \Subset U$ on U . We prove (i), (ii) simultaneously. Let

$$S_{jk} := dd^c u_{jk} \wedge \cdots \wedge dd^c u_{mk} \wedge T_k, \quad S_j := dd^c u_j \wedge \cdots \wedge dd^c u_m \wedge T.$$

We prove by induction in j that

$$u_{(j-1)k} S_{jk} \rightarrow u_{(j-1)} S_j \quad (2.11)$$

k and for every $2 \leq j \leq m+1$ (by convention we put $S_{(m+1)k} := T_k$ and $S_{m+1} := T$). The claim is true for $j = m+1$. Suppose that it holds for $(j+1)$. We need to prove it for j . Let $R_{j\infty}$ be a limit current of $u_{(j-1)k} S_{jk}$ as $k \rightarrow \infty$. By induction hypothesis (2.11) for $(j+1)$ instead of j , $S_{jk} \rightarrow S_j$ as $k \rightarrow \infty$. This combined with the fact that the sequence $(u_{jk})_k$ converges in L^1_{loc} to u_j gives

$$R_{j\infty} \leq u_{j-1} S_j \quad (2.12)$$

(one can see [16, Proposition 3.2]). On the other hand, since $(u_{jk})_k$ is increasing, using Corollary 2.5, we obtain

$$\liminf_{k \rightarrow \infty} u_{(j-1)k} S_{jk} \geq \liminf_{k \rightarrow \infty} u_{(j-1)s} S_{jk} = u_{(j-1)s} S_j$$

for every $s \in \mathbb{N}$. Letting $s \rightarrow \infty$ in the last inequality gives

$$R_{j\infty} \geq (\lim_{s \rightarrow \infty} u_{(j-1)s}) S_j = u_{j-1} S_j + (\lim_{s \rightarrow \infty} u_{(j-1)s} - u_{j-1}) S_j. \quad (2.13)$$

Recall that the set of $x \in U$ with $u_{j-1}(x) > \lim_{s \rightarrow \infty} u_{(j-1)s}(x)$ is empty in the setting of (i) and is a pluripolar set in the setting of (ii). Hence, (2.11) follows from Lemma 2.1, (2.13) and (2.12). We have proved (i) and (ii). We prove (iii) by similar induction. The proof is finished. $\square$

Remark 2.7 By the above proof and Lemma 2.1, Property (ii) of Theorem 2.6 still holds if instead of requiring T has no mass on pluripolar sets, we assume the following two conditions:

- (i) T has no mass on $A_j := \{x \in U : u_j(x) \neq \lim_{k \rightarrow \infty} u_{jk}(x)\}$ for every $1 \leq j \leq m$ and,
- (ii) the set A_j is locally complete pluripolar for every j .

Just by replacing the usual quasi-continuity of psh functions by the stronger one given in Theorem 2.4 for bounded psh functions, we immediately obtain results similar to those in [2]. We state here results we will use later.

Lemma 2.8 (similar to [2, Lemma 4.1]) *Let U be an open subset in $\mathbb{C}^n$. Let T be a closed positive current on U and $u_j, u_{jk}, u'_j, u'_{jk}$ bounded psh functions on U for $k \in \mathbb{N}$ and $1 \leq j \leq m$, where $m \in \mathbb{N}$. Let $q \in \mathbb{N}^*$ and v_j, v'_j bounded psh functions on U for $1 \leq j \leq q$. Put $W := \cap_{j=1}^q \{v_j > v'_j\}$. Assume that*

$$R_k := dd^c u_{1k} \wedge \cdots \wedge dd^c u_{mk} \wedge T \rightarrow R := dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T$$

and

$$R'_k := dd^c u'_{1k} \wedge \cdots \wedge dd^c u'_{mk} \wedge T \rightarrow R' := dd^c u'_1 \wedge \cdots \wedge dd^c u'_m \wedge T$$

as $k \rightarrow \infty$ and

$$\mathbf{1}_W R_k = \mathbf{1}_W R'_k \quad (2.14)$$

for every k . Then we have $\mathbf{1}_W R = \mathbf{1}_W R'$.

Proof The problem is clear if W is open, for example, when v_j is continuous for $1 \leq j \leq q$. In the general case, we will use the strong quasi-continuity to modify v_j . Since the problem is local, we can assume that U is bounded. Let $\epsilon > 0$ be a constant. By Theorem 2.4, we can find bounded continuous functions $\tilde{v}_j$ on U such that $\text{cap}_T(\{\tilde{v}_j \neq v_j\}, U) < \epsilon$. Put $\tilde{W} := \cap_{j=1}^q \{\tilde{v}_j > v'_j\}$ which is an open set. The choice of $\tilde{v}_j$ combined with the definition of cap_T yields that

$$\|\mathbf{1}_W R - \mathbf{1}_{\tilde{W}} R\|_U \leq \epsilon, \quad \|\mathbf{1}_W R_k - \mathbf{1}_{\tilde{W}} R_k\|_U \leq \epsilon.$$

We also have similar estimates for R', R'_k . By this and (2.14), we get $\|\mathbf{1}_{\tilde{W}} R_k - \mathbf{1}_{\tilde{W}} R'_k\|_U \leq 2\epsilon$. This combined with the fact that $\tilde{W}$ is open yields that $\|\mathbf{1}_{\tilde{W}} R - \mathbf{1}_{\tilde{W}} R'\|_U \leq 2\epsilon$. Thus, $\|\mathbf{1}_W R - \mathbf{1}_W R'\|_U \leq 4\epsilon$ for every ϵ . The desired equality follows. This finishes the proof. $\square$

Theorem 2.9 *Let U be an open subset in $\mathbb{C}^n$. Let T be a closed positive current on U and u_j, u'_j bounded psh functions on U for $1 \leq j \leq m$, where $m \in \mathbb{N}$. Let v_j, v'_j be bounded psh functions on U for $1 \leq j \leq q$. Assume that $u_j = u'_j$ on $W := \cap_{j=1}^q \{v_j > v'_j\}$ for $1 \leq j \leq m$. Then we have*

$$\mathbf{1}_W dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = \mathbf{1}_W dd^c u'_1 \wedge \cdots \wedge dd^c u'_m \wedge T. \quad (2.15)$$

Proof We give here a complete proof for the readers' convenience. Let $\epsilon > 0$ be a constant. Put $u''_j := \max\{u_j, u'_j - \epsilon\}$ and $\tilde{W} := \cap_{j=1}^m \{u_j > u'_j - \epsilon\}$. By hypothesis, $W \subset \tilde{W}$. We will prove that

$$\mathbf{1}_{\tilde{W}} dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = \mathbf{1}_{\tilde{W}} dd^c u''_1 \wedge \cdots \wedge dd^c u''_m \wedge T. \quad (2.16)$$

Since the problem is local, we can assume there is a sequence of uniformly bounded smooth psh functions $(u_{jk})_k$ decreasing to u_j for $1 \leq j \leq m$. Since $\tilde{W}_k := \{u_{jk} > u'_j - \epsilon\}$ is open, we have

$$\mathbf{1}_{\tilde{W}_k} dd^c u_{1k} \wedge \cdots \wedge dd^c u_{mk} \wedge T = \mathbf{1}_{\tilde{W}_k} dd^c \max\{u_{1k}, u'_j - \epsilon\} \wedge \cdots \wedge dd^c \{u_{mk}, u'_j - \epsilon\} \wedge T.$$

This together with the inclusion $\tilde{W} \subset \tilde{W}_k$ gives

$$\mathbf{1}_{\tilde{W}} \mathrm{dd}^c u_{1k} \wedge \cdots \wedge \mathrm{dd}^c u_{mk} \wedge T = \mathbf{1}_{\tilde{W}} \mathrm{dd}^c \max\{u_{1k}, u'_j - \epsilon\} \wedge \cdots \wedge \mathrm{dd}^c \{u_{mk}, u'_j - \epsilon\} \wedge T.$$

Using this and Lemma 2.8, we obtain (2.16) by considering $k \rightarrow \infty$. In particular, we get

$$\mathbf{1}_W \mathrm{dd}^c u_1 \wedge \cdots \wedge \mathrm{dd}^c u_m \wedge T = \mathbf{1}_W \mathrm{dd}^c u''_1 \wedge \cdots \wedge \mathrm{dd}^c u''_m \wedge T.$$

Letting $\epsilon \rightarrow 0$ and using Lemma 2.8 again gives

$$\mathbf{1}_W \mathrm{dd}^c u_1 \wedge \cdots \wedge \mathrm{dd}^c u_m \wedge T = \mathbf{1}_W \mathrm{dd}^c \max\{u_1, u'_1\} \wedge \cdots \wedge \mathrm{dd}^c \max\{u_m, u'_m\} \wedge T.$$

The last equality still holds if we replace u_j in the left-hand side by u'_j by using similar arguments. So the desired equality follows. The proof is finished. $\square$

Remark 2.10 Recall that a quasi-psh function u on U is, by definition, locally the sum of a psh function and a smooth one. We can check that results presented above have their analogues for quasi-psh functions.

3 Relative non-pluripolar product

Let X be a complex manifold of dimension n and $T, T_1, \dots, T_m$ closed positive currents on X such that T_j is of bi-degree $(1, 1)$ for $1 \leq j \leq m$. Let U be a local chart of X such that $T_j = \mathrm{dd}^c u_j$ on U for $1 \leq j \leq m$, where $u_1, \dots, u_m$ are psh functions on U . Let $k \in \mathbb{N}$ and $u_{jk} := \max\{u_j, -k\}$ which is a locally bounded psh function. Put $R_k := \mathrm{dd}^c u_{1k} \wedge \cdots \wedge \mathrm{dd}^c u_{mk} \wedge T$. By Theorem 2.9 and the fact that $\{u_j > -k\} = \{u_{jk} > -k\}$, we have

$$\mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k = \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_l \quad (3.1)$$

for every $l \geq k$. As in the case of the usual non-pluripolar products, we have the following basic observation.

Lemma 3.1 *Assume that we have*

$$\sup_{k \in \mathbb{N}} \|\mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k\|_K < \infty \quad (3.2)$$

for every compact K of U . Then the limit current

$$R := \lim_{k \rightarrow \infty} \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k \quad (3.3)$$

is well defined and for every Borel form Φ with bounded coefficients on U such that $\mathrm{Supp} \Phi \Subset U$, we have

$$\langle R, \Phi \rangle = \lim_{k \rightarrow \infty} \langle \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi \rangle. \quad (3.4)$$

Consequently, there holds

$$\mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R = \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \quad \mathbf{1}_{\cup_{j=1}^m \{u_j = -\infty\}} R = 0.$$

Proof By (3.1), we have

$$\begin{aligned} \mathbf{1}_{\cap_{j=1}^m \{u_j > -l\}} R_l &= \mathbf{1}_{\cap_{j=1}^m \{u_j > 0\}} R_l + \sum_{k=1}^l \mathbf{1}_{\cap_{j=1}^m \{-k+1 \geq u_j > -k\}} R_l \\ &= \mathbf{1}_{\cap_{j=1}^m \{u_j > 0\}} R_0 + \sum_{k=1}^l \mathbf{1}_{\cap_{j=1}^m \{-k+1 \geq u_j > -k\}} R_k. \end{aligned}$$

This combined with (3.2) tells us that the mass on a fixed compact of U of the current

$$\mathbf{1}_{\cap_{j=1}^m \{-k \geq u_j > -l\}} R_l = \sum_{k'=k}^l \mathbf{1}_{\cap_{j=1}^m \{-k'+1 \geq u_j > -k'\}} R_{k'}$$

converging to 0 as $l \geq k \rightarrow \infty$. We deduce that $\lim_{k \rightarrow \infty} \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k$ exists and is denoted by R .

Since R, R_k are positive, for every continuous form Φ of compact support in U , we have $\langle R, \Phi \rangle = \lim_{k \rightarrow \infty} \langle R_k, \Phi \rangle$. Let Φ be a Borel form on U such that its coefficients are bounded on U and $\text{Supp} \Phi \Subset U$. Let K be a compact of U containing $\text{Supp} \Phi$ and $U_1 \supset K$ a relatively compact open subset of U . Let $\epsilon > 0$ be a constant. Let k_0 be a positive integer such that

$$\|\mathbf{1}_{\cap_{j=1}^m \{-k \geq u_j > -l\}} R_l\|_{U_1} \leq \epsilon \quad (3.5)$$

for every $l \geq k \geq k_0$. By Lusin's theorem, there exists a continuous form Φ' compactly supported on U_1 such that

$$\|\mathbf{1}_{\cap_{j=1}^m \{u_j > -k_0\}} R_{k_0}\|_{\{x \in U_1 : \Phi'(x) \neq \Phi(x)\}} \leq \epsilon, \quad \|R\|_{\{x \in U_1 : \Phi'(x) \neq \Phi(x)\}} \leq \epsilon. \quad (3.6)$$

Using (3.5) and (3.6) gives

$$|\langle \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi \rangle - \langle R_k, \Phi' \rangle| \lesssim 2\epsilon, \quad |\langle R, \Phi \rangle - \langle R, \Phi' \rangle| \lesssim 2\epsilon.$$

This combined with the fact that $\langle \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi' \rangle \rightarrow \langle R, \Phi' \rangle$ gives

$$\left| \lim_{k \rightarrow \infty} \langle \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi \rangle - \langle R, \Phi \rangle \right| \lesssim 2\epsilon.$$

Letting $\epsilon \rightarrow 0$ gives (3.4). For the other equalities, one just needs to apply (3.4) to suitable Φ . This finishes the proof. $\square$

Lemma 3.2 *Let $T_j = \text{dd}^c \tilde{u}_j + \theta_j$ for $1 \leq j \leq m$, where θ_j is a smooth $(1, 1)$ -form and $\tilde{u}_j$ is θ_j -psh on X . Let*

$$\tilde{u}_{jk} := \max\{\tilde{u}_j, -k\}, \quad \tilde{R}_k := \bigwedge_{j=1}^m (\text{dd}^c \tilde{u}_{jk} + \theta_j) \wedge T.$$

Then the following two properties hold:

- (i) (3.2) holds for every small enough local chart U if and only if we have that for every compact K of X ,

$$\sup_{k \in \mathbb{N}} \|\mathbf{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k\|_K < \infty. \quad (3.7)$$

In this case, if $\tilde{R} := \lim_{k \rightarrow \infty} \mathbf{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$, then $\tilde{R} = R$ on U ,

(ii) $\mathbf{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$ is a positive current.

Proof Firstly observe that $\tilde{R}_k$ is a current of order 0 and of bounded mass on compact subsets of X . Let

$$\tilde{B}_k := \cap_{j=1}^m \{\tilde{u}_j > -k\}.$$

Assume now (3.7). This means $\|\mathbf{1}_{\tilde{B}_k} \tilde{R}_k\|_K$ is uniformly bounded for every compact K . By Remark 2.10, we have $\mathbf{1}_{\tilde{B}_k} \tilde{R}_k = \mathbf{1}_{\tilde{B}_k} \tilde{R}_l$ for every $l \geq k$. Decompose

$$\tilde{B}_l = \tilde{B}_0 \cup \bigcup_{k=1}^l \cap_{j=1}^m \{-k+1 \geq \tilde{u}_j > -k\}$$

which is a disjoint union. Hence, we get

$$\|\mathbf{1}_{\tilde{B}_l} \tilde{R}_l\| = \|\mathbf{1}_{\tilde{B}_0} \tilde{R}_l\| + \sum_{k=1}^l \|\mathbf{1}_{\cap_{j=1}^m \{-k+1 \geq \tilde{u}_j > -k\}} \tilde{R}_l\| = \|\mathbf{1}_{\tilde{B}_0} \tilde{R}_0\| + \sum_{k=1}^l \|\mathbf{1}_{\cap_{j=1}^m \{-k+1 \geq \tilde{u}_j > -k\}} \tilde{R}_k\|,$$

where the masses are measured on some compact K in X . We deduce that the condition $\|\mathbf{1}_{\tilde{B}_k} \tilde{R}_k\|_K$ is uniformly bounded is equivalent to that

$$\|\mathbf{1}_{\cap_{j=1}^m \{-k \geq \tilde{u}_j > -l\}} \tilde{R}_l\|_K = \sum_{k'=k}^l \|\mathbf{1}_{\cap_{j=1}^m \{-k'+1 \geq \tilde{u}_j > -k'\}} \tilde{R}_k\|_K \rightarrow 0 \quad (3.8)$$

as $l \geq k \rightarrow \infty$. Hence, $\mathbf{1}_{\tilde{B}_k} \tilde{R}_k$ converges to a current $\tilde{R}$ and moreover we have

$$\langle \tilde{R}, \Phi \rangle = \lim_{k \rightarrow \infty} \langle \mathbf{1}_{\tilde{B}_k} \tilde{R}_k, \Phi \rangle$$

for every Borel bounded form Φ of compact support in X as in the proof of Lemma 3.1. Consequently, we get

$$\mathbf{1}_{\tilde{B}_k} \tilde{R} = \mathbf{1}_{\tilde{B}_k} \tilde{R}_k, \quad \mathbf{1}_{\cup_{j=1}^m \{\tilde{u}_j = -\infty\}} \tilde{R} = 0. \quad (3.9)$$

Let U, u_j, u_{jk}, R, R_k be as above. We will show that (3.2) is satisfied and $\tilde{R} = R$ on U . Let

$$B_k := \cap_{j=1}^m \{u_j > -k\}.$$

Observe that $u_j = \tilde{u}_j + \tau_j$ for some smooth function τ_j on U with $\mathrm{dd}^c \tau_j = \theta_j$. By shrinking U , we can assume that τ_j is bounded on U and let c_0 be an integer greater than $\sum_{j=1}^m \|\tau_j\|_{L^\infty}$. We have

$$\tilde{u}_{jk} + \tau_j = \max\{\tilde{u}_j + \tau_j, -k + \tau_j\}$$

which is equal to $\max\{\tilde{u}_j + \tau_j, -k\} = u_{jk}$ on the set $\{\tilde{u}_j > -k + c_0\}$. It follows that

$$\mathbf{1}_{B_{k-c_0}} \tilde{R}_k = \mathbf{1}_{B_{k-c_0}} \left(\bigwedge_{j=1}^m \mathrm{dd}^c u_{jk} \wedge T \right) = \mathbf{1}_{B_{k-c_0}} R_k. \quad (3.10)$$

This together with the inclusions $B_{k-2c_0} \subset \tilde{B}_{k-c_0} \subset B_k$ give

$$\mathbf{1}_{B_k - 2c_0} R_{k-2c_0} \leq \mathbf{1}_{\tilde{B}_k - c_0} \tilde{R}_k = \mathbf{1}_{\tilde{B}_k - c_0} \tilde{R}_{k-c_0} \leq \mathbf{1}_{B_k} R_k. \quad (3.11)$$

Hence (3.2) follows. We also deduce from this and (3.9) that $\tilde{R} = R$. Conversely, if (3.2) holds for every U , then, by (3.11), the claim (3.7) holds. Hence, (i) follows. By (3.10), we have

$$\mathbf{1}_{\tilde{B}_k} \tilde{R}_k = \mathbf{1}_{\tilde{B}_k} \tilde{R}_{k+c_0} = \mathbf{1}_{\tilde{B}_k} R_{k+c_0} \geq 0.$$

Thus (ii) follows. The proof is finished. $\square$

Lemma 3.2 applied to U in place of X implies that the condition (3.2) is independent of the choice of u_j and so is the limit R above. As a result, if (3.2) holds for every small enough local chart U as above, then we obtain a positive current R globally defined on X given locally by (3.3). This current is equal to $\tilde{R}$ by Lemma 3.2 again.

Definition 3.3 We say that the non-pluripolar product relative to T of $T_1, \dots, T_m$ is *well defined* if (3.2) holds for every small enough local chart U of X , or equivalently, (3.7) holds. In this case, the non-pluripolar product relative to T (or the T -relative non-pluripolar product) of $T_1, \dots, T_m$, which is denoted by $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$, is defined to be the current $\tilde{R}$ in Lemma 3.2.

Note that $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ is a current of bi-degree $(p+m, p+m)$ if T is of bi-degree (p, p) . If $T_1, \dots, T_m$ are of locally bounded potentials, then $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ is equal to the classical intersection of $T_1, \dots, T_m$ and T , see Proposition 3.6 for more information.

When T is the current of integration along X , we write $\langle T_1 \wedge \dots \wedge T_m \rangle$ for the non-pluripolar product relative to T of $T_1, \dots, T_m$. One can see that this is exactly the usual non-pluripolar product of $T_1, \dots, T_m$ defined in [2, 4, 17]. We note that in general the current $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ can have mass on pluripolar sets in $\cap_{j=1}^m \{u_j > -\infty\}$, see, however, Property (iii) of Proposition 3.5 below. Arguing as in the proof of [4, Proposition 1.6], we obtain the following result.

Lemma 3.4 Assume that X is a compact Kähler manifold. Then, $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ is well defined.

For a closed positive $(1, 1)$ -current R , recall that the *polar locus* I_R of R is the set of points where potentials of R are equal to $-\infty$. Note that by Lemma 3.1, the current $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ has no mass on $\cup_{j=1}^m I_{T_j}$. We collect here some more basic properties of relative non-pluripolar products.

Proposition 3.5

- (i) The product $\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$ is symmetric with respect to $T_1, \dots, T_m$.
- (ii) Given a positive real number λ , we have $\langle (\lambda T_1) \wedge T_2 \wedge \dots \wedge T_m \wedge T \rangle = \lambda \langle T_1 \wedge T_2 \wedge \dots \wedge T_m \wedge T \rangle$.
- (iii) Given a complete pluripolar set A such that T has no mass on A , then $\langle T_1 \wedge T_2 \wedge \dots \wedge T_m \wedge T \rangle$ also has no mass on A .

- (iv) Let T' be a closed positive $(1, 1)$ -current on X and T_j, T as above. Assume that $\langle \bigwedge_{j=1}^m T_j \wedge T \rangle, \langle T'_1 \wedge \bigwedge_{j=2}^m T_j \wedge T \rangle$ are well defined. Then, $\langle (T_1 + T'_1) \wedge \bigwedge_{j=2}^m T_j \wedge T \rangle$ is also well defined and satisfies

$$\left\langle (T_1 + T'_1) \wedge \bigwedge_{j=2}^m T_j \wedge T \right\rangle \leq \left\langle T_1 \wedge \bigwedge_{j=2}^m T_j \wedge T \right\rangle + \left\langle T'_1 \wedge \bigwedge_{j=2}^m T_j \wedge T \right\rangle. \quad (3.12)$$

The equality occurs if T has no mass on $I_{T_1} \cup I_{T'_1}$.

- (v) Let $1 \leq l \leq m$ be an integer. Let T''_j be a closed positive $(1, 1)$ -current on X and T_j, T as above for $1 \leq j \leq l$. Assume that $T''_j \geq T_j$ for every $1 \leq j \leq l$ and T has no mass on $\bigcup_{j=1}^l I_{T''_j - T_j}$. Then, we have

$$\left\langle \bigwedge_{j=1}^l T''_j \wedge \bigwedge_{j=l+1}^m T_j \wedge T \right\rangle \geq \left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle$$

provided that the left-hand side is well defined.

- (vi) Let $1 \leq l \leq m$ be an integer. Assume $R := \langle \bigwedge_{j=l+1}^m T_j \wedge T \rangle$ and $\langle \bigwedge_{j=1}^l T_j \wedge R \rangle$ are well defined. Then, we have $\langle \bigwedge_{j=1}^m T_j \wedge T \rangle = \langle \bigwedge_{j=1}^l T_j \wedge R \rangle$.
- (vii) Let A be a complete pluripolar set. Assume that $\langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \wedge T \rangle$ is well defined. Then, we have

$$\mathbf{1}_{X \setminus A} \langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \wedge T \rangle = \langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \wedge (\mathbf{1}_{X \setminus A} T) \rangle.$$

In particular, the equality

$$\left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle = \left\langle \bigwedge_{j=1}^m T_j \wedge T' \right\rangle$$

holds, where $T' := \mathbf{1}_{X \setminus \bigcup_{j=1}^m I_{T_j}} T$.

Proof Properties (i), (ii) are clear from the definition and the proof of Lemma 3.1. We now check (iii). Let $R := \langle \bigwedge_{j=1}^m T_j \wedge T \rangle$. Since T has no mass on the complete pluripolar set A , using Lemma 2.1 gives $\mathbf{1}_A \bigwedge_{j=1}^m \text{dd}^c u_{jk} \wedge T = 0$ for every k . Using this and Lemma 3.1, we deduce that

$$\mathbf{1}_A \mathbf{1}_{\bigcap_{j=1}^m \{u_j > -k\}} R = \mathbf{1}_A \mathbf{1}_{\bigcap_{j=1}^m \{u_j > -k\}} \bigwedge_{j=1}^m \text{dd}^c u_{jk} \wedge T = 0.$$

Hence $R = 0$ on A and (iii) follows.

We prove (iv). We work on a small local chart U . Write $T'_j = \text{dd}^c u'_j$, $u'_{jk} := \max\{u'_j, -k\}$. Recall $T_j = \text{dd}^c u_j$. We can assume $u_j, u'_j \leq 0$. Let $R' := \langle T'_1 \wedge \bigwedge_{j=2}^m T_j \wedge T \rangle$ and $R'' := \langle (T_1 + T'_1) \wedge \bigwedge_{j=2}^m T_j \wedge T \rangle$. We have

$$\max\{u_1 + u'_1, -k\} = u_{1k} + u'_{1k}$$

on $\{u_1 + u'_1 > -k\}$ because the last set is contained in $\{u_1 > -k\} \cap \{u'_1 > -k\}$. This combined with Lemma 3.1 yields

$$\mathbf{1}_{B_k''} R'' = \mathbf{1}_{B_k'} R + \mathbf{1}_{B_k''} R',$$

where $B_k'' := \{u_1 + u'_1 > -k\} \cap \bigcap_{j=2}^m \{u_j > -k\}$. Letting $k \rightarrow \infty$ in the last equality gives (3.12). Observe that

$$\mathbf{1}_{U \setminus B_k''} R \rightarrow \mathbf{1}_{I_{T_1} \cup I_{T'_1} \cup \bigcup_{j=2}^m I_{T_j}} R$$

as $k \rightarrow \infty$. The last limit is equal to $\mathbf{1}_{I_{T'_1}} R$ because R has no mass on $\bigcup_{j=1}^m I_{T_j}$. Moreover since $I_{T'_1}$ is complete pluripolar and T has no mass on $I_{T'_1}$, by (iii), we have that $\mathbf{1}_{I_{T'_1}} R = 0$. Consequently, $\mathbf{1}_{U \setminus B_k''} R \rightarrow 0$ as $k \rightarrow \infty$ and a similar property of R' also holds. Thus, we obtain the equality in (3.12) if T has no mass on $I_{T_1} \cup I_{T'_1}$. To get (v), we just need to decompose $T_j'' = T_j + T'_j$ for some closed positive current T'_j and use similar arguments as in the proof of (iv).

We prove (vi). We can assume $u_j \leq 0$ for every $1 \leq j \leq m$. Let

$$\psi_k := k^{-1} \max \left\{ \sum_{j=1}^m u_j, -k \right\} + 1.$$

Observe that $0 \leq \psi_k \leq 1$ and

$$\psi_k = 0 \quad \text{on} \quad \bigcup_{j=1}^m \{u_j \leq -k\}.$$

Note that R has no mass on I_{T_j} for $l+1 \leq j \leq m$. This combined with (iii) yields that $\langle \bigwedge_{j=1}^l T_j \wedge R \rangle$ gives no mass on I_{T_j} for $1 \leq j \leq m$. Using this and the fact that $\psi_k \nearrow \mathbf{1}_{X \setminus \bigcup_{j=1}^m I_{T_j}}$ yields

$$\left\langle \bigwedge_{j=1}^l T_j \wedge R \right\rangle = \lim_{k \rightarrow \infty} \psi_k \left\langle \bigwedge_{j=1}^l T_j \wedge R \right\rangle = \lim_{k \rightarrow \infty} \psi_k \bigwedge_{j=1}^l dd^c u_{jk} \wedge R. \quad (3.13)$$

Now since $\psi_k R = \psi_k \bigwedge_{j=l+1}^m dd^c u_{jk} \wedge T$ (Lemma 3.1), we get

$$\psi_k \bigwedge_{j=1}^l dd^c u_{jk} \wedge R = \psi_k \bigwedge_{j=1}^m dd^c u_{jk} \wedge T. \quad (3.14)$$

To see why the last equality is true, notice that it is clear if u_{jk} 's are smooth and in general, we can use sequences of smooth psh functions decreasing to u_{jk} for $1 \leq j \leq l$ and the convergences of Monge–Ampère operators to obtain (3.14). Combining (3.14) with (3.13) gives the desired assertion.

It remains to prove (vii). Let ψ_k be as above. Let $A = \{\varphi = -\infty\}$, where φ is a negative psh function. Define $\psi'_k := k^{-1} \max \left\{ \varphi + \sum_{j=1}^m u_j, -k \right\} + 1$. Since $0 \leq \psi'_k \leq \psi_k$, we get $\{\psi_k = 0\} \subset \{\psi'_k = 0\}$. It follows that

$$\psi'_k \langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \wedge T \rangle = \psi'_k \bigwedge_{j=1}^m dd^c u_{jk} \wedge T = \psi'_k \bigwedge_{j=1}^m dd^c u_{jk} \wedge (\mathbf{1}_{X \setminus A} T)$$

because $T = \mathbf{1}_{X \setminus A} T$ on $\{\psi'_k \neq 0\}$ and the convergence of Monge–Ampère operators. Letting $k \rightarrow \infty$ gives the desired assertion. The proof is finished. $\square$

The following result clarifies the relationship between the relative non-pluripolar product and some other known notions of intersection.

Proposition 3.6

- (i) Let U, u_j, u_{jk}, R_k be as above. Assume that u_j is locally integrable with respect to $T_{j+1} \wedge \cdots \wedge T_m \wedge T$ for $1 \leq j \leq m$ and for every bounded psh function v , the current vR_k converges to $vT_1 \wedge \cdots \wedge T_m \wedge T$ on U as $k \rightarrow \infty$. Then, the current $\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle$ is well defined on U and

$$\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle = \mathbf{1}_{X \setminus \bigcup_{j=1}^m T_j} T_1 \wedge \cdots \wedge T_m \wedge T. \quad (3.15)$$

In particular, if $u_1, \dots, u_{m-1}$ are locally bounded and u_m is locally integrable with respect to T , then we have

$$\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle = \mathbf{1}_{X \setminus I_{T_m}} T_1 \wedge \cdots \wedge T_m \wedge T. \quad (3.16)$$

- (ii) If T is of bi-degree $(1, 1)$, then we have

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \mathbf{1}_{X \setminus I_T} \langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle, \quad (3.17)$$

where the left-hand side is the usual non-pluripolar product of $T_1, \dots, T_m, T$.

We note that in the above (i), the current $T_j \wedge \cdots \wedge T_m \wedge T$ ($1 \leq j \leq m$) is defined inductively as in the classical case (see [1]). The assumption of (i) of Proposition 3.6 is satisfied in well-known classical contexts; we refer to [16] and references therein for details. We notice also that by (vii) of Proposition 3.5, the right-hand side of (3.17) is equal to $\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} (\mathbf{1}_{X \setminus I_T} T) \rangle$.

Proof We prove (i). The question is local. Hence, we can assume $u_j \leq 0$ for every j . Let $\psi_k := k^{-1} \max\{\sum_{j=1}^m u_j, -k\} + 1$. By hypothesis, we get

$$\psi_k R_r \rightarrow \psi_k T_1 \wedge \cdots \wedge T_m \wedge T \quad (3.18)$$

as $r \rightarrow \infty$. Since $\psi_k = 0$ on $\{u_j \leq -k\}$ and $u_{jr} = u_{jk}$ on $\{u_j > -k\}$ for $r \geq k$, using Theorem 2.9, we infer that $\psi_k R_r = \psi_k R_k$. Consequently, the mass $\psi_k R_k$ on compact subsets of U is bounded uniformly in k . Hence, $\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle$ is well defined on U . The above arguments also show that

$$\psi_k R_r = \psi_k R_k = \psi_k \mathbf{1}_{\bigcap_{j=1}^m \{u_j > -k\}} R_r = \psi_k \mathbf{1}_{\bigcap_{j=1}^m \{u_j > -k\}} \langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle$$

which is equal to $\psi_k \langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle$. Letting $r \rightarrow \infty$ in the last equality and using (3.18) give

$$\psi_k T_1 \wedge \cdots \wedge T_m \wedge T = \psi_k \langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle.$$

Letting $k \rightarrow \infty$ gives (3.15). To obtain (3.16), we just need to combine (3.15) with Theorem 2.2.

It remains to check (ii). We work locally. Let u be a local potential of T and $u \leq 0$. Let $u_k := \max\{u, -k\}$ and $\psi'_k := k^{-1} \max\{u + \sum_{j=1}^m u_j, -k\} + 1$. Note that

$$\psi'_k \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \psi'_k \bigwedge_{j=1}^m \mathrm{dd}^c u_{jk} \wedge \mathrm{dd}^c u_k.$$

Since $\{\psi_k = 0\} \subset \{\psi'_k = 0\}$ ($\psi_k \geq \psi'_k \geq 0$), we have

$$\begin{aligned} \psi'_k \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle &= \psi'_k R_k = \psi'_k \bigwedge_{j=1}^m \mathrm{dd}^c u_{jk} \wedge \mathrm{dd}^c u \\ &= \psi'_k \mathbf{1}_{\{u > -k\}} \bigwedge_{j=1}^m \mathrm{dd}^c u_{jk} \wedge \mathrm{dd}^c u \\ &= \psi'_k \mathbf{1}_{\{u > -k\}} \bigwedge_{j=1}^m \mathrm{dd}^c u_{jk} \wedge \mathrm{dd}^c u_k \end{aligned}$$

(one can obtain the last equality as a consequence of (3.16) applied to the case where T is the current of integration along X or alternatively we can use Theorem 2.2 directly). Letting $k \rightarrow \infty$ in the last equality gives (3.17). This finishes the proof. $\square$

As in the case of the usual non-pluripolar products, the relative non-pluripolar products, if well defined, are closed positive currents as showed by the following result.

Theorem 3.7 *Assume that $\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$ is well defined. Then, $\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$ is closed.*

Proof The proof is based on ideas from [4, 21]. We work on a local chart U as above. By shrinking U and subtracting from u_j a suitable constant, we can assume that $u_j \leq 0$. Let

$$\psi_k := k^{-1} \max \left\{ \sum_{j=1}^m u_j, -k \right\} + 1.$$

Observe that $\psi_k = 0$ on $\cup_{j=1}^m \{u_j \leq -k\}$ and $0 \leq \psi_k \leq 1$ increases to $\mathbf{1}_{\cap_{j=1}^m \{u_j > -\infty\}}$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative smooth function such that $g(0) = g'(0) = g'(1) = 0$ and $g(1) = 1$. Let R_k, R be as above. Since $g(1) = 1$ and $g(0) = 0$, we get $g(\psi_k)R \rightarrow R$ as $k \rightarrow \infty$ (recall R has no mass on $\cup_{j=1}^m \{u_j = -\infty\}$). Thus, the desired assertion is equivalent to proving that

$$dR = \lim_{k \rightarrow \infty} d(g(\psi_k)R) = 0. \quad (3.19)$$

Since $g(\psi_k) = g(0) = 0$ on $\cup_{j=1}^m \{u_j \leq -k\}$, we have

$$d(g(\psi_k)R) = d(g(\psi_k)R_k) = g'(\psi_k)d\psi_k \wedge R_k$$

(see [4, Lemma 1.9] or Corollary 2.5 for the second equality). Let $U_1 \Subset U_2$ be relatively compact open subsets of U . By the Cauchy–Schwarz inequality, we have

$$\|g'(\psi_k)d\psi_k \wedge R_k\|_{U_1} \lesssim \|d\psi_k \wedge \mathrm{d}^c \psi_k \wedge R_k\|_{U_1} \|g'(\psi_k)^2 R_k\|_{U_1}.$$

Using $d\psi_k \wedge \mathrm{d}^c \psi_k = \mathrm{dd}^c \psi_k^2 - \psi_k \mathrm{dd}^c \psi_k$ and the Chern–Levine–Nirenberg inequality, one get

$$\|d\psi_k \wedge d^c \psi_k \wedge R_k\|_{U_1} \lesssim \|\psi_k R_k\|_{U_2} \leq \|R_k\|_{\overline{U_2} \cap \cap_{j=1}^m \{u_j > -k\}} \lesssim 1$$

by (3.2). On the other hand, using the equality $g'(\psi_k) = g'(0) = 0$ on $\{u_j \leq -k\}$, we obtain

$$g'(\psi_k)^2 R_k = g'(\psi_k)^2 \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k = g'(\psi_k)^2 R$$

converging to 0 as $k \rightarrow \infty$ because $g'(\psi_k) \rightarrow g'(1) = 0$ on $\cap_{j=1}^m \{u_j > -\infty\}$. Thus, (3.19) follows. This finishes the proof. $\square$

Remark 3.8 Let V be a smooth submanifold of X and T the current of integration along V . If $V \subset \cup_{j=1}^m I_{T_j}$, then the non-pluripolar product relative to T of $T_1, \dots, T_m$ is zero. Consider now the case where $V \not\subset \cup_{j=1}^m I_{T_j}$. In this case, we can define a current T'_j which can be viewed as the intersection of T_j and T as follows. Let $u_1, \dots, u_m$ be local potentials of $T_1, \dots, T_m$, respectively. Let $T'_j := dd^c(u_j|_V)$ for $1 \leq j \leq m$. One can see that T'_j is independent of the choice of u_j ; hence, T'_j is a well-defined closed positive $(1, 1)$ -current on V . Let $\iota : V \hookrightarrow X$ be the natural inclusion. We can check that $\langle \bigwedge_{j=1}^m T_j \wedge T \rangle$ is equal to the pushforward by ι of $\langle \bigwedge_{j=1}^m T'_j \rangle$.

4 Monotonicity

In this section, we present a monotonicity property for relative non-pluripolar products. We begin with the following simple lemma.

Lemma 4.1 *Let U be an open subset of $\mathbb{C}^n$. Let $(u_j^l)_l$ be a sequence of psh functions on U and u_j a psh function on U for $1 \leq j \leq m$. Let $(T_l)_l$ be a sequence of closed positive currents satisfying Condition (*) and T_l converges to a closed positive current T on U as $l \rightarrow \infty$. Assume that one of the following conditions is satisfied:*

- (i) $u_j^l \geq u_j$ and u_j^l converges to u_j in L^1_{loc} as $l \rightarrow \infty$.
- (ii) u_j^l increases to u_j as $l \rightarrow \infty$ almost everywhere and T has no mass on pluripolar sets.

Then, for every smooth weakly positive form Φ with compact support in U , we have

$$\liminf_{l \rightarrow \infty} \int_U \left\langle \bigwedge_{j=1}^m dd^c u_j^l \wedge T_l \right\rangle \wedge \Phi \geq \int_U \left\langle \bigwedge_{j=1}^m dd^c u_j \wedge T \right\rangle \wedge \Phi.$$

When T is the current of integration along X , a related statement in the compact setting was given in [9, Theorem 2.3].

Proof Let Φ be a smooth weakly positive form with compact support in U . Assume (i) holds. Let $u_{jk}^l := \max\{u_j^l, -k\}$ which converges to $u_{jk} := \max\{u_j, -k\}$ in L^1_{loc} as $l \rightarrow \infty$ and $u_{jk}^l \geq u_{jk}$. Put

$$R^l := \left\langle \bigwedge_{j=1}^m \mathrm{dd}^c u_j^l \wedge T_l \right\rangle, \quad R_k^l := \bigwedge_{j=1}^m \mathrm{dd}^c u_{jk}^l \wedge T.$$

Similarly, we define R, R_k by using the formula of R^l, R_k^l , respectively, with u_j^l, u_{jk}^l replaced by u_j, u_{jk} . Since $u_j^l \geq u_j$, we get $\{u_j > -k\} \subset \{u_j^l > -k\}$. It follows that

$$\int_U \mathbf{1}_{\cap_{j=1}^m \{u_j^l > -k\}} R_k^l \wedge \Phi \geq \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k^l \wedge \Phi. \quad (4.1)$$

By Theorem 2.6, we get $R_k^l \rightarrow R_k$ as $l \rightarrow \infty$. Using this together with the fact that when k is fixed, u_{jk}^l is uniformly bounded in l , we see that the strong uniform quasi-continuity for u_{jk} with respect to $(T)_l$ (see Theorem 2.4, and also [2, Corollary 3.3]) implies

$$\begin{aligned} \liminf_{l \rightarrow \infty} \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k^l \wedge \Phi &= \liminf_{l \rightarrow \infty} \int_U \mathbf{1}_{\cap_{j=1}^m \{u_{jk} > -k\}} R_k^l \wedge \Phi \\ &\geq \int_U \mathbf{1}_{\cap_{j=1}^m \{u_{jk} > -k\}} R_k \wedge \Phi = \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k \wedge \Phi \\ &= \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R \wedge \Phi. \end{aligned}$$

This combined with (4.1) yields

$$\begin{aligned} \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} R \wedge \Phi &\leq \liminf_{l \rightarrow \infty} \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j^l > -k\}} R_k^l \wedge \Phi \\ &\leq \liminf_{l \rightarrow \infty} \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j^l > -k\}} R^l \wedge \Phi \leq \liminf_{l \rightarrow \infty} \int_U R^l \wedge \Phi \end{aligned}$$

for every k . Letting $k \rightarrow \infty$ in the last inequality and noticing that R has no mass on $\cup_{j=1}^m \{u_j = -\infty\}$ give the desired assertion.

Now assume (ii). Note that

$$u_j^l \leq u_j^{l+1} \leq u_j$$

for every l, j . Thus, $\{u_j^l > -k\} \subset \{u_j^{l+1} > -k\} \subset \{u_j > -k\}$ and $\tilde{u}_j := \lim_{l \rightarrow \infty} u_j^l \leq u_j$. Using this and arguments similar to those in the previous paragraph, we have

$$\liminf_{l \rightarrow \infty} \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j^l > -k\}} R_k^l \wedge \Phi \geq \int_U \mathbf{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} R \wedge \Phi.$$

Letting $k \rightarrow \infty$ gives

$$\liminf_{l \rightarrow \infty} \int_U R^l \wedge \Phi \geq \int_U \mathbf{1}_{\cap_{j=1}^m \{\tilde{u}_j > -\infty\}} R \wedge \Phi.$$

Recall that $\{u_j > \tilde{u}_j\}$ is pluripolar. This coupled with the fact that T has no mass on pluripolar sets and Property (iii) of Proposition 3.5 yields

$$\liminf_{l \rightarrow \infty} \int_U R^l \wedge \Phi \geq \int_U \mathbf{1}_{\cap_{j=1}^m \{u_j > -\infty\}} R \wedge \Phi = \int_U R \wedge \Phi.$$

This finishes the proof. $\square$

Recall that for closed positive $(1, 1)$ -currents R_1, R_2 on X , we say that R_1 is *less singular* than R_2 if for every local chart U and psh function w_j on U such that $R_j = \text{dd}^c w_j$ on U for $j = 1, 2$, then $w_2 \leq w_1 + O(1)$ on compact subsets of U ; and R_1, R_2 are of the *same singularity type* if $w_1 \leq w_2 + O(1)$ and $w_2 \leq w_1 + O(1)$ on compact subsets of U . The following generalizes [9, Proposition 2.1], see also [19, 22].

Proposition 4.2 *Let X be a compact complex manifold of dimension n . Let m be an integer such that $1 \leq m \leq n$. Let T be a closed positive current of bi-degree (p, p) on X . Let T_j, T'_j be closed positive $(1, 1)$ -currents on X for $1 \leq j \leq m$ such that T_j, T'_j are of the same singularity type and $T_j = \text{dd}^c u_j + \theta_j$, $T'_j := \text{dd}^c u'_j + \theta_j$, where θ_j is a smooth form and u'_j, u_j are θ_j -psh functions, for every $1 \leq j \leq m$. Assume that for every $J, J' \subset \{1, \dots, m\}$ such that $J \cap J' = \emptyset$, the product $\langle \bigwedge_{j \in J} T_j \wedge \bigwedge_{j' \in J'} T'_{j'} \wedge T \rangle$ is well defined. Then, for every dd^c -closed smooth form Φ , we have*

$$\int_X \langle T'_1 \wedge \dots \wedge T'_m \wedge T \rangle \wedge \Phi = \int_X \langle T_1 \wedge \dots \wedge T_m \wedge T \rangle \wedge \Phi. \quad (4.2)$$

Proof By compactness of X , we can assume $u_j, u'_j \leq 0$. By the hypothesis, $\{u_j = -\infty\} = \{u'_j = -\infty\}$ and $w_j := u_j - u'_j$ is bounded outside $\{u_j = -\infty\}$. Without loss of generality, we can assume that

$$|w_j| \leq 1 \quad (4.3)$$

outside $\{u_j = -\infty\}$. Let $A := \bigcup_{j=1}^m \{u_j = -\infty\}$ which is a complete pluripolar set. Put $u_{jk} := \max\{u_j, -k\}$, $u'_{jk} := \max\{u'_j, -k\}$ and

$$\psi_k := k^{-1} \max \left\{ \sum_{j=1}^n (u_j + u'_j), -k \right\} + 1 \quad (4.4)$$

which is quasi-psh and $0 \leq \psi_k \leq 1$, $\psi_k(x)$ increases to 1 for $x \notin A$. We have $\psi_k(x) = 0$ if $u_j(x) \leq -k$ or $u'_j(x) \leq -k$ for some j . Put $w_{jk} := u_{jk} - u'_{jk}$. By (4.3), we have

$$|w_{jk}| \leq 1 \quad (4.5)$$

on X . Let $J, J' \subset \{1, \dots, m\}$ with $J \cap J' = \emptyset$ and

$$R_{JJ'k} := \bigwedge_{j \in J} (\text{dd}^c u_{jk} + \theta_j) \wedge \bigwedge_{j' \in J'} (\text{dd}^c u'_{j'k} + \theta_{j'}) \wedge T.$$

The last current is the difference of two closed positive $(|J| + |J'| + p, |J| + |J'| + p)$ -currents. Hence, $R_{JJ'k}$ might not be positive in general and it is not clear how to control its mass as $k \rightarrow \infty$. This is a subtle point which we need to pay attention to. The relative non-pluripolar product

$$R_{JJ'} := \left\langle \bigwedge_{j \in J} (\text{dd}^c u_j + \theta_j) \wedge \bigwedge_{j' \in J'} (\text{dd}^c u'_{j'} + \theta_{j'}) \wedge T \right\rangle$$

exists by our assumption. Let

$$B_k := \cap_{j \in J} \{u_j > -k\} \cap \cap_{j' \in J'} \{u_{j'} > -k\}.$$

By Lemma 3.2, we get

$$0 \leq \mathbf{1}_{B_k} R_{JJ'} = \mathbf{1}_{B_k} R_{JJ'k}$$

for every J, J', k . Put $\tilde{R}_{JJ'} := \mathbf{1}_{X \setminus A} R_{JJ'}$. The last current is closed (for example see [4, Remark 1.10]) and positive because $R_{JJ'} \geq 0$. Using the fact that $\{\psi_k \neq 0\} \subset B_k \setminus A$, we get

$$\psi_k \tilde{R}_{JJ'} = \psi_k R_{JJ'} = \psi_k R_{JJ'k}. \quad (4.6)$$

Claim. Let $j'' \in \{1, \dots, m\} \setminus (J \cup J')$. Let Φ be dd^c -closed smooth form of bi-degree (p', p') on X , where $p' := n - |J| - |J'| - p - 1$. Then,

$$\lim_{k \rightarrow \infty} \int_X \psi_k \text{dd}^c w_{j''k} \wedge R_{JJ'k} \wedge \Phi = 0. \quad (4.7)$$

We prove Claim. Let ω be a Hermitian metric on X . Let $\eta := \sum_{j=1}^m 2\theta_j$. We have $\text{dd}^c \psi_k + k^{-1}\eta \geq 0$ for every k . By integration by parts and (4.6),

$$\int_X \psi_k \text{dd}^c w_{j''k} \wedge R_{JJ'k} \wedge \Phi = \int_X w_{j''k} \text{dd}^c (\psi_k R_{JJ'k} \wedge \Phi) = \int_X w_{j''k} \text{dd}^c (\psi_k \tilde{R}_{JJ'} \wedge \Phi). \quad (4.8)$$

Observe that

$$\text{dd}^c (\psi_k \tilde{R}_{JJ'} \wedge \Phi) = \text{dd}^c \psi_k \wedge \Phi \wedge \tilde{R}_{JJ'} + 2d\psi_k \wedge d^c \Phi \wedge \tilde{R}_{JJ'}$$

because $\text{dd}^c \Phi = 0$ and Φ is of bi-degree (p', p') . Write $d^c \Phi$ locally as a complex linear combination of forms like $\tau_j \wedge \Phi_j$, where τ_j is a $(0, 1)$ -form or a $(1, 0)$ -form and Φ_j is a positive form. Hence, we can use the Cauchy–Schwarz inequality to obtain

$$\left| \int_X w_{j''k} d\psi_k \wedge d^c \Phi \wedge \tilde{R}_{JJ'} \right| \leq \left(\int_X |w_{j''k}|^2 |d\psi_k \wedge d^c \psi_k \wedge \tilde{R}_{JJ'} \wedge \Phi_0| \right)^{\frac{1}{2}} \left(\int_X \tilde{R}_{JJ'} \wedge \Phi_0 \wedge \omega \right)^{\frac{1}{2}},$$

where $\Phi_0 := \omega^{n-|J|-|J'|-1}$. We deduce that

$$\left| \int_X w_{j''k} d\psi_k \wedge d^c \Phi \wedge \tilde{R}_{JJ'} \right| \lesssim \|\tilde{R}_{JJ'}\|^{\frac{1}{2}} \left(\int_X d\psi_k \wedge d^c \psi_k \wedge \tilde{R}_{JJ'} \wedge \Phi_0 \right)^{\frac{1}{2}}$$

by (4.5). Recall that $\{\lim_{k \rightarrow \infty} \psi_k < 1\}$ is equal to the complete pluripolar set A . Using this, Remark 2.7 and the fact that $\tilde{R}_{JJ'}$ has no mass on A , we get

$$\lim_{k \rightarrow \infty} d\psi_k \wedge d^c \psi_k \wedge \tilde{R}_{JJ'} = \lim_{k \rightarrow \infty} (\text{dd}^c \psi_k^2 - \psi_k \text{dd}^c \psi_k) \wedge \tilde{R}_{JJ'} = 0$$

(we recall that to get Remark 2.7 for $\tilde{R}_{JJ'}$ and ψ_k , we need to use the strong quasi-continuity of ψ_k with respect to the capacity associated to $\tilde{R}_{JJ'}$). Thus, we obtain

$$\lim_{k \rightarrow \infty} \int_X w_{j''k} d\psi_k \wedge d^c \Phi \wedge \tilde{R}_{JJ'} = 0. \quad (4.9)$$

On the other hand, since

$$\int_X w_{j''k} dd^c \psi_k \wedge \Phi \wedge \tilde{R}_{JJ'} = - \int_X dw_{j''k} \wedge d^c \psi_k \wedge \Phi \wedge \tilde{R}_{JJ'} + \int_X w_{j''k} d^c \psi_k \wedge d\Phi \wedge \tilde{R}_{JJ'},$$

using similar arguments, we get

$$\lim_{k' \rightarrow \infty} \left| \int_X w_{j''k} dd^c \psi_k \wedge \Phi \wedge \tilde{R}_{JJ'} \right| = 0$$

Combining this with (4.9) and (4.8) yields (4.7). Claim follows.

Now let $S := \langle T_1 \wedge \cdots T_n \wedge T \rangle - \langle T'_1 \wedge \cdots T'_n \wedge T \rangle$. Using $T_{jk} = T'_{jk} + dd^c w_{jk}$, one can check that

$$\begin{aligned} \int_X \psi_k S \wedge \Phi &= \int_X \psi_k \bigwedge_{j=1}^m T_{jk} \wedge T \wedge \Phi - \int_X \psi_k \bigwedge_{j=1}^m T'_{jk} \wedge T \wedge \Phi \\ &= \sum_{s=1}^m \int_X \psi_k \bigwedge_{j=1}^{s-1} T'_{jk} \wedge dd^c w_{sk} \wedge \bigwedge_{j=s+1}^m T_{jk} \wedge T \wedge \Phi. \end{aligned}$$

This together with Claim yields $\langle S, \Phi \rangle = \lim_{k \rightarrow \infty} \langle \psi_k S, \Phi \rangle = 0$ for every dd^c -closed smooth Φ . This finishes the proof. $\square$

Remark 4.3 Our proof of Proposition 4.2 still works if T'_j, T_j are not in the same cohomology class. In this case, one just needs to modify (4.2) accordingly.

The following result is the monotonicity property of relative non-pluripolar products mentioned in Introduction.

Theorem 4.4 *Let X be a compact Kähler manifold of dimension n . Let T_j, T'_j be closed positive $(1, 1)$ -currents on X for $1 \leq j \leq m$ such that T_j, T'_j are in the same cohomology class for every j and T'_j is less singular than T_j for $1 \leq j \leq m$. Let T be a closed positive current on X . Then, we have*

$$\{\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \wedge T \rangle\}. \quad (4.10)$$

Proof Write $T_j = dd^c u_j + \theta_j$, $T'_j = dd^c u'_j + \theta_j$. Without loss of generality, we can assume that $u'_j \geq u_j$. For $l \in \mathbb{N}$, put $u'_j := \max\{u_j, u'_j - l\}$ which is of the same singularity type as u'_j . Notice that $dd^c u'_j + \theta_j \geq 0$. Since X is Kähler, the current $\langle \bigwedge_{j=1}^m (dd^c u'_j + \theta_j) \wedge T \rangle$ is of mass uniformly bounded in l . Let S be a limit current of $\langle \bigwedge_{j=1}^m (dd^c u'_j + \theta_j) \wedge T \rangle$ as $l \rightarrow \infty$. Since u'_j decreases to u_j as $l \rightarrow \infty$, we can apply Lemma 4.1 to get

$$S \geq \left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle.$$

Consequently, $\{S\} \geq \{\langle \bigwedge_{j=1}^m T_j \wedge T \rangle\}$. Using Proposition 4.2, we see that $\{S\}$ is equal to $\{\langle \bigwedge_{j=1}^m T'_j \wedge T \rangle\}$. The desired assertion, hence, follows. This finishes the proof. $\square$

Remark 4.5 Let the notation be as in Theorem 4.4. Let T be of bi-degree $(1, 1)$ and T' a closed positive $(1, 1)$ -current which is less singular than T . Then, by using arguments similar to those in the proof of Theorem 4.4, we can prove that

$$\{\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \wedge T' \rangle\} \quad (4.11)$$

(here we need to use Lemma 4.1 for a suitable sequence $(T_l)_l$ provided by Theorem 2.2). The inequality (4.11) offers us a way to define a notion of full mass intersection when T is of bi-degree $(1, 1)$. This notion differs from those used below and in [4], albeit all of them are closely related. We will not go into details in this paper.

Consider, from now on, a compact Kähler manifold X with a Kähler form ω . Let T be a closed positive (p, p) -current on X . For every pseudoeffective $(1, 1)$ -class β in X , we define its *polar locus* I_β to be that of a current with minimal singularities in β . This is independent of the choice of a current with minimal singularities.

Let $\alpha_1, \dots, \alpha_m$ be pseudoeffective $(1, 1)$ -classes of X . Let $T_{1,\min}, \dots, T_{m,\min}$ be currents with minimal singularities in the classes $\alpha_1, \dots, \alpha_m$, respectively. By Theorem 4.4 and Lemma 3.4, the class $\{\langle T_{1,\min} \wedge \cdots \wedge T_{m,\min} \wedge T \rangle\}$ is a well-defined pseudoeffective class which is independent of the choice of $T_{j,\min}$. We denote the last class by $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \wedge T \rangle\}$.

Proposition 4.6

- (i) The product $\{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\}$ is symmetric and homogeneous in $\alpha_1, \dots, \alpha_m$.
- (ii) Let α'_1 be a pseudoeffective $(1, 1)$ -class. Assume that T has no mass on $I_{\alpha_1} \cup I_{\alpha'_1}$. Then, we have

$$\{\langle (\alpha_1 + \alpha'_1) \wedge \bigwedge_{j=2}^m \alpha_j \wedge T \rangle\} \geq \{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\} + \{\langle \alpha'_1 \wedge \bigwedge_{j=2}^m \alpha_j \wedge T \rangle\}.$$

- (iii) Let $1 \leq l \leq m$ be an integer. Let $\alpha''_1, \dots, \alpha''_l$ be a pseudoeffective $(1, 1)$ -class such that $\alpha''_j \geq \alpha_j$ for $1 \leq j \leq l$. Assume that T has no mass on $I_{\alpha''_j - \alpha_j}$ for every $1 \leq j \leq l$. Then, we have

$$\{\langle \bigwedge_{j=1}^l \alpha''_j \wedge \bigwedge_{j=l+1}^m \alpha_j \wedge T \rangle\} \geq \{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\}.$$

- (iv) If T has no mass on proper analytic subsets on X , then the product $\{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\}$ is continuous on the set of $(\alpha_1, \dots, \alpha_m)$ such that $\alpha_1, \dots, \alpha_m$ are big.
- (v) If T has no mass on proper analytic subsets on X and $\alpha_1, \dots, \alpha_m$ are big nef, then we have

$$\{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\} = \bigwedge_{j=1}^m \alpha_j \wedge \{T\}.$$

We refer to [4, 5] for related statements in the case where T is the current of integration along X .

Proof The desired assertion (i) follows from Proposition 3.5. We now prove (ii). Let $T_{\min, \alpha_j}, T_{\min, \alpha'_j}$ be currents with minimal singularities in α_j, α'_j , respectively. Observe that $T_{\min, \alpha_j} + T_{\min, \alpha'_j}$ are in $(\alpha_j + \alpha'_j)$. Thus, by Theorem 4.4, we get

$$\{ \langle (\alpha_1 + \alpha'_1) \wedge \bigwedge_{j=2}^m \alpha_j \wedge T \rangle \} \geq \{ \langle (T_{\min, \alpha_1} + T_{\min, \alpha'_1}) \wedge \bigwedge_{j=2}^m T_{\min, \alpha_j} \wedge T \rangle \}.$$

The last class is equal to $\{ \langle \bigwedge_{j=1}^m T_{\min, \alpha_j} \wedge T \rangle \} + \{ \langle T_{\min, \alpha'_1} \wedge \bigwedge_{j=2}^m T_{\min, \alpha_j} \wedge T \rangle \}$ because of the hypothesis and Property (iv) of Proposition 3.5. Hence, (ii) follows. Similarly, we get (iii) by using Property (v) of Proposition 3.5.

We prove (iv). Observe that by a result of Demailly on analytic approximation of currents ([12]), the polar locus I_β of a big class β is contained in a proper analytic subset of X if α is big. Using this, we see that (iv) is a direct consequence of (iii) and the observation that given a constant $\epsilon > 0$, for every α'_j closed enough to α_j , we have that the classes $\alpha'_j - (1 - \epsilon)\alpha_j$ and $(1 + \epsilon)\alpha_j - \alpha'_j$ are big (we use here the bigness of α_j).

It remains to check (v). By (iv) and the bigness of α_j , we get

$$\lim_{\epsilon \rightarrow 0} \{ \langle \bigwedge_{j=1}^m (\alpha_j + \epsilon \{\omega\}) \wedge T \rangle \} = \{ \langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle \}.$$

Since α_j is nef, the limit in the left-hand side of the last equality is equal to $\bigwedge_{j=1}^m \alpha_j \wedge \{T\}$. The proof is finished. $\square$

When T is the current of integration along X , we write $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ for $\{ \langle \alpha_1 \wedge \cdots \wedge \alpha_m \wedge T \rangle \}$. We would like to comment that the class $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ is always bounded from above by the *positive product* of $\alpha_1, \dots, \alpha_m$ defined in [4, Definition 1.17]. They are equal if $\alpha_1, \dots, \alpha_m$ are big by Property (iv) of Proposition 4.6. However, we do not know if they are equal in general, even if $\alpha_1, \dots, \alpha_m$ are nef.

Question 1 Given nef classes $\alpha_1, \dots, \alpha_m$, is $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ defined above equal to the positive product of $\alpha_1, \dots, \alpha_m$ introduced in [4]?

Let $T_1, \dots, T_m$ be closed positive (1, 1)-currents on X . By Theorem 4.4, we have $\{ \langle \bigwedge_{j=1}^m T_j \wedge T \rangle \} \leq \{ \langle \bigwedge_{j=1}^m \{T_j\} \wedge T \rangle \}$. The equality occurs if the masses of these two classes are equal. This is the reason for the following definition.

Definition 4.7 We say that $T_1, \dots, T_m$ are of *full mass (non-pluripolar) intersection relative to T* if we have $\{ \langle \bigwedge_{j=1}^m T_j \wedge T \rangle \} = \{ \langle \bigwedge_{j=1}^m \{T_j\} \wedge T \rangle \}$.

When T is the current of integration along X , we simply say “full mass intersection” instead of “full mass intersection relative to T .” In the last case, we underline that this notion is the one given in [4] if $\{T_1\}, \dots, \{T_m\}$ are big. In general, if $T_1, \dots, T_m$ are of full mass intersection in the sense of [4], then they are so in the sense of Definition 4.7. However, we do not know whether the reversed statement holds.

Let $\mathcal{E}(\alpha_1, \dots, \alpha_m, T)$ be the set of $(T_1, \dots, T_m)$ such that $T_j \in \alpha_j$ for $1 \leq j \leq m$ and

$$\{\langle \bigwedge_{j=1}^m T_j \wedge T \rangle\} = \{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\},$$

or equivalently

$$\int_X \langle \bigwedge_{j=1}^m T_j \wedge T \rangle \wedge \omega^{n-m-p} = \int_X \langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle \wedge \omega^{n-m-p}.$$

Several less general versions of full mass intersections were introduced in [4, 9, 17]. Note that if $T_j = \text{dd}^c u_j + \theta_j$ for some smooth Kähler form θ_j and u_j a θ_j -psh function for every j , then $T_1, \dots, T_m$ are of full mass intersection relative to T if and only if

$$\int_{\bigcup_{j=1}^m \{u_j \leq -k\}} \bigwedge_{j=1}^m T_{jk} \wedge T \wedge \omega^{n-m-p} \rightarrow 0 \quad (4.12)$$

as $k \rightarrow \infty$, where $T_{jk} := \text{dd}^c \max\{u_j, -k\} + \theta_j$ for $1 \leq j \leq m$.

The following result tells us that currents with full mass intersection satisfy the convergence along decreasing or increasing sequences of potentials.

Theorem 4.8 *Let X be a compact Kähler manifold of dimension n . Let T_{jl} be a closed positive $(1, 1)$ -current for $1 \leq j \leq m, l \in \mathbb{N}$ such that T_{jl} converges to T_j as $l \rightarrow \infty$. Let $(U_s)_s$ be a finite covering of X by open subsets such that $T_{jl} = \text{dd}^c u_{j,l,s}$, $T_j = \text{dd}^c u_{j,s}$ on U_s for every s . Assume that*

- (i) $T_1, \dots, T_m$ are of full mass intersection relative to T ,
- (ii) $\{\langle \bigwedge_{j=1}^m \alpha_{jl} \wedge T \rangle\}$ converges to $\{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\}$ as $l \rightarrow \infty$, where $\alpha_{jl} := \{T_{jl}\}$, $\alpha := \{T_j\}$ and one of the following two conditions hold:
- (iii) $u_{j,l,s} \geq u_{j,s}$ and $u_{j,l,s}$ converges to $u_{j,s}$ in L^1_{loc} as $l \rightarrow \infty$,
- (iii') $u_{j,l,s}$ increases to $u_{j,s}$ as $l \rightarrow \infty$ almost everywhere and T has no mass on pluripolar sets.

Then we have

$$\langle T_{1l} \wedge \dots \wedge T_{ml} \wedge T \rangle \rightarrow \langle T_1 \wedge \dots \wedge T_m \wedge T \rangle$$

as $l \rightarrow \infty$.

Proof Observe that $\langle \bigwedge_{j=1}^m T_{jl} \wedge T \rangle$ is of uniformly bounded mass in l by (ii). Let S be a limit current of $\langle \bigwedge_{j=1}^m T_{jl} \wedge T \rangle$ as $l \rightarrow \infty$. By (ii) again, we get $\{S\} \leq \{\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle\}$. By Lemma 4.1 and Condition (iii) or (iii'), we have

$$S \geq \left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle.$$

By (i), the current in the right-hand side is of mass equal to $\langle \bigwedge_{j=1}^m \alpha_j \wedge T \rangle$ which is greater than or equal to the mass of S . So we get the equality $S = \langle \bigwedge_{j=1}^m T_j \wedge T \rangle$. The proof is finished. $\square$

Note that we do not require that $T_{1l}, \dots, T_{ml}$ are of full mass intersection relative to T . However, if T_{jl}, T_j are in the same cohomology class for every j, l , then by Theorem 4.4, Conditions (iii) and (i) of Theorem 4.8 imply that $T_{1l}, \dots, T_{ml}$ are of full mass intersection relative to T for every l . We notice here certain similarity of this result with [14, Proposition 5.4], where a notion of full mass intersection was considered for intersections of currents with analytic sets.

Here are some basic properties of currents with relative full mass intersection.

Lemma 4.9 *Let X be a compact Kähler manifold. Let $T_1, \dots, T_m, T$ be closed positive currents on X such that T_j is of bi-degree $(1, 1)$ and the cohomology class of T_j is Kähler for every j . Then, if $T_1, \dots, T_m$ are of full mass intersection relative to T , then the following three properties hold.*

- (i) T has no mass on $\cup_{j=1}^m I_{T_j}$.
- (ii) Let T'_j be closed positive $(1, 1)$ -currents whose cohomology class are Kähler for $1 \leq j \leq m$ such that T'_j is less singular than T_j for $1 \leq j \leq m$. Then $T'_1, \dots, T'_m$ are of full mass intersection relative to T .
- (iii) For every subset $J = \{j_1, \dots, j_{m'}\} \subset \{1, \dots, m\}$, the currents $T_{j_1}, \dots, T_{j_{m'}}$ are of full mass intersection relative to T . Moreover, if T is the current of integration along X , then T_j has no mass on I_{T_j} for $1 \leq j \leq m$. Moreover, we have
- (iv) $T_1, \dots, T_m$ are of full mass intersection relative to T if and only if $T_1 + C\omega, \dots, T_m + C\omega$ are of full mass intersection relative to T for every constant $C > 0$.

Proof The desired property (i) is a direct consequence of (vii) of Proposition 3.5 and the fact that $\{T_j\}$ is Kähler for every j . Note that if $\{T'_j\} = \{T_j\}$ for every j , then (ii) is a direct consequence of the monotonicity of relative non-pluripolar products (Theorem 4.4). The first claim of the desired property (iii) is deduced from the last assertion applies to $T'_j := T_j$ for $j \in J$ and $T'_j := \theta_j$ for $j \notin J$, where θ_j is a smooth Kähler form in the class $\{T_j\}$. In particular, we have $\langle T_j \rangle = \{T_j\}$. Recall that $\langle T_j \rangle = \mathbf{1}_{X \setminus I_{T_j}} T_j$ (Proposition 3.6). Hence,

$$\|\langle T_j \rangle\| + \|\mathbf{1}_{I_{T_j}} T_j\| = \|T_j\|.$$

It follows that $\mathbf{1}_{I_{T_j}} T_j = 0$ or equivalently, T_j has no mass on I_{T_j} . Hence, (iii) follows.

Now observe that (iv) is a direct consequence of (iii), Property (iv) of Proposition 3.5 and (i). Finally, when $\{T'_j\} \neq \{T_j\}$, Property (iv) allows us to add to T_j, T'_j suitable Kähler forms such that they are in the same cohomology class. So (ii) follows. The proof is finished. $\square$

In the last part of this section, we study Lelong numbers of currents of relative full mass intersection.

Lemma 4.10 *Let V be an irreducible analytic subset of X . Let u be a quasi-psh function on X such that the generic Lelong number of u along V is strictly positive. Let v be a quasi-psh function on X having logarithmic poles along V , i.e., locally near V , we have*

$$v = \lambda \log \sum_{j=1}^l |f_j| + O(1),$$

where $\lambda > 0$ is a constant, and $(f_j)_j$ is a local system of generators of the ideal sheaf associated to V . Then there exists a constant $c > 0$ such that $u \leq cv + O(1)$ on X .

Note that given an analytic set V in X , there always exists a quasi-psh function v as in the hypothesis of Lemma 4.10; see [13, Lemma 2.1].

Proof We can assume that u, v are ω -psh functions. Put $T := dd^c u + \omega$. Note that by hypothesis and Siu's semi-continuity theorem, there is a constant $c_0 > 0$ such that $v(u, x) \geq c_0 > 0$ for every $x \in V$ and $v(u, x) = c_0$ for generic x in V . Consider first the case where V is of codimension 1. Thus, $\mathbf{1}_V T = c_0[V]$ is a nonzero positive current ([12, Lemma 2.17]) and $c_1 v_1 + O(1) \leq v \leq c_1 v_1 + O(1)$ for some constant $c_1 > 0$, where v_1 is a potential of $[V]$. Since $T \geq \mathbf{1}_V T = c_0[V]$, we get $u \leq c_0 v_1 + O(1)$ on X . Thus, $u \leq (c_0/c_1)v + O(1)$ on X .

Consider now $\text{codim } V \geq 2$. By principalizing the ideal sheaf associated to V , we obtain a compact Kähler manifold X' and a surjective map $\rho : X' \rightarrow X$ such that $V' := \rho^{-1}(V)$ is a hypersurface. Note that $v \circ \rho$ is a quasi-psh function having logarithmic poles along an effective divisor supported on V' . Applying the first part of the proof gives the desired assertion. This finishes the proof. $\square$

Recall for every psh function v , the unbounded locus $L(v)$ of v defined to be the set of x such that v is unbounded in every open neighborhood of x . Observe that if v has analytic singularities along V , then $L(v) = V$. For every closed positive $(1, 1)$ -current T , we define $L(T)$ to be the unbounded locus of a potential of T . Here is a necessary condition for currents to be of relative full mass intersection in terms of Lelong numbers.

Theorem 4.11 *Let X be a compact Kähler manifold. Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents on X such that the cohomology class of T_j is Kähler for $1 \leq j \leq m$ and T a closed positive (p, p) -current on X with $p + m \leq n$. Let V be an irreducible analytic subset in X such that the generic Lelong numbers of $T_1, \dots, T_m, T$ are strictly positive. Assume $T_1, \dots, T_m$ are of full mass intersection relative to T . Then, we have $\dim V < n - p - m$.*

Proof Since θ_j is Kähler for every j , we can use Theorem 4.4, Lemma 4.10 and the comment following it to reduce the setting to the case where T_j are currents with analytic singularities along V , for $1 \leq j \leq m$ (hence $L(T_j) = V$). Since T is of bi-degree (p, p) and T has positive Lelong number everywhere on V , we deduce that the dimension of V is at most $n - p$. Let s be a nonnegative integer such that $\dim V = n - p - s$. We need to prove that $s > m$. Suppose on the contrary that $s \leq m$. By Lemma 4.9, the currents $T_1, \dots, T_s$ are of full mass intersection relative to T .

Since $L(T_j) = V$ for every j , we see that for every $J \subset \{1, \dots, s\}$, the Hausdorff dimension of $\cap_{j \in J} L(T_j) \cap \text{Supp } T$ is less than or equal to that of $\cap_{j \in J} L(T_j)$ which is equal to $\dim V = n - p - s \leq n - p - |J|$. Thus, by [11, 16], the intersection of $T_1, \dots, T_s, T$ is classically well defined. By a comparison result on Lelong numbers ([11, Page 169]) and the fact that $T_1, \dots, T_m, T$ have strictly positive Lelong number at every point in V , we see that the Lelong number of $\bigwedge_{j=1}^s T_j \wedge T$ at every point of V is strictly positive. Thus, the current

$\bigwedge_{j=1}^s T_j \wedge T$ has strictly positive mass on V (see [12, Lemma 2.17]). So by Proposition 3.6, $\langle \bigwedge_{j=1}^s T_j \wedge T \rangle$ is not of maximal mass. This is a contradiction. This finishes the proof. $\square$

5 Weighted class of currents of relative full mass intersection

In this section, we introduce the notion of weighted classes of currents with full mass intersection relative to a closed positive current T . We only consider convex weights in the sequel. We refer to [4, 7, 17] for the case where T is the current of integration along X and information about non-convex weights. We also note that [4] considers currents in big classes whereas our setting here is restricted to currents in Kähler classes.

We fix our setting. Let $0 \leq p \leq n$ be an integer and let T be a closed positive current of bi-degree (p, p) on X . For Kähler classes $\alpha_1, \dots, \alpha_m$ ($m \leq n - p$) on X , we denote by $\mathcal{E}(\alpha_1, \dots, \alpha_m, T)$ the set of m -tuple $(T_1, \dots, T_m)$ of closed positive $(1, 1)$ -currents on X such that $T_j \in \alpha_j$ for $1 \leq j \leq m$, and $T_1, \dots, T_m$ are of full mass intersection relative to T (see Definition 4.7).

Let ω be a Kähler form on X . Let $m \in \mathbb{N}^*$. Let $T_j = \text{dd}^c u_j + \theta_j$ be a closed positive $(1, 1)$ -current where θ_j is a smooth Kähler form and u_j is a negative θ_j -psh function for $1 \leq j \leq m$. We assume $p + m \leq n$. This is a minimal assumption because otherwise the relative non-pluripolar product is automatically zero by a bi-degree reason.

For $k \in \mathbb{N}$, put $u_{jk} := \max\{u_j, -k\}$ and $T_{jk} := \text{dd}^c u_{jk} + \theta_j$. Note that $T_{jk} \geq 0$ because $\theta_j \geq 0$. Let β be a Kähler $(1, 1)$ -class and $\mathcal{E}_m(\beta, T)$ the set of closed positive $(1, 1)$ -currents P in the class β such that $(P, \dots, P)$ (m times P) is in $\mathcal{E}(\beta, \dots, \beta, T)$ (m times β). The class $\mathcal{E}_m(\beta)$ was introduced in [17]. We first prove the following result giving the convexity of the class of currents of relative full mass intersection.

Theorem 5.1 *Assume that $T_j, \dots, T_j$ (m times T_j) are of full mass intersection relative to T for $1 \leq j \leq m$. Then, $T_1, \dots, T_m$ are also of full mass intersection relative to T . In particular, for Kähler $(1, 1)$ -classes β, β' , we have*

$$\mathcal{E}_m(\beta, T) + \mathcal{E}_m(\beta', T) \subset \mathcal{E}_m(\beta + \beta', T)$$

and $\mathcal{E}_m(\beta, T)$ is convex.

Proof Note that the second desired assertion is a direct consequence of the first one (recall that $\{T_j\}$'s are Kähler). By Lemma 4.9, T has no mass on I_{T_j} for every $1 \leq j \leq m$. The first desired assertion follows from the following claim.

Claim. Let P_j be one of currents $T_1, \dots, T_m$ for $1 \leq j \leq m$. Then, the currents $P_1, \dots, P_m$ are of full mass intersection relative to T .

Let $\tilde{m}$ be an integer such that there are at least $\tilde{m}$ currents among $P_1, \dots, P_m$ which are equal. We have $0 \leq \tilde{m} \leq m$. We will prove Claim by induction on $\tilde{m}$. When $\tilde{m} = m$, the desired assertion is clear by the hypothesis. Assume that it holds for every $\tilde{m}' > \tilde{m}$. We need to prove it for $\tilde{m}$. By Lemma 4.9, we can assume that θ_j 's are all equal to a form θ .

Without loss of generality, we can assume that $P_j = T_1$ for every $1 \leq j \leq \tilde{m}$ and $P_{(\tilde{m}+1)} = T_2$. If $P_j = T_l$, then we define $v_{jk} := u_{lk}$ and $P_{jk} := T_{lk}$. Put

$$Q := \bigwedge_{j=\tilde{m}+2}^m P_{jk} \wedge T \wedge \omega^{n-m-p}, \quad \tilde{P}_k := \text{dd}^c \max\{\max\{u_1, u_2\}, -k\} + \theta$$

and $\tilde{P} := \text{dd}^c \max\{u_1, u_2\} + \theta$. Since the cohomology class of T_j is Kähler for every j , we can apply (4.12) to P_j . Hence, in order to obtain the desired assertion, we need to check that

$$\int_{B_k} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q \rightarrow 0 \quad (5.1)$$

as $k, l \rightarrow \infty$, where

$$B_k := \{u_{1k} \leq -k\} \cup \{u_{2k} \leq -k\} \cup \bigcup_{j=\tilde{m}+2}^m \{v_{jk} \leq -k\}.$$

By Theorem 2.9, we get

$$\mathbf{1}_{\{u_{1k} < u_{2k}\}} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q = \mathbf{1}_{\{u_{1k} < u_{2k}\}} T_{1k}^{\tilde{m}} \wedge \tilde{P}_k \wedge Q.$$

This implies

$$\mathbf{1}_{\{u_{1k} < u_{2k}\} \cap B_k} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q = \mathbf{1}_{\{u_{1k} < u_{2k}\} \cap B_k} T_{1k}^{\tilde{m}} \wedge \tilde{P}_k \wedge Q.$$

It follows that

$$\int_{\{u_{1k} < u_{2k}\} \cap B_k} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q = \int_{\{u_{1k} < u_{2k}\} \cap B_k} T_{1k}^{\tilde{m}} \wedge \tilde{P}_k \wedge Q \quad (5.2)$$

On the other hand, by induction hypothesis, the currents $T_1, \dots, T_1, P_{\tilde{m}+2}, \dots, P_m$ ($(\tilde{m}+1)$ times T_1) are of full mass intersection relative to T . This combined with Theorem 4.4 implies that $T_1, \dots, T_1, \tilde{P}, P_{\tilde{m}+2}, \dots, P_m$ ($\tilde{m}$ times T_1) are of full mass intersection relative to T because $\tilde{P}$ is less singular than T_1 . Using this, (4.12) and the fact that

$$\{u_{1k} < u_{2k}\} \cap B_k \subset \{u_1 \leq -k\} \cup \{\max\{u_1, u_2\} \leq -k\} \cup \bigcup_{j=\tilde{m}+2}^m \{v_{jk} \leq -k\},$$

we see that the right-hand side of (5.2) converges to 0 as $k \rightarrow \infty$. It follows that

$$\int_{\{u_{1k} < u_{2k}\} \cap B_k} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q \rightarrow 0 \quad (5.3)$$

as $k \rightarrow \infty$.

Now let $\tilde{P}'_k := \text{dd}^c \max\{u_{1k}, u_{2k} - 1\} + \theta$. The last current converges to

$$\tilde{P}' := \text{dd}^c \max\{u_1, u_2 - 1\} + \theta.$$

Observe that $\tilde{P}'$ is less singular than T_2 . By induction hypothesis and Theorem 4.4 as above, we see that the currents $\tilde{P}', \dots, \tilde{P}', T_2, P_{\tilde{m}+2}, \dots, P_m$ ($\tilde{m}$ times $\tilde{P}'$) are of full mass intersection relative to T . Moreover, since T has no mass on I_{T_1} , by (iii) of Proposition 3.5, we get

$$\mathbf{1}_{I_{T_1}} \langle \tilde{P}'^m \wedge T_2 \wedge \bigwedge_{j=\tilde{m}+2}^m P_j \wedge T \rangle = 0 \quad (5.4)$$

Using similar arguments as in the first part of the proof, we obtain

$$\begin{aligned} \int_{\{u_{1k} > u_{2k}-1\} \cap B_k} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q &= \int_{\{u_{1k} > u_{2k}-1\} \cap B_k} \tilde{P}'^{\tilde{m}} \wedge T_{2k} \wedge Q \leq \int_{B_k} \tilde{P}'^{\tilde{m}} \wedge T_{2k} \wedge Q \\ &= \int_X \tilde{P}'^{\tilde{m}} \wedge T_{2k} \wedge Q - \int_{X \setminus B_k} \tilde{P}'^{\tilde{m}} \wedge T_{2k} \wedge Q \\ &= \int_X \tilde{P}'^{\tilde{m}} \wedge T_{2k} \wedge Q - \int_{X \setminus B_k} \langle \tilde{P}'^m \wedge T_2 \wedge \bigwedge_{j=\tilde{m}+2}^m P_j \wedge T \rangle \\ &= \|\langle \tilde{P}'^m \wedge T_2 \wedge \bigwedge_{j=\tilde{m}+2}^m P_j \wedge T \rangle\|_{B_k} \end{aligned} \quad (5.5)$$

because $\tilde{P}', \dots, \tilde{P}', T_2, P_{\tilde{m}+2}, \dots, P_m$ ($\tilde{m}$ times $\tilde{P}'$) are of full mass intersection relative to T . Observe that the right-hand side of (5.5) converges to

$$\|\langle \tilde{P}'^m \wedge T_2 \wedge \bigwedge_{j=\tilde{m}+2}^m P_j \wedge T \rangle\|_{I_{T_1}}$$

as $k \rightarrow \infty$. The last quantity is equal to 0 because of (5.4). Consequently, we get

$$\int_{\{u_{1k} > u_{2k}-1\} \cap B_k} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q \rightarrow 0$$

as $k \rightarrow \infty$. Combining this and (5.3) and the fact that $X = \{u_{1k} < u_{2k}\} \cup \{u_{1k} > u_{2k} - 1\}$ gives (5.1). The proof is finished. $\square$

Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous increasing function such that $\chi(-\infty) = -\infty$ and for every constant c , there exist constants $M, c_1, c_2 > 0$ so that

$$\chi(t+c) \geq c_1 \chi(t) - c_2 \quad (5.6)$$

for $t < -M$. Such a function is called a *weight*. For every function ξ bounded from above on X , let

$$\tilde{E}_\xi(T_1, \dots, T_m, T) := \int_X -\xi \langle T_1 \wedge \dots \wedge T_m \wedge T \rangle \wedge \omega^{n-m-p} \quad (5.7)$$

The following simple lemma, which generalizes [17, Proposition 1.4], will be useful in practice.

Lemma 5.2 *Let $\xi := \chi(\sum_{j=1}^m u_j)$ and $\xi_k := \chi(\max\{\sum_{j=1}^m u_j, -k\})$. Assume that the current $T_1, \dots, T_m$ are of full mass intersection relative to T . Then we have*

$$\tilde{E}_{\xi_k}(T_{1k}, \dots, T_{mk}, T) = \int_X -\xi_k \langle \bigwedge_{j=1}^m T_j \wedge T \rangle \wedge \omega^{n-m-p} \quad (5.8)$$

which increases to $\tilde{E}_{\xi}(T_1, \dots, T_m, T)$ as $k \rightarrow \infty$; and if additionally $\tilde{E}_{\xi}(T_1, \dots, T_m, T) < \infty$, then

$$\left\| \xi_k \bigwedge_{j=1}^m T_{jk} \wedge T - \xi_k \langle \bigwedge_{j=1}^m T_j \wedge T \rangle \right\| \rightarrow 0 \quad (5.9)$$

as $k \rightarrow \infty$.

Proof Put $Q := \langle \bigwedge_{j=1}^m T_j \wedge T \rangle$ and $Q_k := \bigwedge_{j=1}^m T_{jk} \wedge T$. Let ξ_k^0 be the value of ξ_k on $\cup_{j=1}^m \{u_j \leq -k\}$. We have

$$\begin{aligned} \xi_k Q_k &= \xi_k \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} Q + \xi_k^0 \mathbf{1}_{\cup_{j=1}^m \{u_j \leq -k\}} Q_k \\ &= \xi_k \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} Q + \xi_k^0 Q_k - \xi_k^0 \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} Q \\ &= \xi_k Q + \xi_k^0 (Q_k - Q). \end{aligned}$$

Hence (5.8) follows by integrating the last equality over X and using the hypothesis. We check (5.9). Since $Q_k = Q$ on $\cap_{j=1}^m \{u_j > -k\}$, we have

$$\xi_k Q_k - \xi_k Q = \mathbf{1}_{\cup_{j=1}^m \{u_j \leq -k\}} \xi_k Q_k - \mathbf{1}_{\cup_{j=1}^m \{u_j \leq -k\}} \xi_k Q.$$

On the other hand, observe that

$$\|\mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \xi_k Q\| \rightarrow \|\xi Q\| = E_{\xi}(T_1, \dots, T_m, T) < \infty$$

as $k \rightarrow \infty$ because Q has no mass on $\cup_{j=1}^m \{u_j = -\infty\}$. Hence, $\|\mathbf{1}_{\cup_{j=1}^m \{u_j \leq -k\}} \xi_k Q\| \rightarrow 0$ as $k \rightarrow \infty$. Observe

$$\begin{aligned} \int_X \mathbf{1}_{\cup_{j=1}^m \{u_j \leq -k\}} \xi_k Q_k \wedge \omega^{n-m-p} &= \int_X (\xi_k Q_k - \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \xi_k Q_k) \wedge \omega^{n-m-p} \\ &= \int_X (\xi_k Q_k - \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \xi_k Q) \wedge \omega^{n-m-p} \end{aligned}$$

converging to 0 as $k \rightarrow \infty$ by (5.8). Thus, (5.9) follows. The proof is finished. $\square$

Definition 5.3 We say that $T_1, \dots, T_m$ are of full mass intersection relative to T with weight χ if $T_1, \dots, T_m$ are of full mass intersection relative to T and for every nonempty set $J \subset \{1, \dots, m\}$, we have $\tilde{E}_{\xi}(T_j)_{j \in J}, T) < \infty$, where $\xi := \chi(\sum_{j=1}^m u_j)$.

The last definition is independent of the choice of potentials u_j by (5.6). For currents $T_1, \dots, T_m$ of full mass intersection relative to T and $\xi := \chi(\sum_{j=1}^m u_j)$, we can also define the joint ξ -energy relative to T of $T_1, \dots, T_m$ by putting

$$E_{\xi}(T_1, \dots, T_m, T) := \sum_J \int_X -\xi \langle \bigwedge_{j \in J} T_j \wedge T \rangle \wedge \omega^{n-p-|J|},$$

where the sum is taken over every subset J of $\{1, \dots, m\}$. The last energy depends on the choice of potentials but its finiteness does not.

Let $\alpha_1, \dots, \alpha_m$ be Kähler $(1, 1)$ -classes on X . Let $\mathcal{E}_\chi(\alpha_1, \dots, \alpha_m, T)$ be the set of m -tuple $(T_1, \dots, T_m)$ of closed positive $(1, 1)$ -currents such that $T_1, \dots, T_m$ of full mass intersection relative to T with weight χ and $T_j \in \alpha_j$ for $1 \leq j \leq m$.

For pseudoeffective $(1, 1)$ -class β , we define $\mathcal{E}_{\chi,m}(\beta, T)$ to be the subset of $\mathcal{E}_m(\beta, T)$ consisting of P such that $(P, \dots, P)$ is in $\mathcal{E}_\chi(\beta, \dots, \beta, T)$ (m times β). When $m = n$, the class $\mathcal{E}_{\chi,m}(\beta, T)$ was mentioned in [17, Section 5.2.2]. The last class was studied in [4, 17] when T is the current of integration along X and $m = n$.

For $P \in \mathcal{E}_{\chi,m}(\beta, T)$ with $P = dd^c u + \theta$ (θ is Kähler) and $\xi := \chi(mu)$, we put

$$E_\xi(P, T) := E_\xi(P, \dots, P, T)$$

(m times P). The following result explains why our weighted class generalizes that given in [17].

Lemma 5.4

- (i) A current $P \in \mathcal{E}_{\chi,m}(\beta, T)$ if and only if $P \in \mathcal{E}_m(\beta, T)$ and ξ is integrable with respect to $\langle P^m \wedge T \rangle$.
- (ii) Assume that T is the current of integration along X and $m = n$. Then, the energy $E_{\chi,m}(P, T)$ is equivalent to the energy associated to $\tilde{\chi}(t) := \chi(nt)$ given in [17].

Proof This is a direct consequence of computations in the proof of [4, Proposition 2.8 (i)] and Lemma 5.2: one first consider the case where χ is smooth and use Lemma 5.7 below; the general case follows by regularizing χ . The proof is finished. $\square$

The following result is obvious.

Lemma 5.5 Let η_j be a positive closed $(1, 1)$ -form for $1 \leq j \leq m$. Then $T_1, \dots, T_m$ are of full mass intersection relative to T with weight χ if and only if $(T_1 + \eta_1), \dots, (T_m + \eta_m), T$ are of full mass intersection relative to T with weight χ .

From now on, we focus on convex weights. Let $\mathcal{W}^-$ be the set of convex increasing functions $\chi : \mathbb{R} \rightarrow \mathbb{R}$ with $\chi(-\infty) = -\infty$. Observe that if $\chi \in \mathcal{W}^-$, then χ is automatically continuous. Basic examples of χ are $-(-t)^r$ for $0 < r \leq 1$. We have the following important observation which, in particular, implies that $\mathcal{W}^-$ is a set of weights in the sense given above.

Lemma 5.6 Let $\chi \in \mathcal{W}^-$. Let g be a smooth radial cut-off function on $\mathbb{R}$, i.e., $g(t) = g(-t)$ for $t \in \mathbb{R}$, g is of compact support, $0 \leq g \leq 1$ and $\int_{\mathbb{R}} g(t) dt = 1$. Put $g_\epsilon(t) := \epsilon^{-1} g(\epsilon t)$ for every constant $\epsilon > 0$ and $\chi_\epsilon := \chi * g_\epsilon$ (the convolution of χ with g_ϵ). Then, $\chi_\epsilon \in \mathcal{W}^-$, $\chi_\epsilon \geq \chi$ and

$$0 \geq t\chi'_\epsilon(t) \geq \chi_\epsilon(t) - \chi_\epsilon(0) \quad (5.10)$$

for $t \leq 0$. Consequently, (5.6) holds for $\chi \in \mathcal{W}^-$, or in other words, χ is a weight.

Clearly, we always have that χ_ϵ converges uniformly to χ because of continuity of χ .

Proof By definition,

$$\chi_\epsilon(t) = \int_{\mathbb{R}} \chi(t-s)g_\epsilon(s)ds = \int_{\mathbb{R}^+} (\chi(t-s) + \chi(t+s))g_\epsilon(s)ds.$$

We deduce that since χ is convex, $\chi_\epsilon \geq \chi$ for every ϵ . The inequality (5.10) is a direct consequence of the convexity. It remains to prove the last desired assertion. We can assume χ is smooth by the previous part of the proof. Since χ is increasing, it is enough to prove the desired assertion for $c \geq 0$. Fix a constant $c \geq 0$. Consider $t < -|c| + M$ for some big constant M . Write $\chi(t+c) = \chi(t) + \int_t^{t+c} \chi'(r)dr$. Using (5.10), we obtain that there is a constant $c_1 > 0$ independent of c such that

$$\chi(t+c) - \chi(t) \geq - \int_{|t|}^{|t|+c} [-\chi(-r) + c_1]/rdr \geq (\chi(t) - c_1)(\log(|t|+c) - \log|t|).$$

Thus the desired assertion follows. The proof is finished. $\square$

We will need the following computation which seems to be used implicitly in the literature.

Lemma 5.7 *Let $\chi \in \mathcal{C}^3(\mathbb{R})$ and w_1, w_2 bounded psh functions on an open subset U of $\mathbb{C}^n$. Let Q be a closed positive current of bi-dimension $(1, 1)$ on U . Then we have*

$$\text{dd}^c \chi(w_2) \wedge Q = \chi''(w_2)dw_2 \wedge d^c w_2 \wedge Q + \chi'(w_2)\text{dd}^c w_2 \wedge Q \quad (5.11)$$

and the operator $w_1 \text{dd}^c \chi(w_2) \wedge Q$ is continuous (in the usual weak topology of currents) under decreasing sequences of smooth psh functions converging to w_1, w_2 . Consequently, if f is a smooth function with compact support in U , then the equality

$$\int_U f w_1 \text{dd}^c \chi(w_2) \wedge Q = \int_U \chi(w_2) \text{dd}^c (f w_1) \wedge Q \quad (5.12)$$

holds. Moreover, for f as above, we also have

$$\int_U f \chi(w_2) \text{dd}^c w_1 \wedge Q = - \int_U \chi(w_2) df \wedge d^c w_1 \wedge Q - \int_U f \chi'(w_2) dw_2 \wedge d^c w_1 \wedge Q. \quad (5.13)$$

Proof Clearly, all of three desired equalities follows from the integration by parts if w_1, w_2 are smooth. The arguments below essentially say that both sides of these equalities are continuous under sequences of smooth psh functions decreasing to w_1, w_2 . This is slightly non-standard due to the presence of Q even when χ is convex.

First observe that (5.12) is a consequence of the second desired assertion because both sides of (5.12) are continuous under a sequence of smooth psh functions decreasing to w_2 . We prove (5.11). The desired equality (5.11) clearly holds if w_2 is smooth. In general, let $(w_2^\epsilon)_\epsilon$ be a sequence of standard regularizations of w_2 . Recall that $\text{dd}^c \chi(w_2) \wedge Q$ is defined to be $\text{dd}^c(\chi(w_2)Q)$ which is equal to the limit of $\text{dd}^c(\chi(w_2^\epsilon)Q)$ as $\epsilon \rightarrow 0$. By (5.11) for

w_2^ϵ in place of w_2 , we see that $\text{dd}^c(\chi(w_2^\epsilon)Q)$ is of uniformly bounded mass. As a result, $\text{dd}^c\chi(w_2) \wedge Q$ is of order 0. Thus, $w_1\text{dd}^c\chi(w_2) \wedge Q$ is well defined. Put

$$I(w_1, w, w_2) := w_1\chi''(w)dw_2 \wedge d^c w_2 \wedge Q + w_1\chi'(w)\text{dd}^c w_2 \wedge Q.$$

Recall that $I(1, w_2^\epsilon, w_2^\epsilon) \rightarrow \text{dd}^c\chi(w_2) \wedge Q$. By Corollary 2.5, we have

$$I(w_1, w_2, w_2^\epsilon) \rightarrow I(w_1, w_2, w_2) \quad (5.14)$$

as $\epsilon \rightarrow 0$. On the other hand, since χ'' is in $\mathcal{C}^1$, we get

$$|\chi''(w_2^\epsilon) - \chi''(w_2)| \lesssim (w_2^\epsilon - w_2), \quad |\chi'(w_2^\epsilon) - \chi'(w_2)| \lesssim (w_2^\epsilon - w_2).$$

This combined with the convergence of Monge–Ampère operators under decreasing sequences tells us that

$$(I(w_1, w_2^\epsilon, w_2^\epsilon) - I(w_1, w_2, w_2^\epsilon)) \rightarrow 0 \quad (5.15)$$

as $\epsilon \rightarrow 0$. Combining (5.15) and (5.14) gives that $I(w_1, w_2^\epsilon, w_2^\epsilon) \rightarrow I(w_1, w_2, w_2)$ as $\epsilon \rightarrow 0$. Letting $w_1 \equiv 1$ in the last limit, we get (5.11). The second desired assertion also follows. We prove (5.13) similarly. The proof is finished. $\square$

Here is a monotonicity property of weighted classes.

Theorem 5.8 (Monotonicity of weighted classes) *Let $\chi \in \mathcal{W}^-$ with $|\chi(0)| \leq 1$. Let T'_j be a closed positive $(1, 1)$ -current whose cohomology class is Kähler for $1 \leq j \leq m$. Assume that T'_j is less singular than T_j and $T_1, \dots, T_m$ are of full mass intersection relative to T with weight χ . Then, $T'_1, \dots, T'_m$ are also of full mass intersection relative to T with weight χ and for $\xi := \chi(\sum_{j=1}^m u_j)$, we have*

$$E_\xi(T'_1, \dots, T'_m, T) \leq c_1 E_\xi(T_1, \dots, T_m, T) + c_2, \quad (5.16)$$

for some constants $c_1, c_2 > 0$ independent of χ .

Proof Let $T'_j = \text{dd}^c u'_j + \theta'_j$ for some smooth Kähler forms θ'_j . Put $v := \sum_{j=1}^m u_j$, $v_k := \max\{v, -k\}$, $\xi := \chi(v)$ and $\xi_k := \chi(v_k)$. Let $u_{jk} := \max\{u_j, -k\}$, $T_{jk} := \text{dd}^c u_{jk} + \theta_j$. Define u'_{jk}, T'_{jk} similarly. We have $u_j \leq u'_j$. Observe that

$$(T'_j)_{j \in J'}, (T_j)_{j \in J} \text{ are of full mass intersection relative to } T \quad (5.17)$$

by Lemma 4.9. Hence, in order to obtain the first desired assertion, it suffices to check (5.16) because $-\chi(\sum_{j=1}^m u'_j) \leq -\xi$. The desired inequality (5.16) follows by letting $k \rightarrow \infty$ in the following claim and using Lemma 5.2 and (5.17).

Claim. For every $J, J' \subset \{1, \dots, m\}$ with $J \cap J' = \emptyset$, we have

$$\int_X \xi_k \bigwedge_{j \in J'} T'_{jk} \bigwedge_{j \in J} T_{jk} \wedge T \wedge \omega^{n-p-|J|-|J'|} > -c_1 E_{\xi_k}(T_{1k}, \dots, T_{mk}, T) - c_2,$$

for some constants c_1, c_2 independent of χ .

It remains to prove Claim now. We observe that it is enough to prove Claim for χ smooth by Lemma 5.6. Consider, from now on, *smooth* χ . We can also assume that $u_j, u'_j < -M$ for some big constant M .

We prove Claim by induction on $|J'|$. When $J' = \emptyset$, this is clear. Assume Claim holds for every J' with $|J'| < m'$. We need to prove that it holds for J' with $|J'| = m'$. Without loss of generality, we can assume that $1 \in J'$. Put

$$Q_k := \bigwedge_{j \in J' \setminus \{1\}} T'_{jk} \wedge \bigwedge_{j \in J} T_{jk} \wedge T \wedge \omega^{n-p-|J|-|J'|}, \quad Q := \langle \bigwedge_{j \in J' \setminus \{1\}} T'_j \bigwedge_{j \in J} T_j \wedge T \rangle \wedge \omega^{n-p-|J|-|J'|}.$$

By integration by parts, we obtain

$$I_k := \int_X \xi_k T'_{1k} \wedge Q_k = \int_X u'_{1k} dd^c \xi_k \wedge Q_k + \int_X \xi_k \theta'_1 \wedge Q_k. \quad (5.18)$$

Denote by $I_{k,1}, I_{k,2}$ the first and second terms in the right-hand side of the last equality. By induction hypothesis, we get

$$I_{k,2} \geq -c_1 E_{\xi_k}(T_{1k}, \dots, T_{mk}, T) - c_2$$

for some constants $c_1, c_2 > 0$ depending only on the masses of T_j, T'_j for $1 \leq j \leq m$.

It remains to treat $I_{k,1}$. Put $\theta := \sum_{j=1}^m \theta_j$. Note that $dd^c v_k + \theta \geq 0$. By Lemma 5.7, the current $(dd^c \xi_k + \chi'(v_k)\theta) \wedge Q_k$ is positive. This combined with the inequality $u_{1k} \leq u'_{1k}$ gives

$$\begin{aligned} I_{k,1} &= \int_X u'_{1k} (dd^c \xi_k + \chi'(v_k)\theta) \wedge Q_k - \int_X u'_{1k} \chi'(v_k)\theta \wedge Q_k \\ &\geq \int_X u_{1k} (dd^c \xi_k + \chi'(v_k)\theta) \wedge Q_k - \int_X u'_{1k} \chi'(v_k)\theta \wedge Q_k \\ &= \int_X \xi_k T_{1k} \wedge Q_k - \int_X \xi_k \theta_1 \wedge Q_k + \int_X (u_{1k} - u'_{1k}) \chi'(v_k)\theta \wedge Q_k \\ &\geq \int_X \xi_k T_{1k} \wedge Q_k + \int_X \xi_k (\theta - \theta_1) \wedge Q_k - \chi(0) \int_X \theta \wedge Q_k \end{aligned}$$

because

$$(u_{1k} - u'_{1k}) \chi'(v_k) \geq u_{1k} \chi'(v_k) \geq v_k \chi'(v_k) \geq \chi(v_k) - \chi(0) = \xi_k - \chi(0)$$

(see (5.10)). This combined with induction hypothesis gives

$$I_{k,2} \geq -c_1 E_{\xi_k}(T_{1k}, \dots, T_{mk}, T) - c_2.$$

Hence, Claim follows. The proof is finished. $\square$

By the above proof, one can check that if we fix $\theta_1, \dots, \theta_m$ and $\theta'_1, \dots, \theta'_m$, then the constants c_1 and c_2 in (5.16) can be chosen to be independent of $T_1, \dots, T_m$. Here is a convexity property for weighted classes.

Theorem 5.9 *Let $\chi \in \mathcal{W}^-$. Assume that $T_j, \dots, T_j$ (m times T_j) are of full mass intersection relative to T with weight χ for $1 \leq j \leq m$. Then, $T_1, \dots, T_m$ are also of full mass intersection relative to T with weight χ . In particular, for Kähler $(1, 1)$ -classes β, β' , we have*

$$\mathcal{E}_{\chi, m}(\beta, T) + \mathcal{E}_{\chi, m}(\beta', T) \subset \mathcal{E}_{\chi, m}(\beta + \beta', T)$$

and $\mathcal{E}_{\chi, m}(\beta, T)$ is convex.

Proof The second and third desired assertions are direct consequences of the first one. We prove the first desired assertion. Firstly, by Theorem 5.1, we have

$$T_1, \dots, T_m \text{ are of full mass intersection relative to } T. \quad (5.19)$$

By Lemma 5.5, we can assume that $\theta_j = \theta$ for every $1 \leq j \leq m$. The desired assertion is a consequence of the following claim.

Claim. Let P_j be one of currents $T_1, \dots, T_m$ for $1 \leq j \leq m$. Then, the currents $P_1, \dots, P_m$ are of full mass intersection relative to T with weight χ .

Let $\tilde{m}$ be an integer such that there are at least $\tilde{m}$ currents among $P_1, \dots, P_m$ which are equal. We have $0 \leq \tilde{m} \leq m$. We will prove Claim by induction on $\tilde{m}$. When $\tilde{m} = m$, the desired assertion is clear by the hypothesis. Assume that it holds for every number $\tilde{m}' > \tilde{m}$. We need to prove it for $\tilde{m}$.

Without loss of generality, we can assume that $P_j = T_1$ for every $1 \leq j \leq \tilde{m}$ and $P_{(\tilde{m}+1)} = T_2$. If $P_j = T_l$, then we define $v_{jk} := u_{lk}$ and $P_{jk} := T_{lk}$. Put $Q := \bigwedge_{j=\tilde{m}+2}^m P_{jk} \wedge T \wedge \omega^{n-m-p}$, $\tilde{P}_k := \text{dd}^c \max\{u_{1k}, u_{2k}\} + \theta$ and

$$w_k := \sum_{j=\tilde{m}+2}^m v_{jk}.$$

By Lemma 5.2 and (5.19), we need to check that

$$\int_X -\chi(\tilde{m}u_{1k} + u_{2k} + w_k) T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q \leq C \quad (5.20)$$

for some constant C independent of k . Actually, we need to verify a stronger statement that for every $J \subset \{1, \dots, \tilde{m}\}$, then

$$\int_X -\chi(\tilde{m}u_{1k} + u_{2k} + w_k) \bigwedge_{j \in J} P_{jk} \wedge T \wedge \omega^{n-p-|J|}$$

is uniformly bounded in k , but the proof will be similar to that of (5.20).

Observe that $X = \{u_{1k} < u_{2k}\} \cup \{u_{1k} > u_{2k} - 1\}$. Using Theorem 2.9, we get

$$\mathbf{1}_{\{u_{1k} < u_{2k}\}} T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q = \mathbf{1}_{\{u_{1k} < u_{2k}\}} T_{1k}^{\tilde{m}} \wedge \tilde{P}_k \wedge Q.$$

This combined with the fact that $-\chi(\tilde{m}u_{1k} + u_{2k} + w_k) \leq -\chi((\tilde{m}+1)u_{1k} + w_k)$ on $\{u_{1k} < u_{2k}\}$ implies

$$\begin{aligned} & \int_{\{u_{1k} < u_{2k}\}} -\chi(\tilde{m}u_{1k} + u_{2k} + w_k)T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q = \\ & \int_{\{u_{1k} < u_{2k}\}} -\chi(\tilde{m}u_{1k} + u_{2k} + w_k)T_{1k}^{\tilde{m}} \wedge \tilde{P}_k \wedge Q \end{aligned}$$

which is $\leq$

$$\int_X -\chi((\tilde{m} + 1)u_{1k} + w_k)T_{1k}^{\tilde{m}} \wedge \tilde{P}_k \wedge Q.$$

By the fact that $\tilde{P}_k$ is less singular than T_{1k} and Claim in the proof of Theorem 5.8, we see that the right-hand side of the last inequality is bounded uniformly in k because $T_1, \dots, T_1, P_{\tilde{m}+2}, \dots, P_m$ ($(\tilde{m} + 1)$ times T_1) are of full mass intersection relative to T with weight χ (this is a consequence of the induction hypothesis). It follows that

$$\int_{\{u_{1k} < u_{2k}\}} -\chi(\tilde{m}u_{1k} + u_{2k} + w_k)T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q \leq C \quad (5.21)$$

for some constant C independent of k . Similarly, for $\tilde{P}'_k := \text{dd}^c \max\{u_{1k}, u_{2k} - 1\} + \theta$, we have

$$\begin{aligned} & \int_{\{u_{1k} > u_{2k}-1\}} -\chi(\tilde{m}u_{1k} + u_{2k} + w_k)T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q \leq \\ & c_1 \int_X -\chi((\tilde{m} + 1)u_{2k} + w_k)(\tilde{P}'_k)^{\tilde{m}} \wedge T_{2k} \wedge Q + c_2 \end{aligned}$$

for some constants $c_1, c_2 > 0$ independent of k (we used (5.6) here). Using the induction hypothesis again and the fact that $\tilde{P}'_k$ is less singular than T_{2k} , we obtain

$$\int_{\{u_{1k} > u_{2k}-1\}} -\chi(\tilde{m}u_{1k} + u_{2k} + w_k)T_{1k}^{\tilde{m}} \wedge T_{2k} \wedge Q \leq C \quad (5.22)$$

for some constant C independent of k . Combining (5.22) and (5.21) gives (5.20). The proof is finished. $\square$

We now give a continuity property of weighted classes which generalizes the second part of [4, Theorem 2.17]) in the case where the cohomology classes of currents are Kähler.

Proposition 5.10 *Let $\chi \in \mathcal{W}^-$. Let $T_{j,l} = \text{dd}^c u_{j,l} + \theta_j$ be closed positive $(1, 1)$ -currents, for every $1 \leq j \leq m$, $l \in \mathbb{N}$ such that $u_{j,l} \rightarrow u_j$ as $l \rightarrow \infty$ in L^1 and $u_{j,l} \geq u_j$ for every j, l . Assume that $T_1, \dots, T_m$ are of full mass intersection relative to T with weight χ . Then, we have*

$$\chi\left(\sum_{j=1}^m u_{j,l}\right) \left\langle \bigwedge_{j=1}^m T_{j,l} \wedge T \right\rangle \rightarrow \chi\left(\sum_{j=1}^m u_j\right) \left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle$$

as $l \rightarrow \infty$.

Proof By Theorem 5.8, the currents $T_{1,l}, \dots, T_{m,l}$ are of full mass intersection relative to T with weight χ for every l . The desired convergence now follows by using arguments similar

to those in the proof of [4, Theorem 2.17] (notice also Lemma 5.2 and the comment after Theorem 5.8). The proof is finished. $\square$

Remark 5.11 As in [17], we can check that $\mathcal{E}(\alpha_1, \dots, \alpha_m, T) = \cup_{\chi \in \mathcal{W}} \mathcal{E}_{\chi}(\alpha_1, \dots, \alpha_m, T)$ for Kähler classes $\alpha_1, \dots, \alpha_m$ on X .

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