r -Harmonic and Complex Isoparametric Functions on the Lie Groups $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$

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Abstract

In this paper we introduce the notion of complex isoparametric functions on Riemannian manifolds. These are then employed to devise a general method for constructing proper r -harmonic functions. We then apply this to construct the first known explicit proper r -harmonic functions on the Lie group semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$, where $\mathbb{H}^{2n+1}$ denotes the classical $(2n + 1)$ -dimensional Heisenberg group. In particular, we construct such examples on all the simply connected irreducible four-dimensional Lie groups.

Keywords Biharmonic functions · Solvable Lie groups

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1 Introduction

Biharmonic functions are important in physics. Aside from continuum mechanics and elasticity theory, the biharmonic equation also makes an appearance in two-dimensional hydrodynamics problems involving Stokes flows of incompressible Newtonian fluids. A comprehensive review of this fascinating history of biharmonic functions can be found in the article [11].

On this subject, the literature is vast. With only very few exceptions, the domains are either surfaces or open subsets of flat Euclidean space, see, for example, [2]. The development of the very last years has changed this and can be traced, e.g., in the following publications: [4–10]. There the authors develop methods for constructing explicit r -harmonic functions on the classical Lie groups and even some symmetric spaces.

In this paper, we introduce the notion of complex isoparametric functions on a Riemannian manifold (M, g) , see Definition 3.1. It turns out that together with the so-called eigenfunctions they provide us with a method for manufacturing complex-valued proper r -harmonic functions on (M, g) , see Sect. 3.

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We then apply our new method to construct proper r -harmonic functions on the solvable Lie group semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$, where $\mathbb{H}^{2n+1}$ denotes the classical $(2n+1)$ -dimensional Heisenberg group. The study of these particular Lie groups is motivated by the fact that all four-dimensional irreducible Lie groups are, up to isomorphism, semidirect products of one of these two types.

2 Preliminaries

Let (M, g) be a smooth manifold equipped with a Riemannian metric g . We complexify the tangent bundle TM of M to $T^{\mathbb{C}}M$ and extend the metric g to a complex bilinear form on $T^{\mathbb{C}}M$. Then, the gradient $\nabla\phi$ of a complex-valued function $\phi : (M, g) \rightarrow \mathbb{C}$ is a section of $T^{\mathbb{C}}M$. In this situation, the well-known linear *Laplace--Beltrami* operator (alt. *tension* field) τ on (M, g) acts locally on ϕ as follows:

$$\tau(\phi) = \operatorname{div}(\nabla\phi) = \frac{1}{\sqrt{\det g}} \sum_{ij} \frac{\partial}{\partial x_j} \left(g^{ij} \sqrt{\det g} \frac{\partial \phi}{\partial x_i} \right).$$

For two complex-valued functions $\phi, \psi : (M, g) \rightarrow \mathbb{C}$, we have the following well-known product rule

$$\tau(\phi \cdot \psi) = \tau(\phi) \cdot \psi + 2\kappa(\phi, \psi) + \phi \cdot \tau(\psi), \quad (2.1)$$

where the *conformality* operator κ is given by

$$\kappa(\phi, \psi) = g(\nabla\phi, \nabla\psi).$$

Moreover, if $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic function defined on an open set U containing $\phi(M)$, then we have the chain rule

$$\tau(f \circ \phi) = \kappa(\phi, \phi) f''(\phi) + \tau(\phi) f'(\phi). \quad (2.2)$$

For a positive integer r the iterated Laplace--Beltrami operator τ^r is defined inductively by

$$\tau^0(\phi) = \phi, \quad \tau^r(\phi) = \tau(\tau^{r-1}(\phi)).$$

Definition 2.1 Let r be a positive integer. Then, a smooth complex-valued function $\phi : (M, g) \rightarrow \mathbb{C}$ is said to be

- (a) *r*-harmonic if $\tau^r(\phi) = 0$,
- (b) *proper r*-harmonic if $\tau^r(\phi) = 0$ and $\tau^{(r-1)}(\phi)$ does not vanish identically.

It should be noted that the *harmonic* functions are exactly the 1-harmonic and the *biharmonic* functions are the 2-harmonic ones. In some texts, the *r*-harmonic functions are also called *polyharmonic* of order r . We also note that if a function is *r*-harmonic, then it is also *s*-harmonic for any $s \geq r$. Hence, one is usually interested in studying functions which are *proper r*-harmonic.

3 Complex isoparametric functions

A method for constructing proper r -harmonic functions on Riemannian manifolds has recently been developed in [10]. The r -harmonic functions that the authors consider are of the form $f \circ \phi$, where $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic function defined on an open set U containing $\phi(M)$, and $\phi : (M, g) \rightarrow \mathbb{C}$ is an *eigenfunction*, i.e., a smooth complex-valued function such that

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu \cdot \phi^2, \quad (3.1)$$

for some constants $\lambda, \mu \in \mathbb{C}$. The construction from [10] can be adapted to the more general setting when ϕ is a complex isoparametric function, as we will now demonstrate.

Classically, isoparametric functions on Riemannian manifolds are defined as real-valued functions $\phi : (M, g) \rightarrow \mathbb{R}$ such that the tension field τ and the conformality operator κ satisfy

$$\tau(\phi) = \Phi \circ \phi \quad \text{and} \quad \kappa(\phi, \phi) = \Psi \circ \phi, \quad (3.2)$$

for some smooth functions Φ, Ψ . These have been extensively studied due to their beautiful geometric properties, see, e.g., [15]. As we are mainly interested in complex-valued functions, the following complex-valued analogue of the classical real-valued isoparametric functions will turn out to be useful.

Definition 3.1 Let (M, g) be a Riemannian manifold. Then, a smooth complex-valued function $\phi : M \rightarrow \mathbb{C}$ is said to be *complex isoparametric* on M if there exist holomorphic functions $\Phi, \Psi : U \rightarrow \mathbb{C}$ defined on some open set $U \subset \mathbb{C}$ containing $\phi(M)$, such that the tension field τ and the conformality operator κ satisfy

$$\tau(\phi) = \Phi \circ \phi \quad \text{and} \quad \kappa(\phi, \phi) = \Psi \circ \phi. \quad (3.3)$$

Remark 3.2 Note that the term *complex isoparametric* is used here mainly for aesthetical reasons, as the defining conditions (3.2) for real isoparametric functions and (3.3) for complex ones are identical. Despite this similarity, there does not seem to be a direct relation between these two concepts. For example, for a complex-valued function $\phi : M \rightarrow \mathbb{C}$ to be complex isoparametric, it is neither necessary nor sufficient that its real and imaginary parts are real isoparametric. This is caused not only by the restrictive condition of Φ and Ψ being holomorphic, but also by the fact that the relation for the conformality operator may fail to be satisfied, as demonstrated by the functions

$$\mathbb{R} \ni t \mapsto e^{(a+ib)t} \quad \text{and} \quad \mathbb{R} \ni t \mapsto e^{at} + ie^{bt}, \quad a, b \in \mathbb{R}.$$

Indeed, the former is complex isoparametric, but its real and imaginary parts are not real isoparametric, while the latter is not complex isoparametric when $a \neq b$ but its real and imaginary parts are real isoparametric.

Remark 3.3 We also note that other different definitions of complex-valued isoparametric functions have been proposed. For example, in [1] the author defines a function $\phi : M \rightarrow \mathbb{C}$ to be complex isoparametric if

$$\tau(\phi) = \Phi \circ \phi \quad \text{and} \quad |d\phi|^2 = \kappa(\phi, \bar{\phi}) = \Psi \circ \phi$$

for a smooth complex-valued function Φ and a smooth real-valued function Ψ . This definition is less restrictive than ours due to the weaker regularity assumptions on Φ and Ψ , but it is also essentially different because of the assumed condition for the conformality operator κ . Note that the value of $\kappa(\phi, \phi)$ contains no information about the value of $\kappa(\phi, \bar{\phi})$, and vice versa.

A plethora of examples of eigenfunctions (which are a fortiori complex isoparametric) on classical semisimple Lie groups can be found in Table 1 of [10]. Examples of complex isoparametric functions which are not eigenfunctions shall be studied later in this work, see Proposition 5.7.

As in [10], the gist of constructing r -harmonic functions using isoparametric functions is to study compositions of the form $f \circ \phi$, where $\phi : (M, g) \rightarrow \mathbb{C}$ is complex isoparametric and $f : U \rightarrow \mathbb{C}$ is a holomorphic function defined on some open set $U \subset \mathbb{C}$ containing $\phi(M)$. By the chain rule (2.2), the tension field of such compositions satisfies

$$\tau(f \circ \phi) = \kappa(\phi, \phi)f''(\phi) + \tau(\phi)f'(\phi) = (\Psi f'' + \Phi f') \circ \phi.$$

With this in hand, one can first obtain a harmonic function $f_1 \circ \phi$ by solving

$$\tau(f_1 \circ \phi) = 0,$$

which, by the chain rule, reduces to the complex ordinary differential equation

$$\Psi f_1'' + \Phi f_1' = 0.$$

Continuing inductively, we assume that a proper $(r-1)$ -harmonic function of the form $f_{r-1} \circ \phi$ has already been constructed. We can then obtain a proper r -harmonic function $f_r \circ \phi$ by solving the problem

$$\tau(f_r \circ \phi) = f_{r-1} \circ \phi,$$

which again reduces to a complex ordinary differential equation, namely

$$\Psi f_r'' + \Phi f_r' = f_{r-1}.$$

More explicitly, we get the following result.

Theorem 3.4 *Let (M, g) be a Riemannian manifold and $\phi : M \rightarrow \mathbb{C}$ be a complex isoparametric function on M with*

$$\tau(\phi) = \Phi \circ \phi \quad \text{and} \quad \kappa(\phi, \phi) = \Psi \circ \phi,$$

for some holomorphic functions $\Phi, \Psi : U \rightarrow \mathbb{C}$ defined on an open set U of $\mathbb{C}$ containing $\phi(M)$. Suppose that one of the following situations holds.

- (1) *If Ψ vanishes identically, let $\hat{U}$ be an open simply connected subset of $U \setminus \Phi^{-1}(\{0\})$ and define the holomorphic functions $f_r : \hat{U} \rightarrow \mathbb{C}$ for $r \geq 1$ by*

$$f_r(z) = c \left(\int^z \frac{d\zeta}{\Phi(\zeta)} \right)^{r-1},$$

where $c \in \mathbb{C}$ is nonzero.

- (2) If Ψ does not vanish identically, let $\hat{U}$ be an open simply connected subset of $U \setminus \Psi^{-1}(\{0\})$, put

$$\Lambda(z) = \exp \left(- \int^z \frac{\Phi(\zeta)}{\Psi(\zeta)} d\zeta \right), \quad z \in \hat{U}$$

and define the holomorphic functions $f_r : \hat{U} \rightarrow \mathbb{C}$ for $r \geq 1$ by

$$\begin{aligned} f_1(z) &= c_1 \int^z \Lambda(\zeta) d\zeta + c_2, \\ f_r(z) &= \int^z \Lambda(\eta) \int^\eta \frac{f_{r-1}(\zeta)}{\Lambda(\zeta)\Psi(\zeta)} d\zeta d\eta, \quad r > 1, \end{aligned}$$

where $c = (c_1, c_2) \in \mathbb{C}^2$ is nonzero.

Then, in both cases, the composition

$$f_r \circ \phi : \phi^{-1}(\hat{U}) \rightarrow \mathbb{C}$$

is proper r -harmonic on its open domain $\phi^{-1}(\hat{U})$ in M for all $r \geq 1$.

Remark 3.5 If the isoparametric function ϕ is real valued, then one can weaken the assumptions in Theorem 3.4 by only requiring that Φ and Ψ are smooth functions of a real variable and $\hat{U}$ can be taken to be an interval. In the complex-valued case, the requirement that $\hat{U}$ is simply connected is needed to ensure that the holomorphic antiderivatives are well defined.

Upon applying Theorem 3.4 to the particular case when the isoparametric function ϕ is an eigenfunction on (M, g) satisfying (3.1), one sees that the composition $f_r \circ \phi$ is r -harmonic if

$$f_r(z) = \begin{cases} c \log(z)^{r-1} & \text{if } \mu = 0, \lambda \neq 0 \\ c_1 \log(z)^{2r-1} + c_2 \log(z)^{2r-2} & \text{if } \mu \neq 0, \lambda = \mu \\ c_1 z^{1-\frac{\lambda}{\mu}} \log(z)^{r-1} + c_2 \log(z)^{r-1} & \text{if } \mu \neq 0, \lambda \neq \mu \end{cases} \quad (3.4)$$

which has already been obtained in the aforementioned paper [10].

4 The Semidirect Products $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$

Low-dimensional Lie groups, particularly of dimension three and four, are of great importance in physics. Probably most notably such Lie groups are used as models of spacetime in the theory of general relativity. By Lie's third theorem, there is a bijection between real finite-dimensional Lie algebras and connected simply connected Lie groups of the same dimension. Conveniently, real Lie algebras of dimension less than or equal to six have been classified, see, e.g., the recent book [14]. As a consequence, one also obtains classifications of low-dimensional connected simply connected Lie groups.

The study conducted in this paper was initially meant to be a study of r -harmonic functions on the four-dimensional connected simply connected Lie groups. For our purposes, only the Lie groups whose Lie algebras are indecomposable, i.e., not direct

products of lower-dimensional Lie algebras, are of interest. The reason for this is that the Lie groups whose Lie algebras are decomposable are themselves direct products of lower-dimensional Lie groups, and the theory of p -harmonic functions on product manifolds is well known, see, e.g., [5].

According to the classification given, for example, in [12] and [3], all four-dimensional indecomposable real Lie algebras are semidirect products of one of the following types:

$$\mathfrak{r} \ltimes \mathfrak{r}^3, \quad \mathfrak{r}^2 \ltimes \mathfrak{r}^2, \quad \mathfrak{r} \ltimes \mathfrak{h}^3,$$

where $\mathfrak{r}^n$ denotes the n -dimensional abelian algebra and $\mathfrak{h}^{2n+1}$ denotes the $(2n+1)$ -dimensional Heisenberg algebra. For the reader's convenience, we list these Lie algebras in Table 1. We thus see that the corresponding four-dimensional connected simply connected Lie groups are semidirect products of the form

$$\mathbb{R} \ltimes \mathbb{R}^3, \quad \mathbb{R}^2 \ltimes \mathbb{R}^2, \quad \mathbb{R} \ltimes H^3,$$

where H^{2n+1} is the $(2n+1)$ -dimensional Heisenberg group. Linear representations of these four-dimensional Lie groups can be found in Table 2. This motivates our interest in studying the more general semidirect products

$$\mathbb{R}^m \ltimes \mathbb{R}^n \quad \text{and} \quad \mathbb{R}^m \ltimes H^{2n+1}.$$

In particular, we note that such semidirect products are automatically solvable and hence diffeomorphic to the vector space of the corresponding dimension.

4.1 The semidirect products $\mathbb{R}^m \ltimes \mathbb{R}^n$

Let $\mu : \mathbb{R}^m \rightarrow \text{Aut}(\mathbb{R}^n) = \text{GL}_n(\mathbb{R})$ be the smooth homomorphism

$$\mu(t) = \text{Exp} \left(\sum_{k=1}^m A_k t_k \right),$$

for some family $\mathcal{A} = (A_k)_{k=1}^m$ of commuting matrices in $\mathfrak{gl}_n(\mathbb{R})$. Then, the semidirect product $\mathbb{R}^m \ltimes_{\mu} \mathbb{R}^n$ is by definition the smooth manifold $\mathbb{R}^m \times \mathbb{R}^n$ equipped with the Lie group operation

$$(t, x)(s, y) = (t + s, x + \mu(t)y), \quad (t, x), (s, y) \in \mathbb{R}^m \times \mathbb{R}^n.$$

Note that in this case, the family $\mathcal{A}$ completely determines the semidirect product $\mathbb{R}^m \ltimes_{\mu} \mathbb{R}^n$, so that it is natural to use the more suggestive notation $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$.

The Lie group $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$ has a faithful linear representation as the matrix group

$$\left\{ \begin{bmatrix} \mu(t) & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{Exp}(\sum_k D_k t_k) \end{bmatrix} \mid (t, x) \in \mathbb{R}^m \times \mathbb{R}^n \right\},$$

where $(D_k)_{ij} = \delta_{ik} \delta_{jk}$. The m -dimensional block

$$\text{Exp}\left(\sum_{k=1}^m D_k t_k\right)$$

of the representation is needed in general since μ may not be injective, but (parts of) this block may often be omitted. This linear representation induces a natural basis for the corresponding Lie algebra $\mathfrak{r}^m \ltimes_{\mathcal{A}} \mathfrak{r}^n$, namely that consisting of the elements

$$\left.\frac{\partial}{\partial t_k}\right|_0 = \begin{bmatrix} A_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_k \end{bmatrix}, \quad \left.\frac{\partial}{\partial x_i}\right|_0 = \begin{bmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $1 \leq k \leq m$, $1 \leq i \leq n$, and e_i are the canonical unit vectors in $\mathbb{R}^n$.

We now define an inner product on the Lie algebra $\mathfrak{r}^m \ltimes_{\mathcal{A}} \mathfrak{r}^n$ by requiring that this basis becomes orthonormal, and we extend this inner product to a left-invariant Riemannian metric on the Lie group $\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n$. A simple calculation shows that in the coordinates $(t, x) \in \mathbb{R}^m \times \mathbb{R}^n$, this metric is given by

$$g_{(t,x)} = \begin{bmatrix} I_m & 0 \\ 0 & \mu(-t)^T \mu(-t) \end{bmatrix}.$$

It is easy to see that if $\phi, \psi : (\mathbb{R}^m \ltimes_{\mathcal{A}} \mathbb{R}^n, g) \rightarrow \mathbb{C}$ are two complex-valued functions, then the Laplace--Beltrami operator satisfies

$$\tau(\phi) = \sum_{k=1}^m \left(\frac{\partial^2 \phi}{\partial t_k^2} - \text{Tr}(A_k) \frac{\partial \phi}{\partial t_k} \right) + \sum_{i,j=1}^n (\mu(t) \mu(t)^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \quad (4.1)$$

while the conformality operator is given by

$$\kappa(\phi, \psi) = \sum_{k=1}^m \frac{\partial \phi}{\partial t_k} \frac{\partial \psi}{\partial t_k} + \sum_{i,j=1}^n (\mu(t) \mu(t)^T)_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j}. \quad (4.2)$$

4.2 The semidirect products $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$

Throughout this section, we let J_{2n} denote the block diagonal $2n \times 2n$ matrix defined by

$$J_{2n}x = (-x_2, x_1, \dots, -x_{2n}, x_{2n-1}).$$

Note that this is the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

We recall that the Heisenberg group $\mathbb{H}^{2n+1}$ can be seen as the space $\mathbb{R} \times \mathbb{R}^{2n}$ equipped with the Lie group operation

$$(\xi, x) \boxplus (\eta, y) = (\xi + \eta + \frac{1}{2} \langle J_{2n}x, y \rangle, x + y).$$

Its Lie algebra $\mathfrak{h}^{2n+1}$ is nilpotent and has one-dimensional center. More explicitly for a basis $\{\Xi, X_1, \dots, X_{2n}\}$ of $\mathfrak{h}^{2n+1}$ the nonzero Lie brackets are given by

$$[X_{2i-1}, X_{2i}] = \Xi, \quad 1 \leq i \leq n.$$

In what follows, we consider semidirect products $\mathbb{R}^m \ltimes_{\hat{\mu}} H^{2n+1}$ with respect to a specific class of homomorphisms $\hat{\mu} : \mathbb{R}^m \rightarrow \text{Aut}(H^{2n+1})$. The reader interested in more details about this can find them in [13]. Let $\hat{\mu} : \mathbb{R}^m \rightarrow \text{Aut}(H^{2n+1})$ be the smooth homomorphism

$$\hat{\mu}(t) = \begin{bmatrix} a(t) & 0 \\ 0 & \mu(t) \end{bmatrix},$$

where $a : \mathbb{R}^m \rightarrow \mathbb{R}$ and $\mu : \mathbb{R}^m \rightarrow \mathbf{GL}_{2n}(\mathbb{R})$ are given by

$$a(t) = \exp\left(\frac{1}{n} \sum_{k=1}^m \text{Tr}(A_k) t_k\right) \quad \text{and} \quad \mu(t) = \text{Exp}\left(\sum_{k=1}^m A_k t_k\right),$$

for some family $\mathcal{A} = (A_k)_{k=1}^m$ of commuting matrices in $\mathfrak{gl}_{2n}(\mathbb{R})$ of the form

$$\begin{bmatrix} A_{(1,1)} & -\text{adj}(A_{(2,1)}) & -\text{adj}(A_{(3,1)}) & \dots & -\text{adj}(A_{(n,1)}) \\ A_{(2,1)} & A_{(2,2)} & -\text{adj}(A_{(3,2)}) & \dots & -\text{adj}(A_{(n,2)}) \\ A_{(3,1)} & A_{(3,2)} & A_{(3,3)} & \dots & -\text{adj}(A_{(n,3)}) \\ \vdots & & & \ddots & \vdots \\ A_{(n,1)} & A_{(n,2)} & A_{(n,3)} & \dots & A_{(n,n)} \end{bmatrix}, \quad (4.3)$$

where $A_{(i,j)} \in \mathbb{R}^{2 \times 2}$ are such that $\text{Tr} A_{(i,i)} = a$ for $1 \leq i \leq n$, and $\text{adj}(A_{(i,j)})$ denotes the adjugate, i.e., the transpose of the cofactor matrix of $A_{(i,j)}$. In this situation, the Lie group semidirect product $\mathbb{R}^m \ltimes_{\hat{\mu}} H^{2n+1}$ is the manifold $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n}$ equipped with the Lie group operation

$$\begin{aligned} (t, \xi, x)(s, \eta, y) &= (t + s, (\xi, x) \boxplus (a(t)\eta, \mu(t)y)) \\ &= \left(t + s, \xi + a(t)\eta + \frac{1}{2} \langle J_{2n}x, \mu(t)y \rangle, x + \mu(t)y\right), \end{aligned}$$

where $(t, \xi, x), (s, \eta, y) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n}$. As in the previous section, such semidirect products are completely determined by the family $\mathcal{A}$, so that it is natural to use the more suggestive notation $\mathbb{R}^m \ltimes_{\mathcal{A}} H^{2n+1}$. A faithful linear representation of the Lie group $\mathbb{R}^m \ltimes_{\mathcal{A}} H^{2n+1}$ is given by the matrix group

$$\left\{ \begin{bmatrix} a(t) & \frac{1}{2}(J_{2n}x)^T \mu(t) & \xi & 0 \\ 0 & \mu(t) & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \text{Exp}\left(\sum_k D_k t_k\right) \end{bmatrix} \mid (t, \xi, x) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n} \right\},$$

where as before $(D_k)_{ij} = \delta_{ik}\delta_{jk}$. Just as in the previous section, the m -dimensional block

$$\text{Exp}\left(\sum_{k=1}^m D_k t_k\right)$$

of the representation is needed in general, but (parts of) it can often be removed.

We equip the Lie group $\mathbb{R}^m \ltimes_{\mathcal{A}} H^{2n+1}$ with the left-invariant Riemannian metric induced by the inner product on its Lie algebra $\mathfrak{r}^m \ltimes_{\mathcal{A}} \mathfrak{h}^{2n+1}$ defined by requiring that the basis

$$\left. \frac{\partial}{\partial t_k} \right|_0 = \begin{bmatrix} \frac{1}{n} \operatorname{Tr}(A_k) & 0 & 0 & 0 \\ 0 & A_k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_k \end{bmatrix},$$

$$\left. \frac{\partial}{\partial \xi} \right|_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0_{2n} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_m \end{bmatrix}, \quad \left. \frac{\partial}{\partial x_i} \right|_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0_{2n} & e_i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_m \end{bmatrix}$$

on $\mathbf{r}^m \times_{\mathcal{A}} \mathfrak{h}^{2n+1}$ becomes orthonormal. An elementary but long calculation shows that this metric can be expressed in the coordinates $(t, \xi, x) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{2n}$ as

$$g_{(t, \xi, x)} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} J_{2n} x & I_{2n} \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & a(-t)^2 & 0 \\ 0 & 0 & \mu(-t)^T \mu(-t) \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & -\frac{1}{2} (J_{2n} x)^T \\ 0 & 0 & I_{2n} \end{bmatrix}$$

$$= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu(-t)^T \mu(-t) \end{bmatrix} + a(-t)^2 \begin{bmatrix} 0_m & 0 & 0 \\ 0 & 1 & -\frac{1}{2} (J_{2n} x)^T \\ 0 & -\frac{1}{2} J_{2n} x & \frac{1}{4} J_{2n} x (J_{2n} x)^T \end{bmatrix}.$$

With this in hand, it is easy to see that if $\phi, \psi : (\mathbb{R}^m \times_{\mathcal{A}} \mathfrak{h}^{2n+1}, g) \rightarrow \mathbb{C}$ are two complex-valued functions, then the Laplace–Beltrami operator satisfies

$$\begin{aligned} \tau(\phi) &= \sum_{k=1}^m \left(\frac{\partial^2 \phi}{\partial t_k^2} - \frac{n+1}{n} \operatorname{Tr}(A_k) \frac{\partial \phi}{\partial t_k} \right) + \left(a(t)^2 + \frac{1}{4} \langle \mu(t)^T J_{2n} x, \mu(t)^T J_{2n} x \rangle \right) \frac{\partial^2 \phi}{\partial \xi^2} \\ &\quad + \sum_{i=1}^{2n} (\mu(t) \mu(t)^T J_{2n} x)_i \frac{\partial^2 \phi}{\partial \xi \partial x_i} + \sum_{i,j=1}^{2n} (\mu(t) \mu(t)^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \end{aligned} \quad (4.4)$$

and the conformality operator satisfies

$$\begin{aligned} \kappa(\phi, \psi) &= \sum_{k=1}^m \frac{\partial \phi}{\partial t_k} \frac{\partial \psi}{\partial t_k} + \left(a(t)^2 + \frac{1}{4} \langle \mu(t)^T J_{2n} x, \mu(t)^T J_{2n} x \rangle \right) \frac{\partial \phi}{\partial \xi} \frac{\partial \psi}{\partial \xi} \\ &\quad + \frac{1}{2} \sum_{i=1}^{2n} (\mu(t) \mu(t)^T J_{2n} x)_i \left(\frac{\partial \phi}{\partial \xi} \frac{\partial \psi}{\partial x_i} + \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial \xi} \right) \\ &\quad + \sum_{i,j=1}^{2n} (\mu(t) \mu(t)^T)_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j}, \end{aligned} \quad (4.5)$$

5 r -Harmonic Functions on $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$

As we have seen in the preceding sections, there are a lot of similarities between the Laplace--Beltrami operators and the conformality operators on the Lie groups $\mathbb{R}^m \ltimes \mathbb{R}^n$ and $\mathbb{R}^m \ltimes \mathbb{H}^{2n+1}$ so that here we will study these in parallel. To simplify the statements of our results, we fix the following notation.

Notation 5.1 Let G be a Lie group, either $\mathbb{R}^n$ or $\mathbb{H}^{2n+1}$. If $G = \mathbb{R}^n$, the indices i, j are assumed to satisfy $1 \leq i, j \leq n$ and in the case $G = \mathbb{H}^{2n+1}$ the condition $1 \leq i, j \leq 2n$. In both cases, we have $1 \leq k \leq m$, unless otherwise specified. Throughout this section, $\mathcal{A} = (A_k)_k$ will denote a commuting family of invertible real matrices which are of dimensions $n \times n$ if $G = \mathbb{R}^n$ and of dimensions $2n \times 2n$ if $G = \mathbb{H}^{2n+1}$. In the latter case, we also assume that each member of the family $\mathcal{A}$ is of the form (4.3). Finally, we will use the following notation

$$\mathbb{R}^m \ni \omega = \begin{cases} (\operatorname{Tr}(A_1), \dots, \operatorname{Tr}(A_m)) & \text{if } G = \mathbb{R}^n, \\ \frac{n+1}{n} (\operatorname{Tr}(A_1), \dots, \operatorname{Tr}(A_m)) & \text{if } G = \mathbb{H}^{2n+1}. \end{cases} \quad (5.1)$$

Now if $\phi, \psi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ depend only on the coordinates t and x , i.e., if they are independent of ξ , in the case when $G = \mathbb{H}^{2n+1}$, then the Laplace--Beltrami operator satisfies

$$\tau(\phi) = \sum_k \left(\frac{\partial^2 \phi}{\partial t_k^2} - \omega_k \frac{\partial \phi}{\partial t_k} \right) + \sum_{ij} (\mu(t) \mu(t)^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

and the conformality operator is given by

$$\kappa(\phi, \psi) = \sum_k \frac{\partial \phi}{\partial t_k} \frac{\partial \psi}{\partial t_k} + \sum_{ij} (\mu(t) \mu(t)^T)_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j},$$

regardless of the choice of the Lie group G .

Our first result is now the following one of general variable separation.

Proposition 5.2 *Let $\phi, \psi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ be two complex-valued functions such that ϕ depends only on $t \in \mathbb{R}^m$ while $\tau^\alpha(\psi)$ is independent of t for all $\alpha \geq 0$. Then, the tension field of their product $\phi \cdot \psi$ satisfies the identity*

$$\tau^r(\phi \cdot \psi) = \sum_{\alpha=0}^r \binom{r}{\alpha} \tau^\alpha(\phi) \tau^{r-\alpha}(\psi).$$

In particular, if ϕ and ψ are proper p -harmonic and proper r -harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$, respectively, then their product $\phi \cdot \psi$ is proper $(p + r - 1)$ -harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$.

Remark 5.3 This result is reminiscent of the variable separation statement on product manifolds, which can be found in Lemma 6.1 of [5]. For our result here we need the additional assumption that $\tau^\alpha(\psi)$ is independent of t for all $\alpha \geq 0$. This assumption is superfluous for direct products of manifolds, but it is essential in our cases.

Proof Since ϕ is a function of t , we see by the formulae (4.1) and (4.4) for the Laplace–Beltrami operator, that $\tau^\beta(\phi)$ remains to be a function of t for all $\beta \geq 0$. Since by assumption, the tension field $\tau^\alpha(\psi)$ is independent of t for all $\alpha \geq 0$, it follows from the relations (4.2) and (4.5) for the conformality operator, that

$$\kappa(\tau^\beta(\phi), \tau^\alpha(\psi)) = 0,$$

for all $\alpha, \beta \geq 0$. The first part of our statement now follows easily by induction combined with the product rule (2.1) for the Laplace–Beltrami operator. For the final claim of the result notice that the identity we have just proven implies that

$$\tau^{p+r-2}(\phi \cdot \psi) = \binom{p+r-2}{p-1} \tau^{p-1}(\phi) \tau^{r-1}(\psi) \neq 0 \quad \text{and} \quad \tau^{p+r-1}(\phi \cdot \psi) = 0,$$

so that the product $\phi \cdot \psi$ is indeed proper $(p+r-1)$ -harmonic. $\square$

Constructing proper r -harmonic functions $\psi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ such that $\tau^\alpha(\psi)$ is independent of t for $\alpha \geq 0$ seems difficult in general, if not impossible. However, we note that any harmonic function on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ which is independent of t trivially satisfies this condition. On the other hand, we can use our main Theorem 3.4 to construct proper r -harmonic functions depending only on t . We summarize these thoughts in the following statement.

Proposition 5.4

(i) Define the complex-valued function $\psi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by

$$\begin{cases} \psi(x) = a + \sum_i v_i x_i + \sum_{ij} B_{ij} x_i x_j & \text{if } G = \mathbb{R}^n, \\ \psi(\xi, x) = a + b\xi + \sum_i v_i x_i + \sum_{ij} B_{ij} x_i x_j & \text{if } G = \mathbb{H}^{2n+1}, \end{cases}$$

where $a, b, v_i, B_{ij} \in \mathbb{C}$ are not all zero and the coefficients B_{ij} form a symmetric matrix such that

$$\text{Tr}(\mu(t) \mu(t)^T B) = 0, \quad t \in \mathbb{R}^m.$$

Then, ψ is proper harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$.

(ii) Let $(r_k)_{k=1}^m$ be a collection of positive integers and define the complex-valued functions $\phi_k : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by

$$\phi_k(t_k) = \begin{cases} c_1 t_k^{r_k-1} e^{\omega_k t_k} + c_2 t_k^{r_k-1} & \text{if } \omega_k \neq 0, \\ c_1 t_k^{2r_k-1} + c_2 t_k^{2r_k-2} & \text{if } \omega_k = 0, \end{cases}$$

where $c = (c_1, c_2) \in \mathbb{C}^2$ is nonzero. Then, for each $k = 1, \dots, m$, the function ϕ_k is proper r_k -harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$. Furthermore, their product

$$\prod_{k=1}^m \phi_k(t_k)$$

is proper $(r_1 + \dots + r_m - m + 1)$ -harmonic on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$.

Proof To prove part (i) we note that since B is symmetric we have

$$\frac{\partial^2 \psi}{\partial x_i \partial x_j} = 2 B_{ij}$$

so that

$$\tau(\psi) = 2 \sum_{ij} (\mu(t) \mu(t)^T)_{ij} B_{ij} = 2 \operatorname{Tr}(\mu(t) \mu(t)^T B),$$

confirming the result for both choices of the Lie group G .

For part (ii) we note that

$$\tau(t_k) = -\omega_k \quad \text{and} \quad \kappa(t_k, t_k) = 1,$$

so that for each $k = 1, \dots, m$ the coordinate function t_k is isoparametric. The first claim then follows from Theorem 3.4 after an easy calculation. The second claim follows by noting that

$$\tau^r \left(\prod_{k=1}^m \phi_k \right) = \sum_{j_1 + \dots + j_m = r} \binom{r}{j_1, \dots, j_m} \prod_{k=1}^m \tau^{j_k}(\phi_k),$$

which can be proven analogously to Proposition 5.2. $\square$

The matrix B in part (i) of our Proposition 5.4 can of course be taken to be zero. The following examples show though that in important cases this is not necessary.

Example 5.5 Consider the four-dimensional Lie group

$$G_{4.4} = \left\{ \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t & x_1 \\ 0 & e^t & te^t & x_2 \\ 0 & 0 & e^t & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid (t, x) \in \mathbb{R} \times \mathbb{R}^3 \right\}.$$

This is the semidirect product $\mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$ with respect to the family $\mathcal{A}$ consisting of the single matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, following Proposition 5.2, combined with Proposition 5.4, the function

$$(t, x) \mapsto (c_1 t^{r-1} e^{-3t} + c_2 t^{r-1})(a_1 + a_2 x_1 + a_3 x_2 + a_4(x_2^2 - x_3^3 - 2x_1 x_3))$$

is proper r -harmonic on $G_{4.4}$ for any nonzero elements $c \in \mathbb{C}^2$ and $a \in \mathbb{C}^4$. $\triangle$

Example 5.6 For $\alpha \in [-1, 1]$ consider the following four-dimensional Lie group

$$G_{4.8}^\alpha = \left\{ \begin{bmatrix} e^{(1+\alpha)t} & -\frac{x_2}{2}e^t & \frac{x_1}{2}e^{\alpha t} & \xi \\ 0 & e^t & 0 & x_1 \\ 0 & 0 & e^{\alpha t} & x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid (t, \xi, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \right\}.$$

This is the semidirect product $\mathbb{R} \ltimes_{\mathcal{A}} H^3$ with respect to the family $\mathcal{A}$ consisting of the single matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}.$$

Then, Proposition 5.2, in conjunction with Proposition 5.4, implies that the function defined by

$$(t, \xi, x) \mapsto \begin{cases} (c_1 t^{2r-1} + c_2 t^{2r-2})(a_1 + a_2 \xi + a_3 x_1 + a_4 x_2 + a_5 x_1 x_2) & \text{if } \alpha = -1, \\ (c_1 t^{r-1} e^{2(1+\alpha)t} + c_2 t^{r-1})(a_1 + a_2 \xi + a_3 x_1 + a_4 x_2 + a_5 x_1 x_2) & \text{if } \alpha \neq -1 \end{cases}$$

is proper r -harmonic on $G_{4.8}^\alpha$ for any nonzero elements $c \in \mathbb{C}^2$ and $a \in \mathbb{C}^5$. $\triangle$

We now proceed by constructing a non-trivial class of complex isoparametric functions on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$. These functions will depend only on the variables t and x , i.e., they will be independent of the variable ξ if $G = H^{2n+1}$. Here we will use the well-known fact that the elements of a commuting family of matrices always possess a common eigenvector.

Proposition 5.7 *Let v be a common eigenvector of the commuting family $\mathcal{A}^T = (A_k^T)_k$ and $\lambda = (\lambda_1, \dots, \lambda_m)$ be the vector consisting of the corresponding eigenvalues. Then, the complex-valued function $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ defined by*

$$\phi(t, x) = e^{-\langle \lambda, t \rangle} \langle v, x \rangle$$

is complex isoparametric on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ with

$$\tau(\phi) = \langle \lambda, \lambda + \omega \rangle \phi \quad \text{and} \quad \kappa(\phi, \phi) = \langle \lambda, \lambda \rangle \phi^2 + \langle v, v \rangle.$$

Remark 5.8 Here, $\langle \cdot, \cdot \rangle$ denotes the complex bilinear product of vectors given by

$$\langle v, w \rangle = \sum_i v_i w_i, \quad v, w \in \mathbb{C}^n.$$

Proof We have

$$\frac{\partial \phi}{\partial t_k} = -\lambda_k \phi, \quad \frac{\partial^2 \phi}{\partial t_k^2} = \lambda_k^2 \phi, \quad \frac{\partial \phi}{\partial x_i} = e^{-\langle \lambda, t \rangle} v_i \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0.$$

Thus,

$$\tau(\phi) = \sum_k (\lambda_k^2 + \omega_k \lambda_k) \phi = \langle \lambda, \lambda + \omega \rangle \phi,$$

as well as

$$\begin{aligned}
\kappa(\phi, \phi) &= \sum_k \lambda_k^2 \phi^2 + e^{-2\langle \lambda, t \rangle} \sum_{ij} (\mu(t) \mu(t)^T)_{ij} v_i v_j \\
&= \langle \lambda, \lambda \rangle \phi^2 + e^{-2\langle \lambda, t \rangle} \langle \mu(t)^T v, \mu(t)^T v \rangle \\
&= \langle \lambda, \lambda \rangle \phi^2 + \langle v, v \rangle,
\end{aligned}$$

where the final equality follows from the fact that v is an eigenvector of

$$\mu(t)^T = \text{Exp} \left(\sum_k A_k^T t_k \right)$$

with eigenvalue $e^{\langle \lambda, t \rangle}$ by assumption. $\square$

Having constructed complex isoparametric functions, we can now apply Theorem 3.4 to generate examples of r -harmonic functions. As one might suspect, the antiderivatives from Theorem 3.4 cannot be computed explicitly in general. In what follows, we consider several specific cases where this is possible.

Example 5.9 The four-dimensional Lie group

$$G_{4,1} = \left\{ \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & x_1 \\ 0 & 1 & t & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid (t, x) \in \mathbb{R} \times \mathbb{R}^3 \right\}$$

is the semidirect product $\mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$, where $\mathcal{A}$ consists of the single matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

All the eigenvalues of A^T are 0 and the eigenspace is the one-dimensional space spanned by the unit vector e_3 . Thus, the function ϕ from Proposition 5.7 is simply the coordinate function $(t, x) \mapsto x_3$ satisfying

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 1.$$

A simple application of Theorem 3.4 then shows that the function

$$(t, x) \mapsto a_1 x_3^{2r-1} + a_2 x_3^{2r-2}$$

is proper r -harmonic on $G_{4,1}$. By construction this function also satisfies the condition from Proposition 5.2 in that its Laplacian of any order is independent of t . Hence, if p, r, q are positive integers such that $r + q - 1 = p$, Proposition 5.2 combined with part (ii) Proposition 5.4 implies that the function defined by

$$(t, x) \mapsto (c_1 t^{2r-1} + c_2 t^{2r-2})(a_1 x_3^{2q-1} + a_2 x_3^{2q-2})$$

is proper p -harmonic on $G_{4,1}$ for any nonzero $a, c \in \mathbb{C}^2$. $\triangle$

Example 5.10 The celebrated Thurston geometry Sol^3 is the solvable Lie group

$$\mathbf{Sol}^3 = \left\{ \begin{bmatrix} e^t & 0 & x_1 \\ 0 & e^{-t} & x_2 \\ 0 & 0 & 1 \end{bmatrix} \mid (t, x) \in \mathbb{R} \times \mathbb{R}^2 \right\}.$$

Examples of proper r -harmonic functions on this Lie group have already been constructed in [5] and [9]. We can view this Lie group as the semidirect product $\mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^2$, where $\mathcal{A}$ consists of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of A^T are 1 and -1 , and the corresponding eigenvectors are e_1 and e_2 , respectively. In view of Proposition 5.7 we get an isoparametric function for each eigenvector, namely

$$\phi_1(t, x) = e^{-t}x_1 \quad \text{and} \quad \phi_2(t, x) = e^tx_2,$$

both of which satisfy

$$\tau(\phi_i) = \phi_i \quad \text{and} \quad \kappa(\phi_i, \phi_i) = \phi_i^2 + 1.$$

Applying Theorem 3.4 to these isoparametric functions, we find that the functions

$$\begin{aligned} (t, x) &\mapsto c_1 \operatorname{arsinh}(e^{-t}x_1)^{2r-1} + c_2 \operatorname{arsinh}(e^{-t}x_1)^{2r-2}, \\ (t, x) &\mapsto c_1 \operatorname{arsinh}(e^tx_2)^{2r-1} + c_2 \operatorname{arsinh}(e^tx_2)^{2r-2} \end{aligned}$$

are proper r -harmonic on $\mathbf{Sol}^3$ for any nonzero $c \in \mathbb{C}^2 \setminus \Delta$.

The reader should note that in both Examples 5.9 and 5.10 the eigenvector v is non-isotropic, i.e., $\langle v, v \rangle \neq 0$. This is of course expected since the eigenvector is real in both cases. However, if the eigenvector v turns out to be complex, then it could happen that it is isotropic. In this case we see directly that the function ϕ from Proposition 5.7 becomes an eigenfunction cf. (3.1). In fact, a quick inspection of the proof of Proposition 5.7 shows that in this case the vector λ can be taken arbitrarily, rather than requiring it to consist of eigenvalues.

Corollary 5.11 *Let v be a common eigenvector of the commuting family $\mathcal{A}^T = (A_k^T)_k$ and suppose that v is isotropic. Then, for any complex vector $v \in \mathbb{C}^m$ the function $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ defined by*

$$\phi(t, x) = e^{-\langle v, t \rangle} \langle v, x \rangle$$

is an eigenfunction on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ satisfying

$$\tau(\phi) = \langle v, v + \omega \rangle \phi \quad \text{and} \quad \kappa(\phi, \phi) = \langle v, v \rangle \phi^2.$$

For this situation the antiderivatives from Theorem 3.4 have already been computed in [10], cf. also (3.4).

Example 5.12 Consider the following interesting four-dimensional Lie group

$$G_{4,10} = \left\{ \begin{bmatrix} e^{-t_1} \cos(t_2) & e^{-t_1} \sin(t_2) & x_1 & 0 \\ -e^{-t_1} \sin(t_2) & e^{-t_1} \cos(t_2) & x_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{t_2} \end{bmatrix} \mid (t, x) \in \mathbb{R}^2 \times \mathbb{R}^2 \right\}.$$

This is the semidirect product $\mathbb{R}^2 \ltimes_{\mathcal{A}} \mathbb{R}^2$, where the family $\mathcal{A}$ consists of the two commuting matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Their two common eigenvectors are $v_{\pm} = (1, \pm i)$, both of which are isotropic. Hence, we see from Corollary 5.11 that for any $v = (v_1, v_2) \in \mathbb{C}^2$, the functions

$$\phi_{\pm}(t, x) = e^{-(v_1 t_1 + v_2 t_2)}(x_1 \pm ix_2)$$

are eigenfunctions on $G_{4,10}$ with

$$\tau(\phi_{\pm}) = (v_1^2 - 2v_1 + v_2^2) \phi_{\pm} \quad \text{and} \quad \kappa(\phi_{\pm}, \phi_{\pm}) = (v_1^2 + v_2^2) \phi_{\pm}^2.$$

□

In the case when v is isotropic, we can take the real and the imaginary part of the function ϕ from Proposition 5.7 to obtain even more isoparametric functions.

Proposition 5.13 *Let v be a common eigenvector of the commuting family $A^T = (A_k^T)_k$ and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be the vector consisting of the corresponding eigenvalues. Further suppose that the eigenvector v is isotropic and define the function $\phi : \mathbb{R}^m \ltimes_{\mathcal{A}} G \rightarrow \mathbb{C}$ by*

$$\phi(t, x) = e^{-\langle \Re \lambda, t \rangle} \langle v, x \rangle.$$

Then, the real part $\phi_1 = \Re \phi$ and the imaginary part $\phi_2 = \Im \phi$ of ϕ are isoparametric functions on $\mathbb{R}^m \ltimes_{\mathcal{A}} G$ with

$$\begin{aligned} \tau(\phi_i) &= \langle \Re \lambda, \Re \lambda + \omega \rangle \phi_i, \\ \kappa(\phi_i, \phi_i) &= \langle \Re \lambda, \Re \lambda \rangle \phi_i^2 + \frac{1}{2} \langle v, \bar{v} \rangle. \end{aligned}$$

Proof The result follows by direct calculations and Proposition 5.7. □

Example 5.14 For $\alpha \geq 0$, consider the four-dimensional Lie group $G_{4,9}^{\alpha}$ given by

$$\begin{bmatrix} e^{2\alpha t} & -\frac{x_2 \cos(t) + x_1 \sin(t)}{2} e^{\alpha t} & \frac{x_1 \cos(t) - x_2 \sin(t)}{2} e^{\alpha t} & \xi & 0 \\ 0 & e^{\alpha t} \cos(t) & e^{\alpha t} \sin(t) & x_1 & 0 \\ 0 & -e^{\alpha t} \sin(t) & e^{\alpha t} \cos(t) & x_2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^t \end{bmatrix}$$

where $(t, \xi, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$. We can view this group as the semidirect product $\mathbb{R} \ltimes_{\mathcal{A}} \mathbb{H}^3$ with respect to the family $\mathcal{A}$ consisting of the single matrix

$$A = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}.$$

The two eigenvectors of A^T are $v_{\pm} = (1, \pm i)$ and the corresponding eigenvalues are $\lambda_{\pm} = \alpha \pm i$. Note that the eigenvectors $v_{\pm}$ are isotropic. In what follows, it suffices to only consider the eigenvector $v = (1, i)$ and its eigenvalue $\lambda = \alpha + i$. The functions ϕ_1 and ϕ_2 from Proposition 5.13 are then given by

$$\phi_i(t, \xi, x) = e^{-\alpha t} x_i,$$

satisfying

$$\tau(\phi_i) = 3\alpha^2 \phi_i \quad \text{and} \quad \kappa(\phi_i, \phi_i) = \alpha^2 \phi_i^2 + 1.$$

We now apply Theorem 3.4 to these isoparametric functions.

In the case when $\alpha = 0$ an easy calculation of the antiderivatives from Theorem 3.4 shows that for any positive integer r and any nonzero $c \in \mathbb{C}^2$, the functions

$$(t, \xi, x) \mapsto c_1 x_i^{2r-1} + c_2 x_i^{2r-2}, \quad i = 1, 2$$

are proper r -harmonic on $G_{4,9}^0$. By construction, these functions also satisfy the condition from Proposition 5.2 so we may also multiply them by a function depending only on t to obtain even more examples of proper r -harmonic functions cf. Example 5.9.

In the case when $\alpha \neq 0$ the functions f_p from Theorem 3.4 do not seem to possess a nice closed formula. However, let us at least note that for any nonzero $c \in \mathbb{C}^2$ the functions given by

$$(t, \xi, x) \mapsto c_1 \frac{2(\alpha e^{-\alpha t} x_i)^3 + 3\alpha e^{-\alpha t} x_i}{\{(\alpha e^{-\alpha t} x_i \}^2 + 1\}^{3/2}} + c_2, \quad i = 1, 2$$

are proper harmonic on $G_{4,9}^{\alpha}$ and the functions given by

$$\begin{aligned} (t, \xi, x) \mapsto & c_1 \left(\operatorname{arsinh}(\alpha e^{-\alpha t} x_i) + \frac{(\alpha e^{-\alpha t} x_i)^3}{3\{(\alpha e^{-\alpha t} x_i)^2 + 1\}^{3/2}} \right) \\ & + c_2 \left(\frac{2(\alpha e^{-\alpha t} x_i)^3 + 3\alpha e^{-\alpha t} x_i}{\{(\alpha e^{-\alpha t} x_i)^2 + 1\}^{3/2}} \operatorname{arsinh}(\alpha e^{-\alpha t} x_i) - \frac{1}{(\alpha e^{-\alpha t} x_i)^2 + 1} \right), \quad i = 1, 2 \end{aligned}$$

are proper biharmonic on $G_{4,9}^{\alpha}$.

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Appendix A. Tables

Table 1 [3, 12] The (indecomposable) four-dimensional Lie algebras

Lie algebra	Parameters	Lie brackets, basis $\{E_1, E_2, E_3, E_4\}$		
$\mathfrak{g}_{4.1}$		$[E_4, E_1] = 0,$	$[E_4, E_2] = E_1,$	$[E_4, E_3] = E_2,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = 0,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.2}^\alpha$	$\alpha \neq 0$	$[E_4, E_1] = \alpha E_1,$	$[E_4, E_2] = E_2,$	$[E_4, E_3] = E_2 + E_3,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = 0,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.3}$		$[E_4, E_1] = E_1,$	$[E_4, E_2] = 0,$	$[E_4, E_3] = E_2,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = 0,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.4}$		$[E_4, E_1] = E_1,$	$[E_4, E_2] = E_1 + E_2,$	$[E_4, E_3] = E_2 + E_3,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = 0,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.5}^{\alpha\beta\gamma}$	$\alpha\beta\gamma \neq 0$	$[E_4, E_1] = \alpha E_1,$	$[E_4, E_2] = \beta E_2,$	$[E_4, E_3] = \gamma E_3,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = 0,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.6}^{\alpha\beta}$	$\alpha > 0,$ $\beta \in \mathbb{R}$	$[E_4, E_1] = \alpha E_1,$	$[E_4, E_2] = \beta E_2 - E_3,$	$[E_4, E_3] = E_2 + \beta E_3,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = 0,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.7}$		$[E_4, E_1] = 2E_1,$	$[E_4, E_2] = E_2,$	$[E_4, E_3] = E_2 + E_3,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = -E_1,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.8}^\alpha$	$\alpha \in [-1, 1]$	$[E_4, E_1] = (1 + \alpha)E_1,$	$[E_4, E_2] = E_2,$	$[E_4, E_3] = \alpha E_3,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = -E_1,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.9}^\alpha$	$\alpha \geq 0$	$[E_4, E_1] = 2\alpha E_1,$	$[E_4, E_2] = \alpha E_2 - E_3,$	$[E_4, E_3] = E_2 + \alpha E_3,$
		$[E_3, E_1] = 0,$	$[E_3, E_2] = -E_1,$	$[E_2, E_1] = 0$
$\mathfrak{g}_{4.10}$		$[E_4, E_1] = -E_2,$	$[E_4, E_2] = E_1,$	$[E_4, E_3] = 0,$
		$[E_3, E_1] = -E_1,$	$[E_3, E_2] = -E_2,$	$[E_2, E_1] = 0$

Table 2 [3, 12] Linear representations of the four-dimensional Lie groups

Lie group	Parameters	Family $\mathcal{A}$	Linear representation
$G_{4,1} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$		$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & x_1 \\ 0 & 1 & t & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,2}^{\alpha} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3 \quad \alpha \neq 0$		$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} & 0 & 0 & x_1 \\ 0 & e^t & te^t & x_2 \\ 0 & 0 & e^t & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,3} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} e^t & 0 & 0 & x_1 \\ 0 & 1 & t & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,4} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3$		$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t & x_1 \\ 0 & e^t & te^t & x_2 \\ 0 & 0 & e^t & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,5}^{\alpha\beta\gamma} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3 \quad \alpha\beta\gamma \neq 0$		$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} & 0 & 0 & x_1 \\ 0 & e^{\beta t} & 0 & x_2 \\ 0 & 0 & e^{\gamma t} & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,6}^{\alpha\beta} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{R}^3 \quad \alpha > 0, \beta \in \mathbb{R}$		$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 1 \\ 0 & -1 & \beta \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} & 0 & 0 & x_1 \\ 0 & e^{\beta t} \cos(t) & e^{\beta t} \sin(t) & x_2 \\ 0 & -e^{\beta t} \sin(t) & e^{\beta t} \cos(t) & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,7} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{H}^3$		$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^{2t} & -\frac{x_2}{2}e^t & \frac{x_1 - tx_2}{2}e^t & \xi \\ 0 & e^t & te^t & x_1 \\ 0 & 0 & e^t & x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,8}^{\alpha} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{H}^3 \quad \alpha \in [-1, 1]$		$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$	$\begin{bmatrix} e^{(1+\alpha)t} & -\frac{x_2}{2}e^t & \frac{x_1}{2}e^{\alpha t} & \xi \\ 0 & e^t & 0 & x_1 \\ 0 & 0 & e^{\alpha t} & x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$G_{4,9}^{\alpha} = \mathbb{R} \ltimes_{\mathcal{A}} \mathbb{H}^3 \quad \alpha \geq 0$		$\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$	$\begin{bmatrix} e^{2\alpha t} & -\frac{x_2 \cos(t) + x_1 \sin(t)}{2}e^{\alpha t} & \frac{x_1 \cos(t) - x_2 \sin(t)}{2}e^{\alpha t} & \xi & 0 \\ 0 & e^{\alpha t} \cos(t) & e^{\alpha t} \sin(t) & x_1 & 0 \\ 0 & -e^{\alpha t} \sin(t) & e^{\alpha t} \cos(t) & x_2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^t \end{bmatrix}$
$G_{4,10} = \mathbb{R}^2 \ltimes_{\mathcal{A}} \mathbb{R}^2$		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} e^{-t_1} \cos(t_2) & e^{-t_1} \sin(t_2) & x_1 & 0 \\ -e^{-t_1} \sin(t_2) & e^{-t_1} \cos(t_2) & x_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{t_2} \end{bmatrix}$

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