



Global properties of toric nearly Kähler manifolds

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Abstract

We study toric nearly Kähler manifolds, extending the work of Moroianu and Nagy. We give a description of the global geometry using multi-moment maps. We then investigate polynomial and radial solutions to the toric nearly Kähler equation.

Keywords Nearly Kähler manifolds · Moment maps · Monge–Ampère equation

Mathematics Subject Classification 53C26 · 53D20

1 Introduction

A nearly Kähler manifold is an almost Hermitian manifold (M, g, J) such that ∇J is skew symmetric: $(\nabla_X J)X = 0$ for every vector field X on M . Each of these can be decomposed as a Riemannian product of nearly Kähler manifolds which are either Kähler, 6-dimensional, homogeneous, or twistor spaces over quaternionic Kähler manifolds of positive scalar curvature [14]. We will focus on the case of 6-dimensional nearly Kähler manifolds that are *strict* in the sense that they are not Kähler. These are characterized by being the links of metric cones with holonomy G_2 , which makes them Einstein with positive scalar curvature [2].

A main challenge is to construct complete examples of 6-dimensional strictly nearly Kähler manifolds (which will be referred to simply as nearly Kähler manifolds in the rest of the paper). There are homogeneous nearly Kähler structures on S^6 , $S^3 \times S^3$, $\mathbb{C}P^3$, and the flag manifold $SU(3)/T^2$. In [3], these are shown to be the only homogeneous examples. In [6], cohomogeneity one examples are constructed on S^6 and $S^3 \times S^3$. No other complete examples are known. The cohomogeneity two case has been studied in [9], which shows that the infinitesimal symmetry group must be $\mathfrak{u}(2)$.

We will skip to cohomogeneity three in exchange for having an abelian symmetry group by studying nearly Kähler manifolds which are toric in the sense that the automorphism group contains a 3-torus. The homogeneous nearly Kähler structure on $S^3 \times S^3$ is the only known example. The general case has been studied in [12], where the local theory is shown to be equivalent to a Monge–Ampère type equation which

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we will refer to as the toric nearly Kähler equation. The current paper represents the author's efforts to build on this work. One of the main ingredients is the idea of a multi-moment map, which was introduced in [11] as a multi-symplectic generalization of the moment map associated to a symplectic form. A toric nearly Kähler manifold (M, g, J, T) is equipped with two multi-moment maps,

$$\mu : M \rightarrow \Lambda^2 \mathfrak{t} \cong \mathbb{R}^3, \quad \varepsilon : M \rightarrow \Lambda^3 \mathfrak{t} \cong \mathbb{R},$$

which are defined in Sect. 2, where \mathfrak{t} is the Lie algebra of the torus T acting on M . A foundational result in (symplectic) toric geometry [1, 7] states that the moment map of a compact toric manifold has connected fibres, and its image is a convex polytope. The first condition implies that the moment map induces a homeomorphism from the space of orbits onto its image. We prove a nearly Kähler analogue of this:

Theorem 1 *Let M be a complete toric nearly Kähler manifold with the action of a torus T^3 with Lie algebra \mathfrak{t} . Then the multi-moment maps induce a homeomorphism*

$$(\bar{\mu}, \bar{\varepsilon}) : M/T \cong S^3.$$

The T action is free away from a finite number of orbits in the equator $\varepsilon^{-1}(0)$. Moreover the two orbits in $\mu^{-1}(0)$ are Lagrangian.

This theorem generalizes previous work by the author in [5], which describes multi-moment maps of the homogeneous nearly Kähler structure on $S^3 \times S^3$. In that case, $\mu(\varepsilon^{-1}(0))$ is Cayley's nodal cubic surface, whose 4 nodal singular points correspond to the singular T orbits. By studying the topological consequences of this theorem, we prove the following:

Corollary 1 *Any complete toric nearly Kähler manifold has at least 4 torus orbits where the action is not free.*

As a consequence of this, radial solutions to the toric nearly Kähler equation cannot give complete metrics. By studying the corresponding ODE, we see that the singularity that forms must occur at the Lagrangian orbit.

We also study the case when a hypothetical solution to the toric nearly Kähler equation is polynomial in the natural multi-moment map coordinates. The homogeneous nearly Kähler structure on $S^3 \times S^3$ corresponds to a cubic solution φ_0 shown in Equation (4). Using an old theorem of Hesse [8], we prove:

Theorem 2 *Every polynomial solution of the toric nearly Kähler equation with degree at most 5 is equivalent to the cubic solution φ_0 up to coordinate transformation.*

The toric nearly Kähler equation restricted to the space of polynomials is overdetermined for polynomials of degree greater than three, so it is unlikely that there will be other polynomial solutions. However, to show this explicitly is computationally difficult even in the quintic case.

2 Local theory

In this section we review the local theory of toric nearly Kähler manifolds from [12], although we will use a coordinate invariant treatment in order to make clear the invariance properties of the expressions.

First we introduce $SU(3)$ structures, which are a convenient framework for studying nearly Kähler manifolds:

Definition 1 An $SU(3)$ structure $(\omega, \psi = \psi^+ + i\psi^-)$ on a 6-manifold M is a pair of forms $\omega \in \Omega^2(M)$ and $\psi \in \Omega^3(M, \mathbb{C})$ satisfying

$$2\omega^3 = 3\psi^+ \wedge \psi^-, \quad \omega \wedge \psi = 0.$$

We will refer to these equations as the $SU(3)$ structure equations.

Theorem 3 ([4]) A nearly Kähler structure is equivalent to an $SU(3)$ structure (ω, ψ^+) satisfying

$$d\omega = 3\psi^+, \quad d\psi^- = -2\omega \wedge \omega.$$

We will refer to these equations as the nearly Kähler structure equations.

Definition 2 A toric nearly Kähler manifold (M, ω, ψ, T) is a 6-manifold M equipped with a nearly Kähler structure $SU(3)$ structure $(\omega, \psi = \psi^+ + i\psi^-)$ which is invariant under the effective action of a 3-torus T . We will denote by (g, J) the associated Hermitian structure.

Let $K : \mathfrak{t} \rightarrow \Gamma(TM)$ be the linear map which sends elements of the Lie algebra \mathfrak{t} of T to the vector field generating the corresponding action. Let \mathfrak{t}_M denote the image of K . The exact forms $d\omega$ and $d\psi^-$ are T -invariant, so by [11] admit natural multi-moment maps

$$\mu := \omega \circ (\Lambda^2 K) : M \rightarrow \Lambda^2 \mathfrak{t}^*, \quad \varepsilon := \psi^- \circ (\Lambda^3 K) : M \rightarrow \Lambda^3 \mathfrak{t}^*.$$

Let $\mathring{M} := M \setminus \varepsilon^{-1}(0)$. By the definition of ε , K is injective on \mathring{M} . Thus we can define $\theta \in \Omega^1(\mathring{M}, \mathfrak{t})$ such that $\theta \circ K = \text{Id}$ and $\theta|_{\mathfrak{t}_M} = 0$. Define

$$\gamma := J\theta = -\theta \circ J \in \Omega^1(\mathring{M}, \mathfrak{t}).$$

Since ψ is a $(3, 0)$ -form, \mathring{M} is also described as the set of points where the image \mathfrak{t}_M of K intersects transversally with $J\mathfrak{t}_M$. This allows us to write $T\mathring{M} = \mathfrak{t}_M \oplus J\mathfrak{t}_M$ with frame $(\theta, \gamma) \in \Omega^1(\mathring{M}, \mathfrak{t} \oplus \mathfrak{t})$.

Since $\theta + i\gamma$ gives a framing of $\Lambda^{(1,0)}\mathring{M}$, we can write $\psi = i\varepsilon\Lambda^3(\theta + i\gamma)$, so that

$$\psi^+ = \varepsilon(\Lambda^3\gamma - \gamma \wedge \Lambda^2\theta), \quad \psi^- = \varepsilon(\Lambda^3\theta - \theta \wedge \Lambda^2\gamma).$$

Similarly, the rest of the structures can be given in terms of the multi-moment maps (μ, ε) , the frame (θ, γ) , and a matrix

$$C = g \circ (\odot^2 K) : M \rightarrow \text{Sym}^2 \mathfrak{t}^*.$$

For example,

$$\omega = \mu(\Lambda^2\theta + \Lambda^2\gamma) + \theta \wedge \mathbf{c},$$

where $\mathbf{c} = \gamma \lrcorner C \in \Omega^1(\overset{\circ}{M}, \mathfrak{t}^*)$.

Lemma 1 ([12]) *Using this framework, the $SU(3)$ structure equations are equivalent to*

$$\det C = \varepsilon^2 + C(V, V), \tag{1}$$

while the nearly Kähler structure equations are equivalent to

$$d(\varepsilon\mathbf{c}) = 0, \quad \frac{1}{4}\varepsilon d\theta = \mathbf{c} \wedge \mathbf{c} - \mu\mu \cdot (\gamma \wedge \gamma),$$

where $V \in \Gamma(\overset{\circ}{M}, \Lambda^3\mathfrak{t}^* \otimes \mathfrak{t})$ is the element corresponding to μ via the natural isomorphism $\natural : \Lambda^3\mathfrak{t}^* \otimes \mathfrak{t} \cong \Lambda^2\mathfrak{t}^*$.

Here, by $\det C \in \Gamma(\overset{\circ}{M}, (\Lambda^3\mathfrak{t}^*)^{\otimes 2})$, we mean the square of the volume form on \mathfrak{t} induced by C . This agrees with the usual determinant in coordinates.

Since the functions ε , V , and C are T -invariant, they descend to $\overset{\circ}{M}/T$, which can be locally identified with $\Lambda^2\mathfrak{t}^*$ via μ . Since we can think of μ as giving coordinates on $\Lambda^2\mathfrak{t}^*$, we can think of ε , V , and C as functions locally given in these coordinates on some $U \subset \Lambda^2\mathfrak{t}^*$.

These coordinates allow explicit computations of several expressions in terms of a potential function:

Theorem 4 ([12]) *There exists a function $\varphi : U \rightarrow (\Lambda^3\mathfrak{t}^*)^2$ whose Hessian in μ coordinates is C . We also have*

$$\varepsilon^2 = \frac{8}{3}(1 - \partial_r)\varphi, \quad C(V, V) = (\partial_r^2 - \partial_r)\varphi,$$

where ∂_r is the Euler vector field for $\Lambda^2\mathfrak{t}^*$ (so that in coordinates $\partial_r = \mu^i\partial_{\mu_i}$).

Combining this with Equation (1) gives the Monge–Ampère type equation

$$\det \circ \text{Hess } \varphi = \left(\frac{8}{3} - \frac{11}{3}\partial_r + \partial_r^2 \right) \varphi, \tag{\star}$$

which we will refer to as the *toric nearly Kähler equation* or just (\star) .

Note that with respect to the frame (θ, γ) , g is represented by the matrix

$$D := \begin{pmatrix} \text{Hess } \varphi & -\mu \\ \mu & \text{Hess } \varphi \end{pmatrix} \in \Gamma(U, \text{Sym}^2(\mathfrak{t} \oplus \mathfrak{t}^*)),$$

where $\mu \in \Gamma(U, \Lambda^2\mathfrak{t}^*)$ is the inclusion (identity) map.

The above theorem has a partial converse:

Theorem 5 ([12]) *Every solution of the toric nearly Kähler equation on some open set U of $\Lambda^2\mathfrak{t}^*$ defines in a canonical way a nearly Kähler structure with 3 linearly independent commuting infinitesimal automorphisms on $U_0 \times T^3$, where*

$$U_0 = \{x \in U : (1 - \partial_r)\varphi > 0 \text{ and } D \text{ is positive definite}\}.$$

Note that if φ is given by a toric nearly Kähler structure, then $(1 - \partial_r)\varphi$ is proportional to the ε^2 , and D is the expression of g in the frame (θ, γ) . Now consider the following set with an a priori weaker constraint than U_0 :

$$\hat{U}_0 := \{x \in U : (1 - \partial_r)\varphi > 0 \text{ and Hess } \varphi \text{ is positive definite}\}.$$

However, this constraint is not weaker:

Lemma 2 $U_0 = \hat{U}_0$.

Proof Since D being positive definite implies that Hess φ is positive definite, we find that $U_0 \subseteq \hat{U}_0$. It remains to show that D has no null vectors in \hat{U}_0 , which implies the reverse inclusion.

Let $C = \text{Hess } \varphi$ and $\varepsilon^2 = \frac{8}{3}(1 - \partial_r)\varphi$. Defining $j = C^{-1}\mu$, we find that any null vector for D is of the form $(v, w) \in \mathfrak{t} \oplus \mathfrak{t}$ at some point $p \in \hat{U}_0$ with

$$jw = v, \quad jv = -w.$$

Thus v and w are eigenvectors of j^2 at p with eigenvalue -1 . Thus it suffices to show that $j^2 \in \Gamma(U_0, \text{End}(\mathfrak{t}))$ never attains an eigenvalue -1 .

Choosing a basis for \mathfrak{t} so that C_p is diagonal at any chosen $p \in \hat{U}_0$ allows one to verify that $C^{-1}\mu C^{-1} = \left(\frac{CV}{\det C}\right)^\natural$, where we abuse notation by using \natural to also denote the isomorphism $\Lambda^3\mathfrak{t} \otimes \mathfrak{t}^* \cong \Lambda^2\mathfrak{t}$. Then

$$j^2 = C^{-1}\mu C^{-1}\mu = \left(\frac{CV}{\det C}\right)^\natural V^\natural = \frac{-C(V, V)\text{Id} + (CV) \otimes V}{\det C},$$

where throughout this computation we’ve been using juxtaposition to denote ‘matrix multiplication’, or contraction of a single $\mathfrak{t}, \mathfrak{t}^*$ index pair. Since $V \in \mu^{-1}(0) \subseteq j^{-1}(0)$, we find that j^2 has eigenvalues 0 with multiplicity 1 and $-\frac{C(V, V)}{\det C}$ with multiplicity 2. By the toric nearly Kähler equation, -1 is an eigenvalue only when $\varepsilon = 0$, which is impossible on \hat{U}_0 by definition. □

This lemma can be used to interpret what goes wrong when trying to find a completion of a local toric nearly Kähler manifold. If some connected \bar{M} is maximal in the sense that it is not properly contained in a toric nearly Kähler manifold where ε doesn’t vanish, what is happening at the boundary? Using μ , we can interpret this boundary as a set of points in $\Lambda^2\mathfrak{t}^*$. By (1), C is going to remain positive definite as long as ε doesn’t vanish. Thus the previous lemma shows that if ε does not limit to 0 at the boundary point, then the local solution φ to the toric nearly Kähler equation cannot be extended to the boundary point. In Sect. 6, we show that local radial solutions can be extended to have the radius defined between 0 and some finite r_0 . The differential equation is singular at 0, while ε vanishes when the radius is r_0 .

3 Relation to toric G_2 manifolds

For a strict nearly Kähler manifold $(M, \omega, \psi^+ + i\psi^-)$ with metric g , consider the Riemannian cone $(N = M \times (0, \infty), g_N = r^2g + dr^2)$, where $r \in (0, \infty)$ is the radial coordinate. It is well known that N admits a parallel G_2 structure given by

$$\phi := d\left(\frac{r^3\omega}{3}\right), \quad * \phi := -d\left(\frac{r^4\psi^-}{4}\right).$$

If M is toric, then the torus action lifts to a multi-Hamiltonian action on N with respect to the forms ϕ and $* \phi$. This makes N a toric G_2 manifold as studied in [10]. The corresponding multi-moment maps for ϕ and $* \phi$, respectively, are

$$\nu_N := \frac{1}{3}r^3\mu : N \rightarrow \Lambda^2\mathfrak{t}^*, \quad \varepsilon_N := -\frac{1}{4}r^4\varepsilon : N \rightarrow \Lambda^3\mathfrak{t}^*. \tag{2}$$

From [10], $\nu_N \oplus \varepsilon_N$ maps the set of singular orbits S of N to a graph in $\Lambda^2\mathfrak{t}^* \oplus \Lambda^3\mathfrak{t}^* \cong \mathbb{R}^3 \oplus \mathbb{R}$. Moreover, ε_N is constant on each connected component of S . In the case when $N = M \times (0, \infty)$ is the cone over a toric nearly Kähler manifold, then the radial symmetries of (2) imply that ε_N vanishes on the graph, and moreover each edge of the graph is a radial ray shining out from the origin in $\Lambda^2\mathfrak{t}^*$. Since points on the edge of the graph correspond to torus orbits where a single circle collapses, we immediately find

Corollary 2 *On a toric nearly Kähler manifold, the torus action is free away from a finite set of orbits where a single circle collapses and ε vanishes.*

4 Global properties

Let (M, ω, ψ, T) be a connected complete toric nearly Kähler 6-manifold. In this section we will prove the properties of the multi-moment maps claimed in Theorem 1. Recall that we define $\mathring{M} = M \setminus \varepsilon^{-1}(0)$.

Lemma 3 $\mu|_{\mathring{M}}$ is a submersion.

Proof Lemma 4.1(i) in [12] gives $d\mu|_{\mathring{M}} = -4\varepsilon \cdot \gamma$. The result follows since γ has full rank and ε does not vanish on \mathring{M} . □

Now we can show that $\mu(\mathring{M})$ is star-shaped around 0:

Lemma 4 Every $p \in \mathring{M}$ is contained in some path ℓ on \mathring{M} such that $\mu|_{\ell} : \ell \rightarrow \Lambda^2\mathfrak{t}^*$ is an injective map whose image is a line segment between 0 and $\mu(p)$.

Proof If $\mu(p) = 0$, there is nothing to show. Otherwise, there exists some maximal connected subset L of the line segment $\overline{0\mu(p)}$ which lifts to a path ℓ on \mathring{M} through p . By Lemma 3, L is non-empty and open. If L is closed, then $L = \overline{0\mu(p)}$ as required. Otherwise, since M is complete, ℓ has a limiting point $p' \in M \setminus \mathring{M}$. Hence L is a radial line segment travelling outward from $\mu(p')$, a point at which ε vanishes by the definitions of p' and \mathring{M} . However, using Theorem 4, we compute the radial directional derivative

$$\partial_r(\varepsilon^2) = \frac{8}{3}\partial_r(1 - \partial_r)\varphi = -\frac{8}{3}C(V, V) \leq 0.$$

This implies that ε^2 is non-positive along L , contradicting $\varepsilon \neq 0$ at $p \in \mathring{M}$. □

Since μ and ε are T -invariant, they induce maps $\bar{\mu} : M/T \rightarrow \Lambda^2\mathfrak{t}^*$ and $\bar{\varepsilon} : M/T \rightarrow \Lambda^3\mathfrak{t}^*$, which are called *orbital multi-moment maps*.

Lemma 5 *For any connected component M_0 of \mathring{M} , $\bar{\mu}|_{M_0/T}$ is injective.*

Proof For any $p \in M_0$, let ℓ be the path between p and some $p' \in \mu^{-1}(0)$ guaranteed by the previous lemma. The map

$$F : M_0/T \rightarrow \bar{\mu}^{-1}(0) : Tp \mapsto Tp'$$

is clearly well defined and continuous. Since M_0/T is connected and $\bar{\mu}^{-1}(0)$ is discrete, the image of F is a single orbit which we will denote by $o_0 \in M_0/T$.

Now let $o_1, o_2 \in M_0/T$ with $\bar{\mu}(o_1) = \bar{\mu}(o_2)$. For each $i \in \{1, 2\}$, the previous lemma can be used to construct a path ℓ_i between o_0 and o_i in M_0/T which is a lift of the line segment L between 0 and $\bar{\mu}(o_1) = \bar{\mu}(o_2)$. Since $\mu|_{\mathring{M}}$ is a submersion, $\bar{\mu}|_{\mathring{M}/T}$ is a local homeomorphism. In particular, $T_{o_0}\ell_1 = \bar{\mu}^{-1}(T_0L) = T_{o_0}\ell_2$, implying that $\ell_1 = \ell_2$. Thus $o_1 = o_2$ as required. \square

By this lemma, $\bar{\mu}$ gives global coordinates on M_0/T . This allows us to view the multi-moment image $(\mu, \varepsilon)(M_0)$ as the graph of $\bar{\varepsilon}|_{M_0/T}$. Since ε vanishes on ∂M_0 , we find that M/T is recovered by gluing together two of these graphs. We will first need a lemma:

Lemma 6 *Along the set $\{\varepsilon = 0\}$, $d\varepsilon$ only vanishes on a finite set of orbits.*

Proof Let $p \in \varepsilon^{-1}(0)$, so that $\psi^-|_{\Lambda^3\mathfrak{t}_M}$ vanishes at p . Note that by T invariance of the closed form $d\mu = (\Lambda^2K)_\lrcorner(3\psi^+)$, we have $\psi^+|_{\Lambda^3\mathfrak{t}_M} = 0$. Thus $\psi|_{\Lambda^3\mathfrak{t}_M}$ vanishes at p . Since ψ is a complex volume form, this implies that $\Lambda^3\mathfrak{t}_M|_p \subseteq T_p^{1,1}M \wedge T_pM$. Thus for any $0 \neq \Xi \in \Lambda^3\mathfrak{t}$, we can write $\Xi_M|_p = P \wedge X \in T_p^{1,1}M \wedge T_pM$. We compute

$$d\varepsilon(\Xi)_p = \Xi_M|_p \lrcorner d\varphi_p^- = (P \wedge X)_\lrcorner(-2\omega \wedge \omega) = -2\omega(P)(X)_\lrcorner\omega.$$

By the non-degeneracy of ω , this only vanishes when Ξ_M does, which are the points where the torus action is not free. By Corollary 2, this vanishing is at a finite set of orbits. \square

Theorem 6 $(\bar{\mu}, \bar{\varepsilon}) : M/T \rightarrow \Lambda^2\mathfrak{t}^* \oplus \Lambda^3\mathfrak{t}^*$ is injective with image a 3-sphere. Moreover, the component of $(\Lambda^2\mathfrak{t}^* \oplus \Lambda^3\mathfrak{t}^*) \setminus (\mu, \varepsilon)(M)$ containing 0 is star-shaped about 0 .

Proof Let M_+ be a connected component of \mathring{M} . By Lemma 4, $U := \mu(M_+)$ is star shaped around the origin. In particular, ∂M_+ has one connected component. Since $\varepsilon \neq 0$ on \mathring{M} , ε has a sign on M_+ . Since ε vanishes on the 2-dimensional $\partial M_+ \subseteq \partial \mathring{M}$, the previous lemma implies that ε changes sign when crossing this boundary. Thus there is some other connected component M_- of \mathring{M} with the opposite sign of ε and $\partial M_+ \subseteq \partial M_-$. But $\mu(M_-)$ must also be star shaped around 0 with boundary ∂U , so $\mu(M_-) = U = \mu(M_+)$. Thus $\partial M_- = \partial M_+$, so that we have $M = \overline{M_+} \cup \overline{M_-}$. In particular, $\bar{\mu} : M \rightarrow U$ is a double cover ramified over ∂U , with the sign of $\bar{\varepsilon}$ distinguishing the points in each $\bar{\mu}$ fibre. Thus $(\bar{\mu}, \bar{\varepsilon})$ is injective. Since U is diffeomorphic to a 3-ball, the image of $(\bar{\mu}, \bar{\varepsilon})$ is a 3-sphere.

By relabelling if necessary, we can assume that $\pm\varepsilon$ is positive on M_\pm . The component of $(\Lambda^2\mathfrak{t}^* \oplus \Lambda^3\mathfrak{t}^*) \setminus (\mu, \varepsilon)(M)$ containing 0 can be written as $D_+ \cup D_-$, where

$$D_{\pm} = \left\{ (x, y) \in \Lambda^2 \mathfrak{t}^* \oplus \Lambda^3 \mathfrak{t}^* : x \in \bar{\mu}(\overline{M_{\pm}}), \pm y \in [0, \pm \varepsilon(y)] \right\}.$$

Now each D_{\pm} is star-shaped around 0, since $\bar{\mu}(\overline{M_{\pm}})$ is and $\pm \varepsilon$ is decreasing in radial directions. Thus $D_+ \cup D_-$ is star-shaped around 0 as required. □

We can now wrap up the proof of the main theorem:

Proof of Theorem 1 Corollary 2 combined with Theorem 6 gives most of the claim. T orbits in $\mu^{-1}(0)$ must be Lagrangian by definition, and there are two of them, since $\bar{\mu}$ is a double cover ramified at $\varepsilon^{-1}(0)$. □

5 Some topology

We apply the results from the previous section to prove Corollary 1. The obstruction we use to prove this comes from Myers’ theorem [13], which asserts that if a complete Riemannian manifold has Ricci curvature positive and bounded away from zero, then the diameter must be bounded. Since the same must be true for the universal cover, the fundamental group must be finite. In particular, the first Betti number must vanish.

Proposition 1 *Let (M, ω, ψ, T) be a connected complete toric nearly Kähler 6-manifold. Then the action of T is not free.*

Proof Assume that the action of T is free, so that M is a T^3 bundle over S^3 . It follows that we have the Wang long exact sequence [16]

$$H_*(T^3) \rightarrow H_*(M) \rightarrow H_{*-3}(T^3) \xrightarrow{[-1]},$$

which shows that $H_1(M) \cong H_1(T^3) \cong \mathbb{Z}^3$ has positive rank. This contradicts Myers’ theorem. □

Consider the set $S \in M/T$ of T orbits where the action is not free. By Corollary 2, each orbit $s \in S$ has a one-dimensional isotropy group whose Lie algebra is given by a line $\mathfrak{t}(s) \in \mathbb{P}\mathfrak{t}$. We will need the following lemma:

Lemma 7 *The map $\iota : S \rightarrow \mathbb{P}\mathfrak{t}$ is at most two to one.*

Proof Since C is the metric on the torus orbits, S is the vanishing locus of $\det C$. Thus (1) implies that the non-negative functions ε^2 and $C(V, V)$ both vanish on S . Thus $S \subset \partial \dot{M}/T$, and for any $s \in S$, $\mathfrak{t}(s) = (\mathbb{P}V)(s)$, where $\mathbb{P}V \in \Gamma(M/T, \mathbb{P}\mathfrak{t})$ is the projectivication of V . Note that the identification $\Lambda^2 \mathfrak{t}^* \cong \Lambda^3 \mathfrak{t}^* \otimes \mathfrak{t}$ induces an identification $\mathbb{P}\Lambda^2 \mathfrak{t}^* \cong \mathbb{P}\mathfrak{t}$ under which $\mathbb{P}\mu$ is identified with $\mathbb{P}V$. But by Theorem 6, $\bar{\mu}$ is injective on $\partial \dot{M}/T \supset S$. Thus it suffices to show that $\bar{\mu}(S) \subset \mu(\partial \dot{M})$ intersects the line $\mathbb{P}\mu(s)$ at most two times. But $\mu(M)$ is star-shaped around 0, so $\mu(\partial \dot{M})$ intersects the line $\mathbb{P}\mu(s)$ at two antipodal points. □

Consider the case where S has k orbits. We will consider decompositions $M = A \cup B$, where A and B are both unions of T -orbits and $A \cap B \cap S = \emptyset$. To see that such a decomposition exists, note that if H is a hyperplane in $\Lambda^2 \mathfrak{t}^* \oplus \Lambda^3 \mathfrak{t}^*$ disjoint from $(\mu, \varepsilon)(S)$, then

there exists a neighbourhood U of H also disjoint from $(\mu, \epsilon)(S)$. Now it is clear that we can find A and B as claimed with $(\mu, \epsilon)(A \cap B) = U$. Moreover, we see that no two orbits in $S \cap A$ (respectively $S \cap B$) correspond to the same element of \mathfrak{t} , since by the previous lemma, they would correspond to antipodal points in $\mu(\epsilon^{-1}(0))$, which are avoided by this construction.

Now we have $A \cong D_i$ and $B \cong D_j$ where $i + j = k$ and D_i is a T^3 fibration over the three-ball D^3 with i orbits where circles collapse. Moreover, these circles are different, in the sense that the collapsing directions correspond to different vectors in \mathfrak{t} . Using \simeq to denote homotopy equivalence, we will compute these for the first few D_i :

Lemma 8 $D_0 \simeq T^3, D_1 \simeq T^2, D_2 \simeq S^3 \times S^1$.

Proof Since every bundle over D^3 is trivial, $D_0 \cong T^3 \times D^3 \simeq T^3$.

D_1 is a neighbourhood of the collapsed orbit. Thus $D_1 \cong T^2 \times \mathbb{R}^4$ by identifying \mathbb{R}^4 with D_1/T and T^2 the quotient of T with the circle that collapses.

D_2/T is a neighbourhood of a curve C connecting the two collapsing orbits. Thus D_2 retracts to some \tilde{D}_2 such that $\tilde{D}_2/T \cong C$. Since the circles that collapse are different, we can write $\tilde{D}_2 \cong (S^1 \times S^1 \times [0, 1])/\sim$, where \sim collapses the first circle at 0 and the second circle at 1. Now

$$\tilde{D}_2 \rightarrow \mathbb{C}^2 \times S^1 : [\theta_1, \theta_2, \theta_3, x] \mapsto \left(\sin\left(\frac{\pi}{2}x\right)e^{i\theta_1}, \cos\left(\frac{\pi}{2}x\right)e^{i\theta_2}, \theta_3 \right)$$

identifies $\tilde{D}_2 \cong S^3 \times S^1$. □

Before we proceed to applying Mayer–Vietoris to the decomposition $M = A \cup B$, we still need to understand the equatorial region $E := A \cap B$.

Lemma 9 $h_1(E) = h_4(E) \in [2, 3]$.

Proof E must be a T^3 bundle over $U \simeq S^2$. Thus E retracts to a T^3 bundle \bar{E} over S^2 . Since \bar{E} is compact, we have the duality $h_1(\bar{E}) = h_4(\bar{E})$. Part of the Wang sequence is

$$\rightarrow H_{2-2}(T^3) \rightarrow H_1(T^3) \rightarrow H_1(\bar{E}) \rightarrow 0.$$

Thus $h_1(E) = h_1(\bar{E}) \in h_1(T^3) - [0, h_0(T^3)] = 3 - [0, 1] = [2, 3]$. □

We can now work with the Mayer–Vietoris sequence with respect to the decomposition $M = A \cup B = D_i \cup D_j$. Since $A \cap B = E$, this sequence is

$$H_*(E) \rightarrow H_*(D_i) \oplus H_*(D_j) \rightarrow H_*(M) \xrightarrow{[-1]} .$$

We are now ready to prove the main result of this section:

Proof of Corollary 1 The Mayer–Vietoris sequence at $\bullet = 1$ gives

$$h_1(D_i) + h_1(D_j) \leq h_1(E) + h_1(M) \leq 3,$$

where the second inequality uses the previous lemma and Myers’ theorem. But by Lemma 8, $h_1(D_i) = 3 - i$ or $i \in \{0, 1, 2\}$. In particular, for $k = i + j < 3$, we have $h_1(D_i) + h_1(D_j) = 6 - (i + j) > 3$, contradicting our upper bound.

For $k = 3$, choose $A \cong D_1$ and $B \cong D_2$. Since $h_1(M) = 0$, the Mayer–Vietoris sequence at $\bullet = 1$ gives $h_1(E) \geq h_1(D_1) + h_1(D_2) = 3$. Combining this with the previous lemma gives $h_1(E) = h_4(E) = 3$. Since $h_5(M) = h_1(M) = 0$, the Mayer–Vietoris sequence at $\bullet = 4$ gives the contradiction

$$3 = h_4(E) \leq h_4(D_1) + h_4(D_2) = 1.$$

□

6 Radial solutions

In this section, we study solutions of the form $\varphi(\mu) = x(t)$, where $t = \frac{1}{2}\|\mu\|^2$ is a radial coordinate, and $\|\cdot\|$ is the Euclidean metric on $\Lambda^2\mathfrak{t}^*$. These were studied in [12], where they show that the nearly toric equation simplifies to the ODE

$$0 = \mathcal{D}(x) := 3(x'^2 - 2t)(x' + 2tx'') - 8(x - 2tx')$$

subject to the constraint

$$x > 2tx' > 2t\sqrt{2t}, \tag{3}$$

where the derivatives are taken with respect to t . The main result is that such a radial solution cannot be complete:

Theorem 7 *If (M, ω, ψ, T) is a connected complete toric nearly Kähler 6-manifold corresponding to a solution φ to the toric nearly Kähler equation, then φ is not radially symmetric.*

Proof Assume that φ is radially symmetric. Combining this symmetry with Theorem 6, $\mu(M)$ must be a closed 3-disc Δ centred at the origin. Now consider the set of points S in M where the torus action is not free. By Corollary 2, $\mu(S)$ is a discrete set of points in $\partial\Delta$. But by radial symmetry, $\mu(S)$ must be either empty or all of $\partial\Delta$. But $\mu(S)$ is a discrete set, so it can’t be $\partial\Delta$. Thus $\mu(S)$, and hence S is empty. This contradicts Proposition 1. □

We now investigate what goes wrong with the ODE to prevent completeness. Local existence of solutions to ODE’s will give a local solution $x(t)$ to $\mathcal{D}(x) = 0$ near any prescribed initial 1-jet $(t_0, x(t_0), x'(t_0))$ satisfying the constraints (3). Let (t_-, t_+) be the maximal open interval on which the solution can be extended while satisfying the constraints.

t_{\pm} must be either a point where $x(t)$ blows up or a boundary point of the constraints. By Lemma 2, $\varepsilon^2 = \frac{8}{3}(x - 2tx') > 0$ implies the other constraint $x' > \sqrt{2t}$. Thus the boundary condition is simply $\varepsilon^2 = 0$.

Lemma 10 *$x(t)$ does not blow up at $t_+ < \infty$.*

Proof First note that $\varepsilon^2 > 0$ implies that $(\log x)' < \frac{1}{2t}$. Integrating this implies that $x(t) < x_0 \sqrt{\frac{t}{t_0}}$. Since $x(t)$ is also positive, it cannot blow up in finite time.

On the other hand, integrating $x' > \sqrt{2t}$ gives $x - x_0 > \frac{\sqrt{2t} - \sqrt{2t_0}}{3}$. This lower bound for x grows faster as t increases than the upper bound for x in the previous paragraph. Thus t_+ is finite. \square

We compute

$$\varepsilon^2 = \frac{8}{3}(x - 2tx'), \quad 2tx'' = \frac{\varepsilon^2}{x^2 - 2t} - x', \quad (\varepsilon^2)' = -\frac{8}{3} \frac{\varepsilon^2}{x^2 - 2t}.$$

Note that by Lemma 2, the constraints can be rewritten as $0 < \varepsilon^2 \propto x - 2tx'$.

Thus the constraints imply that $x^2 - 2t > 0$, so the ODE is regular when the constraints hold and $t > 0$.

Lemma 11 $t_- = 0$.

Proof Since t_- is the boundary point of a maximal domain of an ODE subject to the constraint $\varepsilon^2 > 0$, at t_- either the ODE is singular and the solution $x(t)$ becomes unbounded or ε^2 vanishes. Since the ODE is singular at $t = 0$, we must have $t_- \geq 0$. Since x is positive and increasing, it must be bounded in (t_-, t_+) . Since ε^2 is decreasing, it cannot vanish at t_- . Thus t_- must a singular point of the ODE, in particular the only one: \square

By Theorem 7, there must be some singularity for $x(t)$ in $[0, t_+]$, and by the previous two lemmas it must be at $t = 0$.

Note that the estimate in Lemma 10 doesn't essentially require radial symmetry: it only uses $\varepsilon^2 \propto \varphi - \partial_r \varphi > 0$. In particular, continuing the discussion following Lemma 2, φ should not become unbounded as one tries to extend solutions in radial directions away from the origin.

7 Polynomial solutions

In this section we will try to understand polynomial solutions to the toric nearly Kähler equation. As described in [12], the toric nearly Kähler structure on $S^3 \times S^3$ corresponds to the solution of the toric nearly Kähler equation

$$\varphi_0 := 3 + \sum_j \mu_j^2 + \frac{1}{\sqrt{3}} \prod_j \mu_j, \quad (4)$$

where $\{\mu_j\}_{j=1}^3$ are coordinates on $\Lambda^2 t^*$ induced by the multi-moment map μ . We will prove Theorem 2 by treating each degree of polynomial separately. First we will introduce some notation. If E is an equation or expression, and m is a monomial in $\mathbb{R}[\mu_1, \mu_2, \mu_3]$, then $[m]E$ and $(m)E$ will refer, respectively, to the coefficient of m in E , and the part of E which is a multiple of m . We will use ∇ to denote the gradient in $\{\mu_j\}_{j=1}^3$ coordinates, and abuse notation by not distinguishing it from its transpose, or a restricted gradient to an context-appropriate subset of the coordinates. Similarly, ∇^2 will denote the Hessian, where the set of coordinates may depend on context.

Proposition 2 *Every cubic solution to the toric nearly Kähler equation is equivalent to φ_0 up to linear changes in coordinates.*

Proof Let φ be some cubic solution of the toric nearly Kähler equation (\star) . Write $\varphi = \sum_{j=0}^3 \varphi^j$, where each φ^k is a degree k homogeneous polynomial in $\{\mu_j\}_{j=1}^3$. As noted in [12], φ^1 can be chosen to be 0. The degree 3 term of (\star) gives

$$|\nabla^2 \varphi^3| = \left(\frac{8}{3} - \frac{11}{3} \partial_r + \partial_r^2\right) \varphi^3 = \left(\frac{8}{3} - 11 + 9\right) \varphi^3 = \frac{2}{3} \varphi^3.$$

Since $|\nabla^2 \varphi^3| \propto \varphi^3$, the plane algebraic curve $V(\varphi^3) \subset \mathbb{C}P^2$ is a union of lines [15], so that φ^3 is a product of linear factors. We can choose coordinates along these lines so that $\varphi^3 = \lambda \prod_{j=1}^3 \mu_j$ for some $\lambda \in \mathbb{R}$. We can compute $|\nabla^2 \varphi^3| = 2\lambda^2 \varphi^3$, so we must have $\lambda^2 = \frac{1}{3}$. Again, by changing coordinates, we may assume that $\lambda = \frac{1}{\sqrt{3}}$.

Writing $\varphi^2 = \varphi_d^2 + \varphi_o^2$, where $\nabla^2 \varphi_d^2$ is a diagonal matrix, while $\nabla^2 \varphi_o^2$ vanishes on the diagonal, one can compute that the degree 2 term of $|\nabla^2 \varphi|$ is $\frac{2}{3}(\varphi_o^2 - \varphi_d^2)$. Thus the degree 2 term of (\star) is

$$-\frac{2}{3} \varphi^2 = \frac{2}{3}(\varphi_o^2 - \varphi_d^2),$$

so that $\varphi_o = 0$. Thus $\nabla^2 \varphi^2$ is a diagonal matrix. We still have the freedom to scale the coordinates so that $\nabla^2 \varphi^2 = 2 \text{Id}$, or equivalently $\varphi^2 = \sum_{j=1}^3 \mu_j^2$. The degree 0 term of (\star) now gives $\frac{8}{3} \varphi^0 = |\nabla^2 \varphi^2| = 8$, so that $\varphi^0 = 3$. Thus $\varphi = \varphi_0$. □

For higher degree polynomial solutions, the toric nearly equation becomes overdetermined, since $|\nabla^2 \varphi|$ is formally a polynomial of degree $3(\text{deg}(\varphi) - 2)$. This suggests that the cubic solution might be the only polynomial solution. By diagonalizing $\nabla^2 \varphi^2$, we can always choose a basis so that $\varphi^2 = \sum_{j=1}^3 \mu_j^2$. As discussed in the previous proof, we will also get $\varphi^0 = 3$ and $\varphi^1 = 0$. The degree 1 term of (\star) tells us that φ^3 is harmonic.

Our main tool will be the following theorem, which we will refer to as Hesse’s theorem since Hesse originally claimed the result:

Theorem 8 ([8]) *If f is a homogeneous polynomial in 4 or less variables over an algebraically closed field, then $\det \circ \text{Hess}(f) = 0$ if and only if f is independent of one of the variables after a suitable homogeneous coordinate change.*

Remark 1 More geometrically, this theorem shows that $\det \circ \text{Hess}(f)$ vanishes whenever the directional derivative of f vanishes along some vector X . The coordinate change corresponds to any linear map which sends X to a generator of a coordinate axis. In particular, we can choose such a linear map to be orthogonal, meaning that the quadratic function $\sum_{j=1}^3 \mu_j^2$ is preserved. For the rest of the paper, whenever we use Hesse’s theorem, we will use such an orthogonal coordinate change.

Proposition 3 *There are no quartic solutions to the toric nearly Kähler equation.*

Proof Let $\varphi = \sum_{j=0}^4 \varphi^j$ be a solution to (\star) , where each φ^j is a homogeneous polynomial of degree j . Assume that $\varphi^4 \neq 0$, so that φ is quartic. We can choose coordinates so that the

quadratic part of φ is $3 + x^2 + y^2 + z^2$, where (x, y, z) is a relabelling of the coordinates $(\mu_j)_{j=1}^3$.

The degree 6 term of (\star) is $|\nabla^2 \varphi_4| = 0$. By Hesse's theorem, this is equivalent to φ_4 being a function of two variables. Thus, we can choose coordinates so that $\varphi^4 = \varphi^4(x, y)$.

Lemma 12 *After some rotation of the $y - z$ coordinates, $\varphi_{zz}^3 = 0$.*

Proof Assume that the claim is false, so that $\varphi_{zz}^3 \neq 0$. The degree 5 term of (\star) gives $|\nabla_{xy}^2 \varphi^4| \varphi_{zz}^3 = 0$, so that $|\nabla_{xy}^2 \varphi^4| = 0$. Using Hesse's theorem, we can choose coordinates so that φ^4 depends only on x . Now the degree 4 term of (\star) gives

$$4\varphi^4 = \varphi_{xx}^4 |\nabla_{yz}^2 \varphi^3|.$$

Write $\varphi^3 = x^3 B^0 + x^2 B^1 + x B^2 + B^3$, where each B^j is a degree j homogeneous polynomial in y and z . The x^2 term of the above equation gives $0 = |\nabla^2 B^3|$. Using a similar argument as in Remark 1, we can use Hesse's theorem to rotate the $y - z$ coordinates (preserving $y^2 + z^2$ while fixing x) so that $B^3 \propto y^3$. Note that our assumption still gives $\varphi_{zz}^3 \neq 0$, hence $B_{zz}^2 \neq 0$. The x^3 term of the above equation then gives $B_{yy}^3 B_{zz}^2 = 0$, implying $B_{zz}^3 = 0$. The x^4 term of the above equation gives $|\nabla^2 B^2| = \frac{1}{3}$.

Now $\Delta \varphi^3 = 0$ implies that $B^1 = 0$. The degree 3 term of $|\nabla^2 \varphi|$ modulo x^3 is

$$|\nabla^2(x B^2)| = -2x B^2 |\nabla^2 B^2| = -\frac{2}{3} x B^2,$$

where the first equality can be verified directly since B^2 is a homogeneous quadratic polynomial. Thus the coefficient of x of the degree 3 term of (\star) is $\frac{2}{3} B^2 = -\frac{2}{3} B^2$, so that $B^2 = 0$. This contradicts $|\nabla^2 B^2| = \frac{1}{3}$. \square

Since $\varphi_{zz}^3 = 0$, we can write $\varphi^3 = B^3(x, y) + z B^2(x, y)$. We have

$$\varphi = 3 + x^2 + y^2 + z^2 + B^3 + z B^2 + \varphi^4.$$

Since φ^3 is harmonic, so are B^2 and B^3 .

$$\nabla^2 \varphi = \begin{pmatrix} 2 \text{Id} + \nabla^2(B^3 + \varphi^4) & \nabla B^2 \\ \nabla B^2 & 2 \end{pmatrix} + z \begin{pmatrix} \nabla^2 B^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now $[z^2](\star)$ gives $2|\nabla^2 B^2| = -\frac{2}{3}$. Combining this with $\Delta B^2 = 0$, we can rotate coordinates so that $B^2 = \frac{xy}{\sqrt{3}}$. Now $(z)(\star)$ gives

$$0 = \frac{4}{\sqrt{3}} (B^3 + \varphi^4)_{xy}.$$

Thus $0 = \varphi_{xy}^4 = B_{xy}^3$. Since B^3 is harmonic, it must vanish. Note that we now have $\varphi = \varphi_0 + \varphi^4$. Now it is easy to see that the degree 2 part of (\star) gives $\Delta \varphi^4 = 0$, since it is the only term depending on φ^4 . Combining this with $\varphi_{xy}^4 = 0$ shows that $\varphi^4 = 0$, a contradiction. \square

Working quite a bit harder we can establish the quintic case:

Proposition 4 *There are no quintic solutions to the toric nearly Kähler equation.*

Proof Let $\varphi = \sum_{j=0}^5 \varphi^j$ be a solution to (\star) with $\varphi^5 \neq 0$. The degree 9 term of (\star) gives $|\nabla^2 \varphi^5| = 0$. Using Hesse’s theorem, we can change variables so that $\varphi_z^5 = 0$.

Lemma 13 $\varphi_{zz}^4 = 0$

Proof Assume $\varphi_{zz}^4 \neq 0$. The degree 8 term of (\star) gives $0 = \varphi_{zz}^4 |\nabla_{xy}^2 \varphi^5|$. Thus $|\nabla_{xy}^2 \varphi^5| = 0$. By Hesse’s theorem, we may assume that $\varphi_y^5 = 0$. Write $\varphi^4 = \sum_j x^j C^{4-j}$, where each C^j is a degree j homogeneous polynomial in y and z . The degree 7 term of (\star) gives

$$\begin{aligned} 0 &= \left| \nabla_{yz}^2 \varphi^4 \right| = \left| \nabla^2 C^4 + x \nabla^2 C^3 + x^2 \nabla^2 C^2 \right| \\ &= \left| \nabla^2 C^4 \right| + x \langle \nabla^2 C^4, \nabla^2 C^3 \rangle \\ &\quad + x^2 \left(\left| \nabla^2 C^3 \right| + \langle \nabla^2 C^4, \nabla^2 C^2 \rangle \right) \\ &\quad + x^3 \langle \nabla^2 C^3, \nabla^2 C^2 \rangle + x^4 \left| \nabla^2 C^2 \right|, \end{aligned}$$

where $\langle N, M \rangle := 2 \operatorname{adj}(N) \cdot M = [t]|N + tM|$. If any C^j vanishes, then we have

$$0 = \left| \nabla^2 C^k \right| = \left| \nabla^2 C^\ell \right| = \langle \nabla^2 C^k, \nabla^2 C^\ell \rangle,$$

where $\{j, k, \ell\} = \{2, 3, 4\}$. This implies that C^k and C^ℓ are both proportional to powers of the same linear term, which we may choose to be y by changing coordinates. This contradicts $\varphi_{zz}^4 \neq 0$. Thus no C^j vanishes. Since $|\nabla^2 C^2| = 0$, we may use Hesse’s theorem to choose coordinates so that $C^2 \propto y^2$. Since $\langle \nabla^2 C^3, \nabla^2 C^2 \rangle = 0$, we must have $C_{zz}^3 = 0$. Thus $C^4 = \varphi_{zz}^4 \neq 0$. Combining this with $|\nabla^2 C^4| = 0$, we may choose coordinates so that $C^4 \propto z^4$. Then $0 = \langle \nabla^2 C^4, \nabla^2 C^3 \rangle$ implies that $C_{yy}^3 = 0$. Thus $C^3 = 0$, a contradiction \square

Lemma 14 $\varphi_{zzz} = 0$.

Proof Assume $\varphi_{zzz} \neq 0$. Write $\varphi = \alpha + z\beta + z^2\gamma + z^3\delta$, where α, β, γ , and δ are polynomials in x and y of degrees 5, 3, 1, and 0, respectively. Again, we will use exponents to denote the corresponding homogeneous parts. We are assuming $\delta \neq 0$. We have

$$\nabla^2 \varphi = \begin{pmatrix} \nabla^2 \alpha & \nabla \beta \\ \nabla \beta & 2\gamma \end{pmatrix} + z \begin{pmatrix} \nabla^2 \beta & 2\nabla \gamma \\ 2\nabla \gamma & 6\delta \end{pmatrix}.$$

The degree 7 term of (\star) with a factor of z gives $|\nabla^2 \alpha^5| = 0$. The degree 6 term with a factor z^2 gives $\langle \nabla^2 \alpha^5, \nabla^2 \beta^3 \rangle = 0$. The degree 5 term with a factor of z^3 gives $|\nabla^2 \beta^3| = 0$. Hesse’s theorem allows us to interpret this as α^5 and β^3 each depending on only one variable, which must be the same due to the cross-term. Thus we can assume that α^5 and β^3 are both functions of x .

The degree 6 term with a factor of z gives $0 = \alpha_{xx}^5 \alpha_{yy}^4 \delta$, so that $\alpha_{yy}^4 = 0$. Thus we have $\nabla_{yz}^2 \varphi^4 = 0$. Note that we now have no distinguished direction in the $y - z$ plane.

We write $\varphi = A + B + C + D$, where A, B, C , and D have degree in $[y, z]$, respectively 0, 1, 2, and 3 and total degrees, respectively, 5, 4, 3 and 3. The degree 5 term of (\star) with a factor of x^3 gives $0 = A_{xx}^5 |\nabla_{yz}^2 D^3|$. Since $A^5 = \varphi^5 \neq 0$, this shows that $|\nabla_{yz}^2 D^3| = 0$. Using Hesse’s theorem, we can write D^3 as a function of y . This contradicts $\varphi_{zzz} \neq 0$. \square

We will continue to use the decomposition with Greek letters. The previous lemma shows that $\delta = 0$.

Lemma 15 $\gamma = 1$.

Proof Assume $\gamma \neq 1$, so that $\nabla\gamma \neq 0$. We can rotate the x - y coordinates to write $\gamma = 1 + cx$ for some $0 \neq c \in \mathbb{R}$. Thus $(z^3)(\star)$ gives $0 = -\beta_{yy}(2c)^2$, so that $\beta_{yy} = 0$. We compute

$$(z^2) \left| \nabla^2 \varphi \right| = -4c^2 \alpha_{yy} - 2\gamma (\beta_{xy})^2 + 4c \beta_{xy} \beta_y = -4c^2 \alpha_{yy} - 2(\beta_{xy})^2 + 2c \beta_{xy} (2 - \partial_r) \beta_y.$$

Thus $(z^2y)(\star)$ gives $\alpha_{yyy} = 0$. Now $[z^2x^3](\star)$ gives $\alpha_{yy}^5 = 0$. We compute

$$(yz) \left| \nabla^2 \varphi \right| = 2 \begin{vmatrix} 0 & \alpha_{xyy} & \beta_{xy} \\ \beta_{xy} & \alpha_{yy} & \beta_y \\ 2c & \beta_y & 2\gamma \end{vmatrix}.$$

This is a polynomial in x , whose quartic term is proportional to $(\beta_{xy}^3)^3$. Thus $[x^4yz](\star)$ gives $\beta_y^3 = 0$. Now $[x^2z^2](\star)$ gives $\alpha_{yy}^4 = 0$. Now the degree 7 term of (\star) gives $0 = c(\alpha_{xy}^5)^2$, so that $\alpha_{xy}^5 = 0$.

Now $[z^2](\star)$ gives $\frac{2}{3} = \beta_{xy}^2 + c^2$, so that $[xz^2](\star)$ gives $0 = \alpha_{xyy}^3 + c$. But φ is harmonic, giving

$$0 = \Delta\varphi^3 = \Delta\alpha^3 + 2c + z\Delta\beta^2 = \alpha_{xx}^3 + z\beta_{xx}^2.$$

Thus $\alpha_{xx}^3 = 0 = \beta_{xx}^2$. Now $[x^3](\star)$ gives $\alpha^5 = 0$, contradicting $\varphi^5 \neq 0$. □

Now that $\gamma = 1$, $(z^2)(\star)$ gives $2|\nabla^2\beta| = -2/3$. Using Hesse’s theorem, we can change coordinates so that $\beta^3 = bx^3$, for some $b \in \mathbb{R}$. Combining these equations with $\Delta\beta^2 = 0$ (from $\nabla\varphi^3 = 0$), we can choose coordinates so that $\beta^2 = \frac{xy}{\sqrt{3}}$, independent of whether or not b vanishes. We will show that b does indeed vanish. $(z)(\star)$ gives

$$\begin{aligned} \frac{2xyz}{3\sqrt{3}} + 4bz^3 &= [z] \left| \nabla^2 \varphi \right| = z \left(2 \langle \nabla^2 \alpha, \nabla^2 \beta \rangle + \left| \begin{matrix} \nabla^2 \beta & \nabla \beta \\ \nabla \beta & 0 \end{matrix} \right| \right) \\ &= z \left(12bx \alpha_{yy} - \frac{4}{\sqrt{3}} \alpha_{xy} - 6bx (\beta_y)^2 + \frac{2}{\sqrt{3}} \beta_x \beta_y \right) \\ &= z \left(12bx \alpha_{yy} - \frac{4}{\sqrt{3}} \alpha_{xy} - 2bx^3 + \frac{2}{3} x \left(3bx^2 + \frac{y}{\sqrt{3}} \right) \right), \end{aligned}$$

so that

$$0 = 12bx \alpha_{yy} - \frac{4}{\sqrt{3}} \alpha_{xy} - 4bx^3.$$

Taking the coefficients of x^0 , x^1 , and x^2 , respectively, of this equation gives

$$[x^1]\alpha_y = 0, \quad [x^2]\alpha_y = \frac{3\sqrt{3}}{2}b[x^0]\alpha_{yy}, \quad [x^3]\alpha_y = \sqrt{3}b[x^1]\alpha_{yy}.$$

In particular, $6\sqrt{3}b = \alpha_{xy}^3 = -\alpha_{yyy}^3$, where the second inequality comes from α^3 being harmonic. Now we compute

$$[x^0z^0]|\nabla^2\varphi| = [x^0] \begin{vmatrix} \alpha_{xx} & 0 & \frac{y}{\sqrt{3}} \\ 0 & \alpha_{yy} & 0 \\ \frac{y}{\sqrt{3}} & 0 & 2 \end{vmatrix} = \alpha_{yy} \left(2\alpha_{xx} - \frac{y^2}{3} \right).$$

In particular, $[y^2](\star)$ gives

$$0 = [y^2]\alpha_{yy}\alpha_{xx} = \alpha_{yyyy}^4 + \alpha_{yyy}\alpha_{xxy} + \alpha_{xxy} = \alpha_{yyyy}^4 - 162b^2.$$

Now $[x^3yz](\star)$ gives

$$0 = b\alpha_{xxyy} \propto b^2\alpha_{yyyy}^4 \propto b^4,$$

so that $b = 0$. Now $[z](\star)$ is simply $\alpha_{xy} = 0$. The remainder of (\star) is

$$\left(\frac{8}{3} - \frac{11}{3}\partial_r + \partial_r^2 \right) \alpha = \begin{vmatrix} \alpha_{xx} & 0 & \frac{y}{\sqrt{3}} \\ 0 & \alpha_{yy} & \frac{x}{\sqrt{3}} \\ \frac{y}{\sqrt{3}} & \frac{x}{\sqrt{3}} & 2 \end{vmatrix} = 2\alpha_{xx}\alpha_{yy} + \frac{1}{3}(\partial_r - \partial_r^2)\alpha.$$

Thus $(xy)(\star)$ gives $0 = \alpha_{xxt}\alpha_{yyy}$. Without loss of generality, by changing coordinates we have $\alpha_{yyy} = 0$. Thus (\star) becomes

$$0 = \left(\frac{8}{3} - 4\partial_r + \frac{4}{3}\partial_r^2 \right) \alpha - 4\alpha_{xx}.$$

In particular, $0 = -\frac{32}{3}\alpha^5$, contradicting $\varphi^5 = \alpha^5 \neq 0$. □

Combining the previous three propositions gives Theorem 2.

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References

1. Atiyah, M.F.: Convexity and commuting Hamiltonians. *Bull. Lond. Math. Soc.* **14**(1), 1–15 (1982)
2. Bär, C.: Real killing spinors and holonomy. *Commun. Math. Phys.* **154**(3), 509–521 (1993)
3. Butruille, J.B.: Homogeneous nearly Kähler manifolds. *Handbook of pseudo-Riemannian geometry and supersymmetry. IRMA Lect. Math. Theor. Phys.* **16**, 399–423 (2010)
4. Carrión, R.R., Salamon, S.: A survey of nearly Kähler manifolds. *Gac. R. Soc. Mat. Esp.* **2**(1), 40–49 (1999)
5. Dixon, K.: The multi-moment maps of the nearly Kähler $S^3 \times S^3$. *Geom. Dedicata.* **200**(1), 351–362 (2019)
6. Foscolo, L., Haskins, M.: New G_2 -holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$. *Ann. Math.* **185**(1), 59–130 (2017)
7. Guillemin, V., Sternberg, S.: Convexity properties of the moment mapping. *Invent. Math.* **67**(3), 491–513 (1982)
8. Lossen, C.: When does the Hessian determinant vanish identically? *Bull. Braz. Math. Soc. New Ser.* **35**(1), 71–82 (2004)
9. Madnick, J.: Nearly-Kähler 6-manifolds of cohomogeneity two: local theory. Preprint [arXiv :1710.05254](https://arxiv.org/abs/1710.05254) (2017)
10. Madsen, T., Swann, A.: Toric geometry of G_2 -manifolds. *Geom. Topol.* **23**(7), 3459–3500 (2019)
11. Madsen, T.B., Swann, A.: Closed forms and multi-moment maps. *Geom. Dedicata.* **165**(1), 25–52 (2013)
12. Moroianu, A., Nagy, P.A.: Toric nearly Kähler manifolds. *Ann. Glob. Anal. Geom.* **55**(4), 703–717 (2019)
13. Myers, S.B.: Riemannian manifolds with positive mean curvature. *Duke Math. J.* **8**(2), 401–404 (1941)
14. Nagy, P.A.: Nearly Kähler geometry and Riemannian foliations. *Asian J. Math.* **6**(3), 481–504 (2002)
15. Stieger, A.: Inflection points on plane algebraic curves. Bachelor's thesis, ETH Zürich (2008)
16. Wang, H.C.: The homology groups of the fibre bundles over a sphere. *Duke Math. J.* **16**(1), 33–38 (1949)

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