Quaternionic contact $4n + 3$ -manifolds and their $4n$ -quotients

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Abstract

We study some types of qc -Einstein manifolds with zero qc -scalar curvature introduced by S. Ivanov and D. Vassilev. Secondly, we shall construct a family of quaternionic Hermitian metrics $(g_a, \{J_\alpha\}_{\alpha=1}^3)$ on the domain Y of the standard quaternion space $\mathbb{H}^n$ one of which, say (g_a, J_1) is a Bochner flat Kähler metric. To do so, we deform conformally the standard *quaternionic contact structure* on the domain X of the *quaternionic Heisenberg Lie group* $\mathcal{M}$ to obtain quaternionic Hermitian metrics on the quotient Y of X by $\mathbb{R}^3$.

Keywords Quaternionic contact structure · Quaternionic Hermitian structure · HyperKähler structure · Quaternionic Heisenberg Lie group · Spherical CR structure · Bochner geometry

Mathematics Subject Classification 53C55 · 57S25 · 51M10

1 Introduction

We study the quaternionic contact structure [3] (qc -structure for short) on $4n + 3$ -manifolds X to construct quaternionic Hermitian $4n$ -manifolds as their quotients. In the previous paper [2], we studied a qc -structure (D, Q) whose $\text{Im } \mathbb{H}$ -valued (globally defined) 1-form ω representing D satisfies that each distribution defined by $d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma = 0$ ($(\alpha, \beta, \gamma) \sim (1, 2, 3)$) has the three-dimensional common kernel on X . (D, ω, Q) is called a *quaternionic CR-structure* (cf. [2, Definition 2.1]). It has shown in [1, 2] that every positive definite quaternionic CR-structure $(X, (\omega, Q))$ induces a 3-Sasaki manifold. Then, X admits a (local) principal $\text{Sp}(1)$ -bundle $: \text{Sp}(1) \rightarrow X \rightarrow X/\text{Sp}(1)$ over a quaternionic Kähler orbifold $X/\text{Sp}(1)$. In particular, according to the results [8, 9] of Biquard's connection [3], X is a qc -Einstein manifold with *nonzero* qc -scalar curvature. For the remaining case of vanishing qc -scalar curvature, there is no nondegenerate quaternionic CR-structure on X since the integrability of *quaternionic CR-structure* does not hold. Taking into account these results, we shall interpret a qc -Einstein manifold with vanishing qc -scalar curvature in terms of the

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differentiable equations of the contact forms ω_α ($\alpha = 1, 2, 3$). Given a quaternionic contact manifold X , let

$$E = \{\xi \in TX \mid d\omega_1(\xi, \nu) = d\omega_2(\xi, \nu) = d\omega_3(\xi, \nu) = 0, \forall \nu \in TX\} \quad (1.1)$$

be the distribution on X . If E has the three-dimensional kernel, then we call (D, ω, Q) a *strict qc-structure* on X .

When X is a *qc-Einstein manifold* with vanishing *qc-scalar curvature*, it follows from Lemma 6.4 [8] (also (1) of Proposition 6.3) that the Reeb fields $\{\xi_\alpha\}_{\alpha=1,2,3}$ of ω are *Killing* and generate a (local) abelian Lie group (that is, $[\xi_\alpha, \xi_\beta] = 0$), it is easy to see that $E = \{\xi_\alpha\}_{\alpha=1,2,3}$. Thus, a *qc-Einstein manifold* with vanishing *qc-scalar curvature* is a strict *qc-manifold*. Conversely, if X is a strict *qc-manifold*, then we prove in Proposition 2.5 of Sect. 2 that E generates a three-dimensional local abelian Lie group $\mathcal{R}$ and if $\mathcal{R}$ extends to a global $\mathbb{R}^3$ -action on X , then there is a principal bundle $\mathbb{R}^3 \rightarrow X \xrightarrow{p} X/\mathbb{R}^3$ over the hyper-Kähler manifold $X/\mathbb{R}^3$. (This holds always locally over an appropriate neighborhood of X in case $\mathcal{R}$ is a local *qc-action*.) Since $X/\mathbb{R}^3$ is hyperKähler (locally in general), using the pullback by p , both *qc-Ricci tensor* and *qc-scalar curvature* of X vanish by the definition (cf. [8]), so X is a *qc-Einstein manifold* with vanishing *qc-scalar curvature*. Thus, a strict quaternionic contact manifold is the same as a *qc-Einstein manifold* with vanishing *qc-scalar curvature*. Indeed, we owe a lot to the referee who pointed out *this equivalence* in our earlier draft.

If a Lie group G admits a left invariant strict *qc-structure*, then G is called a *strict qc-group*. An example is the quaternionic Heisenberg nilpotent Lie group $\mathcal{M}$ with the standard *qc-structure* admitting a nontrivial central extension $1 \rightarrow \mathbb{R}^3 \rightarrow \mathcal{M} \rightarrow \mathbb{H}^n \rightarrow 1$ (cf. Sect. 3.2). We construct a family of simply connected strict *qc solvable* Lie subgroups $\mathcal{M}(k, \ell)$ of $\mathcal{M} \rtimes T^n$ where $k + \ell = n$, $T^n \leq \text{Sp}(n)$, (see Sect. 3.3, cf. [4]).

Theorem A *If G is a contractible unimodular strict qc-group, then G is isomorphic to $\mathcal{M}(k, \ell)$.*

A $4n + 3$ -dimensional *qc-manifold* X is *uniformizable (or spherical)* if X is locally modeled over $(\text{PSp}(n + 1, 1), S^{4n+3})$. (This is the case $W^{qc} = 0$, see [10] also.) The pair $(\text{PSp}(n + 1, 1), S^{4n+3})$ is obtained from projective compactification of the complete simply connected quaternionic hyperbolic space $\mathbb{H}_{\mathbb{H}}^{n+1}$ with $\text{Isom}(\mathbb{H}_{\mathbb{H}}^{n+1}) = \text{PSp}(n + 1, 1)$.

Denote by $\text{Aut}_{qc}(X)$ the group of *qc-transformations* of X . If there exists a discrete subgroup $\Gamma \leq \text{Aut}_{qc}(X)$ acting properly with compact quotient X/Γ , then X is said to be *divisible* (cf. Definition 4.3). The following result [13, Theorem 1.1] was proved for the compact case.

Theorem B *Let M be a $(4n + 3)$ -dimensional compact uniformizable strict qc-manifold. Then, M is qc-conformal to the quaternionic infranilmanifold $\mathcal{M}/\Gamma$ (some finite cover of which is a principal T^3 -bundle over the quaternionic flat torus $T_{\mathbb{H}^n}^n$)*

The following uniqueness theorem characterizes especially the noncompact case (cf. Theorem 4.4).

Theorem C *Let $(X, D, \omega, \{J_\alpha\}_{\alpha=1}^3)$ be a noncompact simply connected uniformizable strict qc-manifold. Put $E = \{\xi_1, \xi_2, \xi_3\}$. Suppose X is divisible by Γ .*

(1) If $\text{Aut}_{qc}(X)$ leaves E invariant, then the developing pair reduces to the equivariant immersion:

$$(\rho, \text{dev}) : (\text{Aut}_{qc}(X), X) \rightarrow (\text{Aut}_{qc}(\mathcal{M}), \mathcal{M}).$$

In addition,

(2) For any $\gamma \in \Gamma$ and $\alpha = 1, 2, 3$, suppose $\gamma_* \xi_\alpha = \sum_{\beta=1}^3 a_{\alpha\beta} \xi_\beta$ for some function $a_{\alpha\beta} : X \rightarrow \text{SO}(3)$. Then,

- (i) $\text{dev} : X \rightarrow \mathcal{M}$ is a qc -diffeomorphism so that $\mathcal{R} = \mathbb{R}^3$.
- (ii) There exists a strict qc -structure $(D, \eta, \{J'_\alpha\}_{\alpha=1}^3)$ qc -conformal to $(\omega, \{J_\alpha\}_{\alpha=1}^3)$. The quotient $(X/\mathbb{R}^3, \{\Theta_\alpha, \hat{J}'_\alpha\}_{\alpha=1}^3)$ is a hyperKähler manifold isometric to $\mathbb{H}^n$.

For the difference between Theorem B and Theorem C, we remark that in Theorem B there is a T^3 -action on X/Γ which lifts to X an $\mathbb{R}^3$ -action centralizing Γ , while in Theorem C X is divisible by Γ , but the intersection $\mathbb{R}^3 \cap \Gamma$ is not necessarily uniform in $\mathbb{R}^3$, which does not imply to induce a T^3 -action on X/Γ .

The second part of this paper treats the *quaternionic Hermitian quotient* in place of the *hyperKähler quotient*. We construct a noncompact qc -manifold to obtain a quaternionic Hermitian manifold $(Y, \{\hat{\Omega}_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3)$ such that one of $(\hat{\Omega}_\alpha, \hat{J}_\alpha)$'s is Kähler. (See Theorem 6.5, Theorem 7.2, Corollary 7.3.)

Theorem D *There exists a uniformizable noncompact qc -manifold X whose quotient by the $\mathbb{R}^3$ -action gives a $4n$ -dimensional quaternionic Hermitian manifold $(Y, \hat{g}, \{\hat{\Omega}_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3)$. Moreover,*

1. $(Y, \hat{g}, \{\hat{\Omega}_1, \hat{J}_1\})$ is a Bochner flat complex Kähler manifold.
2. $(Y, \hat{g})$ is not Einstein. In particular, Y is not isometric to any domain of the quaternionic euclidean space $\mathbb{H}^n$.
3. The quaternionic Hermitian isometry group $\text{Isom}_{qH}(Y, \hat{g}, \{\hat{J}_\alpha\}_{\alpha=1}^3)$ is isomorphic to a k -torus T^k for some k where $n+1 \leq k \leq 2n$.

The paper is organized as follows In Sect. 2, we give some basic facts on *strict qc-structure*. The fundamental property of strict qc -manifolds is proved in Proposition 2.5 which produces hyperKähler structures on their $\mathbb{R}^3$ -quotients as mentioned. In Sect. 3, we review quaternionic Heisenberg nilpotent Lie group $\mathcal{M}$ where the group structure and qc -structure are explained explicitly. We give a nontrivial *strict qc-group* as a qc manifold in Theorem 3.3. From another viewpoint, we discuss strict qc manifolds in connection with spherical (uniformizable) qc geometry $(\text{PSp}(n+1, 1), S^{4n+3})$ in Sect. 4. Theorem 4.4 gives a sufficient condition for a divisible group Γ of the qc -automorphism group $\text{Aut}_{qc}(X)$ characterizing that the quotient $X/\mathbb{R}^3$ may be isometric to $\mathbb{H}^n$ as the standard hyperKähler manifold. In Sect. 5, we relax the condition *strict* on E in order to get a quaternionic Hermitian structure $(\hat{g}, \{\hat{\Omega}_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3)$ on the quotient domain $Y = X/\mathbb{R}^3$ of $\mathbb{H}^n$. This can be achieved by the conformal change of the $\text{Im } \mathbb{H}$ -valued one-form ω_0 which represents the standard qc -structure on $\mathcal{M}$. We can show that one of them, say $(\hat{g}, \hat{\Omega}_1, \hat{J}_1)$ is

a Kähler metric on Y . Moreover, in Sect. 6 a prominent property of this construction is that $(Y, \hat{\Omega}_1, \hat{J}_1)$ admits a Bochner flat Kähler structure. In particular, Y is not locally isometric to any domain of the flat space $\mathbb{H}^n$. In Sect 7, we discuss the quaternionic isometry group $\text{Isom}_{qH}(Y, \hat{g}, \{\hat{\Omega}_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3)$. In course of discussion, we obtain a strictly pseudoconvex spherical pseudo-Hermitian structure $\{\hat{\omega}_1, J_1\}$ on the $(4n+1)$ -quotient $X/\mathbb{R}^2$ such that the pseudo-Hermitian transformation group $\text{Psh}(X/\mathbb{R}^2)$ is isomorphic to $\mathbb{R} \times T^{2n}$. Theorem D is a consequence of the results of Sects. 6 and 7.1.

2 Strict quaternionic contact manifolds

The hypercomplex structure $\{J_\alpha, J_\beta, J_\gamma\}$ on D is defined by the following equation $((\alpha, \beta, \gamma) \sim (1, 2, 3))$:

$$d\omega_\alpha(u, v) = d\omega_\beta(J_\gamma u, v) \quad (\forall u, v \in D). \quad (2.1)$$

There is the reciprocity on D :

$$d\omega_\alpha(J_\alpha u, v) = d\omega_\beta(J_\beta u, v) = d\omega_\gamma(J_\gamma u, v) \quad ((\alpha, \beta, \gamma) \sim (1, 2, 3)). \quad (2.2)$$

It is easy to see from (2.2)

$$d\omega_\alpha(J_\alpha u, J_\alpha v) = d\omega_\alpha(u, v) \quad (\alpha = 1, 2, 3). \quad (2.3)$$

2.1 Strict qc-manifolds

Let $(X, D, \omega, \{J_\alpha\}_{\alpha=1}^3)$ be a strict qc-manifold with distribution E (cf. (1.1)).

Lemma 2.1 E generates a three-dimensional local abelian Lie group $\mathcal{R}$.

Proof Since E is of dimension 3 and transverse to D , it follows $\omega(E) = \text{Im } \mathbb{H}$. There exist vector fields $\{\xi_\alpha\}_{\alpha=1,2,3} \subset E$ such that

$$\omega_\alpha(\xi_\beta) = \delta_{\alpha\beta}. \quad (2.4)$$

(Equivalently $\omega(\xi_\alpha) = \omega_1(\xi_\alpha)i + \omega_2(\xi_\alpha)j + \omega_3(\xi_\alpha)k = \delta_{1\alpha}i + \delta_{2\alpha}j + \delta_{3\alpha}k$.) By (1.1), $2d\omega_\alpha(\xi_\beta, \xi_\gamma) = -\omega_\alpha([\xi_\beta, \xi_\gamma]) = 0$ and so $[\xi_\beta, \xi_\gamma] \in D$. For any $v \in D$, $0 = 2d\omega_\alpha(\xi_\beta, v) = -\omega_\alpha([\xi_\beta, v])$ ($\alpha = 1, 2, 3$) so $[\xi_\beta, v] \in D$ ($(\alpha, \beta, \gamma) \sim (1, 2, 3)$). Using the Jacobi identity,

$$\begin{aligned} 2d\omega_\alpha([\xi_\beta, \xi_\gamma], v) &= -\omega_\alpha([\xi_\beta, \xi_\gamma], v) = \omega_\alpha([\xi_\gamma, v], \xi_\beta) + \omega_\alpha([v, \xi_\beta], \xi_\gamma) \\ &= -2d\omega_\alpha([\xi_\gamma, v], \xi_\beta) - 2d\omega_\alpha([v, \xi_\beta], \xi_\gamma) = 0. \end{aligned}$$

By the non-degeneracy of $d\omega_\alpha$ on D , it follows $[\xi_\beta, \xi_\gamma] = 0$ for any β, γ . Thus, $E = \{\xi_\alpha, \alpha = 1, 2, 3\}$ generates a local abelian Lie group. $\square$

Proposition 2.2 Denote by $\mathcal{L}_\xi$ the Lie derivative of a vector field ξ on X .

- (1) $\mathcal{L}_{\xi_\alpha} \omega_\beta = 0, \mathcal{L}_{\xi_\alpha} d\omega_\beta = 0$ ($\alpha, \beta = 1, 2, 3$). In particular, $\mathcal{L}_{\xi_\alpha} D = D$.
 (2) $\mathcal{L}_{\xi_\alpha} J_\beta = 0$ ($\alpha, \beta = 1, 2, 3$).

Proof First note that $\mathcal{L}_{\xi_\beta} \omega_\alpha = (d\iota_{\xi_\beta} + \iota_{\xi_\beta} d)\omega_\alpha = \iota_{\xi_\beta} d\omega_\alpha = 0$, $\mathcal{L}_{\xi_\alpha} d\omega_\beta = d\mathcal{L}_{\xi_\alpha} \omega_\beta = 0$ from (2.4), (1.1). For any $v \in D$, $0 = (\mathcal{L}_{\xi_\alpha} \omega)(v) = -\omega(\mathcal{L}_{\xi_\alpha} v)$ so $\mathcal{L}_{\xi_\alpha} v \in D$. Thus, $(\mathcal{L}_{\xi_\alpha} J_\beta)v = \mathcal{L}_{\xi_\alpha}(J_\beta v) - J_\beta(\mathcal{L}_{\xi_\alpha} v) \in D$. For $\forall u, v \in D$, $(\mathcal{L}_{\xi_\alpha} d\omega_\gamma)(J_\alpha u, v) = 0$, which equals

$$\mathcal{L}_{\xi_\alpha}(d\omega_\gamma(J_\alpha u, v)) - d\omega_\gamma(\mathcal{L}_{\xi_\alpha}(J_\alpha u), v) - d\omega_\gamma(J_\alpha u, \mathcal{L}_{\xi_\alpha} v) = 0.$$

$$\begin{aligned} d\omega_\gamma((\mathcal{L}_{\xi_\alpha} J_\alpha)u, v) &= d\omega_\gamma(\mathcal{L}_{\xi_\alpha}(J_\alpha u), v) - d\omega_\gamma(J_\alpha \mathcal{L}_{\xi_\alpha} u, v) \\ &= \mathcal{L}_{\xi_\alpha}(d\omega_\gamma(J_\alpha u, v)) - d\omega_\gamma(J_\alpha u, \mathcal{L}_{\xi_\alpha} v) - d\omega_\gamma(J_\alpha \mathcal{L}_{\xi_\alpha} u, v) \\ &= \mathcal{L}_{\xi_\alpha}(d\omega_\beta(u, v)) - d\omega_\beta(u, \mathcal{L}_{\xi_\alpha} v) - d\omega_\beta(\mathcal{L}_{\xi_\alpha} u, v) \quad (2.1) \\ &= (\mathcal{L}_{\xi_\alpha} d\omega_\beta)(u, v) = 0. \end{aligned}$$

Similarly, it follows $d\omega_\alpha((\mathcal{L}_{\xi_\alpha} J_\beta)u, v) = 0$. By the non-degeneracy of $d\omega_\alpha$ on D , it follows $\mathcal{L}_{\xi_\alpha} J_\alpha = 0$, $\mathcal{L}_{\xi_\alpha} J_\beta = 0$.

Let $\mathcal{R}$ denote the local abelian group obtained from Lemma 2.1.

Proposition 2.3 Suppose that $\mathcal{R}$ generates a global abelian group of a strict qc-manifold X . Then, $\mathcal{R}$ acts properly on X as qc-transformations, that is a closed subgroup $\mathcal{R} \leq \text{Aut}_{qc}(X)$.

Proof By (1), (2) of Proposition 2.2, it follows $t^* \omega_\alpha = \omega_\alpha$, $t_* \circ J_\alpha = J_\alpha \circ t_*$ for any $t \in \mathcal{R}$ ($\alpha = 1, 2, 3$). Define a Riemannian metric on X by

$$g(A, B) = \sum_{i=1}^3 \omega_i(A) \cdot \omega_i(B) + d\omega_1(J_1 A, B) \quad (\forall A, B \in TX). \quad (2.5)$$

(We may choose whichever $d\omega_\alpha \circ J_\alpha$ from the reciprocity $d\omega_1 \circ J_1 = d\omega_2 \circ J_2 = d\omega_3 \circ J_3$.) Then, note that $\mathcal{R} \leq \text{Isom}(X, g) \leq \text{Aut}_{qc}(X)$. If $\overline{\mathcal{R}}$ is the closure of $\mathcal{R}$ in $\text{Isom}(X, g)$, then it acts properly on X . Let τ be a vector field induced by a one-parameter subgroup of $\overline{\mathcal{R}}$. Then, there is a sequence of vector fields $\{\xi^{(n)}\} \subset E$ such that $d\omega_1(\tau, A) = \lim_{n \rightarrow \infty} d\omega_1(\xi^{(n)}, A) = 0$ ($\forall A \in TX$) by (1.1). And so $\tau \in E$. This implies $\overline{\mathcal{R}} = \mathcal{R}$.

For example, if X is complete with respect to g of (2.5), then $\mathcal{R}$ extends to a global action of X . If a strict qc-manifold $(X, \omega, D, \{J_\alpha\}_{\alpha=1}^3)$ admits a global $\mathbb{R}^3$ -action induced by E , then $\mathbb{R}^3$ acts properly by Proposition 2.3 and hence freely on X . There is a principal bundle over a $4n$ -dimensional manifold $Y = X/\mathbb{R}^3$: $\mathbb{R}^3 \rightarrow X \xrightarrow{\pi} Y$. We will show that Y admits a hyperKähler metric. Since each $t \in \mathbb{R}^3$ satisfies $J_\alpha \cdot t_* = t_* \cdot J_\alpha$ on D by (2) of Proposition 2.2, $\mathbb{R}^3$ induces a well-defined almost complex structure $\hat{J}_\alpha$ on Y such that $\pi_* \cdot J_\alpha = \hat{J}_\alpha \cdot \pi_* : D \rightarrow TY$ at each point of X . $\{\hat{J}_\alpha\}_{\alpha=1}^3$ constitutes a quaternionic structure on Y . Define a 2-form Ω_α ($\alpha = 1, 2, 3$) on Y to be

$$\pi^* \Omega_\alpha = d\omega_\alpha. \quad (2.6)$$

Proposition 2.4 The 2-form Ω_α is a well-defined closed 2-form ($\alpha = 1, 2, 3$) satisfying the following equality:

$$\Omega_1(\hat{J}_1\hat{A}, \hat{B}) = \Omega_2(\hat{J}_2\hat{A}, \hat{B}) = \Omega_3(\hat{J}_3\hat{A}, \hat{B}) \quad (\hat{A}, \hat{B} \in TY).$$

Moreover,

$$g(\hat{A}, \hat{B}) = \Omega_\alpha(\hat{J}_\alpha\hat{A}, \hat{B}) \quad (\alpha = 1, 2, 3)$$

is a hyperKähler metric on $(Y, \{\hat{J}_\alpha\}_{\alpha=1,2,3})$.

Proof Let $A = V + u, B = W + v \in TX$ ($\exists V, W \in E = \mathbb{R}^3, \exists u, v \in D$). (Similarly, $A' = V' + u', B' = W' + v'.$) Suppose $\pi_*A_p = \pi_*A'_q, \pi_*B_p = \pi_*B'_q$. Then, $q = tp \in X$ for some $t \in \mathbb{R}^3$ in which $A' = t_*T + t_*A, B' = t_*T' + t_*B$ ($\exists T, T' \in \mathbb{R}_p^3 = E_p$). Since each element t leaves ω_α invariant by (1) of Proposition 2.2, (1.1) shows $d\omega_\alpha(A', B') = d\omega_\alpha(A, B)$, thus (2.6) is well-defined. In particular,

$$d\omega_\alpha = 0 \text{ on } Y.$$

Furthermore, as $V, W \in E, d\omega_\alpha(A, B) = d\omega_\alpha(u, v)$. Since $\hat{A} = \pi_*A = \pi_*u, \hat{B} = \pi_*B = \pi_*v$, it follows $d\omega_\alpha(J_\alpha u, v) = \pi^*\Omega_\alpha(J_\alpha u, v) = \Omega_\alpha(\hat{J}_\alpha\hat{A}, \hat{B})$. Since $d\omega_1(J_1u, v) = d\omega_2(J_2u, v) = d\omega_3(J_3u, v)$ is positive definite on D [cf. (2.2)], we have a positive definite 2-form on Y :

$$g(\hat{A}, \hat{B}) = \Omega_1(\hat{J}_1\hat{A}, \hat{B}) = \Omega_2(\hat{J}_2\hat{A}, \hat{B}) = \Omega_3(\hat{J}_3\hat{A}, \hat{B}) \quad (\hat{A}, \hat{B} \in TY).$$

By (2.6), $\Omega_\alpha(\hat{J}_\alpha\hat{A}, \hat{J}_\alpha\hat{B}) = \Omega_\alpha(\hat{A}, \hat{B})$. It follows

$$g(\hat{J}_\alpha\hat{A}, \hat{J}_\alpha\hat{B}) = g(\hat{A}, \hat{B}) \text{ on } Y \quad (\alpha = 1, 2, 3).$$

By the definition, g is a hyperKähler metric on Y . □

In summary, we obtain the result implied in Introduction.

Proposition 2.5 *Let $(X, D, (\omega, \{J_\alpha\}_{\alpha=1}^3))$ be a strict qc-manifold. Let $\mathcal{R}$ be a local abelian group generated by the distribution E . If $\mathcal{R}$ extends to a global action of $\mathbb{R}^3$ on X , then the quotient manifold $Y = X/\mathbb{R}^3$ supports a hyperKähler structure $(g, \{\Omega_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3)$.*

3 Quaternionic Heisenberg Lie group $\mathcal{M}$

3.1 Quick review of quaternionic parabolic geometry

We recall *parabolic quaternionic group* derived from the quaternionic hyperbolic group. The quaternionic hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{H}}$ has a (projective) compactification whose boundary is diffeomorphic to S^{4n+3} . The isometric action of the quaternionic hyperbolic group $\text{Isom}(\mathbb{H}^{n+1}_{\mathbb{H}}) = \text{PSp}(n+1, 1)$ extends to an analytic action on S^{4n+3} , which we may call a *quaternionic contact* action on S^{4n+3} . Let ∞ be the point at infinity of S^{4n+3} . The standard sphere S^{4n+3} with ∞ removed admits a qc-structure isomorphic to the quaternionic Heisenberg Lie group $\mathcal{M}$ with $\text{Aut}_{qc}(\mathcal{M}) = \mathcal{M} \rtimes (\text{Sp}(n) \cdot \text{Sp}(1) \times \mathbb{R}^+)$. Recall the definition of $\mathcal{M}$ from [2]. Put $t = (t_1, t_2, t_3), s = (s_1, s_2, s_3) \in \mathbb{R}^3 = \text{Im } \mathbb{H}$, and $z = {}^t(z_1, \dots, z_n), w = {}^t(w_1, \dots, w_n) \in \mathbb{H}^n$ and so on. Then, $\mathcal{M}$ is the product $\mathbb{R}^3 \times \mathbb{H}^n$ with group law:

$$(t, z) \cdot (s, w) = (t + s - \operatorname{Im} \langle z, w \rangle, z + w)$$

where $\langle z, w \rangle = {}^t \bar{z} w$ is the Hermitian inner product. $\mathcal{M}$ is a nilpotent Lie group such that the center is the commutator subgroup $[\mathcal{M}, \mathcal{M}] = \mathbb{R}^3$ consisting of elements $(t, 0)$.

Each element $h = ((t, v), \sqrt{u}A \cdot a) \in \operatorname{Aut}_{qc}(\mathcal{M})$ acts on $\mathcal{M}$ as

$$h(s, z) = (t + uas\bar{a} - \operatorname{Im} \langle v, \sqrt{u}Az\bar{a} \rangle, v + \sqrt{u}Az\bar{a}) \quad (\forall (s, z) \in \mathcal{M}). \quad (3.1)$$

In particular, $E(\mathcal{M}) = \mathcal{M} \rtimes (\operatorname{Sp}(n) \cdot \operatorname{Sp}(1))$ is a normal subgroup of $\operatorname{Aut}_{qc}(\mathcal{M})$ acting properly and transitively on $\mathcal{M}$ in the manner of (3.1).

3.2 The qc -structure of $\mathcal{M}$

The $\operatorname{Im} \mathbb{H}$ -valued 1-form on $\mathcal{M}$ is defined by

$$\omega_0 = dt_1 i + dt_2 j + dt_3 k + \operatorname{Im} \langle z, dz \rangle. \quad (3.2)$$

Put $\omega_0 = \omega_1 i + \omega_2 j + \omega_3 k$. Let $D_0 = \bigcap_{i=1}^3 \ker \omega_i = \ker \omega_0$ which denotes the codimension 3-subbundle on $\mathcal{M}$ satisfying $T\mathcal{M} = D_0 \oplus \langle \frac{d}{dt_1}, \frac{d}{dt_2}, \frac{d}{dt_3} \rangle$. Let x_α ($\alpha = 1, \dots, 4n$) be a real number so that $\mathbb{R}^{4n}$ is identified with $\mathbb{H}^n = (x_1 + ix_2 + jx_3 + kx_4; \dots; x_{4n-3} + ix_{4n-2} + jx_{4n-1} + kx_{4n})$. A direct calculation shows

$$\begin{aligned} \omega_1 &= dt_1 + (x_1 dx_2 - x_2 dx_1) + (x_4 dx_3 - x_3 dx_4) + \dots \\ &\quad \dots + (x_{4n-3} dx_{4n-2} - x_{4n-2} dx_{4n-3}) + (x_{4n} dx_{4n-1} - x_{4n-1} dx_{4n}). \\ \omega_2 &= dt_2 + (x_1 dx_3 - x_3 dx_1) + (x_2 dx_4 - x_4 dx_2) + \dots \\ &\quad \dots + (x_{4n-3} dx_{4n-1} - x_{4n-1} dx_{4n-3}) + (x_{4n-2} dx_{4n} - x_{4n} dx_{4n-2}). \\ \omega_3 &= dt_3 + (x_1 dx_4 - x_4 dx_1) + (x_3 dx_2 - x_2 dx_3) + \dots \\ &\quad \dots + (x_{4n-3} dx_{4n} - x_{4n} dx_{4n-3}) + (x_{4n-1} dx_{4n-2} - x_{4n-2} dx_{4n-1}). \end{aligned} \quad (3.3)$$

The hypercomplex structure $\{J_1, J_2, J_3\}$ on D_0 is given as in (2.1). Alternatively if $\pi : \mathcal{M} \rightarrow \mathbb{H}^n$ is the canonical projection (homomorphism), then $\pi_* : D_0 \rightarrow T\mathbb{H}^n$ is an isomorphism at each point of $\mathcal{M}$ for which each J_α on D_0 is defined by the commutative rule:

$$\pi_* \circ J_\alpha = I_\alpha \circ \pi_* \quad (3.4)$$

where $\{I_\alpha\}_{\alpha=1}^3$ of the right hand side is the *standard quaternionic structure* $\{i, j, k\}$ on $\mathbb{H}^n$, respectively.

Proposition 3.1 $(\mathcal{M}, (D_0, \omega_0, \{J_\alpha\}_{\alpha=1}^3))$ is a strict qc -manifold for which

- (1) $E_0 = \langle \frac{d}{dt_1}, \frac{d}{dt_2}, \frac{d}{dt_3} \rangle$ generates the center $\mathbb{R}^3$ of $\mathcal{M}$, transverse to D_0 .
- (2) There is a principal bundle: $\mathbb{R}^3 \rightarrow \mathcal{M} \xrightarrow{\pi} \mathbb{H}^n$ whose qc -structure $(\omega_0, \{J_\alpha\}_{\alpha=1}^3)$ induces the standard hyperKähler structure on $\mathbb{H}^n$.

Proof It follows $d\omega_\alpha\left(\frac{d}{dt_\beta}\right) = \delta_{\alpha\beta}$, $d\omega_\alpha\left(\frac{d}{dt_\beta}, X\right) = 0$ ($\forall X \in T\mathcal{M}$) by (3.3). And so $E_0 = \left\langle \frac{d}{dt_1}, \frac{d}{dt_2}, \frac{d}{dt_3} \right\rangle$. The remaining follows from Proposition 2.5. Explicitly, if $g_{\mathbb{H}}$ is the standard quaternionic euclidean metric on $\mathbb{H}^n$, then (3.4), (3.2) show $d\omega_\alpha(J_\alpha X, Y) = g_{\mathbb{H}}(\pi_* X, \pi_* Y)$ ($\forall X, Y \in D_0$). $\square$

Remark 3.2 There is a canonical equivariant Riemannian submersion :

$$\mathbb{R}^3 \rightarrow (E(\mathcal{M}), \mathcal{M}, g_\omega) \xrightarrow{\pi} (E(\mathbb{H}^n), \mathbb{H}^n, g_{\mathbb{H}}) \quad (3.5)$$

where $g_\omega = \sum_{\alpha=1}^3 \omega_\alpha \cdot \omega_\alpha + d\omega_1 \circ J_1$ is a Riemannian metric (cf. (2.5)). Note that this metric is not a 3-Sasaki metric globally defined on $\mathcal{M}$. Here, $E(\mathbb{H}^n) = \mathbb{H}^n \rtimes \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ is the quaternionic isometry group $\mathrm{Isom}(\mathbb{H}^n, g_{\mathbb{H}})$.

From (3.1) (cf. [2]), we see that any element $h = (t, v, \sqrt{u}A \cdot a) \in \mathrm{Aut}_{qc}(\mathcal{M})$ satisfies

$$h^* \omega_0 = u \cdot a \omega_0 \bar{a} \quad (3.6)$$

which thus preserves D_0 . Thus, for $h \in \mathcal{M} \rtimes \mathrm{Sp}(n)$, it follows

$$h^* \omega_0 = \omega_0 \text{ on } \mathcal{M}, \quad h_* J_\alpha = J_\alpha h_* \text{ on } D_0. \quad (3.7)$$

In particular, note that $t_* J_\alpha = J_\alpha t_*$ ($\forall t \in \mathbb{R}^3 = C(\mathcal{M})$).

3.3 Strict qc-group

Let $(D, \omega, \{J_\alpha\}_{\alpha=1}^3)$ be a qc-structure on X . Put

$$\mathrm{Psh}_{qc}(X) = \{h \in \mathrm{Diff}(X) \mid h^* \omega = \omega, h_* J_\alpha = J_\alpha h_*|_D, \alpha = 1, 2, 3\}. \quad (3.8)$$

It is a subgroup of $\mathrm{Aut}_{qc}(X)$. We apply the similar constriction for Sasaki groups (cf. [4]). Let $\rho : \mathbb{H}^\ell \rightarrow \mathrm{Sp}(k)$ be a non-trivial homomorphism ($k + \ell = n$). Define $\mathbb{H}(k, \ell)$ to be the semidirect product $\mathbb{H}^k \rtimes_\rho \mathbb{H}^\ell$ which is canonically embedded to the group of hyperkähler isometries $\mathbb{H}^n \rtimes \mathrm{Sp}(n)$ of flat quaternionic space $\mathbb{H}^n$. Since $\mathbb{H}(k, \ell)$ acts simply transitively on $\mathbb{H}^n$, it is a flat hyperKähler group. (In fact, in view of [7, Theorem II], every flat hyperKähler Lie group contained in $\mathbb{H}^n \rtimes \mathrm{Sp}(n)$ may be conjugate to some $\mathbb{H}(k, \ell)$.) Let $\mathrm{Psh}_{qc}(\mathcal{M}) = \mathcal{M} \rtimes \mathrm{Sp}(n)$ be the normal subgroup of $E(\mathcal{M})$. Take the pull-back $\mathcal{M}(k, \ell)$ of $\mathbb{H}(k, \ell)$ in the following central extension :

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathcal{M} \rtimes \mathrm{Sp}(n) & \xrightarrow{\pi} & \mathbb{H}^n \rtimes \mathrm{Sp}(n) \longrightarrow 1 \\ & & \parallel & & \cup & & \cup \\ 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathcal{M}(k, \ell) & \xrightarrow{p} & \mathbb{H}(k, \ell) \longrightarrow 1. \end{array} \quad (3.9)$$

Here, $\mathcal{M}(n, 0) = \mathcal{M}$. Then, $\mathcal{M}(k, \ell)$ is a simply connected solvable Lie group acting simply transitively by qc-transformations on the strict qc-manifold $\mathcal{M}$. Thus, $\mathcal{M}(k, \ell)$ admits a strict qc structure as a Lie group.

Theorem 3.3 Let G be a contractible unimodular strict qc Lie group. Then, G is isomorphic to $\mathcal{M}(k, \ell)$.

Proof G is viewed as a strict qc -manifold endowed with a left invariant strict qc -structure $(\omega, \{J_\alpha\}_{\alpha=1}^3)$. Then, $G \leq \text{Psh}_{qc}(G)$ by (3.8). If $\mathbb{R}^3$ is the abelian group generated by $E = \langle \xi_\alpha, \alpha = 1, 2, 3 \rangle$, then $\mathbb{R}^3 \leq \text{Psh}_{qc}(G)$ by Proposition 2.2. Let $\text{Isom}_{hk}(G/\mathbb{R}^3) = \{h \in \text{Diff}(G/\mathbb{R}^3) \mid h^*\Omega_\alpha = \Omega_\alpha, h_*\hat{J}_\alpha = \hat{J}_\alpha h_*, \alpha = 1, 2, 3\}$ be a subgroup of isometries of the hyperKähler manifold $G/\mathbb{R}^3$ as in Proposition 2.5. Denote by $\text{Isom}_h(G/\mathbb{R}^3) = \{h \in \text{Diff}(G/\mathbb{R}^3) \mid h^*\Omega_1 = \Omega_1, h_*\hat{J}_1 = \hat{J}_1 h_*\}$ the holomorphic isometry group of $G/\mathbb{R}^3$ as a Kähler manifold. Recall that $\text{Psh}(G/\mathbb{R}^2) = \{h \in \text{Diff}(G/\mathbb{R}^2) \mid h^*\omega_1 = \omega_1, h_*J_1 = J_1 h_*\}$ is the group of strictly pseudoconvex pseudo-Hermitian transformations of $(G/\mathbb{R}^2, (\omega_1, J_1))$. There is the commutative diagram of central group extensions (cf. [4, Proposition 3.4]):

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \text{Psh}_{qc}(G) & \longrightarrow & \text{Isom}_{hk}(G/\mathbb{R}^3) \\ & & \mathbb{R}^2 / \downarrow & & \mathbb{R}^2 / \downarrow & & \downarrow \iota \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \text{Psh}(G/\mathbb{R}^2) & \longrightarrow & \text{Isom}_h(G/\mathbb{R}^3) \longrightarrow 1 \end{array} \quad (3.10)$$

It follows from [4, Theorem 2] that $\text{Isom}_h(G/\mathbb{R}^3)^0 = (\mathbb{C}^k \rtimes \text{U}(k)) \times S_0$ for which S_0 is a semisimple Lie group of noncompact type. It acts transitively on the Kähler manifold $G/\mathbb{R}^3$ holomorphically isometric to the product $\mathbb{C}^k \times D$ where D is the bounded symmetric domain.

Consider (1) $\mathbb{R}^3$ is normal in G . Then, $G/\mathbb{R}^3$ is a flat hyperKähler group by Hano's theorem [7] and so $G/\mathbb{R}^3 = \mathbb{H}(k, \ell)$ as above. Then, the pull back of this in (3.10) gives $G = \mathcal{M}(k, \ell)$. Otherwise, (2) $\mathbb{R}^k$ is normal in G where $k = 0, 1, 2$. Case (i) If $\mathbb{R}^2$ is normal in G , then the quotient group $\hat{G} = G/\mathbb{R}^2$ is a Sasaki group for which $S^1 = \mathbb{R}/\mathbb{Z} \rightarrow \hat{G} = \hat{G}/\mathbb{Z} \rightarrow G/\mathbb{R}^3$ is a pseudo-Hermitian (Sasaki) bundle. As in the proof of [4, Theorem 2], $G/\mathbb{R}^3 = \hat{G}/S^1$ is a bounded symmetric domain so that $G/\mathbb{R}^3 = \mathbb{H}_{\mathbb{R}}^2$ with $G = \text{PSL}(2, \mathbb{R})$. It is impossible for $G/\mathbb{R}^3$ to admit a quaternionic structure. Case (ii) If $\mathbb{R}$ is normal in G , then put $\hat{G} = G/\mathbb{R}$. The principal bundle $\mathbb{R}^2/\mathbb{Z}^2 \rightarrow \hat{G} = \hat{G}/\mathbb{Z}^2 \rightarrow G/\mathbb{R}^3$ becomes a principal bundle of homogeneous space $T^2 \rightarrow \hat{G} \rightarrow \hat{G}/T^2 = G/\mathbb{R}^3$. As $\hat{G}/T^2$ is a bounded symmetric domain, $\hat{G}$ is locally isomorphic to $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$, which is impossible since $\hat{G}/T^2 = \text{PSL}(2, \mathbb{R})/S^1 \times \text{PSL}(2, \mathbb{R})/S^1 = \mathbb{H}_{\mathbb{R}}^2 \times \mathbb{H}_{\mathbb{R}}^2$ has a positive scalar curvature which is not hyperKähler. Finally, Case (iii) $\hat{G} = G/\mathbb{Z}^3$ and $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ such that $G/\mathbb{R}^3 = \hat{G}/T^3$ is a bounded symmetric domain. Thus, $\hat{G}$ is locally isomorphic to $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ and $G/\mathbb{R}^3 = \mathbb{H}_{\mathbb{R}}^2 \times \mathbb{H}_{\mathbb{R}}^2 \times \mathbb{H}_{\mathbb{R}}^2$, which cannot be a quaternionic manifold. $\square$

4 Spherical qc -manifolds

The following theorem is a supporting example to Proposition 2.5 which is implied by Schoen's result [14]. (Compare [6, 12] for the proofs of the quaternionic case.)

Theorem 4.1 *Let $(X, D, \{J_\alpha\}_{\alpha=1}^3)$ be a noncompact quaternionic contact manifold. If $\text{Aut}_{qc}(X)$ does not act properly on X , then X admits the spherical qc -structure qc -conformal to the quaternionic Heisenberg Lie group $\mathcal{M}$.*

Let $(\tilde{\omega}, D_0, \{I_\alpha\}_{\alpha=1}^3)$ be the standard qCR -structure on S^{4n+3} such that $\ker \tilde{\omega} = D_0$ (cf. [2]).

Definition 4.2 A qc -manifold $(M, D, (\omega, \{J_\alpha\}_{\alpha=1}^3))$ is *spherical (or uniformizable)* with respect to $(\mathrm{PSp}(n+1, 1), S^{4n+3})$ if there exists a ρ -equivariant developing immersion from the universal covering manifold X of M :

$$(\rho, \mathrm{dev}) : (\mathrm{Aut}_{qc}(X), X) \rightarrow (\mathrm{PSp}(n+1, 1), S^{4n+3})$$

such that

- (i) $\mathrm{dev}^* \tilde{\omega} = \lambda \cdot \omega \cdot \bar{\lambda} = u \cdot a \omega \bar{a}$ for some map $\lambda = \sqrt{u} \cdot a : X \rightarrow \mathbb{R}^+ \times \mathrm{Sp}(1)$ ($u \in \mathbb{R}^+, a \in \mathrm{Sp}(1)$).
- (ii) If the conjugate by the map $a : X \rightarrow \mathrm{Sp}(1)$ represents the matrix $(a_{\alpha\beta}) : X \rightarrow \mathrm{SO}(3)$, then $\mathrm{dev}_* \circ J_\alpha = \sum_\beta a_{\alpha\beta} I_\beta \circ \mathrm{dev}_* : D \rightarrow D_0$.
- (iii) $\rho : \mathrm{Aut}_{qc}(X) \rightarrow \mathrm{PSp}(n+1, 1)$ is the holonomy homomorphism such that $\mathrm{dev} \circ h = \rho(h) \circ \mathrm{dev}$ ($\forall h \in \mathrm{Aut}_{qc}(X)$).

Definition 4.3 A qc -manifold X is *divisible* if there exists a discrete subgroup $\Gamma \leq \mathrm{Aut}_{qc}(X)$ which acts *properly discontinuously with compact quotient*.

Theorem 4.4 Let $(X, D, \omega, \{J_\alpha\}_{\alpha=1}^3)$ be a simply connected noncompact uniformizable strict qc -manifold. Put $E = \{\xi_\alpha, \alpha = 1, 2, 3\}$. Suppose X is divisible by Γ .

(1) If $\mathrm{Aut}_{qc}(X)$ leaves E invariant, then the developing pair reduces to the equivariant immersion:

$$(\rho, \mathrm{dev}) : (\mathrm{Aut}_{qc}(X), X) \rightarrow (\mathrm{Aut}_{qc}(\mathcal{M}), \mathcal{M}).$$

(2) For any $\gamma \in \Gamma$ and $\alpha = 1, 2, 3$, suppose $\gamma_* \xi_\alpha = \sum_{\beta=1}^3 a_{\alpha\beta} \xi_\beta$ for some function $a_{\alpha\beta} : X \rightarrow \mathrm{SO}(3)$. Then,

- (i) $\mathrm{dev} : X \rightarrow \mathcal{M}$ is a qc -diffeomorphism so that $\mathcal{R} = \mathbb{R}^3$.
- (ii) There exists a strict qc -structure $(\eta, \{J'_\alpha\}_{\alpha=1}^3, \mathbb{R}^3)$ qc -conformal to $(\omega, \{J_\alpha\}_{\alpha=1}^3)$. The quotient $(X/\mathbb{R}^3, \{\Theta_\alpha, J'_\alpha\}_{\alpha=1}^3)$ is a hyperKähler manifold isometric to $\mathbb{H}^n$.

The method of proof is based on that of [13] by taking into account the results of [14] (cf. [6, 12]).

Proof Put $\mathrm{Aut}_{qc}(X) = \overline{\mathrm{Aut}(X)}$. Let $G = \overline{\rho(\mathrm{Aut}(X))}^0$ be the identity component of the closure of the holonomy image $\rho(\mathrm{Aut}(X))$ in $\mathrm{PSp}(n+1, 1)$. We first show that (i) G is not compact.

Case 1. Suppose G is compact. If G has no fixed point on S^{4n+3} , then G has the unique fixed point at the origin 0 in $\mathbb{H}_{\mathbb{H}}^{n+1}$ where $S^{4n+3} = \partial \mathbb{H}_{\mathbb{H}}^{n+1}$. As in the proof of [13], $\mathrm{dev} : X \rightarrow S^{4n+3}$ is shown to be an isometry, which is excluded by the non-compactness of X . So G has the fixed point set F in S^{4n+3} . We may assume that $\mathrm{Aut}(X)$ acts properly on X by Theorem 4.1, so dev misses F . It reduces to an immersion $\mathrm{dev} : X \rightarrow S^{4n+3} - F$. As $\mathrm{Aut}(S^{4n+3} - F)$ acts properly on $S^{4n+3} - F$ by the result of [14], there is a Riemannian

metric on $S^{4n+3} - F$ invariant under $\text{Aut}(S^{4n+3} - F)$. Since X is divisible, X is complete with respect to the pullback metric, $\text{dev} : X \rightarrow S^{4n+3} - F$ is a covering map. On the other hand, if we note that the action of G is linear on S^{4n+3} , F must be a subsphere S^k ($0 \leq k < 4n+3$) such that the complement $S^{4n+3} - F$ is unknotted, that is, homeomorphic to $\mathbb{R}^{k+1} \times S^{4n+2-k}$. Moreover, it is shown in [13, Lemma 3.1] (also [5, p.77]) that $S^{4n+3} - F$ is either one of the following:

- (1) $S^{4n+3} - S^{m-1}$ w h e r e $F = S^{m-1} = \partial \mathbb{H}_{\mathbb{R}}^m$ $(1 \leq m \leq n+1)$.
 $\text{Aut}(S^{4n+3} - S^{m-1}) = P(\text{O}(m, 1) \cdot \text{Sp}(1) \times \text{Sp}(n-m+1)).$
- (2) $S^{4n+3} - S^{2m-1}$ w h e r e $F = S^{2m-1} = \partial \mathbb{H}_{\mathbb{C}}^m$ $(1 \leq m \leq n+1)$.
 $\text{Aut}(S^{4n+3} - S^{2m-1}) = P(\text{U}(m, 1) \cdot \text{U}(1) \times \text{Sp}(n-m+1)).$
- (3) $S^{4n+3} - S^{4m-1}$ w h e r e $F = S^{4m-1} = \partial \mathbb{H}_{\mathbb{H}}^m$ $(1 \leq m \leq n)$.
 $\text{Aut}(S^{4n+3} - S^{4m-1}) = \text{Sp}(m, 1) \cdot \text{Sp}(n-m+1).$
- (4) $S^{4n+3} - S^2$ where $S^2 = \partial \mathbb{H}_{\text{Im}}$. $\text{Aut}(S^{4n+3} - S^2) = \text{SL}(2, \mathbb{C}) \cdot \text{Sp}(n)$. (This case reduces to (3).)

In particular, $S^{4n+3} - F$ is simply connected in each case. Hence, $\text{dev} : X \rightarrow S^{4n+3} - F$ is diffeomorphic, so ρ is an isomorphism. However, this case does not occur since $\text{Aut}(X)^0 \cong G = \rho(\text{Aut}(X)) = \text{Aut}(S^{4n+3} - F)^0$ which is noncompact (in fact $\text{Aut}(S^{4n+3} - F)^0$ contains a noncompact subgroup $\text{O}(m, 1)$, $\text{U}(m, 1)$ ($1 \leq m \leq n+1$), $\text{SL}(2, \mathbb{C})$ or $\text{Sp}(m, 1)$ ($1 \leq m \leq n$), respectively.) As a consequence, the case G is compact does not occur.

Case 2. Suppose G is noncompact. Then, either G has a common fixed point $\{\infty\}$ in S^{4n+3} or G leaves invariant a totally geodesic subspace of $\mathbb{H}_{\mathbb{H}}^{n+1}$. (See [5, Theorem 4.4.1].) In the latter case, G (possibly $\text{PSp}(n+1, 1)$) is either one of the identity component of the above groups (1), (2), (3), (4). On the other hand, let $\mathcal{R}$ be the local abelian group generated by E as before. The holonomy homomorphism ρ maps $\mathcal{R}$ into $\text{PSp}(n+1, 1)$. By Liouville's theorem, $\rho(\mathcal{R})$ extends globally to a subgroup of $\text{PSp}(n+1, 1)$ on S^{4n+3} . As $\text{Aut}_{qc}(X)$ leaves E invariant, $\text{Aut}(X)^0$ normalizes $\mathcal{R}$. Thus, G normalizes $\rho(\mathcal{R})$ also. In particular, the radical of G is nontrivial, so G is not semisimple. On the other hand, G is *semisimple* except for the case $m=1$ of (1) such that $\text{Aut}(S^{4n+3} - S^0)^0 = \text{SO}(1, 1)^0 \times \text{Sp}(1) \cdot \text{Sp}(n) \cong \mathbb{R}^+ \times \text{Sp}(n) \cdot \text{Sp}(1)$. For this case, G has exactly two fixed points $\{0, \infty\}$. Noting from Theorem 4.1, as in the argument of (i), $\text{dev} : X \rightarrow S^{4n+3} - \{0, \infty\} = S^{4n+2} \times \mathbb{R}^+$ is a diffeomorphism so that ρ maps the radical of $\text{Aut}(X)^0$ isomorphically onto the radical $\mathbb{R}^+$. Since the radical contains $\rho(\mathcal{R})$ of dimension three, it is impossible. As a consequence, by the non-ellipticity of elements in $\text{PSp}(n+1, 1)$, G has a *unique* common fixed point $\{\infty\}$. Noting $\text{Aut}(X)$ acts properly on X , dev misses $\{\infty\}$. This proves (1).

(2). Suppose some $\rho(\gamma) \in \rho(\Gamma)$ has a nontrivial summand in $\mathbb{R}^+$ of $\text{Aut}(\mathcal{M})$. It follows from (3.6) that $\rho(\gamma)^* \omega_0 = v \cdot b \omega_0 \bar{b}$ where $\lambda = \sqrt{v} \cdot b \in \text{Sp}(1) \times \mathbb{R}^+$. On the other hand, by our hypothesis (2), $\gamma_* \xi_\alpha = \sum_{\beta=1}^3 a_{\alpha\beta} \xi_\beta$ for some function $a_{\alpha\beta} : X \rightarrow \text{SO}(3)$. Put $\text{dev}_* \xi_\alpha = \tilde{\xi}_\alpha$ on $\mathcal{M}$. As $\text{dev} \circ \gamma = \rho(\gamma) \circ \text{dev}$, letting $(i, j, k) \sim (i_\alpha, i_\beta, i_\gamma)$, it follows

$$\rho(\gamma)_*(\tilde{\xi}_\alpha i_\alpha) = a(\tilde{\xi}_\alpha i_\alpha) \bar{a}$$

where the conjugate of a represents the matrix $(a_{\alpha\beta}) \in \text{SO}(3)$. Calculate

$$\begin{aligned} \omega_0(\rho(\gamma)_* \tilde{\xi}_\alpha i_\alpha) &= a \omega_0(\tilde{\xi}_\alpha i_\alpha) \bar{a} \\ \rho(\gamma)^* \omega_0(\tilde{\xi}_\alpha i_\alpha) &= u \cdot b \omega_0(\tilde{\xi}_\alpha i_\alpha) \bar{b}. \end{aligned} \quad (4.1)$$

Taking the norm in $\mathbb{H}$, it follows

$$|\omega_0(\tilde{\xi}_\alpha i_\alpha)| = |a\omega_0(\tilde{\xi}_\alpha i_\alpha)\bar{a}| = |u \cdot b\omega_0(\tilde{\xi}_\alpha i_\alpha)\bar{b}| = u|\omega_0(\tilde{\xi}_\alpha i_\alpha)|.$$

Hence, $u = 1$ on X . This implies $\rho(\gamma) \in E(\mathcal{M})$ so that $\rho(\Gamma) \leq E(\mathcal{M})$. As usual, there is the $E(\mathcal{M})$ -invariant Riemannian metric on $\mathcal{M}$. Since X is divisible, X is complete with respect to the pullback metric. Thus, $(\rho, \text{dev}) : (\Gamma, X) \rightarrow (E(\mathcal{M}), \mathcal{M})$ is an equivariant isometry. As $\rho : \text{Aut}_{qc}(X) \rightarrow \text{Aut}_{qc}(\mathcal{M})$ is an isomorphism, and $\mathcal{R}$ is normalized by $\text{Aut}_{qc}(X)$, so does $\rho(\mathcal{R})$ in $\text{Aut}_{qc}(\mathcal{M})$. By the action of (3.1) and the group structure of $\text{Aut}_{qc}(\mathcal{M})$ we note $\rho(\mathcal{R}) = \mathbb{R}^3$ which is the center of $\mathcal{M}$. This proves (i). In particular, $\text{dev}_* E = E_0$.

(ii). Let $(\omega_0, \{J_\alpha\}_{\alpha=1}^3)$ be the standard (spherical) qc -structure on $\mathcal{M}$ where $\omega_0 = \omega_1 i + \omega_2 j + \omega_3 k$. By the definition, it satisfies $\text{dev}_* \omega_0 = u \cdot a\omega\bar{a}$ for some $u \in \mathbb{R}^+$, $a \in \text{Sp}(1)$. When a represents $(a_{\alpha\beta}) \in \text{SO}(3)$ as before, it follows $\text{dev}_* J_\alpha = \sum_{\beta} a_{\alpha\beta} J_\beta \text{dev}_*$ (cf. Definition 4.2). Put

$$\eta = \text{dev}_* \omega_0 (\eta_\alpha = \text{dev}_* \omega_\alpha), \quad J'_\alpha = \sum_{\beta=1}^3 a_{\beta\alpha} J_\beta (\alpha = 1, 2, 3). \quad (4.2)$$

We check $(D, \eta, \{J'_\alpha\}_{\alpha=1}^3)$ is a qc -structure qc -conformal to $(D_0, \omega, \{J_\alpha\}_{\alpha=1}^3)$. For this, let $X, Y \in D$ so that $\text{dev}_* X, \text{dev}_* Y \in D_0$. Note that $J_\gamma \text{dev}_* = \text{dev}_* \sum_{\mu} a_{\mu\gamma} J_\mu$ as above. Let $\tilde{J}_\gamma = (d\eta_\beta)^{-1} \circ d\eta_\alpha$ as well as $J_\alpha = (d\omega_\beta)^{-1} \circ d\omega_\alpha$. By calculation,

$$\begin{aligned} d\omega_\beta(J_\gamma(\text{dev}_* u), \text{dev}_* v) &= d\omega_\alpha(\text{dev}_* u, \text{dev}_* v) = d\eta_\alpha(u, v) \\ &= d\omega_\beta(\text{dev}_*(\sum_{\mu} a_{\mu\gamma} J_\mu u), \text{dev}_* v) = d\eta_\beta(J'_\gamma u, v). \end{aligned}$$

Noting $d\eta_\beta(\tilde{J}_\gamma u, v) = d\eta_\alpha(u, v)$, the non-degeneracy of $d\eta_\beta$ implies $\tilde{J}_\gamma = J'_\gamma$. The equations $\text{dev}_* \omega_0 = \eta$, $\text{dev}_* E = E_0$ imply $d\eta(E, A) = 0$ ($\forall A \in TX$). Thus, $(D, \eta, \{J'_\alpha\}_{\alpha=1}^3)$ is a strict qc -structure. Noting that R acts properly and freely on X , we have a smooth manifold $Y = X/R$. As $\text{dev} : X \rightarrow \mathcal{M}$ is ρ -equivariant, dev induces a diffeomorphism $\hat{\text{dev}} : Y \rightarrow \mathbb{H}^n$ with the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\text{dev}} & \mathcal{M} \\ \pi \downarrow & & \pi \downarrow \\ Y & \xrightarrow{\hat{\text{dev}}} & \mathbb{H}^n. \end{array}$$

Applying Proposition 2.5 (cf. (2.6)), the 2-form Θ_α on Y is defined by

$$\pi^* \Theta_\alpha = d\eta_\alpha (\alpha = 1, 2, 3). \quad (4.3)$$

A quaternionic structure $\{\hat{J}'_\alpha\}_{\alpha=1}^3$ on Y is also induced by

$$\hat{J}'_\alpha \pi_* = \pi_* J'_\alpha. \quad (4.4)$$

Using the reciprocity $J'_\gamma = (d\eta_\beta)^{-1} \circ d\eta_\alpha$, it follows for any $\hat{A}, \hat{B} \in TY$

$$\Theta_\alpha(\hat{J}'_\alpha \hat{A}, \hat{J}'_\alpha \hat{B}) = \Theta_\alpha(\hat{A}, \hat{B}) (\alpha = 1, 2, 3).$$

Thus, Θ_α is a Kähler form on $(Y, \{\hat{J}'_\alpha\}_{\alpha=1}^3)$. The quaternionic Hermitian metric

$$g(\hat{A}, \hat{B}) = \Theta_\alpha(\hat{J}'_\alpha \hat{A}, \hat{B}) = \Theta_\beta(\hat{J}'_\beta \hat{A}, \hat{B}) = \Theta_\gamma(\hat{J}'_\gamma \hat{A}, \hat{B}) \quad (4.5)$$

is a hyperKähler metric on $(Y, \{\hat{J}'_\alpha\}_{\alpha=1}^3)$. Let $(\mathbb{H}^n, g_{\mathbb{H}}, \{I_\alpha\}_{\alpha=1}^3)$ be the standard euclidean metric as in Remark 3.2. Noting $d\omega_\alpha \circ J_\alpha = \pi^* g_{\mathbb{H}}$ and (4.2), a calculation shows

$$\hat{\text{dev}}^* g_0 = g, \hat{\text{dev}}_* \circ \hat{J}'_\alpha = I_\alpha \circ \hat{\text{dev}}_*. \quad (4.6)$$

This gives an isometry of $(Y, g, \{\hat{J}'_\alpha\}_{\alpha=1}^3)$ onto $(\mathbb{H}^n, g_{\mathbb{H}}, \{I_\alpha\}_{\alpha=1}^3)$. $\square$

Remark 4.5 The new Kähler form Θ_α and $\Theta = \Theta_1 i + \Theta_2 j + \Theta_3 k$ are related to the original forms Ω and Ω_α as follows. For some constant $c > 0$,

$$\Theta = c \cdot a \Omega \bar{a}, \quad \Theta_\alpha = c \cdot \sum_{\beta=1}^3 a_{\alpha\beta} \Omega_\beta.$$

In fact, as we put $\eta = \text{dev}^* \omega_0 = u \cdot a \omega \bar{a}$, it follows $d\eta = u \cdot a \cdot d\omega \cdot \bar{a} \mid_D$ so that $\Theta = \hat{u} \cdot \alpha \Omega \bar{\alpha}$ where $\hat{u}, \alpha$ are induced functions on Y . As usual, $\Theta^2 = \hat{u}^2 \Omega^2$ which shows that $\hat{u}$ is a constant $c > 0$. We have $\Theta_\alpha = c \sum_{\beta=1}^3 \alpha_{\alpha\beta} \Omega_\beta$. ($\pi^* \alpha = a$, $\pi^* \alpha_\beta = a_{\alpha\beta}$).

5 Quotient quaternionic Hermitian manifolds

For the strict qc -structure $(D_0, \omega_0, \{J_\alpha\}_{\alpha=1}^3)$ on the quaternionic Heisenberg Lie group $\mathcal{M}$, we consider a qc -structure $\eta = \eta_1 i + \eta_2 j + \eta_3 k$ which is qc -conformal to ω . Take a one-form, say η_1 to define a distribution:

$$E_1 = \{\xi \mid d\eta_1(\xi, A) = 0, \quad \forall A \in TX\}. \quad (5.1)$$

E_1 does not induce a distribution such as E . When E_1 generates a three-dimensional abelian Lie group $\mathcal{R}$, we shall show that there is an invariant domain X such that the quotient $X/\mathcal{R}$ admits a special kind of *quaternionic Hermitian* structure.

Choose numbers $a_1, \dots, a_n$ such that

$$0 < a_1 < a_2 < \dots < a_n. \quad (5.2)$$

Let A_t be the diagonal matrix

$$\text{diag}(e^{ia_1 t}, e^{ia_2 t}, \dots, e^{ia_n t}) \in T^n \leq \text{Sp}(n). \quad (5.3)$$

Define a homomorphism $\rho_1 : \mathbb{R}^3 \rightarrow (\mathbb{R}^3, 0) \times \text{Sp}(n) \leq \mathcal{M} \rtimes \text{Sp}(n)$ to be

$$\begin{aligned} \rho_1(t_1) &= (((t_1, 0, 0), 0), A_{t_1}), \\ \rho_1(t_2) &= (((0, t_2, 0), 0), I), \quad \rho_1(t_3) = (((0, 0, t_3), 0), I). \end{aligned} \quad (5.4)$$

More precisely, this action is defined on $\mathcal{M} = \mathbb{R}^3 \times \mathbb{H}^n$ as

$$\begin{aligned}
\rho_1(t_1)((s_1, s_2, s_3), z_1, \dots, z_n) &= ((s_1 + t_1), s_2, s_3), \\
&\quad e^{ia_1 t_1} z_1, \dots, e^{ia_n t_1} z_n), \\
\rho_1(t_2)((s_1, s_2, s_3), z_1, \dots, z_n) &= (s_1, (s_2 + t_2), s_3), z_1, \dots, z_n), \\
\rho_1(t_3)((s_1, s_2, s_3), z_1, \dots, z_n) &= (s_1, s_2, (s_3 + t_3)), z_1, \dots, z_n).
\end{aligned} \tag{5.5}$$

In view of (5.4), the group $\langle \rho_1(t_1), \rho_1(t_2), \rho_1(t_3) \rangle$ forms a 3-dimensional abelian Lie group $\rho_1(\mathbb{R}^3)$. If ξ_1 is the vector field induced by $\{\rho_1(t_1)\}_{t_1 \in \mathbb{R}}$, then it follows

$$\begin{aligned}
\xi_1 &= \frac{\rho_1(t_1)}{dt_1} \Big|_{t_1=0} = \frac{d}{dt_1} + (a_1 i z_1, \dots, a_n i z_n) = \frac{d}{dt_1} + \\
&\quad \sum_{k=1}^n a_k (-x_{4k-2} \frac{d}{dx_{4k-3}} + x_{4k-3} \frac{d}{dx_{4k-2}} - x_{4k} \frac{d}{dx_{4k-1}} + x_{4k-1} \frac{d}{dx_{4k}}).
\end{aligned} \tag{5.6}$$

Letting $z_k = u_k + v_k j = (x_{4k-3} + i x_{4k-2}) + (x_{4k-1} + i x_{4k})j \in \mathbb{H}$ ($k = 1, \dots, n$), a calculation using (3.3) shows that

$$\begin{aligned}
\omega_1((\xi_1)_p) &= 1 + \sum_{k=1}^n a_k (x_{4k-3}^2 + x_{4k-2}^2 - x_{4k}^2 - x_{4k-1}^2) \\
&= 1 + \sum_{k=1}^n a_k (|u_k|^2 - |v_k|^2) \quad \text{at } p = ((s_1, s_2, s_3), z_1, \dots, z_n).
\end{aligned} \tag{5.7}$$

The singular set $S = \{p \in \mathcal{M} \mid \omega_1((\xi_1)_p) = 0\}$ is not entirely equal to $\mathcal{M}$ from (5.7). Denote the domain of $\mathcal{M}$ by

$$X = \{p \in \mathcal{M} \mid \omega_1((\xi_1)_p) \neq 0\} \tag{5.8}$$

(if necessary taking the component containing the origin $(0, 0) \in \mathcal{M}$). Since $\rho_1(\mathbb{R}^3)$ leaves S invariant, so does X . Put $Y = X/\rho_1(\mathbb{R}^3)$. Then, there is a commutative diagram of principal bundles.

$$\begin{array}{ccccc}
\rho_1(\mathbb{R}^3) & \longrightarrow & \mathcal{M} & \xrightarrow{\pi_1} & \mathbb{H}^n \\
|| & & \cup & & \cup \\
\rho_1(\mathbb{R}^3) & \longrightarrow & X & \xrightarrow{\pi_1} & Y.
\end{array} \tag{5.9}$$

The image $\pi_1 S = \{\pi_1(p) \in \mathbb{H}^n \mid \sum_{k=1}^n a_k (|u_k|^2 - |v_k|^2) = -1\}$ is a real hypersurface in $\mathbb{R}^{4n} = \mathbb{H}^n$. Noting $Y = \mathbb{H}^n - \pi_1 S$, it follows each component of Y is simply connected in $\mathbb{H}^n$.

5.1 Conformal change of ω_0

Let $\omega_0 = \omega_1 i + \omega_2 j + \omega_3 k$ be the qc -form on $\mathcal{M}$ (cf. Sect. 3.2). We introduce new 1-forms on X :

$$\eta_\alpha = \frac{1}{\omega_1(\xi_1)} \omega_\alpha \quad (\alpha = 1, 2, 3). \tag{5.10}$$

Put $\eta = \eta_1 i + \eta_2 j + \eta_3 k$ on X . Since η is conformal to ω_0 , it follows $\ker \eta = \ker \omega_0 = D_0$ on X . As $\rho_1(\mathbb{R}^3)$ leaves ω_α invariant, so does η_α . The hypercomplex structure of

η , $\{\tilde{J}_\gamma = (d\eta_\beta|_{D_0})^{-1} \circ (d\eta_\alpha|_{D_0})\}_{\gamma=1}^3$ coincides with $\{J_\gamma\}_{\gamma=1}^3$ of ω_0 on D_0 . Noting $\rho_1(t)_* J_\alpha = J_\alpha \rho_1(t)_*$ ($t \in \mathbb{R}^3$) from (3.7), $\{J_\alpha\}$ induces a quaternionic structure $\{\hat{J}_\alpha\}_{\alpha=1}^3$ on Y such that

$$\pi_{1*} J_\alpha(v) = \hat{J}_\alpha \pi_{1*}(v) \quad (\forall v \in D_0). \quad (5.11)$$

Proposition 5.1 $(Y, \{\hat{J}_\alpha\}_{\alpha=1}^3)$ is a quaternionic Hermitian manifold.

Proof Define $\hat{\Omega}_\alpha$ ($\alpha = 1, 2, 3$) to be

$$\hat{\Omega}_\alpha(\pi_{1*} u, \pi_{1*} v) = d\eta_\alpha(u, v) \quad (\forall u, v \in D_0). \quad (5.12)$$

Since η_α is $\rho_1(\mathbb{R}^3)$ -invariant and the distribution by $\rho_1(\mathbb{R}^3)$ is transverse to D_0 , $\hat{\Omega}_\alpha$ is well-defined on Y (cf. Lemma 5.2). Put

$$\hat{g}(\hat{u}, \hat{v}) = \hat{\Omega}_\alpha(\hat{J}_\alpha \hat{u}, \hat{v}) \quad (\forall \hat{u}, \hat{v} \in TY). \quad (5.13)$$

As $\hat{\Omega}_\alpha$ is invariant under $\hat{J}_\alpha$, it follows $\hat{g}(\hat{J}_\alpha \hat{u}, \hat{J}_\alpha \hat{v}) = \hat{g}(\hat{u}, \hat{v})$. Thus, $\hat{g}$ is a quaternionic Hermitian metric on $(Y, \{\hat{J}_\alpha\}_{\alpha=1}^3)$.

As in (5.4), the distribution $\left\{ \xi_1, \frac{d}{dt_2}, \frac{d}{dt_3} \right\}$ generates $\rho_1(\mathbb{R}^3) \leq \mathcal{M} \rtimes \mathrm{Sp}(n)$. Note from (3.3) that

$$\eta_1(\xi_1) = 1, \eta_1\left(\frac{d}{dt_2}\right) = 0, \eta_1\left(\frac{d}{dt_3}\right) = 0. \quad (5.14)$$

Lemma 5.2 $E_1 = \langle \xi_1, \frac{d}{dt_2}, \frac{d}{dt_3} \rangle$.

Proof For any $A \in TX$, we prove $d\eta_1(\xi_1, A) = 0$, $d\eta_1(\frac{d}{dt_\beta}, A) = 0$ ($\beta = 2, 3$). If $A \in D_0$, then $[\xi_1, A] \in D_0$. Since the distribution $\langle \xi_1, \frac{d}{dt_2}, \frac{d}{dt_3} \rangle$ generates $\mathbb{R}^3$, $[\xi_1, \frac{d}{dt_\beta}] = 0$ ($\beta = 2, 3$). Then, it is easy to see that $d\eta_1(\xi_1, A) = d\eta_1(\xi_1, \frac{d}{dt_\beta}) = 0$. As $\frac{d}{dt_\beta}$ ($\beta = 2, 3$) are induced from the central subgroup $(0, \mathbb{R}^2)$ of $\mathbb{R}^3$, (5.14) shows $d\eta_1(\frac{d}{dt_\beta}, B) = 0$, $d\eta_1(\frac{d}{dt_\beta}, \frac{d}{dt_\gamma}) = 0$ ($\forall B \in D_0$). $\square$

Lemma 5.3 Let $\hat{\Omega}_1$ be the 2-form on Y as in (5.12). Then, $d\hat{\Omega}_1 = 0$.

Proof It suffices to show

$$\pi_1^* \hat{\Omega}_1 = d\eta_1 \quad \text{on } Y. \quad (5.15)$$

$\pi_1^* \hat{\Omega}_1 = d\eta_1$ on D_0 from (5.12). For any $\xi \in E_1$ and $A \in TX$, $d\eta_1(\xi, A) = 0$ by Lemma 5.2. As $E_1 \oplus D_0 = TX$, it follows $\pi_1^* \hat{\Omega}_1 = d\eta_1$ on X . $\square$

6 Pseudo-Hermitian structure

6.1 Heisenberg Lie group $\mathcal{N}$

Let $\mathcal{N}$ be the $4n+1$ -dimensional Heisenberg Lie group which has a central group extension $1 \rightarrow \mathbb{R} \rightarrow \mathcal{N} \xrightarrow{p} \mathbb{C}^{2n} \rightarrow 1$. A pseudo-Hermitian structure $(\omega_{\mathcal{N}}, J_{\mathcal{N}})$ consists of a contact form

$$\omega_{\mathcal{N}} = dt + \Im \langle (z, w), (dz, dw) \rangle = dt + \Im({}^t \bar{z} dz + {}^t \bar{w} dw) \quad ((z, w) \in \mathbb{C}^{2n})$$

together with a complex structure $J_{\mathcal{N}}$ on $\ker \omega_{\mathcal{N}}$ which is isomorphic to the standard complex structure on $\mathbb{C}^{2n}$ at each point of $\mathcal{N}$ (cf. [11]). As $\mathbb{R}^2 = \{(0, t_2, t_3)\}$ is a central subgroup of $\mathbb{R}^3 = C(\mathcal{M})$, there is a quotient nilpotent Lie group $\mathcal{M}/\mathbb{R}^2$ with central group extension $1 \rightarrow \mathbb{R} \rightarrow \mathcal{M}/\mathbb{R}^2 \xrightarrow{\pi} \mathbb{H}^n \rightarrow 1$. We shall find an explicit isomorphism to identify $\mathcal{M}/\mathbb{R}^2$ with $\mathcal{N}$. For our use, let $z + wj \in \mathbb{H}^n$ such that $z, w \in \mathbb{C}^n$. Then, $\mathcal{M}/\mathbb{R}^2$ is the product $\mathbb{R} \times \mathbb{H}^n = \text{Im } \mathbb{H} \pmod{\mathbb{R}j + \mathbb{R}k} \times \mathbb{H}^n$ with group law:

$$(a, z + wj) \cdot (b, z' + w'j) = (a + b - \Im({}^t \bar{z} z' + {}^t w \bar{w}'), z + z' + (w + w')j).$$

Define a diffeomorphism $\varphi : \mathcal{M}/\mathbb{R}^2 \rightarrow \mathcal{N} = \mathbb{R} \times \mathbb{C}^{2n}$ to be

$$\varphi(a, (z + wj)) = (a, (z, \bar{w})). \quad (6.1)$$

As we see that $\varphi((a, z + wj) \cdot (b, z' + w'j)) = (a, (z, \bar{w})) \cdot (b, (z', \bar{w}'))$,

Lemma 6.1 *φ is a Lie group isomorphism of $\mathcal{M}/\mathbb{R}^2$ onto $\mathcal{N}$.*

Consider the projection

$$\mathbb{R}^2 = \mathbb{R}j + \mathbb{R}k \longrightarrow \mathcal{M} \xrightarrow{p_1} \mathcal{M}/\mathbb{R}^2 \quad (6.2)$$

for which the subbundle $\langle \frac{d}{dt_2}, \frac{d}{dt_3} \rangle$ is tangent to the fiber $\mathbb{R}^2$. For $\omega_0 = \omega_1 i + \omega_2 j + \omega_3 k$, noting (3.3), ω_1 induces a 1-form $\hat{\omega}_1$ on $\mathcal{M}/\mathbb{R}^2$ such that

$$p_1^* \hat{\omega}_1 = \omega_1 \quad \text{on } \mathcal{M}. \quad (6.3)$$

As $\hat{\omega}_1 = dt_1 + \Im \langle (z, \bar{w}), (dz, d\bar{w}) \rangle$ from (3.2) ($z + wj \in \mathbb{H}^n$), (6.1) shows

$$\varphi^* \omega_{\mathcal{N}} = \hat{\omega}_1 \quad \text{on } \mathcal{M}/\mathbb{R}^2. \quad (6.4)$$

Let $\hat{\varphi} : \mathbb{H}^n \rightarrow \mathbb{C}^{2n}$ be a diffeomorphism defined by

$$\hat{\varphi}(z + wj) = (z, \bar{w}), \quad (6.5)$$

with the following commutative diagram from (6.1):

$$\begin{array}{ccc} \mathcal{M}/\mathbb{R}^2 & \xrightarrow{\varphi} & \mathcal{N} \\ \hat{\pi} \downarrow & & p \downarrow \\ \mathbb{H}^n & \xrightarrow{\hat{\varphi}} & \mathbb{C}^{2n}. \end{array} \quad (6.6)$$

Take the standard complex structure J_1 on $\mathbb{H}^n$ such that $J_1 u = u\bar{i}$. As $u\bar{i} = (z + wj)\bar{i} = (\bar{i}z + iwj)$, it follows $\hat{\varphi}(u\bar{i}) = \bar{i}\hat{\varphi}(u)$. If we take the anti-complex structure $J'_\mathbb{C}$ on $\mathbb{C}^{2n}$ such as $J'_\mathbb{C}(v) = \bar{i}v$, it follows

$$\hat{\varphi}_* \circ J_1 = J'_\mathbb{C} \circ \hat{\varphi}_* : T\mathbb{H}^n \rightarrow T\mathbb{C}^{2n}. \quad (6.7)$$

As in (3.4) of Sect. 3.2, the almost complex structure J_1 on D_0 induces an almost complex structure J_1 on $p_{1*}D_0 = \ker \hat{\omega}_1$ such that

$$\hat{\pi}_* \circ J_1 = J_1 \circ \hat{\pi}_* : p_{1*}D_0 \rightarrow T\mathbb{H}^n. \quad (6.8)$$

Similarly, if $J'_\mathcal{N}$ is the anti-complex structure on $\ker \omega_\mathcal{N}$ of $\mathcal{N}$, it follows

$$p_* \circ J'_\mathcal{N} = J'_\mathbb{C} \circ p_* : \ker \omega_\mathcal{N} \rightarrow T\mathbb{C}^{2n}. \quad (6.9)$$

Then, (6.7), (6.8) and (6.9) imply

$$\varphi_* \circ J_1 = J'_\mathcal{N} \circ \varphi_* \text{ on } \ker \hat{\omega}_1 (= p_{1*}D_0). \quad (6.10)$$

In particular, J_1 is a complex structure on $p_{1*}D_0 = \ker \hat{\omega}_1$. We have the principal bundle induced from (5.9):

$$\rho_1(\mathbb{R}) \longrightarrow X/\mathbb{R}^2 \xrightarrow{\hat{\pi}_1} Y. \quad (6.11)$$

There is a CR-structure $(\ker \hat{\omega}_1, J_1)$ on $X/\mathbb{R}^2 \subset \mathcal{M}/\mathbb{R}^2$ as above. Let $\hat{J}_1$ be the almost complex structure on Y as in (5.11), that is $\hat{\pi}_{1*}J_1 = \hat{J}_1\hat{\pi}_{1*} : \ker \hat{\omega}_1 \rightarrow TY$.

Lemma 6.2 $\hat{J}_1$ is integrable on Y .

Proof Let $\ker \hat{\omega}_1 \otimes \mathbb{C} = P^{1,0} \oplus P^{0,1}$ be the eigenspace decomposition for J_1 . We have an isomorphism $\hat{\pi}_{1*} : \ker \hat{\omega}_1 \otimes \mathbb{C} \rightarrow TY \otimes \mathbb{C} = Q^{1,0} \oplus Q^{0,1}$ (the eigenspace decomposition for $\hat{J}_1$, respectively). Since J_1 is integrable, $[u, v] \in P^{1,0}$ for $u, v \in P^{1,0}$. Then, $\hat{\pi}_{1*}([u, v]) = [\hat{\pi}_{1*}(u), \hat{\pi}_{1*}(v)]$. It follows $[\hat{\pi}_{1*}(u), \hat{\pi}_{1*}(v)] \in Q^{1,0}$. Thus, $\hat{J}_1$ is integrable. $\square$

Combining this lemma with Lemma 5.3, we obtain

Proposition 6.3 $(Y, \{\hat{\Omega}_1, \hat{J}_1\})$ is a Kähler manifold.

6.2 Bochner flat structure on $(Y, \hat{J}_1)$

Put $e^{iat}z = (e^{ia_1t}z_1, \dots, e^{ia_nt}z_n)$ for short, similarly for $e^{-iat}w$ for $a = (a_1, \dots, a_n)$ satisfying (5.2). Let $\mathbb{R}$ act on $\mathcal{N}$ by

$$\rho(t)(s, (z, w)) = (t + s, (e^{iat}z, e^{-iat}w)) \quad (6.12)$$

such that $\rho(\mathbb{R}) \leq \mathbb{R} \times \mathrm{U}(2n) \leq \mathcal{N} \rtimes \mathrm{U}(2n) \leq \mathrm{Aut}_{CR}(\mathcal{N})$. There induces another principal bundle $\rho(\mathbb{R}) \rightarrow \mathcal{N} \xrightarrow{q} \mathbb{C}^{2n}$. By (6.1) and (5.5),

$$\begin{aligned}\varphi(\rho_1(t)(s, (z + wj))) &= \varphi(s + t, (e^{iat}z + e^{iat}wj)) \\ &= (s + t, (e^{iat}z, e^{-iat}\bar{w})) = \rho(t)\varphi(s, (z + wj)),\end{aligned}$$

there is the bundle isomorphism (cf. (5.9)):

$$\begin{array}{ccc}\rho_1(\mathbb{R}) & \longrightarrow & \rho(\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathcal{M}/\mathbb{R}^2 & \xrightarrow{\varphi} & \mathcal{N} \\ \hat{\pi}_1 \downarrow & & q \downarrow \\ \mathbb{H}^n & \xrightarrow{\hat{\varphi}} & \mathbb{C}^{2n}.\end{array}\quad (6.13)$$

Let ξ be the vector field induced by $\rho(\mathbb{R})$ on $\mathcal{N}$. Put $p_{1*}\xi_1 = \hat{\xi}_1$ from (5.6), (6.2), which is the Reeb field of $(\mathcal{M}/\mathbb{R}^2, (\hat{\omega}_1, J_1))$. We have $\varphi_*\hat{\xi}_1 = \xi$. At $p = ((s_1, s_2, s_3), z_1, \dots, z_n) \in \mathcal{M}$ with $z_k = u_k + v_kj$, (6.4), (6.3) and (5.7) imply

$$\omega_{\mathcal{N}}(\xi) = \hat{\omega}_1(\hat{\xi}_1) = \omega_1(\xi_1) = 1 + \sum_{k=1}^n a_k(|u_k|^2 - |v_k|^2). \quad (6.14)$$

Corresponding to $X/\mathbb{R}^2 = \{x \in p_1(X) \mid \hat{\omega}_1((\hat{\xi}_1)_x) \neq 0\}$, the bundle isomorphism φ maps $X/\mathbb{R}^2$ onto the domain $\mathcal{N}_1 = \{z \in \mathcal{N} \mid \omega_{\mathcal{N}}(\xi_z) \neq 0\}$ of $\mathcal{N}$. As in (5.10), define the contact forms to be

$$\hat{\eta}_1 = \frac{1}{\hat{\omega}_1(\hat{\xi}_1)}\hat{\omega}_1 \text{ on } X/\mathbb{R}^2, \quad \eta_{\mathcal{N}_1} = \frac{1}{\omega_{\mathcal{N}}(\xi)}\omega_{\mathcal{N}} \text{ on } \mathcal{N}_1. \quad (6.15)$$

In particular, $p_{1*}D_0 = \ker \hat{\eta}_1$. Noting (6.14), (6.4) shows that

$$\varphi^*\eta_{\mathcal{N}_1} = \hat{\eta}_1.$$

Put $J_{\mathcal{N}_1} = J_{\mathcal{N}}|_{\mathcal{N}_1}$. Since $(\eta_{\mathcal{N}_1}, J_{\mathcal{N}_1})$ represents a spherical pseudo-Hermitian structure on $\mathcal{N}_1$, this equation together with (6.10) implies (cf. [11])

Proposition 6.4 *The pseudo-Hermitian structure $(X/\mathbb{R}^2, \hat{\eta}_1, J_1)$ is anti-holomorphically isomorphic to the spherical CR-structure $(\mathcal{N}_1, \eta_{\mathcal{N}_1}, J_{\mathcal{N}_1})$.*

The projection $\hat{\pi}_1 : X/\mathbb{R}^2 \rightarrow Y$ of (6.11) induces the following from (5.15), (5.11):

$$\begin{aligned}\hat{\pi}_1^*\hat{\Omega}_1 &= d\hat{\eta}_1 \text{ on } X/\mathbb{R}^2, \\ \hat{J}_\alpha \circ \hat{\pi}_{1*} &= \hat{\pi}_{1*} \circ J_\alpha \text{ on } p_{1*}D_0 \ (\alpha = 1, 2, 3).\end{aligned}\quad (6.16)$$

Theorem 6.5 *$(Y, \hat{g}, \{\hat{\Omega}_1, \hat{J}_1\})$ is an anti-holomorphic Bochner flat Kähler manifold for the quaternionic Hermitian manifold $(Y, \hat{g}, \{\hat{\Omega}_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3)$.*

Proof Since $d\hat{\eta}_1 = \hat{\pi}_1^*\hat{\Omega}_1$ from $X/\mathbb{R}^2$ by (6.16), $\rho_1(\mathbb{R}) \rightarrow X/\mathbb{R}^2 \xrightarrow{\hat{\pi}_1} Y$ is a pseudo-Hermitian bundle with the Reeb field $\hat{\xi}_1$, the Bochner curvature tensor of the Kähler manifold $(Y, \{\hat{\Omega}_1, \hat{J}_1\})$ coincides with the Chern–Moser curvature tensor on $(X/\mathbb{R}^2, \hat{\eta}_1, J_1)$ by the

pull-back of $\hat{\pi}_1$ in (6.16) (cf. [16]). As $X/\mathbb{R}^2$ is *spherical* CR by Proposition 6.4, the Chern–Moser curvature tensor is zero and thus $(Y, \{\hat{\Omega}_1, \hat{J}_1\})$ is a Bochner flat manifold. $\square$

7 Pseudo-Hermitian group $\text{Psh}(X/\mathbb{R}^2)$

We determine the holomorphic isometry group $\text{Isom}_h(Y) = \{f \in \text{Diff}(Y) \mid f^* \hat{\Omega}_1 = \hat{\Omega}_1, f_* \circ \hat{J}_1 = \hat{J}_1 \circ f_*\}$ of the Kähler manifold $(Y, (\hat{\Omega}_1, \hat{J}_1))$. In order to do so, consider the pseudo-Hermitian group of the pseudo-Hermitian manifold $(X/\mathbb{R}^2, \hat{\eta}_1, J_1)$ (cf. [11]):

$$\text{Psh}(X/\mathbb{R}^2) = \{\tilde{f} \in \text{Diff}(X/\mathbb{R}^2) \mid \tilde{f}^* \hat{\eta}_1 = \hat{\eta}_1, \tilde{f}_* \circ J_1 = J_1 \circ \tilde{f}_*\}.$$

As $H^1(Y; \mathbb{R}) = 0$ (see the remark below (5.9)), it associates the exact sequence from [4, Proposition 3.4]:

$$1 \longrightarrow \rho_1(\mathbb{R}) \xrightarrow{l} \text{Psh}(X/\mathbb{R}^2) \xrightarrow{\phi} \text{Isom}_h(Y) \longrightarrow 1. \quad (7.1)$$

Since $\rho_1(\mathbb{R})$ induces the Reeb field $\hat{\xi}_1$ of $\hat{\eta}_1$, $\text{Psh}(X/\mathbb{R}^2)$ itself is the centralizer of $\rho_1(\mathbb{R})$ in $\text{Psh}(X/\mathbb{R}^2)$ (cf. [4, 11]). If we recall the representation of $\rho(\mathbb{R})$ from (6.12):

$$\rho(t) = (t, (e^{ia_1 t}, \dots, e^{ia_n t}, e^{-ia_1 t}, \dots, e^{-ia_n t})) \in \mathbb{R} \times \text{U}(2n) \leq \mathcal{N} \rtimes \text{U}(2n),$$

under the equivariant diffeomorphism φ of (6.13), the condition (5.2) implies that

$$\text{Psh}(X/\mathbb{R}^2) = \mathbb{R} \times T^n \times T^n \leq \mathbb{R} \times \text{U}(2n). \quad (7.2)$$

We have from (7.1) that

Proposition 7.1 $\text{Isom}_h(Y) = T^{2n}$.

Theorem 7.2 *The quaternionic Hermitian manifold $(Y, \hat{g}, \{\hat{J}_\alpha\}_{\alpha=1}^3)$ is not Einstein. Moreover, it is never holomorphically isometric to any domain of the quaternionic euclidean space $\mathbb{H}^n$.*

Proof When the Bochner flat manifold $(Y, \hat{g}, \{\hat{\Omega}_1, \hat{J}_1\})$ is Einstein, it is of constant holomorphic curvature by Tachibana’s result [15]. Thus, Y is locally holomorphically isometric to the flat space $\mathbb{C}^{2n}$. We may assume that the origin $\mathbf{0}$ belongs to Y (cf. (5.7)). Then, the stabilizer at $\mathbf{0}$ is the maximal compact subgroup isomorphic to $\text{U}(2n)$. Since T^{2n} is the full holomorphic isometry group of $(Y, \hat{g}, \hat{J}_1)$ by Proposition 7.1, it is impossible. $(Y, \hat{g}, \hat{J}_1)$ is not holomorphically isometric to any domain of $(\mathbb{H}^n, g_{\mathbb{H}})$ with the standard euclidean metric $g_{\mathbb{H}}$. $\square$

7.1 Quaternionic Hermitian isometry group of $(Y, \hat{g})$

The *quaternionic Hermitian isometry group* of the quaternionic Hermitian manifold $(Y, \hat{g}, \{\hat{\Omega}_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3)$ may be denoted naturally by the following:

$$\text{Isom}_{qH}(Y) = \{ \hat{h} \in \text{Diff}(Y) \mid \hat{h}^* \hat{\Omega}_\alpha = \sum_{\beta=1}^3 \hat{\Omega}_\beta \hat{a}_{\beta\alpha}, \hat{h}_* \circ \hat{J}_\alpha = \sum_{\beta=1}^3 \hat{a}_{\alpha\beta} \hat{J}_\beta \circ \hat{h}_* \},$$

where $(\hat{a}_{\alpha\beta})_{\alpha,\beta=1,2,3} : Y \rightarrow \text{SO}(3)$ are smooth maps.

For the abelian group $\rho_1(\mathbb{R}^3)$ defined by (5.4), let $N_{\text{Aut}(\mathcal{M})}(\rho_1(\mathbb{R}^3))$ be the normalizer of $\rho_1(\mathbb{R}^3)$ in $\text{Aut}(\mathcal{M})$. By the formula (5.3) of A_t , the normalizer of $\{A_t\}$ in $\text{Sp}(n)$ is isomorphic to T^n . Since the only subgroup $S^1 = \langle e^{i\theta} \rangle$ of $\text{Sp}(1)$ normalizes $\rho_1(\mathbb{R}^3)$ in view of the actions (3.1) and (5.4), it follows that

$$N_{\text{Aut}(\mathcal{M})}(\rho_1(\mathbb{R}^3)) = (\mathbb{R}^3, 0) \times T^n \cdot S^1 \leq \mathcal{M} \rtimes \text{Sp}(n) \cdot \text{Sp}(1) \quad (7.3)$$

where $T^n \cdot S^1 = T^n \times_{\{\pm 1\}} S^1$. Recall every element of $\text{Aut}(X)$ extends to an element of $\text{Aut}(\mathcal{M})$. Since each element of $N_{\text{Aut}(\mathcal{M})}(\rho_1(\mathbb{R}^3))$ preserves both ω_1 and ξ_1 from (5.8), it follows $N_{\text{Aut}(X)}(\rho_1(\mathbb{R}^3)) = N_{\text{Aut}(\mathcal{M})}(\rho_1(\mathbb{R}^3))$.

Let $\eta = \eta_1 i + \eta_2 j + \eta_3 k$ be as before (cf. (5.10)). Note $h^* \eta = a \cdot \eta \cdot \bar{a}$ for $h = ((t, 0), A \cdot a) \in N_{\text{Aut}(X)}(\rho_1(\mathbb{R}^3))$. By (7.3), the projection $\pi_1 : X \rightarrow Y$ of (5.9) induces an element $\hat{h} : Y \rightarrow Y$. Since $\pi_1^* \hat{\Omega} = d\eta$ for $\hat{\Omega} = \hat{\Omega}_1 i + \hat{\Omega}_2 j + \hat{\Omega}_3 k$, we have $\hat{h}^* \hat{\Omega} = a \cdot \hat{\Omega} \cdot \bar{a}$ for $a \in S^1$. Thus, it assigns an element $\hat{h} \in \text{Isom}_{qH}(Y, \hat{g}, \{\hat{\Omega}_\alpha, \hat{J}_\alpha\}_{\alpha=1}^3) = \text{Isom}_{qH}(Y)$. Letting $a = (a_{\alpha\beta})_{\alpha,\beta=2,3} \in \text{SO}(2)$, $\text{Isom}_{qH}(Y)$ can be described as

$$\{ \hat{h} \in \text{Diff}(Y) \mid \hat{h}^* \hat{\Omega}_1 = \hat{\Omega}_1, \hat{h}_* \circ \hat{J}_1 = \hat{J}_1 \circ \hat{h}_*, \\ \hat{h}^* \hat{\Omega}_\alpha = \sum_{\beta=2,3} \hat{\Omega}_\beta a_{\beta\alpha}, \hat{h}_* \circ \hat{J}_\alpha = \sum_{\beta=2,3} a_{\alpha\beta} \hat{J}_\beta \circ \hat{h}_* \}. \quad (7.4)$$

Setting $\tilde{\phi}(h) = \hat{h}$, (7.3) gives an exact sequence:

$$1 \longrightarrow \rho_1(\mathbb{R}^3) \longrightarrow N_{\text{Aut}(X)}(\rho_1(\mathbb{R}^3)) \xrightarrow{\tilde{\phi}} T^n \cdot S^1 \leq \text{Isom}_{qH}(Y). \quad (7.5)$$

If $\iota : \text{Isom}_{qH}(Y) \rightarrow \text{Isom}_h(Y) = T^{2n}$ (cf. Proposition 7.1) is the natural inclusion (that is, forgetting the quaternionic structure but leaving the holomorphic structure as it is), then under the equivariant diffeomorphism $\hat{\phi}$ of (6.5), it follows $\iota(T^n \cdot S^1) = \{(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) \times (e^{-i\theta}, \dots, e^{-i\theta})\} \leq T^{2n}$ where $z_i \in S^1$ ($i = 1, \dots, n$). We obtain

Corollary 7.3 *The quaternionic Hermitian isometry group $\text{Isom}_{qH}(Y)^0$ is isomorphic to the torus T^k for some k ($n+1 \leq k \leq 2n$).*

Remark 7.4 Since the forms $\hat{\Omega}_2, \hat{\Omega}_3$ are not Kähler, the equation $\pi_1^* \hat{\Omega}_\alpha = d\eta_\alpha$ does not hold on X (only on D_0), the method of [4, Propositions 3.4, 3.1, 3.2] cannot be applied to show the surjectivity of $\tilde{\phi}$ in (7.5).

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