On VT -harmonic maps

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Abstract

VT -harmonic maps generalize the standard harmonic maps, with respect to the structure of both domain and target. These can be manifolds with natural connections other than the Levi-Civita connection of Riemannian geometry, like Hermitian, affine or Weyl manifolds. The standard harmonic map semilinear elliptic system is augmented by a term coming from a vector field V on the domain and another term arising from a 2-tensor T on the target. In fact, this geometric structure then also includes other geometrically defined maps, for instance magnetic harmonic maps. In this paper, we treat VT -harmonic maps and their parabolic analogues with PDE tools. We establish a Jäger–Kaul type maximum principle for these maps. Using this maximum principle, we prove an existence theorem for the Dirichlet problem for VT -harmonic maps. As applications, we obtain results on Weyl/affine/Hermitian harmonic maps between Weyl/affine/Hermitian manifolds, as well as on magnetic harmonic maps from two-dimensional domains. We also derive gradient estimates and obtain existence results for such maps from noncompact complete manifolds.

Keywords VT -harmonic maps · Maximum principle · Uniqueness · Existence

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1 Introduction

Let (M^m, g) be a compact Riemannian manifolds with nonempty boundary ∂M and $(N^n, \tilde{g})$ a complete Riemannian manifold without boundary. Let $d : N \times N \rightarrow \mathbb{R}$ be the distance function on N and $B_{(1+\sigma)R}(p) := \{q \in N : d(p, q) \leq (1 + \sigma)R\}$ a regular ball in N , that is, disjoint from the cut locus of its center p and of radius $(1 + \sigma)R < \frac{\pi}{2\sqrt{\kappa}}$, where $\kappa = \max\{0, \sup_{B_{(1+\sigma)R}(p)} K_N\}$ and $\sup_{B_{(1+\sigma)R}(p)} K_N$ is an upper bound of the sectional curvature K of N on $B_{(1+\sigma)R}(p)$, and $\sigma > 0$ is any given constant.

Let $V \in \Gamma(TM)$, $T \in \Gamma(\otimes^{1,2}TN)$. We call a map $u : M \rightarrow N$ a VT -harmonic map if u satisfies

$$\tau(u) + du(V) + \text{Tr}_g T(du, du) = 0, \quad (1.1)$$

where $\tau(u) = \text{tr} Ddu$ is the tension field of the map u . This is a generalization the notion of a V -harmonic map that has been studied in recent years as a common framework including Hermitian, affine and Weyl harmonic maps into Riemannian manifolds, that is, the domain possessed a connection different from the Levi-Civita connection, but the target was a Riemannian manifold with its Levi-Civita connection. This generalized the standard harmonic map system $\tau(u) = 0$ to a system of the form $\tau(u) + du(V) = 0$ with a vector field V on the domain. Here, we want to consider targets that are of the same type as the domain. That leads to the system (1.1) with an additional term arising from a 2-tensor T on the target. As this new term $\text{Tr}_g T(du, du)$, in contrast to the term $du(V)$, is analytically of the same weight as the elliptic operator $\tau(u)$ (which includes the Laplace–Beltrami operator of the domain), this makes the analysis more difficult and subtle. This is the problem that we are addressing in this paper.

In local coordinates $\{x^\alpha\}$ on M and $\{y^i\}$ on N , respectively, we can write (1.1) as

$$\Delta_M u^i + \Gamma_{jk}^i(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} + V^\alpha \frac{\partial u^i}{\partial x^\alpha} + T_{jk}^i(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} = 0, \quad (1.2)$$

where Δ_M is the Laplacian on (M, g) , Γ_{jk}^i stands for the Christoffel symbols of $(N, \tilde{g})$, $V := V^\alpha \frac{\partial}{\partial x^\alpha}$ and $T := T_{jk}^i \frac{\partial}{\partial y^j} \otimes dy^j \otimes dy^k$. This is a second-order semilinear elliptic system on the manifold (M, g) .

As is already the case for V -harmonic maps, in general, (1.1) is neither in divergence form, nor has a variational structure. Chen et al. [5] established a Jäger–Kaul type maximum principle for V -harmonic maps by using the method of [8], and combining this with the continuity method, the existence of V -harmonic maps into a regular ball could be proved. Therefore, it is natural to ask whether a maximum principle holds for VT -harmonic maps. However, the case of VT -harmonic maps is harder to deal with than V -harmonic maps since we now have an additional quadratic term arising from the tensor T . Due to this additional structure, the construction of the elliptic operator in [5] is no longer valid in our case. To overcome this difficulty, we use another construction as in [7] to compensate this term and obtain the following maximum principle for VT -harmonic maps:

Theorem 1 *Let $u_1, u_2 \in C^0(M, N)$ be two VT -harmonic maps into a geodesic ball $B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if*

$$\max |\nabla T| + \max |T| \leq C_0, \quad (1.3)$$

then the function $\Theta : M \rightarrow \mathbb{R}$ defined by

$$\Theta := \frac{q_{\frac{\kappa}{4}}(\rho)}{(q_{\kappa}((1+\sigma)R) - q_{\kappa}(\rho_1))^{\frac{1}{2}} \cdot (q_{\kappa}((1+\sigma)R) - q_{\kappa}(\rho_2))^{\frac{1}{2}}} \quad (1.4)$$

satisfies the maximum principle, namely

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$

Here the expression of q_{κ} is given in Sect. 2, and $\rho := d(u_1, u_2)$, $\rho_i := d(p, u_i)$, $i = 1, 2$.

In particular, if $u_1 = u_2$ on the boundary ∂M , then $u_1 \equiv u_2$ on M .

Remark The explicit expression of the constant C_0 in the above and in the subsequent results can be seen in (3.5). Importantly, $C_0 \rightarrow \infty$ for $R \rightarrow 0$. Thus, we can also satisfy the condition on T by making the target ball sufficiently small.

For the heat flow of VT -harmonic maps, an analogous result holds. For $T > 0$, we set

$$M_T := M \times [0, T]$$

and denote the parabolic boundary of M_T by

$$\partial_p M_T := (M \times \{0\}) \cup (\partial M \times [0, T]).$$

We consider the heat flow of VT -harmonic maps

$$\partial_t u = \tau(u) + du(V) + \text{Tr}_g T(du, du) \quad (1.5)$$

and have

Theorem 2 Let $u_1, u_2 \in C^0(M, N)$ be two solutions of heat flow Eq. (1.5) for VT -harmonic maps into a geodesic ball $B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if

$$\max |\nabla T| + \max |T| \leq C_0,$$

then the function $\Theta : M_T \rightarrow \mathbb{R}$ defined by (1.4) with M replaced by M_T satisfies the maximum principle:

$$\max_{M_T} \Theta \leq \max_{\partial_p M_T} \Theta.$$

In particular, if $u_1 = u_2$ on the boundary $\partial_p M_T$, then $u_1 \equiv u_2$ on M_T .

As an application of the above maximum principle, we obtain the existence of VT -harmonic maps into a geodesic ball.

Theorem 3 Let $M, N, V, T, B_R(p)$ be as in Theorem 2. Suppose $u_0 \in H^{2,q}(M, N)$ ($q > m$) with $u_0(M) \subset B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if

$$\max |\nabla T| + \max |T| \leq C_0,$$

then the initial boundary value problem

$$\begin{cases} \partial_t u = \tau(u) + du(V) + \text{Tr}_g T(du, du), \\ u - u_0 \in H_0^{2,q}(M, N), \quad u(0) = u_0, \quad u(M \times [0, \infty)) \subset B_R(p), \end{cases} \quad (1.6)$$

admits a unique global solution u which subconverges to a unique solution $u \in H^{2,q}(M, N)$ of the Dirichlet problem

$$\begin{cases} \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0, \\ u - u_0 \in H_0^{2,q}(M, N), \end{cases} \quad (1.7)$$

such that $u(M) \subset B_R(p)$.

Furthermore, based on Theorem 3, we shall also establish the existence of VT -harmonic maps from complete noncompact Riemannian manifolds by using a gradient estimate and the compact exhaustion method.

Theorem 4 *Let (M^m, g) be a complete noncompact Riemannian manifold and $(N^n, \tilde{g})$ be a complete Riemannian manifold with sectional curvature bounded above by a positive constant κ . Let $B_R(p)$ be a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$ and $u_0 : M \rightarrow N$ a smooth map with $u_0(M) \subset B_R(p)$. Suppose $\|V\|_{L^\infty} < +\infty$.*

For appropriate σ and R , there exists a constant C'_0 depending only on κ, σ, R and the geometry of N , such that if

$$\max |\nabla T| + \max |T| \leq C'_0,$$

then there exists a VT -harmonic map $u \in C^\infty(M, N)$ homotopic to u_0 such that $u(M) \subset B_R(p)$.

2 Preliminaries

Let us first give some notations:

$$\begin{aligned} s_\kappa(t) &:= \begin{cases} t & \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t & \kappa > 0, \end{cases} & q_\kappa(t) &:= \begin{cases} \frac{t^2}{2} & \kappa = 0 \\ \frac{1}{\kappa} (1 - \cos \sqrt{\kappa} t) & \kappa > 0. \end{cases} \\ a_\kappa(t) &:= \begin{cases} 0 & \text{if } t = 0 \\ \frac{1 - s'_\kappa(t)}{s_\kappa(t)} & \text{if } t > 0, \end{cases} & b_\kappa(t) &:= \begin{cases} 0 & \text{if } t = 0 \\ \frac{1 - s'_\kappa(t)}{2s_\kappa(t)} \left(1 + \frac{t}{s_\kappa(t)}\right) & \text{if } t > 0. \end{cases} \end{aligned}$$

In local coordinates $\{x^\alpha\}$ on M and $\{y^i\}$ on N , respectively, the energy density of u is

$$e(u) := g^{\alpha\beta} \tilde{g}_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Assume the metric of N satisfies:

$$0 < \tilde{\lambda}(y)(\delta_{ij}) \leq (\tilde{g}_{ij}(y)) \leq \tilde{\Lambda}(y)(\delta_{ij}), \quad \forall y \in N.$$

Denote $\lambda := \min_N \tilde{\lambda}$ and $\Lambda := \max_N \tilde{\Lambda}$

$\forall y_1, y_2 \in B_R(p)$, there exists a unit speed geodesic $\gamma : [0, \rho] \rightarrow B_R(p) \subset N$ with $\gamma(0) = y_1, \gamma(\rho) = y_2$, where $\rho = \text{dist}(y_1, y_2)$. For any $v_j \in T_{y_j} N, j = 1, 2$, let X be the unique Jacobi field along γ with $X(0) = v_1, X(\rho) = v_2$. Then, we define a pseudo-distance

$$\delta(v_1, v_2) := \begin{cases} \left(\rho \int_0^\rho |\dot{X}|^2 \right)^{\frac{1}{2}} & \text{if } \rho > 0, \\ |v_1 - v_2| & \text{if } \rho = 0. \end{cases}$$

Another pseudo-distance is given by

$$\delta_0(v_1, v_2) := |v_1 - \bar{v}_2|,$$

where $\bar{v}_2 \in T_{y_1}N$ stands for the vector obtained by parallel displacement of $v_2 \in T_{y_2}N$ along γ . Let $L(T_xM, T_yN)$ be the space of all linear maps from T_xM to T_yN . The pseudo-distance δ on the tangent bundle can be extended to a pseudo-distance on the fibers, that is, for $q_1, q_2 \in \cup_{y \in B_R(p)} L(T_xM, T_yN)$ (disjoint union), we define their pseudo-distance as

$$\delta(q_1, q_2) := \left(\sum_{\alpha=1}^m \delta^2(q_1(e_\alpha), q_2(e_\alpha)) \right)^{\frac{1}{2}},$$

where $\{e_1, \dots, e_m\}$ is an orthonormal base for T_xM .

We have the following relationship between these two pseudo-distances:

Lemma 1 ([4]) *There is a positive constant C depending only on $B_R(p)$ and the geometry of N such that for any $y_j \in B_R(p)$ and $v_j \in T_{y_j}N$, $j = 1, 2$, we have*

$$\delta_0^2(v_1, v_2) - C(|v_1|^2 + |v_2|^2)\rho^2 \leq \delta^2(v_1, v_2) \leq \delta_0^2(v_1, v_2) + C(|v_1|^2 + |v_2|^2)\rho^2.$$

Remark 1 In fact, by the proof in [4] and using a well-known expression of the curvature operator (see, e.g., Lemma 4.3.3 in [12]), it is not hard to see that if the sectional curvature K on $B_R(p)$ satisfies $\theta \leq K|_{B_R(p)} \leq \kappa$ for a constant $\theta < 0$, then the constant C can be expressed as $14(\kappa - \theta)$.

The following estimates will also be important for us:

Lemma 2 ([7]) *Let (M, g) be a compact Riemannian manifolds with nonempty boundary ∂M and $(N, \tilde{g})$ a complete manifold without boundary and $B_R(p)$ a regular ball in N . Let*

$$\begin{aligned} g_1 &:= q_\kappa \circ d(p, \cdot) : B_R(p) \rightarrow \mathbb{R}, \\ h &:= q_{\frac{\kappa}{4}} \circ d : B_R(p) \times B_R(p) \rightarrow \mathbb{R}. \end{aligned}$$

Then,

$$\nabla^2 g_1(u, u) \geq s'_\kappa(\tau)|u|^2 \quad (2.1)$$

hold for $u \in T_xN$, $x \in B_R(p)$ and $\tau := d(p, x)$.

$$\nabla^2 h(v, v) \geq -s'_{\frac{\kappa}{4}}(\rho)a_\kappa(\rho) \sum_{i=1}^2 |v_i|^2 \quad (2.2)$$

and

$$\nabla^2 h(v, v) \geq s'_{\frac{\kappa}{4}}(\rho)\delta^2(v_1, v_2) - s_{\frac{\kappa}{4}}(\rho)b_\kappa(\rho) \sum_{i=1}^2 |v_i|^2 \quad (2.3)$$

holds for $v = v_1 \oplus v_2$, $v_j \in T_{y_j}N$, $y_j \in B_R(p)$, $j = 1, 2$, $\rho = \text{dist}(y_1, y_2)$.

3 The maximum principle

Proof of Theorem 1 Let

$$\begin{aligned}\psi(x) &:= q_{\frac{\kappa}{4}} \circ d(u_1(x), u_2(x)), \\ \psi_i(x) &:= q_{\kappa} \circ d(p, u_i(x)), \quad i = 1, 2, \\ \Phi(x) &:= \frac{1}{2} \sum_{i=1}^2 \omega \circ \psi_i(x), \quad \text{where } \omega(t) := -\log(q_{\kappa}((1 + \sigma)R) - t).\end{aligned}$$

We consider the operator

$$\mathcal{L}_V(\cdot) := e^{\Phi} \cdot \operatorname{div}(e^{-2\Phi} \nabla \cdot) + e^{-\Phi} \cdot V(\cdot).$$

By direct computation, we obtain

$$\begin{aligned}\mathcal{L}_V(\Theta) &= \mathcal{L}_V(e^{\Phi} \cdot \psi) = \Delta \psi + \psi(\Delta \Phi - |\nabla \Phi|^2) + \psi V(\Phi) + V(\psi) \\ &= \Delta_V \psi + \psi(\Delta_V \Phi - |\nabla \Phi|^2).\end{aligned}$$

Define $U, U_1, U_2 : M \rightarrow N \times N$ by

$$U(x) := (u_1(x), u_2(x)), \quad U_i(x) := (p, u_i(x)), \quad i = 1, 2.$$

Let $v := \frac{\kappa}{4}$, $h := q_v \circ d$, $\phi := q_{\kappa} \circ d$, then $\psi = h \circ U$, $\psi_i = \phi \circ U_i$.

For any $x \in M$, we let $\tilde{\gamma}$ be the unique geodesic connecting $u_1(x)$ and $u_2(x)$. Choosing a parallel orthonormal frame $\{E_i(t)\}$ along $\tilde{\gamma}$ with $E_1 = \tilde{\gamma}'$, and a local orthonormal frame $\{e_{\alpha}\}_{\alpha=1}^m$ around x , assuming that $\frac{\partial}{\partial y^i} := a_i^j E_j$, we have

$$\begin{aligned}& \delta_0(T(du_1(e_{\alpha}), du_1(e_{\alpha})), T(du_2(e_{\alpha}), du_2(e_{\alpha}))) \\ &= \delta_0 \left((u_1)_{\alpha}^i (u_1)_{\alpha}^j T_{ij}^r(u_1) \frac{\partial}{\partial y^r}(u_1), (u_2)_{\alpha}^i (u_2)_{\alpha}^j T_{ij}^r(u_2) \frac{\partial}{\partial y^r}(u_2) \right) \\ &= \delta_0 \left((u_1)_{\alpha}^i (u_1)_{\alpha}^j T_{ij}^r(u_1) a_r^{\mu}(u_1) E_{\mu}(u_1), (u_2)_{\alpha}^i (u_2)_{\alpha}^j T_{ij}^r(u_2) a_r^{\mu}(u_2) E_{\mu}(u_2) \right) \\ &\leq \sum_{\mu} \left| T_{ij}^r(u_1) a_r^{\mu}(u_1) (u_1)_{\alpha}^i (u_1)_{\alpha}^j - T_{ij}^r(u_2) a_r^{\mu}(u_2) (u_2)_{\alpha}^i (u_2)_{\alpha}^j \right| \\ &\leq \sum_{\mu} \left| (T_{ij}^r(u_1) - T_{ij}^r(u_2)) a_r^{\mu}(u_1) (u_1)_{\alpha}^i (u_1)_{\alpha}^j \right| \\ &\quad + \sum_{\mu} \left| T_{ij}^r(u_2) (a_r^{\mu}(u_1) - a_r^{\mu}(u_2)) (u_1)_{\alpha}^i (u_1)_{\alpha}^j \right| \\ &\quad + \sum_{\mu} \left| T_{ij}^r(u_2) a_r^{\mu}(u_2) (u_1)_{\alpha}^j \left((u_1)_{\alpha}^i - (u_2)_{\alpha}^i \right) \right| \\ &\quad + \sum_{\mu} \left| T_{ij}^r(u_2) a_r^{\mu}(u_2) (u_2)_{\alpha}^i \left((u_1)_{\alpha}^j - (u_2)_{\alpha}^j \right) \right|.\end{aligned}$$

Denote $A = (a_i^k)$, then $AA^T = G := (\tilde{g}_{ik})$. Since

$$\sum_{i,k} |a_i^k|^2 = \|A\|^2 = \operatorname{tr}(AA^T) = \operatorname{tr}(G) \leq n\Lambda, \quad \left| (u_1)_{\alpha}^i (u_1)_{\alpha}^j \right| \leq \frac{1}{\lambda} |du_1|^2,$$

we then obtain

$$\begin{aligned}
 & \delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) \\
 & \leq \sum_{\mu=1}^n \left(\frac{\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T| |du_1|^2 + \frac{\tilde{C}_1}{\lambda} \rho \max |T| |du_1|^2 \right. \\
 & \quad \left. + \frac{\sqrt{n\Lambda}}{\lambda} \max |T| (|du_1| + |du_2|) |du_1 - du_2| \right) \\
 & = \frac{n\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T| |du_1|^2 + \frac{n\tilde{C}_1}{\lambda} \rho \max |T| |du_1|^2 \\
 & \quad + \frac{n\sqrt{n\Lambda}}{\lambda} \max |T| (|du_1| + |du_2|) |du_1 - du_2|,
 \end{aligned}$$

where $\tilde{C}_1 > 0$ is a constant depending only on the bound of (da_r^μ) on $B_{\frac{\pi}{2\sqrt{\kappa}}}(p)$. By Lemma 1, we get

$$\begin{aligned}
 & \delta(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) \\
 & \leq \delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) + \sqrt{C} \rho (|T(du_1(e_\alpha), du_1(e_\alpha))| \\
 & \quad + |T(du_2(e_\alpha), du_2(e_\alpha))|) \\
 & \leq \frac{n\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T| \sum_{i=1}^2 |du_i|^2 + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \rho \max |T| \sum_{i=1}^2 |du_i|^2 \\
 & \quad + \frac{n\sqrt{n\Lambda}}{\lambda} \max |T| (|du_1| + |du_2|) |du_1 - du_2|,
 \end{aligned}$$

where $C = 14(\kappa - \theta)$, and the constant θ is a lower bound of the sectional curvature of N on $B_{\frac{\pi}{2\sqrt{\kappa}}}(p)$. The Cauchy inequality implies that

$$\begin{aligned}
 & \frac{n\sqrt{n\Lambda}}{\lambda} \max |T| |du_i| |du_1 - du_2| s_v(\rho) \leq \frac{\varepsilon_1 n^2}{2} |du_1 - du_2|^2 \\
 & \quad + \frac{n\Lambda}{2\varepsilon_1 \lambda^2} s_v^2(\rho) \max |T|^2 |du_i|^2.
 \end{aligned}$$

By using the formula (2.13) in [7], it follows that

$$\begin{aligned}
 & \langle (\nabla h) \circ U, -T(dU(e_\alpha), dU(e_\alpha)) \rangle \\
 & = -s_v(\rho) \langle (\nabla d) \circ U, T(dU(e_\alpha), dU(e_\alpha)) \rangle \\
 & = -s_v(\rho) \langle \tilde{e}_1(U) \oplus \tilde{e}_2(U), T(du_1(e_\alpha), du_1(e_\alpha)) \oplus T(du_2(e_\alpha), du_2(e_\alpha)) \rangle \\
 & \geq -s_v(\rho) \delta(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) \\
 & \geq -s_v(\rho) \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T| \sum_{i=1}^2 |du_i|^2 + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \rho \max |T| \sum_{i=1}^2 |du_i|^2 \right\} \\
 & \quad - \varepsilon_1 n^2 |du_1 - du_2|^2 - \frac{n\Lambda}{2\varepsilon_1 \lambda^2} s_v^2(\rho) \max |T|^2 \sum_{i=1}^2 |du_i|^2,
 \end{aligned}$$

where $\tilde{e}_1 = -\tilde{\gamma}'(0)$, $\tilde{e}_2 = \tilde{\gamma}'(\rho)$. Since

$$\begin{aligned}
 \delta_0^2(du_1(e_\alpha), du_2(e_\alpha)) &= \delta_0^2\left((u_1)_\alpha^i \frac{\partial}{\partial y^i}(u_1), (u_2)_\alpha^i \frac{\partial}{\partial y^i}(u_2)\right) \\
 &= \delta_0^2\left((u_1)_\alpha^i a_i^j(u_1) E_j(u_1), (u_2)_\alpha^i a_i^j(u_2) E_j(u_2)\right) = \sum_j \left[(u_1)_\alpha^i a_i^j(u_1) - (u_2)_\alpha^i a_i^j(u_2)\right]^2 \\
 &= \sum_j \left[(u_1)_\alpha^i (a_i^j(u_1) - a_i^j(u_2)) + a_i^j(u_2) ((u_1)_\alpha^i - (u_2)_\alpha^i)\right]^2 \\
 &\geq \frac{1}{2} \sum_j \left[a_i^j(u_2) ((u_1)_\alpha^i - (u_2)_\alpha^i)\right]^2 - \sum_j \left[(u_1)_\alpha^i (a_i^j(u_1) - a_i^j(u_2))\right]^2 \\
 &= \frac{1}{2} \sum_j \sum_{ik} \tilde{g}_{ik}(u_2) ((u_1)_\alpha^i - (u_2)_\alpha^i) ((u_1)_\alpha^k - (u_2)_\alpha^k) - \sum_j \left[(u_1)_\alpha^i (a_i^j(u_1) - a_i^j(u_2))\right]^2 \\
 &\geq \frac{n\lambda}{2} |(du_1 - du_2)(e_\alpha)|^2 - n\tilde{C}_1 \rho^2 |du_1(e_\alpha)|^2,
 \end{aligned}$$

then by Lemma 1, we have

$$\begin{aligned}
 \delta^2(du_1(e_\alpha), du_2(e_\alpha)) &\geq \delta_0^2(du_1(e_\alpha), du_2(e_\alpha)) - C(|du_1(e_\alpha)|^2 + |du_2(e_\alpha)|^2) \rho^2 \\
 &\geq \frac{n\lambda}{2} |(du_1 - du_2)(e_\alpha)|^2 - (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i(e_\alpha)|^2.
 \end{aligned}$$

Namely,

$$\delta^2(du_1, du_2) \geq \frac{n\lambda}{2} |du_1 - du_2|^2 - (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2. \quad (3.1)$$

Therefore,

$$\begin{aligned}
 &\langle (\nabla h) \circ U, -T(dU(e_\alpha), dU(e_\alpha)) \rangle \\
 &\geq -s_v(\rho) \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T| \sum_{i=1}^2 |du_i|^2 + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \rho \max |T| \sum_{i=1}^2 |du_i|^2 \right\} \\
 &\quad - \frac{2n\varepsilon_1}{\lambda} \left[\delta^2(du_1, du_2) + (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2 \right] - \frac{n\Lambda}{2\varepsilon_1 \lambda^2} s_v(\rho) \rho \max |T|^2 \sum_{i=1}^2 |du_i|^2 \\
 &= -s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| + \frac{n\Lambda}{2\varepsilon_1 \lambda^2} \max |T|^2 \right\} \sum_{i=1}^2 |du_i|^2 \\
 &\quad - \frac{2n\varepsilon_1}{\lambda} \left[\delta^2(du_1, du_2) + (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2 \right].
 \end{aligned}$$

The above inequality and (2.3) imply that

$$\begin{aligned}
 \Delta_V \psi &= \Delta_V (h \circ U) = \sum_{\alpha} \nabla^2 h(dU(e_{\alpha}), dU(e_{\alpha})) + \langle (\nabla h) \circ U, \tau_V(U) \rangle. \\
 &= \sum_{\alpha} \nabla^2 h(dU(e_{\alpha}), dU(e_{\alpha})) + \sum_{\alpha} \langle (\nabla h) \circ U, -T(dU(e_{\alpha}), dU(e_{\alpha})) \rangle \\
 &\geq \left(s'_v(\rho) - \frac{2n\varepsilon_1}{\lambda} \right) \delta^2(du_1, du_2) - s_v(\rho) b_{\kappa}(\rho) \sum_{i=1}^2 |du_i|^2 \\
 &\quad - s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right. \\
 &\quad \left. + \frac{n\Lambda}{2\varepsilon_1 \lambda^2} \max |T|^2 \right\} \sum_{i=1}^2 |du_i|^2 \\
 &\quad - \frac{2n\varepsilon_1}{\lambda} (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2.
 \end{aligned}$$

Choosing $\varepsilon_1 = \frac{\lambda}{2n} \cos(\sqrt{\kappa}R) > 0$, then we obtain

$$\begin{aligned}
 \Delta_V \psi &\geq -s_v(\rho) b_{\kappa}(\rho) \sum_{i=1}^2 |du_i|^2 - \cos(\sqrt{\kappa}R) (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2 \\
 &\quad - s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right. \\
 &\quad \left. + \frac{n^2\Lambda}{\lambda^3 \cos(\sqrt{\kappa}R)} \max |T|^2 \right\} \sum_{i=1}^2 |du_i|^2.
 \end{aligned} \tag{3.2}$$

It follows from (2.1) that

$$\begin{aligned}
 \Delta_V \psi_i &= \Delta_V (\phi \circ U_i) = \sum_{\alpha} \nabla^2 \phi(dU_i(e_{\alpha}), dU_i(e_{\alpha})) + \langle (\nabla \phi) \circ U_i, \tau_V(U_i) \rangle \\
 &\geq s'_\kappa(\rho_i) |du_i|^2 + s_{\kappa}(\rho_i) \langle (\nabla d) \circ U_i, 0 \oplus (-T(du_i(e_{\alpha}), du_i(e_{\alpha}))) \rangle \\
 &\geq s'_\kappa(\rho_i) |du_i|^2 - s_{\kappa}(\rho_i) |T(du_i(e_{\alpha}), du_i(e_{\alpha}))| \\
 &\geq s'_\kappa(\rho_i) |du_i|^2 - s_{\kappa}(\rho_i) \max |T| |du_i|^2.
 \end{aligned} \tag{3.3}$$

It is easy to check that

$$\omega'' = \omega'^2, \quad \omega' \circ \psi_i = \frac{1}{q_{\kappa}((1 + \sigma)R) - q_{\kappa}(\rho_i)}.$$

Therefore,

$$\begin{aligned}
 \mathcal{L}_V(\Theta) &= \Delta_V \psi + \psi(\Delta_V \Phi - |\nabla \Phi|^2) \\
 &= \Delta_V \psi + \frac{1}{2} \psi \sum_{i=1}^2 (\omega' \circ \psi_i) \Delta_V \psi_i + \sum_{i=1}^2 \left[\frac{1}{2} \psi (\omega'' \circ \psi_i) - \frac{1}{4} \psi (\omega' \circ \psi_i)^2 \right] |\nabla \psi_i|^2 \\
 &= \Delta_V \psi + \frac{1}{2} \psi \sum_{i=1}^2 (\omega' \circ \psi_i) \Delta_V \psi_i + \frac{\psi}{4} \sum_{i=1}^2 (\omega' \circ \psi_i)^2 |\nabla \psi_i|^2 \\
 &\geq -s_v(\rho) b_\kappa(\rho) \sum_{i=1}^2 |du_i|^2 - \cos(\sqrt{\kappa} R) (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2 \\
 &\quad - s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right. \\
 &\quad \left. + \frac{n^2 \Lambda}{\lambda^3 \cos(\sqrt{\kappa} R)} \max |T|^2 \right\} \sum_{i=1}^2 |du_i|^2 \\
 &\quad + \frac{1}{2} \psi \sum_{i=1}^2 \frac{1}{q_\kappa((1+\sigma)R) - q_\kappa(\rho_i)} [s'_\kappa(\rho_i) |du_i|^2 - s_\kappa(\rho_i) \max |T| |du_i|^2] \\
 &= \psi \sum_{i=1}^2 \left\{ \frac{s'_\kappa(\rho_i)}{2(q_\kappa((1+\sigma)R) - q_\kappa(\rho_i))} - \frac{b_\kappa(\rho) s_v(\rho)}{q_v(\rho)} \right. \\
 &\quad - \frac{s_v(\rho) \rho}{q_v(\rho)} \left[\frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right. \\
 &\quad \left. + \frac{n^2 \Lambda}{\lambda^3 \cos(\sqrt{\kappa} R)} \max |T|^2 \right] - \frac{s_\kappa(\rho_i)}{2(q_\kappa((1+\sigma)R) - q_\kappa(\rho_i))} \max |T| \\
 &\quad \left. - \frac{\rho^2}{q_v(\rho)} \cos(\sqrt{\kappa} R) (n\tilde{C}_1 + C) \right\} |du_i|^2 \\
 &\geq \psi \sum_{i=1}^2 \left\{ \frac{s'_\kappa(\rho_i)}{2(q_\kappa((1+\sigma)R) - q_\kappa(\rho_i))} - \frac{b_\kappa(\rho) s_v(\rho)}{q_v(\rho)} \right. \\
 &\quad - \left[\frac{2n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{2n\tilde{C}_1}{\lambda} + 2\sqrt{C} \right) \max |T| \right. \\
 &\quad \left. + \frac{2n^2 \Lambda}{\lambda^3 \cos(\sqrt{\kappa} R)} \max |T|^2 \right] - \frac{\kappa R^2}{1 - \cos(\sqrt{\kappa} R)} \cos(\sqrt{\kappa} R) (n\tilde{C}_1 + C) \\
 &\quad \left. - \frac{\sqrt{\kappa} \sin(\sqrt{\kappa} R)}{2 [\cos(\sqrt{\kappa} R) - \cos((1+\sigma)\sqrt{\kappa} R)]} \max |T| \right\} |du_i|^2,
 \end{aligned} \tag{3.4}$$

where we have used the fact that $\frac{s_v(\rho)\rho}{q_v(\rho)}$ is nonincreasing in $(0, 2R]$ and $\frac{\rho^2}{q_v(\rho)}$ is increasing in $(0, 2R]$. Direct computation gives us

$$\frac{b_\kappa(\rho) s_v(\rho)}{q_v(\rho)} = \frac{\kappa}{4} \left(1 + \frac{1}{s'_\kappa(\rho)} \right) \left(1 + \frac{\rho}{s_\kappa(\rho)} \right) =: \alpha(\rho),$$

and $\alpha(t)$ is increasing in $[0, 2R]$ and $\beta_R(t) = \frac{s'_\kappa(t)}{2(q_\kappa((1+\sigma)R) - q_\kappa(t))}$ is increasing in $[0, R]$. Hence, we obtain

$$\begin{aligned} & \frac{s'_\kappa(\rho_i)}{2(q_\kappa((1+\sigma)R) - q_\kappa(\rho_i))} - \frac{b_\kappa(\rho)s_\nu(\rho)}{q_\nu(\rho)} \\ & \geq \frac{1}{2q_\kappa((1+\sigma)R)} - \frac{\kappa}{4} \left(1 + \frac{1}{s'_\kappa(\frac{2R}{4})} \right) \left(1 + \frac{2R}{s_\kappa(2R)} \right) \\ & = \frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1 + \cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R + \sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{L}_V(\Theta) \geq & \psi \sum_{i=1}^2 \left\{ \frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1 + \cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R + \sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)} \right. \\ & - \frac{2n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| \\ & - \left(\frac{2n\tilde{C}_1}{\lambda} + 2\sqrt{C} + \frac{\sqrt{\kappa}\sin(\sqrt{\kappa}R)}{2[\cos(\sqrt{\kappa}R) - \cos((1+\sigma)\sqrt{\kappa}R)]} \right) \max |T| \\ & \left. - \frac{2n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)} \max |T|^2 - \frac{(n\tilde{C}_1 + C)\kappa R^2\cos(\sqrt{\kappa}R)}{1 - \cos(\sqrt{\kappa}R)} \right\} |du_i|^2. \end{aligned}$$

Clearly, by choosing appropriate σ and R , we obtain

$$\begin{aligned} & \frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1 + \cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R + \sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)} \\ & - \frac{(n\tilde{C}_1 + C)\kappa R^2\cos(\sqrt{\kappa}R)}{1 - \cos(\sqrt{\kappa}R)} > 0. \end{aligned}$$

Hence, if

$$\begin{aligned} & \frac{2n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{2n\tilde{C}_1}{\lambda} + 2\sqrt{C} + \frac{\sqrt{\kappa}\sin(\sqrt{\kappa}R)}{2[\cos(\sqrt{\kappa}R) - \cos((1+\sigma)\sqrt{\kappa}R)]} \right) \max |T| \\ & + \frac{2n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)} \max |T|^2 \\ & \leq \frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1 + \cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R + \sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)} \\ & - \frac{(n\tilde{C}_1 + C)\kappa R^2\cos(\sqrt{\kappa}R)}{1 - \cos(\sqrt{\kappa}R)}, \end{aligned} \quad (3.5)$$

then we have

$$\mathcal{L}_V(\Theta) \geq 0.$$

(For $\sqrt{\kappa}R \rightarrow 0$, we use the Taylor expansions of $\sin$ and $\cos$ to obtain positive values on the right-hand side of (3.5).) It is easy to see that there exists a constant C_0 depending only on κ , σ , R and the geometry of N , so that if

$$\max |\nabla T| + \max |T| \leq C_0, \quad (3.6)$$

then (3.5) holds true; consequently, $\mathcal{L}_V(\Theta) \geq 0$. Applying the ordinary maximum principle, we have

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$

□

Proof of Theorem 2 We consider a parabolic operator of the form

$$\tilde{\mathcal{L}}_V := \mathcal{L}_V - e^{-\Phi} \partial_t.$$

By using

$$\begin{aligned} (\Delta_V - \partial_t) \psi &= (\Delta_V - \partial_t) (h \circ U) = \sum_{\alpha} \nabla^2 h(dU(e_{\alpha}), dU(e_{\alpha})) \\ &\quad + \langle (\nabla h) \circ U, \tau_V(U) - \partial_t U \rangle \end{aligned}$$

and

$$\begin{aligned} (\Delta_V - \partial_t) \psi_i &= (\Delta_V - \partial_t) (\phi \circ U_i) = \sum_{\alpha} \nabla^2 \phi(dU_i(e_{\alpha}), dU_i(e_{\alpha})) \\ &\quad + \langle (\nabla \phi) \circ U_i, \tau_V(U_i) - \partial_t U_i \rangle, \end{aligned}$$

as in the proof of Theorem 1, we can conclude that $\tilde{\mathcal{L}}_V(\Theta) \geq 0$ on M_T . From the parabolic maximum principle, we have

$$\max_{M_T} \Theta \leq \max_{\partial_p M_T} \Theta.$$

4 Existence results

Using the maximum principle obtained in the last section, we shall prove the existence of solutions of the Dirichlet problem for VT -harmonic maps.

Proof of Theorem 3 Let us choose normal coordinates $\{y^i\}_{i=1,2,\dots,n}$ centered at p , then any VT -harmonic map $u : M \rightarrow B_R(p) \subset N$ can be written as

$$u = (u^1, \dots, u^n) \in (H^{2,q}(M))^n$$

which satisfies the elliptic system

$$\Delta_M u^i + \Gamma_{jk}^i(u) \frac{\partial u^j}{\partial x^{\alpha}} \frac{\partial u^k}{\partial x^{\beta}} g^{\alpha\beta} + V^{\alpha} \frac{\partial u^i}{\partial x^{\alpha}} + T_{jk}^i(u) \frac{\partial u^j}{\partial x^{\alpha}} \frac{\partial u^k}{\partial x^{\beta}} g^{\alpha\beta} = 0, \quad i = 1, 2, \dots, n.$$

For simplicity of notation, we write it in a concise form

$$\Delta u + \Gamma(du, du) + du(V) + \text{Tr}_g T(du, du) = 0.$$

Now we consider the initial boundary value problem for the heat flow of VT -harmonic maps

$$\begin{cases} \partial_t u = \Delta u + \Gamma(du, du) + du(V) + \text{Tr}_g T(du, du), \\ u - u_0 \in H_0^{2,q}(M, N), \quad u(0) = u_0, \\ u(M \times [0, +\infty)) \subset B_R(p). \end{cases} \quad (4.1)$$

As in the proof of Theorem 3 in [5], by a continuity method that rests on the maximum principle Theorem 2, we can conclude the global existence of a solution $u(x, t)$ of the above flow (4.1). This solution satisfies

$$\|u(\cdot, t)\|_{1+\alpha} \leq C(q, M, V, T, N, u_0, R), \quad \forall t \in (0, +\infty)$$

for some $\alpha > 0$. Consequently, by the parabolic regularity theory, we have the uniform estimate

$$\|u\|_{C^{1+\alpha, 2+\alpha}(M)} \leq C. \quad (4.2)$$

For $u_1(x, t) = u(x, t)$, $u_2(x, t) = u(x, t + \sigma_1)$, $\sigma_1 > 0$, $\forall (x, t) \in M \times (0, +\infty)$, as in the proof of Theorem 2, the function Θ satisfies

$$\begin{cases} (\Delta - \partial_t)(\frac{\Theta}{\sigma_1^2}) + \langle V - 2\nabla\Phi, \nabla(\frac{\Theta}{\sigma_1^2}) \rangle \geq 0, \\ \Theta|_{\partial M} = 0. \end{cases}$$

By the ordinary maximum principle for functions, it follows that (see pp.178–179 in [17])

$$\left(\frac{\Theta}{\sigma_1^2}\right) \leq C(t - t_0)^{-k}, \quad \forall t \geq t_0$$

for any positive integer k and some $t_0 > 0$. Letting $\sigma_1 \rightarrow 0$, then we obtain $|u_t| \rightarrow 0$ as $t \rightarrow +\infty$, from which together with (4.2), we have u subconverges to a V T -harmonic map u_∞ satisfying (1.7) and $u_\infty(M) \subset B_R(p)$. $\square$

With the Schauder and higher regularity estimates, we can improve Theorem 3 to the following

Theorem 5 *Let $M, N, V, T, B_R(p)$ be as in Theorem 1. Suppose $u_0 \in C^0(M, N)$ with $u_0(M) \subset B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if*

$$\max |\nabla T| + \max |T| \leq C_0, \quad (4.3)$$

then the Dirichlet problem

$$\begin{cases} \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0, \\ u|_{\partial M} = u_0|_{\partial M} \end{cases}$$

admits a unique solution $u \in C^\infty(M, N) \cap C^0(\overline{M}, N)$ such that $u(M) \subset B_R(p)$.

5 Applications

5.1 Weyl harmonic maps (c.f. [14])

Let $(M, [g], {}^W\nabla)$ be a Weyl manifold. According to the definition, there exists a 1-form Θ such that ${}^Wg = \Theta \otimes g$ for any $g \in [g]$. Equivalently, ${}^W\nabla$ is defined by

$${}^W\nabla_X Y = \nabla_X Y - \frac{1}{2}\Theta(X)Y - \frac{1}{2}\Theta(Y)X + \frac{1}{2}g(X, Y)\Theta^\sharp, \quad \forall X, Y \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection and $\Theta^\sharp$ the vector field dual to Θ w.r.t. g . Let $\Gamma_{\alpha\beta}^\gamma, {}^W\Gamma_{\alpha\beta}^\gamma$ be the Christoffel symbols corresponding to ∇ and ${}^W\nabla$, respectively.

Let $(N, [\tilde{g}], {}^W\tilde{\nabla})$ be also a Weyl manifold, and correspondingly, we denote by $\tilde{\Theta}$ the 1-form, and $\tilde{\Gamma}_{ij}^k, {}^W\tilde{\Gamma}_{ij}^k$ are the Christoffel symbols for the Levi-Civita connection $\tilde{\nabla}$ and Weyl connection ${}^W\tilde{\nabla}$, respectively. Let $u : (M, [g], {}^W\nabla) \rightarrow (N, [\tilde{g}], {}^W\tilde{\nabla})$ be the usual smooth map.

Let

$$V := (\Gamma_{\alpha\beta}^\gamma - {}^W\Gamma_{\alpha\beta}^\gamma) g^{\alpha\beta} \frac{\partial}{\partial x^\gamma},$$

$$T(\tilde{X}, \tilde{Y}) := -\frac{1}{2} \tilde{\Theta}(\tilde{X}) \tilde{Y} - \frac{1}{2} \tilde{\Theta}(\tilde{Y}) \tilde{X} + \frac{1}{2} g(\tilde{X}, \tilde{Y}) \tilde{\Theta}^\sharp, \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(TN).$$

Then, we have

$$\begin{aligned} \tau(g, {}^W\nabla, {}^W\tilde{\nabla}) &= g^{\alpha\beta} (u_{\alpha\beta}^k + {}^W\tilde{\Gamma}_{ij}^k u_\alpha^i u_\beta^j - {}^W\Gamma_{\alpha\beta}^\sigma u_\sigma^k) \partial_{y^k} \\ &= [g^{\alpha\beta} (u_{\alpha\beta}^k + \Gamma_{ij}^k u_\alpha^i u_\beta^j + T_{ij}^k u_\alpha^i u_\beta^j - \Gamma_{\alpha\beta}^\sigma u_\sigma^k) + V^\sigma u_\sigma^k] \partial_{y^k} \\ &= \tau(g, \nabla, \tilde{\nabla}) + du(V) + \text{Tr}_g(du, du). \end{aligned}$$

Hence,

$$\tau(g, {}^W\nabla, {}^W\tilde{\nabla}) = 0 \quad \text{iff} \quad \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0.$$

Corollary 1 *Let $(M, [g], {}^W\nabla)$ be a compact Weyl manifold with nonempty boundary ∂M and $(N, [\tilde{g}], {}^W\tilde{\nabla})$ a complete Weyl manifold with sectional curvature bounded from above by $\kappa \geq 0$. Let $u_0 : M \rightarrow N$ be a continuous map with $u_0(M) \subset B_R(p)$, a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if*

$$\max |\nabla \tilde{\Theta}| + \max |\tilde{\Theta}| \leq C_0,$$

then there exists a unique Weyl harmonic map $u : M \rightarrow B_R(p) \subset N$ with $u = u_0$ on ∂M .

5.2 Affine harmonic maps (c.f. [9, 10])

Let $(M, g, \tilde{\nabla}), (N, h, \tilde{\nabla}')$ both be affine manifolds, where $\tilde{\nabla}$ is a global flat and torsion-free connection on M and $\tilde{\nabla}'$ is a torsion-free connection on N . Then, we have

$$\tau(g, \tilde{\nabla}, \tilde{\nabla}') = g^{\alpha\beta} (u_{\alpha\beta}^k + \tilde{\Gamma}_{ij}^k u_\alpha^i u_\beta^j) \partial_{y^k},$$

where $\tilde{\Gamma}_{ij}^k$ are the Christoffel symbols of $\tilde{\nabla}'$.

Regarding (M, g) and (N, h) as Riemannian manifolds, let $\Gamma_{\alpha\beta}^\gamma$ and Γ_{jk}^i be the Christoffel symbols of the Levi-Civita connections ∇ and ∇' of (M, g) and (N, h) , respectively. We then have the usual tension field

$$\tau(g, \nabla, \nabla') = g^{\alpha\beta} (u_{\alpha\beta}^k - \Gamma_{\alpha\beta}^\gamma u_\gamma^k + \Gamma_{ij}^k u_\alpha^i u_\beta^j) \partial_{y^k}.$$

Let

$$\begin{aligned} V &:= g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma \partial_{x^\gamma}, \\ T_{ij}^k &:= \tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k. \end{aligned}$$

Then, we have

$$\tau(g, \tilde{\nabla}, \tilde{\nabla}') = \tau(g, \nabla, \nabla') + du(V) + \text{Tr}_g(du, du).$$

Therefore,

$$\tau(g, \tilde{\nabla}, \tilde{\nabla}') = 0 \quad \text{iff} \quad \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0.$$

Corollary 2 *Let $(M, g, \tilde{\nabla})$ be a compact affine manifold with nonempty boundary ∂M and $(N, h, \tilde{\nabla}')$ a complete affine manifold with sectional curvature bounded from above by $\kappa \geq 0$, where $\tilde{\nabla}$ is a global flat and torsion-free connection on M and $\tilde{\nabla}'$ is a torsion-free connection on N . Let $u_0 : M \rightarrow N$ be a continuous map with $u_0(M) \subset B_R(p)$, a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$. Denote $T_{ij}^k := \tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k$, where $\tilde{\Gamma}_{ij}^k$ and Γ_{jk}^i stand for the Christoffel symbols of $\tilde{\nabla}'$ and ∇' , respectively. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if*

$$\max |\nabla T| + \max |T| \leq C_0,$$

then there exists a unique affine harmonic map $u : M \rightarrow B_R(p) \subset N$ with $u = u_0$ on ∂M .

5.3 Hermitian harmonic maps (c.f. [11, 15])

Let $(M^m, g, \tilde{\nabla})$, $(N^n, h, \tilde{\nabla}')$ are both Hermitian manifolds, where $\tilde{\nabla}$ and $\tilde{\nabla}'$ are holomorphic torsion-free connections on M and N , respectively. Direct calculation gives us

$$\tau(g, \tilde{\nabla}, \tilde{\nabla}') = g^{\alpha\bar{\beta}} \left(\frac{\partial^2 u^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma_{jk}^i \frac{\partial u^j}{\partial z^\alpha} \frac{\partial u^k}{\partial \bar{z}^\beta} \right) \frac{\partial}{\partial w^i} + \overline{g^{\alpha\bar{\beta}}} \left(\frac{\partial^2 u^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma_{jk}^i \frac{\partial u^j}{\partial z^\alpha} \frac{\partial u^k}{\partial \bar{z}^\beta} \right) \frac{\partial}{\partial \bar{w}^i},$$

where Γ_{jk}^i are the Christoffel symbols of $\tilde{\nabla}'$.

Let J be the almost complex structure, and $\{e_A\} = \{e_1, \dots, e_m, Je_1, \dots, Je_m\}$ a local basis of M . Let ∇, ∇' be the Levi-Civita connections on M and N , respectively, and Γ_{jk}^i the Christoffel symbols of ∇' . Set

$$V := \tilde{\nabla}_{e_A} e_A - \nabla_{e_A} e_A \quad \text{and} \quad T_{jk}^i := \Gamma_{jk}^i - \Gamma_{jk}^i,$$

then we have

$$\tau(g, \tilde{\nabla}, \tilde{\nabla}') = \tau(u) + du(V) + \text{Tr}_g T(du, du).$$

Namely,

$$\tau(g, \tilde{\nabla}, \tilde{\nabla}') = 0 \quad \text{iff} \quad \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0.$$

Corollary 3 *Let $(M^m, g, \tilde{\nabla})$ be a compact Hermitian manifold with nonempty boundary ∂M and $(N^n, h, \tilde{\nabla}')$ a complete Hermitian manifold with sectional curvature bounded from above by $\kappa \geq 0$, where $\tilde{\nabla}$ and $\tilde{\nabla}'$ are holomorphic torsion-free connections on M and N , respectively. Let $u_0 : M \rightarrow N$ be a continuous map with $u_0(M) \subset B_R(p)$, a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$. Denote $T_{ij}^k := \tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k$, where $\tilde{\Gamma}_{ij}^k$ and Γ_{jk}^i stand for the Christoffel symbols of $\tilde{\nabla}'$ and ∇' , respectively. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if*

$$\max |\nabla T| + \max |T| \leq C_0,$$

then there exists a unique Hermitian harmonic map $u : M \rightarrow B_R(p) \subset N$ with $u = u_0$ on ∂M .

5.4 Magnetic harmonic maps

We now consider a case that, in contrast to the previous ones, does not arise from a structure different from the Riemannian, but from an additional structure on a Riemannian manifold. Let (Σ^m, g) be an m -dimensional compact oriented Riemannian manifold with nonempty boundary, $(N, \tilde{g})$ a Riemannian manifold of dimension n . Let $u : (\Sigma^m, g) \rightarrow (N, \tilde{g})$ be a map and $Z \in \Gamma(\text{Hom}(\Lambda^m T\Sigma, TN)) \cong \Gamma(\Lambda^m T^*\Sigma \otimes TN)$.

Consider the following system:

$$\tau(u) + Z(du(e_1) \wedge \cdots \wedge du(e_m)) = 0, \quad (5.1)$$

where $\{e_1, \dots, e_m\}$ is a positively oriented local orthonormal frame of Σ^m . In string theory, it can be interpreted as the motion equation of an $(m-1)$ -brane under an extrinsic magnetic force (c.f. [13]). In [13], the author obtained the global existence of the heat flow in one-dimensional case.

Using a similar method as above, in the two-dimensional case, we can obtain the following

Theorem 6 *Let $u_1, u_2 \in C^0(\Sigma^2, N)$ be two magnetic harmonic maps into a geodesic ball $B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if*

$$\max |\nabla Z| + \max |Z| \leq C_0,$$

then the function $\Theta : \Sigma^2 \rightarrow \mathbb{R}$ defined by

$$\Theta := \frac{q_{\frac{\kappa}{4}}(\rho)}{(q_{\kappa}((1+\sigma)R) - q_{\kappa}(\rho_1))^{\frac{1}{2}} \cdot (q_{\kappa}((1+\sigma)R) - q_{\kappa}(\rho_2))^{\frac{1}{2}}} \quad (5.2)$$

satisfies the maximum principle, namely

$$\max_{\Sigma^2} \Theta \leq \max_{\partial \Sigma^2} \Theta.$$

Here $\rho := d(u_1, u_2)$, $\rho_i := d(p, u_i)$, $i = 1, 2$.

In particular, if $u_1 = u_2$ on the boundary $\partial \Sigma^2$, then $u_1 \equiv u_2$ on Σ^2 .

For the heat flow of magnetic harmonic maps, an analogous result holds. For $T > 0$, we set

$$\Sigma_T^2 := \Sigma^2 \times [0, T]$$

and denote the parabolic boundary of Σ_T^2 by

$$\partial_p \Sigma_T^2 := (\Sigma^2 \times \{0\}) \cup (\partial \Sigma^2 \times [0, T]).$$

For the heat flow of magnetic harmonic maps

$$\partial_t u = \tau(u) + Z(du(e_1) \wedge du(e_2)), \quad (5.3)$$

we have

Theorem 7 *Let $u_1, u_2 \in C^0(\Sigma^2, N)$ be two solutions of heat flow Eq. (5.3) for magnetic harmonic maps into a geodesic ball $B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if*

$$\max |\nabla Z| + \max |Z| \leq C_0,$$

then the function $\Theta : \Sigma_T^2 \rightarrow \mathbb{R}$ defined by (5.2) with Σ^2 replaced by Σ_T^2 satisfies the maximum principle:

$$\max_{\Sigma_T^2} \Theta \leq \max_{\partial_p \Sigma_T^2} \Theta.$$

In particular, if $u_1 = u_2$ on the boundary $\partial_p \Sigma_T^2$, then $u_1 \equiv u_2$ on Σ_T^2 .

As an application of the above maximum principle, we obtain the existence of magnetic harmonic maps into a geodesic ball.

Theorem 8 Let Σ^2 , N , Z , $B_R(p)$ be as in Theorem 7. Suppose $u_0 \in H^{2,q}(\Sigma^2, N)$ ($q > 2$) with $u_0(\Sigma^2) \subset B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ , σ , R and the geometry of N , such that if

$$\max |\nabla Z| + \max |Z| \leq C_0,$$

then the initial boundary value problem

$$\begin{cases} \partial_t u = \tau(u) + Z(du(e_1) \wedge du(e_2)), \\ u - u_0 \in H_0^{2,q}(\Sigma^2, N), \quad u(0) = u_0, \quad u(\Sigma^2 \times [0, \infty)) \subset B_R(p), \end{cases} \quad (5.4)$$

admits a unique global solution u which subconverges to a unique solution $u \in H^{2,q}(\Sigma^2, N)$ of the Dirichlet problem

$$\begin{cases} \tau(u) + Z(du(e_1) \wedge du(e_2)) = 0, \\ u - u_0 \in H_0^{2,q}(\Sigma^2, N), \end{cases} \quad (5.5)$$

such that $u(\Sigma^2) \subset B_R(p)$.

Remark 2 In local coordinates $\{x^\alpha\}$ on M and $\{y^j\}$ on N , respectively, the term $\text{Tr}_g T(du, du)$ can be written as $T_{jk}^i(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta}$, where $T := T_{jk}^i \frac{\partial}{\partial y^i} \otimes dy^j \otimes dy^k$. Correspondingly, the term $Z(du(e_1) \wedge du(e_2))$ in (5.3) could be written as $(Z_{jk}^i - Z_{kj}^i) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta}$. Using $Z_{jk}^i - Z_{kj}^i$ in place of T_{jk}^i in the proof of the results for VT -harmonic maps, we can conclude the above theorems for magnetic harmonic maps.

6 VT-harmonic maps from complete manifolds into geodesic balls

In this section, we shall establish the existence of VT -harmonic maps from complete non-compact manifolds into geodesic balls in complete Riemannian manifolds with sectional curvature bounded above by a positive constant.

Before proving the existence theorem, we first give the following Bochner formula:

Lemma 3 Let (M^m, g) and $(N^n, \tilde{g})$ be Riemannian manifolds. Let $\text{Ric}_V := \text{Ric} - \frac{1}{2} L_V g$, where Ric is the Ricci curvature of M and L_V is the Lie derivative. Suppose u is a VT -harmonic map from M to N , then

$$\begin{aligned}
\frac{1}{2} \Delta_V e(u) &= |\nabla du|^2 + \sum_{\alpha=1}^m \langle du(\text{Ric}_V(e_\alpha)), du(e_\alpha) \rangle \\
&\quad - \sum_{\alpha, \beta=1}^m R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)) \\
&\quad - \sum_{\alpha, \beta=1}^m \langle (\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)), du(e_\alpha) \rangle \\
&\quad - \sum_{\alpha, \beta=1}^m \langle 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)), du(e_\alpha) \rangle,
\end{aligned} \tag{6.1}$$

where $\{e_\alpha\}$ is a local orthonormal frame of M .

Proof By the proof of Proposition 1.3.5 in [18] (c.f. also in [1]), we have

$$\begin{aligned}
\frac{1}{2} \Delta e(u) &= |\nabla du|^2 + \sum_{\alpha=1}^m \langle \nabla_{e_\alpha} \tau(u), du(e_\alpha) \rangle + \sum_{\alpha=1}^m \langle du(\text{Ric}(e_\alpha)), du(e_\alpha) \rangle \\
&\quad - \sum_{\alpha, \beta=1}^m R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)).
\end{aligned}$$

Let $\{e_\alpha\}$ be a local orthonormal normal frame of M at the considered point. Since

$$\begin{aligned}
\sum_{\alpha=1}^m \langle \nabla_{e_\alpha} du(V), du(e_\alpha) \rangle &= \sum_{\alpha=1}^m \langle (\nabla_{e_\alpha} du)(V) + du(\nabla_{e_\alpha} V), du(e_\alpha) \rangle \\
&= \sum_{\alpha=1}^m \langle (\nabla_V du)(e_\alpha) + du(\nabla_{e_\alpha} V), du(e_\alpha) \rangle = \sum_{\alpha=1}^m \langle \nabla_V du(e_\alpha) + du(\nabla_{e_\alpha} V), du(e_\alpha) \rangle \\
&= \frac{1}{2} V|du|^2 + \sum_{\alpha, \beta=1}^m \langle \nabla_{e_\alpha} V, e_\beta \rangle \langle du(e_\beta), du(e_\alpha) \rangle \\
&= \frac{1}{2} V|du|^2 + \frac{1}{2} \sum_{\alpha, \beta=1}^m (L_V g)(e_\alpha, e_\beta) \langle du(e_\alpha), du(e_\beta) \rangle
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{e_\alpha} (\text{Tr}_g T(du, du)) &= \nabla_{e_\alpha} (T(du(e_\beta), du(e_\beta))) \\
&= (\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)) + 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{1}{2} \Delta e(u) &= |\nabla du|^2 - \frac{1}{2} V|du|^2 - \frac{1}{2} \sum_{\alpha, \beta=1}^m (L_V g)(e_\alpha, e_\beta) \langle du(e_\alpha), du(e_\beta) \rangle \\
&\quad - \sum_{\alpha, \beta=1}^m \langle (\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)) + 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)), du(e_\alpha) \rangle \\
&\quad + \sum_{\alpha=1}^m \langle du(\text{Ric}(e_\alpha)), du(e_\alpha) \rangle - \sum_{\alpha, \beta=1}^m R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)),
\end{aligned}$$

which implies that (6.1) holds. $\square$

Using the above Bochner formula and the estimate of $\Delta_V r$ in [6] (here r denotes the distance function on M), we establish the gradient estimate for VT -harmonic maps.

Theorem 9 *Let (M^m, g) be a complete noncompact Riemannian manifold with*

$$\text{Ric}_V := \text{Ric} - \frac{1}{2}L_V g \geq -A,$$

where $A \geq 0$ is a constant, Ric is the Ricci curvature of M and L_V is the Lie derivative. Let $(N^n, \tilde{g})$ be a complete Riemannian manifold with sectional curvature bounded from above by a positive constant κ . Let $u : M \rightarrow N$ be a VT -harmonic map such that $u(M) \subset B_{\tilde{R}}(p)$, where $B_{\tilde{R}}(p)$ is a regular ball in N , i.e., disjoint from the cut locus of p and $\tilde{R} < \frac{\pi}{2\sqrt{\kappa}}$. Suppose $\|V\|_{L^\infty} < +\infty$, $\|T\|_{L^\infty} < +\infty$, $\|\nabla T\|_{L^\infty} < +\infty$ and

$$\left(1 + (m+1)^2 - \frac{1}{(m+1)^2}\right) \|T\|_{L^\infty}^2 + \frac{\sqrt{\kappa}}{\cos(\sqrt{\kappa}\tilde{R})} \|T\|_{L^\infty} < \frac{\kappa}{\min\{m, n\}}. \quad (6.2)$$

Then, we have

$$|\nabla u| \leq C_6(\sqrt{A} + 1),$$

where $C_6 > 0$ is a constant depending only on $m, n, \kappa, \tilde{R}, V, T$.

Proof Let r, ρ be the respective distance functions on M and N from some fixed points $\tilde{p} \in M, p \in N$. Let $B_a(\tilde{p})$ be a geodesic ball of radius a around $\tilde{p}$. Define $\varphi := \cos(\sqrt{\kappa}\rho)$. Then, the Hessian comparison theorem implies

$$\text{Hess}^N(\varphi) \leq -\kappa(\cos(\sqrt{\kappa}\rho))\tilde{g}. \quad (6.3)$$

Define $f : B_a(\tilde{p}) \rightarrow \mathbb{R}$ by

$$f := (a^2 - r^2) \frac{|\nabla u|}{\varphi \circ u}.$$

Denote $\psi := \frac{|\nabla u|}{\varphi \circ u}$. Clearly, f achieves its maximum at some interior point of $B_a(\tilde{p})$, say q . WLOG, we assume that $\nabla u(q) \neq 0$. Then, from

$$\begin{aligned} \nabla f(q) &= 0, \\ \Delta_V f(q) &\leq 0, \end{aligned}$$

we obtain at q :

$$\frac{\nabla r^2}{a^2 - r^2} = \frac{\nabla \psi}{\psi}, \quad (6.4)$$

$$\frac{\Delta_V \psi}{\psi} - \frac{\Delta_V r^2}{a^2 - r^2} - \frac{2\langle \nabla r^2, \nabla \psi \rangle}{(a^2 - r^2)\psi} \leq 0. \quad (6.5)$$

It follows from the above two inequalities that

$$\frac{\Delta_V \psi}{\psi} - \frac{\Delta_V r^2}{a^2 - r^2} - \frac{2|\nabla r^2|^2}{(a^2 - r^2)^2} \leq 0. \quad (6.6)$$

By formula (2.4) in [6] (see also [16]), we have

$$\Delta_V r^2 = 2r\Delta_V r + 2|\nabla r|^2 \leq 2r(A(r - r_0) + \tilde{C}_0) + 2, \quad (6.7)$$

where $r_0 > 0$ is a sufficiently small constant and $\tilde{C}_0 := \max_{\partial B_{r_0}(\bar{p})} \Delta_V r$.

Let $\{e_\alpha\}$ be a local orthonormal frame field of M and s the rank of u at the point. We shall compute in normal coordinates at the considered point of N . By Newton's inequality, we get

$$\begin{aligned} \sum_{\alpha, \beta} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)) &= \sum_{\alpha \neq \beta} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)) \\ &\leq 2\kappa \sum_{1 \leq \alpha < \beta \leq s} \left(\sum_i (u_\alpha^i)^2 \right) \left(\sum_j (u_\beta^j)^2 \right) \leq 2\kappa \cdot \frac{s(s-1)}{2} \cdot \frac{1}{s^2} \left(\sum_{\alpha, i} (u_\alpha^i)^2 \right)^2 \\ &= \frac{s-1}{s} \kappa |du|^4 \leq \frac{s_0-1}{s_0} \kappa |du|^4, \end{aligned}$$

where we have used the fact that $s_0 := \min\{m, n\} \geq s$ in the third " $\leq$ ".

The Cauchy–Schwarz inequality gives us

$$\begin{aligned} |(\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta), du(e_\alpha))| &\leq \|\nabla T\|_{L^\infty} |du|^2 \cdot |du| \\ &\leq \varepsilon_2 e(u) + \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 e(u)^2, \\ |2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta), du(e_\alpha))| &\leq 2\|T\|_{L^\infty} |\nabla du| |du| \cdot |du| \leq \varepsilon_3 |\nabla du|^2 \\ &\quad + \frac{1}{\varepsilon_3} \|T\|_{L^\infty}^2 e(u)^2. \end{aligned}$$

The formula (3.12) in [2] (see also [3]) implies that for any $\epsilon > 0$

$$|\nabla du|^2 \geq \frac{1-\epsilon}{m-1} |\tau(u)|^2 + \left(\frac{m}{m-1} - \frac{1}{(m-1)\epsilon} \right) |\nabla \sqrt{e(u)}|^2.$$

Choosing $\epsilon = m$, then we have

$$|\nabla du|^2 \geq -|\tau(u)|^2 + \left(1 + \frac{1}{m} \right) |\nabla \sqrt{e(u)}|^2.$$

By the VT -harmonic map equation (1.1), it is easy to see that

$$|\tau(u)|^2 \leq e(u) \|V\|_{L^\infty}^2 + e(u)^2 \|T\|_{L^\infty}^2.$$

Hence, from the Bochner formula (6.1), we obtain

$$\begin{aligned} \frac{1}{2} \Delta_V e(u) &\geq (-(1-\varepsilon_3) \|V\|_{L^\infty}^2 - A - \varepsilon_2) e(u) + (1-\varepsilon_3) \left(1 + \frac{1}{m} \right) |\nabla \sqrt{e(u)}|^2 \\ &\quad + \left(-(1-\varepsilon_3) \|T\|_{L^\infty}^2 - \kappa \left(1 - \frac{1}{s_0} \right) - \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - \frac{1}{\varepsilon_3} \|T\|_{L^\infty}^2 \right) e(u)^2. \end{aligned}$$

Since

$$\frac{1}{2} \Delta_V e(u) = \frac{1}{2} \Delta_V |\nabla u|^2 = |\nabla |\nabla u||^2 + |\nabla u| \Delta_V |\nabla u|,$$

therefore

$$\begin{aligned} \Delta_V |\nabla u| &\geq \left[(1-\varepsilon_3) \left(1 + \frac{1}{m} \right) - 1 \right] \frac{|\nabla |\nabla u||^2}{|\nabla u|} + (-(1-\varepsilon_3) \|V\|_{L^\infty}^2 - A - \varepsilon_2) |\nabla u| \\ &\quad + \left[-(1-\varepsilon_3) \|T\|_{L^\infty}^2 - \kappa \left(1 - \frac{1}{s_0} \right) - \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - \frac{1}{\varepsilon_3} \|T\|_{L^\infty}^2 \right] |\nabla u|^3. \end{aligned}$$

Choose $\varepsilon_3 = \frac{1}{(m+1)^2}$, then

$$\begin{aligned} \Delta_V |\nabla u| &\geq \frac{1}{m+1} \frac{|\nabla |\nabla u||^2}{|\nabla u|} + \left[- \left(1 - \frac{1}{(m+1)^2} \right) \|V\|_{L^\infty}^2 - A - \varepsilon_2 \right] |\nabla u| \\ &\quad + \left[- \left(1 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 - \kappa \left(1 - \frac{1}{s_0} \right) \right. \\ &\quad \left. - \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - (m+1)^2 \|T\|_{L^\infty}^2 \right] |\nabla u|^3. \end{aligned} \quad (6.8)$$

For simplicity in the following computation, we denote

$$\begin{aligned} C_1 &:= \frac{1}{m+1}, \quad C_2 := - \left(1 - \frac{1}{(m+1)^2} \right) \|V\|_{L^\infty}^2 - A - \varepsilon_2, \\ C_3 &:= - \left(1 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 - \kappa \left(1 - \frac{1}{s_0} \right) - \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - (m+1)^2 \|T\|_{L^\infty}^2. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} \nabla \psi &= \frac{\nabla |\nabla u|}{\varphi \circ u} - \frac{|\nabla u| \nabla(\varphi \circ u)}{(\varphi \circ u)^2}, \\ \Delta_V \psi &= \frac{\Delta_V |\nabla u|}{\varphi \circ u} - \frac{|\nabla u| \Delta_V(\varphi \circ u)}{(\varphi \circ u)^2} - \frac{2}{\varphi \circ u} \langle \nabla \psi, \nabla(\varphi \circ u) \rangle \\ &\geq C_1 \frac{|\nabla |\nabla u||^2}{(\varphi \circ u) |\nabla u|} + C_2 \frac{|\nabla u|}{\varphi \circ u} + C_3 \frac{|\nabla u|^3}{\varphi \circ u} - \frac{\psi \Delta_V(\varphi \circ u)}{\varphi \circ u} - \frac{2 \langle \nabla \psi, \nabla(\varphi \circ u) \rangle}{\varphi \circ u}. \end{aligned}$$

The Cauchy–Schwarz implies that

$$\begin{aligned} & - \frac{2 \langle \nabla \psi, \nabla(\varphi \circ u) \rangle}{\varphi \circ u} \\ &= -(2 - 2C_1) \frac{\langle \nabla \psi, \nabla(\varphi \circ u) \rangle}{\varphi \circ u} - 2C_1 \frac{\langle \nabla \psi, \nabla(\varphi \circ u) \rangle}{\varphi \circ u} \\ &= -(2 - 2C_1) \frac{\langle \nabla \psi, \nabla(\varphi \circ u) \rangle}{\varphi \circ u} - 2C_1 \frac{\langle \nabla |\nabla u|, \nabla(\varphi \circ u) \rangle}{(\varphi \circ u)^2} + 2C_1 \frac{|\nabla(\varphi \circ u)|^2 |\nabla u|}{(\varphi \circ u)^3} \\ &\geq -(2 - 2C_1) \frac{\langle \nabla \psi, \nabla(\varphi \circ u) \rangle}{\varphi \circ u} - C_1 \frac{|\nabla |\nabla u||^2}{(\varphi \circ u) |\nabla u|} + C_1 \frac{|\nabla(\varphi \circ u)|^2 |\nabla u|}{(\varphi \circ u)^3}. \end{aligned}$$

From the above two inequalities and (6.4), we get

$$\frac{\Delta_V \psi}{\psi} \geq C_2 + C_3 |\nabla u|^2 - \frac{\Delta_V(\varphi \circ u)}{\varphi \circ u} + C_1 \frac{|\nabla(\varphi \circ u)|^2}{(\varphi \circ u)^2} - (2 - 2C_1) \frac{\langle \nabla r^2, \nabla(\varphi \circ u) \rangle}{(a^2 - r^2)(\varphi \circ u)}. \quad (6.9)$$

Since

$$\left| \frac{\langle \nabla r^2, \nabla(\varphi \circ u) \rangle}{(a^2 - r^2)(\varphi \circ u)} \right| \leq \frac{2r |\nabla(\varphi \circ u)| \cdot |\nabla r|}{(a^2 - r^2)(\varphi \circ u)} \leq \frac{2r |\nabla(\varphi \circ u)|}{(a^2 - r^2)(\varphi \circ u)} \leq \frac{2\sqrt{\kappa} r |\nabla u|}{(a^2 - r^2)(\varphi \circ u)} \quad (6.10)$$

and from (6.3),

$$\begin{aligned}\Delta_V(\varphi \circ u) &= g^{\alpha\beta} \nabla^2 \varphi(\partial_\alpha u, \partial_\beta u) + \langle (\nabla \varphi) \circ u, \tau(u) + du(V) \rangle \\ &= g^{\alpha\beta} u_{x^\alpha}^j u_{x^\beta}^k \text{Hess}^N(\varphi) \left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k} \right) + \langle (\nabla \varphi) \circ u, -\text{Tr}_g T(du, du) \rangle \quad (6.11) \\ &\leq (-\kappa \cos(\sqrt{\kappa} \rho) + \sqrt{\kappa} \|T\|_{L^\infty}) |\nabla u|^2.\end{aligned}$$

Therefore, from (6.6), (6.7), (6.9)–(6.11), we obtain

$$\begin{aligned}C_2 + \left(C_3 + \kappa - \frac{\sqrt{\kappa} \|T\|_{L^\infty}}{\cos(\sqrt{\kappa} \rho \circ u)} \right) |\nabla u|^2 - \frac{4(1 - C_1)\sqrt{\kappa} r}{(a^2 - r^2)(\varphi \circ u)} |\nabla u| \\ - \frac{2Ar^2 - 2Ar_0r + 2\tilde{C}_0r + 2}{a^2 - r^2} - \frac{8r^2}{(a^2 - r^2)^2} \leq 0.\end{aligned} \quad (6.12)$$

Since the condition (6.2) tells us

$$\left(1 + (m+1)^2 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 + \frac{\sqrt{\kappa}}{\cos(\sqrt{\kappa} \tilde{R})} \|T\|_{L^\infty} < \frac{\kappa}{s_0},$$

there exists a constant $\varepsilon_0 > 0$, such that

$$\left(1 + (m+1)^2 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 + \frac{\sqrt{\kappa}}{\cos(\sqrt{\kappa} \tilde{R})} \|T\|_{L^\infty} + \varepsilon_0 < \frac{\kappa}{s_0}.$$

Choosing $\varepsilon_2 = \frac{\|\nabla T\|_{L^\infty}^2 + 1}{4\varepsilon_0}$, then we have

$$\begin{aligned}C_3 + \kappa - \frac{\sqrt{\kappa} \|T\|_{L^\infty}}{\cos(\sqrt{\kappa} \rho \circ u)} &= - \left(1 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 - \kappa \left(1 - \frac{1}{s_0} \right) \\ &\quad - \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - (m+1)^2 \|T\|_{L^\infty}^2 + \kappa - \frac{\sqrt{\kappa} \|T\|_{L^\infty}}{\cos(\sqrt{\kappa} \rho \circ u)} \\ &\geq \frac{\kappa}{s_0} - \left(1 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 - \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - (m+1)^2 \|T\|_{L^\infty}^2 - \frac{\sqrt{\kappa} \|T\|_{L^\infty}}{\cos(\sqrt{\kappa} \tilde{R})} \\ &= \frac{\kappa}{s_0} - \left[\left(1 + (m+1)^2 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 + \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 + \frac{\sqrt{\kappa} \|T\|_{L^\infty}}{\cos(\sqrt{\kappa} \tilde{R})} \right] \\ &> \frac{\kappa}{s_0} - \left[\left(1 + (m+1)^2 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 + \varepsilon_0 + \frac{\sqrt{\kappa} \|T\|_{L^\infty}}{\cos(\sqrt{\kappa} \tilde{R})} \right] =: C_4 > 0.\end{aligned}$$

Therefore, it follows from (6.12) that

$$\begin{aligned}C_4 |\nabla u|^2 - \frac{4(1 - C_1)\sqrt{\kappa} r}{(a^2 - r^2)(\varphi \circ u)} |\nabla u| - \left(\frac{2Ar^2 - 2Ar_0r + 2\tilde{C}_0r + 2}{a^2 - r^2} + \frac{8r^2}{(a^2 - r^2)^2} - C_2 \right) \\ \leq 0.\end{aligned}$$

Note the elementary fact that if $ax^2 - bx - c \leq 0$ with a, b, c all positive, then

$$x \leq \frac{b}{a} + \sqrt{\frac{c}{a}}.$$

Hence, at the point q ,

$$|\nabla u| \leq \frac{4(1 - C_1)\sqrt{\kappa} r}{C_4(a^2 - r^2) \cos(\sqrt{\kappa} \tilde{R})} + \sqrt{\frac{1}{C_4} \left(\frac{2Ar^2 - 2Ar_0r + 2\tilde{C}_0r + 2}{a^2 - r^2} + \frac{8r^2}{(a^2 - r^2)^2} - C_2 \right)}.$$

From this, we can derive the upper bound of f , and it is easy to conclude that at every point of $B_{\frac{a}{2}}(\tilde{p})$, we have

$$\begin{aligned} |\nabla u| &\leq C_5 \left(\sqrt{A + \frac{\|\nabla T\|_{L^\infty}^2 + 1}{4\varepsilon_0}} + \left(1 - \frac{1}{(m+1)^2}\right) \|V\|_{L^\infty} + \frac{1}{a} + \frac{1}{\sqrt{a}} \right) \\ &\leq C_6 \left(\sqrt{A} + 1 + \frac{1}{a} + \frac{1}{\sqrt{a}} \right), \end{aligned} \quad (6.13)$$

where $C_6 > 0$ is a constant depending only on $m, n, \kappa, \tilde{R}, V, T$.

For any fixed $x \in M$, letting $a \rightarrow \infty$ in (6.13), we obtain $|\nabla u| \leq C_6(\sqrt{A} + 1)$. $\square$

Proof of Theorem 4 We first choose a constant C'_0 depending only on κ, σ, R and the geometry of N , such that if

$$\max |\nabla T| + \max |T| \leq C'_0,$$

then both the condition (4.3) in Theorem 5 and the condition (6.2) in Theorem 9 hold. Let $\{\Omega_i\}$ be a compact exhaustion of M . By Theorem 5, we have a sequence of maps $\{u_i\}$ which solve the Dirichlet problem

$$\begin{cases} \tau(u_i) + du_i(V) + \text{Tr}_g T(du_i, du_i) = 0, \\ u_i|_{\partial\Omega_i} = u_0|_{\partial\Omega_i}, \\ u_i \text{ homotopic to } u_0 \text{ rel. } \partial\Omega_i, \end{cases}$$

where $u_i \in C^\infty(\Omega_i, N) \cap C^0(\overline{\Omega_i}, N)$ such that $u_i(\Omega_i) \subset B_R(p)$.

For any compact set $K \subset M$, there exists an integer $i_0 > 0$, such that $K \subset \Omega_i$ for $i > i_0$. Then, by (6.8),

$$\begin{aligned} \Delta_V |\nabla u_i| &\geq \frac{1}{m+1} \frac{|\nabla |\nabla u_i||^2}{|\nabla u_i|} + \left[- \left(1 - \frac{1}{(m+1)^2}\right) \|V\|_{L^\infty}^2 - \tilde{A} - \frac{\|\nabla T\|_{L^\infty}^2 + 1}{4\varepsilon_0} \right] |\nabla u_i| \\ &\quad + \left[- \left(1 + (m+1)^2 - \frac{1}{(m+1)^2}\right) \|T\|_{L^\infty}^2 - \kappa \left(1 - \frac{1}{s_0}\right) - \varepsilon_0 \right] |\nabla u_i|^3, \end{aligned} \quad (6.14)$$

where $\tilde{A}$ is a positive constant depending only on the bounds for Ricci curvature of K and $\|V\|_{C^1(K)}$.

Since K is compact, there exist finitely many such geodesic balls $\{B_{a_j}(\tilde{p}_j)\}_{j=1}^{k_0} \subset M$, such that $\bigcup_{j=1}^{k_0} B_{a_j}(\tilde{p}_j) \supset K$. Hence, for any $q \in K$, there is a geodesic ball, say $B_{a_{j_0}}(\tilde{p}_{j_0})$ ($1 \leq j_0 \leq k_0$), containing q . Then, by Theorem 9, we can conclude that

$$|\nabla u_i|(q) \leq C_{7j_0}(\sqrt{\tilde{A}} + 1).$$

Hence,

$$\sup_K |\nabla u_i| \leq \max_{1 \leq j_0 \leq k_0} \left\{ C_{7j_0}(\sqrt{\tilde{A}} + 1) \right\} =: C_8,$$

where C_8 is a positive constant independent of i . Then, by the standard elliptic theory, u_i subconverges to a VT -harmonic map $u \in C^\infty(M, N)$ with $u(M) \subset B_R(p)$ and u is homotopic to u_0 . $\square$

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