On certain classes of $\mathbf{Sp}(4, \mathbb{R})$ symmetric G_2 structures

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Abstract

We find two different families of $\mathbf{Sp}(4, \mathbb{R})$ symmetric G_2 structures in seven dimensions. These are G_2 structures with G_2 being the split real form of the simple exceptional complex Lie group G_2 . The first family has $\tau_2 \equiv 0$, while the second family has $\tau_1 \equiv \tau_2 \equiv 0$, where τ_1, τ_2 are the celebrated G_2 -invariant parts of the intrinsic torsion of the G_2 structure. The families are different in the sense that the first one lives on a homogeneous space $\mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_l$, and the second one lives on a homogeneous space $\mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$. Here $\mathbf{SL}(2, \mathbb{R})_l$ is an $\mathbf{SL}(2, \mathbb{R})$ corresponding to the $\mathfrak{sl}(2, \mathbb{R})$ related to the long roots in the root diagram of $\mathfrak{sp}(4, \mathbb{R})$, and $\mathbf{SL}(2, \mathbb{R})_s$ is an $\mathbf{SL}(2, \mathbb{R})$ corresponding to the $\mathfrak{sl}(2, \mathbb{R})$ related to the short roots in the root diagram of $\mathfrak{sp}(4, \mathbb{R})$.

Keywords Homogeneous G_2 structures · Skew symmetric torsion · Split signature metric

1 Introduction: a question of Maciej Dunajski

Recently, together with Hill [5], we uncovered an $\mathbf{Sp}(4, \mathbb{R})$ symmetry of the nonholonomic kinematics of a car. I talked about this at the Abel Symposium in Ålesund, Norway, in June 2019. After my talk Maciej Dunajski, intrigued by the root diagram of $\mathfrak{sp}(4, \mathbb{R})$ which appeared in the talk, asked me if using it I can see a G_2 structure on a 7-dimensional homogeneous space $M = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})$.

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My immediate answer was: ‘I can think about it, but I have to know which of the $\mathbf{SL}(2, \mathbb{R})$ subgroups of $\mathbf{Sp}(4, \mathbb{R})$ I shall use to build M .’ The reason for the ‘but’ word in my answer was that there are at least two $\mathbf{SL}(2, \mathbb{R})$ subgroups of $\mathbf{Sp}(4, \mathbb{R})$, which lie quite differently in there. One can see them in the root diagram above: the first $\mathbf{SL}(2, \mathbb{R})$ corresponds to the *long* roots, as, for example, E_1 and E_{10} , whereas the second one corresponds to the *short* roots, as, for example, E_2 and E_9 . Since Maciej never told me which $\mathbf{SL}(2, \mathbb{R})$ he wants, I decided to consider both of them and to determine what kind of G_2 structures one can associate with the respective choice of a subgroup.

I emphasize that in the below considerations I will use the *split real form* of the simple exceptional Lie group G_2 . Therefore, the corresponding G_2 structure metrics will *not* be Riemannian.¹ They will have signature (3, 4).

2 The Lie algebra $\mathfrak{sp}(4, \mathbb{R})$

The Lie algebra $\mathfrak{sp}(4, \mathbb{R})$ is given by the 4×4 matrices

$$E = (E^\alpha_\beta) = \begin{pmatrix} a_5 & a_7 & a_9 & 2a_{10} \\ -a_4 & a_6 & a_8 & a_9 \\ a_2 & a_3 & -a_6 & -a_7 \\ -2a_1 & a_2 & a_4 & -a_5 \end{pmatrix},$$

where the coefficients a_I , $I = 1, 2, \dots, 10$, are real constants. The Lie bracket in $\mathfrak{sp}(4, \mathbb{R})$ is the usual commutator $[E, E'] = E \cdot E' - E' \cdot E$ of two matrices E and E' . We start with the following basis (E_I),

$$E_I = \frac{\partial E}{\partial a_I}, \quad I = 1, 2, \dots, 10,$$

in $\mathfrak{sp}(4, \mathbb{R})$.

In this basis, modulo the antisymmetry, we have the following nonvanishing commutators: $[E_1, E_5] = 2E_1$, $[E_1, E_7] = -2E_2$, $[E_1, E_9] = -2E_4$, $[E_1, E_{10}] = 4E_5$, $[E_2, E_4] = E_1$, $[E_2, E_5] = E_2$, $[E_2, E_6] = E_2$, $[E_2, E_7] = 2E_3$, $[E_2, E_8] = E_4$, $[E_2, E_9] = -E_5 - E_6$, $[E_2, E_{10}] = -2E_7$, $[E_3, E_4] = -E_2$, $[E_3, E_6] = 2E_3$, $[E_3, E_8] = -E_6$, $[E_3, E_9] = -E_7$,

¹ For some of the Riemannian counterparts of the structures considered here, see for example, [6].

$$[E_4, E_5] = E_4, [E_4, E_6] = -E_4, [E_4, E_7] = E_5 - E_6, [E_4, E_9] = -2E_8, [E_4, E_{10}] = -2E_9, \\ [E_5, E_7] = E_7, [E_5, E_9] = E_9, [E_5, E_{10}] = 2E_{10}, [E_6, E_7] = -E_7, [E_6, E_8] = 2E_8, [E_6, E_9] = E_9, \\ [E_7, E_8] = E_9, [E_7, E_9] = E_{10}.$$

We see that there are at least *two* $\mathfrak{sl}(2, \mathbb{R})$ Lie algebras here. The first one is

$$\mathfrak{sl}(2, \mathbb{R})_l = \text{Span}_{\mathbb{R}}(E_1, E_5, E_{10}),$$

and the second is

$$\mathfrak{sl}(2, \mathbb{R})_s = \text{Span}_{\mathbb{R}}(E_2, E_5 + E_6, E_9).$$

The reason for distinguishing these two is as follows:

The eight 1-dimensional vector subspaces $\mathfrak{g}_I = \text{Span}(E_I)$, $I = 1, 2, 3, 4, 7, 8, 9, 10$, of $\mathfrak{sp}(4, \mathbb{R})$ are the *root spaces* of this Lie algebra. They correspond to the Cartan subalgebra of $\mathfrak{sp}(4, \mathbb{R})$ given by $\mathfrak{h} = \text{Span}(E_5, E_6)$. It follows that the pairs (E_I, E_J) of the root vectors, such that $I + J = 11$, $I, J \neq 5, 6$, correspond to the *opposite roots* of $\mathfrak{sl}(2, \mathbb{R})$. Knowing the Killing form for $\mathfrak{sl}(2, \mathbb{R})$, which in the basis (E_I) , and its dual basis (E^I) , $E_I \lrcorner E^J = \delta^J_I$, is

$$K = \frac{1}{12} K_{IJ} E^I \odot E^J = -4E^1 \odot E^{10} + 2E^2 \odot E^9 + E^3 \odot E^8 \\ - 2E^4 \odot E^7 + E^5 \odot E^5 + E^6 \odot E^6,$$

one can see that the roots corresponding to the root vectors (E_1, E_{10}) and (E_3, E_8) are *long*, and the roots corresponding to the root vectors (E_2, E_9) and (E_4, E_7) are *short*; compare the Euclidian lengths of these roots as drawn on the G_2 root diagram presented at the beginning of this article.² Thus, the Lie algebra $\mathfrak{sl}(2, \mathbb{R})_l$ containing root vectors (E_1, E_{10}) corresponding to the *long* roots lies quite different in $\mathfrak{sp}(4, \mathbb{R})$ than the Lie algebra $\mathfrak{sl}(2, \mathbb{R})_s$ containing the root vectors (E_2, E_9) corresponding to the *short* roots.

3 G_2 structures on $\text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_l$

3.1 Compatible pairs (g, ϕ) on M_l

To consider the homogeneous space $M_l = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_l$, it is convenient to change the basis (E_I) in $\mathfrak{sp}(4, \mathbb{R})$ to a new one, (e_I) , in which the last three vectors span $\mathfrak{sl}(2, \mathbb{R})_l$. Thus, we take:

$$e_1 = E_2, e_2 = E_3, e_3 = E_4, e_4 = E_6, e_5 = E_7, e_6 = E_8, e_7 = E_9, e_8 = E_1, e_9 = E_5, e_{10} = E_{10}.$$

If now, one considers (e_I) as the basis of the Lie algebra of left invariant vector fields on the Lie group $\text{Sp}(4, \mathbb{R})$ then the dual basis (e^I) , $e_I \lrcorner e^J = \delta^J_I$, of the left invariant forms on $\text{Sp}(4, \mathbb{R})$ satisfies:

² We hope that the reader noticed that we use the same symbol E_I for ‘root vectors’ spanning 1-dimensional ‘root spaces’ of $\mathfrak{g}_2$, as well as for the ‘roots’ E_I of $\mathfrak{g}_2$ depicted on the root diagram.

$$\begin{aligned}
de^1 &= -e^1 \wedge (e^4 + e^9) + e^2 \wedge e^3 - 2e^5 \wedge e^8 \\
de^2 &= -2e^1 \wedge e^5 - 2e^2 \wedge e^4 \\
de^3 &= -e^1 \wedge e^6 + e^3 \wedge (e^4 - e^9) - 2e^7 \wedge e^8 \\
de^4 &= e^1 \wedge e^7 + e^2 \wedge e^6 + e^3 \wedge e^5 \\
de^5 &= 2e^1 \wedge e^{10} + e^2 \wedge e^7 + e^5 \wedge (e^9 - e^4) \\
de^6 &= 2e^3 \wedge e^7 - 2e^4 \wedge e^6 \\
de^7 &= 2e^3 \wedge e^{10} - e^5 \wedge e^6 + e^7 \wedge (e^4 + e^9) \\
de^8 &= -e^1 \wedge e^3 - 2e^8 \wedge e^9 \\
de^9 &= e^1 \wedge e^7 - e^3 \wedge e^5 - 4e^8 \wedge e^{10} \\
de^{10} &= -e^5 \wedge e^7 - 2e^9 \wedge e^{10}.
\end{aligned} \tag{3.1}$$

Here we used the usual formula relating the structure constants c^I_{JK} , from $[e_J, e_K] = c^I_{JK}e_I$, to the differentials of the Maurer–Cartan forms (e^I), $de^I = -\frac{1}{2}c^I_{JK}e^J \wedge e^K$.

In this basis, the Killing form on $\mathbf{Sp}(4, \mathbb{R})$ is

$$K = \frac{1}{12}c^I_{JK}c^K_{LI}e^J \odot e^L = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7 + (e^9)^2 - 4e^8 \odot e^{10}.$$

Here, we have used the notation $e^I \odot e^J = \frac{1}{2}(e^I \otimes e^J + e^J \otimes e^I)$, $(e^I)^2 = e^I \odot e^I$.

One can now use equations (3.1) to see that the homogeneous space $M_I = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_I$ is the leaf space of a certain integrable rank 3 distribution D_I on $\mathbf{Sp}(4, \mathbb{R})$, establishing explicitly that $\mathbf{Sp}(4, \mathbb{R})$ has, in particular, the structure of the principal $\mathbf{SL}(2, \mathbb{R})$ fiber bundle $\mathbf{SL}(2, \mathbb{R})_I \rightarrow \mathbf{Sp}(4, \mathbb{R}) \rightarrow M_I = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_I$.

Indeed, the 3-dimensional distribution D_I , generated by the vector fields X on $\mathbf{Sp}(4, \mathbb{R})$ annihilating the span of the 1-forms $(e^1, e^2, \dots, e^7)$, is integrable, $de^\mu \wedge e^1 \wedge e^2 \dots \wedge e^7 \equiv 0$, $\mu = 1, 2, \dots, 7$, so that we have a well-defined 7-dimensional leaf space M_I of the corresponding foliation. Moreover, the Maurer–Cartan equations (3.1), restricted to a leaf defined by $(e^1, e^2, \dots, e^7) \equiv 0$, reduce to $de^8 = -2e^8 \wedge e^9$, $de^9 = -4e^8 \wedge e^{10}$, $de^{10} = -2e^9 \wedge e^{10}$, showing that each leaf can be identified with the Lie group $\mathbf{SL}(2, \mathbb{R})_I$. Thus, the projection $\mathbf{Sp}(4, \mathbb{R}) \rightarrow M_I$ from the Lie group $\mathbf{Sp}(4, \mathbb{R})$ to the leaf space M_I is the projection to the homogeneous space $M_I = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_I$.

In this section, I will use from now on Greek indices μ, ν , etc., to run from 1 to 7. They number the first seven basis elements in the bases (e_I) and (e^I) .

Now, I look for all bilinear symmetric forms $g = g_{\mu\nu}e^\mu \odot e^\nu$ on $\mathbf{Sp}(4, \mathbb{R})$, with constant coefficients $g_{\mu\nu} = g_{\nu\mu}$, which are constant along the leaves of the foliation defined by D_I . Technically, I search for those g whose Lie derivative with respect to any vector field X from D_I vanishes,

$$\mathcal{L}_X g = 0 \text{ for all } X \text{ in } D_I. \tag{3.2}$$

I have the following proposition:

Proposition 3.1 *The most general $g = g_{\mu\nu}e^\mu \odot e^\nu$ satisfying condition (3.2) is*

$$\begin{aligned}
g &= g_{22}(e^2)^2 + 2g_{24}e^2 \odot e^4 + g_{44}(e^4)^2 + 2g_{35}(e^3 \odot e^5 - e^1 \odot e^7) \\
&\quad + 2g_{26}e^2 \odot e^6 + 2g_{46}e^4 \odot e^6 + g_{66}(e^6)^2.
\end{aligned}$$

Thus, I have a 7-parameter family of bilinear forms on $\mathbf{Sp}(4, \mathbb{R})$ that *descend* to well-defined pseudo-Riemannian metrics on the leaf space M_l . Note that the restriction of the Killing form K to the space where $(e^8, e^9, e^{10}) \equiv 0$ is in this family. This corresponds to $g_{22} = g_{24} = g_{46} = 0$ and $g_{44} = 2g_{26} = -g_{35} = 1$.

Since the aim of my note is *not* to be exhaustive, but rather to show how to produce G_2 structures on $\mathbf{Sp}(4, \mathbb{R})$ homogeneous spaces, from now on I will restrict myself to only one $\mathbf{SL}(2, \mathbb{R})_l$ invariant bilinear form g on $\mathbf{Sp}(4, \mathbb{R})$, namely to

$$g_K = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7, \quad (3.3)$$

coming from the restriction of the Killing form. It follows from Proposition 3.1 that this form is a well-defined (3, 4) signature metric on the quotient space $M_l = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_l$.

I now look for the 3-forms $\phi = \frac{1}{6}\phi_{\mu\nu\rho}e^\mu \wedge e^\nu \wedge e^\rho$ on $\mathbf{Sp}(4, \mathbb{R})$ that are constant along the leaves of the distribution D_l , i.e., such that

$$\mathcal{L}_X \phi = 0 \text{ for all } X \text{ in } D_l. \quad (3.4)$$

Then, I have the following proposition.

Proposition 3.2 *There is a 10-parameter family of 3-forms $\phi = \frac{1}{6}\phi_{\mu\nu\rho}e^\mu \wedge e^\nu \wedge e^\rho$ on $\mathbf{Sp}(4, \mathbb{R})$ which satisfy condition (3.4). The general formula for them is:*

$$\begin{aligned} \phi = & fe^{125} + a(e^{235} - e^{127}) + pe^{145} + q(e^{147} + e^{345}) \\ & + se^{156} + t(e^{356} - e^{167}) + he^{237} + be^{246} + re^{347} + ue^{367}. \end{aligned}$$

Here $e^{\mu\nu\rho} = e^\mu \wedge e^\nu \wedge e^\rho$, and $a, b, f, h, p, q, r, s, t$ and u are real constants.

Thus there is a 10-parameter family of 3-forms ϕ that descends from $\mathbf{Sp}(4, \mathbb{R})$ to the $\mathbf{Sp}(4, \mathbb{R})$ homogeneous space $M_l = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_l$.

Now, I introduce an important notion of *compatibility* of a pair (g, ϕ) where g is a metric and ϕ is a 3-form on a 7-dimensional oriented manifold M . The pair (g, ϕ) on M is *compatible* if and only if

$$(X \lrcorner \phi) \wedge (X \lrcorner \phi) \wedge \phi = 3g(X, Y) \text{vol}(g), \quad \forall X, Y \in \text{TM}.$$

Here $\text{vol}(g)$ is a *volume form* on M related to the metric g .

Restricting, as I did, to the $\mathbf{Sp}(4, \mathbb{R})$ invariant metric g_K on M_l as in (3.3), I now ask which of the 3-forms ϕ from Proposition 3.2 are compatible with the metric (3.3). In other words, I now look for the constants $a, b, f, h, p, q, r, s, t$ and u such that

$$(e_\mu \lrcorner \phi) \wedge (e_\nu \lrcorner \phi) \wedge \phi = 3g_K(e_\mu, e_\nu) e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \wedge e^7, \quad (3.5)$$

for $g = g_K$ given in (3.3).

I have the following proposition.

Proposition 3.3 *The general solution to the Eq. (3.5) is given by*

$$b = \frac{1}{2}, f = \frac{ap}{1-q}, h = \frac{a(q-1)}{p}, r = \frac{q^2-1}{p}, s = \frac{p(1-q)}{4a}, t = \frac{1-q^2}{4a}, u = \frac{(q^2-1)(q+1)}{4ap}.$$

This leads to the following corollary.

Corollary 3.4 *The most general pair (g_K, ϕ) on M_I compatible with the $\mathbf{Sp}(4, \mathbb{R})$ invariant metric*

$$g_K = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7,$$

coming from the Killing form in $\mathbf{Sp}(4, \mathbb{R})$, is a 3-parameter family with ϕ given by:

$$\begin{aligned} \phi = & \frac{ap}{1-q} e^{125} + a(e^{235} - e^{127}) + pe^{145} + q(e^{147} + e^{345}) + \frac{p(1-q)}{4a} e^{156} \\ & + \frac{1-q^2}{4a} (e^{356} - e^{167}) + \frac{a(q-1)}{p} e^{237} + \frac{1}{2} e^{246} + \frac{q^2-1}{p} e^{347} + \frac{(q^2-1)(q+1)}{4ap} e^{367}. \end{aligned}$$

Here $a \neq 0$, $p \neq 0$, $q \neq 1$ are free parameters, and $e^{\mu\nu\rho} = e^\mu \wedge e^\nu \wedge e^\rho$ as before.

3.2 G_2 structures in general

Compatible pairs (g, ϕ) on 7-dimensional manifolds are interesting since they give examples of G_2 structures [2]. In general, a G_2 structure consists of a compatible pair (g, ϕ) of a metric g and a 3-form ϕ on a 7-dimensional manifold M , and it is in addition assumed that the 3-form ϕ is *generic*, meaning that at every point of M it lies in one of the two *open* orbits of the natural action of $\mathbf{GL}(7, \mathbb{R})$ on 3-forms in $\mathbb{R}^7$. The simple exceptional Lie group G_2 appears here as the common stabilizer in $\mathbf{GL}(7, \mathbb{R})$ of both g and ϕ .

It follows (from compatibility) that the G_2 structures can have metrics g of only two signatures: the Riemannian ones and (3, 4) signature ones. If the signature of g is Riemannian, the corresponding G_2 structure is related to the compact real form of the simple exceptional complex Lie group G_2 , and in the (3, 4) signature case the corresponding G_2 structure is related to the noncompact (split) real form of the complex group G_2 . In this sense, our Corollary 3.4 provides a 3-parameter family of split real form G_2 structures on M_I .

G_2 structures can be classified according to the properties of their *intrinsic torsion* [1, 2]. Making a long story short, this torsion is totally determined by finding *four* p -forms τ_p on M , $p = 0, 1, 2, 3$, each belonging to one of four different irreducible representations of G_2 . Before telling on how to find these forms given a G_2 structure (g, ϕ) , we need some preparation.

We recall that the group G_2 acts in $\mathbb{R}^7$, and this induces its action on spaces $\bigwedge^p$ of p -forms in $\mathbb{R}^7$. Of course the 1-dimensional space $\bigwedge^0$ is G_2 irreducible, as well as is the space of 1-forms $\bigwedge^1 = \bigwedge^1_7$. The G_2 irreducible decompositions of the spaces of 2- and 3-forms look like $\bigwedge^2 = \bigwedge^2_7 \oplus \bigwedge^2_{14}$ and $\bigwedge^3 = \bigwedge^3_1 \oplus \bigwedge^3_7 \oplus \bigwedge^3_{27}$. Here we use the convention that the lower index i in $\bigwedge^p_i$ denotes the *dimension* of the corresponding representation. It is further convenient to introduce the Hodge dual, $*$, which is defined on p -forms λ by

$$*\lambda(e_{\mu_1}, \dots, e_{\mu_{7-p}}) \text{vol}(g) = \lambda \wedge g(e_{\mu_1}) \wedge \dots \wedge g(e_{\mu_{7-p}}), \quad X \lrcorner g(e_\mu) = g(e_\mu, X).$$

By the Hodge duality, the decomposition of $\bigwedge^4$ into G_2 irreducible components is similar to this for $\bigwedge^3$. We further mention that the 7-dimensional representations $\bigwedge^1_7$, $\bigwedge^2_7$ and $\bigwedge^3_7$ are all G_2 equivalent. Also, one can see that, e.g., $\bigwedge^3_{27} = \{\alpha \in \bigwedge^3 \text{ s.t. } \alpha \wedge \phi = 0 \text{ \& } \alpha \wedge * \phi = 0\}$.

The intrinsic torsion components τ_0 , τ_1 , τ_2 and τ_3 have values in the following G_2 irreducible modules: the 3-form τ_3 has values in the 27-dimensional *irreducible* representation $\bigwedge^3_{27}$, the 2-form τ_2 has values in the 14-dimensional *irreducible* representation $\bigwedge^2_{14}$, the 1-form τ_1 has values in the 7-dimensional *irreducible* representation $\bigwedge^1_7$, and the 0-form τ_0 has values in the trivial representation $\bigwedge^0_1$.

The result of Bryant [1, 2] states that for every G_2 structure (g, ϕ) on M there exist unique forms τ_0 , τ_1 , τ_2 and τ_3 on M , with values in the above-mentioned representations, such that

$$\begin{aligned} d\phi &= \tau_0 * \phi + 3\tau_1 \wedge \phi + * \tau_3 \\ d * \phi &= 4\tau_1 \wedge * \phi + \tau_2 \wedge \phi. \end{aligned} \quad (3.6)$$

Thus, Eq. (3.6) enable to determine all the intrinsic torsion components τ_0 , τ_1 , τ_2 and τ_3 of a given G_2 structure (g, ϕ) . They are called Bryant's [1, 2] equations. It follows that vanishing or not of each of the forms τ_p is a G_2 invariant property of a G_2 structure.

3.3 All $\text{Sp}(4, \mathbb{R})$ symmetric G_2 structures on M_l with the metric coming from the Killing form

The below theorem characterizes the G_2 structures corresponding to compatible pairs (g_K, ϕ) from Corollary 3.4; it summarizes the already obtained results and, in addition, provides formulas for the intrinsic torsion which are needed for the characterization.

Theorem 3.5 *Let g_K be the $(3, 4)$ signature metric on $M_l = \text{Sp}(4, \mathbb{R})/\text{SL}(2, \mathbb{R})_l$ arising as the restriction of the Killing form K from $\text{Sp}(4, \mathbb{R})$ to M_l ,*

$$g_K = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7.$$

Then the most general G_2 structure associated with such g_K is a 3-parameter family (g_K, ϕ) with the 3-form

$$\begin{aligned} \phi &= \frac{ap}{1-q} e^{125} + a(e^{235} - e^{127}) + pe^{145} + q(e^{147} + e^{345}) + \frac{p(1-q)}{4a} e^{156} \\ &\quad + \frac{1-q^2}{4a} (e^{356} - e^{167}) + \frac{a(q-1)}{p} e^{237} + \frac{1}{2} e^{246} + \frac{q^2-1}{p} e^{347} + \frac{(q^2-1)(q+1)}{4ap} e^{367}. \end{aligned}$$

For this structure, the torsions τ_μ solving the Bryant's equations (3.6) are:

$$\begin{aligned}
\tau_0 &= \frac{6}{7} \frac{(2a-p)^2 q - (2a+p)^2}{ap}, \\
\tau_1 &= \frac{1}{4} (2a-p) \left(-e^2 + \frac{1}{2} \frac{(2a+p)(q-1)}{ap} e^4 + \frac{1}{2} \frac{q^2-1}{ap} e^6 \right), \\
\tau_2 &= 0, \\
\tau_3 &= \left(\frac{3}{28} (2a-p)^2 + \frac{8ap}{7(q-1)} \right) e^{125} + \frac{11p^2 + 16ap - 12a^2 + 3q(2a-p)^2}{28p} e^{127} \\
&\quad - \frac{44a^2 + 16ap - 3p^2 + 3q(2a-p)^2}{28a} e^{145} \\
&\quad + \frac{(7-4q)(2a+p)^2 - 3q^2(2a-p)^2}{28ap} e^{147} \\
&\quad + \frac{3p^2(q-1)^2 - 12ap(q^2-1) + 4a^2(31+22q+3q^2)}{112a^2} e^{156} \\
&\quad - \frac{(q^2-1)(44a^2 + 16ap - 3p^2 + 3q(2a-p)^2)}{112a^2 p} e^{167} \\
&\quad + \frac{12a^2 - 16ap - 11p^2 - 3q(2a-p)^2}{28p} e^{235} \\
&\quad - \frac{12a^2(q-1)^2 - 12ap(q^2-1) + p^2(31+22q+3q^2)}{28p^2} e^{237} \\
&\quad + \frac{4ap(6-q) + (4a^2 + p^2)(q-1)}{14ap} e^{246} \\
&\quad + \frac{(7-4q)(2a+p)^2 - 3q^2(2a-p)^2}{28ap} e^{345} \\
&\quad + \frac{(q^2-1)(12a^2 - 16ap - 11p^2 - 3q(2a-p)^2)}{28ap^2} e^{347} \\
&\quad + \frac{(q^2-1)(44a^2 + 16ap - 3p^2 + 3q(2a-p)^2)}{112a^2 p} e^{356} \\
&\quad + \frac{(q^2-1)(q+1)(12a^2 - 44ap + 3p^2 - 3q(2a-p)^2)}{112a^2 p^2} e^{367},
\end{aligned}$$

where as usual $e^{\mu\nu} = e^\mu \wedge e^\nu$ and $e^{\mu\nu\rho} = e^\mu \wedge e^\nu \wedge e^\rho$.

Thus, the 3-parameter family of G_2 structures on M_l described in this theorem have the entire 14-dimensional torsion $\tau_2 = 0$. This means that *all* these G_2 structures are *integrable* in the terminology of [3, 4], or what is the same, this means that they all have the *totally skew symmetric torsion*.

4 G_2 structures on $\mathrm{Sp}(4, \mathbb{R})/\mathrm{SL}(2, \mathbb{R})_s$

Now we consider the homogeneous space $M_s = \mathrm{Sp}(4, \mathbb{R})/\mathrm{SL}(2, \mathbb{R})_s$. Since $\mathfrak{sl}(2, \mathbb{R})$ is spanned by $E_2, E_5 + E_6, E_9$, it is convenient to put these vectors at the end of the new basis of the Lie algebra $\mathfrak{sp}(4, \mathbb{R})$. We choose this new basis (f_l) in $\mathfrak{sp}(4, \mathbb{R})$ as:

$$\begin{aligned} f_1 &= E_1, f_2 = E_3, f_3 = E_4, f_4 = E_6 - E_5, f_5 = E_7, f_6 = E_8, \\ f_7 &= E_{10}, f_8 = E_2, f_9 = E_5 + E_6, f_{10} = E_9. \end{aligned}$$

If now, one considers (f_i) as the basis of the Lie algebra of invariant vector fields on the Lie group $\mathbf{Sp}(4, \mathbb{R})$, then the dual basis (f^I) , $f_i \lrcorner f^I = \delta^I_i$, of the left invariant forms on $\mathbf{Sp}(4, \mathbb{R})$, satisfies:

$$\begin{aligned} df^1 &= 2f^1 \wedge (f^4 - f^9) + f^3 \wedge f^8 \\ df^2 &= -2f^2 \wedge (f^4 + f^9) + 2f^5 \wedge f^8 \\ df^3 &= 2f^1 \wedge f^{10} + 2f^3 \wedge f^4 + f^6 \wedge f^8 \\ df^4 &= 2f^1 \wedge f^7 + \frac{1}{2}f^2 \wedge f^6 + f^3 \wedge f^5 \\ df^5 &= f^2 \wedge f^{10} + 2f^4 \wedge f^5 - 2f^7 \wedge f^8 \\ df^6 &= 2f^3 \wedge f^{10} - 2(f^4 + f^9) \wedge f^6 \\ df^7 &= 2(f^4 - f^9) \wedge f^7 - f^5 \wedge f^{10} \\ df^8 &= 2f^1 \wedge f^5 + f^2 \wedge f^3 - 2f^8 \wedge f^9 \\ df^9 &= -2f^1 \wedge f^7 + \frac{1}{2}f^2 \wedge f^6 + f^8 \wedge f^{10} \\ df^{10} &= 2f^3 \wedge f^7 - f^5 \wedge f^6 - 2f^9 \wedge f^{10}. \end{aligned} \quad (4.1)$$

In this basis, the Killing form on $\mathbf{Sp}(4, \mathbb{R})$ is

$$\begin{aligned} K &= \frac{1}{12} c^I_{JK} c^K_{LI} f^J \odot f^L \\ f^L &= 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7 + 2(f^9)^2 + 2f^8 \odot f^{10}, \end{aligned}$$

where as usual the structure constants c^I_{JK} are defined by $[f_I, f_J] = c^K_{IJ} f_K$.

Using the same arguments, as in the case of M_l , we again see that $\mathbf{Sp}(4, \mathbb{R})$ has the structure of the principal $\mathbf{SL}(2, \mathbb{R})$ fiber bundle $\mathbf{SL}(2, \mathbb{R})_s \rightarrow \mathbf{Sp}(4, \mathbb{R}) \rightarrow M_s = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$ over the homogeneous space $M_s = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$. In particular, we have a foliation of $\mathbf{Sp}(4, \mathbb{R})$ by integral leaves of an integrable distribution D_s spanned by the annihilator of the forms $(f^1, f^2, \dots, f^7)$. As before, also in this section, we will use Greek indices μ, ν , etc., to run from 1 to 7. They now number the first seven basis elements in the bases (f_i) and (f^I) .

Repeating the procedure from the previous sections, I now search for all bilinear symmetric forms $g = g_{\mu\nu} f^\mu \odot f^\nu$ on $\mathbf{Sp}(4, \mathbb{R})$, with constant coefficients $g_{\mu\nu} = g_{\nu\mu}$, whose Lie derivative with respect to any vector field X from D_l vanishes,

$$\mathcal{L}_X g = 0 \text{ for all } X \text{ in } D_s. \quad (4.2)$$

I have the following proposition.

Proposition 4.1 *The most general $g = g_{\mu\nu} f^\mu \odot f^\nu$ satisfying condition (4.2) is*

$$\begin{aligned} g &= g_{33}((f^3)^2 - 2f^1 \odot f^6) + g_{44}(f^4)^2 + g_{55}((f^5)^2 + 2f^2 \odot f^7) \\ &\quad + 2g_{26}(-2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7). \end{aligned}$$

Thus, this time, I only have a 4-parameter family of bilinear forms on $\mathbf{Sp}(4, \mathbb{R})$ that descend to well-defined pseudo-Riemannian metrics on the leaf space M_s . Note that the

restriction of the Killing form K to the space where $(f^8, f^9, f^{10}) \equiv 0$ is in this family. This corresponds to $g_{33} = g_{55} = 0$ and $g_{44} = 2, g_{26} = 1/2$.

Again for simplicity reasons, I will solve the problem of finding $\mathbf{Sp}(4, \mathbb{R})$ invariant G_2 structures on M_s restricting to only those pairs (g, ϕ) for which $g = g_K$, where

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7, \quad (4.3)$$

which again means that I only will consider one metric, the one coming from the restriction of the Killing form of $\mathbf{Sp}(4, \mathbb{R})$ to M_s . It is a well-defined $(3, 4)$ signature metric on the quotient space $M_s = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$.

I now look for the 3-forms $\phi = \frac{1}{6}\phi_{\mu\nu\rho}f^\mu \wedge f^\nu \wedge f^\rho$ on $\mathbf{Sp}(4, \mathbb{R})$ which are such that

$$\mathcal{L}_X \phi = 0 \text{ for all } X \text{ in } D_s. \quad (4.4)$$

I have the following proposition.

Proposition 4.2 *There is precisely a 5-parameter family of 3-forms $\phi = \frac{1}{6}\phi_{\mu\nu\rho}f^\mu \wedge f^\nu \wedge f^\rho$ on $\mathbf{Sp}(4, \mathbb{R})$ which satisfies condition (4.4). The general formula for ϕ is:*

$$\begin{aligned} \phi = & a(4f^{147} + f^{246} + 2f^{345}) + b(2f^{156} + f^{236} - 4f^{137}) + qf^{136} \\ & + h(f^{256} - 4f^{157} - 2f^{237}) + pf^{257}. \end{aligned}$$

Here $f^{\mu\nu\rho} = f^\mu \wedge f^\nu \wedge f^\rho$, and a, b, q, h and p are real constants.

Solving for all 3-forms ϕ from this 5-parameter family that are compatible, as in (3.5), with the metric g_K from (4.3), I arrive at the following proposition.

Proposition 4.3 *The general solution to the equations (3.5) is given by*

$$a = \frac{1}{2}, \quad b = h = 0, \quad p = \frac{1}{q}.$$

This leads to the following corollary.

Corollary 4.4 *The most general pair (g_K, ϕ) on M_s compatible with the $\mathbf{Sp}(4, \mathbb{R})$ invariant metric*

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7,$$

coming from the Killing form in $\mathbf{Sp}(4, \mathbb{R})$, is a 1-parameter family with ϕ given by:

$$\phi = 2f^{147} + \frac{1}{2}f^{246} + f^{345} + qf^{136} + \frac{1}{q}f^{257}.$$

Here $q \neq 0$ is a free parameter, and $f^{\mu\nu\rho} = f^\mu \wedge f^\nu \wedge f^\rho$ as before.

4.1 All $\mathbf{Sp}(4, \mathbb{R})$ symmetric G_2 structures on M_s with the metric coming from the Killing form

Similarly as in Sect. 3.3 we now summarize the already obtained results about the considered $Sp(2, \mathbb{R})$ symmetric G_2 structures on M_s in a theorem; it is given below and has also a new part consisting of the formulas for the intrinsic torsion.

Theorem 4.5 *Let g_K be the $(3, 4)$ signature metric on $M_s = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$ arising as the restriction of the Killing form K from $\mathbf{Sp}(4, \mathbb{R})$ to M_s ,*

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7.$$

Then, the most general G_2 structure associated with such g_K is a 1-parameter family (g_K, ϕ) with the 3-form

$$\phi = 2f^{147} + \frac{1}{2}f^{246} + f^{345} + qf^{136} + \frac{1}{q}f^{257}.$$

For this structure

$$d * \phi = 0,$$

i.e., the torsions

$$\tau_1 = \tau_2 = 0.$$

The rest of the torsions solving Bryant's equations (3.6) are:

$$\begin{aligned} \tau_0 &= -\frac{18}{7}, \\ \tau_3 &= \frac{2}{7} (4f^{147} + f^{246} + 2f^{345}) - \frac{3}{7} (qf^{136} + \frac{1}{q}f^{257}). \end{aligned}$$

where, as usual $f^{\mu\nu\rho} = f^\mu \wedge f^\nu \wedge f^\rho$; $q \neq 0$.

So on $M_s = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$ there exists a 1-parameter family of the above G_2 structures which is *coclosed*. Therefore, in particular, it is *integrable*

I note that formally I can also obtain coclosed G_2 structures on M_l , using Theorem 3.5. It is enough to take $p = 2a$ in the solutions of this Theorem. The question if in the resulting 2-parameter family of the coclosed G_2 structures there is a 1-parameter subfamily equivalent to the structures I have on M_s via Theorem 4.5 needs further investigation. However, I doubt that the answer to this question is positive, since it is visible from the root diagram for $\mathbf{Sp}(4, \mathbb{R})$ that the spaces M_l and M_s are geometrically quite different. Indeed, apart from the $\mathbf{Sp}(4, \mathbb{R})$ invariant G_2 structures, which I have just introduced in this note, the spaces M_l and M_s have quite different *additional* $\mathbf{Sp}(4, \mathbb{R})$ invariant structures. A short look at the root diagram on page 1 of this note shows that M_l has *two* well-defined $\mathbf{Sp}(4, \mathbb{R})$ invariant rank 3-distributions, corresponding to the pushforwards from $\mathbf{Sp}(4, \mathbb{R})$ to M_l of the vector spaces $D_{l1} = \text{Span}_{\mathbb{R}}(E_2, E_3, E_7)$ and $D_{l2} = \text{Span}_{\mathbb{R}}(E_4, E_8, E_9)$. Likewise M_s , in addition to the discussed G_2 structures, has also a well defined pair of $\mathbf{Sp}(4, \mathbb{R})$ invariant rank 3-distributions, corresponding to the pushforwards from $\mathbf{Sp}(4, \mathbb{R})$ to M_s of the vector spaces $D_{s1} = \text{Span}_{\mathbb{R}}(E_1, E_4, E_8)$ and $D_{s2} = \text{Span}_{\mathbb{R}}(E_3, E_7, E_{10})$. The problem is that these two sets of pairs of $\mathbf{Sp}(4, \mathbb{R})$ invariant distributions are quite different. The distributions on M_l have

constant growth vector $(2, 3)$, while the distributions on M_s are *integrable*. These pairs of distributions constitute an immanent ingredient of the geometry on the corresponding spaces M_l and M_s and, since they are diffeomorphically nonequivalent and they make the G_2 geometries there quite different. I believe that this fact makes the G_2 structures obtained on M_l and M_s really nonequivalent.

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