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Skew Killing spinors in four dimensions

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Abstract
This paper is devoted to the classification of 4-dimensional Riemannian spin manifolds carrying skew Killing spinors. A skew Killing spinor ψ is a spinor that satisfies the equation $\nabla_X \psi = AX \cdot \psi$ with a skew-symmetric endomorphism A. We consider the degenerate case, where the rank of A is at most two everywhere and the non-degenerate case, where the rank of A is four everywhere. We prove that in the degenerate case the manifold is locally isometric to the Riemannian product $\mathbb{R} \times N$ with N having a skew Killing spinor and we explain under which conditions on the spinor the special case of a local isometry to $\mathbb{S}^2 \times \mathbb{R}^2$ occurs. In the non-degenerate case, the existence of skew Killing spinors is related to doubly warped products whose defining data we will describe.

Keywords Generalised Killing spinors · Doubly warped product · Hodge operator

Mathematics Subject Classification 53C25 · 53C27

1 Introduction

Let (M^n, g) be an n-dimensional Riemannian spin manifold. A generalised Killing spinor on M is a section ψ of the spinor bundle ΣM of M satisfying the overdetermined differential equation $\nabla_X \psi = AX \cdot \psi$ for some symmetric endomorphism field A of TM. Here and as usual, "·" denotes the Clifford multiplication on ΣM. Numerous papers have been devoted to the classification of Riemannian spin manifolds carrying such spinors. Several results have

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been obtained for particular A but it is still an open problem to get a complete classification for general A. Let us quote some of these results. First, recall that when A is the zero tensor field, that is, the corresponding spinor is parallel, then M. Wang [24] showed that such manifolds can be characterised by their holonomy groups which can be read off the Berger classification. The case where A is a nonzero real multiple of the identity is that of classical real Killing spinors. It was shown by C. Bär [3] that real Killing spinors correspond to parallel spinors on the (irreducible) cone over the manifold, to which then M. Wang's result applies. Furthermore, in dimension $n \leq 8$, there are several results on a classification up to isometry [6,18]. When the tensor A is parallel [20], or a Codazzi tensor [5] or both A and g are analytic [2] (see also [10]), it is shown that the manifold M is isometrically embedded into another spin manifold of dimension $n+1$ carrying a parallel spinor and that the tensor A is the half of the second fundamental form of the immersion. We also cite the partial classification of generalised Killing spinors on the round sphere [21,23] and on 4-dimensional Einstein manifolds of positive scalar curvature [22] where in some cases the generalised Killing spinor turns out to be a Killing spinor.

In this paper, we are interested in an equation dual to the generalised Killing one, which we call *skew Killing spinor equation*. More precisely, on a given Riemannian spin manifold (M^n, g), a spinor field ψ is called a skew Killing spinor if it satisfies for some *skew-symmetric* endomorphism field A of TM the differential equation

$$\nabla_X \psi = AX \cdot \psi \tag{1}$$

for all $X \in TM$. This equation was originally defined in [16]. Each skew Killing spinor is a parallel section with respect to the modified metric connection $\nabla - A \otimes \mathrm{Id}$, in particular it has constant length. Moreover, for a given skew symmetric endomorphism field A of TM, the space of skew Killing spinors is a complex vector space of dimension at most $\mathrm{rk}_{\mathbb{C}}(\Sigma M) = 2^{[n/2]}$.

Very few examples of Riemannian spin manifolds (M^n, g) carrying skew Killing spinors are known for which $A \neq 0$. For 2-dimensional manifolds, apart from $\mathbb{R}^2$ or quotients thereof with trivial spin structure, only the round sphere of constant curvature can carry such spinors and in that case they correspond to restrictions of Killing spinors from $\mathbb{S}^3$ onto totally geodesic $\mathbb{S}^2$ [16]. In that case, the tensor A coincides with the standard complex structure J induced by the conformal class of $\mathbb{S}^2$ or with $-J$ depending on the sign of the Killing constant chosen on $\mathbb{S}^3$. Each skew Killing spinor on $\mathbb{S}^2$ immediately gives rise to a three-dimensional example, namely to a skew Killing spinor on $\mathbb{S}^2 \times \mathbb{R}$, where $A = \pm J$ on $\mathbb{S}^2$ is trivially extended to the $\mathbb{R}$-factor. More generally, for a manifold of dimension $n = 3$ the following is known [16, Prop. 4.3]. If M^3 admits a skew Killing spinor ψ, then, locally, ψ can be transformed into a parallel spinor by a suitable conformal change of the metric. In particular, M^3 is locally conformally flat. If, in addition, M^3 is simply-connected, then this conformal change is defined globally. Conversely, if (M^3, g) admits a nonzero parallel spinor, then for any conformal change of g, there exists a skew Killing spinor with respect to the new metric. See Sect. 4.1 for more detailed information.

In dimensions 6 and 7, there are lots of examples provided by SU(3)- resp. G_2-structures on M, see e.g. types χ_1, χ_2, χ_4 in [1, Lemma 3.5] and type $\mathcal{W}_2, \mathcal{W}_4$ in [1, Lemma 4.5] respectively.

Obvious examples in four dimensions can be obtained as products $N \times \mathbb{R}$, where N is a three-dimensional manifold admitting a skew Killing spinor, see Example 4.1. A special case of this construction is the product $\mathbb{S}^2 \times \mathbb{R}^2$, see Example 4.2. For each of the endomorphisms $A^{\pm} := \pm J \oplus 0$, this manifold admits the maximal number of skew Killing spinors.

The main purpose of this work is to establish a classification result when the dimension of M is four. Note that the pointwise rank of A is either zero, two or four. We will split the classification into two parts. In Sect. 4 we will study the degenerate case, where the rank of A is at most two everywhere. In Sect. 5 we will consider the case where $\mathrm{rk}(A) = 4$ on all of M. Note that the two analysed classes of skew Killing spinors in dimension 4 do not cover all the possible cases. There might exist nontrivial skew Killing spinors such that the rank of A is 4 on some nonempty subset of the manifold and strictly smaller on another nonempty subset. Before we start the classification, we determine the general integrability conditions in arbitrary dimensions arising from the existence of a skew Killing spinor, see Sect. 2. In Sects. 3 and 4, we specify these conditions to four dimensions, especially to the degenerate case. We use that the spinor bundle ΣM splits into the eigenspaces $\Sigma^+ M$ and $\Sigma^- M$ of the volume form and the bundle of two-forms splits into those of self-dual and of anti-self-dual forms, which act on $\Sigma^\pm M$. We also adapt some techniques used in [22] but for a skew-symmetric endomorphism A. We use the integrability conditions to achieve the following classification result in case that the Killing map is degenerate everywhere.

Theorem A *Let (M^4, g) Letbe a connected Riemannian spin manifold carrying a skew Killing spinor ψ, where the rank of the corresponding skew-symmetric tensor field A is at most two everywhere. Then either ψ is parallel on M or, around every point of M, we have a local Riemannian splitting $\mathbb{R} \times N$ with N having a skew Killing spinor. If, in addition, the length of the summand ψ^+ in the decomposition $\psi = \psi^+ + \psi^- \in \Sigma^+ M \oplus \Sigma^- M$ is not constant, then we are in the second case with $N = \mathbb{R} \times \mathbb{S}^2$, that is, (M, g) is a local Riemannian product $\mathbb{S}^2 \times \mathbb{R}^2$ around every point.*

For a more detailed formulation see Theorem 4.12, where we also discuss the global structure of (M, g) if M is complete.

Let us turn to the case where the Killing map is non-degenerate everywhere. In Sect. 5.1 we will prove that, essentially, the existence of a skew Killing spinor ψ with non-degenerate Killing map A is equivalent to the existence of a Killing vector field η and an almost complex structure J satisfying certain conditions, see Proposition 5.1 for a detailed formulation. The spinor ψ and the data η and J are related by the equations $J(X) \cdot \psi^- = iX \cdot \psi^-$ and $g(\eta, X) = \langle X \cdot \psi^+, \psi^- \rangle / |\psi|^2$ for all $X \in TM$.

In Sect. 5.2, we consider the special case where $A\eta$ is parallel to $J\eta$. Then $AJ = JA$ holds and J is integrable, see Remark 5.3. Manifolds with skew Killing spinors satisfying these conditions are related to doubly warped products. A *doubly warped product* is a Riemannian manifold (M, g) of the form $(I \times \hat{M}, dt^2 \oplus \rho(t)^2 \hat{g}_{\hat{\eta}} \oplus \sigma(t)^2 \hat{g}_{\hat{\eta}^\perp})$, where $(\hat{M}, \hat{g})$ is a Riemannian manifold with unit Killing vector field $\hat{\eta}$, and $\hat{g}_{\hat{\eta}}$, $\hat{g}_{\hat{\eta}^\perp}$ are the components of the metric $\hat{g}$ along $\mathbb{R}\hat{\eta}$ and $\hat{\eta}^\perp$, respectively, $I \subset \mathbb{R}$ is an open interval and $\rho, \sigma : I \to \mathbb{R}$ are smooth positive functions on I. Locally, doubly warped products can be equivalently described as local DWP-structures, see the appendix. On $\hat{M}$, we define a function $\hat{\tau}$ by $\hat{\nabla}_X \hat{\eta} = \hat{\tau} \hat{J}(X)$ for $X \in \hat{\eta}^\perp$, where $\hat{J}$ is a fixed Hermitian structure on $\eta^\perp$. Locally, $(\hat{M}, \hat{g})$ is a Riemannian submersion over a two-dimensional base manifold B. Let $\hat{K}$ denote the Gaussian curvature of B. We obtain the following result, see Theorem 5.5 and Corollary 5.8.

Theorem B *Let (M, g) admit a skew Killing spinor such that $A\eta || J\eta$ and $|\eta| \notin \{0, 1/2\}$ everywhere. Then M is locally isometric to a doubly warped product for which the data $\hat{K}$ and $\hat{\tau}$ are constant and ρ and σ satisfy the differential equations*

$$(\sigma^2)' = -\frac{2}{\sqrt{1 - 4\rho^2}} \rho \hat{\tau}, \qquad (\sigma^2)' \frac{\rho'}{\rho} = \hat{K} - 2\frac{\rho^2}{\sigma^2} \hat{\tau}^2.$$

Conversely, if M is isometric to a simply-connected doubly warped product for which the data $\hat{K}$ and $\hat{\tau}$ are constant and ρ and σ satisfy the above differential equations, then (M, g) admits a skew Killing spinor such that $A\eta || J\eta$.

The differential equations in Theorem B can be locally solved and one obtains explicit formulas for the doubly warped product. Let us finally mention that the skew Killing spinors on $M = I \times \hat{M}$ are related to quasi Killing spinors in the sense of [12] on $\hat{M}$, see Remark 5.10.

2 General integrability conditions for skew Killing spinors

In this section we give a few necessary conditions for the existence of nonzero skew Killing spinors. Before we state the main result, we recall some facts from Riemannian and spin geometry, see e.g. [6, Chap. 1], [19, Chap. 2] or [9, Chap. 1].

In all this paper we identify, on a Riemannian manifold (M^n, g), one-forms with vector fields via the metric g. Recall that the Hodge star operator is defined by

$$\omega \wedge *\omega' = \langle \omega, \omega' \rangle \mathrm{vol}_g$$

for all differential p-forms ω, ω' on M, where vol_g is the volume form of M (giving its orientation). The Hodge star operator satisfies $*^2 = (-1)^{p(n-p)}$ on p-forms and has the following useful properties

$$X \wedge *\omega = (-1)^{p+1} * (X \lrcorner \omega) \quad \text{and} \quad X \lrcorner * \omega = (-1)^p * (X \wedge \omega) \tag{2}$$

for any vector field X. Recall also that the Clifford multiplication between a vector field X and a differential p-form ω is defined as

$$X \cdot \omega = X \wedge \omega - X \lrcorner \omega \quad \text{and} \quad \omega \cdot X = \omega \wedge X + (-1)^p X \lrcorner \omega, \tag{3}$$

from which the identity $X \cdot Y \cdot + Y \cdot X \cdot = -2g(X, Y)$ follows for any vector fields X and Y.

From now on, we assume M to be spin with fixed spin structure. In that case, there exists a Hermitian vector bundle $\Sigma M \to M$, called the spinor bundle, on which the tangent bundle TM acts by Clifford multiplication, $TM \otimes \Sigma M \to \Sigma M$; $X \otimes \psi \mapsto X \cdot \psi$. We will write $XY \cdot \psi$ instead of $X \cdot Y \cdot \psi$. Recall that a real p-form also acts by Clifford multiplication in a formally self- or skew-adjoint way according to its degree: for any p-form ω and any spinors φ, ψ, we have

$$\langle \omega \cdot \varphi, \psi \rangle = (-1)^{\frac{p(p+1)}{2}} \cdot \langle \varphi, \omega \cdot \psi \rangle.$$

The Levi-Civita connection ∇ on M defines a metric connection, also denoted by ∇, on ΣM with respect to the Hermitian product $\langle \cdot , \cdot \rangle$ and that preserves Clifford multiplication. In other words, for all $X, Y \in \Gamma(TM)$, the identities

$$X(\langle \psi, \varphi \rangle) = \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle, \qquad \nabla_X (Y \cdot \varphi) = \nabla_X Y \cdot \varphi + Y \cdot \nabla_X \varphi$$

are satisfied for all spinor fields ψ, φ. If we denote by $R_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ the curvature tensor associated with the connection ∇, the spinorial Ricci identity states that, for all ψ and X,

$$-\frac{1}{2}\mathrm{Ric}(X) \cdot \psi = \sum_{j=1}^{n} e_j \cdot R_{X,e_j} \psi, \tag{4}$$

see e.g. [6, Eq. 1.13].

In the following, we will assume the manifold M to carry a skew Killing spinor field ψ with corresponding skew-symmetric endomorphism A. We make A into a 2-form via the metric g, that is, we consider $(X,Y)\mapsto g(AX,Y)$, which we still denote by A. In a pointwise orthonormal basis $\{e_i\}_{i=1,\cdots,n}$ of TM, we have $A=\frac{1}{2}\sum_{j=1}^n e_j\wedge Ae_j$ (mind the factor $\frac{1}{2}$). In particular, Clifford multiplication of any spinor field ψ by A is given by

$$A\cdot\psi=\tfrac{1}{2}\sum_{j=1}^{n} e_j\cdot Ae_j\cdot\psi\,. \tag{5}$$

In the next proposition, we compute the curvature data arising from the existence of such a spinor. These integrability equations will play a crucial role for the classification in the 4-dimensional case.

Proposition 2.1 *Let ψ be any spinor field solving*

$$\nabla_X\psi=AX\cdot\psi,\qquad \forall X\in TM$$

for some skew-symmetric endomorphism field A of the tangent bundle of a spin manifold (M^n,g). Then the following identities hold for $X,Y\in\Gamma(TM)$:

1. $R_{X,Y}\psi=((\nabla_XA)(Y)-(\nabla_YA)(X)+2AY\wedge AX)\cdot\psi.$
2. $-\frac{1}{2}\mathrm{Ric}(X)\cdot\psi=\big(\nabla_XA+X\lrcorner dA+(\delta A)(X)+4A\wedge AX+2A^2X\big)\cdot\psi$, *where d is the exterior derivative and δ is the codifferential w.r.t. the metric g.*
3. $\mathrm{S}\psi=4\big(2dA+\delta A+4A\wedge A+|A|^2\big)\cdot\psi$, *where S denotes the scalar curvature of (M,g) and $|A|^2:=\sum_{j=1}^n|Ae_j|^2$ written in any pointwise orthonormal basis $(e_j)_{1\le j\le n}$ of TM.*

Proof We differentiate (1) and take suitable traces of the identities obtained. First, if $x\in M$ and $X,Y\in\Gamma(TM)$ such that $\nabla X=\nabla Y=0$ at x, then

$$\begin{aligned}\nabla_X\nabla_Y\psi=\nabla_X(AY\cdot\psi)\;&=\;(\nabla_XA)(Y)\cdot\psi+AY\cdot\nabla_X\psi\\ &=(\nabla_XA)(Y)\cdot\psi+AY\cdot AX\cdot\psi\end{aligned}$$

at x. Therefore, applying Eqs. (3), we write

$$\begin{aligned}R_{X,Y}\psi&=\nabla_X\nabla_Y\psi-\nabla_Y\nabla_X\psi\\ &=\big((\nabla_XA)(Y)-(\nabla_YA)(X)+AY\cdot AX-AX\cdot AY\big)\cdot\psi\\ &=\big((\nabla_XA)(Y)-(\nabla_YA)(X)+2AY\wedge AX-g(AY,AX)+g(AX,AY)\big)\cdot\psi\\ &=\big((\nabla_XA)(Y)-(\nabla_YA)(X)+2AY\wedge AX\big)\cdot\psi,\end{aligned}$$

which is the first identity.

Next we fix a local orthonormal basis of TM, which we denote by $(e_j)_{1\le j\le n}$. Using the spinorial Ricci formula (4) and the identities (3), we compute

$$
\begin{aligned}
-\frac{1}{2}\mathrm{Ric}(X)\cdot\psi &= \sum_{j=1}^{n} e_j \cdot R_{X,e_j}\psi \\
&= \sum_{j=1}^{n} e_j \cdot \big((\nabla_X A)(e_j) - (\nabla_{e_j} A)(X) + 2Ae_j \wedge AX\big)\cdot\psi \\
&= \Big(\sum_{j=1}^{n} e_j \cdot (\nabla_X A)(e_j) - \sum_{j=1}^{n} e_j \wedge (\nabla_{e_j} A)(X) + \sum_{j=1}^{n} e_j \lrcorner (\nabla_{e_j} A)(X) \\
&\quad + 2\sum_{j=1}^{n} e_j \cdot (Ae_j \wedge AX)\Big)\cdot\psi.
\end{aligned}
$$

Now we compute each term separately. First, $\sum_{j=1}^{n} e_j \cdot (\nabla_X A)(e_j)\cdot\psi = 2\nabla_X A\cdot\psi$ by (5), where we see $\nabla_X A$ as a 2-form on M. The second sum can be computed in terms of the exterior and the covariant derivatives of A. Namely

$$
-\sum_{j=1}^{n} e_j \wedge (\nabla_{e_j} A)(X) = \Big(\sum_{j=1}^{n} e_j \wedge \nabla_{e_j} A\Big)(X) - \sum_{j=1}^{n} g(X,e_j)\nabla_{e_j} A \;=\; X\lrcorner dA - \nabla_X A.
$$

The third sum can be expressed in terms of the codifferential of A:

$$
\sum_{j=1}^{n} e_j \lrcorner (\nabla_{e_j} A)(X) = \sum_{j=1}^{n} (\nabla_{e_j} A)(X, e_j) = -\sum_{j=1}^{n} (\nabla_{e_j} A)(e_j, X) = (\delta A)(X).
$$

It remains to notice that, by Eqs. (3), we have

$$
\begin{aligned}
\sum_{j=1}^{n} e_j \cdot (Ae_j \wedge AX)\cdot\psi &= \sum_{j=1}^{n} e_j \cdot Ae_j \cdot AX \cdot\psi + \sum_{j=1}^{n} g(Ae_j, AX)e_j\cdot\psi \\
&= (2A\cdot AX - A^2X)\cdot\psi \;=\; (2A\wedge AX + A^2X)\cdot\psi.
\end{aligned}
$$

This shows the second equation.

To obtain the scalar curvature, we take the trace of the spinorial Ricci identity. For a given local orthonormal basis $(e_j)_{1\le j\le n}$ of TM, we write

$$
\begin{aligned}
\frac{\mathrm{S}}{2}\psi &= -\frac{1}{2}\sum_{j=1}^{n} e_j \cdot \mathrm{Ric}(e_j)\cdot\psi \\
&= \sum_{j=1}^{n} e_j \cdot \big(\nabla_{e_j} A + e_j \lrcorner dA + (\delta A)(e_j) + 4A\wedge Ae_j + 2A^2 e_j\big)\cdot\psi \\
&\overset{(3)}{=} \sum_{j=1}^{n} \Big(e_j \wedge \nabla_{e_j} A - e_j \lrcorner \nabla_{e_j} A\Big)\cdot\psi + \sum_{j=1}^{n}\Big(e_j \wedge (e_j \lrcorner dA) - \underbrace{e_j \lrcorner (e_j \lrcorner dA)}_{0}\Big)\cdot\psi \\
&\quad + \sum_{j=1}^{n} (\delta A)(e_j)e_j\cdot\psi + 4\sum_{j=1}^{n}\Big(e_j \wedge A \wedge Ae_j - \underbrace{e_j \lrcorner (A\wedge Ae_j)}_{0}\Big)\cdot\psi \\
&\quad + 2\sum_{j=1}^{n}\Big(\underbrace{e_j \wedge A^2 e_j}_{0} - g(A^2 e_j, e_j)\Big)\cdot\psi
\end{aligned}
$$

$$= \left(dA + \delta A + 3dA + \delta A + 8A \wedge A + 2|A|^2\right) \cdot \psi$$
$$= \left(4dA + 2\delta A + 8A \wedge A + 2|A|^2\right) \cdot \psi,$$

which is the last identity. Here, we use the identity $\sum_{j=1}^n e_j \wedge (e_j \lrcorner \omega) = p\omega$, which holds for any p-form ω. □

3 The vector fields η and ξ in four dimensions

In this section we consider a 4-dimensional spin manifold (M, g) that carries a skew Killing spinor. On spin manifolds of even dimension $2m$, the complex volume form $(\mathrm{vol}_g)_{\mathbb{C}} := i^m e_1 \cdot e_2 \ldots \cdot e_{2m}$, where $(e_j)_{j=1,\cdots,2m}$ is an arbitrary orthonormal frame, splits the spinor bundle into two orthogonal subbundles that correspond to the eigenvalues ± 1 of $(\mathrm{vol}_g)_{\mathbb{C}}$. Hence, on our four-dimensional manifold (M, g), we have $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, where

$$\Sigma^\pm M := \{\psi \in \Sigma M \mid (\mathrm{vol}_g)_{\mathbb{C}} \cdot \psi = \pm\psi\}.$$

The spaces $\Sigma^\pm M$ are preserved by the connection ∇ of the spinor bundle and are interchanged by Clifford multiplication by tangent vectors. According to this decomposition, we write any spinor field ψ as $\psi = \psi^+ + \psi^-$ and we set $\bar{\psi} := \psi^+ - \psi^-$. Recall now that differential forms act on the spinor bundle ΣM as follows: for any differential p-form ω on M and $\psi \in \Gamma(\Sigma M)$

$$\omega \cdot \psi = *\omega \cdot \bar{\psi} \quad \text{for} \quad p = 1, 2 \quad \text{and} \quad \omega \cdot \psi = -(*\omega) \cdot \bar{\psi} \quad \text{for} \quad p = 3, 4. \tag{6}$$

Let $\bigwedge^2_\pm M = \{\omega \in \bigwedge^2 M \mid *\omega = \pm\omega\}$ be the spaces of self-dual and anti-self-dual forms on M. For $\omega \in \bigwedge^2 M$ we denote by $\omega_\pm$ the projections of ω to these spaces. Then, one can easily see from Eqs. (6) that $\bigwedge^2_\pm M$ acts trivially on $\Sigma^\mp M$ and that the maps

$$\bigwedge\nolimits^2_- M \longrightarrow \Sigma^- M \cap (\psi^-)^\perp, \ \omega_- \longmapsto \omega_- \cdot \psi^-,$$
$$\bigwedge\nolimits^2_+ M \longrightarrow \Sigma^+ M \cap (\psi^+)^\perp, \ \omega_+ \longmapsto \omega_+ \cdot \psi^+ \tag{7}$$

are isomorphisms if $\psi^+ \neq 0$ and $\psi^- \neq 0$.

Now assume that ψ is a skew Killing spinor of norm one. By decomposing ψ into ψ^+ and ψ^- as above, we obtain isomorphisms (7) on the open set $M' := M_0 \cap M_1$, with

$$M_0 := \{x \in M \mid \psi^-(x) \neq 0\} \quad \text{and} \quad M_1 := \{x \in M \mid \psi^+(x) \neq 0\}.$$

Equation (1) can be written as $\nabla_X \psi^\pm = AX \cdot \psi^\mp$. We define a vector field η on M and a vector field ξ on M_0 by

$$g(\eta, X) := \langle X \cdot \psi^+, \psi^- \rangle, \quad \psi^+ =: \xi \cdot \psi^-, \tag{8}$$

where the definition of ξ uses that the map $T_p M \to \Sigma^+_p M$, $X \mapsto X \cdot \psi^-$ is bijective at each $p \in M_0$. Then, clearly $\eta = -|\psi^-|^2 \xi$ holds on M_0 and $1 = |\psi^+|^2 + |\psi^-|^2 = |\psi^-|^2(1 + |\xi|^2)$. We define

$$f := 1 - 2|\psi^-|^2, \quad \rho := |\eta| \leq 1/2.$$

Then

$$\rho = \frac{|\xi|}{1 + |\xi|^2}, \quad f = \frac{|\xi|^2 - 1}{|\xi|^2 + 1}, \quad f^2 = 1 - 4\rho^2, \quad \eta = \frac{1}{2}(f - 1)\xi \tag{9}$$

holds, where these functions are defined.

We collect some properties of η and ξ that will be used later on.

Lemma 3.1 *On M, we have*

$$df = 4A\eta \tag{10}$$
$$\nabla_X \eta = fAX \tag{11}$$
$$d\eta = 2fA, \quad \delta\eta = 0 \tag{12}$$
$$f dA = -4A\eta \wedge A\,. \tag{13}$$

Proof Differentiating the function $|\psi^-|^2$ along any vector field $X \in TM$ gives

$$X(|\psi^-|^2) = 2\langle\nabla_X\psi^-, \psi^-\rangle = 2\langle AX\cdot\psi^+, \psi^-\rangle = 2g(\eta, AX) = -2g(A\eta, X).$$

This proves (10). To prove (11), we consider two vector fields X and Y that can be assumed to be parallel at some point $x \in M$ to compute

$$\begin{aligned} g(\nabla_X\eta, Y) &= X(g(\eta, Y)) \ = \ X(\langle Y\cdot\psi^+, \psi^-\rangle) \\ &= \langle Y\cdot AX\cdot\psi^-, \psi^-\rangle + \langle Y\cdot\psi^+, AX\cdot\psi^+\rangle \\ &= -g(Y, AX)|\psi^-|^2 + g(Y, AX)|\psi^+|^2 \\ &= (1-2|\psi^-|^2)g(AX, Y) \end{aligned}$$

at x, which is (11). Moreover,

$$d\eta(X, Y) = (\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) = 2fA(X, Y),$$

which yields the first part of (12). The divergence of η is clearly zero by (11) and the fact that A is skew-symmetric. Finally,

$$0 = dd\eta = 2f dA + 2df \wedge A,$$

which together with (10) gives (13). □

Remark 3.2 It follows from Lemma 3.1 that $\nabla\eta$ is skew-symmetric on M which means that η is a Killing vector field on M.

The open sets M_0 and M_1 are dense in $\{p \in M \mid A_p \neq 0\}$. Indeed, if, e.g., ψ^- vanishes on some open set $U \subset \{p \in M \mid A_p \neq 0\}$, then so does its covariant derivative and therefore $AX\cdot\psi^+ = 0$ on U. Hence $A = 0$ on U, which contradicts the assumption on A.

With the notation introduced above, we have $M_0 = \{x \in M \mid f(x) \neq 1\}$ and $M_1 = \{x \in M \mid f(x) \neq -1\}$. Then $M' = M_0 \cap M_1 = \{x \in M \mid f(x) \neq \pm 1\} = \{x \in M \mid \rho(x) \neq 0\}$. We define also the set

$$\begin{aligned} M'' &:= \left\{x \in M \mid \rho(x) \notin \{0, \tfrac{1}{2}\}\right\} = M' \cap \left\{x \in M \mid \rho(x) \neq \tfrac{1}{2}\right\} \\ &= \{x \in M \mid f(x) \notin \{0, \pm 1\}\}. \end{aligned}$$

By Lemma 3.1, equation (10), the open set M'' is dense in $\{p \in M \mid A_p(\eta) \neq 0\}$. In particular, $M'' \subset M$ is dense if A is non-degenerate everywhere. The case where $\rho = 1/2$ on an open set will be treated in Proposition 4.3.

Remark 3.3 Let us change the orientation of M and denote by $\hat{\Sigma}M$ the spinor bundle with respect to the new orientation. Then we can identify $\hat{\Sigma}M$ with ΣM via $\hat{\Sigma}^+M = \Sigma^-M$ and $\hat{\Sigma}^+M = \Sigma^-M$ Accordingly, we define a section $\hat{\psi}$ of $\hat{\Sigma}M$ by $\hat{\psi}^+ = \psi^-$, $\hat{\psi}^- = \psi^+$. With

ψ also $\hat{\psi}$ is a skew Killing spinor and the vector fields $\hat{\xi}$ and $\hat{\eta}$ associated with $\hat{\psi}$ are equal to $\hat{\xi} = -\xi/|\xi|^2$ and $\hat{\eta} = -\eta$, respectively.

On M'', we have $|\xi| \neq 1$. Hence, if there exists a skew Killing spinor on M and if $M = M''$, then we always may assume that $|\xi| > 1$ up to a possible change of orientation on each connected component of M. If $|\xi| > 1$, then f is positive, therefore $f = \sqrt{1 - 4\rho^2}$.

4 The degenerate case

In this section we assume that $\mathrm{rk}(A) \leq 2$ everywhere on M^4, which is equivalent to suppose that the kernel of A is at every point either 4- or 2-dimensional. Then $AX \wedge A = 0$ for all $X \in TM$. In particular, $dA = 0$ on M'' by Lemma 3.1.

4.1 Examples

Example 4.1 If (N, h) is a 3-dimensional spin manifold with a skew Killing spinor $\varphi \neq 0$, then $(N \times \mathbb{R}, h \oplus dt^2)$ admits a skew Killing spinor $\psi \neq 0$ for which $|\psi^+| = |\psi^-|$ holds.

Let us prove the above statement. Recall that the spinor bundle of $M = N \times \mathbb{R}$ is given by $\Sigma M = \Sigma N \oplus \Sigma N$ and the Clifford multiplication on M is related to the one on N by [4]

$$(X \cdot_N \oplus - X \cdot_N)\psi = X \cdot \partial_t \cdot \psi.$$

where ∂_t is the unit vector field on $\mathbb{R}$ and $X \in TN$. Now we set $\psi := \varphi + \partial_t \cdot \varphi$ according to the above decomposition. Let A denote the Killing map associated with ψ. Then we can easily check that $\nabla_{\partial_t}\psi = 0$ and, for $X \in TN$,

$$\begin{aligned} \nabla_X \psi &= \nabla_X \varphi + \partial_t \cdot \nabla_X \varphi \\ &= AX \cdot_N \varphi + \partial_t \cdot (AX \cdot_N \varphi) \\ &= AX \cdot \partial_t \cdot \varphi + \partial_t \cdot AX \cdot \partial_t \cdot \varphi \\ &= AX \cdot \psi. \end{aligned}$$

Hence ψ is a skew Killing spinor on M. The vector field ξ in this example is just $-\partial_t$ which is parallel. Since $|\partial_t| = 1$, we have $|\psi^+| = |\psi^-|$.

Let us recall at this point, what is known about three-dimensional manifolds with skew Killing spinors. As already mentioned in the introduction, each skew Killing spinor on $\mathbb{S}^2$ immediately gives rise to a three-dimensional example, namely to a skew Killing spinor on $\mathbb{S}^2 \times \mathbb{R}$. Furthermore, if $\dim N = 3$ and if (N, g) admits a skew Killing spinor ψ, then N is locally conformally flat [16, Prop. 4.3]. Indeed, locally, there exists a function u such that ψ transforms into a parallel spinor $\bar{\psi}$ with respect to the metric $\bar{g} := e^{2u}g$ and three-dimensional Riemannian manifolds with a non-trivial parallel spinor field are flat. If N is simply-connected, then u is globally defined. In the latter case the metric $\bar{g}$ is not necessarily complete even if (N, g) is.

Conversely, if (N, g) admits a nonzero parallel spinor, then for any conformal change of the metric on the manifold N there exists a skew Killing spinor with respect to the new metric. We conclude this overview with the flat case $N = \mathbb{R}^3$. If $\psi \neq 0$ is a solution of (1) on $N = \mathbb{R}^3$ endowed with the flat metric, then $A = 0$ and ψ is a parallel spinor field. Indeed, as mentioned above, there exists a globally defined function u on $\mathbb{R}^3$ such that the metric $\bar{g} := e^{2u}g$ admits a parallel spinor. Hence, $\bar{g}$ is also flat. In particular, the scalar curvature $\bar{\mathrm{S}}$ vanishes. On the other hand, $\bar{\mathrm{S}} = 8e^{-2u}e^{-u/2}\Delta e^{u/2}$ since $\bar{g}$ arises by conformal change from the flat metric g. Therefore $\Delta(e^{u/2}) = 0$, that is, $e^{u/2}$ is a harmonic function on $\mathbb{R}^3$. But since $e^{u/2} \geq 0$, Liouville's theorem implies that $e^{u/2}$ – and so u itself – is constant. This shows that $A = 0$.

Example 4.2 We consider $M = \mathbb{S}^2 \times \mathbb{R}^2$. Let J denote the standard complex structure on $\mathbb{S}^2$. We define endomorphisms $A^\pm := \pm J \oplus 0$ on $TM = T\mathbb{S}^2 \oplus T\mathbb{R}^2$. For each of these endomorphisms, the space of skew Killing spinors is four-dimensional. It is spanned by elements with non-vanishing $A\eta$ and it is also spanned by elements for which $A\eta = 0$ holds.

Let us prove this statement. The spinor bundle of $\mathbb{S}^2 \times \mathbb{R}^2$ is pointwise given by $\Sigma(\mathbb{S}^2 \times \mathbb{R}^2) = \Sigma\mathbb{S}^2 \otimes \Sigma\mathbb{R}^2$ and the Clifford multiplication on $\mathbb{S}^2 \times \mathbb{R}^2$ is [4]

$$X \cdot (\varphi \otimes \sigma) = (X \cdot_{\mathbb{S}^2} \varphi) \otimes \bar{\sigma}, \qquad Y \cdot (\varphi \otimes \sigma) = \varphi \otimes (Y \cdot_{\mathbb{R}^2} \sigma),$$

for $X \in T\mathbb{S}^2$ and $Y \in T\mathbb{R}^2$. Now, we consider on $\mathbb{S}^2$ a skew Killing spinor φ, corresponding to the standard complex structure J, and a parallel spinor σ in $\Sigma^+(\mathbb{R}^2)$ of norm 1. The spinor field $\psi := \varphi \otimes \sigma$ is clearly a skew Killing spinor, since in the $\mathbb{S}^2$-direction we have

$$\nabla_X \psi = (\nabla_X \varphi) \otimes \sigma = (JX \cdot_{\mathbb{S}^2} \varphi) \otimes \sigma = JX \cdot (\varphi \otimes \sigma) = JX \cdot \psi$$

and $\nabla_Y \psi = 0$ in the $\mathbb{R}^2$-direction. The same computation holds when replacing J by $-J$ and choosing $\sigma \in \Sigma^-(\mathbb{R}^2)$. As the spaces of skew Killing spinors φ corresponding to the standard complex structure J or its opposite on $\mathbb{S}^2$ are each complex 2-dimensional, we deduce that the space of skew Killing spinors with Killing map A^+ is at least – and therefore exactly – 4-dimensional. The same holds for A^-. In particular, each skew Killing spinor on $\mathbb{S}^2 \times \mathbb{R}^2$ is a linear combination with constant coefficients of skew Killing spinors for A^+ and also one of skew Killing spinors for A^-. Note that the vector field ξ, associated to the above-defined skew Killing spinor ψ, is the one coming from the spinor φ on $\mathbb{S}^2$, since $T\mathbb{S}^2 \simeq \Sigma^+\mathbb{S}^2$ and

$$\psi^+ = \varphi^+ \otimes \sigma = (\xi_{\mathbb{S}^2} \cdot \varphi^-) \otimes \sigma = \xi_{\mathbb{S}^2} \cdot (\varphi^- \otimes \sigma) = \xi_{\mathbb{S}^2} \cdot \psi^-.$$

Therefore, $\xi = \xi_{\mathbb{S}^2}$ and $A^2\xi = J^2\xi_{\mathbb{S}^2} = -\xi_{\mathbb{S}^2}$, which cannot vanish on the sphere. This shows that $A\eta \neq 0$. If we consider instead of the above constructed ψ the spinor $\psi + Y \cdot \bar{\psi}$ for a parallel vector field Y on $\mathbb{R}^2$ with $|Y| = 1$, we obtain a skew Killing spinor with $\xi = -Y$, hence $A\eta = 0$.

4.2 Classification

Let us first assume that $\rho = 1/2$ on an open set. By definition of ρ, this condition is equivalent to $|\psi^+| = |\psi^-|$. We prove that, under this assumption, the manifold is locally isometric to that in Example 4.1.

Proposition 4.3 *Let ψ be a nonzero skew Killing spinor on M^4 and assume that $|\psi^+| = |\psi^-|$ on an open set U. Then U is a local Riemannian product of a line by a 3-dimensional Riemannian manifold carrying a skew Killing spinor.*

Proof Let ψ be a skew Killing spinor of norm one such that $|\psi^+| = |\psi^-|$. Then $f = 0$ by definition of f. Therefore η is parallel by Lemma 3.1. In this case $\eta^\perp$ is integrable and the spinor ψ restricts to a skew Killing spinor on the integral manifolds. In fact, for any given integral manifold N, its spinor bundle is identified with $\Sigma^+ M$, so the spinor $\varphi = \psi^+$ restricts to a skew Killing spinor on N. Indeed,

$$\nabla^N_X \varphi = \nabla^M_X \psi^+ = AX \cdot \psi^- = -AX \cdot \xi \cdot \psi^+ = -AX \cdot_N \varphi,$$

which proves the assertion. □

In the next part of the section, we want to exclude the case $\rho = 1/2$ and make the stronger assumption

$$\begin{array}{l}(M^4, g) \text{ is a Riemannian spin manifold} \\ \text{carrying a skew Killing spinor such that } M = M'' \text{ and } \mathrm{rk}(A) = 2 \text{ everywhere.}\end{array} \tag{GA}$$

Due to the orthogonal splitting of the spinor bundle $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ we can decompose further the equations in Proposition 2.1 in order to get more integrability conditions. Namely,

Proposition 4.4 *Under assumption (GA), we have*

$$\delta A = 0 \tag{14}$$
$$\mathrm{S} = 4|A|^2 \tag{15}$$
$$\mathrm{Ric}(\eta) = -4A^2\eta \tag{16}$$
$$\nabla_\eta A = 0 \tag{17}$$
$$(\nabla_X A)(\eta) = -f\left(\tfrac{1}{4}\mathrm{Ric}(X) + A^2 X\right) \tag{18}$$
$$\nabla_X(A\eta) = -\frac{f}{4}\mathrm{Ric}(X), \tag{19}$$
$$\eta \lrcorner \nabla_X(*A) = \nabla_X((*A)\eta) = \frac{1}{4}\mathrm{Ric}(X) + A^2 X \tag{20}$$
$$0 = \left(\tfrac{1}{2}\mathrm{Ric}(X) \wedge \xi + 2A^2 X \wedge \xi + \nabla_X A\right)_- \tag{21}$$

for every $X \in TM$.

Proof We take the orthogonal projection of the formulas in Proposition 2.1 to $\Sigma^+ M$ and $\Sigma^- M$. This gives, after using $\psi^+ = \xi \cdot \psi^-$, $dA = 0$ and $A \wedge AX = 0$ that

$$0 = \left(\tfrac{1}{2}\mathrm{Ric}(X) + 2A^2 X\right) \cdot \psi^- + (\nabla_X A + (\delta A)(X))\, \xi \cdot \psi^- \tag{22}$$
$$0 = \left(\tfrac{1}{2}\mathrm{Ric}(X) + 2A^2 X\right) \cdot \xi \cdot \psi^- + (\nabla_X A + (\delta A)(X)) \cdot \psi^- \tag{23}$$

for all $X \in TM$ and

$$0 = \left(|A|^2 - \tfrac{1}{4}\mathrm{S}\right) \cdot \xi \cdot \psi^- + (\delta A) \cdot \psi^- \tag{24}$$
$$0 = \left(|A|^2 - \tfrac{1}{4}\mathrm{S}\right) \cdot \psi^- + (\delta A) \cdot \xi \cdot \psi^-, \tag{25}$$

respectively. Equation (22) gives

$$0 = \left(\tfrac{1}{2}\mathrm{Ric}(X) + 2A^2 X + \xi \wedge \nabla_X A + \xi \lrcorner \nabla_X A + (\delta A)(X)\xi\right) \cdot \psi^-.$$

Hence, by formula (6), we obtain

$$0 = \tfrac{1}{2}\mathrm{Ric}(X) + 2A^2 X + *(\xi \wedge \nabla_X A) + \xi \lrcorner \nabla_X A + (\delta A)(X)\xi. \tag{26}$$

Equation (23) yields

$$0 = \left(\tfrac{1}{2}\mathrm{Ric}(X) \wedge \xi - \tfrac{1}{2}\mathrm{Ric}(X, \xi) + 2A^2 X \wedge \xi - 2g(A^2 X, \xi) + \nabla_X A + (\delta A)(X)\right) \cdot \psi^-.$$

Now, by taking the scalar product with ψ^- and identifying the real part, the 0-th order term must vanish. This gives

$$0 = \tfrac{1}{2}\mathrm{Ric}(\xi) + 2A^2\xi - \delta A. \tag{27}$$

Also, we have

$$\left(\tfrac{1}{2}\mathrm{Ric}(X)\wedge\xi+2A^2X\wedge\xi+\nabla_XA\right)\cdot\psi^-=0.$$

The isomorphism from $\bigwedge^2_- M$ to the orthogonal complement $(\psi^-)^\perp$ yields equality (21) from the above identity. Equation (24) gives

$$0=\delta A+(|A|^2-\tfrac{1}{4}\mathrm{S})\xi. \tag{28}$$

Furthermore, equation (25) yields

$$0=\left(|A|^2-\tfrac{1}{4}\mathrm{S}-(\delta A)(\xi)-\xi\wedge\delta A\right)\cdot\psi^-.$$

Taking the Hermitian product with ψ^-, we obtain

$$0=\left(\xi\wedge\delta A\right)_- \tag{29}$$

$$0=-(\delta A)(\xi)+|A|^2-\tfrac{1}{4}\mathrm{S} \tag{30}$$

after identifying the real parts. By (29), we have

$$\xi\wedge\delta A=*(\xi\wedge\delta A)=-\xi\lrcorner*\delta A.$$

Hence, the interior product with ξ yields $0=\xi\lrcorner(\xi\wedge\delta A)=|\xi|^2\delta A-(\delta A)(\xi)\cdot\xi$. Now, applying equation (28) to ξ gives

$$0=(\delta A)(\xi)+\left(|A|^2-\tfrac{1}{4}\mathrm{S}\right)|\xi|^2,$$

which, after combining with (30), leads to $0=(1+|\xi|^2)(\delta A)(\xi)$, which gives (14). Now (28) yields (15). Equation (16) now follows from (27) and (14).

From (21), we get

$$*\left(\tfrac{1}{2}\mathrm{Ric}(X)\wedge\xi+2A^2X\wedge\xi+\nabla_XA\right)=\tfrac{1}{2}\mathrm{Ric}(X)\wedge\xi+2A^2X\wedge\xi+\nabla_XA,$$

which, by equation (2), is equivalent to

$$\tfrac{1}{2}\xi\lrcorner*(\mathrm{Ric}(X))+2\xi\lrcorner*(A^2X)+*\nabla_XA=\tfrac{1}{2}\mathrm{Ric}(X)\wedge\xi+2A^2X\wedge\xi+\nabla_XA.$$

Taking the interior product by ξ, this gives

$$\xi\lrcorner*\nabla_XA=\xi\lrcorner\left(\left(\tfrac{1}{2}\mathrm{Ric}(X)+2A^2X\right)\wedge\xi\right)+\xi\lrcorner\nabla_XA,$$

therefore

$$*(\xi\wedge\nabla_XA)=\left(-\tfrac{1}{2}\mathrm{Ric}(X)-2A^2X\right)|\xi|^2+\xi\lrcorner\nabla_XA$$

by Eqs. (2) and (16). On the other hand, Eqs. (26) and (14) give

$$*(\xi\wedge\nabla_XA)=-\tfrac{1}{2}\mathrm{Ric}(X)-2A^2X-\xi\lrcorner\nabla_XA.$$

Substracting and adding the latter two equations and replacing ξ by $-(1+|\xi|^2)\eta$ yields (18) and the identity $\eta\lrcorner\nabla_X(*A)=\frac{1}{4}\mathrm{Ric}(X)+A^2X$ for all $X\in\Gamma(TM)$. The last equation yields (20) since $(*A)(\nabla_X\eta)=f(*A)(AX)=*fAX\wedge A=0$. Furthermore, equation (18) shows that the expression $(\nabla_XA)(\eta,Y)$ is symmetric in X and Y. Therefore

$$0=(\nabla_XA)(\eta,Y)-(\nabla_YA)(\eta,X)=-dA(X,Y,\eta)+(\nabla_\eta A)(X,Y)=(\nabla_\eta A)(X,Y)$$

by $dA=0$. This proves (17). Equation (19) follows from (18) together with $\nabla_X\eta=fAX$. □

Remark 4.5 We can prove integrability conditions analogous to those in Proposition 4.4 also for arbitrary rank of A. These general conditions are more involved. Since we will not use them in the present paper, we do not state them here.

Lemma 4.6 *Under assumption (GA), the set* $\{p \in M \mid A\eta|_p \neq 0\}$ *is dense in* M.

Proof Assume that $A\eta = 0$ on an open set U. We know that η is a Killing vector field on M. Moreover, by Lemma 3.1, the vector field η has constant length on U. Indeed, for every $X \in TM$,

$$X\left(|\eta|^2\right) = 2g\left(\nabla_X \eta, \eta\right) = 2fg(AX, \eta) = -2fg(A\eta, X) = 0.$$

By [7, Thm. 4], since (16) implies $\mathrm{Ric}(\eta) = 0$, we can conclude that η is parallel on U. But this contradicts equation (11) in Lemma 3.1 since $f \neq 0$ and $A \neq 0$ everywhere by assumption. □

In the following, we will often assume that $A\eta \neq 0$ on all of M. If $A\eta \neq 0$, then we have $A^2\eta \neq 0$ everywhere, from which it follows that the vectors $\frac{A\eta}{|A\eta|}$ and $\frac{A^2\eta}{|A^2\eta|}$ form an orthonormal basis of the image of A. As A is of rank 2, we obtain

$$A = \frac{1}{|A\eta|^2} A\eta \wedge A^2\eta. \tag{31}$$

Furthermore, note that (31) already implies

$$A^3\eta = -\frac{|A^2\eta|^2}{|A\eta|^2} A\eta = -\frac{\mathrm{S}}{8} A\eta, \tag{32}$$

where the last equality comes from the identity (15). Obviously, $A^3\eta = -\frac{\mathrm{S}}{8} A\eta$ holds also if $A\eta = 0$.

Since $df = 4A\eta$ by Lemma 3.1, (19) implies

$$\nabla df = -f\mathrm{Ric}. \tag{33}$$

This equation has been extensively studied in [14]. Using this formula, we now express the Ricci tensor of the vector field $A\eta$. □

Lemma 4.7 *If (GA) holds, then the Ricci tensor satisfies*

$$\mathrm{Ric}(A\eta) = \frac{\mathrm{S}}{2} A\eta + \frac{f}{16} d\mathrm{S}, \qquad \mathrm{Ric}((*A)\eta) = \frac{1}{16} d\mathrm{S}. \tag{34}$$

In particular, we have

$$(A\eta)(\mathrm{S}) = f((*A)\eta)(\mathrm{S}). \tag{35}$$

Proof By Bochner's formula for 1-forms, $\Delta(df) - \mathrm{Ric}(df) = \nabla^*\nabla(df)$ holds. Since $\nabla df = -f\mathrm{Ric}$ is symmetric and since $\nabla^* = \delta$ on symmetric (0, 2)-tensors, this gives

$$\Delta(df) - \mathrm{Ric}(df) = \delta\nabla df = \delta(-f\mathrm{Ric}) = \mathrm{Ric}(df) - f\delta(\mathrm{Ric}) = \mathrm{Ric}(df) + \frac{f}{2} d\mathrm{S},$$

where we used the well-known identity $d\mathrm{S} = -2\delta\mathrm{Ric}$. Hence, we deduce

$$\Delta(df) = 2\mathrm{Ric}(df) + \frac{f}{2} d\mathrm{S}.$$

But (33) also gives $\Delta f = -\mathrm{tr}_g(\nabla df) = f\mathrm{S}$, so that $\Delta(df) = d(\Delta f) = d(f\mathrm{S})$. Therefore

$$\mathrm{Ric}(df) = \frac{1}{2}\mathrm{S}df + \frac{f}{4}d\mathrm{S}.$$

The first equation in (34) now follows from the equality $df = 4A\eta$.

In the following, we will compute the Ricci curvature of the vector field $(*A)\eta$. Notice first that $(*A)\eta = \eta\lrcorner(*A) = *(\eta \wedge A)$. Hence, this vector field belongs to the kernel of A as

$$g(AX, (*A)\eta)\mathrm{vol}_g = AX \wedge *^2(\eta \wedge A) = -AX \wedge \eta \wedge A = 0$$

for any $X \in TM$. Based on the fact $A\eta\lrcorner(*A) = *(A\eta \wedge A) = 0$, we first compute

$$A\eta\lrcorner\nabla_X(*A) = -(*A)(\nabla_X A\eta) = \frac{f}{4}(*A)(\mathrm{Ric}(X)) = *\frac{f}{4}(\mathrm{Ric}(X) \wedge A). \tag{36}$$

This gives

$$\begin{aligned}\eta\lrcorner(A\eta\lrcorner\nabla_X(*A)) &= - * \frac{f}{4}(\eta \wedge \mathrm{Ric}(X) \wedge A) = -\frac{f}{4}\mathrm{Ric}(X)\lrcorner * (\eta \wedge A)\\ &= -\frac{f}{4}\mathrm{Ric}((*A)\eta, X).\end{aligned}$$

On the other hand, by (20) and (32), we have

$$A\eta\lrcorner(\eta\lrcorner\nabla_X(*A)) = A\eta\lrcorner\Big(\frac{1}{4}\mathrm{Ric}(X) + A^2X\Big) = g\Big(\frac{1}{4}\mathrm{Ric}(A\eta) + A^3\eta, X\Big) = \frac{f}{64}g(d\mathrm{S}, X).$$

Comparing the two identities gives the second equation in (34). Equation (35) can be deduced from computing $\mathrm{Ric}(A\eta, (*A)\eta))$ in two ways from (34) taking the scalar product by $(*A)\eta$ in the first formula and by $A\eta$ in the second one. Remember that $(*A)\eta$ lies in the kernel of A. □

In the following, we will establish and prove three technical lemmas (Lemmas 4.8, 4.9 and 4.10), which will show that the kernel and the image of the endomorphism A are integrable and totally geodesic. Then the proof of Theorem A will follow from the de Rham theorem.

Lemma 4.8 *Assume that (GA) holds. Then we have the identity*

$$\nabla_{A\eta}A^2\eta = -\frac{f}{4}\mathrm{Ric}(A^2\eta) - \frac{f^2}{32}A(d\mathrm{S}). \tag{37}$$

Proof By continuity, it suffices to prove the assertion on the set $\{p \in M \mid A\eta|_p \neq 0\}$ since this set is dense in M by Lemma 4.6. Therefore we may assume that $A\eta \neq 0$ everywhere. For any $X \in TM$, we have

$$d(|A\eta|^2) = 2g(\nabla(A\eta), A\eta) = -\frac{f}{2}\mathrm{Ric}(A\eta), \tag{38}$$

where we use equation (19) in the last equality. Therefore, from Lemma 4.7, we find

$$d\left(\frac{1}{|A\eta|^2}\right) = \frac{f}{2|A\eta|^4}\mathrm{Ric}(A\eta) = \frac{f}{4|A\eta|^4}\big(\mathrm{S}\,A\eta + \frac{f}{8}d\mathrm{S}\big).$$

Moreover, $\delta(A^2\eta) = 0$. Indeed, for any two-form ω in four dimensions and any vector X, the formula $\delta(X\lrcorner\omega) = *(dX \wedge *\omega) - \delta\omega(X)$ holds. Using $\delta A = 0$ and $4d(A\eta) = ddf = 0$, this yields

$$\delta(A^2\eta) = \delta(A\eta\lrcorner A) = *(d(A\eta) \wedge *A) - (\delta A)(A\eta) = 0.$$

Now, by taking the divergence of both sides of (31), we compute

$$
\begin{aligned}
0 = \delta A \;&=\; \delta\left(\frac{1}{|A\eta|^2}A\eta\wedge A^2\eta\right) \;=\; -d\left(\frac{1}{|A\eta|^2}\right)\lrcorner(A\eta\wedge A^2\eta)+\frac{1}{|A\eta|^2}\delta(A\eta\wedge A^2\eta)\\
&= -\frac{f}{4|A\eta|^2}(\mathrm{S}\,A\eta+\frac{f}{8}d\mathrm{S})\lrcorner A+\frac{1}{|A\eta|^2}\big(\delta(A\eta)A^2\eta+\nabla_{A^2\eta}A\eta-\nabla_{A\eta}A^2\eta-\delta(A^2\eta)A\eta\big),
\end{aligned}
$$

where we use the formula $\delta(X\wedge Y) = (\delta X)Y+\nabla_Y X-\nabla_X Y-(\delta Y)X$, valid for any $X, Y\in TM$. Furthermore, the divergence of $A\eta$ is equal to $f\mathrm{S}/4$ as an easy consequence from taking the trace of equation (19). This finally gives (37). □

The following technical lemma expresses a partial trace of the Ricci tensor.

Lemma 4.9 *Assume that (GA) holds and that* $A\eta\neq 0$ *everywhere. Then the following identity holds:*

$$
\frac{1}{|A\eta|^2}\mathrm{Ric}(A\eta, A\eta)+\frac{1}{|A^2\eta|^2}\mathrm{Ric}(A^2\eta, A^2\eta) = \mathrm{S}-\frac{2}{f\mathrm{S}}A\eta(\mathrm{S}).
$$

Proof The proof relies on taking the scalar product of $\mathrm{Ric}(A^2\eta)$ in Lemma 4.8 with the vector field $A^2\eta$. Indeed, we have

$$
\begin{aligned}
\mathrm{Ric}(A^2\eta, A^2\eta) &= -\frac{4}{f}\Big(g(\nabla_{A\eta}A^2\eta+\frac{f^2}{32}A(d\mathrm{S}), A^2\eta)\Big)\\
&= -\frac{2}{f}A\eta(|A^2\eta|^2)+\frac{f}{8}g(d\mathrm{S}, A^3\eta)\\
&\overset{(32)}{=} -\frac{2}{f}A\eta\Big(\frac{\mathrm{S}}{8}|A\eta|^2\Big)-\frac{f\mathrm{S}}{64}A\eta(\mathrm{S})\\
&\overset{(38)}{=} \frac{\mathrm{S}}{8}\mathrm{Ric}(A\eta, A\eta)-\Big(\frac{|A\eta|^2}{4f}+\frac{f\mathrm{S}}{64}\Big)A\eta(\mathrm{S}).
\end{aligned}
$$

Hence, again by (32), we find

$$
\frac{\mathrm{Ric}(A^2\eta, A^2\eta)}{|A^2\eta|^2} = \frac{\mathrm{Ric}(A\eta, A\eta)}{|A\eta|^2}-\left(\frac{2}{f\mathrm{S}}+\frac{f}{8|A\eta|^2}\right)A\eta(\mathrm{S}).
$$

Finally, the identity

$$
\frac{\mathrm{Ric}(A\eta, A\eta)}{|A\eta|^2} = \frac{\mathrm{S}}{2}+\frac{f}{16|A\eta|^2}A\eta(\mathrm{S}),
$$

which follows from Lemma 4.7, leads to the required equality. □

Lemma 4.10 *If (GA) holds, then the scalar curvature is constant and* $\mathrm{Ric}+4A^2 = 0$.

Proof As in the proof of Lemma 4.8, we may assume that $A\eta\neq 0$ everywhere. By Lemma 4.7 we know that

$$
\mathrm{Ric}(A\eta)-\frac{\mathrm{S}}{2}A\eta = f\mathrm{Ric}((*A)\eta).
$$

We take the divergence of both sides. We start with the left hand side. Note that for any vector field $X\in\Gamma(TM)$ the formula $\delta(\mathrm{Ric}(X)) = g(\delta\mathrm{Ric}, X)-\sum_{i=1}^4 g(\mathrm{Ric}(e_i), \nabla_{e_i}X)$

holds, where $e_1, \dots, e_4$ is any pointwise orthonormal basis. Using this and $\delta(A\eta) = \frac{f\mathrm{S}}{4}$, we compute

$$\delta(\mathrm{Ric}(A\eta) - \frac{\mathrm{S}}{2}A\eta) = g(\delta\mathrm{Ric}, A\eta) - \sum_{i=1}^{4} g(\mathrm{Ric}(e_i), \nabla_{e_i} A\eta) - \frac{1}{2}\big(- g(d\mathrm{S}, A\eta) + \mathrm{S}\,\delta(A\eta)\big)$$
$$= -\frac{1}{2}g(d\mathrm{S}, A\eta) + \frac{f}{4}\sum_{i=1}^{4} g(\mathrm{Ric}(e_i), \mathrm{Ric}(e_i)) + \frac{1}{2}g(d\mathrm{S}, A\eta) - \frac{f}{8}\mathrm{S}^2$$
$$= \frac{f}{4}|\mathrm{Ric}|^2 - \frac{f}{8}\mathrm{S}^2. \tag{39}$$

To get the divergence of the right-hand side, we first compute that of the vector field $\mathrm{Ric}((*A)\eta)$. For this, we use the same formula as above and again $d\mathrm{S} = -2\delta\mathrm{Ric}$ to write

$$\delta(\mathrm{Ric}((*A)\eta)) = -\frac{1}{2}((*A)\eta)(\mathrm{S}) - \sum_{i=1}^{4} g(\mathrm{Ric}(e_i), \nabla_{e_i}((*A)\eta))$$
$$= -\frac{1}{2f}(A\eta)(\mathrm{S}) - \sum_{i=1}^{4} g(\mathrm{Ric}(e_i), \nabla_{e_i}((*A)\eta)). \tag{40}$$

In the last equality, we used (35). Inserting (20) into (40), we find

$$\delta(\mathrm{Ric}((*A)\eta)) = -\frac{1}{2f}(A\eta)(\mathrm{S}) - \frac{1}{4}|\mathrm{Ric}|^2 - \sum_{i=1}^{4} g(\mathrm{Ric}(e_i), A^2 e_i),$$

which in turn gives

$$\delta(f\mathrm{Ric}((*A)\eta)) = -g(df, \mathrm{Ric}((*A)\eta)) + f \cdot \delta(\mathrm{Ric}((*A)\eta)))$$
$$= -\frac{3}{4}(A\eta)(\mathrm{S}) - \frac{f}{4}|\mathrm{Ric}|^2 - f\sum_{i=1}^{4} g(\mathrm{Ric}(e_i), A^2 e_i) \tag{41}$$

by (34). Comparing Eqs. (39) and (41), we obtain

$$\sum_{i=1}^{4} g(\mathrm{Ric}(e_i), A^2 e_i) = -\frac{3}{4f}(A\eta)(\mathrm{S}) - \frac{1}{2}|\mathrm{Ric}|^2 + \frac{1}{8}\mathrm{S}^2.$$

On the other hand, this sum can be computed on the particular orthonormal frame $\frac{A\eta}{|A\eta|}, \frac{A^2\eta}{|A^2\eta|}, e_3, e_4$ with e_3, e_4 in the kernel of A as follows: using Lemma 4.9, we write

$$\sum_{i=1}^{4} g(\mathrm{Ric}(e_i), A^2 e_i) = \frac{1}{|A\eta|^2}\mathrm{Ric}(A\eta, A^3\eta) + \frac{1}{|A^2\eta|^2}\mathrm{Ric}(A^2\eta, A^4\eta)$$
$$\overset{(32)}{=} -\frac{\mathrm{S}}{8}\left(\frac{1}{|A\eta|^2}\mathrm{Ric}(A\eta, A\eta) + \frac{1}{|A^2\eta|^2}\mathrm{Ric}(A^2\eta, A^2\eta)\right)$$
$$= -\frac{\mathrm{S}^2}{8} + \frac{1}{4f}A\eta(\mathrm{S}). \tag{42}$$

Comparing these two computations yields

$$4A\eta(\mathrm{S}) = f(\mathrm{S}^2 - 2|\mathrm{Ric}|^2). \tag{43}$$

The Cauchy-Schwarz Inequality gives

$$\sum_{i=1}^{4} g(\mathrm{Ric}(e_i), A^2 e_i) \leq |\mathrm{Ric}||A^2|. \tag{44}$$

We take the square of this inequality. Then we use (42) and (43) to express the left- and right-hand sides, respectively. We obtain

$$\Big(-\frac{\mathrm{S}^2}{8}+\frac{1}{4f}A\eta(\mathrm{S})\Big)^2 \leq \Big(\frac{\mathrm{S}^2}{2}-\frac{2}{f}A\eta(\mathrm{S})\Big)\frac{\mathrm{S}^2}{32} = \frac{\mathrm{S}^4}{64}-\frac{\mathrm{S}^2}{16f}A\eta(\mathrm{S}),$$

where besides (15), which says that $\mathrm{S} = 4|A|^2$, we used $|A^2|^2 = (|A|^2)^2/2$, which follows from the fact that A is skew-symmetric of rank two. This inequality is only true if $A\eta(\mathrm{S}) = 0$. But then (44) is an equality. Hence, Ric is a multiple of A^2 at every point of M^4. Since $\mathrm{Tr}\,\mathrm{Ric} = \mathrm{S}$ and $\mathrm{Tr}A^2 = -|A|^2 = -\mathrm{S}/4$, we obtain $\mathrm{Ric} = -4A^2$. As the vector field $(*A)\eta$ lies in the kernel of A, the second equation in (34) implies that the scalar curvature is constant. This ends the proof. □

Lemma 4.11 *If (GA) is satisfied, then (M, g) is locally isometric to $\mathbb{R}^2 \times \mathbb{S}^2$.*

Proof We show that the two orthogonal distributions $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$ – which are both of rank two by assumption – are parallel. If this is proved to be true, then we get a local Riemannian product by the de Rham decomposition theorem. Clearly, it suffices to show that $\mathrm{Im}(A)$ is parallel since $\mathrm{Ker}(A) = \mathrm{Im}(A)^\perp$. Let us first consider the open subset $V := \{p \in M \mid A\eta|_p \neq 0\}$. On V, the image of A is spanned by $A\eta$ and $A^2\eta$. Note that $\nabla_X A\eta = fA^2X$ by (19) and Lemma 4.10. Therefore $\nabla_X A\eta$ is contained in $\mathrm{Im}(A)$ for all $X \in TM$. Furthermore, by equation (36) and Lemma 4.10, we have $A\eta \lrcorner \nabla_X(*A) = 0$ for all $X \in TM$. Equation (21) now gives $A\eta \lrcorner \nabla_X A = 0$. Therefore $\nabla_X A^2\eta = A(\nabla_X(A\eta)) = fA^3(X)$. In particular, also $\nabla_X A^2\eta$ is contained in $\mathrm{Im}(A)$ for all $X \in TM^4$. This proves that $\mathrm{Im}(A)$ is parallel.

We want to extend this splitting of TM into two parallel distributions to all of M. To this end, we observe that, on V, the Ricci map has constant eigenvalues $0, 0, \mathrm{S}/2, \mathrm{S}/2 > 0$ and $\mathrm{Ker}(A)$ and $\mathrm{Im}(A)$ are the eigendistributions. Since $V \subset M$ is dense by Lemma 4.6, these are also the eigenvalues of Ric on all of M and the two-dimensional eigendistributions of Ric are parallel on all of M. We deduce that (M, g) is locally isometric to the Riemannian product $\mathbb{R}^2 \times \mathbb{S}^2$. □

Now we can prove the main result of this section. In particular, it says that, in the degenerate case, the skew Killing spinor is parallel or (M, g) is locally isometric to one of the examples discussed in Section 4.1.

Theorem 4.12 *Let (M^4, g) be a connected Riemannian spin manifold carrying a skew Killing spinor ψ, where the rank of the corresponding skew-symmetric tensor field A is ≤ 2 everywhere. Then either ψ is parallel (i.e., $A = 0$) on M or, around every point of M, we have a local Riemannian splitting $\mathbb{R} \times N$ with N having a skew Killing spinor. If, moreover, $|\psi^+|$ (hence also $|\psi^-|$) is not constant, then (M, g) is a local Riemannian product $\mathbb{S}^2 \times \mathbb{R}^2$ around every point and the Killing map equals $\pm J \oplus 0$.*

If, in addition, (M, g) is complete, then (M, g) is globally isometric to the Riemannian product $\mathbb{S}^2 \times \Sigma^2$, where Σ^2 is either flat $\mathbb{R}^2$, a flat cylinder with trivial spin structure or a flat 2-torus with trivial spin structure.

Proof We define $U := \{p \mid A_p \neq 0\}$ and $U' := U \cap M', U'' := U \cap M''$. Recall that $U' \subset U$ is dense. We know that equation (33) holds on the open set U''. We claim that it holds on all of M. Obviously, it is true on the closure $\overline{U''}$ of U''. It also holds on $U' \setminus \overline{U''}$ since this set is open with $f \equiv 0$. Consequently, it holds on U', therefore on U since $U' \subset U$ dense. Hence it is true on $\mathrm{supp}(A) = \overline{U}$. Furthermore, on the complement of $\mathrm{supp}(A)$, we have $df = 0$ and $\mathrm{Ric} = 0$, which implies that (33) holds on M. Now we can apply Prop. 1.2 in [17], which shows that either $f \equiv 0$ on M or $\mathrm{supp}(f) = M$. If $f \equiv 0$, then Proposition 4.3 applies. Assume now that $\mathrm{supp}(f) = M$. Then M'' is dense in M. Let U and U'' be defined as above. On U'', the assumption (GA) is satisfied. As we have seen, the eigenvalues of Ric are 0 and $\mathrm{S}/2$ and the eigendistributions of Ric on U'' are parallel. Therefore this holds also on $\overline{U}'' = \overline{U}$. If $\overline{U} = M$, then we are done by Lemma 4.11. If $\overline{U} = \emptyset$, then ψ is parallel. Assume that $\overline{U}$ were non-empty and not equal to M. Then the complement W of $\overline{U}$ is open and not empty with $A_{|W} = 0$. Thus ψ is parallel on W, hence $\mathrm{Ric} = 0$ on W, therefore also on $\overline{W}$. Since M is connected, $\overline{U} \cap \overline{W}$ is non-empty. Hence we can choose a point p in this intersection. But then $p \in \overline{U}$ would imply that $\mathrm{S}/2 > 0$ is an eigenvalue of Ric_p and $p \in \overline{W}$ would imply that $\mathrm{Ric}_p = 0$, a contradiction.

Note that, as we already noticed in [14, Theorem 2.4], the manifold (M, g) must be globally isometric to the product $\mathbb{S}^2 \times \Sigma^2$, where Σ^2 is a quotient of flat $\mathbb{R}^2$. The reason is that the fundamental group of M can act on the $\mathbb{S}^2$-factor only in a trivial way. It remains to recall that a parallel spinor descends from $\mathbb{R}^2$ to a nontrivial quotient (flat cylinder or torus) if and only if the fundamental group acts on the spin structure of $\mathbb{R}^2$ in a trivial way, that is, the quotient Σ^2 carries the trivial spin structure. □

We end this section with the question—asked by Ilka Agricola— whether skew Killing spinors can be seen as parallel spinors w.r.t. a covariant derivative induced by some metric connection on (TM, g).

Proposition 4.13 *Let (M^4, g) be any Riemannian spin manifold and ψ be any nonzero skew Killing spinor on M. Assume that, w.r.t. the splitting $\psi = \psi^+ + \psi^-$, neither $\psi^\pm$ vanish on M. Assume there exists a metric connection ∇' on (TM, g) such that ψ is parallel w.r.t. the covariant derivative induced by ∇' on ΣM.*
Then $A\xi = 0$, in particular $|\psi^+| = |\psi^-|$. Moreover, $\nabla'_X = \nabla_X + 2\Big((AX \wedge \frac{\xi}{|\xi|^2})_+ - (AX \wedge \xi)_-\Big)$ for all $X \in TM$.

Proof Write $\nabla' = \nabla - B$ for some unknown $B \in T^*M \otimes \Lambda^2 T^*M$. Recall that, for any $X \in TM$, $BX \in \mathrm{End}(TM)$ must be skew-symmetric because of both ∇, ∇' being metric. Then for any section $\varphi \in \Sigma M$ and any $X \in TM$,

$$\nabla'_X \varphi = \nabla_X \varphi - \frac{1}{2} BX \cdot \varphi,$$

where we see BX as a two-form acting by Clifford multiplication on ΣM. Since by assumption ψ^+ does not vanish anywhere, ξ is a nowhere vanishing vector field on M. The question is now whether B exists such that

$$\frac{1}{2} BX \cdot \psi = AX \cdot \psi$$

holds for all $X \in TM$. Using the splitting $\psi = \psi^+ + \psi^-$, we obtain the following equivalent systems:

$$\begin{cases} \frac{1}{2}BX \cdot \psi^+ = AX \cdot \psi^- \\ \\ \frac{1}{2}BX \cdot \psi^- = AX \cdot \psi^+ \end{cases} \iff \begin{cases} \frac{1}{2}BX \cdot \psi^+ = -AX \cdot \frac{\xi}{|\xi|^2} \cdot \psi^+ \\ \\ \frac{1}{2}BX \cdot \psi^- = AX \cdot \xi \cdot \psi^- \end{cases}$$
$$\iff \begin{cases} \frac{1}{2}BX \cdot \psi^+ = -(AX \wedge \frac{\xi}{|\xi|^2}) \cdot \psi^+ + \langle AX, \frac{\xi}{|\xi|^2} \rangle \psi^+ \\ \\ \frac{1}{2}BX \cdot \psi^- = (AX \wedge \xi) \cdot \psi^- - \langle AX, \xi \rangle \psi^- \end{cases}$$
$$\iff \begin{cases} \left(\frac{1}{2}BX + (AX \wedge \frac{\xi}{|\xi|^2})\right) \cdot \psi^+ = \langle AX, \frac{\xi}{|\xi|^2} \rangle \psi^+ \\ \\ \left(\frac{1}{2}BX - (AX \wedge \xi)\right) \cdot \psi^- \quad = -\langle AX, \xi \rangle \psi^- . \end{cases}$$

Recall that a real 2-form acts in a skew-Hermitian way on ΣM, therefore we obtain $\langle AX, \xi \rangle = 0$ for all $X \in TM$, which implies that $A\xi = 0$. Moreover, since self-dual resp. anti-self-dual 2-forms kill negative resp. positive half spinors, the preceding system is equivalent to

$$\begin{cases} \left(\frac{1}{2}BX + (AX \wedge \frac{\xi}{|\xi|^2})\right)_+ \cdot \psi^+ = 0 \\ \\ \left(\frac{1}{2}BX - (AX \wedge \xi)\right)_- \cdot \psi^- \quad = 0 . \end{cases}$$

On the other hand, as we have seen above, the maps $\bigwedge^2_- M \longrightarrow \Sigma^- M \cap (\psi^-)^\perp$, $\omega_- \longmapsto \omega_- \cdot \psi^-$ and $\bigwedge^2_+ M \longrightarrow \Sigma^+ M \cap (\psi^+)^\perp$, $\omega_+ \longmapsto \omega_+ \cdot \psi^+$ are isomorphisms if $\psi^+ \neq 0$ and $\psi^- \neq 0$. Therefore we can deduce that $\left(\frac{1}{2}BX + (AX \wedge \frac{\xi}{|\xi|^2})\right)_+ = 0$ and $\left(\frac{1}{2}BX - (AX \wedge \xi)\right)_- = 0$, which yields $BX = -2\left((AX \wedge \frac{\xi}{|\xi|^2})_+ - (AX \wedge \xi)_-\right)$ and concludes the proof of Proposition 4.13. □

With other words, only a special subcase of the degenerate case can be considered with that ansatz, namely that considered in Proposition 4.3. As a consequence, the general classification of 4-dimensional Riemannian spin manifolds with skew Killing spinors cannot be obtained that way.

5 Skew Killing spinors with non-degenerate Killing map *A*

This section is devoted to the case where we have a skew Killing spinor ψ whose Killing map A is non-degenerate everywhere. Recall that ψ defines a vector field η by (8). As above, we put $\rho := |\eta|$. Here, we want to assume that $M'' = \{x \in M \mid \rho(x) \notin \{0, 1/2\}\} = \{x \in M \mid f(x) \notin \{0, \pm 1\}\}$ is equal to M. This is a sensible restriction since M'' is dense in M if A is non-degenerate everywhere, see Sect. 3. Working on M'' has the advantage that we do not have to care about the sign of f. Indeed, as explained in Remark 3.3, up to a possible change of orientation on each connected component we may assume that $f > 0$. In particular, f is defined by $\rho = |\eta|$ via $f = \sqrt{1 - 4\rho^2}$, which will be important for the reverse direction of Proposition 5.1.

We remark that the main results of this section are also true if A is degenerate as long as we consider only the (not necessarily dense) subset $M'' \subset M$. However, here the degenerate case is less interesting since it has already been thoroughly studied in the preceding section.

5.1 Equivalent description by complex structures

Let M be a manifold and A be a skew-symmetric endomorphism field on M. Define a tensor field C_A on M by $C_A(X, Y) := (\nabla_X A)(Y) - (\nabla_Y A)(X)$. In the first part of the next proposition we will assume that $M = M''$ holds. As already noted, this is reasonable if A is non-degenerate everywhere.

Proposition 5.1 *Let M be a four-dimensional spin manifold and A be a skew-symmetric endomorphism field on M. Put $C := C_A$.*

If (M, g) admits a skew Killing spinor ψ associated with A such that $M = M''$, then there exist an almost Hermitian structure J and a nowhere vanishing vector field η of length $|\eta| =: \rho < 1/2$ such that

$$(\nabla_Y J)(X) = \frac{4}{f-1} X \lrcorner \big(J\eta \wedge AY + \eta \wedge JAY\big), \tag{45}$$

$$\nabla \eta = fA, \tag{46}$$

$$g(C(\eta, X), J\eta) = \rho^2 f g(C_P, X) \tag{47}$$

$$g(C(J\eta, Z), J\eta) = *(C_P \wedge Z \wedge \eta \wedge J\eta), \quad Z \in P := \{\eta, J\eta\}^\perp, \tag{48}$$

where $f := \sqrt{1 - 4\rho^2}$ and $C_P := C(s, Js)$ for any unit vector $s \in P$, and such that the sectional curvature K_P in direction P satisfies

$$K_P = -\rho^{-2} g(C_P, J\eta) + 4A_P^2, \tag{49}$$

where $A_P := g(As, Js)$ for any unit vector $s \in P$.

If M is simply-connected, then also the converse statement is true.

Lemma 5.2 *Assume that J, A and η satisfy equations (45) and (46). Then*

$$g(C(X, Y), \eta) = 0, \tag{50}$$

$$R(X, Y)\eta = fC(X, Y) - 4\eta \lrcorner (AX \wedge AY), \tag{51}$$

$$R(X, Y)J\eta = -JC(X, Y) + \frac{4}{f-1} g(C(X, Y), J\eta)\eta - 4J(\eta) \lrcorner (AX \wedge AY). \tag{52}$$

Proof Note first that $X(f^2) = X(1 - 4|\eta|^2) = -8g(\nabla_X \eta, \eta) = -8fg(AX, \eta)$. This implies $X(f) = -4g(AX, \eta)$, which we will use in the following. Let X and Y be vector fields on M and assume that $\nabla X = \nabla Y = 0$ holds at a point $p \in M$. At p, we have

$$\nabla_X \nabla_Y \eta = \nabla_X (fAY) = -4g(AX, \eta)AY + f(\nabla_X A)Y.$$

Therefore

$$R(X, Y)\eta = -4\big(g(AX, \eta)AY - g(AY, \eta)AX\big) + fC(X, Y),$$

which gives Eq. (51). In particular, this yields $0 = R(X, Y, \eta, \eta) = g(C(X, Y), \eta)$, which proves (50) since $f \neq 0$ everywhere.

In the following computation, the sign '≡' means equality up to a term $S(X, Y)$ for some symmetric bilinear map S. We compute

$$
\begin{aligned}
\nabla_X \nabla_Y (J\eta) &= \nabla_X\big((\nabla_Y J)(\eta) + J(\nabla_Y \eta)\big)\\
&= \nabla_X\Big(\frac{4}{f-1}\big(-g(\eta, AY)J\eta + \rho^2 JAY - g(\eta, JAY)\eta\big) + J(\nabla_Y\eta)\Big)\\
&\equiv \frac{16}{(f-1)^2}\, g(AX,\eta)\big(\rho^2 JAY - g(\eta, JAY)\eta\big)\\
&\quad + \frac{4}{f-1}\Big(-g(\eta,(\nabla_X A)Y)J\eta - g(\eta, AY)(\nabla_X J)\eta - fg(\eta, AY)JAX\\
&\quad + 2fg(AX,\eta)JAY + \rho^2(\nabla_X J)(AY) + \rho^2 J(\nabla_X A)Y - fg(AX, JAY)\eta\\
&\quad - g(\eta, (\nabla_X J)AY)\eta\\
&\quad - g(\eta, J(\nabla_X A)Y)\eta - fg(\eta, JAY)AX\Big)\\
&\quad + f(\nabla_X J)(AY) + J(\nabla_X\nabla_Y\eta)\\
&\equiv \frac{16}{(f-1)^2}\, g(AX,\eta)\big(\rho^2 JAY - g(\eta, JAY)\eta\big)\\
&\quad + \frac{4}{f-1}\Big(-g(\eta,(\nabla_X A)Y)J\eta - g(\eta, AY)(\nabla_X J)\eta + 2fg(AX,\eta)JAY + \rho^2(\nabla_X J)AY\\
&\quad + \rho^2 J(\nabla_X A)Y - g(\eta,(\nabla_X J)AY)\eta - g(\eta, J(\nabla_X A)Y)\eta - 2fg(\eta, JAY)AX\Big)\\
&\quad + J(\nabla_X\nabla_Y\eta)\\
&\equiv \frac{16\rho^2}{(f-1)^2}\big(g(AX,\eta)JAY + g(J\eta, AY)AX\big)\\
&\quad + \frac{4}{f-1}\Big(-g(\eta,(\nabla_X A)Y)J\eta + 2fg(AX,\eta)JAY + \rho^2 J(\nabla_X A)Y\\
&\quad - g(\eta, J(\nabla_X A)Y)\eta - 2fg(\eta, JAY)AX\Big)\\
&\quad + J(\nabla_X\nabla_Y\eta)\\
&= 4g(AX,\eta)JAY + 4g(AY, J\eta)AX - \frac{4}{f-1}\big(g(\eta,(\nabla_X A)Y)J\eta + g(\eta, J(\nabla_X A)Y)\eta\big)\\
&\quad - (f+1)J(\nabla_X A)Y + J(\nabla_X\nabla_Y\eta).
\end{aligned}
$$

This implies

$$
\begin{aligned}
R(X,Y)J\eta \;=\; & 4g(AX,\eta)JAY - 4g(AY,\eta)JAX + 4g(AY, J\eta)AX - 4g(AX, J\eta)AY\\
& - \frac{4}{f-1}\big(g(\eta, C(X,Y))J\eta - g(J\eta, C(X,Y))\eta\big) - (f+1)JC(X,Y) + J(R(X,Y)\eta).
\end{aligned}
$$

Using Eqs. (50) and (51) we obtain (52). □

Proof of Prop. 5.1 Before we start the proof of the two directions of the assertion, let us first suppose that, on M, we are given a Hermitian structure J and a nowhere vanishing vector field η of length $\rho < 1/2$. We want to define a vector field ξ such that the identities $\xi = -(|\xi|/\rho)\cdot\eta$ and $\rho = |\xi|/(1+|\xi|^2)$ hold according to equation (9). Since this leads to a quadratic equation, we have to choose one of the solutions. Here we use our assumption $M = M''$ and define $f = \sqrt{1-4\rho^2}$ and $\xi = 2(f-1)^{-1}\eta$, compare Remark 3.3, which motivates this choice. Assume that the orientation on M is such that orthonormal bases of the form s_1, Js_1, s_2, Js_2 are negatively oriented. We define a one-dimensional subbundle E of ΣM by

$$E := \{\varphi \mid J(X)\cdot\varphi^- = iX\cdot\varphi^- \text{ for all } X \in TM,\ \varphi^+ = \xi\cdot\varphi^-\}. \tag{53}$$

We want to show that E is parallel with respect to $\hat{\nabla}$ defined by $\hat{\nabla}_X\varphi := \nabla_X\varphi - AX\cdot\varphi$ if and only if J and η satisfy (45) and (46). Let X and Y be vector fields satisfying $\nabla X = \nabla Y = 0$ at $p \in M$. Then we have at $p \in M$

$$\begin{aligned}
J(X)\cdot(\hat{\nabla}_Y\varphi)^- &= J(X)\cdot(\nabla_Y\varphi - AY\cdot\varphi)^-\\
&= J(X)\cdot(\nabla_Y\varphi^- - AY\cdot\varphi^+)\\
&= \nabla_Y(J(X)\cdot\varphi^-) - (\nabla_Y J)(X)\cdot\varphi^- - J(X)A(Y)\cdot\varphi^+\\
&= \nabla_Y(iX\cdot\varphi^-) - (\nabla_Y J)(X)\cdot\varphi^- + A(Y)J(X)\xi\cdot\varphi^- + 2g(JX, AY)\varphi^+\\
&= iX\cdot\nabla_Y\varphi^- - (\nabla_Y J)(X)\cdot\varphi^- - iA(Y)\xi X\cdot\varphi^- - 2g(JX,\xi)AY\cdot\varphi^-\\
&\quad +2g(JX, AY)\varphi^+\\
&= iX\cdot\nabla_Y\varphi^- - (\nabla_Y J)(X)\cdot\varphi^- - iXA(Y)\xi\cdot\varphi^- + 2ig(\xi, X)AY\cdot\varphi^-\\
&\quad -2ig(AY, X)\xi\cdot\varphi^- - 2g(JX,\xi)AY\cdot\varphi^- + 2g(JX, AY)\xi\cdot\varphi^-\\
&= iX\cdot\nabla_Y\varphi^- - (\nabla_Y J)(X)\cdot\varphi^- - iXA(Y)\cdot\varphi^+ + 2g(\xi, X)JA(Y)\cdot\varphi^-\\
&\quad -2g(AY, X)J(\xi)\cdot\varphi^- - 2g(JX,\xi)AY\cdot\varphi^- + 2g(JX, AY)\xi\cdot\varphi^-.
\end{aligned}$$

This equals $iX\cdot(\hat{\nabla_Y}\varphi)^-$ if and only if $(\nabla_Y J)(X) = 2X\lrcorner\big(J\xi\wedge AY + \xi\wedge JAY\big)$ holds, which is equivalent to equation (45). Furthermore,

$$\begin{aligned}
(\hat{\nabla_X}\varphi)^+ &= \nabla_X\varphi^+ - AX\cdot\varphi^- \;=\; \nabla_X(\xi\cdot\varphi^-) - AX\cdot\varphi^-\\
&= (\nabla_X\xi)\cdot\varphi^- + \xi\cdot\nabla_X\varphi^- - AX\cdot\varphi^-\\
&= (\nabla_X\xi)\cdot\varphi^- + \xi A(X)\cdot\varphi^+ + \xi\cdot(\hat{\nabla_X}\varphi)^- - AX\cdot\varphi^-\\
&= \big(\nabla_X\xi - (1-|\xi|^2)AX + 2g(A\xi, X)\xi\big)\cdot\varphi^- + \xi\cdot(\hat{\nabla_X}\varphi)^-.
\end{aligned}$$

This equals $\xi\cdot(\hat{\nabla_X}\varphi)^-$ if and only if $\nabla_X\xi = (1-|\xi|^2)AX - 2g(A\xi, X)\xi$ holds, which is equivalent to (46). Consequently, E is parallel with respect to $\hat{\nabla}$ if and only if J and η satisfy (45) and (46).

Assume that $\hat{\nabla}$ reduces to a connection $\hat{\nabla}^E$ on E. Then Eqs. (45) and (46), and therefore also (50), (51) and (52) hold. We will show that the curvature $\hat{R}$ of $\hat{\nabla}^E$ vanishes if and only if the Riemannian curvature R of M equals the tensor B defined by

$$B(X, Y) := \rho^{-2}\big(*(C(X,Y)\wedge\eta) - fC(X,Y)\wedge\eta\big) - 4AX\wedge AY \tag{54}$$

for all vector fields X and Y on M. By an easy calculation similar to that in the proof of Proposition 2.1, we get

$$\hat{R}_{X,Y}\varphi = \tfrac{1}{2}R(X,Y)\cdot\varphi - C(X,Y)\cdot\varphi + 2\big(AX\wedge AY\big)\cdot\varphi.$$

This shows that $\hat{R}$ vanishes if and only if

$$R(X,Y)\cdot\varphi = 2C(X,Y)\cdot\varphi - 4\big(AX\wedge AY\big)\cdot\varphi \tag{55}$$

for all vector fields X and Y and all sections φ of E. In the following, we will use that $\bigwedge^2_\pm M$ acts trivially on $\Sigma^\mp M$ and that, for any nowhere vanishing section $\varphi^\pm$ of $\Sigma^\pm M$, the maps defined by (7) are isomorphisms. Let φ be a section of E such that $\varphi^+(x) \neq 0$, $\varphi^-(x) \neq 0$ for all $x \in M$ (here we use that ξ does not vanish). Then

$$\begin{aligned} 2C(X,Y)\cdot\varphi &= 2C(X,Y)\cdot\big(\xi\cdot\varphi^- - |\xi|^{-2}\xi\cdot\varphi^+\big) \\ &\overset{(50)}{=} 2(C(X,Y)\wedge\xi)\cdot\varphi^- - 2|\xi|^{-2}(C(X,Y)\wedge\xi)\cdot\varphi^+ \\ &= 2(C(X,Y)\wedge\xi)_-\cdot\varphi - 2|\xi|^{-2}(C(X,Y)\wedge\xi)_+\cdot\varphi \\ &= \frac{4}{f-1}(C(X,Y)\wedge\eta)_-\cdot\varphi - \frac{f-1}{\rho^2}(C(X,Y)\wedge\eta)_+\cdot\varphi \\ &= \rho^{-2}\big(*(C(X,Y)\wedge\eta) - fC(X,Y)\wedge\eta\big)\cdot\varphi. \end{aligned}$$

Therefore (54) and (55) show that $\hat{R}$ vanishes if and only if $R = B$. The latter condition is equivalent to the system of equations

$$R(X,Y)\eta = B(X,Y)\eta \tag{56}$$

$$R(X,Y)J\eta = B(X,Y)J\eta \tag{57}$$

$$R(s,Js,s,Js) = g(B(s,Js)s,Js) \tag{58}$$

$$g(B(\eta,X)s,Js) = g(B(s,Js)\eta,X) \tag{59}$$

$$g(B(J\eta,Z)s,Js) = g(B(s,Js)J\eta,Z) \tag{60}$$

for all $X, Y \in \mathfrak{X}(M)$ and all $Z \in \Gamma(P)$. Recall that (50) holds in our situation, which we will use in the following computations. Equations (59) and (60) are equivalent to the two equations

$$\begin{aligned} g\big(s\lrcorner\big(*(C(\eta,X)\wedge\eta) - fC(\eta,X)\wedge\eta\big),Js\big) &= g\big(\eta\lrcorner\big(*(C_P\wedge\eta) - fC_P\wedge\eta\big),X\big), \\ g\big(s\lrcorner\big(*(C(J\eta,Z)\wedge\eta) - fC(J\eta,Z)\wedge\eta\big),Js\big) &= g\big(J\eta\lrcorner\big(*(C_P\wedge\eta) - fC_P\wedge\eta\big),Z\big), \end{aligned}$$

which are equivalent to (47) and (48), respectively. Because of

$$\eta\lrcorner\big(*(C(X,Y)\wedge\eta) - fC(X,Y)\wedge\eta\big) = f\rho^2 C(X,Y),$$

and

$$\begin{aligned} J(\eta)\lrcorner\big(*(C(X,Y)\wedge\eta) - fC(X,Y)\wedge\eta\big) &= *(C(X,Y)\wedge\eta\wedge J\eta) - fg(C(X,Y),J\eta)\eta \\ &= -\rho^2 g(C(X,Y),s)Js + \rho^2 g(C(X,Y),Js)s - fg(C(X,Y),J\eta)\eta \\ &= -\rho^2\big(g(JC(X,Y),Js)Js + g(JC(X,Y),s)s\big) - fg(C(X,Y),J\eta)\eta \\ &= -\rho^2 JC(X,Y) - (f+1)g(C(X,Y),J\eta)\eta \\ &= -\rho^2 JC(X,Y) + \frac{4\rho^2}{f-1}g(C(X,Y),J\eta)\eta, \end{aligned}$$

Lemma 5.2 shows that equation (56) is equivalent to (51) and (57) is equivalent to (52). Recall that (51) and (52) are satisfied in our situation. Finally,

$$g\big(s\lrcorner\big(*(C_P\wedge\eta) - fC_P\wedge\eta\big),Js\big) = g\big(*(s\wedge C_P\wedge\eta),Js\big) = g(C_P,J\eta),$$

which implies that (58) is equivalent to (49). Consequently, the curvature $\hat{R}$ of $\hat{\nabla}$ vanishes if and only if the Eqs. (47), (48) and (49) hold.

Now we can prove both directions of the proposition. Suppose that there exists a spinor field ψ on M satisfying $\nabla_X\psi = AX\cdot\psi$ for all $X \in TM$ such that $M = M''$. The latter condition means that the vector field η defined in (8) satisfies $0 < \rho = |\eta| < 1/2$. In particular, $\psi^- \neq 0$ everywhere and we can define an almost Hermitian structure J by $J(X)\cdot\psi^- = iX\cdot\psi^-$.

Therefore we may apply our above considerations. If we define $E \subset \Sigma M$ and $\hat{\nabla}$ as above, then ψ is a $\hat{\nabla}$-parallel section of E. In particular, $\hat{\nabla}$ reduces to a connection $\hat{\nabla}^E$ and the curvature of $\hat{\nabla}^E$ vanishes therefore (45)– (49) hold.

Conversely, if we are given an almost Hermitian structure J and a nowhere vanishing vector field η of length $0 < \rho = |\eta| < 1/2$ such that (45)–(49) are satisfied. Then we can define a one-dimensional subbundle $E \subset \Sigma M$ by (53) together with a flat covariant derivative $\hat{\nabla}$ on E. If M is simply-connected, then E admits a parallel section, which is a skew Killing spinor. □

Remark 5.3 Let J be an almost Hermitian structure on a four-dimensional manifold M such that (45) and (46) hold for a skew-symmetric endomorphism field A and a vector field E. Then J defines a reduction of the SO(4)-bundle SO(M) to U(2). Here we want to give the intrinsic torsion of this bundle in the special case where A and J commute. The two components of the intrinsic torsion of this bundle are the Nijenhuis tensor N of J and the differential $d\Omega$ of the Kähler form $\Omega := g(J\cdot, \cdot)$. A direct calculation using (45) and (46) shows that under the assumption $AJ = JA$ these components are given by $N = 0$ and $d\Omega = -2A \wedge (\xi \lrcorner \Omega)$.

5.2 The case where $A\eta$ is parallel to $J\eta$

Let us assume again that the Killing map A is non-degenerate everywhere. We want to consider the case where $A\eta$ is parallel to $J\eta$ in more detail. We will see that, in this situation, the existence of skew Killing spinors is related to doubly warped products and to local DWP-structures. These notions and their basic properties are explained in the “appendix”.

Lemma 5.4 *Assume that M admits a skew Killing spinor with nowhere vanishing Killing map A that satisfies $A\eta = uJ\eta$ for some function u. Then $A^2\eta = -u^2\eta$. In particular, $AJ = JA$.*

Proof Note first that Eqs. (13) and (50) give

$$\begin{aligned} 0 = f(dA)(X, Y, \eta) &= f\big(g((\nabla_X A)Y, \eta) - g((\nabla_Y A)X, \eta) + g((\nabla_\eta A)X, Y)\big) \\ = fg(C(X, Y), \eta) &+ fg((\nabla_\eta A)X, Y) = fg((\nabla_\eta A)X, Y) \end{aligned}$$

for all $X, Y \in TM$. Consequently, $f\nabla_\eta A = 0$. Because of

$$\eta \lrcorner (J\eta \wedge A\eta + \eta \wedge JA\eta) = |\eta|^2 JA\eta - g(\eta, JA\eta)\eta = -u|\eta|^2\eta + u|\eta|^2\eta = 0,$$

Eq. (45) gives $(\nabla_\eta J)\eta = 0$. Now, by differentiating the equality $A\eta = uJ\eta$ in the direction of η, we get

$$\nabla_\eta A\eta = (\nabla_\eta A)\eta + A(\nabla_\eta \eta) = \eta(u)J\eta + u(\nabla_\eta J)\eta + uJ(\nabla_\eta \eta) = \eta(u)J\eta + uJ(\nabla_\eta \eta).$$

Finally, using the fact that $\nabla_\eta \eta = fA\eta$ and $f\nabla_\eta A = 0$, we get that $\eta(u) = 0$ and $f^2A^2\eta = -u^2 f^2\eta$. The latter equation implies $A^2\eta = -u^2\eta$ since $\mathrm{supp}(f) = M$. □

Let $(\hat{M}^3, \hat{g}, \hat{\eta})$ be a minimal Riemannian flow, i.e., an orientable three-dimensional Riemannian manifold together with a unit Killing vector field $\hat{\eta}$. Then, locally, $(\hat{M}, \hat{g})$ is a Riemannian submersion over a two-dimensional base manifold B. Let us fix a Hermitian structure $\hat{J}$ on $\hat{\eta}^\perp$ and put $\omega := \hat{g}(\cdot, \hat{J}\cdot)$. We define a function $\hat{\tau}$ on $\hat{M}$ which is constant along the fibres by $\hat{\nabla}_X\hat{\eta} = \hat{\tau}\cdot\hat{J}(X)$ for $X \in \hat{\eta}^\perp$. Furthermore, let $\hat{K}$ denote the Gaussian curvature of B. Now consider the metric $g_{rs} = r^2\hat{g}_{\hat{\eta}} \oplus s^2\hat{g}_{\hat{\eta}^\perp}$ on $\hat{M}$, where $\hat{g}_{\hat{\eta}}, \hat{g}_{\hat{\eta}^\perp}$ are the components of the metric $\hat{g}$ along $\mathbb{R}\hat{\eta}$ and $\hat{\eta}^\perp$, respectively and r, s are positive real parameters. Then $(\hat{M}, g_{rs}, r^{-1}\hat{\eta})$ is again a minimal Riemannian flow and we obtain new functions $\hat{\tau}$ and $\hat{K}$, say $\hat{\tau}_{rs}$ and $\hat{K}_{rs}$. These functions satisfy

$$\hat{\tau}_{rs} = rs^{-2}\hat{\tau}, \quad \hat{K}_{rs} = s^{-2}\hat{K}. \tag{61}$$

If our four-dimensional manifold M is endowed with a DWP-structure, then every three-dimensional leaf associated with this structure can be understood as a minimal Riemannian flow. In this way, we obtain functions τ and K on M.

In the first part of the next theorem we will assume that $\rho = |\eta| \notin \{0, 1/2\}$ holds everywhere, i.e., that $M = M''$. As noted in the introduction to Sect. 5, this is a reasonable assumption if A is non-degenerate everywhere.

Theorem 5.5 *Assume that M admits a skew Killing spinor such that $A\eta || J\eta$ and that $\rho = |\eta| \notin \{0, 1/2\}$ everywhere. Then $(\nu := -\rho^{-1}J\eta, \eta)$ is a local DWP-structure on M such that*

$$f\mu = \tau, \quad K = 2\mu\lambda + 2\tau^2, \tag{62}$$

for $f := \sqrt{1-4\rho^2}$, where λ and μ are the eigenvalues of the Weingarten map $W = -\nabla\nu$ on $\mathbb{R}\eta$ and $\eta^\perp \cap \nu^\perp$, respectively.

Conversely, suppose that M is simply-connected and admits a local DWP-structure (ν, η) on M such that the length ρ of η satisfies $0 < \rho < 1/2$. Moreover, assume that K and τ satisfy (62) for $f := \sqrt{1-4\rho^2}$. Then M admits a skew Killing spinor such that η is associated with ψ according to (8) and such that $A\eta || J\eta$.

Proof Assume first that M admits a skew Killing spinor such that $A\eta || J\eta$ and $0 < \rho < 1/2$ everywhere. We define a vector field ν and functions A_E and A_P by

$$\nu = -\rho^{-1}J\eta, \quad AJ\eta = -A_E\eta, \quad AJZ = -A_P Z, \; Z \in \{\eta, \nu\}^\perp.$$

Then η is a Killing vector field, see Remark 3.2. Equation (46) yields

$$\nu(\rho) = fA_E. \tag{63}$$

We want to show that (ν, η) is a DWP-structure. The next Lemma will prove all properties of such a structure except the conditions for the Weingarten map $W = -\nabla\nu$ and its eigenvalues. □

Lemma 5.6 *Assume that M admits a skew Killing spinor such that $A\eta || J\eta$ and $|\eta| \notin \{0, 1/2\}$ everywhere. Then*

1. *$\nu^\perp$ is integrable,*
2. *the vector field η has constant length on the integral manifolds of $\nu^\perp$,*
3. *the unit vector field ν is geodesic.*

Proof Take $X, Y \perp J\eta$. Using $JA = AJ$ we obtain

$$\begin{aligned} g([X,Y], J\eta) &= g(\nabla_X Y, J\eta) - g(\nabla_Y X, J\eta) \;=\; -g(Y, \nabla_X(J\eta)) + g(X, \nabla_Y(J\eta)) \\ &= -g(Y, (\nabla_X J)\eta) - g(Y, J(\nabla_X\eta)) + g(X, (\nabla_Y J)\eta) + g(X, J(\nabla_Y\eta)) \\ &= -4(f-1)^{-1} g\big(Y, \eta\lrcorner\big(J\eta\wedge AX + \eta\wedge JAX\big)\big) - fg(Y, JAX) \\ &\quad +4(f-1)^{-1} g\big(X, \eta\lrcorner\big(J\eta\wedge AY + \eta\wedge JAY\big)\big) + fg(X, JAY) \\ &= 4(f-1)^{-1}\big(-g\big(Y, \rho^2 JAX - g(\eta, JAX)\eta\big) \\ &\quad +g\big(X, \rho^2 JA(Y) - g(\eta, JAY)\eta\big)\big) \\ &= 0 \end{aligned}$$

since $JA\eta$ is a multiple of η. This proves the first claim. For $X \perp J\eta$, we have

$$Xg(\eta, \eta) = 2g(\nabla_X \eta, \eta) = 2fg(AX, \eta) = 0$$

since $A\eta || J\eta$. This shows the second assertion. The third one follows from (45), (46) and (63). □

We compute the eigenvalues of the Weingarten map $-\nabla\nu$, where we use that $\rho = |\eta|$ is constant on the integral manifolds of $\nu^\perp$:

$$\begin{aligned} -\nabla_\eta \nu &= \rho^{-1}\nabla_\eta(J\eta) \;=\; \rho^{-1}(\nabla_\eta J)(\eta) + \rho^{-1}J(\nabla_\eta \eta) \\ &= 4(f-1)^{-1}\rho^{-1}\eta \lrcorner (J\eta \wedge A\eta + \eta \wedge JA\eta) + f\rho^{-1}JA\eta \\ &= -f\rho^{-1}A_E\eta, \end{aligned} \tag{64}$$

$$\begin{aligned} -\nabla_Z \nu &= \rho^{-1}\nabla_Z(J\eta) \;=\; \rho^{-1}(\nabla_Z J)(\eta) + \rho^{-1}J(\nabla_Z \eta) \\ &= 4(f-1)^{-1}\rho^{-1}\eta \lrcorner (J\eta \wedge AZ + \eta \wedge JAZ) + f\rho^{-1}JAZ \\ &= \rho^{-1}A_P Z \end{aligned} \tag{65}$$

for $Z \in \nu^\perp \cap \eta^\perp$. Therefore

$$\lambda = -f\rho^{-1}A_E, \quad \mu = \rho^{-1}A_P \tag{66}$$

are the eigenvalues of $-\nabla\nu$. We fix a local section s in $P = \{\eta, \nu\}^\perp$ and put

$$s_1 := -\eta/\rho, \quad s_2 := Js_1 = \nu, \quad s_3 := s, \quad s_4 := Js. \tag{67}$$

Let $s^1, \ldots, s^4$ denote the dual local basis of T^*M. By (46), (64) and (65), the coefficients $\theta_{ij} := g(\nabla s_i, s_j)$ of the Levi Civita connection satisfy

$$\begin{aligned} &\theta_{12} = -f\rho^{-1}A_E\, s^1,\ \theta_{13} = f\rho^{-1}A_P\, s^4,\ \theta_{14} = -f\rho^{-1}A_P\, s^3, \\ &\theta_{23} = -\rho^{-1}A_P\, s^3, \quad \theta_{24} = -\rho^{-1}A_P\, s^4\,. \end{aligned} \tag{68}$$

This gives

$$C_P = -2\rho^{-1}\big(A_P^2 + fA_PA_E\big)s_2 - s_3(A_P)s_3 - s_4(A_P)s_4.$$

Indeed, (50) shows that $g(C_P, s_1) = 0$. Furthermore,

$$\begin{aligned} g(C_P, s_2) &= g((\nabla_{s_3}A)(s_4) - (\nabla_{s_4}A)(s_3), s_2) \\ &= s_3\big(g(As_4, s_2)\big) - g(A(\nabla_{s_3}s_4), s_2) - g(As_4, \nabla_{s_3}s_2) \\ &\quad - s_4\big(g(As_3, s_2)\big) + g(A(\nabla_{s_4}s_3), s_2) + g(As_3, \nabla_{s_4}s_2) \\ &= g(\nabla_{s_3}s_4, As_2) - g(As_4, \nabla_{s_3}s_2) - g(\nabla_{s_4}s_3, As_2) + g(As_3, \nabla_{s_4}s_2) \\ &= A_E\big(\theta_{14}(s_3) - \theta_{13}(s_4)\big) + A_P\big(\theta_{23}(s_3) + \theta_{24}(s_4)\big), \\ g(C_P, s_3) &= g((\nabla_{s_3}A)(s_4) - (\nabla_{s_4}A)(s_3), s_3) \;=\; g((\nabla_{s_3}A)(s_4), s_3) \\ &= g(\nabla_{s_3}(As_4) - A(\nabla_{s_3}s_4), s_3) \\ &= -g(\nabla_{s_3}(A_Ps_3), s_3) + g(\nabla_{s_3}s_4, A_Ps_4) \\ &= -s_3(A_P). \end{aligned}$$

Analogously, $g(C_P, s_4) = -s_4(A_P)$. Equations (68) imply

$$\begin{aligned} &g(C(J\eta, Z), J\eta) = \rho^2 g(C(s_2, Z), s_2) = \rho^2\big(s_2(g(AZ, s_2)) \\ &\quad -g(A(\nabla_{s_2}Z), s_2) - g(AZ, \nabla_{s_2}s_2)\big) = 0 \end{aligned}$$

for $Z \in \{s_3, s_4\}$ and, similarly,

$$\begin{aligned} g(C(\eta, X), J\eta) &= \rho^2\big(s_1(A_E g(X, s_1)) - A_E g(\nabla_{s_1} X, s_1) - g(AX, \nabla_{s_1} s_2) - X(A_E)\big) \\ &= \rho^2\big(s_1(A_E) g(X, s_1) + A_E g(X, \nabla_{s_1} s_1) + g(X, A(\nabla_{s_1} s_2)) - X(A_E)\big) \\ &= \rho^2\Big(s_1(A_E) g(X, s_1) - \frac{f A_E^2}{\rho} g(X, s_2) + \frac{f A_E^2}{\rho} g(X, s_2) - X(A_E)\Big) \\ &= -\rho^2 X(A_E) \end{aligned}$$

for $X \in \{s_2, s_3, s_4\}$. Furthermore, $g(C(\eta, s_1), J\eta) = 0$ since C is antisymmetric. Hence, under the assumption that (68) holds, Eqs. (47), (48) and (49) are equivalent to the system of equations

$$s_2(A_E) = 2f\rho^{-1}A_P^2 + 2f^2\rho^{-1}A_E A_P \tag{69}$$

$$s_j(A_P) = s_j(A_E) = 0, \quad j = 3, 4, \tag{70}$$

$$K_P = -2f\rho^{-2}A_E A_P - 2(\rho^{-2} - 2)\, A_P^2. \tag{71}$$

We also have

$$s_1(A_E) = s_1(A_P) = 0.$$

Indeed, (50) implies $g\big(C(s_1, s_2), \eta\big) = 0$, therefore we obtain

$$0 = g\big((\nabla_{s_1} A)s_2 - (\nabla_{s_2} A)s_1, \eta\big) = g\big((\nabla_{s_1} A)s_2, \eta\big) = g\big(\nabla_{s_1}(As_2), \eta\big) = \rho s_1(A_E),$$

which gives $s_1(A_E) = 0$. Using (69) and taking into account that $[s_1, s_2]$ is a multiple of s_1, we get

$$0 = s_1(s_2(A_E)) = 2s_1\big(f\rho^{-1}A_P^2 + f^2\rho^{-1}A_E A_P\big) = 2f\rho^{-1}\big(2A_P + fA_E\big)s_1(A_P).$$

Assume that $s_1(A_P)(x) \neq 0$ at $x \in M$. Then $s_1(A_P) \neq 0$ in an open neighbourhood U of x. But then $2A_P = -fA_E$ on U, which would imply $s_1(A_P) = 0$, a contradiction.

Hence we proved that besides ρ also A_E and A_P are constant on the integral manifolds of $\nu^\perp$. Therefore also μ and λ are constant along these leaves. Consequently, (ν, η) is a local DWP-structure on M. By (66), the associated function τ satisfies

$$\tau = \rho^{-1} g(\nabla_s \eta, Js) = \rho^{-1} g(fAs, Js) = f\mu,$$

where $s \in \{\eta, \nu\}^\perp$ is of length one. This proves the first equation in (62).

It remains to prove that also the second equation in (62) is true. Let N be an integral manifold of $\nu^\perp$. Then, locally, N is a Riemannian submersion over a base manifold B. The following lemma will relate the sectional curvature K_P in direction of $P = \mathrm{span}\{s_3, s_4\}$ to the Gaussian curvature K of B, which will almost finish the proof of the forward direction of Theorem 5.5.

Lemma 5.7 *Let (ν, η) be a local DWP-structure such that the coefficients of the Levi-Civita connection satisfy (68) with respect to an orthonormal frame $s_1 = -\eta/\rho$, $s_2 = \nu$, s_3, s_4. Then the Gaussian curvature K of B equals*

$$K = K_P + (1 + 3f^2)\rho^{-2}A_P^2.$$

Proof The second fundamental form α of $N \subset M$ satisfies

$$\alpha(s_3, s_3) = \alpha(s_4, s_4) = \rho^{-1}A_P s_2, \quad \alpha(s_3, s_4) = 0,$$

which follows from (68). Hence the Gauss equation gives

$$\begin{aligned} K_P = R(s_3, s_4, s_4, s_3) &= R^N(s_3, s_4, s_4, s_3) - g\big(\alpha(s_3, s_3), \alpha(s_4, s_4)\big) \\ &= R^N(s_3, s_4, s_4, s_3) - \rho^{-2} A_P^2. \end{aligned} \tag{72}$$

Let $\mathcal{A}$ denote the fundamental tensor used in O'Neill's formulas. We have

$$\mathcal{A}_{s_j} s_j = g(\nabla_{s_j} s_j, s_1) s_1 = 0, \quad j = 3, 4$$

and

$$\mathcal{A}_{s_3} s_4 = -\mathcal{A}_{s_4} s_3 = g(\nabla_{s_3} s_4, s_1) s_1 = -\theta_{14}(s_3) s_1 = f \rho^{-1} A_P s_1.$$

The O'Neill formula for R^N now gives

$$R^N(s_3, s_4, s_4, s_3) = K - 3|\mathcal{A}_{s_3} s_4|^2 = K - 3 f^2 \rho^{-2} A_P^2,$$

which combined with (72) implies the assertion. □

Lemma 5.7 together with (66) and (71) finally shows that B has constant curvature

$$K = -2 f \rho^{-2} A_E A_P + 2 f^2 \rho^{-2} A_P^2 = 2\mu\lambda + 2\tau^2.$$

Now let M be simply-connected and let (ν, η) be a local DWP-structure on M such that $0 < |\eta| < 1/2$ and such that (62) holds. Note that $f = \sqrt{1 - 4\rho^2}$ is smooth since $\rho = |\eta| < 1/2$. By assumption, ρ is constant on the integral leaves of $\nu^{\perp}$. We write ∂_t for the derivative in direction ν. By (78), we have $\rho' = -\lambda\rho$, which implies

$$f' = 4\lambda\rho^2 / f = \lambda(-f + 1/f). \tag{73}$$

We define the functions

$$A_E := -\lambda\rho f^{-1}, \quad A_P := \mu\rho, \tag{74}$$

which are all constant along the integral leaves of $\nu^{\perp}$. We consider a local orthonormal frame

$$s_1 := -\eta/\rho, \quad s_2 = \nu, \quad s_3, \quad s_4$$

such that s_3, s_4 is a positively oriented basis of $\{\eta, \nu\}^{\perp}$. The assumption that (ν, η) is a local DWP-structure with eigenvalues λ and μ together with the assumption $\tau = f\mu$ imply that the local coefficients of the Levi-Civita connection satisfy Eqs. (68). Indeed,

$$\nabla_{s_2} s_2 = 0, \quad \nabla_{s_1} s_2 = -\lambda s_1, \quad \nabla_{s_j} s_2 = -\mu s_j, \ j = 3, 4$$

implies

$$\begin{aligned} \theta_{12} &= -g(\nabla s_2, s_1) = -g(\nabla_{s_1} s_2, s_1) s^1 = \lambda s^1, \\ \theta_{23} &= g(\nabla s_2, s_3) = g(\nabla_{s_3} s_2, s_3) s^3 = -\mu s^3, \end{aligned}$$

and (74) gives the formulas for θ_{12} and θ_{23}. Similarly, we get θ_{24}. On span$\{s_3, s_4\}$, we fix the Hermitian structure J that maps s_3 to s_4. Recall that τ is defined by $\nabla_X \eta = \tau J X$ for all $X \in$ span$\{s_3, s_4\}$. Since η is a Killing vector field and ρ is constant along the integrals leaves, we obtain

$$\begin{aligned}\theta_{13} &= g(\nabla_{s_1}s_1, s_3)s^1 + \cdots + g(\nabla_{s_4}s_1, s_3)s^4 \\ &= -\rho^{-1}\big(g(\nabla_{s_1}\eta, s_3)s^1 + g(\nabla_{s_2}\eta, s_3)s^2 + g(\nabla_{s_4}\eta, s_3)s^4\big) \\ &= -\rho^{-1}\big(-g(s_1, \nabla_{s_3}\eta)s^1 - g(s_2, \nabla_{s_3}\eta)s^2 + g(\nabla_{s_4}\eta, s_3)s^4\big) \\ &= -g(s_1, \nabla_{s_3}s_1)s^1 - g(s_2, \nabla_{s_3}s_1)s^2 + \tau s^4 \\ &= f\mu s^4 \ = \ f\rho^{-1}A_P s^4,\end{aligned}$$

where we used the already proven equation $\theta_{12}(s_3) = 0$. Analogously, we obtain θ_{14}. Now we define skew-symmetric maps A and J by

$$\begin{gathered}A(s_1) = A_E s_2, \quad A(s_2) = -A_E s_1, \quad A(s_3) = A_P s_4, \quad A(s_4) = -A_P s_3, \\ J(s_1) = s_2, \quad J(s_2) = -s_1, \quad J(s_3) = s_4, \quad J(s_4) = -s_3.\end{gathered}$$

Note that J extends the above defined map J on $\mathrm{span}\{s_3, s_4\}$. A few lines above, we proved that (68) holds in our situation. Using this equation, we obtain

$$\begin{aligned}\nabla_{s_1}\eta &= -\rho\nabla_{s_1}s_1 \ = \ fA_E s_2 \ = \ fA(s_1), \\ \nabla_{s_2}\eta &= -\rho' s_1 - \rho\nabla_{s_2}s_1 \ = \ \lambda\rho s_1 \ = \ -fA_E s_1 \ = \ fA(s_2), \\ \nabla_{s_3}\eta &= -\rho\nabla_{s_3}s_1 \ = \ fA_P s_4 \ = \ fA(s_3), \\ \nabla_{s_4}\eta &= -\rho\nabla_{s_4}s_1 \ = \ -fA_P s_3 \ = \ fA(s_4).\end{aligned}$$

Hence, η satisfies (46). By definition of J and A, Eq. (45) is equivalent to the system of equations

$$\begin{aligned}&\nabla_\eta J = \nabla_{J\eta} J = 0, \\ &(\nabla_s J)(\eta) = (f+1)A_P s, \qquad (\nabla_s J)(J\eta) = -(f+1)A_P J(s), \\ &(\nabla_s J)(s) = 4(f-1)^{-1}A_P\eta, \ (\nabla_s J)(Js) = -4(f-1)^{-1}A_P J(\eta),\end{aligned}$$

for all $s \in \{\eta, J\eta\}^\perp$, $|s| = 1$, which indeed can be verified using (68). Finally, we prove that (47), (48) and (49) hold. We already have seen that these equations are equivalent to (69), (70) and (71). Now we use Lemma 5.7. Together with our assumption (62) and equation (74), it implies

$$K_P = 2\mu\lambda + 2\tau^2 - \rho^{-2}(1+3f^2)A_P^2 = -2f\rho^{-2}A_P A_E - \rho^{-2}(1+f^2)A_P^2,$$

which is equivalent to (71). Also (70) holds since ρ, λ and μ are constant on the leaves by assumption. It remains to prove (69). Locally, (M, g) is isometric to a doubly warped product $(I \times \hat{M}, \rho(t)^2\hat{g}_{\hat{\eta}} \oplus \sigma(t)^2\hat{g}_{\hat{\eta}^\perp})$. In particular, $\sigma' = -\mu\sigma$ by (78). Furthermore, $\tau = \hat{\tau}\rho\sigma^{-2}$ and $K = \hat{K}\sigma^{-2}$ by (61) for some constants $\hat{\tau}$ and $\hat{K}$. By assumption (62),

$$\mu f = \rho\sigma^{-2}\hat{\tau}.$$

Taking the absolute value and then the logarithm on both sides and differentiating, we obtain

$$\frac{\mu'}{\mu} + \frac{f'}{f} = \frac{\rho'}{\rho} - 2\frac{\sigma'}{\sigma} = -\lambda + 2\mu$$

and therefore, by (73),

$$\mu' = (1-f^{-2})\lambda\mu - \lambda\mu + 2\mu^2 = -f^{-2}\lambda\mu + 2\mu^2$$

holds (globally) on M. By assumption,

$$\hat{K} = K\sigma^2 = (2\mu\lambda + 2\tau^2)\sigma^2 = 2\mu(\lambda + \mu f^2)\sigma^2.$$

Differentiating, using $\sigma' = -\mu\sigma$ and dividing by $2\mu\sigma^2$ yields

$$\begin{aligned} 0 &= \frac{\mu'}{\mu}(\lambda + \mu f^2) + \lambda' + \mu' f^2 + 2\mu f f' - 2\mu\lambda - 2\mu^2 f^2 \\ &= (-\lambda f^{-2} + 2\mu)(\lambda + \mu f^2) + \lambda' + (-\lambda\mu f^{-2} + 2\mu^2) f^2 + 2(1 - f^2)\lambda\mu \\ &\quad -2\mu\lambda - 2\mu^2 f^2, \end{aligned}$$

therefore $\lambda' = \lambda^2 f^{-2} - 2 f^2(\mu^2 - \lambda\mu)$, which gives (69) by Eqs. (73) and (74). Consequently, we proved that Eqs. (45)–(49) hold. Now Proposition 5.1 shows the existence of a skew Killing spinor. □

Corollary 5.8 *Let (M, g) admit a skew Killing spinor such that $A\eta || J\eta$ and $|\eta| \notin \{0, 1/2\}$ everywhere. Then M is locally isometric to a doubly warped product $(I \times \hat{M}, dt^2 \oplus \rho(t)^2 \hat{g}_{\hat{\eta}} \oplus \sigma(t)^2 \hat{g}_{\hat{\eta}^\perp})$ for which the data $\hat{K}$ and $\hat{\tau}$ are constant and ρ and σ satisfy the differential equations*

$$(\sigma^2)' = -\frac{2}{\sqrt{1 - 4\rho^2}}\rho\hat{\tau} \tag{75}$$

$$(\sigma^2)' \frac{\rho'}{\rho} = \hat{K} - 2\frac{\rho^2}{\sigma^2}\hat{\tau}^2. \tag{76}$$

Conversely, if M is isometric to a doubly warped product $(I \times \hat{M}, dt^2 \oplus \rho(t)^2 \hat{g}_{\hat{\eta}} \oplus \sigma(t)^2 \hat{g}_{\hat{\eta}^\perp})$ for which the data $\hat{K}$ and $\hat{\tau}$ are constant and ρ and σ satisfy the differential equations (75) and (76) and if $\hat{M}$ is simply-connected, then (M, g) admits a skew Killing spinor such that $A\eta || J\eta$.

Proof The condition $\mu \cdot f = \tau$ is equivalent to $-\frac{\sigma'}{\sigma} \cdot f = \frac{\rho}{\sigma^2}\hat{\tau}$, hence to (75), and $K = 2\mu\lambda + 2\tau^2$ is equivalent to $\frac{\hat{K}}{\sigma^2} = 2\frac{\rho'}{\rho}\frac{\sigma'}{\sigma} + 2\left(\frac{\rho}{\sigma^2}\hat{\tau}\right)^2$, hence to (76). □

Remark 5.9 Locally, Eqs. (75) and (76) can be solved explicitly to get solutions σ and ρ.

Remark 5.10 Let us study the restriction of a skew Killing spinor ψ to N. The restriction $(\Sigma M)|_N$ can be understood using an isomorphism

$$\phi : (\Sigma M)|_N \longrightarrow \Sigma N \oplus \Sigma N = \phi((\Sigma^+ M)|_N) \oplus \phi((\Sigma^- M)|_N)$$

which is compatible with the Clifford multiplication in the following sense. If $\phi(\varphi) = (u, v)$, then

$$\phi(\nu \cdot \varphi) = (-v, u), \quad \phi(\nu \cdot X \cdot \varphi) = (-X \cdot_N u, X \cdot_N v),$$

where $\nu = s_2$ is a normal vector of N, X is a tangent vector of N and '$\cdot_N$' denotes the Clifford multiplication on ΣN. In particular, $s_1 s_3 s_4 \cdot_N u = u$ for all $u \in \Sigma N$. By the spinorial O'Neill formulas, we obtain

$$\nabla^N_\eta \phi(\psi^\pm) = -\frac{\lambda}{2f}\eta \cdot_N \phi(\psi^\pm), \quad \nabla^N_Z \phi(\psi^\pm) = -\frac{\mu f}{2} Z \cdot_N \phi(\psi^\pm) \tag{77}$$

for all $Z \in TN \cap \eta^\perp$. Up to rescaling, these are Sasakian quasi-Killing spinors on N, which we will explain in the following.

Up to rescaling of the metric, each integral manifold N in our construction has a Sasakian structure, see [8] for a definition of such structures. Indeed, η restricted to N is a Killing vector field of constant length and $\nabla\eta$ restricted to $\eta^\perp$ equals $|\eta|\tau J|_{\eta^\perp}$, where also τ is constant. The Nijenhuis tensor of $J|_{\eta^\perp}$ vanishes since $\eta^\perp$ is two-dimensional. Consequently, $\tilde{\xi} := \eta/(\tau|\eta|)$ is the Reeb vector field of a Sasakian structure on $(N, \tilde{g} := \tau^2 g)$. The scalar curvature of $(N, \tilde{g})$ equals $\mathrm{S} = 4\lambda/(f\tau) + 2$.

A spinor field ψ on a Sasakian manifold $(\tilde{M}, \tilde{\xi}, \tilde{g})$ with Reeb vector field $\tilde{\xi}$ is called a Sasakian quasi-Killing spinor of type (a, b) if it satisfies $\nabla_X\psi = aX \cdot \psi$ for $X \in \tilde{\xi}^\perp$ and $\nabla_{\tilde{\xi}}\psi = (a+b)\tilde{\xi} \cdot \psi$ for $a, b \in \mathbb{R}$. If $(\tilde{M}, \tilde{\xi}, \tilde{g})$ admits a Sasakian quasi-Killing spinor of type (a, b), then the scalar curvature S is constant and given by $\mathrm{S} = 8m(2m + 1)a^2 + 16mab$, see [12], Lemma 6.4. In the following sense, in three dimensions, the converse is true. Let $(\tilde{M}, \tilde{g}, \tilde{\xi})$ be a simply-connected three-dimensional Sasakian spin manifold with constant scalar curvature S. Then there exist two linear independent Sasakian quasi-Killing spinors of type $(-1/2, 3/4 - \mathrm{S}/8)$, see [12], Theorem 8.4.

We identify the spinor bundle $\tilde{\Sigma}N$ of $(N, \tilde{g})$ with ΣN by $\Sigma N \to \tilde{\Sigma}N, \varphi \mapsto \tilde{\varphi}$ such that a section φ in ΣN satisfies

$$(X \cdot_N \varphi)^\sim = \tilde{X} \cdot_N \tilde{\varphi}, \quad (\nabla^N_X \varphi)^\sim = \tilde{\nabla}^N_X \tilde{\varphi},$$

where $\tilde{X} := X/|\tau|$ and $\tilde{\nabla}^N$ denotes the Levi-Civita connection on $\tilde{\Sigma}N$. Now we consider the restriction of the skew Killing spinor $\psi = \psi^+ + \psi^-$ to N. We will write $\psi^\pm$ instead of $\phi(\psi^\pm)$. Then, by (77),

$$\tilde{\nabla}^N_\eta \tilde{\psi}^\pm = (\nabla^N_\eta \tilde{\psi}^\pm)^\sim = (-\frac{\lambda}{2f}\eta \cdot_N \psi^\pm)^\sim = -\frac{\lambda}{2f|\tau|}\eta \cdot_N (\psi^\pm)^\sim,$$
$$\tilde{\nabla}^N_Z \tilde{\psi}^\pm = (\nabla^N_Z \tilde{\psi}^\pm)^\sim = (-\frac{\mu f}{2} Z \cdot_N \psi^\pm)^\sim = -\frac{\mu f}{2|\tau|} Z \cdot_N (\psi^\pm)^\sim = -\frac{\mathrm{sgn}(\tau)}{2} Z \cdot_N (\psi^\pm)^\sim$$

for $Z \in \eta^\perp$. Hence, $\tilde{\psi}^\pm$ is a Sasakian quasi-Killing spinor with $a = -\mathrm{sgn}(\tau)/2$ and

$$b = -\frac{\lambda}{2f|\tau|} + \frac{\mathrm{sgn}(\tau)}{2} = \mathrm{sgn}(\tau)\Big(-\frac{\lambda}{2f\tau} + \frac{1}{2}\Big) = \mathrm{sgn}(\tau)\Big(\frac{3}{4} - \frac{\mathrm{S}}{8}\Big).$$

Therefore we are up to a change of orientation exactly in the situation described above.

In dimension three Sasakian quasi-Killing spinors of type $(-1/2, 3/4 - \mathrm{S}/8)$ can also be understood as transversal Killing spinors, see [13] for a definition. If we return to our original metric g on N, this means that the restrictions of $\psi^\pm$ to N are transversal Killing spinors. Indeed,

$$\begin{aligned}
\bar{\nabla}_\eta \psi^\pm &= \nabla^N_\eta \psi^\pm - \frac{1}{2}\tau|\eta|s_3 s_4 \cdot_N \psi^\pm \;=\; \nabla^N_\eta \psi^\pm + \frac{1}{2}\tau|\eta|s_1 \cdot_N \psi^\pm \\
&= \Big(-\frac{\lambda}{2f} - \frac{1}{2}\tau\Big)\eta \cdot_N \psi^\pm, \\
\bar{\nabla}_Z \psi^\pm &= \nabla^N_Z \psi^\pm - \frac{1}{2}\tau s_1 J(Z) \cdot_N \psi^\pm \;=\; -\frac{\mu f}{2} Z \cdot_N \psi^\pm + \frac{\tau}{2} Z \cdot_N \psi^\pm \;=\; 0
\end{aligned}$$

holds for the transversal covariant derivative $\bar{\nabla}$ on N.

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A Appendix: Doubly warped products

Definition A.1 A *doubly warped product* is a Riemannian manifold (M, g) of the form

$$(I \times \hat{M}, dt^2 \oplus \rho(t)^2 \hat{g}_{\hat{\eta}} \oplus \sigma(t)^2 \hat{g}_{\hat{\eta}^\perp}),$$

where $(\hat{M}, \hat{g})$ is a Riemannian manifold with unit Killing vector field $\hat{\eta}$ and $\hat{g}_{\hat{\eta}}$, $\hat{g}_{\hat{\eta}^\perp}$ are the components of the metric $\hat{g}$ along $\mathbb{R}\hat{\eta}$ and $\hat{\eta}^\perp$, respectively, $I \subset \mathbb{R}$ is an open interval and $\rho, \sigma : I \to \mathbb{R}$ are smooth positive functions on I.

Definition A.2 Let (M, g) be a Riemannian manifold. A local DWP-structure $(\nu, \hat{\eta})$ on (M, g) consists of

1. a unit geodesic vector field ν whose orthogonal complement distribution is integrable,
2. a nontrivial Killing vector field $\hat{\eta}$ on (M, g) that is pointwise orthogonal to ν and whose length is constant along any integral leaf of $\nu^\perp$

with the property that the Weingarten map $W := -\nabla\nu$ of each integral leaf of $\nu^\perp$ has two eigenspaces, $\mathbb{R}\hat{\eta}$ and $\hat{\eta}^\perp \cap \nu^\perp$ and the corresponding eigenvalues λ and μ are constant along the leaf.

In Definition A.2, DWP stands for "doubly warped product". Both notions of doubly-warped products appear to be equivalent at a local scale:

Proposition A.3 *If (M, g) is isometric to a doubly warped product, then (M, g) admits a local DWP-structure. Conversely, if (M, g) has a local DWP-structure, then it is locally isometric to a doubly-warped product.*

Proof First assume that (M, g) is isometric to a doubly warped product, thus $(M, g) = (I \times \hat{M}, dt^2 \oplus \rho(t)^2 \hat{g}_{\hat{\eta}} \oplus \sigma(t)^2 \hat{g}_{\hat{\eta}^\perp})$. Then we have the following expressions for the Levi-Civita connection ∇ of (M, g), see e.g. [15, Sect. 3] (mind that our $\hat{\eta}$ here corresponds to $\hat{\xi}$ in [15] and that our ρ and σ correspond to $\rho\sigma$ and ρ, respectively). For all sections X, Y of $\pi_2^* Q$, where $Q := \eta^{\perp\,\hat{}} \to \hat{M}$,

$$\begin{aligned}
&\nabla_{\partial_t}\partial_t = 0, \qquad \nabla_{\partial_t}\hat{\eta} = \frac{\rho'}{\rho}\hat{\eta}, \qquad \nabla_{\partial t} X = \frac{\partial X}{\partial t} + \frac{\sigma'}{\sigma} X,\\
&\nabla_{\hat{\eta}}\partial_t = \frac{\rho'}{\rho}\hat{\eta}, \quad \nabla_{\hat{\eta}}\hat{\eta} = -\rho\rho'\partial_t, \ \nabla_{\hat{\eta}} X = \hat{\nabla}_{\hat{\eta}} X + \frac{\rho^2}{\sigma^2}\hat{h}X,\\
&\nabla_X \partial_t = \frac{\sigma'}{\sigma} X, \ \nabla_X \hat{\eta} = \frac{\rho^2}{\sigma^2}\hat{h}X, \ \nabla_X Y = \hat{\nabla}_X Y - \frac{1}{\sigma^2} g(\hat{h}X, Y)\xi - \sigma' g(X, Y)\partial_t,
\end{aligned} \tag{78}$$

where $\hat{h} := \nabla^{\hat{M}}\hat{\eta} \in \Gamma(\mathrm{End}(Q))$. It is straightforward to see that $\nu := \partial_t$ is a geodesic vector field with $\nu^\perp = T\hat{M}$, the vector field $\hat{\eta}$ (seen as a section of $\pi_2^* T\hat{M} \subset TM$) is Killing on (M, g) with constant length along each $\{t\} \times \hat{M}$ and that $W := -\nabla\partial_t = -\frac{\rho'}{\rho}\mathrm{Id}_{\mathbb{R}\hat{\eta}} \oplus -\frac{\sigma'}{\sigma}\mathrm{Id}_Q$.

Conversely, let $(\nu, \hat{\eta})$ be a local DWP-structure on (M, g). Let p be a point in M. Then we find a local leaf $\hat{M}$ of $\nu^\perp$ such that the integral curves of ν starting from $\hat{M}$ are defined at least on an interval $(-t_0, t_0)$. We denote by $\hat{g}$ the induced metric on $\hat{M}$. Up to rescaling $\hat{\eta}$ by a nonzero constant, we may assume that $\hat{g}(\hat{\eta}, \hat{\eta}) = 1$ along $\hat{M}$. Consider the map $F : (-t_0, t_0) \times \hat{M} \to M$ given by $F(t, x) := F_t(x)$, where $(F_t)_t$ is the flow of the vector field ν. The map F is clearly a local diffeomorphism. Next we identify the pull-back metric F^*g on $(-t_0, t_0) \times \hat{M}$. For any given $(t, x) \in (-t_0, t_0) \times \hat{M}$ and $X \in T_x\hat{M}$, we have

$$(F^*g)_{(t,x)}(\partial_t, X) = g_{F(t,x)}(\nu, d_xF_t(X)) = g_{F(t,x)}(d_xF_t(\nu), d_xF_t(X)) = (F_t^*g)_x(\nu, X).$$

Since ν is geodesic of constant length, $(\mathcal{L}_\nu g)(\nu, Y) = g(\nabla_\nu\nu, Y) + g(\nabla_Y\nu, \nu) = 0$ holds for all $Y \in TM$. Consequently, the derivative

$$\begin{aligned}\frac{\partial}{\partial t}(F_t^*g)_x(\nu, X) &= \frac{\partial}{\partial s}(F_{t+s}^*g)_x(\nu, X)\big|_{s=0}\\ &= (\mathcal{L}_\nu g)_{F(t,x)}((F_t)_*\nu, (F_t)_*X) = (\mathcal{L}_\nu g)_{F(t,x)}(\nu, (F_t)_*X)\end{aligned}$$

vanishes, therefore $(F_t^*g)_x(\nu, X) = (F_0^*g)_x(\nu, X) = g_x(\nu, X) = 0$ for all $(t, x) \in (-t_0, t_0) \times \hat{M}$. This proves the splitting $F^*g = dt^2 \oplus g_t$, where $g_t := (F_t^*g)|_{T\hat{M}\times T\hat{M}}$. As a next step, we compute g_t more precisely along each of the distributions $\mathbb{R}\hat{\eta}$ and Q of $T\hat{M}$. We first notice that $\hat{\eta}$ is invariant under the flow of ν. Namely, we write $W = \lambda\mathrm{Id}_{\mathbb{R}\eta} \oplus \mu\mathrm{Id}_Q$ for functions $\lambda, \mu : \mathbb{R} \to \mathbb{R}$, which are constant along each integral leaf of $\nu^\perp$ by assumption. Since $\hat{\eta}$ is Killing, we have $g(\nabla_\nu\hat{\eta}, \nu) = 0$. Moreover, because of $\hat{\eta} \perp \nu$,

$$g(\nabla_\nu\hat{\eta}, \hat{\eta}) = -g(\nabla_{\hat{\eta}}\hat{\eta}, \nu) = g(\nabla_{\hat{\eta}}\nu, \hat{\eta}) = -g(W\hat{\eta}, \hat{\eta}) = -\lambda g(\hat{\eta}, \hat{\eta}).$$

Note that this proves in particular that, if $\hat{\eta}$ vanishes at a point, then it must vanish on the corresponding integral leaf of $\nu^\perp$ and therefore identically on the image of F since $g(\hat{\eta}, \hat{\eta})$ satisfies the ODE $\nu(g(\hat{\eta}, \hat{\eta})) = -2\lambda g(\hat{\eta}, \hat{\eta})$. Furthermore, for every $X \in Q$,

$$g(\nabla_\nu\hat{\eta}, X) = -g(\nabla_X\hat{\eta}, \nu) = g(\nabla_X\nu, \hat{\eta}) = -\mu g(\hat{\eta}, X) = 0.$$

As a first consequence, $\nabla_\nu\hat{\eta} = -\lambda\hat{\eta}$. This implies $\mathcal{L}_\nu\hat{\eta} = [\nu, \hat{\eta}] = \nabla_\nu\hat{\eta} - \nabla_{\hat{\eta}}\nu = 0$, so that $(F_t)_*\hat{\eta} = \hat{\eta}$ for every $t \in \mathbb{R}$. For any $X, Y \in \nu^\perp$, we have

$$(\mathcal{L}_\nu g)(X, Y) = g(\nabla_X\nu, Y) + g(\nabla_Y\nu, X) = -2g(WX, Y).$$

Therefore, in particular,

$$\begin{aligned}\frac{\partial}{\partial s}(F_s^*g)(\hat{\eta}, Y)\big|_{s=t} &= (\mathcal{L}_\nu g)_{F(t,x)}((F_t)_*\hat{\eta}, (F_t)_*Y) = (\mathcal{L}_\nu g)_{F(t,x)}(\hat{\eta}, (F_t)_*Y)\\ &= -2g(W\hat{\eta}, (F_t)_*Y) = -2(\lambda \circ F) \cdot g(\hat{\eta}, (F_t)_*Y)\\ &= -2(\lambda \circ F)(F_t^*g)(\hat{\eta}, Y). \end{aligned} \tag{79}$$

Consequently, for fixed $Y \in T_x\hat{M} \cap \nu^\perp$, the function $\varphi(t) := (F_t)^*g(\hat{\eta}, Y)$ satisfies the differential equation

$$\varphi'(t) = -2\lambda(F(t, x)) \cdot \varphi(t).$$

For $Y \in Q$, we have $\varphi(0) = 0$, hence $\varphi = 0$. This means that the flow of ν preserves the distribution $\hat{\eta^{\perp}}$. For $Y = \hat{\eta}$, we have $\varphi(0) = 1$, therefore

$$(F_t)^* g(\hat{\eta}, \hat{\eta}) = \varphi(t) = \exp(-2 \int_0^t \lambda \circ F_s ds) =: \rho(t)^2.$$

Finally, for $X, Y \in Q$, a computation analogous to (79) shows that

$$\frac{\partial}{\partial s}(F_s^* g)(X, Y)\Big|_{s=t} = -2(\mu \circ F)(F_t^* g)(X, Y),$$

which yields

$$(F_t^* g)(X, Y) = \exp(-2 \int_0^t \mu \circ F_s ds) \cdot \hat{g}(X, Y) =: \sigma(t)^2 \hat{g}(X, Y).$$

It remains to notice that $\hat{\eta}$ must be a Killing vector field along $(\hat{M}, \hat{g})$ since it is already Killing on (M, g) and is tangent to $\hat{M}$. On the whole, we obtain the doubly warped product metric as required. □

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