Lower semicontinuity for the Helfrich problem

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Abstract

We minimise the Canham–Helfrich energy in the class of closed immersions with prescribed genus, surface area, and enclosed volume. Compactness is achieved in the class of oriented varifolds. The main result is a lower-semicontinuity estimate for the minimising sequence, which is in general false under varifold convergence by a counter example by Große-Brauckmann. The main argument involved is showing partial regularity of the limit. It entails comparing the Helfrich energy of the minimising sequence locally to that of a biharmonic graph. This idea is by Simon, but it cannot be directly applied, since the area and enclosed volume of the graph may differ. By an idea of Schygulla we adjust these quantities by using a two parameter diffeomorphism of $\mathbb{R}^3$.

Keywords Closed immersions · Canham–Helfrich energy · Lower semicontinuity · Variational problem · Oriented varifolds

Mathematics Subject Classification 26B15 · 35J35 · 49J45 · 49Q20 · 58J90

1 Introduction

This article deals with minimising the Helfrich energy for closed oriented smooth connected two-dimensional immersions $f : \Sigma \rightarrow \mathbb{R}^3$, which is defined as

$$W_{H_0}(f) := \int_{\Sigma} (\bar{H} - H_0)^2 d\mu_g. \quad (1.1)$$

Here $\bar{H}$ is the scalar mean curvature, i.e. the sum of the principal curvatures with respect to a chosen unit normal of f . We call μ_g the area measure on Σ induced by f and the euclidean metric of $\mathbb{R}^3$. Furthermore, $H_0 \in \mathbb{R}$ is called spontaneous curvature. This energy was introduced by Helfrich [20] and Canham [3] to model the shape of blood cells. Hence it is

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called the Canham–Helfrich or short Helfrich energy. We recover the Willmore energy by setting $H_0 = 0$ and multiplying by $\frac{1}{4}$, i.e.

$$W_{\text{Will}}(f) := \frac{1}{4} \int_{\Sigma} |\bar{H}|^2 dA = \frac{1}{4} W_0(f). \quad (1.2)$$

The Willmore energy goes back to Thomsen [40]. He denoted critical points of the Willmore energy as conformal minimal surfaces. Willmore later revived the mathematical discussion in [41]. Please note, that for the Willmore energy an orientation is not needed, contrary to the Helfrich energy. Hence we need a fixated normal. We assume Σ to have a continuous orientation τ , given as a 2-form. Then we set $\nu_f := * (df(\tau)) \in \partial B_1(0) \subset \mathbb{R}^3$ as the unit normal of f . Here $*$ denotes the Hodge- $*$ -Operator. Furthermore, we like to prescribe the area and enclosed volume of f . Therefore we set

$$\text{Area}(f) := \int_{\Sigma} d\mu_g, \quad \text{Vol}(f) := \frac{1}{3} \int_{\Sigma} f(x) \cdot \nu_f(x) d\mu_g(x). \quad (1.3)$$

Please note that, if f is an embedding and ν_f the outer normal, $\text{Vol}(f)$ would be the volume of the set enclosed by f . In the general case, $\text{Vol}(f)$ may become negativ dependent on the orientation.

Minimisers of such a problem satisfy the following Euler–Lagrange equation (see, e.g. [31, Eq. (31)])

$$2\Delta_f \bar{H} + 4\bar{H} \left(\frac{1}{4} \bar{H}^2 - K \right) - 2H_0 K - H_0^2 \bar{H} - \lambda_A \bar{H} + \lambda_V = 0. \quad (1.4)$$

Here, $\lambda_A \in \mathbb{R}$ and $\lambda_V \in \mathbb{R}$ correspond to Lagrange multipliers for the prescribed area and enclosed volume, respectively. This differential equation is highly nonlinear since the Laplace–Beltrami Δ_f depends on the unknown immersion f . Furthermore, it is of fourth order, hence standard techniques like the maximum principle are not applicable. Nevertheless a lot of important results concerning such problems have been achieved: Existence of closed Willmore surfaces of arbitrary genus has been shown in the papers [39] and [2]. If the ambient space becomes a general 3-manifold, the problem has been examined in, e.g. [30].

Prescribing additional conditions and showing existence has also been very successful, i.e. the isoperimetric ratio in [23] and [36], the area in [25], boundary conditions in [6, 32, 35] and [9]. Finally, the Willmore conjecture was shown to be true in [28]. Further research and references for Willmore problems are summarized in the surveys [21] resp. [19]. The class of axisymmetric surfaces is especially important for modelling purposes; see, e.g. [11]. Also some existence results can be more readily achieved in this class, see, e.g. [7, 12] or [14] and solutions to (1.4) can be analysed in greater depths, e.g. the behaviour of singularities in [16].

Furthermore, the addition of the orientation complicates and changes the situation in the Helfrich setting. For example, the class of invariances is notably smaller (see, e.g. [8]) and lower semi-continuity is in general false under varifold convergence; see [18, p. 550, Remark (ii)] for a counterexample. Nevertheless, some progress has been made: Existence of Helfrich surfaces with prescribed surface area near the sphere has been achieved in [24] by examining the corresponding L^2 -flow. This result has been extended to a general existence result for spherical Helfrich immersions with prescribed area and enclosed volume in [29] by variational parametric methods. The axisymmetric case has been handled independently in [5] and [4]. A more general approach was used in [10] by working with

Gauss-graphs. There lower semicontinuity was shown, but the limit has to be in C^2 , which is a priori not clear.

In general, this is a hard problem, since the Helfrich energy for oriented varifolds lacks a variational characterisation (cf. (2.3) and cf. [13, p. 3]).

In this paper, we will show a lower-semicontinuity estimate for minimising sequences with an arbitrary but fixed topology by an ambient approach, i.e. geometric measure theory. In the case of spherical topology this has been achieved with a parametric approach in [29, Thm. 3.3]. The case of embeddings still remains open, since we do not have a Li-Yau type inequality. This prevents our arguments to be adaptable to this situation as explained in Remark 5.2.

For our minimising procedure, we introduce the following set for a given two-dimensional smooth connected manifold Σ without boundary, which fixes the topology.

$$M_{Area_0, Vol_0} := \left\{ f : \Sigma \rightarrow \mathbb{R}^3, \text{ 2-dimensional oriented smooth immersion, } \right. \\ \left. Area(\Sigma) = Area_0, Vol(\Sigma) = Vol_0, 0 \in f(\Sigma) \right\} \quad (1.5)$$

for a positive parameter $Area_0 > 0$ and a nontrivial real parameter $Vol_0 \in \mathbb{R} \setminus \{0\}$. Furthermore let us call

$$E_{H_0, Area_0, Vol_0} := \inf \{ W_{H_0}(f), f \in M_{Area_0, Vol_0} \}. \quad (1.6)$$

Now we can state our main result in case $M_{Area_0, Vol_0} \neq \emptyset$ (For unknown terminology please consult Sect. 2):

Theorem 1.1 *Let $f_k \in M_{Area_0, Vol_0}$ be a minimising sequence, i.e. it satisfies $W_{H_0}(f_k) \rightarrow E_{H_0, Area_0, Vol_0}$. Let also V_k^0 be the sequence of oriented integral 2 varifolds corresponding to f_k by (2.4). Then there exists an oriented integral 2 varifold V^0 on $\mathbb{R}^3$ and a subsequence $V_{k_j}^0$, such that*

$$V_{k_j} \rightarrow V^0 \text{ as oriented varifolds,} \\ Area(V^0) = Area_0, \quad Vol(V^0) = Vol_0 \neq 0.$$

More importantly, this subsequence enjoys a lower-semicontinuity estimate:

$$W_{H_0}(V^0) \leq \liminf_{j \rightarrow \infty} V_{k_j}^0.$$

Furthermore, there exist at most finitely many bad points, such that V^0 is locally a union of $C^{1,\beta} \cap W^{2,2}$ graphs outside these bad points. For a precise statement of the graphical decomposition please refer to Lemma 5.1.

The proof works in two major steps. The first is to show some preliminary regularity of V^0 (see Lemma 5.1). After this the lower-semicontinuity estimate follows precisely as in [13, Lemma 4.1], hence we will not include this step here.

The initial regularity is shown by comparing the Helfrich energy of the minimising sequence to a biharmonic replacement. This idea was first used by Simon in [39]. Since the biharmonic replacement does not have the same area and/or enclosed volume, we will correct these parameters by an idea of Schygulla [36]. We generalise Schygulla's techniques to the case of immersions and to two prescribed quantities. This idea essentially is adjusting these quantities of the biharmonic replacement outside of the replacement region by a two parameter diffeomorphism.

The paper is build up as follows: Sect. 2 is concerned with compactness in the class of oriented varifolds. Here the Helfrich energy, the area and the enclosed volume are formulated for these varifolds. Next in Sect. 3 we examine our definition of enclosed volume for oriented varifolds in greater detail and for example calculate an Euler–Lagrange equation. Afterwards we construct the aforementioned two parameter diffeomorphism and analyse these in 4. In Sect. 5 we finally show the initial $C^{1,\beta}$ -regularity, from which the lower-semicontinuity estimate follows. In Appendix A, we collect some usefull results for our reasoning.

2 Compactness

In this chapter, we recall the necessary results and objects to obtain compactness for a minimising sequence in measure theoretic terms:

Since the Helfrich energy depends on the orientation, we will work with oriented varifolds in our variational framework. Oriented varifolds were introduced by Hutchinson in [22, Sect. 3]. We recall the necessary definitions here (see also [13, Appendix B]). First

$$G^0(2, 3) = \{ \tau^1 \wedge \tau^2 \in \Lambda_2 \mathbb{R}^3 : \tau^1, \tau^2 \in \mathbb{R}^3, |\tau^1| = |\tau^2| = 1, \tau^1 \perp \tau^2 \}$$

is called the oriented Grassmannian manifold of two-dimensional oriented linear subspaces in $\mathbb{R}^3$. Since we need to connect orientations with normals, we also need the Hodge star operator $*$: $G^0(2, 3) \rightarrow \partial B_1(0)$. In our setting, $*$ becomes just the cross product, i.e. $*(\tau^1 \wedge \tau^2) = \tau^1 \times \tau^2$. An oriented integral 2-varifold on an open set $\Omega \subset \mathbb{R}^3$ is given by a countable 2-rectifiable set $M \subset \Omega$, $\mathcal{H}^2$ -measurable densities $\theta_+, \theta_- : M \rightarrow \mathbb{N}_0$ and an orientation $\xi : M \rightarrow G^0(2, 3)$, such that $*(\xi(x)) \perp T_x M$ for $\mathcal{H}^2$ a.e. $x \in M$. Then the corresponding oriented varifold is a Radon measure on $\Omega \times G^0(2, 3)$ given for $\Phi \in C_0^0(\Omega \times G^0(2, 3))$ by

$$V^0[M, \theta_+, \theta_-, \xi](\Phi) := \int_M \Phi(x, \xi(x)) \theta_+(x) + \Phi(x, -\xi(x)) \theta_-(x) d\mathcal{H}^2(x). \quad (2.1)$$

Furthermore, we need to define the Helfrich energy for such an oriented integral varifold. Hence we need to make sense of a mean curvature. For this, let $\pi^0 : \mathbb{R}^3 \times G^0(2, 3)$ be given by $\pi^0(x, \xi) = x$. Then the mass of V^0 is defined as

$$\mu_{V^0} := \pi^0(V^0) = (\theta_+ + \theta_-) \mathcal{H}^2 \llcorner M,$$

which is also an integral varifold in the sense of [38, Sect. 15]. The first variation of V^0 is defined as the first variation of μ_{V^0} , i.e. $\delta V^0 := \delta \mu_{V^0}$. Thus we define the mean curvature vector of V^0 to be the mean curvature vector of μ_{V^0} (cf. [38, Sect. 16]). We say V^0 has a mean curvature vector $H_{V^0} \in L^2(\mu_{V^0})$, if H_{V^0} is square integrable w.r.t. μ_{V^0} and for every $X \in C_0^1(\Omega, \mathbb{R}^3)$ we have

$$\delta V^0(X) = - \int_{\Omega} H_{V^0} \cdot X d\mu_{V^0}.$$

This means that μ_{V^0} does not have a generalized boundary in the sense of [38, Sect. 39].

The integral 2 current associated with V^0 is given by

$$[|V^0|](\omega) := \int_{G^0(\Omega)} \langle \omega(x), \xi \rangle dV^0(x, \xi), \quad \omega \in C_0^\infty(\Omega, \Lambda^2 \mathbb{R}^3).$$

We choose the same notations for currents as in [38, Chapter 6].

For an integral oriented varifolds $V^0 = V^0[M, \theta_+, \theta_-, \xi]$ the current satisfies

$$[|V^0|](\omega) = \int_M \langle \omega(x), \xi(x) \rangle (\theta_+(x) - \theta_-(x)) d\mathcal{H}^2$$

and is therefore integral as well. We will call $\partial[|V^0|]$ the boundary in the sense of currents of V^0 . Here ∂ is the boundary operator for currents (see, e.g. [38, Eq. (26.3)]).

Convergence of oriented varifolds is defined as weak convergence of the corresponding Radon measures. That is, we say a sequence of oriented integral varifolds V_k^0 on $\Omega \subset \mathbb{R}^3$ converges weakly to an oriented integral varifold V^0 , iff for all $\Phi \in C_0^0(\Omega \times G^0(2, 3))$ we have $V_k^0(\Phi) \rightarrow V^0(\Phi)$. Then [22, Theorem 3.1] gives us, that the following set is sequentially compact with respect to oriented varifold convergence:

$$\begin{aligned} \{V^0 \text{ oriented integral varifold} : \forall \Omega' \subset\subset \Omega \exists C(\Omega') < \infty : \\ \mu_{V^0}(\Omega') + \|\delta \mu_{V^0}\|(\Omega') + M_{\Omega'}(\partial[|V^0|]) \leq C(\Omega')\} \end{aligned} \quad (2.2)$$

Here, $M_{\Omega'}(\cdot)$ denotes the mass in the sense of currents.

The Helfrich energy of V^0 is (see also [13, Eq. (2.1)])

$$\begin{aligned} W_{H_0}(V^0) &= \int_M |H_{V^0}(x) - H_0(*\xi(x))|^2 \theta_+(x) \\ &\quad + |H_{V^0}(x) + H_0(*\xi(x))|^2 \theta_-(x) d\mathcal{H}^2(x) \\ &= \int |H_{V^0}(x) - H_0(*\xi)|^2 dV^0(x, \xi). \end{aligned} \quad (2.3)$$

Furthermore, we need to define the enclosed volume and the area of such oriented varifolds:

$$\text{Area}(V^0) := \mu_{V^0}(\mathbb{R}^3), \quad \text{Vol}(V^0) := \frac{1}{3} \int \langle x, (*\xi) \rangle dV^0(x, \xi).$$

Now let Σ be a smooth oriented two-dimensional manifold and $f : \Sigma \rightarrow \mathbb{R}^3$ a smooth immersion. To employ the compactness criterion (2.2) we need to define a corresponding oriented integral varifold (see [13, Sect. 2]): Let $\xi_f : f(\Sigma) \rightarrow G^0(2, 3)$ be an $\mathcal{H}^2$ measurable orientation. Then the corresponding oriented integral varifold is

$$V_f^0 := V^0(f(\Sigma), \theta_+, \theta_-, \xi_f). \quad (2.4)$$

To define the densities, let us denote the chosen continuous orientation of $T_x \Sigma$ by $\tau(x)$. Then $\theta_+, \theta_- : f(\Sigma) \rightarrow \mathbb{N}_0$ are defined by

$$\begin{aligned} \theta_+(y) &= \sum_{x \in f^{-1}(y)} \text{sign}_+(df(\tau(x)) \wedge (\xi_f(y))), \\ \theta_-(y) &= \sum_{x \in f^{-1}(y)} \text{sign}_-(df(\tau(x)) \wedge (\xi_f(y))). \end{aligned} \quad (2.5)$$

Here

$$\text{sign}_+(df(\tau(x))/(\xi_f(y))) = \begin{cases} 1, & \text{if } \xi_f(y) \text{ is the same orientation as } df(\tau(x)) \\ 0, & \text{else.} \end{cases}$$

Analogously, $\text{sign}_-(df(\tau(x))/(\xi_f(y))) = 1$, if $\xi_f(y)$ is the opposite orientation of $df(\tau(x))$. Please note that these densities are only well defined $\mathcal{H}^2 \llcorner f(\Sigma)$ almost everywhere, which is enough to obtain a well defined oriented varifold.

Let $f_k \in M_{Area_0, Vol_0}$ be a minimising sequence as in Theorem 1.1 with orientation $\xi_k : f_k(\Sigma) \rightarrow G^0(2, 3)$ and no boundary, i.e. closed. Furthermore let

$$V_k^0 := V^0(f_k(\Sigma), \theta_+^k, \theta_-^k, \xi_k), \quad (2.6)$$

with $\theta_\pm^k$ defined as in (2.5). Let $H_k : f_k(\Sigma) \rightarrow \mathbb{R}^3$ be the mean curvature vector of V_k^0 . Then the Helfrich energy of f_k becomes

$$W_{H_0}(f_k) = \int_{f_k(\Sigma)} |H_k(x) - H_0(*\xi_k(x))|^2 \theta_+^k + |H_k(x) + H_0(*\xi_k(x))|^2 \theta_-^k d\mathcal{H}^2.$$

By Cauchy–Schwartz’s and Young’s inequality we get for $\varepsilon > 0$

$$\begin{aligned} & \int_{f_k(\Sigma)} |H_k(x) - H_0(*\xi_k(x))|^2 \theta_+^k d\mathcal{H}^2 \\ &= \int_{f_k(\Sigma)} (|H_k|^2 - 2H_0 \langle H_k, (*\xi_k) \rangle + H_0^2) \theta_+^k d\mathcal{H}^2 \\ &\geq \int_{f_k(\Sigma)} (|H_k|^2 - 2|H_0||H_k| + H_0^2) \theta_+^k d\mathcal{H}^2 \\ &\geq \int_{f_k(\Sigma)} \left(|H_k|^2 - \varepsilon |H_k|^2 - \frac{1}{\varepsilon} H_0^2 + H_0^2 \right) \theta_+^k d\mathcal{H}^2 \\ &= (1 - \varepsilon) \int_{f_k(\Sigma)} |H_k|^2 \theta_+^k d\mathcal{H}^2 + \left(1 - \frac{1}{\varepsilon} \right) H_0^2 \int_{f_k(\Sigma)} \theta_+^k d\mathcal{H}^2. \end{aligned}$$

By repeating the argument with θ_-^k , we obtain for $\varepsilon \in (0, 1)$

$$\frac{W_{H_0}(f_k) + \left(\frac{1}{\varepsilon} - 1\right) H_0^2 Area_0}{1 - \varepsilon} \geq \int_{\Sigma_k} |H_k|^2 (\theta_+^k + \theta_-^k) d\mathcal{H}^2 = \int |H_k|^2 d\mu_k. \quad (2.7)$$

Hence there is a $C = C(H_0) > 0$, such that

$$\int |H_k|^2 d\mu_k \leq C \cdot (W_{H_0}(f_k) + Area(f_k)). \quad (2.8)$$

By the definition of M_{Area_0, Vol_0} , we also have

$$0 \in f_k(\Sigma), \quad (2.9)$$

for every $k \in \mathbb{N}$. Since the f_k are closed, we have $\partial[|V_k^0|] = 0$. Hence, equation (2.6) allows us to employ Hutchinsons compactness result for oriented integral varifolds [22, Thm. 3.1] resp. (2.2) and obtain an oriented integral 2-varifold $V^0 = V^0[M, \theta_+, \theta_-, \xi]$, such that after extracting a subsequence and relabeling we have

$$V_k^0 \rightarrow V^0 \text{ as oriented varifolds.}$$

Furthermore let A_k be the second fundamental form of f_k . By (2.6) and since the topology of f_k is fixed, we get

$$\int_{f_k(\Sigma)} |A_k|^2 d\mathcal{H}^2 \leq C$$

for some constant $C > 0$ independent of k (see also [35, Eq. (1.1)]). Hence, by [22, Thm. 5.3.2] $\mu := \mu_{V^0}$ has a weak second fundamental form $A_\mu \in L^2(\mu)$. By possibly extracting another subsequence, we obtain

$$\begin{aligned} V_k^0 &\rightarrow V^0 \text{ as oriented varifolds,} \\ \mu_k &\rightarrow \mu \text{ weakly as varifolds,} \\ |A_k|^2 \mu &\rightarrow \nu \text{ weakly as Radon measures,} \\ |A_\mu|^2 \mu &\leq \nu \text{ and } \nu(\mathbb{R}^3) \leq C < \infty. \end{aligned} \quad (2.10)$$

Without loss of generality we also have $M = \text{spt}(\mu)$.

Before we can proceed, we need to ensure that the limit V^0 satisfies $\text{Area}(V^0) = A_0$ and $\text{Vol}(V^0) = \text{Vol}_0$. For the first one, we use the following lemma

Lemma 2.1 (see [13], Lemma 2.1) *$\text{spt}(\mu)$ is compact.*

Proof The proof is analouge to [13, Lemma 2.1]. For the reader's convenience we include it here. Simon's diameter estimate [39, Lemma 1.1] and the bound on the Willmore energy yield

$$\text{diam}(f_k(\Sigma)) \leq C \sqrt{\mu_k(\mathbb{R}^3) \cdot W_{\text{Will}}(f_k)} \leq C. \quad (2.11)$$

Now let $x \in \text{spt}(\mu)$. For an arbitrary $\rho > 0$ we obtain by, e.g. [27, Prop. 4.26] and the definition of the support of a Radon measure

$$0 < (\mu)(B_\rho(x)) \leq \liminf_{k \rightarrow \infty} (\mu_k)(B_\rho(x)).$$

Hence $\text{spt}(\mu_k) \cap B_\rho(x) \neq \emptyset$ for k big enough. Therefore, we can find $x_k \in \text{spt}(\mu_k)$ such that $x_k \rightarrow x$. By (2.11), we finally obtain

$$\text{diam}(\text{spt}(\mu)) \leq C$$

and the lemma is proven. $\square$

Furthermore, (2.11) and (2.9) yield a constant $N > 0$, such that for all k we have

$$\text{spt}(\mu), f_k(\Sigma) \subset B_N(0). \quad (2.12)$$

Now choose a smooth cut-off function $\varphi \in C_0^\infty(B_{2N}(0))$, such that $\varphi = 1$ on $B_N(0)$. The varifold convergence of μ_k now yields

$$\text{Area}_0 = \lim_{k \rightarrow \infty} \mu_k(\mathbb{R}^3) = \lim_{k \rightarrow \infty} \int \varphi d\mu_k = \int \varphi d\mu = \mu(\mathbb{R}^3).$$

Also $(x, \xi) \mapsto \varphi(x) \langle x, *(\xi) \rangle$ defines a continuous function with compact support on $\mathbb{R}^3 \times G^0(2, 3)$. Hence, the oriented varifold convergence yields

$$\begin{aligned} Vol_0 &= Vol(f_k) = \frac{1}{3} \int \varphi(x) \langle x, *(\xi) \rangle dV_k^0(x, \xi) \\ &\rightarrow \frac{1}{3} \int \varphi(x) \langle x, *(\xi) \rangle dV^0(x, \xi) = Vol(V^0). \end{aligned}$$

The enclosed volume will need more attention, since we have to calculate a first variation. We will do this in Sect. 3

3 First Variation of enclosed volume

In this section, we derive a suitable formula for the first variation of the enclosed volume with respect to a smooth vectorfield. Let $V^0 = V^0[M, \theta_+, \theta_-, \xi]$ be an oriented 2-integral varifold on $\mathbb{R}^3$ with compact support, finite mass and empty boundary, i.e. $\partial[|V^0|] = 0$. Then the isoperimetric inequality for currents [38, Thm 30.1] yields an integral 3-current $R \in D_3(\mathbb{R}^3)$ with

$$\partial R = [|V^0|], \quad (M(R))^{\frac{2}{3}} \leq CM([|V^0|]) \leq C\mu_{V^0}(\mathbb{R}^3) < \infty \quad (3.1)$$

for some constant $C > 0$ independent of R or $[|V^0|]$ and such that R has compact support. Since R is three-dimensional in $\mathbb{R}^3$, [38, Remark 26.28] yields a function $\theta_R : \mathbb{R}^3 \rightarrow \mathbb{Z}$ of bounded variation, such that for every $\omega \in D^3(\mathbb{R}^3)$ we have

$$R(\omega) = \int \langle \omega, e_1 \wedge e_2 \wedge e_3 \rangle \theta_R d\mathcal{L}^3. \quad (3.2)$$

Now we claim the following equation, which we will prove afterwards:

$$Vol(V^0) = - \int \theta_R d\mathcal{L}^3. \quad (3.3)$$

As a short remark: If V^0 would be given by an embedded closed surface, R would represent the open and bounded set with boundary V^0 .

Since θ_R is of bounded variation, we find a Borel measure $|\nabla \theta_R|$ and a Borel measurable function $\sigma_R : \mathbb{R}^3 \rightarrow \partial B_1(0)$, such that for every smooth $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with compact support we have (see, e.g. [15, Section 5.1])

$$\int \operatorname{div} g \theta_R d\mathcal{L}^3 = - \int \langle g, \sigma_R \rangle d|\nabla \theta_R|.$$

We now claim

$$\int \langle g, \sigma_R \rangle d|\nabla \theta_R| = \int \langle g(x), *(\xi) \rangle dV^0(x, \xi) \quad (3.4)$$

for every $g \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$. The proof is as follows: We define $\omega := -g_1 dx_2 \wedge dx_3 + g_2 dx_1 \wedge dx_3 - g_3 dx_1 \wedge dx_2$ and then [38, Remark 26.28] yields:

$$\begin{aligned}
\int \langle g, \sigma_R \rangle d|\nabla \theta_R| &= - \int \theta_R \operatorname{div}(g) d\mathcal{L}^3 \\
&= -R(d\omega) = -(\partial R)(\omega) = -[|V^0|](\omega) \\
&= - \int \langle \omega(x), \xi \rangle dV^0(x, \xi).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\xi &= \xi_1 e_2 \wedge e_3 + \xi_2 e_1 \wedge e_3 + \xi_3 e_1 \wedge e_2 \\
\Rightarrow * \xi &= \xi_1 e_1 - \xi_2 e_2 + \xi_3 e_3.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \omega, \xi \rangle &= -g_1 \xi_1 + g_2 \xi_2 - g_3 \xi_3 \\
\Rightarrow \langle g, * (\xi) \rangle &= g_1 \xi_1 - g_2 \xi_2 + g_3 \xi_3 = -\langle \omega, \xi \rangle.
\end{aligned}$$

which yields (3.4).

Since R has compact support and by multiplying with a smooth cutoff function, (3.4) is valid for every smooth vectorfield. Hence, we can apply (3.4) to $g(x) = \frac{1}{3}x$ and obtain

$$\begin{aligned}
\int 1 \cdot \theta_R d\mathcal{L}^3 &= \frac{1}{3} \int \operatorname{div}(x) \theta_R(x) d\mathcal{L}^3(x) = -\frac{1}{3} \int \langle x, \sigma_R \rangle d|\nabla \theta_R| \\
&= -\frac{1}{3} \int \langle x, * \xi \rangle dV^0(x, \xi) = -\operatorname{Vol}(V^0),
\end{aligned} \tag{3.5}$$

which is (3.2).

Under our assumptions, i.e. finite mass and compact support, the current R is unique. Let us prove this claim: Assume we have $R_1, R_2 \in D_3(\mathbb{R}^3)$ with compact support and finite mass satisfying

$$\partial R_1 = [|V^0|], \quad \partial R_2 = [|V^0|].$$

Then

$$\partial(R_1 - R_2) = 0$$

and

$$M(R_1 - R_2) \leq M(R_1) + M(R_2) < \infty.$$

Now the constancy theorem for currents (see, e.g. [38, Thm. 26.27]) yields a $c \in \mathbb{R}$, such that

$$R_1 - R_2 = c[|\mathbb{R}^3|].$$

Since $R_1 - R_2$ has finite mass, $c = 0$, i.e.

$$R_1 = R_2. \tag{3.6}$$

If we want to calculate the first derivative of the enclosed volume, we need to make sense of mapping an oriented integral varifold by a diffeomorphism. So let $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a diffeomorphism. Then we define $g_{\#} V^0 := V^0[\tilde{M}, \tilde{\theta}_+, \tilde{\theta}_-, \tilde{\xi}]$ by

$$\begin{aligned}
\tilde{M} &:= g(M) \\
\tilde{\theta}_{\pm}(x) &:= \theta_{\pm}(g^{-1}(x)) \\
\tilde{\xi}(x) &:= \left(\frac{d^M g_{y\#} \xi(y)}{|d^M g_{y\#} \xi(y)|} \right) \Big|_{y=g^{-1}(x)}.
\end{aligned} \tag{3.7}$$

Then we have for every $\omega \in D^2(\mathbb{R}^3)$

$$\begin{aligned}
&[|g_{\#} V^0|](\omega) \\
&= \int_{g(M)} \left\langle \omega(x), \left(\frac{d^M g_{y\#} \xi(y)}{|d^M g_{y\#} \xi(y)|} \right) \Big|_{y=g^{-1}(x)} \right\rangle (\theta_+(g^{-1}(x)) - \theta_-(g^{-1}(x))) d\mathcal{H}^2
\end{aligned}$$

By [38, Remark 27.2] we also have

$$\begin{aligned}
&(g_{\#}[|V^0|])(\omega) \\
&= \int_{g(M)} \left\langle \omega(x), \left(\frac{d^M g_{y\#} \xi(y)}{|d^M g_{y\#} \xi(y)|} \right) \Big|_{y=g^{-1}(x)} \right\rangle (\theta_+(g^{-1}(x)) - \theta_-(g^{-1}(x))) d\mathcal{H}^2
\end{aligned}$$

and hence

$$g_{\#}[|V^0|] = [|g_{\#} V^0|]. \tag{3.8}$$

Now we are ready to calculate the first variation of the enclosed volume. Let $X \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be a vectorfield and $\Phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the corresponding flow, i.e. $\partial_t(\Phi(t, x)) = X(\Phi(t, x))$ and $\Phi(0, x) = x$ for all t, x . Since $x \mapsto \Phi(t, x)$ is a diffeomorphism for every t , $\Phi(t, \cdot)_{\#}[|V^0|]$ is well defined. Equation (3.8) now yields with [38, Remark 26.21]

$$\partial([|\Phi(t, \cdot)_{\#} V^0|]) = \partial(\Phi(t, \cdot)_{\#}[|V^0|]) = \Phi(t, \cdot)_{\#}(\partial[|V^0|]) = \Phi(t, \cdot)_{\#} R. \tag{3.9}$$

Hence the uniqueness property of R and (3.3) give us the first equality

$$Vol(\Phi(t, \cdot)_{\#} V^0) = - \int \theta_{\Phi(t, \cdot)_{\#} R} d\mathcal{L}^3 = - \int \theta_R(\Phi(t, \cdot)^{-1}(x)) d\mathcal{L}^3(x). \tag{3.10}$$

The second equality follows from [38, Remark 27.2] and the fact, that $x \mapsto \Phi(t, x)$ is orientation preserving for every $t \in \mathbb{R}$. If we now decompose θ_R into positiv and negativ parts, we can employ the calculation for the first variation of varifolds (see, e.g. [38, Sect. 16]) and we finally obtain

$$\left(\frac{\partial}{\partial t} Vol(\Phi(t, \cdot)_{\#} V^0) \right) \Big|_{t=0} = - \int \operatorname{div}(X) \theta_R d\mathcal{L}^3 = \int \langle X, \sigma_R \rangle d|\nabla \theta_R|. \tag{3.11}$$

4 Area and volume correction

In this chapter, we introduce a two parameter diffeomorphism of $\mathbb{R}^3$ to adjust an error appearing in the graphical decomposition method by Simon, see [39, Sect. 3] for the prescribed area and enclosed volume. This diffeomorphism will be constructed by two vector fields and their corresponding flows (cf. Fig. 1).

This idea was introduced by Schygulla in [36] for a one parameter diffeomorphism and prescribed isoperimetric ratio. We will expand this method by using a version of the inverse function theorem (see Theorem A.2) with some explicit bounds on the size of the set of invertibility.

Before we can start working on our diffeomorphism, we have to define currents for the minimising sequence f_k , with which we will be able to calculate the enclosed volume as in Sect. 3. So let $(R_k)_{k \in \mathbb{N}} \subset D_3(\mathbb{R}^3)$ be a sequence of integral currents satisfying (see also (3.1))

$$\begin{aligned} \partial R_k &= [|V_k^0|] \\ \text{and } M(R_k)^{\frac{2}{3}} &\leq CM(|V_k^0|). \end{aligned} \quad (4.1)$$

Here, $C > 0$ is given by the isoperimetric inequality [38, Thm. 30.1]. As in Sect. 3 there are functions of bounded variation $\theta_{R_k} : \mathbb{R}^3 \rightarrow \mathbb{Z}$ representing R_k as in (3.2). Since

$$M(R_k) \leq CM(\partial R_k)^{\frac{3}{2}} = CM(|V_k^0|)^{\frac{3}{2}} \leq C\mu_k(\mathbb{R}^3)^{\frac{3}{2}} = CVol_0^{\frac{3}{2}} < \infty,$$

the compactness theorem for currents [38, Thm. 27.3] resp. for functions of bounded variation [15, Section 5.2.3] yield a convergent subsequence of R_k resp. θ_k in the current resp. BV sense. After relabeling, we can hence assume $R_k \rightarrow R$ weakly as currents and $\theta_{R_k} \rightarrow \theta_R$ in the BV sense. Here, $\theta_R : \mathbb{R}^3 \rightarrow \mathbb{Z}$ satisfies (3.2) with respect to R . Since the boundaries of currents also converge, we have

$$\partial R = [|V^0|].$$

Since R has finite mass, the uniqueness result (3.6) also yields $M(R)^{\frac{2}{3}} \leq CM(|V^0|)$.

Let us define the following sequence of functions corresponding to the area and enclosed volume: For this we need $X, Y \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Let Φ_X resp. Φ_Y be the flow of X resp. Y . Now let

Fig. 1 Correcting Area and Volume outside of a prescribed region

$$\Phi(s, t, x) := \Phi_X(s, \Phi_Y(t, x)) \quad (4.2)$$

for $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^3$. Let

$$F(s, t) := \left(\frac{(\Phi(s, t, \cdot)_\# \mu)(\mathbb{R}^3)}{\text{Vol}(\Phi(s, t, \cdot)_\# V^0)} \right). \quad (4.3)$$

and

$$F_k(s, t) := \left(\frac{(\Phi(s, t, \cdot)_\# \mu_k)(\mathbb{R}^3)}{\text{Vol}(\Phi(s, t, \cdot)_\# V_k^0)} \right). \quad (4.4)$$

Since Φ does not depend on k , we obtain $F_k^1 \rightarrow F^1$ pointwise as $k \rightarrow \infty$. Furthermore, θ_{R_k} converges in the BV-sense and we therefore have L^1 -convergence $\theta_{R_k} \rightarrow \theta_R$. Hence, for all $s, t \in \mathbb{R}$

$$\begin{aligned} \text{Vol}(\Phi(s, t, \cdot)_\# V_k^0) &\stackrel{(3.10)}{=} - \int \theta_{R_k}(\Phi(s, t, \cdot)^{-1}(x)) \, d\mathcal{L}^3(x) \\ &= - \int \theta_{R_k} |\det(D\Phi)| \, d\mathcal{L}^3 \rightarrow - \int \theta_R |\det(D\Phi)| \, d\mathcal{L}^3 \\ &= - \int \theta_R(\Phi^{-1}(s, t, x)) \, d\mathcal{L}^3 \stackrel{(3.10)}{=} \text{Vol}(\Phi(s, t, \cdot)_\# V^0), \end{aligned}$$

which yields the pointwise convergence $F_k \rightarrow F$. Here $^{-1}$ and D refer to the x -variable of Φ .

Since we want to locally invert F and F_k , we have to derive F and find X, Y such that $DF(0, 0)$ is invertible. Equation (3.11) and the usual calculation of the first variation of a varifold (see, e.g. [38, Sect. 16]) yields

$$DF(0, 0) = \begin{pmatrix} \int \text{div}_{T_x \mu} X(x) \, d\mu(x) & \int \text{div}_{T_x \mu} Y(x) \, d\mu(x) \\ - \int \theta_R \text{div} X \, d\mathcal{L}^3 & - \int \theta_R \text{div} Y \, d\mathcal{L}^3 \end{pmatrix} \quad (4.5)$$

To find the necessary vectorfields, we work similarly to [36, Lemma 4] and prove the following two lemmas:

Lemma 4.1 (cf. Lemma 4 in [36]) *There exists an $r > 0$, such that for every $x \in \text{spt}(\mu)$ there exists a point $\eta \in \text{spt}(\mu) \setminus B_r(x)$, such that for every $\varepsilon > 0$ there exists a vectorfield $Y \in C_0^\infty(B_\varepsilon(\eta), \mathbb{R}^3)$ satisfying*

$$\int \theta_R \text{div} Y \, dx = - \int \langle Y, \sigma_R \rangle \, d|\nabla \theta_R| \neq 0.$$

Proof We proceed by contradiction and assume the statement is false. Then there exist sequences $r_k \rightarrow 0$ and $x_k \in \text{spt}(\mu)$, such that for all $\eta \in \text{spt}(\mu) \setminus B_{r_k}(x_k)$ there is an $\varepsilon_\eta > 0$ such that

$$\int \langle Y, \sigma_R \rangle \, d|\nabla \theta_R| = 0.$$

for all $Y \in C_0^\infty(B_{\varepsilon_\eta}(\eta), \mathbb{R}^3)$. By Lemma 2.1 $\text{spt}(\mu)$ is compact and thus we can extract a subsequence and relabel such that $x_k \rightarrow x \in \text{spt}(\mu)$. Hence we obtain

$$\sigma_R = 0 \quad |\nabla\theta_R|-\text{a.e.}$$

Thus by (3.5)

$$\text{Vol}(V^0) = 0,$$

which is a contradiction to $\text{Vol}(V^0) = \text{Vol}_0 \neq 0$. $\square$

Lemma 4.2 (cf. Lemma 4 in [36]) *Either $H_\mu \in L^\infty(\mu)$ or there exists an $r > 0$, such that for every $z \in \text{spt}(\mu)$ there exist two points $\eta_Y, \eta_X \in \text{spt}(\mu) \setminus B_r(z)$, such that for every $\varepsilon_Y, \varepsilon_X > 0$ there exist vectorfields $Y \in C_0^\infty(B_{\varepsilon_Y}(\eta_Y))$ and $X \in C_0^\infty(B_{\varepsilon_X}(\eta_X))$ satisfying*

$$\begin{aligned} & \int \text{div}_{T_x\mu} X(x) \, d\mu(x) \int \langle Y, \sigma_R \rangle \, d|\nabla\theta_R| \\ & - \int \text{div}_{T_x\mu} Y(x) \, d\mu(x) \int \langle X, \sigma_R \rangle \, d|\nabla\theta_R| \neq 0. \end{aligned}$$

Proof Let us assume that we do not find an $r > 0$ as requested. Then we find sequences $r_k \searrow 0$, $z_k \in \text{spt}(\mu)$, such that for every $\eta_Y, \eta_X \in \text{spt}(\mu) \setminus B_{r_k}(z_k)$ there are $\varepsilon_{Y,\eta_Y}, \varepsilon_{X,\eta_X} > 0$, which satisfy

$$\begin{aligned} & \int \text{div}_{T_x\mu} X(x) \, d\mu(x) \int \langle Y, \sigma_R \rangle \, d|\nabla\theta_R| \\ & - \int \text{div}_{T_x\mu} Y(x) \, d\mu(x) \int \langle X, \sigma_R \rangle \, d|\nabla\theta_R| = 0 \end{aligned} \quad (4.6)$$

for every $Y \in C_0^\infty(B_{\varepsilon_{Y,\eta_Y}}(\eta_Y))$ and $X \in C_0^\infty(B_{\varepsilon_{X,\eta_X}}(\eta_X))$. Since $\text{spt}(\mu)$ is compact, we find a convergent subsequence of z_k and after relabeling we have $z_k \rightarrow z \in \text{spt}(\mu)$. Hence for every $\eta_Y, \eta_X \in \text{spt}(\mu) \setminus \{z\}$ there are $\varepsilon_{Y,\eta_Y}, \varepsilon_{X,\eta_X} > 0$, such that every $Y \in C_0^\infty(B_{\varepsilon_{Y,\eta_Y}}(\eta_Y), \mathbb{R}^3)$ and $X \in C_0^\infty(B_{\varepsilon_{X,\eta_X}}(\eta_X), \mathbb{R}^3)$ satisfies (4.6). According to Lemma 4.1, we find an $\eta \in \text{spt}(\mu) \setminus \{z\}$ and a $Y \in C_0^\infty(B_{\varepsilon_{Y,\eta}}(\eta))$ with

$$\int \langle Y, \sigma_R \rangle \, d|\nabla\theta_R| \neq 0$$

From now on this Y is fixated. All in all we find for every $\eta_X \in \text{spt}(\mu) \setminus \{z\}$ a radius ε_{η_X} such that for every $X \in C_0^\infty(B_{\varepsilon_{X,\eta_X}})$ we have

$$\begin{aligned} & \int \text{div}_{T_x\mu} X(x) \, d\mu(x) \int \langle Y, \sigma_R \rangle \, d|\nabla\theta_R| \\ & - \int \text{div}_{T_x\mu} Y(x) \, d\mu(x) \int \langle X, \sigma_R \rangle \, d|\nabla\theta_R| = 0 \end{aligned}$$

Hence

$$\begin{aligned}
\int_{\text{spt}(\mu)} \langle H_\mu, X \rangle (\theta_+ + \theta_-) d\mathcal{H}^2 &= - \int \text{div}_{T_x \mu} X(x) d\mu(x) \\
&= C(\varepsilon_Y) \int \langle X, \sigma_R \rangle d|\nabla \theta_R| \\
&\stackrel{(3.4)}{=} C(\varepsilon_Y) \int \langle X, (*\xi) \rangle dV^0 \\
&= C(\varepsilon_Y) \int_{\text{spt}(\mu)} \langle X(x), (*\xi(x)) \rangle (\theta_+ - \theta_-) d\mathcal{H}^2(x)
\end{aligned}$$

Since η_X is arbitrary and $\mu(\{z\}) = 0$, we have

$$H_\mu(\theta_+ + \theta_-) = C(\varepsilon_Y) \cdot (*\xi)(\theta_+ - \theta_-) \quad \mu\text{-a.e.} \quad (4.7)$$

Furthermore $\theta_+ + \theta_- \geq 1$ μ -a.e. and $\theta_\pm \in \mathbb{N}_0$ finally yield $H_\mu \in L^\infty(\mu)$. $\square$

Since we like to apply Theorem A.2 to F_k , we need to be able to estimate the difference of two values of the first derivative independently of k . To do this, we employ the mean value theorem and hence need to estimate the second derivative:

Lemma 4.3 *For every $X, Y \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and every $T > 0$, there exists a constant $C = C(\|\Phi\|_{C^3(B_T(0) \times \mathbb{R}^3)}, \text{Area}_0, \text{Vol}_0) > 0$, such that for every $s, t \in B_T(0)$ we have for every $k \in \mathbb{N}$:*

$$|D^2 F_k(s, t)| \leq C.$$

Proof We start by estimating F^2 :

Since (3.10) only needs the corresponding map to be a diffeomorphism, we obtain the same result for $x \mapsto \Phi(s, t, x)$. Furthermore, the usual substitution formula yields:

$$\begin{aligned}
F_k^2(s, t) &= - \int \theta_{R_k}(\Phi(s, t, \cdot)^{-1}(x)) d\mathcal{L}^3(x) \\
&= - \int |\det(D_x \Phi(s, t, x))| \theta_{R_k}(x) d\mathcal{L}^3(x).
\end{aligned} \quad (4.8)$$

Now there exists a constant $C > 0$ only dependent on $T > 0$, X , Y and their respective derivatives, such that for every $s, t \in (-T, T)$ we have

$$D_{s,t}^2 |\det(D_x \Phi(s, t, x))|, D_{s,t}^2 |\det(D_x \Phi(s, t, x))| \leq C.$$

Hence, for these s, t the isoperimetric inequality [38, Thm 30.1] yields

$$|D^2 F_k^2(s, t)| \leq C \int |\theta_{R_k}| d\mathcal{L}^3 \leq C(\mu_k(\mathbb{R}^3))^{\frac{3}{2}} = C(\text{Area}_0)^{\frac{3}{2}}.$$

Now let us turn to F^1 :

Since $x \mapsto \Phi(s, t, x)$ is a diffeomorphism, we can apply [38, Eq. 15.7] and obtain

$$F_k^1(s, t) = \int (J_{\mu_k} \Phi(s, t, x)) d\mu_k. \quad (4.9)$$

Furthermore, in the Jacobian $J_{\mu_k} \Phi(s, t, x)$ the measure μ_k is independent of s and t . Also the derivatives $d^{\mu_k} \Phi(s, t, x)$ appearing in the Jacobian, can be estimated by the full derivative of $x \mapsto \Phi(s, t, x)$. Therefore there is a constant $C > 0$ only dependend on $T > 0$, X , Y and their respective derivatives, such that for every $s, t \in (-T, T)$ we have

$$|J_{\mu_k} \Phi(s, t, x)| \leq C.$$

This yields

$$|D^2 F^1(s, t)| \leq C \mu_k(\mathbb{R}^3) = C \cdot \text{Area}_0,$$

which is the desired conclusion. $\square$

If we change the minimising sequence by Φ , we also have to be sure, that the Helfrich energy and the second fundamental form is controlled as well:

Lemma 4.4 (cf. [36] p. 915, Eq. (v)) *For every $X, Y \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and every $T > 0$, there exists a constant $C = C(\|\Phi\|_{C^3(B_r(0) \times \mathbb{R}^3)}, \text{Area}_0, \text{Vol}_0) > 0$ (independent of k), such that for every $s, t \in (-T, T)$ we have*

$$\left| D_{s,t} \int |A_k^{s,t}|^2 d\mu_k \right| \leq C.$$

Here, $A_k^{s,t}$ is the second fundamental form of $\Sigma \ni x \mapsto \Phi(s, t, f_k(x))$.

Proof By a partition of unity on Σ and a rigid motion we can assume, that we can write $x \mapsto \Phi(s, t, f_k(x))$ locally as a smooth graph with small Lipschitz norm. Hence, we have $u_k : \mathbb{R}^2 \supset B_r(0) \rightarrow \mathbb{R}$ smooth with $|\nabla u_k| \leq 1$ and $u_k(0) = 0$, which satisfies for a small open set $U \subset \Sigma$:

$$f_k(U) = \text{graph}(u_k).$$

Now we can calculate the second fundamental form of $\Phi(s, t, f_k(\cdot))$ by using u_k and chain rule:

$$\begin{aligned} \partial_i (\Phi(s, t, (x, u_k(x)))) &= \sum_{j=1}^2 (\partial_j \Phi)(s, t, (x, u_k(x))) \delta_{ij} \\ &\quad + (\partial_3 \Phi)(s, t, (x, u_k(x))) \partial_i u_k(x) \\ &= (\partial_i \Phi)(s, t, (x, u_k(x))) + (\partial_3 \Phi)(s, t, (x, u_k(x))) \partial_i u_k(x). \end{aligned} \quad (4.10)$$

Then the second derivatives are as follows:

$$\begin{aligned}
\partial_{\mathcal{C}} \partial_i (\Phi(s, t, (x, u_k(x)))) &= \sum_{m=1}^2 (\partial_m \partial_i \Phi)(s, t, (x, u_k(x))) \delta_{m\mathcal{C}} \\
&\quad + (\partial_3 \partial_i \Phi)(s, t, (x, u_k(x))) \partial_{\mathcal{C}} u_k(x) \\
&\quad + \sum_{m=1}^2 (\partial_m \partial_3 \Phi)(s, t, (x, u_k(x))) \delta_{m\mathcal{C}} \partial_i u_k(x) \\
&\quad + (\partial_3^2 \Phi)(s, t, (x, u_k(x))) \partial_{\mathcal{C}} u_k(x) \partial_i u_k(x) \\
&\quad + \partial_3 \Phi(s, t, (x, u_k(x))) \partial_{\mathcal{C}} \partial_i u_k(x).
\end{aligned} \tag{4.11}$$

The unit normal can be expressed by

$$n(s, t, x) = \pm \frac{\partial_1 (\Phi(s, t, (x, u_k(x)))) \times \partial_2 (\Phi(s, t, (x, u_k(x))))}{\left| \partial_1 (\Phi(s, t, (x, u_k(x)))) \times \partial_2 (\Phi(s, t, (x, u_k(x)))) \right|}$$

Since $x \mapsto \Phi(s, t, x)$ is a diffeomorphism of $\mathbb{R}^3$, such that it is the identity outside of a ball, we find a constant $C = C(T) > 0$, such that for every $(s, t) \in B_T(0)$ we have

$$|\det D_x \Phi(s, t, x)| \geq C.$$

This yields a noncollapsing of the basis of the tangential space, i.e.

$$\left| \partial_1 (\Phi(s, t, (x, u_k(x)))) \times \partial_2 (\Phi(s, t, (x, u_k(x)))) \right| \geq C > 0$$

for a possibly different constant $C = C(T) > 0$. The second fundamental form can be expressed in these coordinates as

$$(A_k^{s,t})_{i\mathcal{C}} = \langle \partial_{\mathcal{C}} \partial_i (\Phi(s, t, (x, u_k(x)))) , n(s, t, x) \rangle.$$

All in all we therefore have

$$D_{s,t}(|A_k^{s,t}|^2) \leq C(1 + \|\nabla u_k\|_{L^\infty}^4 + \|D^2 u_k\|_{L^\infty}^2)$$

with some constant $C > 0$ independent of k . Since we imposed a small Lipschitz Norm on u_k we deduce by, e.g. [9, p. 5]

$$D_{s,t}(|A_k^{s,t}|^2) \leq C(1 + |A_k|^2).$$

Hence

$$\left| D_{s,t} \int |A_k^{s,t}|^2 d\mu_k \right| \leq C$$

and C does not depend on k . □

Remark 4.5 With the same techniques as in Lemma 4.4 we can also show (cf. (4.8))

$$|D_{s,t} W_{H_0}(f_k^{s,t}) + D_{s,t} \text{Area}(f_k^{s,t})| \leq C$$

for all $(s, t) \in B_T(0)$ and C is independent of k . Here $f_k^{s,t}$ is the immersion $\Sigma \ni x \mapsto \Phi(s, t, f_k(x))$.

5 Partial regularity and lower semicontinuity

In this section, we adapt the partial regularity method introduced by Simon (see [39, Section 3]). This method is based on replacing parts of the minimising sequence with biharmonic graphs and compare the resulting energies. Here we use the idea of Schygulla [36, Lemma 5] to correct the area and enclosed volume of the modified sequence, so that they become competitors for the minimum of the Helfrich energy again.

Let $\varepsilon_0 > 0$ be fixated. In dependence of this ε_0 we say $x_0 \in \text{spt}(\mu)$ is a good point, iff (see (2.10))

$$v(\{x_0\}) < \varepsilon_0^2.$$

In neighbourhoods of these good points we will show $C^{1,\beta}$ regularity and a graphical decomposition of μ . Since we work with immersions with possible self intersections, we also use the ideas of Schätzle (see [35, Prop. 2.2]) to implement Simon's regularity method (cf. [39, Section 3]). The lemma is now as follows (cf. also [13, Lemma 3.1]):

Lemma 5.1 *For any $\varepsilon > 0$ there exist $\varepsilon_0 = \varepsilon_0(H_0, \text{Vol}_0, \text{Area}_0, \varepsilon) > 0$, $\theta = \theta(H_0, \text{Vol}_0, \text{Area}_0, \varepsilon) > 0$, $\rho_0 = \rho_0(H_0, \text{Vol}_0, \text{Area}_0, \varepsilon) > 0$ and $\beta = \beta(H_0, \text{Vol}_0, \text{Area}_0) > 0$, such that for every good point $x_0 \in \text{spt}(\mu)$ and good radius $0 < \rho_{x_0} \leq \rho_0$ satisfying*

$$v(\overline{B_{\rho_{x_0}}(x_0)}) < \varepsilon_0^2, \quad (5.1)$$

μ is a union of $(W^{2,2} \cap C^{1,\beta})$ -graphs in $B_{\theta\rho_{x_0}}(x_0)$ of functions $u_i \in (W^{2,2} \cap C^{1,\beta})(B_{\theta\rho_{x_0}}(x_0) \cap L_i)$. Here, $L_i \subset \mathbb{R}^3$ are two-dimensional affine spaces and $i = 1, \dots, I_{x_0} \leq C(\text{Area}_0, \text{Vol}_0, H_0)$. Furthermore, the u_i satisfy the following estimate

$$\begin{aligned} & (\theta\rho_{x_0})^{-1} \|u_i\|_{L^\infty(B_{\theta\rho_{x_0}}(x_0) \cap L_i)} \\ & + \|\nabla u_i\|_{L^\infty(B_{\theta\rho_{x_0}}(x_0) \cap L_i)} + (\theta\rho_{x_0})^\beta \text{höl}_{B_{\theta\rho_{x_0}}(x_0) \cap L_i, \beta}(\nabla u_i) \leq \varepsilon. \end{aligned} \quad (5.2)$$

Moreover, we have a power-decay for the second fundamental form, i.e. $\forall x \in B_{\frac{\theta\rho_{x_0}}{4}}(x_0)$, $0 < \rho < \frac{\theta\rho_{x_0}}{4}$

$$\int_{B_\rho(x)} |A_\mu|^2 d\mu \leq C(H_0, \text{Area}_0, \text{Vol}_0)(\varepsilon_0^2 + \rho_0)\rho^\beta \rho_{x_0}^{-\beta}. \quad (5.3)$$

Proof We start by applying the graphical decomposition lemma of Simon (see [39, Lemma 2.1]) to the minimising sequence $f_k : \Sigma \rightarrow \mathbb{R}^3$. This lemma is also applicable for immersions by an argument of Schätzle (see [35, p. 280] and cf. [13, Lemma A.6], in the beginning we work as in these papers). We repeat some steps of [35, (2.11)–(2.16)], which we will need later (see also [13, (3.5)–(3.14)]):

The upper-semicontinuity of the weak convergence $|A_k|^2 \mu_k \rightarrow \nu$ for Radon measures (see, e.g. [27, Prop. 4.26]) yields for every ball satisfying (5.1)

$$\limsup_{k \rightarrow \infty} \int_{B_{\rho_{x_0}}(x_0)} |A_k|^2 d\mu_k \leq v(\overline{B_{\rho_{x_0}}(x_0)}) < \varepsilon_0^2. \quad (5.4)$$

Hence, the aforementioned graphical decomposition lemma [13, Lemma A.6] is applicable and we can decompose $f_k^{-1}(B_{\frac{\rho_{x_0}}{2}}(x_0))$ for large k and $\varepsilon_0 > 0$ small enough into closed pairwise disjoint sets $D_{k,i} \subset \Sigma$ (which are topological discs), $i = 1, \dots, I_k$, $I_k \leq CE_{H_0, Area_0, Vol_0}$ (cf. (2.7), (1.6)), i.e.

$$f_k^{-1}\left(\overline{B_{\frac{\rho_{x_0}}{2}}(x_0)}\right) = \bigcup_{i=1}^{I_k} D_{k,i}.$$

Furthermore, for every $i = 1, \dots, I_k$ we have affine two-dimensional planes $L_{k,i} \subset \mathbb{R}^3$, smooth functions $u_{k,i} : \overline{\Omega_{k,i}} \subset L_{k,i} \rightarrow L_{k,i}^\perp$ ($\Omega_{k,i} = \Omega_{k,i}^0 \setminus \cup_m d_{k,i,m}$, $\Omega_{k,i}^0 \subset L_{k,i}$ simply connected, $d_{k,i,m}$ closed pairwise disjoint discs), satisfying

$$\rho_{x_0}^{-1}|u_{k,i}| + |\nabla u_{k,i}| \leq C(E_{H_0, Area_0, Vol_0})\varepsilon_0^{\frac{1}{22}} \quad (5.5)$$

and pairwise disjoint topological discs $P_{k,i,1}, \dots, P_{k,i,J_k} \subset D_{k,i}$, such that

$$f_k\left(D_{k,i} - \bigcup_{j=1}^{J_k} P_{k,i,j}\right) = \text{graph}(u_{k,i}) \cap \overline{B_{\frac{\rho_{x_0}}{2}}(x_0)} \quad (5.6)$$

and

$$\sum_{i=1}^{I_k} \sum_{j=1}^{J_k} \text{diam} f_k(P_{k,i,j}) \leq C(E_{H_0, Area_0, Vol_0})\varepsilon_0^{\frac{1}{2}} \rho_{x_0}. \quad (5.7)$$

The arguments [35, Eq. (2.12)-(2.14)] yield for a chosen $0 < \tau < \frac{1}{2}$ an ε_0 small enough and $0 < \theta < \frac{1}{4}$, such that

$$\frac{\mu_{g_k}(D_{k,i} \cap f_k^{-1}(B_\sigma(x)))}{w_2 \sigma^2} < 1 + \tau \quad (5.8)$$

for $B_\sigma(x) \subset B_{\theta\rho_{x_0}}(x_0)$ arbitrary. Here, w_2 denotes the Hausdorff measure of the two-dimensional euclidean unit ball and μ_{g_k} the area measure on Σ induced by f_k . As in [35, p. 281] resp. [13, Eq. (3.8)] we define Radon measures on $\mathbb{R}^3$, which will lead to a decomposition of μ by Radon measures of Hausdorff density one:

$$\begin{aligned} \mu_{k,i} &:= \mathcal{H}^2 \lfloor f_k(D_{k,i} \cap f_k^{-1}(B_{\theta\rho_{x_0}}(x_0))) \\ &= f_k\left(\mu_{g_k} \lfloor (D_{k,i} \cap f_k^{-1}(B_{\theta\rho_{x_0}}(x_0)))\right). \end{aligned} \quad (5.9)$$

Since the $\mu_{k,i}$ are integer rectifiable, they are of density one, by (5.8). Furthermore we have as in [35, p. 281] resp. [13, Eq. (3.9)]

$$\sum_{i=1}^{I_k} \mu_{k,i} = \mu_k \lfloor B_{\theta\rho_{x_0}}(x_0). \quad (5.10)$$

After taking a subsequence depending on x_0 , $\theta\rho_{x_0}$ and relabeling we can assume $I_k = I$ and

$$\mu_{k,i} \rightarrow \mu_i \text{ weakly as varifolds in } B_{\theta\rho_{x_0}}(x_0). \quad (5.11)$$

As shown in [13, Eq. (3.13)] (see also [35, p. 281]) we get

$$\mu|_{B_{\theta\rho_{x_0}}(x_0)} = \sum_{i=1}^I \mu_i. \quad (5.12)$$

As in [35, Eq. (2.16)] we can pass to the limit in (5.8) and obtain

$$\frac{\mu_i(B_\sigma(x))}{w_2\sigma^2} \leq 1 + \tau \quad \forall \overline{B_\sigma(x)} \subset B_{\theta\rho_{x_0}}(x_0). \quad (5.13)$$

We will apply Allard's regularity theorem A.1 to the μ_i . Inequality (5.13) already takes care of the needed density estimate (A.3). The remainder of the proof will be about showing (A.2) or a similar L^p -bound for the mean curvature. Here we need to make a distinction of cases given by Lemma 4.2.

First we assume $H_\mu \in L^\infty$:

In this case, we will show that $H_{\mu_i} \in L^p$, $p > 2$ arbitrary. By the usual regularity Theorem of Allard (see, e.g. [38, Thm. 24.2]), the μ_i will be $C^{1,\beta}$ graphs.

We need the definition of the tilt and height excess (see, e.g. [33, Eqs. (1.1),(1.2)]):

$$\text{tiltex}_\mu(x, \omega, T) := \omega^{-2} \int_{B_\omega(x)} \|T_\xi \mu - T\|^2 d\mu(\xi), \quad (5.14)$$

$$\text{heightex}_\mu(x, \omega, T) := \omega^{-4} \int_{B_\omega(x)} \text{dist}(\xi - x, T)^2 d\mu(\xi). \quad (5.15)$$

Here $x \in \mathbb{R}^3$, $\omega \in \mathbb{R}$ and $T \subset \mathbb{R}^3$ is a two-dimensional subspace (cf. [38, Sect. 38] for defining a norm on subspaces, i.e. the unoriented Grassmannian). Since $H_\mu \in L^\infty(\mu)$, [33, Thm. 5.1] yields

$$\text{tiltex}_\mu(x, \omega, T), \text{heightex}_\mu(x, \omega, T) \leq O_x(\omega^2). \quad (5.16)$$

By the definition of the height and tilt excess we obtain for every $i = 1, \dots, I$

$$\text{tiltex}_{\mu_i}(x, \omega, T) \leq \text{tiltex}_\mu(x, \omega, T) \leq O_x(\omega^2)$$

and

$$\text{heightex}_{\mu_i}(x, \omega, T) \leq \text{heightex}_\mu(x, \omega, T) \leq O_x(\omega^2),$$

because $\mu_i \leq \mu$. Furthermore by [34, Thm. 3.1] μ_i and μ are C^2 -rectifiable. Hence [34, Cor. 4.3] is applicable and yields

$$H_{\mu_i} = H_\mu, \quad \mu_i\text{-a.e.} \quad (5.17)$$

Since $H_\mu \in L^\infty(\mu)$ and $\mu_i \leq \mu$ we finally obtain

$$H_{\mu_i} \in L^\infty(\mu_i), \quad (5.18)$$

which yields by the usual Allard regularity theorem (see, e.g. [38, Thm. 24.2]), that every μ_i is a $C^{1,\beta}$ -graph with $\beta \in (0, 1)$ arbitrary.

This case is therefore done.

Now we modify Schygulla's argument [36, Lemma 5] to our situation: The beginning of the argument is as in [39, Lemma 3.1]. For the readers convenience and because we need the notation, we repeat these steps here (see also [13, pp. 9–10]):

Let us choose $0 < \rho < \theta \rho_{x_0}$ fixed but arbitrary. We need to apply the graphical decomposition Lemma [39, Lemma 2.1] again to $f_k(D_{k,i}) \cap \bar{B}_\rho(x_0)$. Hence we obtain smooth functions $v_{k,i,\ell} : \bar{\Omega}_{k,i,\ell} \subset \tilde{L}_{k,i,\ell} \rightarrow \tilde{L}_{k,i,\ell}^\perp$, $\tilde{L}_{k,i} \subset \mathbb{R}^3$ two-dimensional planes ($\ell = 1, \dots, N_{k,i} \leq C(E_{H_0, Area_0, Vol_0})$), $\bar{\Omega}_{k,i} = \tilde{\Omega}_{k,i}^0 \setminus \bigcup_m \tilde{d}_{k,i,\ell,m}$, $\tilde{\Omega}_{k,i}^0$ simply connected and $\tilde{d}_{k,i,\ell,m}$ closed pairwise disjoint discs. Furthermore we have

$$\rho^{-1}|v_{k,i,\ell}| + |\nabla v_{k,i,\ell}| \leq C(E_{H_0, Area_0, Vol_0})\varepsilon_0^{\frac{1}{22}} \quad (5.19)$$

and closed pairwise disjoint topological discs $\tilde{P}_{k,i,\ell,1}, \dots, \tilde{P}_{k,i,\ell,J_{k,i,\ell}} \subset \bar{D}_{k,i,\ell}$ ($\bar{D}_{k,i,\ell}$ is a topological disc as well) such that for all ℓ

$$f_k\left(\bar{D}_{k,i,\ell} - \bigcup_{j=1}^{J_{k,i,\ell}} \tilde{P}_{k,i,\ell,j}\right) \cap \overline{B_\rho(x_0)} = \text{graph}(v_{k,i,\ell}) \cap \overline{B_\rho(x_0)} \quad (5.20)$$

These $\tilde{P}_{k,i,\ell,j}$ also satisfy the following estimate

$$\sum_{j=1}^{J_{k,i,\ell}} \text{diam}_{f_k}(\tilde{P}_{k,i,\ell,j}) \leq C(E_{H_0, Area_0, Vol_0})\varepsilon_0^{\frac{1}{2}}\rho \leq \frac{1}{8}\rho, \quad (5.21)$$

if we choose $C(E_{H_0, Area_0, Vol_0})\varepsilon_0^{\frac{1}{2}} < \frac{1}{8}$ (The need for (5.21) is also the reason for applying the graphical decomposition a second time). Let us also introduce the corresponding Radon measures similar to (5.9)

$$\tilde{\mu}_{k,i,\ell} := \mathcal{H}^2 \llcorner f_k(\tilde{D}_{k,i,\ell}) = \mu_{k,i} \llcorner f_k(\tilde{D}_{k,i,\ell}). \quad (5.22)$$

Since $f_k|_{D_{k,i} \cap f_k^{-1}(B_{\theta\rho_{x_0}}(x_0))}$ is an embedding, we also have

$$\sum_{\ell=1}^{N_{k,i}} \tilde{\mu}_{k,i,\ell} = \mu_{k,i} \llcorner B_\rho(x_0). \quad (5.23)$$

Let us define

$$C_\sigma^{k,i,\ell}(x_0) := \left\{ x + y : x \in B_\sigma(x_0) \cap \tilde{L}_{k,i,\ell}, y \in \tilde{L}_{k,i,\ell}^\perp \right\}. \quad (5.24)$$

Inequality (5.21) yields $\mathcal{L}^1$ -measurable sets $S_k \subset (\frac{1}{2}\rho, \frac{3}{4}\rho)$, such that $\forall j = 1, \dots, J_{k,i,\ell}$

$$\mathcal{L}^1(S_k) \geq \frac{1}{8}\rho \text{ and } \forall \sigma \in S_k : \partial C_\sigma^{k,i,\ell}(x_0) \cap f(\tilde{P}_{k,i,\ell,j}) = \emptyset.$$

Therefore $v_{k,i,\ell}|_{\partial B_\sigma(x_0) \cap \tilde{L}_{k,i,\ell}}$ and $\nabla v_{k,i,\ell}|_{\partial B_\sigma(x_0) \cap \tilde{L}_{k,i,\ell}}$ are well defined for any $\sigma \in S_k$. Hence Lemma A.3 is applicable and yields a function $w_{k,i,\ell} : B_\sigma(x_0) \cap \tilde{L}_{k,i,\ell} \rightarrow \tilde{L}_{k,i,\ell}^\perp$ with Dirichlet boundary data given by $v_{k,i,\ell}$ and $\nabla v_{k,i,\ell}$.

This defines a sequence of immersions $f_{k,\sigma}^{\text{graph}} : \Sigma_{k,\sigma} \rightarrow \mathbb{R}^3$ by

$$\Sigma_{k,\sigma} := (\Sigma \setminus (f_k^{-1}(B_\sigma(x_0)) \cap \tilde{D}_{k,i})) \oplus (B_\sigma(x_0) \cap \tilde{L}_{m,i})$$

and

$$f_{k,\sigma}^{graph}(p) = \begin{cases} f_k(p), & p \in \Sigma \setminus (f_k^{-1}(B_\sigma(x_0)) \cap \tilde{D}_{k,i}) \\ w_{k,i,\ell}(p), & p \in (B_\sigma(x_0) \cap \tilde{L}_{m,i}). \end{cases}$$

Since $\tilde{D}_{k,i}$ is topologically a disc, Σ and $\Sigma_{k,\sigma}$ are topologically equivalent. $f_{k,\sigma}^{graph}$ does not have the same area or enclosed volume as f_k , which we like to correct now. Therefore we need some estimates on these properties. We start with the area. Here we use (5.8) and Lemma A.3 to obtain

$$\begin{aligned} |\text{Area}(f_k) - \text{Area}(f_{k,\sigma}^{graph})| &\leq 2 \max\{\text{Area}(f_k|_{\tilde{D}_{k,i}}), \text{Area}(w_{k,i,\ell})\} \\ &\leq 2 \max\left\{\tilde{\mu}_{k,i,\ell}(B_\sigma(x_0)), C \int_{B_\sigma(x_0) \cap \tilde{L}_{k,i,\ell}} d\mathcal{H}^2\right\} \\ &\leq C\sigma^2 \leq C\rho^2. \end{aligned} \quad (5.25)$$

Let us proceed with the enclosed volume. Since $f_k(\Sigma) \subset B_N(0)$, for $N > 0$ big enough and independent of k , we also get $f_{k,\sigma}^{graph}(\Sigma_{k,\sigma}) \subset B_N(0)$. Hence the definition of the enclosed volume and the Cauchy–Schwartz inequality yield

$$\begin{aligned} |\text{Vol}(f_k) - \text{Vol}(f_{k,\sigma}^{graph})| &\leq 2 \max\{|\text{Vol}(f_k|_{\tilde{D}_{k,i}})|, |\text{Vol}(w_{k,i,\ell})|\} \\ &\leq C(N) \max\{\text{Area}(f_k|_{\tilde{D}_{k,i}}), \text{Area}(w_{k,i,\ell})\} \\ &\leq C\sigma^2 \leq C\rho^2. \end{aligned} \quad (5.26)$$

Now we apply Lemma 4.2: Hence, we find an $r > 0$ and points $\eta_X, \eta_Y \in \text{spt}(\mu) \setminus B_r(x_0)$ (without loss of generality we assume $\rho_0, \rho_{x_0} \leq \frac{r}{2}$), which satisfy that for every $\varepsilon_X, \varepsilon_Y > 0$, we find vectorfields $X \in C_0^\infty(B_{\varepsilon_X}(\eta_X))$ and $Y \in C_0^\infty(B_{\varepsilon_Y}(\eta_Y))$, such that

$$\begin{aligned} &\int \text{div}_{T_x\mu} X(x) d\mu(x) \int \langle Y, \sigma_R \rangle d|\nabla\theta_R| \\ &- \int \text{div}_{T_x\mu} Y(x) d\mu(x) \int \langle X, \sigma_R \rangle d|\nabla\theta_R| \neq 0. \end{aligned} \quad (5.27)$$

Here R denotes the 3-current defined in the beginning of Sect. 4 and θ_R the corresponding BV-function. We also fix $\varepsilon_X = \varepsilon_Y = \rho$. Furthermore, let Φ be defined as in (4.2) and F and F_k be as in (4.3) but with respect to $f_{k,\sigma}^{graph}$ instead. The results of Sect. 4 are still valid, since the diffeomorphism Φ does not influence the graphical comparison function (cf. Fig. 1). As in the beginning of Sect. 4 we obtain

$$F_k(s, t) \rightarrow F(s, t).$$

The oriented varifold convergence and the L^1 convergence of θ_{R_k} yield

$$|\det DF_k(0, 0)| \rightarrow |\det DF(0, 0)| \geq c_0 > 0 \quad (5.28)$$

for a fixed constant $c_0 > 0$. Hence, for k big enough, we get

$$|\det DF_k(0, 0)| \geq \frac{c_0}{2}. \quad (5.29)$$

Hence, $DF_k(0, 0)$ is invertible. Furthermore, the mean value theorem and Lemma 4.3 yield for every $T_0 > 0$ a $C = C(T_0) > 0$, such that for every $(s, t), (s', t') \in B_{T_0}(0)$ we have

$$|DF_k(s, t) - DF_k(s', t')| \leq C \|D^2 F_k\|_{L^\infty(\overline{B_r(0)})} |(s, t) - (s', t')| \leq CT \quad (5.30)$$

if we choose $0 < T < T_0$ arbitrary. By the formula for the inverse matrix via the adjoint matrix we obtain for k big enough

$$|(DF_k(0, 0))^{-1}| \leq C. \quad (5.31)$$

The constant C is independent of k . Next we will apply Lemma A.2. Hence, we need to define a functions $\tilde{F}$ and $\tilde{F}_k$ satisfying the assumptions of that Lemma:

$$\begin{aligned} \tilde{F}(s, t) &:= (DF(0, 0))^{-1}(F(s, t) - F(0, 0)) \\ \tilde{F}_k(s, t) &:= (DF_k(0, 0))^{-1}(F_k(s, t) - F_k(0, 0)) \end{aligned} \quad (5.32)$$

Here $^{-1}$ is meant as the matrix inverse. Hence

$$\begin{aligned} \tilde{F}(0, 0) &= 0, & D\tilde{F}(0, 0) &= I \in \mathbb{R}^{2 \times 2}, \\ \tilde{F}_k(0, 0) &= 0, & D\tilde{F}_k(0, 0) &= I \in \mathbb{R}^{2 \times 2}. \end{aligned}$$

Let furthermore $(s, t), (s', t') \in B_T(0)$ for $0 < T < T_0$. Then by (5.30) and (5.31) we have

$$\begin{aligned} |D\tilde{F}_k(s, t) - D\tilde{F}_k(s', t')| &\leq |(DF_k(0, 0))^{-1}| \cdot |DF_k(s, t) - DF_k(s', t')| \\ &\leq CT < CT_0 =: \delta_0 < 1, \end{aligned}$$

if we choose T_0 small enough. So Lemma A.2 is applicable to $\tilde{F}_k$ and therefore we find for every $(\tilde{y}, \tilde{z}) \in B_{(1-\delta_0)T}(0)$ parameters (s_k, t_k) with

$$\tilde{F}_k(s_k, t_k) = (\tilde{y}, \tilde{z}).$$

By (5.32) we obtain

$$F_k(s_k, t_k) = F_k(0, 0) + DF_k(0, 0)(\tilde{y}, \tilde{z}) =: (y, z).$$

Since $DF_k(0, 0)$ is invertible, we obtain a $\gamma > 0$ (by (5.29) only dependend on T_0), such that for every $(y, z) \in B_\gamma(F_k(0, 0))$ we find $(s_k, t_k) \in B_{(1-\delta_0)T}(0)$ satisfying

$$F_k(s_k, t_k) = (y, z).$$

Furthermore we may choose $\gamma > 0$ to be maximal, i.e. satisfying the following property: There is a $(y_0, z_0) \in \partial B_\gamma(0)$ and a $(\tilde{y}_0, \tilde{z}_0) \in \partial B_{(1-\delta_0)T}(0)$ with

$$DF_k(0, 0)(\tilde{y}_0, \tilde{z}_0) = (y_0, z_0).$$

This yields

$$(1 - \delta_0)T = |(\tilde{y}_0, \tilde{z}_0)| = |(DF_k(0, 0))^{-1}(y_0, z_0)| \leq |DF_k(0, 0)^{-1}| |(y_0, z_0)| \leq C\gamma.$$

Hence

$$C(1 - \delta_0)T \leq \gamma. \quad (5.33)$$

Now we choose $T_0 := \rho_0$, $T := \rho$ (by choosing ρ_0 small enough our results are still true). For later purposes we also state that the inverse inequality of (5.34) is true as well, only with a bigger constant of course, i.e. we have with $C_1 < C_2$ independent of k

$$C_1\rho \leq \gamma \leq C_2\rho. \quad (5.34)$$

Hence γ can at most decay linearly in ρ , while (5.25) and (5.26) show a quadratic error in ρ for the area and volume. By choosing ρ_0 small enough, we therefore obtain for k big enough parameters $(s_k, t_k) \in B_\gamma(F_k(0, 0))$ with

$$F_k(s_k, t_k) = (Area_0, Vol_0).$$

Let V_k^{graph} be the oriented varifold induced by $f_{k,\sigma}^{graph}$ and let us call $V_k^{s,t} := \Phi(s, t, \cdot)_\# V_k^{graph}$. The corresponding mass is called $\mu_k^{s,t}$. We denote with $A_k^{s,t}$ the second fundamental form and with $H_k^{s,t}$ the mean curvature vector of $V_k^{s,t}$. The orientation is called $\xi_k^{s,t}$.

Since we cannot replace $v_{k,i,\ell}$ by $w_{k,i,\ell}$ and still have a minimising sequence, we will have to correct the resulting error by the diffeomorphism Φ . By the mean value theorem and Lemma 4.4 the L^2 -norm of the second fundamental form of the Φ -corrected varifolds are controlled:

$$\left| \int |A_k^{s,t}| d\mu_k^{s,t} - \int |A_k^{0,0}| d\mu_k^{s,t} \right| \leq C|(s, t)| \leq C\rho. \quad (5.35)$$

Since F_k is continuously differentiable and the derivative is bounded independently of k (see the proof of Lemma 4.3), the area is controlled as well:

$$\left| \mu_k^{s,t}(\mathbb{R}^3) - \mu_k^{0,0}(\mathbb{R}^3) \right| = |F_k(s, t) - F_k(0, 0)| \leq C|(s, t)| \leq C\rho. \quad (5.36)$$

The Helfrich energy is controlled by Remark 4.5 and again the mean value theorem

$$\left| \int_{\text{spt}(\Phi) \times G^0(2,3)} |H_k^{s,t}(x) - H_0(*\xi)|^2 dV_k^{s,t}(x, \xi) - \int_{\text{spt}(\Phi) \times G^0(2,3)} |H_k^{0,0}(x) - H_0(*\xi)|^2 dV_k^{0,0}(x, \xi) \right| \leq C\rho. \quad (5.37)$$

Let us denote with $A_{k,i,\ell}^w$ the second fundamental form, $H_{k,i,\ell}^w$ the mean curvature vector, $\xi_{w_{k,i,\ell}}$ the orientation and with $K_{k,i,\ell}^w$ the Gauss curvature of $\text{graph}(w_{k,i,\ell})$. By the Gauss-Bonnet Theorem $\int_\Sigma K_k d\mu_{g_k}$ is given entirely by the topology of Σ . Hence f_k is also a minimising sequence for $f \mapsto W_{H_0, A}(f) + \kappa \int_\Sigma K_f d\mu_g$, $\kappa \in \mathbb{R}$ arbitrary under prescribed area and enclosed volume. Here K_f denotes the Gauss curvature of a given immersion $f : \Sigma \rightarrow \mathbb{R}^3$. By [9, Eq. (11)] we have $|A_k|^2 = |H_k|^2 - 2K_k$ and K_k is the Gauss curvature of f_k . The following calculation is an adaptation to our situation from [13, pp. 10–11]:

$$\begin{aligned}
& \int_{B_\sigma(x_0)} |A_k|^2 d\tilde{\mu}_{k,i,\ell} \\
&= \int_{B_\sigma(x_0)} |H_k|^2 d\tilde{\mu}_{k,i,\ell} - 2 \int_{B_\sigma(x_0)} K_k d\tilde{\mu}_{k,i,\ell} \\
&\stackrel{(2.8)}{\leq} C \left(\int_{B_\sigma(x_0)} |H_k - H_0(*\xi_{f_k})|^2 d\tilde{\mu}_{k,i,\ell} + \tilde{\mu}_{k,i,\ell}(B_\sigma(x_0)) \right) \\
&\quad - 2 \int_{B_\sigma(x_0)} K_k d\tilde{\mu}_{k,i,\ell} \\
&\leq C \left(\int_{B_\sigma(x_0)} |H_k - H_0(*\xi_{f_k})|^2 d\tilde{\mu}_{k,i,\ell} - \frac{2}{C} \int_{B_\sigma(x_0)} K_k d\tilde{\mu}_{k,i,\ell} \right) \\
&\quad + C\tilde{\mu}_{k,i,\ell}(B_\sigma(x_0)) \\
&\leq C \left(\int_{\text{graph}(w_{k,i,\ell})} |H_{k,i,\ell}^w - H_0(*\xi_{w_{k,i,\ell}})|^2 d\mathcal{H}^2 - \frac{2}{C} \int_{\text{graph}(w_{k,i,\ell})} K_{k,i,\ell}^w d\mathcal{H}^2 \right) \\
&\quad + \left| \int_{\text{spt}(\Phi) \times G^0(2,3)} |H_k^{S_{k,i,\ell}} - H_0(*\xi)|^2 dV_k^{S_{k,i,\ell}} - \frac{2}{C} \int_{\text{spt}(\Phi)} K_k^{S_{k,i,\ell}} d\mu_k^{S_{k,i,\ell}} \right. \\
&\quad \left. - \int_{\text{spt}(\Phi) \times G^0(2,3)} |H_k^{0,0} - H_0(*\xi)|^2 dV_k^{0,0} + \frac{2}{C} \int_{\text{spt}(\Phi)} K_k^{0,0} d\mu_k^{0,0} \right| \\
&\quad + C\tilde{\mu}_{k,i,\ell}(B_\rho(x_0)) + \varepsilon_k \\
&\leq C \left(\int_{\text{graph}(w_{k,i,\ell})} |H_{k,i,\ell}^w|^2 d\mathcal{H}^2 + \mathcal{H}^2(\text{graph}(w_{k,i,\ell})) \right. \\
&\quad \left. + \frac{2}{C} \int_{\text{graph}(w_{k,i,\ell})} K_{k,i,\ell}^w d\mathcal{H}^2 \right) + C\tilde{\mu}_{k,i,\ell}(B_\rho(x_0)) + C\rho + \varepsilon_k \\
&\stackrel{A.3}{\leq} C \left(\int_{B_\sigma(x_0) \cap \tilde{L}_{k,i,\ell}} |D^2 w_{k,i,\ell}|^2 dx + \sigma \right) + C\tilde{\mu}_{k,i,\ell}(B_\rho(x_0)) + C\rho + \varepsilon_k \\
&\stackrel{A.3, (5.8)}{\leq} C\sigma \int_{\text{graph}(v_{k,i,\ell}|_{\partial B_\sigma(x_0) \cap \tilde{L}_{k,i,\ell}})} |A_{m,i}|^2 d\mathcal{H}^1 + C\sigma + C\rho + \varepsilon_k
\end{aligned}$$

Here $\varepsilon_k \rightarrow 0$ for $k \rightarrow \infty$. The estimates connecting $D^2 w_{k,i,\ell}$ with the corresponding curvatures can be seen by, e.g. [9, Subsection 2.1] and the bound on the gradient of $w_{k,i,\ell}$. Integrating over S_ρ together with the Co-Area formula (see, e.g. [38, Eq. (10.6)]) yields

$$\int_{B_{\frac{\rho}{2}}(x_0)} |A_k|^2 d\tilde{\mu}_{k,i,\ell} \leq C \int_{B_{\frac{3}{4}\rho}(x_0) \setminus B_{\frac{\rho}{2}}(x_0)} |A_k|^2 d\tilde{\mu}_{k,i,\ell} + C\rho + \varepsilon_k.$$

Summing over ℓ yields with the help of (5.22)

$$\int_{B_{\frac{\rho}{2}}(x_0)} |A_k|^2 d\mu_{k,i} \leq C \int_{B_{\frac{3}{4}\rho}(x_0) \setminus B_{\frac{\rho}{2}}(x_0)} |A_k|^2 d\mu_{k,i} + C\rho + \varepsilon_k.$$

Hole filling yields a $0 < \Theta < 1$ independent of k , satisfying

$$\int_{B_{\frac{\rho}{2}}(x_0)} |A_k|^2 d\mu_{k,i} \leq \Theta \int_{B_{\frac{3}{4}\rho}(x_0)} |A_k|^2 d\mu_{k,i} + C\rho + \varepsilon_k.$$

The semi-continuity properties of ν (see, e.g. [27, Prop. 4.26]) for measure convergence yield for $k \rightarrow \infty$

$$\nu_i(B_{\frac{\rho}{2}}(x_0)) \leq \Theta \nu_i(B_{\rho}(x_0)) + C\rho. \quad (5.38)$$

Here $|A_k|^2 \mu_{k,i} \rightarrow \nu_i$ for $k \rightarrow \infty$ as Radon measures. By $\mu_{k,i} \leq \mu_k$ we also get $\nu_i \leq \nu$. Since we only needed the estimate

$$\nu(\overline{B_{\rho_{x_0}}(x_0)}) \leq \varepsilon_0^2$$

to obtain (5.38), we can repeat the argument for every $B_{\rho}(x) \subset B_{\frac{\rho_{x_0}}{4}}(x_0)$. This yields for these balls

$$\nu_i(B_{\frac{\rho}{2}}(x_0)) \leq \Theta \nu_i(B_{\rho}(x)) + C\rho.$$

For example, [17, Lemma 8.23] gives us

$$\nu_i(B_{\rho}(x)) \leq C \left(\frac{\rho}{\rho_{x_0}} \right)^{\beta} \nu_i(B_{\frac{\rho_{x_0}}{4}}(x)) + C\sqrt{\rho\rho_{x_0}}. \quad (5.39)$$

Here $C = C(\Theta) > 0$ and $\beta = \beta(\Theta) > 0$ are constants. Since $B_{\frac{\rho_{x_0}}{4}}(x) \subset B_{\rho_{x_0}}(x_0)$ we also get

$$\nu_i(B_{\rho}(x)) \leq C \left(\frac{\rho}{\rho_{x_0}} \right)^{\beta} \nu_i(B_{\rho_{x_0}}(x_0)) + C\sqrt{\rho\rho_{x_0}}. \quad (5.40)$$

Since $\frac{\rho}{\rho_{x_0}} < 1$ we can choose $\beta < \frac{1}{2}$ and obtain

$$\begin{aligned} \nu_i(B_{\rho}(x)) &\leq C\rho^{\beta}\rho_{x_0}^{-\beta} \left(\nu_i(B_{\rho_{x_0}}(x_0)) + \rho^{\frac{1}{2}-\beta}\rho_{x_0}^{\frac{1}{2}+\beta} \right) \\ &\leq C\rho^{\beta}\rho_{x_0}^{-\beta} \left(\nu(\overline{B_{\rho_0}(x_0)}) + \rho_0 \right) \\ &\leq C\rho^{\beta}\rho_{x_0}^{-\beta} (\varepsilon_0^2 + \rho_0). \end{aligned}$$

By $|A_{\mu_i}|^2 \mu_i \leq \nu_i$ we therefore get

$$\int_{B_{\rho}(x)} |A_{\mu_i}|^2 d\mu_i \leq C\rho^{\beta}\rho_{x_0}^{-\beta} (\varepsilon_0^2 + \rho_0). \quad (5.41)$$

Choosing $C(\varepsilon_0 + \rho_0)$ small enough, Allard's regularity theorem A.1 (cf. (A.2)) yields μ_i to be a $C^{1,\beta}$ graph. By (5.12) μ is a union of $C^{1,\beta}$ graphs in a neighbourhood of x_0 , which all satisfy estimates in the form of (A.5). This finishes the proof. $\square$

Remark 5.2 The proof of Lemma 5.1 does not work, if we would minimise in the class of embeddings, since the lack of a Li-Yau-type inequality prevents us from showing that the $f_{k,\sigma}^{graph}$ are still embeddings. This is a key problem, because we cannot compare the Helfrich energy of $f_{k,\sigma}^{graph}$ to f_k without it.

Next we formulate the lower-semicontinuity property of the minimising sequence:

Lemma 5.3 *The minimising sequence V_k^0 for the Helfrich problem (1.6) satisfies*

$$W_{H_0}(V^0) \leq \liminf_{k \rightarrow \infty} W_{H_0}(V_k^0).$$

Proof Since μ is locally a graph of $C^{1,\beta} \cap W^{2,2}$ graphs outside of finitely many points (see Lemma 5.1), and these graphs are approximated by $\mu_{k,i}$, see (5.11), the proof of the lower-semicontinuity estimate is the same as in [13, Lemma 4.1]. $\square$

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Auxiliary results

For the readers convenience we collect a few needed results:

The following is a variant of Allard's regularity Theorem. A proof of this statement can be found in [39, Section 3] or [37, Korollar 20.3] (see also [35, Theorem B.1]).

Theorem A.1 (Allard's regularity Theorem, see [1], Theorem 8.16) *For $n, m \in \mathbb{N}$, $0 < \beta < 1$, $\alpha > 0$ there exist $\varepsilon_0 = \varepsilon_0(n, m, \alpha, \beta) > 0$, $\gamma = \gamma(n, m, \alpha, \beta)$ and $C = C(n, m, \alpha, \beta)$ such that:*

Let μ be an integral n -varifold in $B_{\rho_0}^{n+m}(0)$, $0 < \rho_0 < \infty$, $0 < \varepsilon < \varepsilon_0$ with locally bounded first variation in $B_{\rho_0}^{n+m}(0)$ satisfying

$$\rho^{1-n} \|\delta\mu\|(B_\rho) \leq \varepsilon^2 (\rho^{-n} \mu(B_\rho))^{1-\alpha} \rho^{2\beta} \rho_0^{-2\beta}, \quad \forall B_\rho \subset B_{\rho_0}(0) \quad (\text{A.1})$$

or weak mean curvature $H_\mu \in L^2(\mu \llcorner B_{\rho_0}^{n+m}(0))$ satisfying

$$(\rho^{2-n}) \left(\int_{B_\rho} |H_\mu|^2 d\mu \right)^{\frac{1}{2}} \leq \varepsilon (\rho^{-n} \mu(B_\rho))^{\frac{1}{2}-\alpha} \rho^\beta \rho_0^{-\beta}, \quad \forall B_\rho \subset B_{\rho_0}(0) \quad (\text{A.2})$$

and

$$0 \in \text{spt}\mu, \quad \rho_0^{-n} \mu(B_{\rho_0}(0)) \leq (1 + \varepsilon) \omega_n. \quad (\text{A.3})$$

Then there exists $u \in C^{1,\beta}(B_{\gamma\varepsilon\rho_0}^n(0), \mathbb{R}^m)$, $u(0) = 0$, such that after rotation

$$\mu \llcorner B_{\gamma\varepsilon\rho_0}^{n+m}(0) = \mathcal{H}^n \llcorner (\text{graph} u \cap B_{\gamma\varepsilon\rho_0}^{n+m}(0)) \quad (\text{A.4})$$

and

$$(\varepsilon \rho_0)^{-1} \|u\|_{L^\infty(B_{\gamma\varepsilon\rho_0}^n(0))} + \|\nabla u\|_{L^\infty(B_{\gamma\varepsilon\rho_0}^n(0))} + (\varepsilon \rho_0)^\beta \operatorname{höl}_{B_{\gamma\varepsilon\rho_0}^n(0),\beta} \nabla u \leq C\varepsilon^{\frac{1}{2(n+1)}}. \quad (\text{A.5})$$

In Sect. 3 we need a version of the inverse function theorem with explicit estimates on the size of domain and codomain on which the function is invertible:

Theorem A.2 (See [26], Chapter XIV Sect. 1, Lemma 1.3) *Let $0 \in U \subset \mathbb{R}^n$ be open and $f \in C^1(U, \mathbb{R}^n)$. Furthermore let $f(0) = 0$, $Df(0) = \operatorname{id}$. Assume $r > 0$ with $\overline{B_r(0)} \subset U$ and let $0 < s < 1$ satisfy*

$$\|Df(z) - Df(x)\| \leq s$$

for all $x, z \in \overline{B_r(0)}$. Here $\|Df(z)\| = \sup_{|x|=1} |Df(z)x|$. If $y \in \mathbb{R}^n$ and $|y| \leq (1-s)r$, then there exists a unique $x \in B_r(0)$, such that $f(x) = y$.

The following lemma provides a suitable comparison function in Sect. 4. It is a generalisation by Schygulla of the biharmonic comparison principle by Simon (see [39, Lemma 2.2]):

Lemma A.3 (See [36], p. 938 Lemma 8) *Let $L \subset \mathbb{R}^3$ be a two-dimensional plane, $x_0 \in L$ and $u \in C^\infty(U, L^\perp)$, where $U \subset L$ is an open neighbourhood of $L \cap \partial B_\rho(x_0)$. Moreover let $|Du| \leq c$ on u . Then there exists a function $w \in C^\infty(\overline{B_\rho(x_0)}, L^\perp)$ such that*

$$\begin{aligned} w &= u, \quad \frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} \text{ on } \partial B_\rho(x_0), \\ \frac{1}{\rho} \|w\|_{L^\infty(B_\rho(x_0))} &\leq c \left(\frac{1}{\rho} \|u\|_{L^\infty(\partial B_\rho(x_0))} + \|Du\|_{L^\infty(\partial B_\rho(x_0))} \right), \\ \|Dw\|_{L^\infty(B_\rho(x_0))} &\leq c \|Du\|_{L^\infty(\partial B_\rho(x_0))}, \\ \int_{B_\rho(x_0)} |D^2 w(x)|^2 dx &\leq c \rho \int_{\operatorname{graph} u_{\partial B_\rho(x_0)}} |A|^2 d\mathcal{H}^1. \end{aligned}$$

Here A denotes the second fundamental form of $\operatorname{graph} u$. Furthermore ν is the outer normal of $L \cap B_\rho(x_0)$ with respect to L .

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