Ann Glob Anal Geom (2018) 54:87–122
https://doi.org/10.1007/s10455-018-9594-4

Variations of the total mixed scalar curvature of a distribution

Vladimir Rovenski[1] · **Tomasz Zawadzki**[2]

Received: 6 May 2017 / Accepted: 3 January 2018 / Published online: 18 January 2018

Abstract We examine the total mixed scalar curvature of a smooth manifold endowed with a distribution as a functional of a pseudo-Riemannian metric. We develop variational formulas for quantities of extrinsic geometry of the distribution and use this key and technical result to find the critical points of this action. Together with the arbitrary variations of the metric, we consider also variations that preserve the volume of the manifold or partially preserve the metric (e.g., on the distribution). For each of those cases, we obtain the Euler–Lagrange equation and its several solutions. Examples of critical metrics that we find are related to various fields of geometry such as contact and 3-Sasakian manifolds, geodesic Riemannian flows, codimension-one foliations, and distributions of interesting geometric properties (e.g., totally umbilical and minimal).

Keywords Pseudo-Riemannian metric · Distribution · Foliation · Variation · Mixed scalar curvature · Contact structure

Mathematics Subject Classification 53C12 · 53C15

1 Introduction

Distributions on manifolds appear in various situations—e.g., as fields of tangent planes of foliations or kernels of differential forms. When the metric of a pseudo-Riemannian manifold is non-degenerate on a distribution, it defines a pseudo-Riemannian almost-product

✉ Tomasz Zawadzki
zawadzki@math.uni.lodz.pl

Vladimir Rovenski
vrovenski@univ.haifa.ac.il

1 Department of Mathematics, University of Haifa, Mount Carmel, 31905 Haifa, Israel

2 Faculty of Mathematics and Computer Science, University of Lodz, ul. Banacha 22, 90-238 Lodz, Poland

structure—i.e., a pair of orthogonal, complementary distributions that span the tangent bundle [8]. The mixed scalar curvature is one of the simplest curvature invariants of a pseudo-Riemannian almost-product structure. It is defined as a sum of sectional curvatures of planes that non-trivially intersect with both of the distributions. (We give the exact definition and formula (1) later.) Its investigation led to multiple results regarding the existence of vector fields [5], foliations [18], and submersions [19] of interesting geometric properties.

The fundamental and open question (similar to the question about existence of Einstein metrics on a manifold) is the following: *What are the best metrics on a smooth manifold endowed with a distribution*? On a manifold with a distribution, we can define the total mixed scalar curvature as a functional on the space of all pseudo-Riemannian metrics that are non-degenerate on the distribution. Every such metric yields a pseudo-Riemannian almost-product structure and hence the mixed scalar curvature of our distribution (and of its metric-dependent orthogonal complement). Since we deal also with non-compact manifolds, we assume that the total mixed scalar curvature is in fact an integral of the mixed scalar curvature over a sufficiently large, relatively compact set. When viewed as a functional of the metric, the total mixed scalar curvature may be considered an analogue of the Einstein–Hilbert action [1], and the best metrics (of the above question) are proposed to be among critical metrics of the action.

The goal of this paper is to examine metrics critical for the total mixed scalar curvature with respect to different kinds of variations of metric. Apart from varying among all the metrics that are non-degenerate on the distribution, we shall also restrict to the case when the varying metric remains fixed on the distribution, and the "complementary" case when metric varies only on the distribution—preserving its orthogonal complement and the metric on it. This approach applies to finding an optimal extension of a metric that is defined only on a distribution—which is a problem of the relationship between sub-Riemannian and Riemannian geometry. Moreover, in analogy to the Einstein–Hilbert action, all variations will be considered in two versions: with and without the additional requirement that the metrics preserve the volume of the manifold [3]. The Euler–Lagrange equations that we obtain for those various cases are actually similar enough to be viewed as slight modifications of one general equation, and hence can be presented in a concise form.

The first part of the paper starts with all necessary definitions, continues to the development of variation formulas for several geometric quantities, and culminates in the formulation of the Euler–Lagrange equation for the total mixed scalar curvature. The equation we obtain is difficult to solve in full generality, although it can be related to Ricci-type curvatures previously described in the literature [12]. It can be decomposed into three independent parts, two of them being symmetric with respect to interchanging the given distribution and its orthogonal complement. Those two equations are also the same as the ones obtained for more restrictive, so-called adapted variations, considered in [2,14]. It is worth noting that the variation formulas for geometric quantities, that we obtain along the way to the Euler–Lagrange equation, can be of use also for many other functionals.

The second part of the paper is dedicated to examples of metrics critical for the total mixed scalar curvature. In Sect. 3.1, we consider the case when the fixed distribution is one-dimensional, i.e., tangent to the flowlines of a unit vector field. (In four-dimensional, general relativity setting, this case corresponds to the one examined in [1].) We rephrase the Euler–Lagrange equation and examine it in the case of geodesic Riemannian flows, comparing the results for different types of variations. In Sect. 3.2, we consider situation "dual" to the one from Sect. 3.1—fixing a distribution tangent to a codimension-one foliation. Then, with the assumption of a special coordinate system, the Euler–Lagrange equation can be in some cases explicitly solved. This setting allows us to find more critical metrics in the case of general variations preserving the volume of the manifold. In Sect. 3.3, we consider the one-dimensional distribution spanned by the Reeb field on a contact manifold, which allows us to give an interpretation of some geometric quantities that appear in the Euler–Lagrange equation. Using the results obtained earlier for geodesic Riemannian flows, we show that the metrics of K-contact structures are critical with respect to all variations that fix the volume and partially preserve the metric (either on the distribution or everywhere else except it), thus generalizing a theorem from [4]. As a different application of the variational formulas obtained earlier, we also examine a measure of non-integrability of the orthogonal complement of the Reeb field, showing that contact metric structures are critical for this functional. The results we obtain for contact manifolds are then generalized to the setting of 3-Sasakian manifolds. Finally, in Sect. 3.4, we consider variation of the total mixed scalar curvature of a manifold endowed with a non-integrable distribution. Because of the complexity of the arising Euler–Lagrange equation, we look only for those critical metrics for which the orthogonal complement of the distribution is integrable. We show that K-contact and 3-Sasakian metrics (when the orthogonal complement of the distribution is integrable and has dimension one or three) are critical with respect to all variations that fix the volume and partially preserve the metric also in this setting. In case of codimension-one distribution, one of the variations that we consider has a particularly interesting geometric interpretation; we give an additional example of metric critical with respect to it.

2 Main results

In this part, we give necessary definitions, develop variation formulas for geometric quantities (that is the most technical and key result), and formulate the Euler–Lagrange equation for the total mixed scalar curvature of a manifold endowed with a distribution.

2.1 Preliminaries

This section recalls definitions of some functions and tensors, used also in [2,14] and introduces several new notions related to geometry of pseudo-Riemannian almost-product manifolds.

Let $\mathrm{Sym}^2(M)$ be the space of all symmetric (0, 2)-tensors tangent to a smooth manifold M. A *pseudo-Riemannian metric* of index q on M is an element $g \in \mathrm{Sym}^2(M)$ such that each g_x $(x \in M)$ is a non-degenerate bilinear form of index q on the tangent space $T_x M$. For $q = 0$ (i.e., g_x is positive definite), g is a Riemannian metric, and for $q = 1$ it is called a Lorentz metric. Let $\mathrm{Riem}(M) \subset \mathrm{Sym}^2(M)$ be the subspace of pseudo-Riemannian metrics of given signature.

Let $R(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]}$ be the curvature tensor of the Levi–Civita connection ∇ of g. At a point $x \in M$, a two-dimensional linear subspace $X \wedge Y$ (called a plane section) of $T_x M$ is *non-degenerate* if $W(X, Y) := g(X, X)\, g(Y, Y) - g(X, Y)\, g(X, Y) \neq 0$. For such section at x, the sectional curvature is the number $K(X \wedge Y) = g(R(X, Y)X, Y)/W(X, Y)$.

A subbundle $\widetilde{\mathcal{D}} \subset TM$ (called a distribution) is *non-degenerate*, if g_x is non-degenerate on $\widetilde{\mathcal{D}}_x \subset T_x M$ for every $x \in M$; in this case, the orthogonal complement of $\widetilde{\mathcal{D}}$, denoted by $\mathcal{D}$, is also non-degenerate [10], and we have $\widetilde{\mathcal{D}}_x \cap \mathcal{D}_x = 0$, $\widetilde{\mathcal{D}}_x \oplus \mathcal{D}_x = T_x M$ for all $x \in M$. A connected manifold M^{n+p} with a pseudo-Riemannian metric g and a pair of complementary orthogonal non-degenerate distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ of ranks $\dim \widetilde{\mathcal{D}}_x = n$ and $\dim \mathcal{D}_x = p$ for every $x \in M$ is called a *pseudo-Riemannian almost-product structure* on M, [8]. Such $(M, \widetilde{\mathcal{D}}, \mathcal{D}, g)$ is also sometimes called a *pseudo-Riemannian almost-product* *manifold*. Let $\mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D}) \subset \mathrm{Riem}(M)$ be the subspace of metrics making $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ orthogonal and non-degenerate.

Let $\mathfrak{X}_M$ be the module over $C^\infty(M)$ of all vector fields on M, and let $\mathfrak{X}_{\mathcal{D}}$ and $\mathfrak{X}_{\widetilde{\mathcal{D}}}$ be the modules of sections of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, respectively. The following convention is adopted for the range of indices:

$$a, b, c \ldots \in \{1 \ldots n\}, \quad i, j, k \ldots \in \{1 \ldots p\}.$$

The "musical" isomorphisms $\sharp$ and $\flat$ will be used for rank one and symmetric rank 2 tensors. For example, if $\omega \in T^1_0 M$ is a 1-form and $X, Y \in \mathfrak{X}_M$, then $\omega(Y) = g(\omega^\sharp, Y)$ and $X^\flat(Y) = g(X, Y)$. For (0, 2)-tensors A and B we have $\langle A, B\rangle = \mathrm{Tr}\,_g(A^\sharp B^\sharp) = \langle A^\sharp, B^\sharp\rangle$.

The sectional curvature $K(X \wedge Y)$ is called *mixed* if $X \in \widetilde{\mathcal{D}}$ and $Y \in \mathcal{D}$. Let $\{E_a, \mathcal{E}_i\}$ be a local orthonormal frame *adapted* to $(\widetilde{\mathcal{D}}, \mathcal{D})$, i.e.,

$$E_a \in \widetilde{\mathcal{D}}, \quad \mathcal{E}_i \in \mathcal{D},$$

and let $\epsilon_i = g(\mathcal{E}_i, \mathcal{E}_i)$, $\epsilon_a = g(E_a, E_a)$. We have $|\epsilon_i| = |\epsilon_a| = 1$ and $W(E_a, \mathcal{E}_i) = \epsilon_a\epsilon_i \neq 0$. The function on M,

$$\mathrm{S}_{\mathrm{mix}} = \sum_{a,i} K(E_a \wedge \mathcal{E}_i) = \sum_{a,i} \epsilon_a\epsilon_i\, g(R(E_a, \mathcal{E}_i)E_a,\, \mathcal{E}_i) \tag{1}$$

is called the *mixed scalar curvature*, see [18], and does not depend on the choice of the adapted orthonormal frame. If a distribution is spanned by a unit vector field N, i.e., $g(N, N) = \epsilon_N \in \{-1, 1\}$, then $\mathrm{S}_{\mathrm{mix}} = \epsilon_N \,\mathrm{Ric}_{N,N}$, where $\mathrm{Ric}_{N,N}$ is the Ricci curvature in the N-direction.

To compute $\mathrm{S}_{\mathrm{mix}}$ on (M, g) we only need to fix one of the distributions, say $\widetilde{\mathcal{D}}$, then we obtain the second distribution as its g-orthogonal complement and the function (1) is well defined. Given a pair $(M, \widetilde{\mathcal{D}})$ of a manifold and a distribution, we shall study pseudo-Riemannian structures non-degenerate on $\widetilde{\mathcal{D}}$ and critical for the functional

$$J_{\mathrm{mix}, \widetilde{\mathcal{D}}, \Omega} :\; g \mapsto \int_\Omega \mathrm{S}_{\mathrm{mix}}(g)\, \mathrm{d}\,\mathrm{vol}_g, \tag{2}$$

where Ω in (2) is a relatively compact domain of M (and $\Omega = M$ when M is closed), containing supports of variations of the metric. The Euler–Lagrange equation for (2), that we shall obtain later, is expressed in terms of extrinsic geometry of the distribution $\widetilde{\mathcal{D}}$ and its orthogonal complement $\mathcal{D}$. In order to understand it, we shall define several notions on a pseudo-Riemannian almost-product manifold $(M, \widetilde{\mathcal{D}}, \mathcal{D}, g)$.

For every $X \in \mathfrak{X}_M$ we have $X = \widetilde{X} + X^\perp$, where $\widetilde{X} \equiv X^\top$ is the $\widetilde{\mathcal{D}}$-component of X (respectively, $X^\perp$ is the $\mathcal{D}$-component of X) with respect to g. We define $g^\perp$ and $g^\top$ by

$$g^\perp(X, Y) = g(X^\perp, Y^\perp), \quad g^\top(X, Y) = g(X^\top, Y^\top), \quad (X, Y \in \mathfrak{X}_M).$$

The symmetric (0, 2)-tensor $r_{\mathcal{D}}$, given by

$$r_{\mathcal{D}}(X, Y) = \sum_a \epsilon_a\, g(R(E_a,\, X^\perp)E_a,\, Y^\perp), \quad (X, Y \in \mathfrak{X}_M),$$

is referred to as the *partial Ricci tensor* adapted for $\mathcal{D}$; see [2,14]. In particular, by (1),

$$\mathrm{Tr}\,_g r_{\mathcal{D}} = \mathrm{S}_{\mathrm{mix}}(g). \tag{3}$$

Note that the partial Ricci curvature $r_{\mathcal{D}}(X, X)$ in the direction of a unit vector $X \in \mathcal{D}$ is the sum of sectional curvatures over all mixed planes containing X.

Let $T, h : \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}} \to \mathcal{D}$ and $\tilde{T}, \tilde{h} : \mathcal{D} \times \mathcal{D} \to \widetilde{\mathcal{D}}$ be the integrability tensors and the second fundamental forms of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$, respectively.

$$T(X, Y) = (1/2)\,[X,\ Y]^{\perp},\quad h(X, Y) = (1/2)\,(\nabla_X Y + \nabla_Y X)^{\perp}\quad (X, Y \in \mathfrak{X}_{\widetilde{\mathcal{D}}}),$$
$$\tilde{T}(X, Y) = (1/2)\,[X,\ Y]^{\top},\quad \tilde{h}(X, Y) = (1/2)\,(\nabla_X Y + \nabla_Y X)^{\top}\quad (X, Y \in \mathfrak{X}_{\mathcal{D}}).$$

Using an orthonormal adapted frame, one may find the formulae

$$\langle \tilde{h}, \tilde{h}\rangle = \sum_{i,j} \epsilon_i \epsilon_j\, g(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j), \tilde{h}(\mathcal{E}_i, \mathcal{E}_j)),$$
$$\langle h, h\rangle = \sum_{a,b} \epsilon_a \epsilon_b\, g(h(E_a, E_b), h(E_a, E_b)),$$
$$\langle \tilde{T}, \tilde{T}\rangle = \sum_{i,j} \epsilon_i \epsilon_j\, g(\tilde{T}(\mathcal{E}_i, \mathcal{E}_j), \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)),$$
$$\langle T, T\rangle = \sum_{a,b} \epsilon_a \epsilon_b\, g(T(E_a, E_b), T(E_a, E_b)).$$

The mean curvature vector fields of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are, respectively,

$$H = \mathrm{Tr}\ _g h = \sum_a \epsilon_a h(E_a, E_a),\quad \tilde{H} = \mathrm{Tr}\ _g \tilde{h} = \sum_i \epsilon_i \tilde{h}(\mathcal{E}_i, \mathcal{E}_i).$$

A distribution $\widetilde{\mathcal{D}}$ is called *totally umbilical*, *minimal*, or *totally geodesic*, if $h = \frac{1}{n} H\, g^{\top},\ H = 0$, or $h = 0$, respectively. There exist minimal, nowhere totally geodesic distributions of any codimension > 1 on Lie groups with left-invariant metrics, see [15]. In the case of foliations, the metric can be chosen to be bundle-like and mixed scalar curvature is leafwise constant.

The Weingarten operator A_Z of $\widetilde{\mathcal{D}}$ with respect to $Z \in \mathcal{D}$, and the operator $T^{\sharp}_Z$ are defined by

$$g(A_Z(X), Y) = g(h(X, Y), Z),\ \ g\left(T^{\sharp}_Z(X), Y\right) = g(T(X, Y), Z)\quad (X, Y \in \mathfrak{X}_{\widetilde{\mathcal{D}}}).$$

Similarly, we define for $N \in \widetilde{\mathcal{D}}$

$$g(\tilde{A}_N(X), Y) = g(\tilde{h}(X, Y), N),\ \ g\left(\tilde{T}^{\sharp}_N(X), Y\right) = g(\tilde{T}(X, Y), N)\quad (X, Y \in \mathfrak{X}_{\mathcal{D}}).$$

For the local orthonormal frame $\{E_i,\ \mathcal{E}_a\}$ (adapted to the distributions), we use the following convention for various (1, 1)-tensors: $\tilde{T}^{\sharp}_a := \tilde{T}^{\sharp}_{E_a},\ A_i := A_{\mathcal{E}_i}$, etc.

The Divergence Theorem states that $\int_M (\mathrm{div}\,\xi)\,\mathrm{d}\,\mathrm{vol}_g = 0$, when M is closed; this is also true if M is open and $\xi \in \mathfrak{X}_M$ is supported in a relatively compact domain $\Omega \subset M$. The $\mathcal{D}^{\perp}$-*divergence* of ξ is defined by $\mathrm{div}^{\perp}\,\xi = \sum_i \epsilon_i\, g(\nabla_{\mathcal{E}_i}\,\xi, \mathcal{E}_i)$; similarly, $\widetilde{\mathrm{div}}\,\xi = \sum_a \epsilon_a\, g(\nabla_{E_a}\,\xi, E_a)$. Thus, the divergence of ξ is $\mathrm{div}\,\xi = \mathrm{Tr}\,(\nabla\xi) = \mathrm{div}^{\perp}\,\xi + \widetilde{\mathrm{div}}\,\xi$. Observe that for $X \in \mathfrak{X}_{\mathcal{D}}$ we get

$$\mathrm{div}^{\perp} X = \mathrm{div}\, X + g(X,\ H). \tag{4}$$

Indeed, using $H = \sum_{a\le n} \epsilon_a\, h(E_a, E_a)$ and $g(X,\ E_a) = 0$, one derives (4):

$$\mathrm{div}\, X - \mathrm{div}^{\perp} X = \sum_a \epsilon_a\, g(\nabla_{E_a} X,\ E_a) = -\sum_a \epsilon_a\, g(h(E_a, E_a), X) = -g(X,\ H).$$

For a (1, 2)-tensor P define a (0, 2)-tensor $\operatorname{div}^\perp P$ by

$$(\operatorname{div}^\perp P)(X, Y) = \sum_i \epsilon_i\, g((\nabla_{\mathcal{E}_i} P)(X, Y), \mathcal{E}_i) \quad (X, Y \in \mathfrak{X}_{\widetilde{\mathcal{D}}}).$$

Then the divergence of P is $\operatorname{div} P = \widetilde{\operatorname{div}}\, P + \operatorname{div}^\perp P$. For a $\mathcal{D}$-valued (1, 2)-tensor P, similarly to (4), we have

$$\sum_a \epsilon_a\, g((\nabla_{E_a} P)(X, Y), E_a) = -g(P(X, Y), H)$$

and

$$\operatorname{div}^\perp P = \operatorname{div} P + \langle P,\ H\rangle\,, \tag{5}$$

where $\langle P,\ H\rangle(X, Y) := g(P(X, Y),\ H)$ is a (0, 2)-tensor. For example, $\operatorname{div}^\perp h = \operatorname{div} h + \langle h,\ H\rangle$.

For any function f on M, we introduce the following notation of the projections of its gradient onto distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$:

$$\nabla^\top f \equiv \widetilde{\nabla} f := (\nabla f)^\top, \quad \nabla^\perp f := (\nabla f)^\perp.$$

The $\widetilde{\mathcal{D}}$-Laplacian of a function f is given by the formula $\widetilde{\Delta} f = \widetilde{\operatorname{div}}\,(\widetilde{\nabla} f)$. The $\mathcal{D}$-*deformation tensor* $\operatorname{Def}_{\mathcal{D}} Z$ of a vector field Z (e.g., $Z = H$) is the symmetric part of ∇Z restricted to $\mathcal{D}$,

$$2\operatorname{Def}_{\mathcal{D}} Z(X, Y) = g(\nabla_X Z, Y) + g(\nabla_Y Z, X) \quad (X, Y \in \mathfrak{X}_{\mathcal{D}}).$$

As in [2,14], we define self-adjoint (1, 1)-tensors: $\mathcal{A} := \sum_i \epsilon_i A_i^2$, called the *Casorati operator* of $\mathcal{D}$, and $\mathcal{T} := \sum_i \epsilon_i (T_i^\sharp)^2$. Similarly, we define $\widetilde{\mathcal{A}} = \sum_a \epsilon_a \tilde{A}_a^2$ and $\widetilde{\mathcal{T}} = \sum_a \epsilon_a (\tilde{T}_a^\sharp)^2$. We also define the symmetric (0, 2)-tensors Ψ and $\widetilde{\Psi}$ by formulas

$$\Psi(X, Y) = \operatorname{Tr}\left(A_Y A_X + T_Y^\sharp T_X^\sharp\right) \quad (X, Y \in \mathfrak{X}_{\mathcal{D}}),$$
$$\widetilde{\Psi}(X, Y) = \operatorname{Tr}\left(\tilde{A}_Y \tilde{A}_X + \tilde{T}_Y^\sharp \tilde{T}_X^\sharp\right) \quad (X, Y \in \mathfrak{X}_{\widetilde{\mathcal{D}}}).$$

The partial Ricci tensor can be presented in terms of the extrinsic geometry. Using its definition and the decomposition of tangent bundle into two orthogonal distributions, similarly as in [2], one can obtain the following lemma, that we prove below for readers' convenience.

Lemma 1 *Let* $g \in \operatorname{Riem}(M,\ \widetilde{\mathcal{D}},\ \mathcal{D})$. *Then the following identity holds:*

$$r_{\mathcal{D}} = \operatorname{div} \tilde{h} + \langle \tilde{h},\ \tilde{H}\rangle - \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat - \Psi + \operatorname{Def}_{\mathcal{D}} H. \tag{6}$$

Proof For $X, Y \in \mathfrak{X}_{\mathcal{D}}$ and $U, V \in \mathfrak{X}_{\widetilde{\mathcal{D}}}$ we have, see [11, Lemma 2.25],

$$g(R^\nabla(U,\ X)V,\ Y) = g(((\nabla_U \tilde{C})_V - \tilde{C}_V \tilde{C}_U)X, Y) + g(((\nabla_X C)_Y - C_Y C_X)U, V), \tag{7}$$

where the conullity tensors $\tilde{C} : \widetilde{\mathcal{D}} \times \mathcal{D} \to \mathcal{D}$ and $C : \mathcal{D} \times \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ are defined by

$$\tilde{C}_U(X) = -(\nabla_X U)^\perp, \quad C_X(U) = -\,(\nabla_U X)^\top\ .$$

Note that $\tilde{C}_U = \tilde{A}_U + \tilde{T}_U^\sharp$, $C_X = A_X + T_X^\sharp$, $\widetilde{\Psi}(U, V) = \operatorname{Tr}_g(\tilde{C}_V \tilde{C}_U)$ and $\Psi(X, Y) = \operatorname{Tr}_g(C_Y C_X)$. We can assume that $\nabla_X Y \in \mathcal{D}_x$ and $\nabla_X E_a \in \mathcal{D}_x$ at a given point $x \in M$. Note that

$$\sum_a \epsilon_a g((\nabla_X C)_Y(E_a),\ E_a) = \nabla_X\left(g\left(\sum_a \epsilon_a h(E_a, E_a),\ Y\right)\right) = g(\nabla_X H,\ Y).$$

Let $\widetilde{\operatorname{div}}\,\tilde{C} = \sum_{a=1}^{n} \epsilon_a (\nabla_a \tilde{C})_a$. Then, tracing (7) over $\widetilde{\mathcal{D}}_x$ yields

$$r_{\mathcal{D}}(X, Y) = g(\widetilde{\operatorname{div}}\,\tilde{C}(X), Y) - g\left(\sum_a \epsilon_a \tilde{C}_a^2(X), Y\right) + g(\nabla_X H, Y) - \operatorname{Tr}_g(C_Y C_X). \quad (8)$$

Using $\operatorname{Tr}_g(A_Y T_X^\sharp) = 0 = \operatorname{Tr}_g(T_Y^\sharp A_X)$ (since h is symmetric and T is antisymmetric), we extract (6) as the symmetric part of (8). □

The extrinsic scalar curvatures of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are defined by

$$\mathrm{S}_{\,\mathrm{ex}} = g(H, H) - \langle h, h\rangle, \quad \widetilde{\mathrm{S}}_{\,\mathrm{ex}} = g(\tilde{H}, \tilde{H}) - \langle \tilde{h}, \tilde{h}\rangle,$$

respectively. Tracing (6) over $\mathcal{D}$ and applying (3) and the equalities

$$\operatorname{Tr} \mathcal{A} = \langle h, h\rangle, \quad \operatorname{Tr} \mathcal{T} = -\langle T, T\rangle,$$
$$\operatorname{Tr}_g \Psi = \operatorname{Tr}(\mathcal{A} + \mathcal{T}) = \langle h, h\rangle - \langle T, T\rangle,$$
$$\operatorname{Tr}_g(\operatorname{div} h) = \operatorname{div} H, \quad \operatorname{Tr}_g(\operatorname{Def}_{\mathcal{D}} H) = \operatorname{div} H + g(H, H)$$

yields the formula (see also [18])

$$\mathrm{S}_{\mathrm{mix}} = \mathrm{S}_{\,\mathrm{ex}} + \widetilde{\mathrm{S}}_{\,\mathrm{ex}} + \langle T, T\rangle + \langle \tilde{T}, \tilde{T}\rangle + \operatorname{div}(H + \tilde{H}), \quad (9)$$

which shows how $\mathrm{S}_{\mathrm{mix}}$ is built of invariants of the extrinsic geometry of the distributions.

We define the following (1, 2)-tensors on $(M, \widetilde{\mathcal{D}}, \mathcal{D}, g)$ for all $X, Y, Z \in \mathfrak{X}_M$:

$$\alpha(X, Y) = \frac{1}{2}\left(A_{X^\perp}(Y^\top) + A_{Y^\perp}(X^\top)\right), \quad \tilde{\alpha}(X, Y) = \frac{1}{2}\left(\tilde{A}_{X^\top}(Y^\perp) + \tilde{A}_{Y^\top}(X^\perp)\right),$$
$$\theta(X, Y) = \frac{1}{2}\left(T^\sharp_{X^\perp}(Y^\top) + T^\sharp_{Y^\perp}(X^\top)\right), \quad \tilde{\theta}(X, Y) = \frac{1}{2}\left(\tilde{T}^\sharp_{X^\top}(Y^\perp) + \tilde{T}^\sharp_{Y^\top}(X^\perp)\right),$$
$$\tilde{\delta}_Z(X, Y) = \frac{1}{2}\left(g\left(\nabla_{X^\top} Z,\, Y^\perp\right) + g(\nabla_{Y^\top} Z,\, X^\perp)\right).$$

For any (0, 2)-tensors P, Q and S on TM, we define a tensor $\Lambda_{P,Q}$ by

$$\langle \Lambda_{P,Q}, S\rangle = \sum_{\lambda,\mu} \epsilon_\lambda \epsilon_\mu [S(P(e_\lambda, e_\mu), Q(e_\lambda, e_\mu)) + S(Q(e_\lambda, e_\mu), P(e_\lambda, e_\mu))],$$

where $\{e_\lambda\}$ is a full orthonormal basis of TM and $\epsilon_\lambda = g(e_\lambda, e_\lambda) \in \{-1, 1\}$. Note that

$$\Lambda_{P,Q} = \Lambda_{Q,P} \quad \text{and} \quad \Lambda_{P,Q_1+Q_2} = \Lambda_{P,Q_1} + \Lambda_{P,Q_2}$$

for all (0, 2)-tensors P, Q, Q_1, Q_2. We also use symmetric (0, 2)-tensors Φ_h and Φ_T defined as in [2,14],

$$\langle \Phi_h,\ S\rangle = S(H,\ H) - \sum_{a,\,b} \epsilon_a\, \epsilon_b\, S(h(E_a, E_b), h(E_a, E_b)),$$
$$\langle \Phi_T,\ S\rangle = -\sum_{a,\,b} \epsilon_a\, \epsilon_b\, S(T(E_a, E_b), T(E_a, E_b))$$

for any symmetric (0, 2)-tensor S. Note that $\Phi_T = -\frac{1}{2}\Lambda_{T,T}$ and $\Phi_h = H^\flat \otimes H^\flat - \frac{1}{2}\Lambda_{h,h}$. Tensors $\Phi_{\tilde{h}}$ and $\Phi_{\tilde{T}}$ are defined analogously.

We define a self-adjoint (1, 1)-tensor (with zero trace)

$$\mathcal{K} = \sum_i \epsilon_i \left[T_i^\sharp, A_i\right] = \sum_i \epsilon_i \left(T_i^\sharp A_i - A_i\, T_i^\sharp\right)$$

and its counterpart $\widetilde{\mathcal{K}} = \sum_a \epsilon_a\,[\tilde{T}_a^\sharp, \tilde{A}_a]$. It is easy to see that all the above tensors defined with the use of an adapted orthonormal frame in fact do not depend on the choice of such frame.

Remark 1 (see [14]) Let us clarify the geometrical sense of Φ_h and $\tilde{\mathcal{K}}$. If g is definite on $\widetilde{\mathcal{D}}$, then $\Phi_h = 0$ if and only if one of the following point-wise conditions holds:

$$\text{(i) } h = 0; \quad \text{(ii) } H \neq 0,\ \ \mathrm{S}_{\rm ex} = 0 \text{ and the image of } h \text{ is spanned by } H\,.$$

If $\mathcal{D}$ is integrable, then $\tilde{T}_a^\sharp = 0$ $(a = 1, \ldots, n)$, hence $\tilde{\mathcal{K}} := \sum_a \epsilon_a\,[\tilde{T}_a^\sharp, \tilde{A}_a] = 0$. If $\mathcal{D}$ is totally umbilical, then every operator $\tilde{A}_a$ is a multiple of identity and $\tilde{\mathcal{K}}$ vanishes as well.

2.2 Variation formulas

Let $(M, \widetilde{\mathcal{D}}, g)$ be a manifold with distribution and a pseudo-Riemannian metric g. We consider smooth 1-parameter variations $\{g_t \in \mathrm{Riem}(M) : |t| < \varepsilon\}$ of the metric $g_0 = g$. We assume that the induced infinitesimal variations, represented by a symmetric $(0, 2)$-tensor $B_t \equiv \partial g_t/\partial t$, are supported in a relatively compact domain Ω in M, i.e., $g_t = g$ outside Ω for all $|t| < \varepsilon$. We adopt the notations

$$\partial_t \equiv \partial/\partial t, \quad B \equiv \partial_t g_{t\,|\,t=0},$$

but we shall also write B instead of B_t to make formulas easier to read, wherever it does not lead to confusion. Since B is symmetric, for any $(0, 2)$-tensor C, we have $\langle C,\, B\rangle = \langle \mathrm{Sym}(C),\, B\rangle$. We denote by $\mathcal{D}(t)$ the g_t-orthogonal complement of $\widetilde{\mathcal{D}}$.

Definition 1 (i) Let $\widetilde{\mathcal{D}}$ be a distribution on (M, g). A family of metrics $\{g_t \in \mathrm{Riem}(M) : |t| < \varepsilon\}$ such that $g_0 = g$ and for all $|t| < \varepsilon$:

$$g_t(X, Y) = g(X, Y) \quad (X, Y \in \mathfrak{X}_{\widetilde{\mathcal{D}}}),$$

will be called $g^\perp$*-variation*. For $g^\perp$-variations the metric on $\widetilde{\mathcal{D}}$ is preserved.

(ii) Let $\widetilde{\mathcal{D}}$ be a distribution on (M, g) and let $\mathcal{D}$ be its g-orthogonal complement. A family of metrics $\{g_t \in \mathrm{Riem}(M) : |t| < \varepsilon\}$ such that $g_0 = g$, for all $|t| < \varepsilon$ the distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ remain orthogonal and

$$g_t(X, Y) = g(X, Y) \quad (X, Y \in \mathfrak{X}_{\mathcal{D}}),$$

will be called $g^\top$*-variation*. For $g^\top$-variations only the metric on $\widetilde{\mathcal{D}}$ changes.

We will now relate the variations defined above to arbitrary variations of g. Let $\mathcal{D} = \mathcal{D}(0)$ be the g-orthogonal complement of $\widetilde{\mathcal{D}}$. While the distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ may not be g_t-orthogonal for $t > 0$, we can assume that they span the tangent bundle. For any $X \in TM$, let $X_{\widetilde{\mathcal{D}}}$ denote the g-orthogonal projection of X onto $\widetilde{\mathcal{D}}$ and let $X_{\mathcal{D}}$ denote the g-orthogonal projection of X onto $\mathcal{D}$. Let $\mathrm{V} = (\mathcal{D} \times \widetilde{\mathcal{D}}) + (\widetilde{\mathcal{D}} \times \mathcal{D})$ be the subspace of $TM \times TM$ spanned by $(\mathcal{D} \times \widetilde{\mathcal{D}}) \cup (\widetilde{\mathcal{D}} \times \mathcal{D})$. Then, given $g \in \mathrm{Riem}(M)$, we have $g_t = g_{t|\,\mathcal{D}\times\mathcal{D}} + g_{t\,|\,\mathcal{D}\times\widetilde{\mathcal{D}}} + g_{t\,|\,\widetilde{\mathcal{D}}\times\mathcal{D}} + g_{t\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}}$, where

$$g_{t\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}}(X, Y) = g_t(X_{\widetilde{\mathcal{D}}}, Y_{\widetilde{\mathcal{D}}}), \quad g_{t|\,\mathcal{D}\times\mathcal{D}}(X, Y) = g_t(X_{\mathcal{D}}, Y_{\mathcal{D}}),$$
$$g_{t\,|\,\mathcal{D}\times\widetilde{\mathcal{D}}}(X, Y) = g_t(X_{\mathcal{D}}, Y_{\widetilde{\mathcal{D}}}), \quad g_{t\,|\,\widetilde{\mathcal{D}}\times\mathcal{D}}(X, Y) = g_t(X_{\widetilde{\mathcal{D}}}, Y_{\mathcal{D}});$$

thus, $g_{t\,|\mathrm{V}}(X, Y) = g_t(X_{\mathcal{D}}, Y_{\widetilde{\mathcal{D}}}) + g_t(X_{\widetilde{\mathcal{D}}}, Y_{\mathcal{D}})$, and we can present g_t in the following form:

$$g_t = \begin{pmatrix} g_{t\,|\,\mathcal{D}\times\mathcal{D}} & g_{t\,|\,\mathcal{D}\times\widetilde{\mathcal{D}}} \\ g_{t\,|\,\widetilde{\mathcal{D}}\times\mathcal{D}} & g_{t\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}} \end{pmatrix}.$$

Similarly, $B_t = B_t^\perp + B_t^/ + \tilde B_t$, where $B_t^\perp = \partial_t g_{t|\,\mathcal{D}\times\mathcal{D}}$, $\tilde B_t = \partial_t g_{t\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}}$ and $B_t^/ = \partial_t g_{t\,|\mathrm{V}}$. For $g^\perp$-variations, $g_t = g_{t|\,\mathcal{D}\times\mathcal{D}} + g_{t\,|\,\mathrm{V}} + g_{0\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}}$ and for $g^\top$-variations $g_t = g_{0|\,\mathcal{D}\times\mathcal{D}} + g_{t\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}}$ (as $g_{t\,|\,\mathrm{V}} = g_{0\,|\,\mathrm{V}} = 0$), we have, respectively,

$$B_t = B_t^\perp + B_t^/ = \begin{pmatrix} B^\perp_{t\,|\,\mathcal{D}\times\mathcal{D}} & B^/_{t\,|\,\mathcal{D}\times\widetilde{\mathcal{D}}} \\ B^/_{t\,|\,\widetilde{\mathcal{D}}\times\mathcal{D}} & 0 \end{pmatrix}, \quad B_t = \tilde B_t = \begin{pmatrix} 0 & 0 \\ 0 & \tilde B_{t\,|\,\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}} \end{pmatrix}.$$

By the above, the derivative B_t of any variation g_t can be decomposed into sum of derivatives of some $g^\perp$- and $g^\top$-variations.

For all $X, Y, Z \in \mathfrak{X}_M$, the Levi–Civita connection ∇^t of g_t ($|t| < \varepsilon$) evolves as, see for example [17],

$$2\, g_t\left(\partial_t\left(\nabla^t_X Y\right), Z\right) = \left(\nabla^t_X B\right)(Y, Z) + \left(\nabla^t_Y B\right)(X, Z) - \left(\nabla^t_Z B\right)(X, Y), \qquad (10)$$

where the first covariant derivative of a (0, 2)-tensor B is expressed as

$$(\nabla_Z B)(Y, V) = Z(B(Y, V)) - B(\nabla_Z Y, V) - B(Y, \nabla_Z V).$$

Let $\mathcal{D}(t)$ be the g_t-orthogonal complement of $\widetilde{\mathcal{D}}$. Let $^\top$ and $^\perp$ denote the g_t-orthogonal projections onto $\widetilde{\mathcal{D}}$ and $\mathcal{D}(t)$, respectively; note that these projections are t-dependent.

Lemma 2 *Let g_t be a $g^\perp$-variation of g with $B_t = \partial_t g_t$. Let $\{E_a, \mathcal{E}_i\}$ be a local $(\widetilde{\mathcal{D}}, \mathcal{D})$-adapted and orthonormal for $t = 0$ frame, that evolves according to*

$$\partial_t E_a = 0, \quad \partial_t \mathcal{E}_i = -(1/2)\left(B_t^\sharp(\mathcal{E}_i)\right)^\perp - \left(B_t^\sharp(\mathcal{E}_i)\right)^\top. \qquad (11)$$

Then, for all t, $\{E_a(t), \mathcal{E}_i(t)\}$ is a g_t-orthonormal frame adapted to $(\widetilde{\mathcal{D}}, \mathcal{D}(t))$.

Proof Since $\partial_t E_a = 0$ and $E_a(0) \in \widetilde{\mathcal{D}}$, we have for $g^\perp$-variation $\partial_t(g_t(E_a, E_b)) = 0$. Also,

$$\begin{aligned}\partial_t(g_t(E_a, \mathcal{E}_i)) &= (\partial_t g_t)(E_a(t), \mathcal{E}_i(t)) + g_t(\partial_t E_a(t), \mathcal{E}_i(t)) + g_t(E_a(t), \partial_t \mathcal{E}_i(t)) \\ &= B_t(E_a(t), \mathcal{E}_i(t)) - \frac{1}{2}\, g_t\left(\left(B_t^\sharp(\mathcal{E}_i(t))\right)^\perp, E_a(t)\right) \\ &\quad - g_t\left(E_a(t), B_t^\sharp(\mathcal{E}_i(t))^\top\right) = 0.\end{aligned}$$

Now that we know that $g_t(E_a, \mathcal{E}_i) = 0$, it follows that $\mathcal{E}_i(t) \in \mathcal{D}(t)$, and for any X, we have $g_t(\mathcal{E}_i, X^\top) = 0$. We can finish the proof by computing

$$\begin{aligned}\partial_t(g_t(\mathcal{E}_i, \mathcal{E}_j)) &= (\partial_t g_t)(\mathcal{E}_i(t), \mathcal{E}_j(t)) + g_t(\partial_t \mathcal{E}_i(t), \mathcal{E}_j(t)) + g_t(\mathcal{E}_i(t), \partial_t \mathcal{E}_j(t)) \\ &= B_t(\mathcal{E}_i(t), \mathcal{E}_j(t)) - \frac{1}{2}\, g_t\left(\left(B_t^\sharp(\mathcal{E}_i(t))\right)^\perp, \mathcal{E}_j(t)\right) \\ &\quad - \frac{1}{2}\, g_t\left(\mathcal{E}_i(t), \left(B_t^\sharp \mathcal{E}_j(t)\right)^\perp\right) = 0.\end{aligned}$$

□

The evolution of $\mathcal{D}(t)$ gives rise to the evolution of both $\widetilde{\mathcal{D}}$- and $\mathcal{D}(t)$-components of any vector X on M.

Lemma 3 *Let g_t be a $g^\perp$-variation of g. Then for any t-dependent vector X on M, we have*

$$\partial_t(X^\top) = (\partial_t X)^\top + (B^\sharp(X^\perp))^\top, \quad \partial_t(X^\perp) = (\partial_t X)^\perp - (B^\sharp(X^\perp))^\top.$$

Proof Using the frame from Lemma 2, we can write

$$X^\top = \sum_a \epsilon_a g_t(X_t, E_a(t))E_a, \quad X^\perp = \sum_i \epsilon_i g_t(X_t, \mathcal{E}_i(t))\mathcal{E}_i(t). \tag{12}$$

We have

$$\begin{aligned} B_t(E_a(t), E_b(t)) &= (\partial_t g_t)(E_a(t), E_b(t)) \\ &= \partial_t(g_t(E_a, E_b)) - g_t(\partial_t E_a(t), E_b(t)) - g_t(E_a(t), \partial_t E_b(t)) = 0; \end{aligned}$$

hence, $(B^\sharp(X^\top))^\top = 0$, which implies

$$(B^\sharp(X))^\top = (B^\sharp(X^\perp))^\top. \tag{13}$$

The proof follows from differentiating (12) and using (11) and (13). □

Remark 2 Let B be a symmetric (0,2)-tensor. The following computations will be used to obtain variation formulas:

$$\begin{aligned} \langle\, \langle \alpha, \tilde H\rangle, B\rangle &= \sum_{a,i} \epsilon_a\epsilon_i g(\alpha(E_a, \mathcal{E}_i), \tilde H)B(E_a, \mathcal{E}_i) \\ &\quad + \sum_{a,i} \epsilon_a\epsilon_i g(\alpha(\mathcal{E}_i, E_a), \tilde H)B(\mathcal{E}_i, E_a) \\ &= 2\sum_{a,i} \epsilon_a\epsilon_i g(\alpha(E_a, \mathcal{E}_i), \tilde H)B(E_a, \mathcal{E}_i) \\ &= \sum_{a,i} \epsilon_a\epsilon_i g(A_i(E_a), \tilde H)B(E_a, \mathcal{E}_i), \\ \langle \Lambda_{\alpha,\theta}, B\rangle &= \sum_{a,i} \epsilon_a\epsilon_i B(A_i(E_a), T_i^\sharp(E_a)). \end{aligned}$$

Later we will also use the fact that for $X \in \widetilde{\mathcal{D}}$, $N \in \mathcal{D}$ we have

$$\Lambda_{\alpha,\tilde\theta}(X, N) = \frac{1}{2}\sum_{a,i} \epsilon_a\epsilon_i\, g(X, A_i E_a)g(N, \tilde T_a^\sharp \mathcal{E}_i).$$

Similar formulas can be obtained for $\Lambda_{\alpha,\tilde\alpha}$, $\Lambda_{\theta,\tilde\alpha}$, etc.

The key and most technical result of this section is the following.

Proposition 1 *Let g_t be a $g^\perp$-variation of g. Then*

$$\partial_t\langle \tilde h, \tilde h\rangle = \langle \operatorname{div}\tilde h - 4\,\Lambda_{\tilde\alpha,\theta} + \widetilde{\mathcal{K}}^\flat,\ B\rangle - \operatorname{div}\langle \tilde h, B\rangle, \tag{14a}$$

$$\partial_t g(\tilde H, \tilde H) = \langle\, (\operatorname{div}\tilde H)\, g^\perp + 4\,\langle \theta, \tilde H\rangle,\ B\rangle - \operatorname{div}((\operatorname{Tr}_{\mathcal{D}} B^\sharp)\tilde H), \tag{14b}$$

$$\partial_t\langle h, h\rangle = 2\operatorname{div}(\langle \alpha, B\rangle) - 2\,\langle (\operatorname{div}\alpha)_{\,|\mathcal{V}} + \Lambda_{\alpha,\tilde\alpha+\tilde\theta} - \frac{1}{2}\Phi_h,\ B\rangle - B(H, H), \tag{14c}$$

$$\begin{aligned} \partial_t g(H, H) &= 2\big\langle\, \langle \tilde\theta - \tilde\alpha,\ H\rangle + \operatorname{Sym}(H^\flat \otimes \tilde H^\flat) - \tilde\delta_H,\ B\big\rangle - B(H, H) \\ &\quad + 2\operatorname{div}((B^\sharp H)^\top), \end{aligned} \tag{14d}$$

$$\partial_t\langle \tilde T, \tilde T\rangle = 2\,\langle\, \tilde{\mathcal{T}}^\flat + \Lambda_{\tilde\theta,\theta-\alpha} - (\operatorname{div}\tilde\theta)_{\,|\mathcal{V}},\ B\,\rangle + 2\operatorname{div}\langle \tilde\theta,\ B\rangle, \tag{14e}$$

$$\partial_t\langle T, T\rangle = -\langle \Phi_T, B\rangle. \tag{14f}$$

Proof In the proof we denote by $(\mathrm{i})_j$ the jth term of the right-hand side of formula (i). We shall use an adapted frame that satisfies $\nabla^t_X \mathcal{E}_j \in \widetilde{\mathcal{D}}$ and $\nabla^t_X E_b \in \mathcal{D}$ for all $X \in T_x M$, at a point $x \in M$ for which all the formulas are considered, and for the value of parameter t at which the variation is computed. (All the results hold true without this assumption, but it simplifies the computations.)

Proof of (14a). We use Lemma 3 to compute $\partial_t \langle \tilde h, \tilde h\rangle$, as a sum of 10 terms in $g(\cdot\,, E_a)$,

$$\begin{aligned}
\partial_t \langle \tilde h, \tilde h\rangle &= \sum_{i,j} \epsilon_i\epsilon_j B(\tilde h(\mathcal{E}_i,\mathcal{E}_j), \tilde h(\mathcal{E}_i,\mathcal{E}_j))\\
&\quad + \sum_{i,j} \epsilon_i\epsilon_j g\left(\tilde h(\mathcal{E}_i,\mathcal{E}_j),\, \partial_t\left(\left(\nabla^t_{\mathcal{E}_i}\mathcal{E}_j + \nabla^t_{\mathcal{E}_j}\mathcal{E}_i\right)^\top\right)\right)\\
&= \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a g(\tilde h(\mathcal{E}_i,\mathcal{E}_j), E_a)\, g\Big(\nabla^t_{(\partial_t\mathcal{E}_i)^\top}\mathcal{E}_j + \nabla^t_{(\partial_t\mathcal{E}_j)^\top}\mathcal{E}_i + \nabla^t_{\mathcal{E}_i}\left((\partial_t\mathcal{E}_j)^\top\right)\\
&\quad + \nabla^t_{\mathcal{E}_j}\left((\partial_t\mathcal{E}_i)^\top\right) + \nabla^t_{\mathcal{E}_i}\left((\partial_t\mathcal{E}_j)^\perp\right) + \nabla^t_{\mathcal{E}_j}\left((\partial_t\mathcal{E}_i)^\perp\right) + \nabla^t_{(\partial_t\mathcal{E}_i)^\perp}\mathcal{E}_j + \nabla^t_{(\partial_t\mathcal{E}_j)^\perp}\mathcal{E}_i\\
&\quad + (\partial_t\nabla^t)_{\mathcal{E}_i}\mathcal{E}_j + (\partial_t\nabla^t)_{\mathcal{E}_j}\mathcal{E}_i,\ E_a\Big),
\end{aligned}\tag{15}$$

where we used $B(\tilde h(\mathcal{E}_i,\mathcal{E}_j), \tilde h(\mathcal{E}_i,\mathcal{E}_j)) = 0$ (since B vanishes on $\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}$). The last two terms $(15)_9$ and $(15)_{10}$ are equal, and their sum can be computed in the following way:

$$\begin{aligned}
2g\left((\partial_t\nabla^t)_{\mathcal{E}_i}\mathcal{E}_j, E_a\right) &= \left(\nabla^t_{\mathcal{E}_i} B\right)(\mathcal{E}_j, E_a) + \left(\nabla^t_{\mathcal{E}_j} B\right)(\mathcal{E}_i, E_a) - \left(\nabla^t_{E_a} B\right)(\mathcal{E}_i,\mathcal{E}_j)\\
&= \nabla^t_{\mathcal{E}_i} B(\mathcal{E}_j, E_a) - B\left(\nabla^t_{\mathcal{E}_i} E_a, \mathcal{E}_j\right) + \nabla^t_{\mathcal{E}_j} B(\mathcal{E}_i, E_a) - B\left(\nabla^t_{\mathcal{E}_j} E_a, \mathcal{E}_i\right)\\
&\quad - \nabla^t_{E_a} B(\mathcal{E}_i,\mathcal{E}_j) + B\left(\nabla^t_{E_a}\mathcal{E}_i, \mathcal{E}_j\right) + B\left(\nabla^t_{E_a}\mathcal{E}_j, \mathcal{E}_i\right).
\end{aligned}$$

Using Lemma 2, we rewrite (15) as

$$\begin{aligned}
\partial_t \langle \tilde h, \tilde h\rangle &= -\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a \nabla^t_{E_a} B(\mathcal{E}_i,\mathcal{E}_j)\, g(\tilde h(\mathcal{E}_i,\mathcal{E}_j), E_a)\\
&\quad - \sum_{i,j} \epsilon_i\epsilon_j\left[g(\tilde h(B^\sharp\mathcal{E}_i,\mathcal{E}_j), \tilde h(\mathcal{E}_i,\mathcal{E}_j)) + g(\tilde h(B^\sharp\mathcal{E}_j,\mathcal{E}_i), \tilde h(\mathcal{E}_i,\mathcal{E}_j))\right]\\
&\quad - \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\nabla^t_{\mathcal{E}_i}\left((B^\sharp\mathcal{E}_j)^\top\right) + \nabla^t_{\mathcal{E}_j}\left((B^\sharp\mathcal{E}_i)^\top\right), E_a\right) g(\tilde h(\mathcal{E}_i,\mathcal{E}_j), E_a)\\
&\quad - \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\nabla^t_{(B^\sharp\mathcal{E}_j)^\top}\mathcal{E}_i + \nabla^t_{(B^\sharp\mathcal{E}_i)^\top}\mathcal{E}_j, E_a\right) g(\tilde h(\mathcal{E}_i,\mathcal{E}_j), E_a)\\
&\quad + \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a \left(\nabla^t_{\mathcal{E}_i} B(\mathcal{E}_j, E_a) + \nabla^t_{\mathcal{E}_j} B(\mathcal{E}_i, E_a)\right) g(\tilde h(\mathcal{E}_i,\mathcal{E}_j), E_a)\\
&\quad - \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a \left(B\left(\nabla^t_{\mathcal{E}_i} E_a, \mathcal{E}_j\right) + B\left(\nabla^t_{\mathcal{E}_j} E_a, \mathcal{E}_i\right)\right) g(\tilde h(\mathcal{E}_i,\mathcal{E}_j), E_a)\\
&\quad + \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a \left(B\left(\nabla^t_{E_a}\mathcal{E}_i, \mathcal{E}_j\right) + B\left(\nabla^t_{E_a}\mathcal{E}_j, \mathcal{E}_i\right)\right) g(\tilde h(\mathcal{E}_i,\mathcal{E}_j), E_a).
\end{aligned}\tag{16}$$

From the definition $2\,\mathrm{Sym}(C) = C + C^*$, we have

$$\Big\langle 2\sum_a \epsilon_a \left(\tilde T_a^\sharp \tilde A_a\right)^\flat,\ B\Big\rangle = 2\left\langle \mathrm{Sym}\left(\sum_a \epsilon_a \tilde T_a^\sharp \tilde A_a\right)^\flat,\ B\right\rangle$$
$$= \left\langle \sum_a \epsilon_a \left[\tilde T_a^\sharp,\ \tilde A_a\right]^\flat,\ B\right\rangle = \langle \widetilde{\mathcal{K}}^\flat,\ B\rangle,$$

and we obtain (14a), using the following computations for all seven lines of (16):

$$\sum_{i,j} \epsilon_i\epsilon_j\, g(\tilde h(B^\sharp\mathcal{E}_i, \mathcal{E}_j), \tilde h(\mathcal{E}_i, \mathcal{E}_j)) = \langle \tilde{\mathcal{A}}^\flat,\ B\rangle,$$

$$\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\nabla^t_{\mathcal{E}_i}\left(B^\sharp(\mathcal{E}_j)^\top\right), E_a\right) g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a) = \mathrm{div}\langle\tilde\alpha, B\rangle - \langle(\mathrm{div}\,\tilde\alpha)_{|\mathrm{V}},\ B\rangle,$$

$$\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\nabla^t_{(B^\sharp\mathcal{E}_j)^\top}\mathcal{E}_i, E_a\right) g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a)$$
$$= -\sum_{i,a} \epsilon_i\,\epsilon_a\, B\left(\tilde A_a(\mathcal{E}_i),\ A_i(E_a) - T_i^\sharp(E_a)\right) = -\langle \Lambda_{\tilde\alpha,\alpha-\theta}, B\rangle,$$

$$\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a)\nabla^t_{\mathcal{E}_i} B(E_a, \mathcal{E}_j) = \mathrm{div}\langle\tilde\alpha,\ B\rangle - \langle(\mathrm{div}\,\tilde\alpha)_{|\mathrm{V}},\ B\rangle,$$

$$\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a)\nabla^t_{E_a} B(\mathcal{E}_i, \mathcal{E}_j) = \mathrm{div}\langle\tilde h, B\rangle - \langle \mathrm{div}\,\tilde h,\ B\rangle,$$

$$\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, B\left(\nabla^t_{\mathcal{E}_i} E_a, \mathcal{E}_j\right) g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a) = -\langle \tilde{\mathcal{A}}^\flat + \widetilde{\mathcal{K}}^\flat/2,\ B\rangle,$$

$$\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a)\, B\left(\nabla^t_{E_a}\mathcal{E}_i, \mathcal{E}_j\right)$$
$$= -\sum_{i,a} \epsilon_i\epsilon_a\, B\left(\tilde A_a(\mathcal{E}_i), A_i(E_a) + T_i^\sharp(E_a)\right) = -\langle \Lambda_{\tilde\alpha,\alpha+\theta}, B\rangle.$$

As an example, we give a detailed computation of the fourth line above:

$$\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a)\nabla^t_{\mathcal{E}_i} B(E_a, \mathcal{E}_j)$$
$$= \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, \nabla^t_i(g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a)B(E_a, \mathcal{E}_j))$$
$$- \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, B(E_a, \mathcal{E}_j)\nabla^t_i(g(\tilde h(\mathcal{E}_i, \mathcal{E}_j), E_a))$$
$$= \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, \nabla^t_i(g(B(E_a, \mathcal{E}_j)\tilde A_a\mathcal{E}_j, \mathcal{E}_i))$$
$$- \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, B(E_a, \mathcal{E}_j)\left(\nabla^t_i g\left(\tilde A_a\mathcal{E}_j, \mathcal{E}_i\right)\right)$$
$$= \sum_i \epsilon_i\, g\left(\nabla^t_i\left(\sum_{j,a}\epsilon_j\epsilon_a B(E_a, \mathcal{E}_j)\tilde A_a\mathcal{E}_j\right), \mathcal{E}_i\right)$$

$$
-\sum_{j,a}\epsilon_j\epsilon_a\, B(E_a,\mathcal{E}_j)g\left(\sum_i \epsilon_i\, \nabla^t_i \tilde{A}_a\mathcal{E}_j,\mathcal{E}_i\right)\Bigg)
$$

$$
= \operatorname{div}^{\perp}\langle B,\tilde{\alpha}\rangle - \langle B_{\,|\mathrm{V}},\operatorname{div}^{\perp}\tilde{\alpha}\rangle = \operatorname{div}\langle\tilde{\alpha},\ B\rangle - \langle(\operatorname{div}\tilde{\alpha})_{\,|\mathrm{V}},\ B\rangle.
$$

Note that while $\tilde{\alpha}=\tilde{\alpha}_{\,|\mathrm{V}}$, its divergence $\operatorname{div}\tilde{\alpha}$ may not vanish on $\mathcal{D}\times\mathcal{D}$ or $\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}$.

Proof of (14b). We compute for any $X\in T_xM$, using Lemmas 2 and 3,

$$
\begin{aligned}
g(\partial_t\tilde{H},X) &= \sum_i \epsilon_i\, g\left(\partial_t\left(\left(\nabla^t_{\mathcal{E}_i}\mathcal{E}_i\right)^{\top}\right),X\right) = \sum_i\epsilon_i\, g\left(\partial_t\left(\nabla^t_{\mathcal{E}_i}\mathcal{E}_i\right),X^{\top}\right)\\
&= \sum_i\epsilon_i\left(\nabla_{(-B^{\sharp}\mathcal{E}_i)^{\top}}\mathcal{E}_i - \nabla_{\left(\frac12 B^{\sharp}\mathcal{E}_i\right)^{\perp}}\mathcal{E}_i - \nabla_{\mathcal{E}_i}\left((B^{\sharp}\mathcal{E}_i)^{\top}\right)\right.\\
&\quad\left. -\nabla_{\mathcal{E}_i}\left(\left(\frac12 B^{\sharp}\mathcal{E}_i\right)^{\perp}\right) + (\partial_t\nabla^t)_{\mathcal{E}_i}\mathcal{E}_i,\, X^{\top}\right).
\end{aligned}\tag{17}
$$

Using known formula (10) for t-derivative of the Levi–Civita connection, see [17], we present $(17)_5$ (i.e., the fifth term in $g(\cdot,X^{\top})$ of (17)) as the following sum (we omit summation by i below):

$$
\begin{aligned}
g\left((\partial_t\nabla^t)_{\mathcal{E}_i}\,\mathcal{E}_i,\ X^{\top}\right) &= \nabla_{\mathcal{E}_i}\left(g\left(B^{\sharp}\mathcal{E}_i,X^{\top}\right)\right) - g\left(B^{\sharp}\mathcal{E}_i,\nabla_{\mathcal{E}_i}X^{\top}\right)\\
&\quad -\frac12\nabla_{X^{\top}}(B(\mathcal{E}_i,\mathcal{E}_i)) - B\left(\left(A_i+T^{\sharp}_i\right)(X^{\top}),\mathcal{E}_i\right)\\
&\quad + B\left((\nabla^t_{X^{\top}}\mathcal{E}_i)^{\perp},\mathcal{E}_i\right) - B\left(\nabla^t_{\mathcal{E}_i}\mathcal{E}_i,X^{\top}\right).
\end{aligned}\tag{18}
$$

The last two terms above, $(18)_5$ and $(18)_6$, vanish by the assumption $(\nabla^t_{X^{\top}}\mathcal{E}_i)^{\perp}=(\nabla^t_{\mathcal{E}_i}\mathcal{E}_i)^{\perp}=0$ and vanishing of B on $\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}$. We present the term $(17)_3$ as the sum of two terms

$$
-g\left(\nabla_{\mathcal{E}_i}((B^{\sharp}\mathcal{E}_i)^{\top}),X^{\top}\right) = -g\left(\nabla_{\mathcal{E}_i}\left(B^{\sharp}\mathcal{E}_i\right),X^{\top}\right) + g\left(\nabla_{\mathcal{E}_i}\left((B^{\sharp}\mathcal{E}_i)^{\perp}\right),X^{\top}\right),\tag{19}
$$

and then rewrite the term $(19)_2$ as

$$
\begin{aligned}
g\left(\nabla_{\mathcal{E}_i}((B^{\sharp}\mathcal{E}_i)^{\perp}),X^{\top}\right) &= -g\left((B^{\sharp}\mathcal{E}_i)^{\perp},\nabla_{\mathcal{E}_i}X^{\top}\right)\\
&= -\sum_j\epsilon_j\, g\left((B^{\sharp}\mathcal{E}_i)^{\perp},\mathcal{E}_j\right)g\left(\mathcal{E}_j,\nabla_{\mathcal{E}_i}X^{\top}\right)\\
&= \sum_j\epsilon_j\, B(\mathcal{E}_i,\mathcal{E}_j)\,g\left(\nabla_{\mathcal{E}_i}\mathcal{E}_j,X^{\top}\right)\\
&= \sum_j\epsilon_j\, B(\mathcal{E}_i,\mathcal{E}_j)\,g(\tilde{h}(\mathcal{E}_i,\mathcal{E}_j),X^{\top}).
\end{aligned}
$$

Note that $(18)_1$ + $(18)_2$ + $(19)_1$ =0. For the sum $(17)_2$ + $(17)_4$, we get

$$
g\left(-\nabla_{\left(\frac12 B^{\sharp}\mathcal{E}_i\right)^{\perp}}\mathcal{E}_i - \nabla_{\mathcal{E}_i}\left(\left(\frac12 B^{\sharp}\mathcal{E}_i\right)^{\perp}\right),X^{\top}\right) = -g\left(\tilde{h}\left(\mathcal{E}_i,B^{\sharp}\mathcal{E}_i\right),X^{\top}\right).
$$

For the term $(17)_1$, we get

$$
-\sum_i\epsilon_i\, g\left(\nabla_{(B^{\sharp}\mathcal{E}_i)^{\top}}\mathcal{E}_i,X^{\top}\right) = -\sum_{i,a}\epsilon_a\epsilon_i\, g\left(\nabla_{E_a}\mathcal{E}_i,X^{\top}\right)g\left(B^{\sharp}\mathcal{E}_i,E_a\right)
$$

$$= \sum_{i,a} \epsilon_a \epsilon_i\, g\left(\left(A_i + T_i^\sharp\right)(E_a), X^\top\right)\, B(\mathcal{E}_i, E_a)$$
$$= \langle\, \langle \alpha + \theta, X^\top \rangle,\ B \rangle.$$

For the term (18)$_3$, we obtain

$$-\sum_i \epsilon_i\, \nabla_{X^\top}(B(\mathcal{E}_i, \mathcal{E}_i)) = -X^\top(\mathrm{Tr}\,_{\mathcal{D}} B).$$

For the term (18)$_4$, we get

$$\begin{aligned}
-\sum_i \epsilon_i\, B\left(\left(A_i + T_i^\sharp\right)(X^\top), \mathcal{E}_i\right) &= -\sum_i \epsilon_i\, g\left(\left(A_i + T_i^\sharp\right)(X^\top), B^\sharp(\mathcal{E}_i)\right)\\
&= -\sum_{i,a} \epsilon_i \epsilon_a\, g\left(B^\sharp\left(\mathcal{E}_i\right), E_a\right)\, g\left(\left(A_i + T_i^\sharp\right)(X^\top), E_a\right)\\
&= -\sum_{i,a} \epsilon_i \epsilon_a\, B\left(\mathcal{E}_i, E_a\right)\, g\left(\left(A_i - T_i^\sharp\right)(E_a), X^\top\right)\\
&= \langle\, \langle \theta - \alpha, X^\top \rangle,\ B \rangle.
\end{aligned}$$

Finally, we collect results: (17) = (17)$_1$ + (17)$_2$ + (19)$_1$ + (19)$_2$ + (17)$_4$ + (18)$_1$ + (18)$_2$ + (18)$_3$ + (18)$_4$ + (18)$_5$ + (18)$_6$ to obtain

$$g(\partial_t \tilde H, X) = \langle\, 2\langle \theta, X^\top \rangle,\ B \rangle - \frac{1}{2}\, X^\top(\mathrm{Tr}\,_{\mathcal{D}} B). \tag{20}$$

Let $X = \tilde H$. Using $B(\tilde H, \tilde H) = 0$ and $\tilde H(\mathrm{Tr}\,_{\mathcal{D}} B^\sharp) = \mathrm{div}((\mathrm{Tr}\,_{\mathcal{D}} B^\sharp)\tilde H) - (\mathrm{Tr}\,_{\mathcal{D}} B^\sharp)\,\mathrm{div}\,\tilde H$, we get

$$\partial_t g(\tilde H, \tilde H) = 2\, g(\partial_t \tilde H, \tilde H) = \langle\, 4\, \langle \theta, \tilde H \rangle,\ B \rangle - \mathrm{div}((\mathrm{Tr}\,_{\mathcal{D}} B^\sharp)\tilde H) + (\mathrm{Tr}\,_{\mathcal{D}} B)\,\mathrm{div}\,\tilde H.$$

Finally note that $\mathrm{Tr}\,_{\mathcal{D}} B = \langle g, B \rangle$, and that completes the proof of (14b).

The computations for h and H are easier, since $B(X, Y) = 0$ for $X, Y \in \widetilde{\mathcal{D}}$.

Proof of (14c). We observe that

$$\partial_t \langle h, h \rangle = \sum_{a,b} \epsilon_a \epsilon_b \left(B(h(E_a, E_b), h(E_a, E_b)) + 2\, g(\partial_t h(E_a, E_b), h(E_a, E_b))\right),$$

where, using Lemma 3 and formula (10) for $\partial_t \nabla^t$, we compute

$$\begin{aligned}
g(\partial_t h(E_a, E_b), h(E_a, E_b)) &= \frac{1}{2}\, g\left(\partial_t\left(\left(\nabla^t_{E_a} E_b + \nabla^t_{E_b} E_a\right)^\perp\right), h(E_a, E_b)\right)\\
&= \frac{1}{2}\, g\left(\partial_t\left(\nabla^t_{E_a} E_b + \nabla^t_{E_b} E_a\right), h(E_a, E_b)\right)\\
&= \frac{1}{2}\, g\left((\partial_t \nabla^t)_{E_a} E_b + (\partial_t \nabla^t)_{E_b} E_a, h(E_a, E_b)\right)\\
&= \frac{1}{2} \sum_i \epsilon_i\, g(h(E_a, E_b), \mathcal{E}_i)\Big(\nabla^t_{E_a} B(\mathcal{E}_i, E_b) + \nabla^t_{E_b} B(\mathcal{E}_i, E_a)\\
&\quad - 2\, B(h(E_a, E_b), \mathcal{E}_i) + B\left(\nabla^t_{\mathcal{E}_i} E_a, E_b\right) + B\left(\nabla^t_{\mathcal{E}_i} E_b, E_a\right)\Big).
\end{aligned}$$

We used in the above

$$
\begin{aligned}
2\, g((\partial_t \nabla^t)_{E_a} E_b, \mathcal{E}_i) = {} & \nabla^t_{E_a} B(\mathcal{E}_i, E_b) + \nabla^t_{E_b} B(\mathcal{E}_i, E_a) \\
& - 2B(h(E_a, E_b), \mathcal{E}_i) + B\left(\nabla^t_{\mathcal{E}_i} E_a, E_b\right) + B\left(\nabla^t_{\mathcal{E}_i} E_b, E_a\right) \\
& - B\left(\left(\nabla^t_{E_a} E_b + \nabla^t_{E_b} E_a\right)^\top, \mathcal{E}_i\right) - \nabla_{\mathcal{E}_i} B(E_a, E_b) - B\left(\nabla_{E_a} \mathcal{E}_i, E_b\right) \\
& - B\left(\nabla_{E_b} \mathcal{E}_i, E_a\right).
\end{aligned}
$$

and the assumptions $\left(\nabla^t_{E_a} E_b\right)^\top = \left(\nabla^t_{E_b} E_a\right)^\top = 0$, $\left(\nabla_{E_b} \mathcal{E}_i\right)^\perp = \left(\nabla_{E_a} \mathcal{E}_i\right)^\perp = 0$ and $B(X, Y) = 0$ for $X, Y \in \widetilde{\mathcal{D}}$, due to which the last four terms in the formula above vanish. Note that

$$
\begin{aligned}
& \sum_{a,b,i} \epsilon_a \epsilon_b \epsilon_i\, g(h(E_a, E_b), \mathcal{E}_i) \nabla^t_{E_a} B(E_b, \mathcal{E}_i) = \operatorname{div}\langle B, \alpha\rangle - \langle (\operatorname{div} \alpha)_{|\mathrm{V}},\ B\rangle, \\
& \sum_{a,b,i} \epsilon_a \epsilon_b \epsilon_i\, g(h(E_a, E_b), \mathcal{E}_i) B(\nabla^t_{\mathcal{E}_i} E_a, E_b) = -\langle \Lambda_{\alpha, \tilde\alpha + \tilde\theta}, B\rangle.
\end{aligned}
$$

Finally, we obtain (14c):

$$
\begin{aligned}
\partial_t \langle h, h\rangle = {} & \sum_{a,b} \epsilon_a \epsilon_b B(h(E_a, E_b), h(E_a, E_b)) + 2\, \operatorname{div}\langle B, \alpha\rangle - 2\, \langle (\operatorname{div} \alpha)_{|\mathrm{V}},\ B\rangle \\
& -2 \sum_{a,b} \epsilon_a \epsilon_b B(h(E_a, E_b), h(E_a, E_b)) - 2\, \langle \Lambda_{\alpha, \tilde\alpha + \tilde\theta},\ B\rangle.
\end{aligned}
$$

Proof of (14d). We observe that

$$
\partial_t g(H, H) = B(H, H) + 2\, g(\partial_t H, H).
$$

For arbitrary $X \in T_x M$, using Lemma 3 and formula (10) for $\partial_t \nabla^t$, we obtain

$$
\begin{aligned}
g(\partial_t H, X) & = \sum_a \epsilon_a g\left(\partial_t \left(\left(\nabla^t_{E_a} E_a\right)^\perp\right), X\right) \\
& = \sum_a \epsilon_a g\left(\left(\partial_t \nabla^t\right)_{E_a} E_a, X^\perp\right) - \sum_a \epsilon_a g\left(B^\sharp\left(\left(\nabla^t_{E_a} E_a\right)^\perp\right), X^\top\right) \\
& = \sum_a \epsilon_a \left(\nabla^t_{E_a} B\right)\left(E_a, X^\perp\right) - \frac{1}{2} \sum_a \epsilon_a \left(\nabla^t_{X^\perp} B\right)(E_a, E_a) - B(H, X^\top).
\end{aligned}
$$

We have

$$
\begin{aligned}
\sum_a \epsilon_a \left(\nabla^t_{E_a} B\right)(E_a, X^\perp) = {} & \sum_a \epsilon_a \nabla^t_{E_a} g(B^\sharp(X^\perp), E_a) \\
& - \sum_a \epsilon_a B\left(\nabla^t_{E_a} E_a, X^\perp\right) - \sum_a \epsilon_a g\left(B^\sharp E_a, \nabla^t_{E_a} X^\perp\right) \\
= {} & \sum_a \epsilon_a g\left(\nabla^t_{E_a} (B^\sharp(X^\perp)), E_a\right) + g\left(B^\sharp(X^\perp), H\right) - B(H, X^\perp) \\
& - \sum_{a,i} \epsilon_a \epsilon_i B(E_a, \mathcal{E}_i) g\left(\nabla^t_{E_a} X^\perp, \mathcal{E}_i\right) \\
= {} & \operatorname{div}(B^\sharp(X^\perp))^\top + g(B^\sharp(X^\perp), \tilde H) - g(B^\sharp(X^\perp), H) - \langle \tilde\delta_{X^\perp}, B\rangle.
\end{aligned}
$$

Also, for $g^\perp$-variations

$$\frac{1}{2}\sum_a \epsilon_a \left(\nabla^t_{X^\perp} B\right)(E_a, E_a) = -\sum_a \epsilon_a\, B\left(\nabla^t_{X^\perp} E_a, E_a\right) = \langle\langle\tilde\alpha - \tilde\theta, X^\perp\rangle, B\rangle;$$

hence,

$$g(\partial_t H, X) = \operatorname{div}(B^\sharp(X^\perp))^\top + g(B^\sharp(X^\perp), \tilde H) - g(B^\sharp(X^\perp), H) - \langle\tilde\delta_{X^\perp}, B\rangle \\ - \langle\langle\tilde\alpha - \tilde\theta, X^\perp\rangle, B\rangle - B(H, X^\top). \tag{21}$$

It follows that

$$\partial_t g(H, H) = B(H, H) + 2\left(\langle\langle\tilde\theta - \tilde\alpha, H\rangle, B\rangle + \operatorname{div}(B^\sharp H)^\top - B(H, H)\right. \\ \left. + B(H, \tilde H) - \langle\tilde\delta_H,\ B\rangle\right)$$

Finally, using $B(H, \tilde H) = \langle\operatorname{Sym}(H^\flat \otimes \tilde H^\flat),\ B\rangle$, we obtain (14d).
Proof of (14e). We compute

$$\partial_t\langle\tilde T, \tilde T\rangle = \sum_{i,j}\epsilon_i\epsilon_j\, \partial_t g(\tilde T(\mathcal{E}_i, \mathcal{E}_j), \tilde T(\mathcal{E}_i, \mathcal{E}_j)) \\ = \sum_{i,j}\epsilon_i\epsilon_j\left(B(\tilde T(\mathcal{E}_i, \mathcal{E}_j), \tilde T(\mathcal{E}_i, \mathcal{E}_j)) + 2\, g(\partial_t\tilde T(\mathcal{E}_i, \mathcal{E}_j), \tilde T(\mathcal{E}_i, \mathcal{E}_j))\right).$$

For the last term of the above, by symmetry $(\partial_t\nabla^t)_{\mathcal{E}_i}\mathcal{E}_j = (\partial_t\nabla^t)_{\mathcal{E}_j}\mathcal{E}_i$ and omitting sum, we get

$$2\, g(\partial_t\tilde T(\mathcal{E}_i, \mathcal{E}_j), \tilde T(\mathcal{E}_i, \mathcal{E}_j)) = g\left(\partial_t\left(\left(\nabla^t_{\mathcal{E}_i}\mathcal{E}_j - \nabla^t_{\mathcal{E}_j}\mathcal{E}_i\right)^\top\right), \tilde T(\mathcal{E}_i, \mathcal{E}_j)\right) \\ = g\left(\partial_t\left(\nabla^t_{\mathcal{E}_i}\mathcal{E}_j - \nabla^t_{\mathcal{E}_j}\mathcal{E}_i\right), \tilde T(\mathcal{E}_i, \mathcal{E}_j)\right) \\ + g\left(B^\sharp\left(\left(\nabla^t_{\mathcal{E}_i}\mathcal{E}_j - \nabla^t_{\mathcal{E}_j}\mathcal{E}_i\right)^\perp\right), \tilde T(\mathcal{E}_i, \mathcal{E}_j)\right) \\ = g\Big(\nabla_{(\partial_t\mathcal{E}_i)^\top}\mathcal{E}_j + \nabla_{(\partial_t\mathcal{E}_i)^\perp}\mathcal{E}_j + \nabla_{\mathcal{E}_i}((\partial_t\mathcal{E}_j)^\top) + \nabla_{\mathcal{E}_i}((\partial_t\mathcal{E}_j)^\perp) \\ - \nabla_{(\partial_t\mathcal{E}_j)^\top}\mathcal{E}_i - \nabla_{(\partial_t\mathcal{E}_j)^\perp}\mathcal{E}_i - \nabla_{\mathcal{E}_j}((\partial_t\mathcal{E}_i)^\top) \\ - \nabla_{\mathcal{E}_j}((\partial_t\mathcal{E}_i)^\perp),\ \tilde T(\mathcal{E}_i, \mathcal{E}_j)\Big), \tag{22}$$

where we have used the assumption $(\nabla^t_{\mathcal{E}_i}\mathcal{E}_j)^\perp = (\nabla^t_{\mathcal{E}_j}\mathcal{E}_i)^\perp = 0$. We will compute eight terms in (22) separately. First we calculate

$$g\left(\tilde T\left((\partial_t\mathcal{E}_i)^\perp, \mathcal{E}_j\right), X^\top\right) = -\frac{1}{2}\, g\left(\tilde T\left(\left(B^\sharp\mathcal{E}_i\right)^\perp, \mathcal{E}_j\right), X^\top\right) \\ = -\frac{1}{2}\sum_a \epsilon_a\, g\left(\tilde T\left(\left(B^\sharp\mathcal{E}_i\right)^\perp, \mathcal{E}_j\right), E_a\right)\, g(E_a, X^\top) \\ = -\frac{1}{2}\sum_a \epsilon_a\, g\left(\tilde T^\sharp_a\left(B^\sharp\mathcal{E}_i\right), \mathcal{E}_j\right)\, g(E_a, X^\top).$$

Then, assuming $X = \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)$, we find the sum (22)$_2$ + (22)$_8$:

$$
\begin{aligned}
&2\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a \left(-\frac{1}{2}\right) g\left(\tilde{T}_a^\sharp(B^\sharp\mathcal{E}_i), \mathcal{E}_j\right) g\left(E_a, \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)\right)\\
&\quad= \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\tilde{T}_a^\sharp\mathcal{E}_j, B^\sharp\mathcal{E}_i\right) g\left(\tilde{T}_a^\sharp\mathcal{E}_i, \mathcal{E}_j\right)\\
&\quad= \sum_{i,j,a,k} \epsilon_i\epsilon_j\epsilon_a\epsilon_k\, g\left(\tilde{T}_a^\sharp\mathcal{E}_j, \mathcal{E}_k\right) B(\mathcal{E}_i, \mathcal{E}_k)\, g\left(\tilde{T}_a^\sharp\mathcal{E}_i, \mathcal{E}_j\right)\\
&\quad= -\sum_{i,a,k} \epsilon_i\epsilon_a\epsilon_k\, g\left(\tilde{T}_a^\sharp\mathcal{E}_k, \tilde{T}_a^\sharp\mathcal{E}_i\right) B(\mathcal{E}_i, \mathcal{E}_k)\\
&\quad= \sum_{i,a,k} \epsilon_i\epsilon_a\epsilon_k\, g\left(\left(\tilde{T}_a^\sharp\right)^2 \mathcal{E}_k, \mathcal{E}_i\right) B(\mathcal{E}_i, \mathcal{E}_k).
\end{aligned}
$$

For (22)$_4$ + (22)$_6$ we have the same, thus (22)$_2$ + (22)$_4$ + (22)$_6$ + (22)$_8$ = $2\,\langle\tilde{\mathcal{T}},\ B\rangle$.

For (22)$_1$, which is equal to (22)$_5$, we have

$$
\begin{aligned}
&-\sum_{i,j} \epsilon_i\epsilon_j\, g\left(\nabla_{(B^\sharp\mathcal{E}_i)^\top}\mathcal{E}_j, \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)\right)\\
&\quad= -\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\nabla_{E_a}\mathcal{E}_j, \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)\right) B(\mathcal{E}_i, E_a)\\
&\quad= \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\left(A_j + T_j^\sharp\right) E_a, \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)\right) B(\mathcal{E}_i, E_a)\\
&\quad= \sum_{i,j,a,b} \epsilon_i\epsilon_j\epsilon_a\epsilon_b\, g\left(\left(A_j + T_j^\sharp\right) E_a, E_b\right) g\left(\tilde{T}_b^\sharp\mathcal{E}_i, \mathcal{E}_j\right) B(\mathcal{E}_i, E_a)\\
&\quad= -\sum_{i,j,b} \epsilon_i\epsilon_j\epsilon_b\, B\left(\mathcal{E}_i, \left(A_j - T_j^\sharp\right) E_b\right) g\left(\tilde{T}_b^\sharp\mathcal{E}_j, \mathcal{E}_i\right)\\
&\quad= -\sum_{j,b} \epsilon_j\epsilon_b\, B\left(\tilde{T}_b^\sharp\mathcal{E}_j, \left(A_j - T_j^\sharp\right) E_b\right) = \langle\Lambda_{\tilde{\theta},\theta-\alpha},\ B\rangle.
\end{aligned}
$$

Thus, (22)$_1$ + (22)$_5$ = $\langle 2\,\Lambda_{\tilde{\theta},\theta-\alpha},\ B\rangle$. For the term (22)$_3$, we have

$$
\begin{aligned}
&\sum_{i,j} \epsilon_i\epsilon_j\, g\left(\nabla_{\mathcal{E}_i}\left(\left(-B^\sharp\mathcal{E}_j\right)^\top\right), \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)\right)\\
&\quad= \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, g\left(\nabla_{\mathcal{E}_i}\left(\left(-B^\sharp\mathcal{E}_j\right)^\top\right), E_a\right) g\left(\tilde{T}(\mathcal{E}_i, \mathcal{E}_j), E_a\right)\\
&\quad= -\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a \left(\nabla_{\mathcal{E}_i} g\left(\left(B^\sharp\mathcal{E}_j\right)^\top, E_a\right) - g\left(\left(B^\sharp\mathcal{E}_j\right)^\top, \nabla_{\mathcal{E}_i} E_a\right)\right) g\left(\tilde{T}_a^\sharp(\mathcal{E}_i), \mathcal{E}_j\right)\\
&\quad= -\sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a \left(\nabla_{\mathcal{E}_i}\left(g(B(\mathcal{E}_j, E_a)\left(-\tilde{T}_a^\sharp\mathcal{E}_j\right), \mathcal{E}_i)\right) - B(\mathcal{E}_j, E_a)\nabla_{\mathcal{E}_i} g\left(-\tilde{T}_a^\sharp\mathcal{E}_j, \mathcal{E}_i\right)\right)\\
&\quad= \operatorname{div}^\perp\langle\tilde{\theta}, B\rangle - \sum_{i,j,a} \epsilon_i\epsilon_j\epsilon_a\, B(\mathcal{E}_j, E_a)\nabla_{\mathcal{E}_i} g\left(\tilde{T}_a^\sharp\mathcal{E}_j, \mathcal{E}_i\right)\\
&\qquad + g\left(\left(B^\sharp\mathcal{E}_j\right)^\top, \nabla^t_{\mathcal{E}_i} E_a\right) g\left(\tilde{T}_a^\sharp\mathcal{E}_i, \mathcal{E}_j\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname{div}^{\perp}\langle\tilde\theta, B\rangle - \langle\, (\operatorname{div}^{\perp}\tilde\theta)_{\,|\mathrm{V}}, B\rangle\\
&= \operatorname{div}\langle\tilde\theta, B\rangle + \langle\,\langle\tilde\theta, B\rangle, H\rangle - \langle(\operatorname{div}\tilde\theta)_{\,|\mathrm{V}}, B\rangle - \langle\,\langle\tilde\theta, B\rangle, H\rangle,
\end{aligned}
$$

where we have used the assumption $\left(\nabla^t_{\mathcal{E}_i} E_a\right)^{\top} = 0$ to remove the second term of the third line above. Thus, (22)$_3$ + (22)$_7$ $= 2\,\operatorname{div}\langle\tilde\theta, B\rangle - 2\,\langle(\operatorname{div}\tilde\theta)_{\,|\mathrm{V}},\ B\rangle$. Using the above, we obtain (14e).

Proof of (14f). We calculate using Lemma 3,

$$
\begin{aligned}
g(\partial_t T(E_a, E_b), X^{\perp}) &= \frac{1}{2}\, g\left(\partial_t\left(\left(\nabla^t_{E_a} E_b - \nabla^t_{E_b} E_a\right)^{\perp}\right), X^{\perp}\right)\\
&= \frac{1}{2}\, g\left(\partial_t\left(\nabla^t_{E_a} E_b - \nabla^t_{E_b} E_a\right), X^{\perp}\right)\\
&\quad -\frac{1}{2}\, g\left(B^{\sharp}\left(\left(\nabla^t_{E_a} E_b - \nabla^t_{E_b} E_a\right)^{\perp}\right), (X^{\perp})^{\top}\right) = 0.
\end{aligned}
$$

Then we obtain (14f):

$$
\begin{aligned}
\partial_t\langle T, T\rangle &= \partial_t \sum_{a,b} \epsilon_a\epsilon_b\, g(T(E_a, E_b), T(E_a, E_b))\\
&= \sum_{a,b} \epsilon_a\epsilon_b \left(B(T(E_a, E_b), T(E_a, E_b)) + 2\, g(\partial_t T(E_a, E_b), T(E_a, E_b))\right)\\
&= \sum_{a,b} \epsilon_a\epsilon_b\, B(T(E_a, E_b), T(E_a, E_b)) = -\langle\Phi_T,\ B\rangle.
\end{aligned}
$$

This completes the proof. □

Corollary 1 *For $g^{\perp}$-variations, we have*

$$
\begin{aligned}
\partial_t\,\widetilde{\mathrm{S}}_{\,\mathrm{ex}} &= \langle(\operatorname{div}\tilde H)\, g^{\perp} + 4\langle\theta,\ \tilde H\rangle - \operatorname{div}\tilde h + 4\,\Lambda_{\tilde\alpha,\theta} - \widetilde{\mathcal{K}}^{\flat},\ B\rangle\\
&\quad + \operatorname{div}(\langle\tilde h, B\rangle - (\operatorname{Tr}_{\mathcal{D}} B)\tilde H), \qquad (23a)\\
\partial_t\,\mathrm{S}_{\,\mathrm{ex}} &= \langle -\Phi_h + 2\langle\tilde\theta - \tilde\alpha,\ H\rangle + 2\,\operatorname{Sym}(H^{\flat}\otimes\tilde H^{\flat}) - 2\,\tilde\delta_H + 2(\operatorname{div}\alpha)_{\,|\mathrm{V}}\\
&\quad + 2\,\Lambda_{\alpha,\tilde\alpha+\tilde\theta},\ B\rangle + 2\operatorname{div}\left((B^{\sharp} H)^{\top} - \langle\alpha, B\rangle\right). \qquad (23b)
\end{aligned}
$$

Proof Formula (23a) follows from (14a) and (14b), and (23b) follows from (14c) and (14d). □

Similarly as Lemma 2, one can prove the following

Lemma 4 *Let $\{E_a,\ \mathcal{E}_i\}$ be a local $(\widetilde{\mathcal{D}},\ \mathcal{D})$-adapted and g-orthonormal frame. For any variation g_t, the frame evolving according to equations:*

$$
\begin{aligned}
E_a(0) = E_a,\quad \partial_t E_a &= -\frac{1}{2}(B^{\sharp} E_a)^{\top},\\
\mathcal{E}_i(0) = \mathcal{E}_i,\quad \partial_t \mathcal{E}_i &= -(B^{\sharp}\mathcal{E}_i)^{\top} - \frac{1}{2}(B^{\sharp}\mathcal{E}_i)^{\perp},
\end{aligned}
$$

where $B = \partial_t g_t$, remains an orthonormal frame adapted to $\widetilde{\mathcal{D}}$ and $\mathcal{D}(t)$.

For any $g^{\top}$-variation, the frame evolving according to equations:

$$E_a(0) = E_a, \quad \partial_t E_a = -\frac{1}{2}(B^\sharp E_a)^\top,$$
$$\mathcal{E}_i(0) = \mathcal{E}_i, \quad \partial_t \mathcal{E}_i = 0,$$

where $B = \partial_t g_t$, remains an orthonormal frame adapted to $\widetilde{\mathcal{D}}$ and $\mathcal{D}$.

Lemma 3 remains true without any changes for both g_t- and $g^\top$-variations, and Proposition 1 has the following analogue.

Proposition 2 *For $g^\top$-variations of g, we have*

$$\partial_t \langle \tilde{h}, \tilde{h} \rangle = \langle \Phi_{\tilde{h}}, B \rangle - B(\tilde{H}, \tilde{H}), \tag{24a}$$
$$\partial_t g(\tilde{H}, \tilde{H}) = -B(\tilde{H}, \tilde{H}), \tag{24b}$$
$$\partial_t \langle h, h \rangle = \langle \operatorname{div} h + \mathcal{K}^\flat,\ B \rangle - \operatorname{div}\langle h, B \rangle, \tag{24c}$$
$$\partial_t g(H, H) = \langle\, (\operatorname{div} H)\, g^\top,\ B \rangle - \operatorname{div}\big((\operatorname{Tr}_{\,\widetilde{\mathcal{D}}} B^\sharp)\, H \big), \tag{24d}$$
$$\partial_t \langle \tilde{T}, \tilde{T} \rangle = -\langle \Phi_{\tilde{T}}, B \rangle, \tag{24e}$$
$$\partial_t \langle T, T \rangle = 2\, \langle\, \mathcal{T}^\flat,\ B\, \rangle. \tag{24f}$$

Proof The claim follows in fact from the computations that we already did in the proof of Proposition 1. Careful comparison of Lemmas 2 and 4 indicates that in order to obtain (24a)–(24f) it is enough to take formulas dual (with respect to interchanging $\widetilde{\mathcal{D}}$ and $\mathcal{D}$) to (14a)–(14f) and assume in them that $B = B_{|\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}}$. □

Remark 3 Note that $g^\top$-variations coincide with one of two families of *adapted variations* considered in a previous paper of the authors [14] and in [2]. The adapted variations are a special case of variations considered in this paper—as they are additionally required to keep the distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}(0)$ orthogonal for all g_t. Here we make no such assumption, allowing the g_t-orthogonal complement of the distribution $\widetilde{\mathcal{D}}$ to vary, which enables us to consider arbitrary variations of the metric. Indeed, one can prove that variation formulas for general variations g_t are sums of the corresponding formulas from Propositions 1 and 2. This follows from the fact that every infinitesimal variation of g can be decomposed into the sum of infinitesimal $g^\perp$- and $g^\top$-variations. Such decomposition would not be possible with the use of adapted variations only.

As the last of technical tools that we shall use, we note the following formula for variation of the volume form, true for any variation of a metric g_t with $B = \partial_t g_{t|t=0}$ [17]:

$$\partial_t \mathrm{d}\,\mathrm{vol}_{g|t=0} = \frac{1}{2}(\operatorname{Tr}_{\,g} B)\, \mathrm{d}\,\mathrm{vol}_g\,. \tag{25}$$

2.3 Euler–Lagrange equation

In this section, we present the Euler–Lagrange equation for the action (2). We consider different kinds of variations of metric. For arbitrary variations of the metric, the Euler–Lagrange equation is simply a condition for vanishing of the gradient of the functional: $\delta J_{\mathrm{mix},\widetilde{\mathcal{D}},\Omega}(g)$, where

$$\frac{\mathrm{d}}{\mathrm{dt}}\, J_{\mathrm{mix},\widetilde{\mathcal{D}},\Omega}(g_t)_{\,|\,t=0} = \int_\Omega \langle \delta J_{\mathrm{mix},\widetilde{\mathcal{D}},\Omega},\, B \rangle \mathrm{d}\,\mathrm{vol}_g$$

for any variation g_t with $B = \partial_t g_{t|t=0}$. In analogue to the Einstein–Hilbert action, one can also consider variations preserving the volume of Ω. For such variations, using (25), we have

$$0 = \partial_t \int_M \mathrm{d\,vol}_g = \int_M \partial_t \mathrm{d\,vol}_g = \int_M \frac{1}{2}(\mathrm{Tr}\ B)\mathrm{d\,vol}_g = \frac{1}{2}\int_\Omega \langle g, B\rangle \mathrm{d\,vol}_g .$$

Hence, metric g is critical for the volume-preserving variations if and only if the condition

$$\int_\Omega \langle \delta J_{\mathrm{mix},\widetilde{\mathcal{D}},\Omega}, B\rangle \mathrm{d\,vol}_g = 0$$

holds for all B satisfying $\int_\Omega \langle g, B\rangle = 0$. It follows that the Euler–Lagrange equation is now

$$\delta J_{\mathrm{mix},\widetilde{\mathcal{D}},\Omega} = \lambda g, \tag{26}$$

where $\lambda \in \mathbb{R}$ is an arbitrary constant [3] (i.e., every metric satisfying it with some constant $\lambda \in \mathbb{R}$ is critical). Unfortunately, in our case the functional $J_{\mathrm{mix},\widetilde{\mathcal{D}},\Omega}$ is not a Riemannian functional (i.e., it is not invariant under all diffeomorphisms of M), hence we cannot take as λ an arbitrary function [3]. Note that the Euler–Lagrange equation for arbitrary variations is a special case of (26), with $\lambda = 0$.

We can also consider volume-preserving $g^\top$- and $g^\perp$-variations. For $g^\top$-variations, B is restricted to $\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$ and for $g^\perp$-variations B vanishes on $\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$. Hence, the Euler–Lagrange equation is still (26), only either restricted to $\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}$ (for $g^\top$-variations) or considered everywhere except $\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$ (for $g^\perp$-variations).

Theorem 1 (Euler–Lagrange equation) *A metric* $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ *is critical for the action* (2) *with respect to volume-preserving* $g^\perp$*-variations if and only if*

$$r_{\mathcal{D}} - \langle \tilde h, \tilde H\rangle + \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat + \Phi_h + \Phi_T + \Psi - \mathrm{Def}_{\mathcal{D}}(H) + \widetilde{\mathcal{K}}^\flat$$
$$-\frac{1}{2}\left(\mathrm{S}_{\mathrm{mix}} + \mathrm{div}(\tilde H - H)\right) g^\perp = \lambda g^\perp, \tag{27a}$$
$$2\langle \theta, \tilde H\rangle + (\mathrm{div}(\alpha - \tilde\theta))_{|\mathrm{V}} + \langle \tilde\theta - \tilde\alpha, H\rangle + \mathrm{Sym}(H^\flat \otimes \tilde H^\flat)$$
$$-\tilde\delta_H + 2\Lambda_{\tilde\alpha,\theta} + \Lambda_{\alpha,\tilde\alpha} + \Lambda_{\theta,\tilde\theta} = 0. \tag{27b}$$

A metric $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ *is critical for the action* (2) *with respect to volume-preserving* $g^\top$*-variations if and only if*

$$r_{\widetilde{\mathcal{D}}} - \langle h, H\rangle + \mathcal{A}^\flat - \mathcal{T}^\flat + \Phi_{\tilde h} + \Phi_{\tilde T} + \widetilde\Psi - \mathrm{Def}_{\widetilde{\mathcal{D}}}(\tilde H) + \mathcal{K}^\flat$$
$$-\frac{1}{2}\left(\widetilde{\mathrm{S}}_{\mathrm{mix}} + \mathrm{div}(H - \tilde H)\right) g^\top = \lambda g^\top. \tag{27c}$$

Proof Let g_t be a $g^\perp$-variation, and let $Q(g) := \mathrm{S}_{\mathrm{mix}} - \mathrm{div}(H + \tilde H)$. Then

$$\frac{\mathrm d}{\mathrm{dt}} J_{\mathrm{mix},\widetilde{\mathcal{D}},\Omega}(g_t)_{\,|\,t=0} = \frac{\mathrm d}{\mathrm{dt}}\int_\Omega Q(g_t)\,\mathrm{d\,vol}_{g_t\,|\,t=0} + \frac{\mathrm d}{\mathrm{dt}}\int_\Omega \mathrm{div}(H + \tilde H)\,\mathrm{d\,vol}_{g_t\,|\,t=0} .$$

Differentiating the formula $\mathrm{div}\,X\cdot\mathrm{d\,vol}_g = \mathcal{L}_X(\mathrm{d\,vol}_g)$, and using (25) we obtain $\partial_t(\,\mathrm{div}\,X) = \mathrm{div}(\partial_t X) + (1/2)\,X(\mathrm{Tr}\ B^\sharp)$ for any t-dependent vector field X. In particular, it follows that

$$\frac{\mathrm d}{\mathrm{dt}}\int_\Omega \mathrm{div}(H + \tilde H)\,\mathrm{d\,vol}_{g_t} = \int_\Omega \partial_t\,(\mathrm{div}(H + \tilde H))\,\mathrm{d\,vol}_{g_t} + \int_\Omega \mathrm{div}(H + \tilde H)\,\partial_t\,(\mathrm{d\,vol}_{g_t})$$
$$= \int_\Omega \mathrm{div}(\partial_t(H + \tilde H))\,\mathrm{d\,vol}_{g_t}$$

$$
\begin{aligned}
&+\int_\Omega (1/2)\,((H+\tilde H)(\operatorname{Tr}\ B^\sharp))\,\mathrm{d}\operatorname{vol}_{g_t}\\
&+\int_\Omega (1/2)(\operatorname{Tr}\ B^\sharp)\operatorname{div}(H+\tilde H)\,(\mathrm{d}\operatorname{vol}_{g_t})\\
=&\int_\Omega \operatorname{div}(\partial_t(H+\tilde H))\,\mathrm{d}\operatorname{vol}_{g_t}+(1/2)\\
&\int_\Omega \operatorname{div}((\operatorname{Tr}\ B^\sharp)\cdot(H+\tilde H))\,\mathrm{d}\operatorname{vol}_{g_t}.
\end{aligned}
$$

For $g^\perp$-variations supported inside Ω, it follows from (20) and (21) that both fields $\partial_t(H+\tilde H)$ and $(\operatorname{Tr}\ B^\sharp)\cdot(H+\tilde H)$ vanish on $\partial\Omega$, and hence $\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \operatorname{div}(H+\tilde H)\,\mathrm{d}\operatorname{vol}_g=0$. We have therefore

$$
\frac{\mathrm{d}}{\mathrm{dt}}\,J_{\mathrm{mix},\widetilde{\mathcal D},\Omega}(g_t)_{\,|\,t=0}=\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega Q(g_t)\,\mathrm{d}\operatorname{vol}_{g_t\,|\,t=0},
$$

and $Q(g)$ can be presented using (9) as

$$
Q(g)=\ \mathrm{S}_{\,\mathrm{ex}}(g)+\widetilde{\mathrm{S}}_{\,\mathrm{ex}}(g)+\langle T,T\rangle_g+\langle\tilde T,\tilde T\rangle_g\,. \tag{28}
$$

Applying Corollary 1 and Proposition 1 to (28), using (5) and removing integrals of divergences of vector fields compactly supported in Ω, we get

$$
\begin{aligned}
\int_\Omega \partial_t Q(g_t)_{\,|\,t=0}\,\mathrm{d}\operatorname{vol}_g=&\int_\Omega\big\langle 4\Lambda_{\tilde\alpha,\theta}-\operatorname{div}\tilde h-\widetilde{\mathcal K}^\flat-\Phi_h-\Phi_T+2\,\tilde{\mathcal T}^\flat+4\,\langle\theta,\ \tilde H\rangle\\
&+(\operatorname{div}\tilde H)g^\perp+2(\operatorname{div}\alpha)_{\,|\mathrm{V}}+2\,\Lambda_{\alpha,\tilde\alpha+\tilde\theta}+2\,\langle\tilde\theta-\tilde\alpha,H\rangle\\
&+2\operatorname{Sym}(H^\flat\otimes\tilde H^\flat)-2\,\tilde\delta_H+2\Lambda_{\tilde\theta,\theta-\alpha}-2(\operatorname{div}\tilde\theta)_{\,|\mathrm{V}},\ B\big\rangle\,\mathrm{d}\operatorname{vol}_g,
\end{aligned}
\tag{29}
$$

where $B=\{\partial_t g_t\}_{\,|\,t=0}$. Since

$$
\frac{\mathrm{d}}{\mathrm{dt}}\,J_{\mathrm{mix},\widetilde{\mathcal D},\Omega}(g_t)_{|\,t=0}=\int_\Omega \partial_t Q(g_t)_{\,|\,t=0}\,\mathrm{d}\operatorname{vol}_g+\int_\Omega Q(g)\,(\partial_t\mathrm{d}\operatorname{vol}_{g_t\,|\,t=0}),
$$

by (29) and (25), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\,J_{\mathrm{mix},\widetilde{\mathcal D},\Omega}(g_t)_{|\,t=0}=&\int_\Omega\big\langle 4\,\Lambda_{\tilde\alpha,\theta}-\operatorname{div}\tilde h-\widetilde{\mathcal K}^\flat-\Phi_h-\Phi_T+2\,\tilde{\mathcal T}^\flat+4\,\langle\theta,\ \tilde H\rangle\\
&+2(\operatorname{div}(\alpha-\tilde\theta))_{\,|\mathrm{V}}+2\Lambda_{\alpha,\tilde\alpha+\tilde\theta}+2\langle\tilde\theta-\tilde\alpha,H\rangle+2\operatorname{Sym}(H^\flat\otimes\tilde H^\flat)\\
&-2\,\tilde\delta_H+2\,\Lambda_{\tilde\theta,\theta-\alpha}+\frac12\big(\,\mathrm{S}_{\,\mathrm{mix}}+\operatorname{div}(\tilde H-H)\big)\,g^\perp,\ B\big\rangle\,\mathrm{d}\operatorname{vol}_g\,.
\end{aligned}
\tag{30}
$$

If g is critical for $J_{\mathrm{mix},\widetilde{\mathcal D},\Omega}$ with respect to $g^\perp$-variations, then the integral in (30) is zero for arbitrary symmetric $(0,2)$-tensor B vanishing on $\widetilde{\mathcal D}\times\widetilde{\mathcal D}$. This yields the Euler–Lagrange equation, that we can decompose into two independent parts: its $\mathcal D\times\mathcal D$ and V-components, obtaining the following:

$$
\operatorname{div}\tilde h+\widetilde{\mathcal K}^\flat+\Phi_h+\Phi_T-2\,\tilde{\mathcal T}^\flat-\frac12\big(\,\mathrm{S}_{\,\mathrm{mix}}+\operatorname{div}(\tilde H-H)\big)\,g^\perp=0, \tag{31a}
$$

$$
\begin{aligned}
&2\,\langle\theta,\ \tilde H\rangle+2\Lambda_{\tilde\alpha,\theta}+(\operatorname{div}(\alpha-\tilde\theta))_{\,|\mathrm{V}}+\Lambda_{\alpha,\tilde\alpha+\tilde\theta}+\langle\tilde\theta-\tilde\alpha,H\rangle\\
&\quad+\operatorname{Sym}(H^\flat\otimes\tilde H^\flat)-\tilde\delta_H+\Lambda_{\tilde\theta,\theta-\alpha}=0.
\end{aligned}
\tag{31b}
$$

For volume-preserving $g^\perp$-variations, the Euler–Lagrange equation will be (31b), and instead of (31a), one needs to consider the following:

$$\operatorname{div}\tilde{h} + \widetilde{\mathcal{K}}^\flat + \Phi_h + \Phi_T - 2\,\tilde{\mathcal{T}}^\flat - \frac{1}{2}\big(\mathrm{S}_{\,\mathrm{mix}} + \operatorname{div}(\tilde{H} - H)\big)\,g^\perp = \lambda g^\perp. \tag{32}$$

Using tensor $r_{\mathcal{D}}$ (Lemma 1) and replacing $\operatorname{div}\tilde{h}$ in (31a) according to (6), we rewrite (32) as (27a). Using the properties $\Lambda_{P,Q} = \Lambda_{Q,P}$ and $\Lambda_{P,Q_1+Q_2} = \Lambda_{P,Q_1} + \Lambda_{P,Q_2}$, we rewrite (31b) as (27b). Finally, using the fact that all variation formulas for $g^\top$-variations are dual to the $\mathcal{D}\times\mathcal{D}$ components of the variation formulas for $g^\perp$-variations, we can take the dual equation to (32) to obtain the following Euler–Lagrange equation for volume-preserving $g^\top$-variations:

$$\operatorname{div}h + \mathcal{K}^\flat + \Phi_{\tilde{h}} + \Phi_{\tilde{T}} - 2\,\mathcal{T}^\flat - \frac{1}{2}\big(\mathrm{S}_{\,\mathrm{mix}} + \operatorname{div}(H - \tilde{H})\big)\,g^\top = \lambda g^\top, \tag{33}$$

as the dual to $\mathrm{S}_{\,\mathrm{mix}}$ is $\mathrm{S}_{\,\mathrm{mix}}$. Using the dual of Lemma 1 yields (27c). □

Remark 4 (i) Equations (27a) and (27c) are dual to each other and coincide with the equations obtained in [14] for *adapted variations* of metric (see Remark 3 with a discussion of their relationship with $g^\perp$- and $g^\top$-variations). However, (27a) corresponds to the variation of the orthogonal complement of $\widetilde{\mathcal{D}}$ and cannot be obtained by means of adapted variations.

(ii) We can relate the Euler–Lagrange equation for different types of variations. To obtain the Euler–Lagrange equation for arbitrary, not necessarily preserving volume of (M, g), $g^\perp$-variations (respectively, $g^\top$-variations), one should merely set $\lambda = 0$ in the Euler–Lagrange equation obtained for volume-preserving $g^\perp$-variations (respectively, $g^\top$-variations). To obtain the Euler–Lagrange equation for arbitrary variations g_t preserving the volume of (M, g), one should consider both Euler–Lagrange equations for volume-preserving $g^\perp$- and $g^\top$-variations, with the same, arbitrary constant $\lambda \in \mathbb{R}$.

In general, it is difficult to find critical points of (2) for arbitrary variations of metric. A trivial example of such metric is the one of the metric product of manifolds, i.e., with both $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ integrable and totally geodesic. A more interesting case (to be considered in further work) are critical left-invariant metrics on Lie groups endowed with left-invariant distributions. There exist, however, many interesting examples of metrics critical with respect to volume-preserving variations, or volume-preserving $g^\perp$- and $g^\top$-variations considered separately—we shall present some of them in further sections. Note that the volume-preserving $g^\perp$- and $g^\top$-variations generalize other variations considered in literature, e.g., the variation among associated metrics on a contact manifold [4], discussed in Sect. 3.3.

3 Particular cases

In this part of the paper, we examine the Euler–Lagrange equations (27a)–(27c), assuming particular (co)dimension of the distribution $\widetilde{\mathcal{D}}$ or the existence of an additional structure on the manifold M. In these special geometric settings, we obtain examples of metrics critical for the action (2), with respect to variations previously discussed.

3.1 Flows

Let $\widetilde{\mathcal{D}}$ be spanned by a nonsingular vector field N, and then it is tangent to the one-dimensional foliation by the flowlines of N. In this case, $\mathrm{S}_{\rm mix} = \epsilon_N \operatorname{Ric}_{N,N}$, $R_N = R(N, \cdot)N$ is the Jacobi operator and the partial Ricci tensor takes a particularly simple form:

$$r_{\widetilde{\mathcal{D}}} = \epsilon_N \operatorname{Ric}_{N,N}\ g^{\top}, \quad r_{\mathcal{D}} = \epsilon_N (R_N)^{\flat}.$$

We have $\tilde h = \tilde h_{sc} N$, where $\tilde h_{sc} = \epsilon_N \langle \tilde h, N\rangle$ is the scalar second fundamental form of $\mathcal{D}$. Let $\tilde A_N$ be the Weingarten operator associated with $\tilde h_{sc}$ and let $\tilde\tau_i = \operatorname{Tr} \tilde A_N^{\,i}$ $(i \ge 0)$. We have $\mathrm{S}_{\rm ex} = g(H,H) - \langle h, h\rangle = g(H,H) - g(H,H) = 0$, $\widetilde{\mathrm{S}}_{\rm ex} = \tilde\tau_1^2 - \tilde\tau_2$ and

$$\operatorname{div} N = \sum_i \epsilon_i\, g(\nabla_{\mathcal{E}_i} N, \mathcal{E}_i) = -g\left(N, \sum_i \epsilon_i \nabla_{\mathcal{E}_i} \mathcal{E}_i\right) = -g(N, \tilde H) = -\tilde\tau_1,$$
$$\operatorname{div}(\tilde\tau_1 N) = N(\tilde\tau_1) + \tilde\tau_1 \operatorname{div} N = N(\tilde\tau_1) - \tilde\tau_1^2.$$

The curvature of the flow lines is $H = \epsilon_N \nabla_N N$. From Theorem 1, we obtain the following.

Corollary 2 (Euler–Lagrange equation) *Let a distribution $\widetilde{\mathcal{D}}$ be spanned by a unit vector field N on a manifold M with respect to $g \in \operatorname{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$. Then g is critical for the action* (2) *with respect to volume-preserving $g^{\perp}$-variations if and only if*

$$\epsilon_N \left(R_N + \tilde A_N^2 - \left(\tilde T_N^{\sharp}\right)^2 + \left[\tilde T_N^{\sharp}, \tilde A_N\right]\right)^{\flat} - \tilde\tau_1 \tilde h_{sc} + H^{\flat} \otimes H^{\flat} - \operatorname{Def}_{\mathcal{D}} H$$
$$-\frac{1}{2}\left(\epsilon_N \operatorname{Ric}_{N,N} + \operatorname{div}(\epsilon_N \tilde\tau_1 N - H)\right) g^{\perp} = \lambda g^{\perp}, \tag{34a}$$
$$\operatorname{div}^{\perp} \tilde T_N^{\sharp}|_{\mathcal{D}} + 2\left(\tilde T_N^{\sharp}(H)\right)^{\flat} = 0; \tag{34b}$$

and the metric g is critical for the action (2) *with respect to volume-preserving $g^{\top}$-variations if and only if*

$$\epsilon_N \operatorname{Ric}_{N,N} - 4\langle \tilde T, \tilde T\rangle - \operatorname{div}(\epsilon_N \tilde\tau_1 N + H) = 2\lambda. \tag{34c}$$

Moreover, the metric g is critical for the action (2) *with respect to all volume-preserving variations if and only if all equations* (34a)–(34c) *hold, with the same constant λ.*

Proof An easy computation shows that

$$\begin{aligned}
&\widetilde{\mathcal{A}} = \epsilon_N \tilde A_N^2, \quad \langle \tilde h_{sc} N, \tilde H\rangle = \tilde\tau_1 \tilde h_{sc}, \quad \Psi = H^{\flat} \otimes H^{\flat}, \quad \widetilde\Psi = (\epsilon_N \tilde\tau_2 - \langle \tilde T, \tilde T\rangle)\, g^{\top},\\
&\mathcal{A} = g(H,H)\,\widetilde{\operatorname{id}}, \quad \mathcal{T} = 0, \quad \langle h, H\rangle = g(H,H)\, g^{\top},\\
&H = \epsilon_N \nabla_N N, \quad h = H\, g^{\top}, \quad \langle h, h\rangle = g(H,H),\\
&\tilde H = \epsilon_N \tilde\tau_1 N, \quad \tilde\tau_1 = \epsilon_N \operatorname{Tr}{}_g \tilde h_{sc}, \quad \langle \tilde h, \tilde h\rangle = \epsilon_N \tilde\tau_2, \quad \operatorname{Def}_{\widetilde{\mathcal{D}}} \tilde H = \epsilon_N N(\tilde\tau_1)\, g^{\top}.
\end{aligned} \tag{35}$$

Notice that $(H^{\flat} \otimes H^{\flat})(X, Y) = g(H, X)\, g(H, Y)$. Substituting

$$\Phi_h = 0 = \mathrm{S}_{\rm ex}, \quad \widetilde{\mathrm{S}}_{\rm ex} = \epsilon_N(\tilde\tau_1^2 - \tilde\tau_2), \quad \widetilde{\mathcal{T}} = \epsilon_N \tilde T_N^{\sharp\,2}$$

into (27a) and using (35) yields (34a). Substituting

$$h = H\, g^{\top}, \quad \Phi_{\tilde h} = \epsilon_N \left(\tilde\tau_1^2 - \tilde\tau_2\right) g^{\top}, \quad \Phi_{\tilde T} = -\langle \tilde T, \tilde T\rangle\, g^{\top}, \quad \mathcal{K}^{\flat} = 0$$

into (27c) and using (35) yields (34c).

Let X be orthogonal to N with $\nabla_Z X \in \widetilde{\mathcal{D}}$ for all $Z \in TM$. We have $\theta = 0$ and since

$$\begin{aligned} 2\,(\operatorname{div}\alpha)(X, N) &= g(\nabla_N H - \tilde\tau_1 H, X),\\ 2\,\langle\tilde\theta - \tilde\alpha, H\rangle(X, N) &= -g\left(\tilde T^\sharp_N(H) + \tilde A_N(H), X\right),\\ 2\operatorname{Sym}(H^\flat \otimes H^\flat)(X, N) &= g(\tilde\tau_1 H,\ X),\\ 2\,\tilde\delta_H(X, N) &= g(\nabla_N H, X),\\ 2\,\Lambda_{\alpha,\tilde\alpha}(X, N) &= g(\tilde A_N(H), X), \end{aligned}$$

the Euler–Lagrange Eq. (27b) reduces to

$$(\operatorname{div}\tilde\theta)_{\,|\mathrm{V}} - \langle\tilde\theta, H\rangle = 0. \tag{36}$$

For $X \in \mathcal{D}$ such that $\nabla_Z X \in \widetilde{\mathcal{D}}$ for all $Z \in TM$, we have

$$\begin{aligned} 2\ \operatorname{div}\tilde\theta(X, N) &= \sum_i \epsilon_i\, g\left(\left(\nabla_{\mathcal{E}_i}\tilde T^\sharp_N\right)(X), \mathcal{E}_i\right) + \epsilon_N\, g\left(\nabla_N\left(\tilde T^\sharp_N(X)\right), N\right)\\ &= \left(\operatorname{div}^\perp \tilde T^\sharp_N\right)(X) + g\left(\tilde T^\sharp_N(H), X\right). \end{aligned}$$

Hence, (36) is written as (34b). □

By (5), we have $\operatorname{div}\tilde h = N(\tilde h_{sc}) - \tilde\tau_1 \tilde h_{sc}$ and $\operatorname{div} h = (\operatorname{div} H)\,\tilde g$. Then, see (6) and (9),

$$\begin{aligned} \epsilon_N\left(R_N + \tilde A_N^2 + \left(\tilde T^\sharp_N\right)^2\right)^\flat &= N(\tilde h_{sc}) - H^\flat \otimes H^\flat + \operatorname{Def}_{\mathcal{D}} H,\\ \epsilon_N \operatorname{Ric}_{N,N} &= \operatorname{div} H + \epsilon_N(N(\tilde\tau_1) - \tilde\tau_2) + \langle\tilde T, \tilde T\rangle. \end{aligned} \tag{37}$$

Remark that $(37)_2$ is simply the trace of $(37)_1$.

A flow of a unit vector N is called *geodesic* if the orbits are geodesics ($h = 0$) and *Riemannian* if the metric is bundle-like ($\tilde h = 0$). A nonsingular Killing vector field clearly defines a Riemannian flow; moreover, a Killing vector field of constant length generates a geodesic Riemannian flow. Restricting Corollary 2 to the case of a geodesic Riemannian flow, we obtain the following.

Corollary 3 *Let $\widetilde{\mathcal{D}}$ be spanned by a unit vector field N that generates a geodesic Riemannian flow on a pseudo-Riemannian manifold (M^{p+1}, g). Then g is critical for the action (2) with respect to volume-preserving $g^\perp$-variations if and only if all the following conditions hold:*

$$R_N = (1/p)\operatorname{Ric}_{N,N}\operatorname{id}^\perp, \tag{38a}$$

$$\operatorname{Ric}_{X,N} = 0 \quad (X \in \mathcal{D}), \tag{38b}$$

$$\operatorname{Ric}_{N,N} = \text{const}; \tag{38c}$$

and the metric g is critical for the action (2) with respect to volume-preserving $g^\top$-variations if and only if (38c) holds.

Proof From $(37)_1$ we obtain $R_N = -\left(\tilde T^\sharp_N\right)^2$ and (34a) takes the form

$$2\epsilon_N(R_N)^\flat - \frac{1}{2}\epsilon_N \operatorname{Ric}_{N,N}\, g^\perp = \lambda g^\perp,$$

which together with $\operatorname{Ric}_{N,N} = \operatorname{Tr}\ R_N$ yields (38a) and (38c).

For a geodesic Riemannian N-flow, (34b) reduces to condition $\operatorname{div}^\perp \tilde{T}^\sharp_N(X) = 0$ for all $X \in \mathcal{D}$, that we shall now examine. A Riemannian geodesic flow locally gives rise to a Riemannian submersion with totally geodesic fibers. Such mappings can be described by the following tensor, introduced by Gray [8] and adjusted here to our notation:

$$\mathcal{O}_X Y = (\nabla_{X^\perp} Y^\top)^\perp + (\nabla_{X^\perp} Y^\perp)^\top \quad (X, Y \in TM).$$

It follows that $\mathcal{O}$ is antisymmetric with respect to g and $\mathcal{O}_X Y = \tilde{T}(X, Y)$ for $X, Y \in \mathcal{D}$. Hence, for $X, Y \in \mathcal{D}$ we have $g(\tilde{T}^\sharp_N X, Y) = g(\tilde{T}(X, Y), N) = g(\mathcal{O}_X Y, N) = -g(\mathcal{O}_X N, Y)$ and we obtain $\tilde{T}^\sharp_N X = -\mathcal{O}_X N$.

Let $X \in \mathcal{D}$ and $\nabla_Z X \in \widetilde{\mathcal{D}}$ for all $Z \in TM$. Using an adapted frame with $\mathcal{E}_i \in \widetilde{\mathcal{D}}$ at a point, the fact that $\nabla_N N = 0$, and the antisymmetry of $\nabla_Z \mathcal{O}$ for all $Z \in TM$, we obtain:

$$\begin{aligned}\left(\operatorname{div}^\perp \tilde{T}^\sharp_N\right)(X) &= \sum_i g\left(\nabla_{\mathcal{E}_i} \tilde{T}^\sharp_N X, \mathcal{E}_i\right) = -\sum_i g\left(\nabla_{\mathcal{E}_i} \mathcal{O}_X N, \mathcal{E}_i\right)\\ &= -\sum_i g\left(\left(\nabla_{\mathcal{E}_i} \mathcal{O}\right)_X N, \mathcal{E}_i\right) = \sum_i g\left(\left(\nabla_{\mathcal{E}_i} \mathcal{O}\right)_X \mathcal{E}_i, N\right).\end{aligned}$$

From the formula (5.37e) from [16], adjusted to our definitions of R and Ric, it follows that

$$(\operatorname{div}^\perp \tilde{T}^\sharp_N)(X) = -\sum_i g(R(\mathcal{E}_i, X)\mathcal{E}_i, N) = -\operatorname{Ric}_{X,N}.$$

Thus, we obtain (38b). Finally, for volume-preserving $g^\top$-variations, we have the Euler–Lagrange Eq. (34c), that for geodesic Riemannian flows takes the form $\epsilon_N \operatorname{Ric}_{N,N} = -\frac{2}{3}\lambda$. □

From Corollary 3, we immediately obtain the following.

Corollary 4 *Let (M^{p+1}, g), with $p > 1$, be an Einstein manifold with a geodesic Riemannian flow. Let $\widetilde{\mathcal{D}}$ be the 1-dimensional distribution tangent to the flowlines. Then g is critical for the action (2) with respect to volume-preserving $g^\perp$ and $g^\top$-variations.*

The following proposition shows that the only manifolds with geodesic Riemannian flows critical for the action (2) with respect to all volume-preserving variations locally are in fact metric products.

Proposition 3 *Let $\widetilde{\mathcal{D}}$ be spanned by a unit vector field N that generates a geodesic Riemannian flow on a pseudo-Riemannian manifold (M^{p+1}, g). If g is a critical metric for the action (2) with respect to all volume-preserving variations then $\mathcal{D}$ is integrable.*

Proof Using Remark 4(ii), we can write the Euler–Lagrange equation for arbitrary volume-preserving variations as follows:

$$\epsilon_N\left(R_N - \left(\tilde{T}^\sharp_N\right)^2\right)^\flat = \frac{1}{2}\left(\epsilon_N \operatorname{Ric}_{N,N} + 2\lambda\right) g^\perp, \tag{39a}$$

$$\operatorname{Ric}_{X,N} = 0 \quad (X \in \mathcal{D}), \tag{39b}$$

$$\epsilon_N \operatorname{Ric}_{N,N} = 4\langle \tilde{T}, \tilde{T}\rangle + 2\lambda, \tag{39c}$$

where $\lambda \in \mathbb{R}$ is an arbitrary constant. Using $R^N = -\left(\tilde{T}^\sharp_N\right)^2$ and $\operatorname{Tr}\left(\tilde{T}^\sharp_N\right)^2 = -\langle \tilde{T}, \tilde{T}\rangle$, we obtain from (39c) $\lambda = -3\langle \tilde{T}, \tilde{T}\rangle/2$. On the other hand, (39a) yields

$$-2\left(\left(\tilde{T}^\sharp_N\right)^2\right)^\flat = \frac{1}{2}\left(\langle \tilde{T}, \tilde{T}\rangle + 2\lambda\right) g^\perp,$$

and taking its trace we obtain $(4-p)\langle\tilde T,\tilde T\rangle = 2\lambda p$. It follows from $n, p\ge 0$ that the two equations for λ have a solution only for $\langle\tilde T,\tilde T\rangle = 0$. □

3.2 Codimension-one foliations

In this section, we consider the action (2), where $\widetilde{\mathcal D}$ is tangent to a codimension-one foliation (of dimension $n>1$). We find metrics critical with respect to volume-preserving $g^\perp$- and $g^\top$-variations, as well as arbitrary volume-preserving variations.

Let $\mathcal F$ be a codimension-one foliation tangent to the distribution $\widetilde{\mathcal D}$. Let h_{sc} be the scalar second fundamental form, and A_N the Weingarten operator of $\mathcal F$; then we define the functions $\tau_i = \operatorname{Tr} A_N^i$ $(i\ge 0)$—the power sums of the principal curvatures k_i of the leaves. The τ's can be expressed using the *elementary symmetric functions* $\sigma_1,\dots,\sigma_n$,

$$\sigma_j = \sum_{i_1<\dots<i_j} k_{i_1}\cdot\dots\cdot k_{i_j}\quad (0\le j\le n),$$

called *mean curvatures* in the literature. For example, $\sigma_0=1=\tau_0$, $\sigma_1=\tau_1$, and $2\sigma_2=\tau_1^2-\tau_2$.

We have $T=0=\tilde T$ and

$$h_{sc}(X,Y)=\epsilon_N\, g(\nabla_X Y,\ N),\quad A_N(X)=-\nabla_X N\quad (X,Y\in T\mathcal F).$$

We define the vector field $(\widetilde{\operatorname{div}}\,A_N)^\sharp\in\mathfrak X_{\widetilde{\mathcal D}}$ by the following equation:

$$g((\widetilde{\operatorname{div}}\,A_N)^\sharp, X) = (\widetilde{\operatorname{div}}\,A_N)(X)\quad (X\in\mathfrak X_{\widetilde{\mathcal D}}).$$

Then we can formulate the following.

Proposition 4 *Let $\widetilde{\mathcal D}$ be the distribution tangent to a codimension-one foliation of a manifold M^{n+1}. Then a metric g on M is critical for the action* (2) *with respect to volume-preserving $g^\perp$-variations if and only if*

$$\tau_1^2-\tau_2 = 2\epsilon_N\,\lambda, \tag{40a}$$
$$(\widetilde{\operatorname{div}}A_N)^\sharp - \nabla^\top\tau_1 = 0, \tag{40b}$$

and g on M is critical for the action (2) *with respect to volume-preserving $g^\top$-variations if and only if:*

$$\nabla_N h_{sc} - \tau_1 h_{sc} = \frac12\left(2\,\epsilon_N\left(N(\tau_1)-\tau_1^2\right)+\epsilon_N\left(\tau_1^2-\tau_2\right)+2\lambda\right)\,g^\top. \tag{40c}$$

Moreover, a metric g on M is critical for the action (2) *with respect to all volume-preserving variations if and only if* (40a)–(40c) *hold with the same constant λ.*

Proof Equations (40a) and (40c) follow from (27a) and (27c). This can be shown by a direct computation, but since the orthogonal complement of $\widetilde{\mathcal D}$ is spanned by a single vector field N, we can use the equations obtained for flows, as *adapted* (i.e., with $\partial_t g$ restricted to $\mathcal D\times\mathcal D$) $g^\perp$-variations in this section correspond to $g^\top$-variations in Sect. 3.1. (See Remarks 3 and 4(i).) Hence, (40a) is dual to (34c) and (40c) is dual to (34a), with the additional assumption that $\widetilde{\mathcal D}$ is integrable. Indeed, using (37) in (34c) and (34a), then taking their duals and setting $T=0$, we obtain (40a) and (40c). For $X\in\widetilde{\mathcal D}$ and $N\in\mathcal D$, we have

$$2\,\langle\tilde\alpha, H\rangle(X,N) = \tau_1\, g(X,\tilde H),$$
$$2\,\operatorname{Sym}(H^\flat\otimes\tilde H^\flat)(X,N) = \tau_1\, g(X,\tilde H),$$

$$
\begin{aligned}
2\,\tilde\delta_H(X,N) &= X(\tau_1),\\
2\,\Lambda_{\alpha,\tilde\alpha}(X,N) &= g(A_N(\tilde H),X),\\
2\,(\operatorname{div}\alpha)(X,N) &= (\operatorname{div}A_N)(X).
\end{aligned}
$$

Using the above equations and the fact that for all $X\in\widetilde{\mathcal D}$ we have

$$
\begin{aligned}
(\operatorname{div}A_N)(X) &= \sum_\lambda \epsilon_\lambda g(\nabla_\lambda(A_N(X)),e_\lambda) - \sum_\lambda \epsilon_\lambda g(A_N(\nabla_\lambda X),e_\lambda)\\
&= \sum_a \epsilon_a\, g(\nabla_{E_a}(A_N(X)),E_a) + \epsilon_N\, g(\nabla_N(A_N(X)),N)\\
&\quad - \sum_a \epsilon_a\, g(A_N(\nabla_{E_a}X),E_a) = (\widetilde{\operatorname{div}}\,A_N)(X) - g(A_N(X),\tilde H),
\end{aligned}
$$

we reduce (27b) to (40b). □

From Proposition 4, we obtain the following.

Corollary 5 *Let $\widetilde{\mathcal D}$ be tangent to a codimension-one foliation of a pseudo-Riemannian manifold (M^{n+1},g), and let the unit normal field N of $\mathcal F$ be complete in M. Then metric g is critical for the action* (2) *with respect to all volume-preserving variations if and only if*

$$\tau_1^2-\tau_2 = 2\epsilon_N\,\lambda,\quad \nabla_N h_{sc}-\tau_1 h_{sc} = \frac{2\lambda}{1-n}\,g^\top, \tag{41}$$

$$(\widetilde{\operatorname{div}}A_N)^\sharp - \nabla^\top\tau_1 = 0, \tag{42}$$

and τ_1 is bounded on M only for $\lambda\ge 0$; moreover, $\tau_1=0$ when $\lambda=0$.

Proof Taking trace of (40c) and using (40a), we obtain

$$N(\tau_1)-\tau_1^2 = \frac{2n\epsilon_N\lambda}{1-n}. \tag{43}$$

Using the above and (40a) in (40c) yields $(41)_2$. Equation (43) has a global bounded solution only for $\lambda\ge 0$, and for $\lambda=0$, this solution is $\tau_1=0$. □

Codimension-one foliations admit *biregular foliated coordinates* $(x_0,\dots,x_n)$, see [7, Section 5.1], i.e., the leaves are the level sets $\{x_0=c\}$ and N-curves are given by $\{x_i=c_i\ (i>0)\}$. From now on, we assume that a foliated pseudo-Riemannian manifold $(M,\mathcal F,g)$ admits *orthogonal biregular foliated coordinates* (hence, $g_{ij}=0$ for $i\ne j$), then $g=g_{00}\,dx_0^2+\sum_{i>0}g_{ii}\,(dx_i)^2$. Denote by $g_{ii,\mu}$ the derivative of g_{ii} in the ∂_μ-direction. We adopt the convention $\mu\in\{0,\dots,n\}$, $a,i,j\in\{1,\dots,n\}$. We have $g_{00}=\epsilon_N|g_{00}|$ and $g_{ii}=\epsilon_i|g_{ii}|$.

Lemma 5 *For a pseudo-Riemannian metric in orthogonal biregular foliated coordinates of a codimension-one foliation on (M,g), one has* [13]

$$
\begin{aligned}
N &= \partial_0/\sqrt{|g_{00}|}\quad\text{(the unit normal)},\\
\Gamma^j_{i0} &= (1/2)\,\delta^j_i\, g_{ii,0}/g_{ii},\quad \Gamma^i_{00} = -(1/2)\,g_{00,i}/g_{ii},\quad \Gamma^0_{ij} = -\delta_{ij}\,g_{ii,0}/(2\,g_{00}),\\
h_{ij} &= \Gamma^0_{ij}\sqrt{g_{00}} = -\frac12\,\epsilon_N\,\delta_{ij}\,g_{ii,0}/\sqrt{|g_{00}|}\quad\text{(the second fundamental form)},\\
A^j_i &= -\Gamma^j_{i0}/\sqrt{|g_{00}|} = -\frac{1}{2\sqrt{|g_{00}|}}\,\delta^j_i\,\frac{g_{ii,0}}{g_{ii}}\quad\text{(the Weingarten operator)},
\end{aligned}
$$

$$\tau_1=-\frac{1}{2\sqrt{|g_{00}|}}\sum_{i>0}\frac{g_{ii,0}}{g_{ii}},\quad \tau_2=\frac{1}{4\,|g_{00}|}\sum_{i>0}\left(\frac{g_{ii,0}}{g_{ii}}\right)^2,\quad \text{etc.}$$

Using the above, one can obtain

$$(\nabla_N\,h_{sc})_{ii}=-\frac{\epsilon_N}{2\,|g_{00}|}\left(g_{ii,00}-\frac{1}{2}\,g_{ii,0}(\log|g_{00}|)_{,\,0}-(g_{ii,0})^2/g_{ii}\right),$$

$$(\widetilde{\operatorname{div}}A_N)(\partial_i)=\partial_i\left(-\frac{1}{2\sqrt{|g_{00}|}}\cdot\frac{g_{ii,\,0}}{g_{ii}}\right)-\frac{1}{2\sqrt{|g_{00}|}}\cdot\frac{g_{ii,\,0}}{g_{ii}}\sum_{a>0}\Gamma^a_{ai}+\frac{1}{2\sqrt{|g_{00}|}}\sum_{a>0}\Gamma^a_{ai}\frac{g_{aa,\,0}}{g_{aa}}$$

for $i=1,\ldots,n$, *where*

$$\Gamma^a_{ai}=\frac{1}{2}\cdot\frac{g_{aa,\,i}}{g_{aa}}. \tag{44}$$

Lemma 6 *Let $\mathcal{F}$ be a codimension-one foliation of a pseudo-Riemannian manifold (M,g) tangent to $\widetilde{\mathcal{D}}$, with a unit normal field N complete in M. Let there exist global orthogonal biregular foliated coordinates $(x_0,x_1,\ldots,x_n)$, with the leaves of $\mathcal{F}$ given by $\{x_0=c\}$, and g of the form*

$$g_{ii}=\epsilon_i\,f_i(x_1,\ldots x_n)\,e^{-2\int\sqrt{|g_{00}|}\,y_i(t,x_1,\ldots,x_n)\,\mathrm{d}t},\quad i=1,\ldots,n, \tag{45}$$

where f_i $(i=1,\ldots n)$ are positive functions. Then (41) is written as the system

$$\tau_1^2-\tau_2=2\epsilon_N\lambda, \tag{46a}$$

$$\partial_0 y_i-\tau_1\sqrt{|g_{00}|}\,y_i-\frac{2\lambda}{1-n}\,\epsilon_N\sqrt{|g_{00}|}=0,\quad i=1,\ldots,n \tag{46b}$$

where $\tau_1=y_1+\cdots+y_n$, $\tau_2=y_1^2+\cdots+y_n^2$, λ is a constant and g_{00} is a smooth function of constant sign. Also, (42) takes the following form:

$$\partial_i y_i+y_i\sum_{a>0}\Gamma^a_{ai}-\sum_{a>0}y_a\Gamma^a_{ai}-\partial_i\sum_{a>0}y_a=0,\quad i=1,\ldots,n,$$

which can be written equivalently as

$$\partial_i\sum_{a>0,\,a\neq i}y_a+\sum_{a>0,\,a\neq i}\Gamma^a_{ai}(y_i-y_a)=0,\quad i=1,\ldots,n. \tag{47}$$

Proof This follows from a straightforward computation, using (45) and Lemma 5. □

The following lemma shows how the Euler–Lagrange Eqs. (46b) and (46a) are related, when the same constant λ is considered in both of them.

Lemma 7 *Let $g_{00}\equiv 1$ and let $y_1,\ldots,y_n$ be functions that satisfy (46b). If at some point $x_0=0$ (46a) holds, then $y_1,\ldots,y_n$ satisfy (46a) for all values $x_0\in\mathbb{R}$.*

Proof Let $y_1,\ldots,y_n$ satisfy (46b). Then

$$\partial_0\left(\tau_1^2-\tau_2 x\right)=\partial_0\left(\sum_{i=1}^n y_i\right)^2-\partial_0\sum_{i=1}^n y_i^2$$

$$= 2\left(\sum_{j=1}^{n} y_j\right) \cdot \left(\sum_{i=1}^{n} \partial_0 y_i\right) - 2\left(\sum_{i=1}^{n} y_i \partial_0 y_i\right)$$

$$= 2\sum_{i=1}^{n} (\partial_0 y_i)(\tau_1 - y_i) = 2\sum_{i=1}^{n} \left(\frac{2\lambda}{1-n}\,\epsilon_N + \tau_1 y_i\right)(\tau_1 - y_i)$$

$$= 2\sum_{i=1}^{n} \left(\frac{2\lambda}{1-n}\,\epsilon_N \tau_1 - \frac{2\lambda}{1-n}\,\epsilon_N y_i + \tau_1^2 y_i - \tau_1 y_i^2\right)$$

$$= 2\left(\frac{2n\lambda}{1-n}\,\epsilon_N \tau_1 - \frac{2\lambda}{1-n}\,\epsilon_N \tau_1 + \tau_1^3 - \tau_1 \tau_2\right) = 2\tau_1\left(-2\lambda\,\epsilon_N + \tau_1^2 - \tau_2\right).$$

Defining $u := -2\lambda\,\epsilon_N + \tau_1^2 - \tau_2$, we can write the above equation as $\partial_0 u = 2\tau_1 \cdot u$. It follows from the uniqueness of solution of this ODE, that the only solution satisfying $u(0) = 0$ is $u \equiv 0$. □

We use the last lemma to give a construction of metric of the form (45) with $g_{00} \equiv 1$, that is critical for the action (2) with respect to arbitrary volume-preserving variations.

Proposition 5 *Assume the following for* $i = 1, \dots, n$*:*

- $\lambda \geq 0$, $g_{00} \equiv 1$ *and let* $\eta_i(x_1, \dots, x_n)$ *be functions on* $\mathbb{R}^n$ *satisfying*

$$\left(\sum_{i=1}^{n} \eta_i\right)^2 - \sum_{i=1}^{n} \eta_i^2 = -2\epsilon_n \lambda\,,$$

- $y_i(x_0, x_1, \dots, x_n)$ *satisfies* (46b) *with* $\tau_1 = \sum_i y_i$ *and* $y_i(0, x_1, \dots, x_n) = \eta_i(x_1, \dots, x_n)$,
- Γ^a_{ai} *are any functions satisfying* (47) *such that* $\Gamma^a_{ai} = \Gamma^a_{ia}$,
- $f_a(x_1, \dots, x_n)$ *are any functions satisfying*

$$\partial_i f_a = 2 f_a \cdot \left(\Gamma^a_{ai} + \partial_i \int y_a(t, x_1, \dots, x_n)\, \mathrm{d}t\right).$$

Then the metric g *given by* (45)*, with* f_i *and* y_i *as above, is critical for the action* (2) *with respect to arbitrary volume-preserving variations.*

Proof Equation (46b) holds by the construction, (46a) holds by Lemma 7, and (47) holds by the construction and Eq. (44). □

Example 1 Let $n = 2$, $\epsilon_N = 1$ and $g_{00} \equiv 1$. Then we have the following solution of (46b) and (46a):

$$y_1 = -\sqrt{|\lambda|}\coth(\sqrt{|\lambda|}x_0 + \sqrt{|\lambda|}c), \quad y_2 = -\sqrt{|\lambda|}\tanh(\sqrt{|\lambda|}x_0 + \sqrt{|\lambda|}c),$$

where c is an arbitrary constant.

It is more difficult to find critical metrics of particular geometric properties.

Proposition 6 *Let* g *be a metric on* M^{n+1}*, with* $n > 1$*, critical for the action* (2) *with respect to all volume-preserving variations and let* $\widetilde{\mathcal{D}}$ *be tangent to a codimension-one foliation. If* $\widetilde{\mathcal{D}}$ *is minimal, then* $\widetilde{\mathcal{D}}$ *is totally geodesic.*

Proof If $\tau_1 = 0$, then (43) yields $\lambda = 0$ and from (46a), we obtain $\tau_2 = 0$. It follows that $y_1 = \cdots = y_n = 0$ and $\widetilde{\mathcal{D}}$ is totally geodesic. □

Proposition 7 *Let g be a metric on M^{n+1}, with $n > 1$, critical for the action* (2) *with respect to all volume-preserving variations and let $\widetilde{\mathcal{D}}$ be tangent to a codimension-one foliation, with unit normal field N. If $\widetilde{\mathcal{D}}$ is totally umbilical, then it has constant mean curvature τ_1.*

Proof Let $y_1 = \cdots = y_n = y$, then $\tau_1 = ny$, $\tau_2 = \tau_1^2/n$ and (46a) yields $2\epsilon_N \lambda = (n-1)\tau_1^2/n$. From (43) we obtain $N(\tau_1) = 0$. For totally umbilical foliations (42) reduces to

$$\frac{n-1}{n}\nabla^{\top}\tau_1 = 0.$$

Since $n > 1$, τ_1 is constant on M. □

3.3 Contact and 3-Sasakian structures

Contact manifolds come with a natural foliation given by the flowlines of the Reeb field. They also admit an (in general, non-unique) *associated metric* of well examined properties; we show that for such metric one of the Euler–Lagrange Eq. (27b), always holds. We use this fact later to show that 3-Sasakian structures are a natural source of the metrics critical for the action (2). Recall [4] that a manifold M^{2n+1} with a 1-form η such that

$$d\eta(\xi, X) = 0 \quad (X \in TM), \quad \eta(\xi) = 1,$$

is called a *contact manifold*, and ξ is called the *characteristic vector field* (or the *Reeb field*). A Riemannian metric g on a contact manifold (M^{2n+1}, η) is *associated* if there exists a $(1, 1)$-tensor ϕ such that for all $X, Y \in TM$

$$\eta(X) = g(\xi, X), \quad d\eta(X, Y) = g(X, \phi(Y)), \quad \phi^2 = -I + \eta \otimes \xi. \tag{48}$$

The above (ϕ, ξ, η, g) is called a *contact metric structure* on M. For all contact manifolds we consider in this section, let $\widetilde{\mathcal{D}}$ be spanned by ξ and let $\mathcal{D}$ denote its orthogonal complement.

Remark 5 While we shall consider only the Riemannian metric in this section, there is a natural way to make a Riemannian contact manifold (M, η, g) a pseudo-Riemannian contact manifold: by setting $\bar{g} = g - 2\,\eta \otimes \eta$ as the new metric [6]. Then $-\bar{g}(X, \xi) = \eta(X)$ for all $X \in TM$ and the remaining equations of (48) hold for $\bar{g}$ without changes. This transformation does not invalidate our main results.

Proposition 8 *Let (ϕ, ξ, η, g) be a contact metric structure on M. Then the Euler–Lagrange Eq.* (34b) *is satisfied for $N = \xi$.*

Proof For a contact metric structure, we have (see [4])

$$H = \nabla_\xi\, \xi = 0. \tag{49}$$

For all X, Y such that $g(X, \xi) = g(Y, \xi) = 0$

$$\begin{aligned} d\eta(X, Y) &= -\frac{1}{2}\,\eta([X, Y]) = -\frac{1}{2}\,g([X, Y], \xi) \\ &= -g\left(\tilde{T}^{\sharp}_{\xi}(X), Y\right) = g\left(X, \tilde{T}^{\sharp}_{\xi}(Y)\right). \end{aligned}$$

Hence, it follows from (48), that

$$g(X, \phi(Y)) = g\left(X, \tilde{T}^{\sharp}_{\xi}(Y)\right),$$

and since $\phi(\xi) = 0$, we obtain the following equality: $\tilde{T}^\sharp_\xi = \phi$. By (49), (34b) for $N = \xi$ reduces to $\operatorname{div}^\perp(\tilde{T}^\sharp_\xi)|_\mathcal{D} = 0$, which takes the following form:

$$(\operatorname{div}^\perp \phi)(Y) = 0 \quad (Y \in \mathcal{D}). \tag{50}$$

For $Y \in \mathcal{D}$, the formula for contact metric structures in [4, Corollary 6.1] yields

$$2\, g((\nabla_{\mathcal{E}_i} \phi)(Y), \mathcal{E}_i) = g([\phi, \phi](Y, \mathcal{E}_i), \phi(\mathcal{E}_i)),$$

where

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi(X), \phi(Y)] - \phi[\phi(X), Y] - \phi[X, \phi(Y)].$$

As in [4, Corollary 6.1], considering an orthonormal ϕ-basis (see [4, p. 44]), i.e., assuming that $\mathcal{E}_{i+p/2} = \phi(\mathcal{E}_i)$ for $i = 1, \ldots, p/2$, we obtain that

$$\sum_{i=1}^{p} g([\phi, \phi](Y, \mathcal{E}_i), \phi(\mathcal{E}_i)) = -\sum_{i=1}^{p} g([\phi, \phi](Y, \phi(\mathcal{E}_i)), \phi^2(\mathcal{E}_i)).$$

Hence, $(\operatorname{div}^\perp \phi)(Y) = 0$ and (34b) is satisfied. □

On any contact manifold, there exists a (non-unique) contact metric structure; see [4]. Among them there is a class particularly interesting from the geometric point of view.

Definition 2 [4] A contact metric structure for which ξ is Killing is called *K-contact*.

Proposition 9 *Any K-contact metric g is critical for the action* (51)*, with respect to both volume-preserving $g^\perp$- and $g^\top$-variations.*

Proof We have already seen in (49) that the integral curves of ξ are geodesics for the contact metric structure. On the other hand, a nonsingular Killing vector field defines a Riemannian flow ($\tilde{h} = 0$). Thus, in case of a K-contact structure, we can use Corollary 3.

By [4, Theorem 7.2], if (M, g) is a K-contact manifold then (38a) and (38c) are satisfied with $\mathrm{Ric}_{N,N} = p$. As was shown in Proposition 8, also (38b) holds. □

In [4], the action (2), which reduces to

$$J_{\mathrm{mix}, \widetilde{\mathcal{D}}, \Omega} :\ g \to \int_\Omega \mathrm{Ric}_{N,N}(g)\, \mathrm{d}\,\mathrm{vol}_g \tag{51}$$

has been studied on the set of metrics associated with a given contact form.

Definition 3 [4, p. 24] A contact structure is *regular* if ξ is regular as a vector field, that is, every point of the manifold has a neighborhood such that any integral curve of ξ passing through the neighborhood passes through only once.

Theorem 2 (see Theorem 10.12 in [4]) *An associated metric g on a compact regular contact manifold (M, η) is critical for the action* (51) *considered on the set of metrics associated with η if and only if it is K-contact.*

We have $g(\xi, \xi) = 1$ for any associated metric and the volume form of associated metric on a contact manifold can be expressed only in terms of η and $d\eta$. Therefore, variations of the metric restricted to the set of all associated metrics form a subclass of the volume-preserving $g^\perp$-variations. Hence, on compact regular contact manifolds, Proposition 9 and Theorem 2 together give the following characteristic of some critical metrics—for a larger space of variations.

Corollary 6 *Let (M, η) be a compact regular contact manifold, and let g be an associated metric. Then g is critical for the action (51) for volume-preserving $g^\perp$-variations if and only if g is K-contact.*

Flowlines of Reeb vector fields on contact manifolds are often described as having "maximally non-integrable" orthogonal distributions. We can give this notion a precise meaning by considering the following action:

$$J_{\tilde{T},\Omega}:\ g \to \int_\Omega \langle \tilde{T}, \tilde{T}\rangle \,\mathrm{d\,vol}_g, \tag{52}$$

and showing that contact metric structures are its critical points. Note that (52) is the total norm of the integrability tensor of the varying orthogonal complement of a fixed distribution $\widetilde{\mathcal{D}}$.

Proposition 10 *A metric $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ is critical for the action (52) with respect to volume-preserving $g^\perp$-variations if and only if*

$$2\,\widetilde{\mathcal{T}}^\flat = -\left(\frac{1}{2}\,\langle \tilde{T}, \tilde{T}\rangle - \lambda\right) g^\perp\,, \tag{53a}$$

$$\Lambda_{\tilde{\theta},\theta-\alpha} = (\operatorname{div}\tilde{\theta})_{\,|\mathrm{V}}. \tag{53b}$$

Proof Let g_t be a $g^\perp$-variation. Using Proposition 1, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\,J_{\tilde{T},\Omega}(g_t)_{\,|\,t=0} = \int_\Omega \langle\, 2\,\widetilde{\mathcal{T}}^\flat + 2\,\Lambda_{\tilde{\theta},\theta-\alpha} - 2\,(\operatorname{div}\tilde{\theta})_{\,|\mathrm{V}} + \frac{1}{2}\,\langle \tilde{T}, \tilde{T},\ \rangle\, g^\perp, B\rangle \,\mathrm{d\,vol}_g\,.$$

Decomposing the resulting Euler–Lagrange equation into parts defined on $\mathcal{D}\times\mathcal{D}$ and V yields (53a) and (53b). □

Note that, as expected, distributions with integrable orthogonal complement are critical for (52). Also, using the results obtained for the contact metric structure, we get the following.

Proposition 11 *Let (ϕ, ξ, η, g) be a contact metric structure on M and let $\widetilde{\mathcal{D}}$ be spanned by ξ. Then g is critical for the action (52) with respect to volume-preserving $g^\perp$-variations.*

Proof Using results from the proof of Proposition 8, we compute

$$\begin{aligned}\langle \tilde{T}, \tilde{T}\rangle &= \sum_{i,j} g(\tilde{T}(\mathcal{E}_i, \mathcal{E}_j), \tilde{T}(\mathcal{E}_i, \mathcal{E}_j)) = \sum_{i,j} g(\tilde{T}^\sharp_\xi(\mathcal{E}_i), \mathcal{E}_j)^2 \\ &= \sum_i g(\phi(\mathcal{E}_i), \phi(\mathcal{E}_i))^2 = p.\end{aligned}$$

We also have $\widetilde{\mathcal{T}} = (\tilde{T}^\sharp_\xi)^2 = -\,\mathrm{id}$; hence, $\widetilde{\mathcal{T}}^\flat = -\,g^\perp$. By the above, (53a) is satisfied; (53b) reduces to $(\operatorname{div}\tilde{\theta})_{|V} = 0$ and holds by Proposition 8. □

In higher dimensions, contact metric structures are generalized by contact 3-structures, defined as follows [9].

Definition 4 A *contact 3-structure* is defined as a set of three contact structures, $\eta_a, a = 1, 2, 3$, with the same associated metric g satisfying

$$\phi_c = \phi_a \circ \phi_b - \eta_a \otimes \xi_b = -\phi_b \circ \phi_a + \eta_b \otimes \xi_a$$

for any cyclic permutation (a, b, c) of $(1, 2, 3)$. If each of them is Sasakian structure, it is called a *Sasakian 3-structure* (some authors call it a *3-Sasakian structure*).

Theorem 3 [9] *A contact 3-structure is necessarily a Sasakian 3-structure.*

For any Sasakian 3-structure, let $\widetilde{\mathcal{D}}$ be the distribution spanned by 3 characteristic vector fields ξ_1, ξ_2, ξ_3, its orthogonal complement will be denoted by $\mathcal{D}$. Since $[\xi_a, \xi_b] = 2\xi_c$ for any cyclic permutation (a, b, c) of $(1, 2, 3)$, we see that $\widetilde{\mathcal{D}}$ is integrable.

Proposition 12 *The metric of a Sasakian 3-structure on M is critical for the action* (2) *(where $\widetilde{\mathcal{D}}$ is spanned by the characteristic vector fields), with respect to both volume-preserving $g^\perp$- and $g^\top$-variations.*

Proof Since every ξ_a defines a Sasakian structure, we can use the following formulas for any unit vectors X, Y orthogonal to ξ_a (so we can also have $X = \xi_b$, etc.):

$$R(X, Y)\xi_a = \eta_a(Y)X - \eta_a(X)Y, \quad R(X, \xi_a)Y = -g(X, Y)\xi_a + \eta_a(Y)X.$$

The above formulas are consistent with their analogues for ξ_b and ξ_c, and yield the following: $r_{\mathcal{D}} = 3g^\perp$ and $r_{\widetilde{\mathcal{D}}} = pg^\top$. We also have

$$\Phi_{\tilde{T}}(\xi_a, \xi_b) = \widetilde{\Psi}(\xi_a, \xi_b) = -pg(\xi_a, \xi_b),$$
$$\tilde{T}^\flat(X, Y) = -3g(X, Y) \quad (X, Y \in \mathcal{D})$$

and $\langle \tilde{T}, \tilde{T}\rangle = 3p$. It follows that (27a) and (27c) are satisfied, but never with the same constant λ. The remaining Euler–Lagrange Eq. (27b) reduces to $(\operatorname{div}\tilde{\theta})_{|V} = 0$. Since g is a K-contact metric for ξ_a, for all $Y \in TM$ we have $\nabla_Y \xi_a = -\phi_a(Y)$ [4] and $\tilde{T}_a^\sharp(Y) = \phi_a(Y)$, and it follows that $(\operatorname{div}\tilde{\theta})(Y, \xi_a) = (\operatorname{div}\phi_a)(Y)$. As for any contact metric structure, we have $\nabla_{\xi_a}\xi_a = 0$ and $\phi_a(\xi_a) = 0$. Any vector $X \in \mathcal{D}$ is orthogonal to ξ_a and in all tensor formulas we can assume that $\nabla_Z X$ is colinear with ξ_a for all $Z \in TM$. Then

$$\begin{aligned}
(\operatorname{div}\tilde{\theta})(X, \xi_a) &= \sum_i g((\nabla_{\mathcal{E}_i}\phi_a)(X), \mathcal{E}_i) + g((\nabla_{\xi_b}\phi_a)(X), \xi_b)\\
&\quad + g((\nabla_{\xi_c}\phi_a)(X), \xi_c) + g((\nabla_{\xi_a}\phi_a)(X), \xi_a)\\
&= \sum_i g((\nabla_{\mathcal{E}_i}\phi_a)(X), \mathcal{E}_i) + g((\nabla_{\xi_b}\phi_a)(X), \xi_b)\\
&\quad + g((\nabla_{\xi_c}\phi_a)(X), \xi_c) - g(\phi_a(X), \nabla_{\xi_a}\xi_a)\\
&= \sum_i g((\nabla_{\mathcal{E}_i}\phi_a)(X), \mathcal{E}_i) + g((\nabla_{\xi_b}\phi_a)(X), \xi_b) + g((\nabla_{\xi_c}\phi_a)(X), \xi_c).
\end{aligned}$$

and similarly for ξ_b, ξ_c. The formula we obtained above, when considered for a contact metric structure $(\phi_a, \xi_a, \eta_a, g)$ on M, is precisely $(\operatorname{div}^\perp\phi_a)(X)$. In the proof of Proposition 8 we showed that (50) holds, and hence (27b) is satisfied. □

3.4 Non-integrable distributions

In this section we examine the action (2) for a fixed, non-integrable distribution $\widetilde{\mathcal{D}}$ on a manifold (M, g). As there is no explicit procedure of solving the Euler–Lagrange equation in this case, we must resort to considering particular examples of critical metrics. For this purpose, we set as $\widetilde{\mathcal{D}}$ the distribution orthogonal to the Reeb fields on contact and 3-Sasakian manifolds. In this setting, dual to the one considered in Sect. 3.3, K-contact and 3-Sasakian metrics are critical for the action (2) for volume-preserving $g^\perp$- and $g^\top$-variations. We show how a K-contact metric can be slightly modified and still remain critical. Finally, we consider $g^\perp$-variations of a codimension-one distribution, and give an example of contact metric structure (that is not K-contact) critical with respect to them.

Proposition 13 *Let $\widetilde{\mathcal{D}}$ be a totally geodesic, non-integrable distribution with totally geodesic, integrable orthogonal complement $\mathcal{D}$ on (M, g). Then g is critical for the action (2) with respect to volume-preserving $g^\perp$-variations if and only if*

$$\Phi_T - \frac{1}{2}\langle T, T\rangle\, g^\perp = \lambda g^\perp; \tag{54a}$$

and g is critical for the action (2) with respect to volume-preserving $g^\top$-variations if and only if

$$-2\mathcal{T}^\flat - \frac{1}{2}\langle T, T\rangle\, g^\top = \lambda g^\top. \tag{54b}$$

Proof The Euler–Lagrange Eq. (27b) is always satisfied, since by the assumptions all its terms vanish. Equation (32) becomes (54a) and (33) takes the form (54b). Notice that $\mathcal{D}$ is tangent to a totally geodesic Riemannian foliation. □

One can show, similarly as in the proof of Proposition 3, that (54a) and (54b) cannot hold together with the same constant λ. Thus, in the setting of Proposition 13, one can find metrics critical for volume-preserving $g^\perp$- and $g^\top$-variations, but only considered separately (each with different constant λ in the corresponding Euler–Lagrange equation).

Note that (54a) yields that the mapping $\mathcal{D} \ni X \mapsto T^\sharp_X \in T^*M \otimes TM$ is conformal with respect to the metric g and the metric induced by g on $T^*M \otimes TM$. Since

$$\Phi_T(X, Y) = -\sum_{a,b} \epsilon_a \epsilon_b g(T(E_a, E_b), X) g(T(E_a, E_b), Y) = \mathrm{Tr}\left(T^\sharp_Y T^\sharp_X\right)$$

can be related to the Killing form on $SO(n)$, it is natural to look for examples of critical metrics on manifolds with the action of this group.

Example 2 (i) Let (M, g) be a K-contact manifold and let $\widetilde{\mathcal{D}}$ be the orthogonal distribution of the Reeb field. Then g is critical for the action (2) with respect to volume-preserving $g^\perp$- and $g^\top$-variations. Indeed, for the Reeb field ξ we have: $\Phi_T = -n g^\perp$ and $\mathcal{T}^\flat = -g^\top$ (in general, $\mathcal{T}^\flat = -\epsilon_N g^\top$, see Remark 5).

(ii) Let (M, g) be a 3-Sasakian manifold and let $\widetilde{\mathcal{D}}$ be the distribution orthogonal to all integral manifolds of the Reeb fields ξ_1, ξ_2, ξ_3. Then g is critical for the action (2) with respect to volume-preserving $g^\perp$- and $g^\top$-variations. Indeed, we have $\Phi_T = -n g^\perp$ and $\mathcal{T}^\flat = -3 \cdot g^\top$.

(iii) Let (M, g) be a K-contact manifold and let $\widetilde{\mathcal{D}}$ be the orthogonal distribution of the Reeb field. Let $\bar g = \phi g^\top + \psi g^\perp$, with positive functions $\phi, \psi \in C^\infty(M)$, then the relations between geometric quantities on $(M, \bar g)$ and (M, g) are following: $\langle T, T\rangle_{\bar g} = \psi\phi^{-2}\langle T, T\rangle$ and $\mathcal{T}^\flat_{\bar g} = \psi\phi^{-1}\mathcal{T}^\flat$. We also have $\tau_1 = -\frac{n}{2}\psi^{-1/2}\phi^{-1}N(\phi)$ on $(M, \bar g)$. It follows that for all positive functions $\phi, \psi \in C^\infty(M)$ satisfying $N(\phi) = 0$ and $\psi^{-1/2}\phi = const$ the metric $\bar g$ is critical for the action (2) with respect to volume-preserving $g^\perp$- and $g^\top$-variations.

Remark 6 Let $\widetilde{\mathcal{D}}$ be a distribution with integrable complement $\mathcal{D}$. Then for $X \in \widetilde{\mathcal{D}}$ and $N \in \mathcal{D}$ we have $\langle \theta, \tilde H\rangle(X, N) = \frac{1}{2} g\left(\tilde H, T^\sharp_N X\right)$ and $\Lambda_{\tilde\alpha,\theta}(X, N) = -\frac{1}{2} g\left(\tilde H, T^\sharp_N X\right)$. It follows that the Euler–Lagrange Eq. (27b) does not depend of the integrability tensor T of $\widetilde{\mathcal{D}}$, only on its extrinsic geometry.

If $\widetilde{\mathcal{D}}$ is a codimension-one distribution, its orthogonal complement is always integrable. Then every $g^\perp$-variation corresponds to a family of foliations (by curves g_t-orthogonal to $\widetilde{\mathcal{D}}$)

that share the same transversal geometry. For example, if $\widetilde{\mathcal{D}}$ is totally geodesic or umbilical, all foliations corresponding to metrics g_t are Riemannian or conformal. Thus, $g^\perp$-variation can be a tool to find the locally "best" (e.g., minimizing a functional) metrics for foliations of some fixed transverse property. Using equations dual to the ones formulated in Sect. 3.1 and some easy computations, we can obtain the following.

Proposition 14 *Let $\widetilde{\mathcal{D}}$ be a codimension one distribution on a manifold M^{n+1} with unit normal field N. A metric g on M is critical for the action (2) with respect to volume-preserving $g^\perp$-variations if and only if:*

$$\epsilon_N \operatorname{Ric}_{N,N} - 4\langle T, T\rangle - \operatorname{div}(\epsilon_N \tau_1 N + \tilde{H}) = 2\lambda, \tag{55a}$$
$$(\widetilde{\operatorname{div}} A_N)^\sharp - \nabla^\top \tau_1 = 0. \tag{55b}$$

A metric g on M is critical for the action (2) with respect to volume-preserving $g^\top$-variations if and only if:

$$\begin{aligned}\epsilon_N \left(R_N + A_N^2 - \left(T_N^\sharp\right)^2 + \left[T_N^\sharp, A_N\right]\right)^\flat - \tau_1 h_{sc} + \tilde{H}^\flat \otimes \tilde{H}^\flat - \operatorname{Def}_{\widetilde{\mathcal{D}}} \tilde{H}\\ -\frac{1}{2}\left(\epsilon_N \operatorname{Ric}_{N,N} + \operatorname{div}(\epsilon_N \tau_1 N - \tilde{H})\right) g^\top = \lambda g^\top.\end{aligned} \tag{55c}$$

A metric g on M is critical for the action (2) with respect to all volume-preserving variations if and only if (55a)–(55c) hold, with the same constant λ.

Example 3 As an example, we can consider the following contact metric structure on $\mathbb{R}^3$, which is not K-contact [4]. Let

$$\eta = \frac{1}{2}(dz - y\,dx), \quad g = \frac{1}{4}\begin{pmatrix} 1+y^2+z^2 & z & -y \\ z & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

Using an adapted orthonormal frame: $E_1 = 2(\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z})$, $E_2 = 2\frac{\partial}{\partial y}$, $E_3 = N = 2\frac{\partial}{\partial z}$, one can show that in $\{E_1, E_2\}$ basis of $\mathcal{D}$ we have

$$\tilde{A}_N = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{T}_N^\sharp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have $\operatorname{Ric}_{N,N} = 0$ [4], $\tilde{H} = 0 = H$, $\tau_1 = 0$ and $(R_N + A_N^2 - (T_N^\sharp)^2 + [T_N^\sharp, A_N])^\flat$ is not conformal, hence (55c) is not satisfied, although (55a) holds. Further computations show that $\nabla_{E_1} E_2 = -2E_3$ and $\nabla_{E_1} E_3 = 2E_2$ are the only non-vanishing derivatives of vector fields E_a from the frame, with respect to that frame. We obtain $(\widetilde{\operatorname{div}} A_N)^\sharp = 0$ and hence also (55b) is satisfied. It follows that g is critical for the action (2) with respect to volume-preserving $g^\perp$-variations.

Acknowledgements This article was partly written during the second author's stay at a postdoctoral fellowship in Department of Mathematics at the University of Haifa. The second author would like to thank the first author and the University of Haifa for their hospitality. Both authors would like to thank the anonymous referee for the thorough review and helpful comments.

References

1. Barletta, E., Dragomir, S., Rovenski, V., Soret, M.: Mixed gravitational field equations on globally hyperbolic spacetimes. Class. Quantum Gravity **30**, 085015 (2013). https://doi.org/10.1088/0264-9381/30/8/085015
2. Barletta, E., Dragomir, S., Rovenski, V.: The Einstein-Hilbert type action on foliations. Balkan J. Geom. Appl. **22**(1), 1–17 (2017)
3. Besse, A.L.: Einstein Manifolds. Springer, Berlin (1987)
4. Blair, D.: Riemannian Geometry of Contact and Symplectic Manifolds. Birkhäuser, Basel (2010)
5. Brito, F., Walczak, P.: On the energy of unit vector fields with isolated singularities. Ann. Pol. Math. **73**(3), 269–274 (2000)
6. Brunetti, L., Pastore, A.M.: On the classification of Lorentzian Sasaki space forms. Publications De L'Institut Mathématique, Nouvelle série, tome **94**(108), 163–168 (2013)
7. Candel, A., Conlon, L.: Foliations, I–II. AMS, Providence (2000)
8. Gray, A.: Pseudo-Riemannian almost-product manifolds and submersions. J. Math. Mech. **16**(7), 715–737 (1967)
9. Kashiwada, T.: On a contact 3-structure. Math. Z. **238**(4), 829–832 (2001)
10. O'Neill, B.: Semi-Riemannian Geometry. Academic Press, New York (1983)
11. Rovenski, V.: Foliations on Riemannian Manifolds and Submanifolds. Birkhäuser, Basel (1998)
12. Rovenski, V.: On solutions to equations with partial Ricci curvature. J. Geom. Phys. **86**, 370–382 (2014)
13. Rovenski, V., Walczak, P.: Topics in extrinsic geometry of codimension-one foliations. In: Springer Briefs in Mathematics. Springer (2011)
14. Rovenski, V., Zawadzki, T.: The Einstein–Hilbert type action on foliated pseudo-Riemannian manifolds. Preprint (2016) arXiv:1604.00985
15. Takagi, R., Yorozu, S.: Minimal foliations on Lie groups. Tohoku Math. J. **36**(4), 541–554 (1984)
16. Tondeur, P.: Geometry of Foliations. Springer, Berlin (1997)
17. Topping, P.: Lectures on the Ricci Flow, LMS Lecture Notes, vol. 325. Cambridge University Press, Cambridge (2006)
18. Walczak, P.: An integral formula for a Riemannian manifold with two orthogonal complementary distributions. Colloq. Math. **58**, 243–252 (1990)
19. Zawadzki, T.: Existence conditions for conformal submersions with totally umbilical fibers. Differe. Geom. Appl. **35**, 69–85 (2014)