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Using the FES framework to derive new physical degrees of freedom for finite element spaces of differential forms

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Abstract
In this paper, we study a geometric approach for constructing physical degrees of freedom for sequences of finite element spaces. Within the framework of finite element systems, we propose new degrees of freedom for the spaces $\mathcal{P}_r\Lambda^k$ of polynomial differential forms and we verify numerically their unisolvence.

Keywords Finite element system · High order · New degrees of freedom · Exterior differential calculus

Mathematics Subject Classification (2010) 65N30

1 Introduction

Given a simplicial mesh of a domain in $\mathbb{R}^n$, there are well-known finite-dimensional spaces of differentials forms that are smooth on each simplex and compatible along interfaces. They are known as Whitney forms [21] and, as analyzed in [6], they correspond with the lowest order finite element spaces presented in [16, 17]. All these

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spaces and many others (such as Lagrange finite elements and discontinuous ones) are instances of spaces of polynomial differential forms belonging to the two families $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$. These families fit into the *finite element exterior calculus* [4] framework, where everything is cast into the language of differential forms and the concepts of differential complexes and commuting diagrams are used to study their properties.

There are two standard choices of degrees of freedom for these discrete spaces: the *harmonic* ones [9] and the *moments* [4]. It is well known that the associated interpolators commute with the exterior derivative. A third possibility for the family $\mathcal{P}_r^- \Lambda^k$ has been proposed in [19]: the *weights*, which are integrals on specific subsimplices of dimension k, called the small k-simplices. These latter degrees of freedom are labeled as *physical* since they represent quantities physically meaningful such as circulations if $k = 1$, fluxes if $k = 2$ or densities if $k = n$, for the considered field intended as a k-form. On the numerical side, these weights are interesting because they are a natural generalization to $k > 0$ of the evaluations at points. The evaluations at the points of the principal lattice in the simplex are used in Lagrange finite elements to define, when $k = 0$, a set of degrees of freedom from which we can reconstruct an approximation of a scalar field, intended as a 0-form. The weights have another remarkable property: for each k and from each simplex T, there is a one-to-one correspondence between the canonical spanning family of $\mathcal{P}_r^- \Lambda^k(T)$ and the small k-simplices; moreover, the subset of the such small k-simplices corresponding to a basis (obtained from the spanning family removing the redundant elements) yields unisolvent degrees of freedom. To the best of the author's knowledge, physical degrees of freedom for the second family $\mathcal{P}_r \Lambda^k$ have never been considered. The aim of these pages is to fill this gap.

To this purpose, here we develop an abstract framework for physical degrees of freedom in the context of finite element systems (FES). We recall that FES were introduced in [10] and allow to construct mixed finite elements, generalizing those of [7, 16, 17, 20]. In this work, in particular,

i We derive three contraints that must be satisfied by a unisolvent system of physical degrees of freedom with commuting interpolator.
ii We recast the existing degrees of freedom for the first family $\mathcal{P}_r^- \Lambda^k$ into this framework.
iii We propose a physical system of degrees of freedom for the second family $\mathcal{P}_r \Lambda^k$ in two dimensions. The studied construction is inspired by the isomorphism

$$\mathring{\mathcal{P}}_{r-k} \Lambda^k(T) \cong \mathcal{P}^-_{r-\dim T} \Lambda^{\dim T - k}(T)$$

for a simplex T. As an interesting consequence, we obtain a geometrical interpretation of the duality pairing between $\mathring{\mathcal{P}}_{r-k} \Lambda^k$ and $\mathcal{P}^-_{r-n} \Lambda^{n-k}$ as realization of Lefschetz duality (see, e.g., [12] or [13]).
iv We verify numerically the unisolvence of the proposed physical degrees of freedom for $\mathcal{P}_r \Lambda^1$ in two dimensions (known in the literature as the Brezzi-Douglas-Marini finite element space in [7]). The numerical validation for $k = 1$ together with Theorem 5.1 allows to conjecture the unisolvence of this new system of degrees of freedom for the second family $\mathcal{P}_r \Lambda^k$ in two dimensions.

The outline of this work is as follows. In Section 2, we recall the notation and the main results of the abstract framework of FES which we will use in the following. In Section 3, we include in this framework the physical degrees of freedom and identify some necessary conditions on the choice of the k-cells to obtain unisolvent degrees of freedom. In Section 4, we show that the overdetermining *small simplices* proposed in [19] and the unisolvent ones suggested in [1] for the space $\mathcal{P}_r^- \Lambda^k$ can be recovered by using this technique. Finally, in Section 5, we propose physical degrees of freedom for the spaces $\mathcal{P}_r \Lambda^k$ in two dimensions and give a numerical evidence for their unisolvence when $k = 1$.

2 Basic notation and abstract framework

In this section, we generalize the notions of simplex and simplicial complex. Then we describe the abstract framework of Finite Element Systems introduced in [10]. This framework includes as a special case both the spaces $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$ for which we are going to define new degrees of freedom.

2.1 Geometrical toolbox

Homological algebra A *differential complex* $V^\bullet = \{(V^k, d^k)_{k\in\mathbb{Z}}\}$ is a (countable) collection of vector spaces $\{V^k\}_{k\in\mathbb{Z}}$ equipped with linear operators $d^k : V^k \to V^{k+1}$, called differentials and satisfying, for each k, the condition $d^{k+1} \circ d^k = 0$. Hence, the range of d^k is contained in the kernel of d^{k+1}. The cohomology group $H^k V^\bullet$ is the vector space defined by

$$H^k V^\bullet = (\ker d^k : V^k \to V^{k+1})/(\operatorname{im} d^{k-1} : V^{k-1} \to V^k).$$

A differential complex $V^\bullet$ is usually represented via a diagram

$$\dots \to V^{k-1} \overset{d^{k-1}}{\to} V^k \overset{d^k}{\to} V^{k+1} \to \dots$$

A complex $V^\bullet$ is said to be exact at an index k if $H^k V^\bullet = \{0\}$, or equivalently, $\ker d^k = \operatorname{im} d^{k-1}$. A *chain map* between differential complexes $V^\bullet = \{V^k, d^k\}_{k\in\mathbb{Z}}$ and $W^\bullet = \{W^k, \delta^k\}$ is a collection of linear maps $F^\bullet = \{F^k : V^k \to W^k\}$ such that, for each k, the following diagram commutes :

$$\begin{array}{ccc} V^k & \xrightarrow{d^k} & V^{k+1} \\ \downarrow g^k & & \downarrow g^{k+1} \\ W^k & \xrightarrow{\delta^k} & W^{k+1}. \end{array}$$

If $g^\bullet : V^\bullet \to W^\bullet$ is a cochain map, it induces a sequence of linear maps

$$\begin{aligned} H^k g^\bullet : H^k V^\bullet &\to H^k W^\bullet, \\ [\omega] &\mapsto [g^k \omega]. \end{aligned}$$

If $H^k g^\bullet$ is an isomorphism for each k, then we say that $g^\bullet$ *induces isomorphisms in cohomology*.

Differential geometry and exterior calculus Let now Ω be a manifold with Lipschitz boundary embedded in $\mathbb{R}^n$. We indicate by $\Lambda^k(\Omega)$ the set of differential k-forms over Ω and by d^k the exterior derivative acting on $\Lambda^k(\Omega)$. It is well known that d^k is linear and $d^{k+1} \circ d^k = 0$. Therefore, the collection $\Lambda^\bullet(\Omega) := \{\Lambda^k(\Omega), d^k\}$ is a differential complex, called the *de Rham complex*. Most often the superscript k in d^k is dropped. As a convention, the arrows starting or ending in 0 are the only possible ones to initiate or finish the diagram. Arrows starting in $\mathbb{R}$ are, unless otherwise specified, the maps taking a value to the corresponding constant function. Arrows ending in $\mathbb{R}$ are integration of forms of maximal degree. Other unspecified arrows are instances of the exterior derivative. Several times we will use *Stokes theorem*:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega, \qquad \omega \in \Lambda^{\dim \Omega - 1}(\Omega).$$

Assume now that Ω is a bounded domain with Lipschitz boundary in $\mathbb{R}^n$ equipped with the standard Euclidean metric. We can then introduce the *Hodge star operator* $*^k : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ (see, e.g., [12]). Moreover, let $\sharp : \Lambda^1(\Omega) \to C^\infty(\Omega, \mathbb{R}^n)$ be the flat operator that, given a metric, transforms a 1-form into a vector. The cases $n = 2$ and $n = 3$ are particularly interesting since we can rephrase the de Rham complex in the language of vector calculus. For $n = 3$, we have the following commuting diagram where the vertical arrows are isomorphisms:

$$\begin{CD}
0 @>>> \Lambda^0(\Omega) @>d>> \Lambda^1(\Omega) @>d>> \Lambda^2(\Omega) @>d>> \Lambda^3(\Omega) @>>> 0 \\
@. @VV\iota V @VV\sharp V @VV\sharp\circ *^2 V @VV*^3 V @. \\
0 @>>> C^\infty(\Omega) @>\text{grad}>> C^\infty(\Omega, \mathbb{R}^3) @>\text{curl}>> C^\infty(\Omega, \mathbb{R}^3) @>\text{div}>> C^\infty(\Omega) @>>> 0.
\end{CD}$$

The case $n = 2$ is trickier (and often source of confusion) since there are *two* ways of passing from 1-forms to vector fields, giving rise to two equivalent de Rham complexes:

$$\begin{CD}
0 @>>> C^\infty(\Omega) @>\text{curl}>> C^\infty(\Omega, \mathbb{R}^2) @>\text{div}>> C^\infty(\Omega) @>>> 0 \\
@. @A\iota AA @A-\sharp\circ *^1 AA @A*^2 AA @. \\
0 @>>> \Lambda^0(\Omega) @>d>> \Lambda^1(\Omega) @>d>> \Lambda^2(\Omega) @>>> 0 \\
@. @VV\iota V @VV\sharp V @VV*^2 V @. \\
0 @>>> C^\infty(\Omega) @>\text{grad}>> C^\infty(\Omega, \mathbb{R}^2) @>\text{curl}_2>> C^\infty(\Omega) @>>> 0.
\end{CD}$$

In the diagram above, $\text{curl} := R \circ \text{grad}$ and $\text{curl}_2 := \text{div} \circ R$ where R is the clockwise rotation of $\pi/2$ in $\mathbb{R}^2$. It follows that there are two Sobolev-de Rham complexes:

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}_2, \Omega) \xrightarrow{\text{curl}_2} L^2(\Omega) \longrightarrow 0,$$

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0.$$

The spaces $H(\mathrm{div}, \Omega)$ and $H(\mathrm{curl}_2, \Omega)$ differ by a rotation of $\pi/2$ and the same holds for the finite element spaces used to approximate them. For example, in [20] and [7], the $H(\mathrm{div})$ case is addressed, but the same spaces, rotated by $\pi/2$, can be used to approximate $H(\mathrm{curl}_2)$.

Let now γ be a smooth oriented curve embedded in $\Omega \subset \mathbb{R}^2$ (denote by ι_γ the embedding), e.g., $\gamma = \partial\Omega$, and let $\boldsymbol{t}_\gamma$ be its tangent vector field. Define the normal vector field as $\boldsymbol{n}_\gamma := R\boldsymbol{t}_\gamma$. For a 1-form $\omega \in \Lambda^1(\Omega)$, it holds

$$\iota_\gamma^*\omega = (\sharp\omega)_{\restriction\gamma} \cdot \boldsymbol{t}_\gamma = (-\sharp *^1 \omega)_{\restriction\gamma} \cdot \boldsymbol{n}_\gamma, \tag{2.1}$$

that is, the pull-back of ω to a curve corresponds with the tangential or the normal component of the associated proxy field, depending on the chosen isomorphism.

Cellular homology Let $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n$ be $n+1$ points in $\mathbb{R}^n$ set in general position. Their convex hull $T = [\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n]$ is an *n-simplex*. Denote by $\lambda_0, \ldots, \lambda_n$ its barycentric coordinates. For each $k = 0, \ldots, n$, for each $0 \le i_0 < i_1 < \cdots < i_k \le n$, the convex hull of $\boldsymbol{x}_{i_0}, \ldots, \boldsymbol{x}_{i_k}$ is a *k-subsimplex* S of T. We will write $S = [\boldsymbol{x}_{i_0}, \ldots, \boldsymbol{x}_{i_k}]$ and $S \le T$ to indicate that S is a subsimplex of T. This motivates the following definition: for some nonnegative integer $k \le n$ define

$$\Sigma_0(k, n) := \{\sigma : \{0, \ldots, k\} \to \{0, \ldots, n\} \mid \sigma(i) < \sigma(i+1)\, \forall i = 0, \ldots, k-1\}.$$

We denote by $\mathcal{R}(\sigma)$ the range of σ. To each $\sigma \in \Sigma_0(k, n)$, we can associate the k-subsimplex of T:

$$S_\sigma := [\boldsymbol{x}_{\sigma(0)}, \ldots, \boldsymbol{x}_{\sigma(k)}].$$

With a mild abuse of notation we will write for $S = S_\sigma$, $\mathcal{R}(S) := \mathcal{R}(\sigma)$. For example, if $T = [\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2]$ is a 2-simplex (i.e., a triangle), then its edges are $S_{(01)} = [\boldsymbol{x}_0, \boldsymbol{x}_1]$, $S_{(02)} = [\boldsymbol{x}_0, \boldsymbol{x}_2]$, and $S_{(12)} = [\boldsymbol{x}_1, \boldsymbol{x}_2]$. Given $\sigma \in \Sigma_0(k, n)$, let $\sigma^c \in \Sigma_0(n-k-1, n)$ be the unique element such that $\mathcal{R}(\sigma) \cup \mathcal{R}(\sigma^c) = \{0, 1, \ldots, n\}$. If $k = n$, then σ^c is just the empty map. For $\sigma \in \Sigma_0(k, n)$, we write

$$d\lambda_\sigma = d\lambda_{\sigma(0)} \wedge d\lambda_{\sigma(1)} \wedge \cdots \wedge d\lambda_{\sigma(k)}.$$

Definition 2.1 Let T be an n-simplex. For $k = 0, \ldots, n$, let $S = S_\sigma \le T$ for $\sigma \in \Sigma_0(k, n)$. We define the associated Whitney k-form $\omega^S = \omega^\sigma$ as

$$\omega^S := \omega^\sigma := k! \sum_{i=0}^{k} (-1)^i \lambda_{\sigma(i)} d\lambda_{\sigma(0)} \wedge \cdots \wedge \widehat{d\lambda_{\sigma(i)}} \wedge \cdots \wedge d\lambda_{\sigma(k)},$$

where the widehat means that the underlying term is omitted.

For a subsimplex S of T, will denote by $\mathrm{tr}_{T,S}$ the pull-back associated with the inclusion $\iota_{S,T} : S \to T$. The generalization of a simplex is a cell, which we now define. Denote by $\mathbb{B}^k$ the unit ball in $\mathbb{R}^k$ and by $\mathbb{S}^{k-1}$ its boundary.

Definition 2.2 Let Ω be a metric space. A k-dimensional *cell* is a closed subset T of Ω for which there is a Lipschitz bijection

$$\mathcal{L} : \mathbb{B}^k \to T$$

with a Lipschitz inverse. For $k \geq 1$, we denote by ∂T the *boundary* of T, that is $\partial T := \mathcal{L}(\mathbb{S}^{k-1})$. We define the *interior* of T as $\mathring{T} := T \setminus \partial T$. For $k = 0$, T is a point, $\partial T = \emptyset$ and $\mathring{T} = T$[1].

Note that the definition of the boundary does not depend on the chosen bi-Lipschitz isomorphism $\mathcal{L}$. In this work, we will consider only "flat," cells, that is, a 0-cell is a point, a 1-cell is an edge, a 2-cell is a polygon, and so on.

Definition 2.3 A *cellular complex* is a pair $(\Omega, \mathcal{T})$ where Ω is a compact metric space and $\mathcal{T}$ is a finite set of cells in Ω such that:

- Distinct cells in $\mathcal{T}$ have disjoint interiors.
- The boundary of any cell in $\mathcal{T}$ is a union of cells in $\mathcal{T}$.
- The union of all cells in $\mathcal{T}$ is Ω.

When the metric space Ω is understood, we will just write $\mathcal{T}$. We assume that each cell $T \in \mathcal{T}$ has been oriented (as a manifold with corners). If T is an oriented cell, we denote by $-T$ the same cell, but with the opposite orientation. Given $T \in \mathcal{T}^{k+1}$ and $S \in \mathcal{T}^k$ define their *relative orientation* or *incidence number* $o(T, S)$ as 1 if S is outward oriented compared to T, -1 if it is inward oriented and 0 otherwise. The following result is Proposition 5.2 in [8].

Lemma 2.1 *Let $\mathcal{T}$ be a cellular complex. The intersection of two cells in $\mathcal{T}$ is a union of cells in $\mathcal{T}$.*

In this work, a *simplicial complex* is a cellular complex such that each cell is a simplex and the intersection of two cells is a cell (not just a union of cells).

Denote by $\mathcal{T}^k$ the set of k-cells in $\mathcal{T}$. A k-chain c is a formal linear combinations (over $\mathbb{R}$) of k-cells, that is

$$c = \sum_{T \in \mathcal{T}^k} c_T T, \quad c_T \in \mathbb{R}.$$

The set $C_k(\mathcal{T})$ of all k-chains is a finite dimensional real vector space and its dimension equals the number of k-cells. The *boundary operator* $\partial_k : C_k(\mathcal{T}) \to C_{k-1}(\mathcal{T})$ is defined on k-cells as

$$\partial_k T := \sum_{S \in \mathcal{T}^{k-1}} o(T, S) S,$$

and extended by linearity on k-chains. The dual space of $C_k(\mathcal{T})$ is denoted by $C^k(\mathcal{T})$ and it is called the space of k-cochains, that is a k-cochain is a linear function $C_k(\mathcal{T}) \to \mathbb{R}$. The *coboundary operator* $\delta^\bullet$ is simply the dual of the boundary operator, i.e., $\delta^k := (\partial_{k+1})^*$. It is easy to check that $C^\bullet(\mathcal{T}) := \{C^k(\mathcal{T}), \delta^k\}_{k \in \mathbb{Z}}$

[1] This is consistent with equipping T with its intrinsic topology, instead of the one inherited from the ambient space

is a differential complex. Let now $(\Omega, \mathcal{T})$ be a cellular complex, with Ω smooth Riemannian manifold with corners. We define the *de Rham map* as

$$\begin{aligned}\mathfrak{R}^{(k)} : \Lambda^k &\to C^k(\mathcal{T}) \\ \omega &\mapsto \left\{c \mapsto \int_c \omega\right\}.\end{aligned}$$

Stokes theorem implies that de Rham map is a chain map from $\Lambda^\bullet(\Omega)$ to $C^\bullet(\mathcal{T})$.

Let now Ω' be a closed submanifold of Ω (e.g., $\Omega' = \partial\Omega$) and define

$$\Lambda^k(\Omega, \Omega') := \{\omega \in \Lambda^k(\Omega) \mid \iota^*_{\Omega'}\omega = 0\}.$$

If $(\Omega', \mathcal{T}')$ is a cellular complex and $\mathcal{T}' \subset \mathcal{T}$, then we can define the space of *relative k-chains* as $C_k(\mathcal{T}, \mathcal{T}') := C_k(\mathcal{T})/C_k(\mathcal{T}')$. We denote its dual space by $C^k(\mathcal{T}, \mathcal{T}')$, that is the space of relative k-cochains. As noted in [15], we can view this space as the subspace of $C^k(\mathcal{T}')$ made of cochains that vanish on each cell of $\mathcal{T}'$, that is the cochains supported by $\mathcal{T} \setminus \mathcal{T}'$. It is easy to check that the coboundary operator maps this space into itself: if $\phi \in C^k(\mathcal{T})$ such that $\phi(T) = 0$ for each cell $T \in \mathcal{T}'$, then ∂T is a chain carried by $\mathcal{T}'$ and therefore

$$(\delta\phi)(T) = \phi(\partial T) = 0.$$

Moreover, we can define a " relative de Rham map,"

$$\begin{aligned}\Lambda^k(\Omega, \Omega') &\to C^k(\mathcal{T}, \mathcal{T}') \\ \omega &\mapsto \left\{[c] \mapsto \int_c \omega\right\}\end{aligned} \tag{2.2}$$

which is well defined and is a chain map.

2.2 Finite element systems

In this section, we make use of the abstract framework of finite element systems [8, 9] (see also Section 7 in [14]).

Definition 2.4 Let Ω be a polyhedral domain in $\mathbb{R}^n$ and let $\mathcal{T}$ be a cellular complex over Ω. A *finite element system* (FES) X is a choice, for each $T \in \mathcal{T}$ and for each $k \in \mathbb{Z}_{\geq 0}$, of a finite dimensional subspace $X^k(T)$ of $\Lambda^k(T)$ such that:

– if $S, T \in \mathcal{T}$, $S \leq T$ then the pull-back $\mathrm{tr}_{T,S} := \iota^*_{S,T}$ by the inclusion $\iota_{S,T} : S \to T$, induces a map
$$\mathrm{tr}_{T,S} : X^k(T) \to X^k(S); \tag{2.3}$$
– for any simplex $T \in \mathcal{T}$, for each $k \in \mathbb{Z}_{\geq 0}$, the exterior derivative maps $X^k(T)$ into $X^{k+1}(T)$;

Let now $\mathcal{T}$ be a simplicial triangulation. The global discrete space is constructed as

$$X^k(\mathcal{T}) := \{\omega \in \bigoplus_{T \in \mathcal{T}} X^k(T) \mid \mathrm{tr}_{T,S}\omega_T = \omega_S,\ \forall S, T \in \mathcal{T},\ S \leq T\}, \tag{2.4}$$

that is, $X^k(\mathcal{T})$ is the subset of single valued elements of $\bigoplus_{T\in\mathcal{T}} X^k(T)$. This definition holds in particular for any subtriangulation $\mathcal{T}'$ of $\mathcal{T}$. An important example is the boundary ∂T of a simplex T.

Definition 2.5 Let X be finite element system, then:

– if, for any simplex $T \in \mathcal{T}$, the space $X^0(T)$ contains the constant functions on T and the complex

$$0 \to \mathbb{R} \xrightarrow{\iota} X^0(T) \xrightarrow{d} \dots \xrightarrow{d} X^{\dim T}(T) \to 0 \tag{2.5}$$

is exact, then we will say that X is *locally exact*.
– if, for each $T \in \mathcal{T}$, for each $k \in \mathbb{Z}_{\geq 0}$, the map $\mathrm{tr}_{T,\partial T} : X^k(T) \to X^k(\partial T)$ is onto, we will say that X *admits extensions*. The kernel of this map is denoted by $\mathring{X}^k(T)$. That is,

$$\mathring{X}^k(T) := \{\omega \in X^k(T) \mid \mathrm{tr}_{T,\partial T}\,\omega = 0\}.$$

We will say that a finite element system X is *compatible* if both conditions hold. The following result is Proposition 2.2. in [9].

Lemma 2.2 *Assume that, for any $T \in \mathcal{T}$ with $S \in \partial T$, for each $k = 0, \dots, \dim S$, there exists a linear mapping*

$$\mathrm{ext}^k_{S,T} : \mathring{X}^k(S) \to X^k(T),$$

which is a right inverse of the trace operator, that is

$$\mathrm{tr}_{T,S}\mathrm{ext}^k_{S,T}\omega = \omega, \quad \forall \omega \in \mathring{X}^k(S),$$

and, for $S, S' \in \partial T$, $\dim S = \dim S'$, $S \neq S'$

$$\mathrm{tr}_{T,S'}\mathrm{ext}^k_{S,T}\omega = 0$$

then X admits extensions. We will call $\mathrm{ext}^k_{S,T}$ an extension operator.

Assume moreover that, for each $S \leq T$, we have extension operators $\mathrm{ext}^k_{S,T}$: $\mathring{X}^k(S) \to X^k(T)$ such that they satisfy for $\omega \in \mathring{X}^k(S)$ and $S' \leq T$ with $\dim(S') = \dim(S)$:

$$\mathrm{tr}_{S'}\mathrm{ext}^k_{S,T}\omega = \begin{cases} \omega \text{ if } S = S', \\ 0 \text{ otherwise.} \end{cases} \tag{2.6}$$

The following result is Proposition 4.20 in [4].

Lemma 2.3 *Fix $k \in \{0, \dots, \dim T\}$. Assume that we have extension operators as in* Eq. 2.6 *and that*

$$\sum_{S\leq T} \mathrm{ext}^k_{S,T}\eta_S = 0, \tag{2.7}$$

for some family of $\eta_S \in \mathring{X}^k(S)$, $S \leq T$. Then each η_S vanishes.

As a consequence, the space $X^k(T)$ admits the following geometrical decomposition

$$X^k(T) = \bigoplus_{S \le T} \mathrm{ext}^k_{S,T} \mathring{X}^k(S)$$

Such decomposition is important in practice since it leads to a local basis for the (on the left-hand side) space $X^k(T)$ consisting of elements $\mathrm{ext}^k_{S,T} w$, where $S \le T$ and w ranges over a basis for the (right-hand-sided) space $\mathring{X}^k(S)$.

Example 2.1 We consider the following two spaces of polynomial differential forms defined in [4]

$$\mathcal{P}_r^- \Lambda^k(T) := \mathcal{P}_{r-1}(T) \cdot \mathtt{W}^k(T),$$
$$\mathcal{P}_{r-k} \Lambda^k(T) := \mathcal{P}_{r-k}(T) \otimes \mathrm{Alt}^k \mathbb{R}^{\dim T},$$

where $\mathtt{W}^k(T)$ is the space of *Whitney forms* and $\mathcal{P}_s(T) = \emptyset$ for $s < 0$. The spaces $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_{r-k} \Lambda^k$ are finite element systems.[2] They are locally exact (see [4] for a proof) and they have extension operators (see, e.g., [14])

$$\mathrm{ext}^k_{S,T} : \mathring{\mathcal{P}}_{r-k} \Lambda^k(S) \to \mathcal{P}_{r-k} \Lambda^k(T),$$
$$\mathrm{ext}^{-,k}_{S,T} : \mathring{\mathcal{P}}_r^- \Lambda^k(S) \to \mathcal{P}_r^- \Lambda^k(T),$$

for each $S, T \in \mathcal{T}$, $S \le T$; therefore, we conclude from Lemma 2.2 that they admit extensions. In particular, for each $T \in \mathcal{T}$, they admit the following geometrical decompositions (see [4] and [14]):

$$\mathcal{P}_r^- \Lambda^k(T) = \bigoplus_{S \le T} \mathrm{ext}^{-,k}_{S,T} \mathring{\mathcal{P}}_r^- \Lambda^k(S), \tag{2.8}$$

$$\mathcal{P}_{r-k} \Lambda^k(T) = \bigoplus_{S \le T} \mathrm{ext}^k_{S,T} \mathring{\mathcal{P}}_{r-k} \Lambda^k(S). \tag{2.9}$$

Example 2.2 We detail the case $k = 0$ (see also Section 2.3 of [5]). For both families of spaces recalled in Example 2.6, we have $X^0(T) = \mathcal{P}_r(T)$. Let $\mathcal{I}(r, n)$ be the set of multi-indices

$$\mathcal{I}(r, n) = \{\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_n)\,,\ \alpha_i \in \mathbb{Z}_{\ge 0},\ \sum_i \alpha_i = r\}\,.$$

and $\mathcal{S}(\boldsymbol{\alpha}) = \{i\,,\ \alpha_i \ne 0\}$. To define the extension operator, for each $S = S_\sigma$ subsimplex of T, let $\lambda_0^S, \dots, \lambda_{\dim S}^S$ be its barycentric coordinates. We will use the notation

$$(\lambda^S)^{\boldsymbol{\alpha}} := (\lambda_0^S)^{\alpha_0} \dots (\lambda_n^S)^{\alpha_n}.$$

Then the set

$$\mathcal{B}\mathring{\mathcal{P}}_r(S) := \{(\lambda^S)^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathcal{I}(r, \dim S),\ \mathcal{S}(\boldsymbol{\alpha}) = \{0, \dots, \dim S\}\}$$

[2] Note that $\mathcal{P}_r \Lambda^k$ is a finite element system too, but it is not locally exact. Therefore, from now on when we refer to the "second family," of polynomial differential forms, we will mean $\mathcal{P}_{r-k} \Lambda^k$.

is a basis for $\mathring{\mathcal{P}}_r(S)$, the space of polynomials of degree r on S that vanish on the boundary of S. In fact, if S is an simplex, then the map $p \mapsto \lambda_0^S \dots \lambda_{\dim S}^S p$ is an isomorphism between $\mathcal{P}_{r-\dim S-1}(S)$ and $\mathring{\mathcal{P}}_r(S)$. In particular, it maps the monomial basis of $\mathcal{P}_{r-\dim S-1}(S)$ onto $\mathcal{B}\mathring{\mathcal{P}}_r(S)$. Then we define the extension operator

$$\begin{aligned} \mathrm{ext}_{S,T} : \mathring{\mathcal{P}}_r(S) &\to \mathcal{P}_r(T), \\ (\lambda^S)^{\boldsymbol{\alpha}} &\mapsto (\lambda^T)^{\sigma \circ \boldsymbol{\alpha}}, \end{aligned} \tag{2.10}$$

where $\sigma \circ \boldsymbol{\alpha}$ is the element in $\mathcal{I}(r, n)$ such that, for $i = 0, \dots, n$:

$$(\sigma \circ \boldsymbol{\alpha})_i = \begin{cases} \alpha_{\sigma(j)} \text{ if } i = \sigma(j) \text{ for some } j = 0, \dots, \dim S, \\ 0 \text{ otherwise.} \end{cases}$$

It follows that

$$\mathrm{ext}_{S,T} \mathcal{B}\mathring{\mathcal{P}}_r(S) = \{\lambda^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathcal{I}(r, n),\ \mathcal{S}(\boldsymbol{\alpha}) = \mathcal{R}(\sigma)\}.$$

Now we verify condition (2.6): if S' is a simplex different from S but with the same dimension, then there exists a vertex $\boldsymbol{x}_j$ that belongs to S but not to S'. It follows that S' is a subsimplex of the face F_{j^*} opposite to $\boldsymbol{x}_j$, which lays on the hyperplane $\lambda_j = 0$. On the other side if $\lambda^{\boldsymbol{\alpha}} \in \mathcal{B}\mathring{\mathcal{P}}_r(S)$, then $\alpha_j > 0$. It follows that $\mathrm{tr}_{S'} \mathrm{ext}_{S,T} \mathcal{B}\mathring{\mathcal{P}}_r(S) = 0$. Therefore, we obtain the geometrical decomposition

$$\mathcal{P}_r(T) = \bigoplus_{S \le T} \mathrm{ext}_{S,T} \mathring{\mathcal{P}}_r(S).$$

We have also the following property (Proposition 5.7 in [8]).

Lemma 2.4 *Assume that we have extension operators. Then the exactness* (2.5) *is equivalent to the combination of the following two properties, for each* $T \in \mathcal{T}$*:*

- *The space* $X^0(T)$ *contains the constant functions;*
- *the complex*

$$0 \to \mathring{X}^0(T) \xrightarrow{d} \dots \xrightarrow{d} \mathring{X}^{\dim T}(T) \xrightarrow{\int} \mathbb{R} \to 0 \tag{2.11}$$

is exact.

2.3 Degrees of freedom

Definition 2.6 A *system of degrees of freedom* (sysdof) W is a choice, for each $T \in \mathcal{T}$, for each $k \in \mathbb{Z}_{\ge 0}$ of a finite dimensional subspace $\mathring{W}^k(T)$ of $\Lambda^k(T)^*$, where the superscript $*$ indicates the dual.

If $S \le T$, then each $\phi \in \mathring{W}^k(S)$ extends trivially to $\Lambda^k(T)^*$ via the map

$$\begin{aligned} \mathring{W}^k(S) &\to \Lambda^k(T)^* \\ \phi &\mapsto \phi \circ \mathrm{tr}_{T,S}. \end{aligned}$$

With an abuse of notation, for each $T \in \mathcal{T}$, we will consider $\mathring{W}^k(T)$ as a subspace of $\Lambda^k(\Omega)^*$. We recall the following definition (e.g., see [8]).

Definition 2.7 A sysdof W is said to be *unisolvent* for a finite element system X if, for each $T \in \mathcal{T}$, for each $k \in \mathbb{Z}_{\geq 0}$, the map

$$\begin{aligned} X^k(T) &\to \bigoplus_{S \leq T} \mathring{W}^k(S)^*, \\ \omega &\mapsto \left\{\phi \mapsto \phi(\mathrm{tr}_{T,S}\omega)\right\}_{S \leq T}, \end{aligned} \tag{2.12}$$

is an isomorphism. If the map (2.12) is injective, but not surjective, then we say that the sysdof is *overdetermining*. Finally, we say that W is *minimal* if

$$\dim X^k(T) = \sum_{S \leq T} \dim \mathring{W}^k(S)^*.$$

We have the following result (Proposition 2.5 in [9])

Lemma 2.5 *Let W be a sysdof and let X be a finite element system. The following are equivalent:*

- *W is unisolvent;*
- *X admits extensions, and for each simplex $T \in \mathcal{T}$, the map*

$$\begin{aligned} \mathring{X}^k(T) &\to \mathring{W}^k(T)^*, \\ \omega &\mapsto \{\phi \mapsto \phi(\omega)\} \end{aligned} \tag{2.13}$$

  is an isomorphism.
- *For each $T \in \mathcal{T}$, the map $\mathring{X}^k(T) \to \mathring{W}^k(T)^*$ is injective and W is minimal.*

Example 2.3 Let T be a simplex *of arbitrary dimension* and let η be a polynomial $(\dim T - k)$-form. Denote by ϕ_η the functional

$$\phi_\eta : \omega \mapsto \int_T \omega \wedge \eta, \quad \omega \in \Lambda^k(T).$$

For the finite element systems $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_{r-k}\Lambda^k$ defined in Example 2.6, the standard systems of degrees of freedom are given respectively by the spaces

$$\mathring{W}_r^{-,k}(T) := \{\phi_\eta \mid \eta \in \mathcal{P}_{r-\dim T+k-1}\Lambda^{\dim T-k}(T)\}, \tag{2.14}$$

$$\mathring{W}_{r-k}^k(T) := \{\phi_\eta \mid \eta \in \mathcal{P}^-_{r-\dim T}\Lambda^{\dim T-k}(T)\}. \tag{2.15}$$

As proved in [4], these sysdofs are unisolvent. As observed in [14], the unisolvence is equivalent to the following isomorphisms:

$$\mathring{\mathcal{P}}_r^- \Lambda^k(T) \simeq \mathcal{P}_{r-\dim T+k-1}\Lambda^{\dim T-k}(T), \tag{2.16}$$

$$\mathring{\mathcal{P}}_{r-k}\Lambda^k(T) \simeq \mathcal{P}^-_{r-\dim T}\Lambda^{\dim T-k}(T). \tag{2.17}$$

Note that for $k = 0$, the spaces on the left-hand side in Eqs. 2.16 and 2.17 are both equal to $\mathring{\mathcal{P}}_r(T)$ and the right-handed ones to $\mathcal{P}_{r-\dim T-1}(T)$.

Example 2.4 In the case $k = 0$, both sysdofs reduce to

$$(\mathring{W}_r^{-,0}(T) =) \quad \mathring{W}_r^0(T) := \left\{\omega \mapsto \int_T \omega\, \eta \operatorname{vol}_T \mid \eta \in \mathcal{P}_{r-\dim T-1}(T)\right\}.$$

To prove the unisolvence, we check the third condition in Lemma 2.5. We start by proving the injectivity of the map $\mathring{\mathcal{P}}_r(T) \to \mathring{W}_r^0(T)^*$. Let $\omega \in \mathring{P}_r(T)$ and assume that

$$\int_T \omega\, \eta \operatorname{vol}_T = 0 \quad \forall \eta \in \mathcal{P}_{r-\dim T-1}(T).$$

Since the restriction of ω to the boundary of T is zero, then the barycentric coordinates of T divide ω, i.e., $\omega = \lambda_0 \ldots \lambda_{\dim T}\varphi$ for some $\varphi \in \mathcal{P}_{r-\dim T-1}(T)$. Then, taking $\eta = \varphi$, it follows that

$$\int_T \lambda_0 \ldots \lambda_{\dim T}\, \varphi^2 \operatorname{vol}_T = 0.$$

Since $\lambda_0 \ldots \lambda_{\dim T}$ is strictly positive in the interior of T, it follows that $\varphi = 0$ and consequently $\omega = 0$. Minimality follows from the dimension count:

$$\begin{aligned}\sum_{S \le T} \dim \mathring{W}_r^0(S) &= \sum_{S \le T} \dim \mathcal{P}_{r-\dim S-1}(S)\\ &= \sum_{j=0}^{n} \binom{n+1}{j+1}\binom{r-1}{j}\\ &= \sum_{j=0}^{n} \binom{n+1}{n-j}\binom{r-1}{j} = \binom{n+r}{n} = \dim \mathcal{P}_r(T).\end{aligned}$$

We compare now briefly the finite element systems with the classical approach by [11] used, e.g., in [4]. In the classical theory of finite elements, one has a local space $X^k(T) \subset \Lambda^k(T)^*$ and a space of functionals $W^k(T) \subset \Lambda^k(T)^*$ only for *top-dimensional* cells T. The degrees of freedom are said to be unisolvent if the map

$$\begin{aligned} X^k(T) &\to W^k(T)^*,\\ \omega &\mapsto \{\phi \mapsto \phi(\omega)\}\end{aligned}$$

is an isomorphism. Notice that we still do not know anything about how to construct the global space $X^k(\mathcal{T})$. If we furthermore assume that the $W^k(T)$ admits a geometrical decomposition

$$W^k(T) = \bigoplus_{S \le T} \mathring{W}^k(S), \tag{2.18}$$

then we can define the global space $X^k(\mathcal{T})$ as the set of forms ω in $L^2\Lambda^k(\Omega)$ such that:

- The restriction $\omega_{\restriction T}$ of ω to a top-dimensional cell T belongs to $X^k(T)$;
- If S is a common subcell of two top-dimensional cells T and T', then, for each $\phi \in \mathring{W}^k(S)$, $\phi(\operatorname{tr}_S \omega_{\restriction T}) = \phi(\operatorname{tr}_S \omega_{\restriction T'})$.

Therefore, it is the geometrical decomposition (2.18) that determines the degree of inter-element continuity. Moreover, we set $X^k(S) := \mathrm{tr}_S X^k(T)$ for each $S \le T$, so that condition (2.3) is true almost by definition. If the geometrical decomposition (2.18) is chosen in such a way that, for each $S \le T$, the map (2.13) is an isomorphism, then (2.18) induces a geometrical decomposition of the space $X^k(T)$ itself:

$$X^k(T) = \bigoplus_{S \le T} \mathrm{ext}^k_{S,T} \mathring{X}^k(S). \tag{2.19}$$

This path seems quite twisted: (i) degrees of freedom should reflect the nature (and global regularity properties) of the fields they represent, (ii) satisfy the geometrical decomposition (2.18), and (iii) the map (2.13) be an isomorphism. On the other side, in finite element systems, the spaces $X^k(T)$ (where now T is a cell of arbitrary dimension) are chosen carefully to satisfy the geometrical decomposition (2.19) and *then* the spaces $\mathring{W}^k(S)$ are chosen in such a way that the map (2.13) is an isomorphism. In particular note that the definition of the global space $X^k(\mathcal{T})$ is intrinsic and does not depend on the degrees of freedom. This observations will be of key importance in the introduction of physical degrees of freedom. We show the differences in the two approaches with an example.

Example 2.5 For $k = 0$, let T be a *top-dimensional* simplex and $X^0(T) = \mathcal{P}_r(T)$. Let $\mathcal{F}^0_r(T)$ be the principal lattice of order r of T, that is, the set of points with barycentric coordinates:

$$\left(\frac{\alpha_0}{r}, \dots, \frac{\alpha_{\dim T}}{r}\right),$$

for $(\alpha_0, \dots, \alpha_{\dim T}) = \boldsymbol{\alpha} \in \mathcal{I}(r, n)$. Let us define $W^0_r(T) \subset C^0(T)^*$ as the space spanned by the evaluations on $\mathcal{F}^0_r(T)$, that is

$$W^0_r(T) := \mathrm{span}\{\omega \mapsto \omega(s) \mid s \in \mathcal{F}^0_r(T)\}.$$

It is well known that a polynomial of degree r on T is uniquely determined by its values on $\mathcal{F}^0_r(T)$ and therefore the map $\mathcal{P}_r(T) \to W^0_r(T)^*$ is an isomorphism. Moreover, the space $W^0_r(T)$ admits the following geometrical decomposition

$$W^0_r(T) := \bigoplus_{S \le T} \mathring{W}^0_r(S),$$

with

$$\mathring{W}^0_r(S) := \mathrm{span}\{\omega \mapsto \omega(s) \mid s \in \mathring{\mathcal{F}}^0_r(S)\}, \quad \mathring{\mathcal{F}}^0_S = \mathcal{F}^0_r(S) \cap \mathring{S}.$$

With this geometrical decomposition, it follows that the global space $\mathcal{P}_r(\mathcal{T})$ is the space of functions ω in $L^2(\Omega)$ such that:

- on each n-simplex T of $\mathcal{T}$, $\omega_{\restriction T} \in \mathcal{P}_r(T)$;
- if S is a common subsimplex of the n-simplices T and T' then, for each $s \in \mathring{\mathcal{F}}^0_r(T)$, then $\omega_{\restriction T}(s) = \omega_{\restriction T'}(s)$.

Therefore, the geometrical decomposition of the degrees of freedom guarantees that a function in $\mathcal{P}_r(\mathcal{T})$ is continuous. Finally, for each subsimplex S of T, the space $\mathcal{P}_r(S)$ is just the restriction of $\mathcal{P}_r(T)$ to S and the map $\mathring{\mathcal{P}}_r(S) \to \mathring{W}^0_r(S)^*$ is an isomorphism.

If we identify $W_r^0(T)$ with $\mathcal{P}_r(T)^*$, then the geometrical decomposition of $W_r^0(T)$ induces a geometrical decomposition of the space $\mathcal{P}_r(T)$ itself.

On the other side, in the framework of finite element systems, for a simplex T (now of arbitrary dimension), we set $X^0(T) = \mathcal{P}_r(T)$. The associated global space $\mathcal{P}_r(\mathcal{T})$ is defined as the set of functions ω in $L^2(\Omega)$ such that:

- for each simplex, of any dimension, of $\mathcal{T}$, $\omega_{\restriction T} \in \mathcal{P}_r(T)$;
- if S is a subsimplex of T, then $(\omega_{\restriction T})_{\restriction S} = \omega_{\restriction S}$.

The two definitions of $\mathcal{P}_r(\mathcal{T})$ are equivalent, in the sense that they produce the same space, but the former involves degrees of freedom, while the latter does not. Moreover, we have seen in Example 2.2 that $\mathcal{P}_r(T)$ admits a geometrical decomposition that does not need degrees of freedom to be defined or proved. Finally, the spaces $\mathring{W}_r^0(T)$ for $T \in \mathcal{T}$ constitute a unisolvent sysdofs.

2.4 Interpolators

For a finite element system X, an *interpolator* is a collection of projection operators $\Pi^k(T) : \Lambda^k(T) \to X^k(T)$, which commute with the trace operator. An unisolvent system of degrees of freedom defines an interpolator by requiring

$$\phi(\Pi^k(T)\omega) = \phi(\omega) \quad \text{for each } \phi \in W^k(T).$$

We are interested in *commuting* (with the exterior derivative) interpolators, that is we want the following diagram to be commutative:

$$\begin{array}{ccc} \Lambda^k(T) & \xrightarrow{d} & \Lambda^{k+1}(T) \\ \big\downarrow \Pi^k(T) & & \big\downarrow \Pi^{k+1}(T) \\ X^k(T) & \xrightarrow{d} & X^{k+1}(T). \end{array}$$

We take the following lemma from [8].

Lemma 2.6 *Assume that X is a finite element system that is locally exact and admits extensions. Assume moreover which W is a unisolvent sysdof. Then the induced interpolators commute with the exterior derivative if and only if for each* $T \in \mathcal{T}$, $k = 1, \ldots, \dim T$, *for each* $\phi \in \mathring{W}^{k+1}(T)$ *we have*

$$\phi \circ d \in \bigoplus_{S \leq T} \mathring{W}^k(S). \tag{2.20}$$

Example 2.6 It is easy to check that the sysdofs Eqs. 2.14 and 2.15 satisfy condition Eq. 2.20. We give the proof for the first one. Let $\eta \in \mathcal{P}_{r-\dim T+k}\Lambda^{\dim T-k-1}(T)$ and let ϕ_η be the associated functional in $\mathring{W}^{k+1}(T)$. Then, Stokes theorem implies that, for $\omega \in \Lambda^k(T)$,

$$\begin{aligned}
\phi_\eta \circ d(\omega) &= \int_T d\omega \wedge \eta \\
&= \sum_{S \in \Delta_{\dim T-1}(T)} o(T,S) \int_S \omega \wedge \mathrm{tr}_S \eta + (-1)^{k+1} \int_T \omega \wedge d\eta \\
&= \sum_{S \in \Delta_{\dim T-1}(T)} o(T,S)\, \phi_{\mathrm{tr}_S \eta}(\omega) + (-1)^{k+1} \phi_{d\eta}(\omega).
\end{aligned}$$

Since we have $\mathrm{tr}_S \eta \in \mathcal{P}_{r+k-1-\dim T+1} \Lambda^{\dim T-k-1}(S) = \mathcal{P}_{r+k-1-\dim S} \Lambda^{\dim S-k}(S)$ and $d\eta \in \mathcal{P}_{r-\dim T+k-1} \Lambda^{\dim T-k}(T)$, then $\phi_{\mathrm{tr}_S \eta} \in \mathring{W}^k(S)$ and $\phi_{d\eta} \in \mathring{W}^k(T)$, so that condition (2.20) is satisfied.

3 Physical degrees of freedom

The natural generalization of the degrees of freedom described, for $k = 0$, in Example 2.5 is to consider, for any $k > 0$, integrals of k-forms on particular sets of k-cells (see an example of such k-cells for $r = 3$ in Fig.1, with $k = 0$ on the left, and $k = 1, 2$ on the right).

Assume then that, for each simplex $T \in \mathcal{T}$, for each k, we have a *finite* set of k-cells $\mathring{\mathcal{F}}^k(T) = \{s_1, s_2, \ldots\}$[3] contained in T and such that distinct cells have disjoint interiors. For each simplex $T \in \mathcal{T}$, define

$$\mathring{W}^k(T) := \mathrm{span}\left\{\phi : \omega \mapsto \int_s \mathrm{tr}_{T,s}\omega \mid s \in \mathring{\mathcal{F}}^k(T)\right\}. \tag{3.1}$$

These degrees of freedom are called *physical* since they have a natural physical interpretation: e.g., for $k = 1$, they are the work done by a force (a 1-form) along a line (a 1-cell).

In this section, we will derive some constraints (i.e., necessary conditions) on the choice of the sets of k-cells that must be satisfied to obtain overdetermining or unisolvent sysdofs such that the associated interpolator commutes with the exterior derivative.

A first condition that the sets $\mathring{\mathcal{F}}^k(T)$ must satisfy is the following one:

Constraint 1 *For each $T \in \mathcal{T}$, for each $k = 0, \ldots, \dim T$*

$$|\mathring{\mathcal{F}}^k(T)| \geq \dim \mathring{X}^k(T),$$

where $|\cdot|$ indicates the cardinality.

We prove now the necessity of this constraint.

[3] In this work, we use the lower case s to denote "small" cells, to distinguish them from the simplices of the mesh $\mathcal{T}$, which are indicated by the capital letters S and T.

Lemma 3.1 *If* $\mathring{W}^k(T)$ *has the form* (3.1), *then* $\dim \mathring{W}^k(T) = |\mathring{\mathcal{F}}^k(T)|$. *In particular, if the sysdof W is overdetermining then Constraint 1 holds. If W is unisolvent, then Constraint 1 holds with the equality.*

Proof Clearly $\dim \mathring{W}^k(T) \leq |\mathring{\mathcal{F}}^k(T)|$. Assume by contradiction that the inequality is strict. Then there exists $s \in \mathring{\mathcal{F}}^k(T)$ such that, for each $\omega \in \Lambda^k(T)$, it holds

$$\int_s \mathrm{tr}_s \omega = \sum_{s' \in \mathring{\mathcal{F}}^k(T) \setminus \{s\}} \beta_{s'} \int_{s'} \mathrm{tr}_{s'} \omega \tag{3.2}$$

for some $\beta_{s'} \in \mathbb{R}$. Let now $\hat{\omega} \in \Lambda^k(T)$ such that

$$\int_{s'} \mathrm{tr}_{s'} \hat{\omega} = \begin{cases} 1 \text{ if } s = s', \\ 0 \text{ otherwise.} \end{cases}$$

Inserting $\hat{\omega}$ into expression (3.2), we obtain $1 = 0$, which is a contradiction. □ □

The following condition is necessary to guarantee that the interpolator associated to the degrees of freedom defined in Eq. 3.1 commutes with the exterior derivative.

Constraint 2 *For each* $T \in \mathcal{T}$, *the union*

$$\mathcal{F}^\bullet(T) := \bigcup_{S \leq T} \bigcup_{k=0}^{\dim S} \mathring{\mathcal{F}}^k(S).$$

is a cellular complex.

Lemma 3.2 *Let W be a unisolvent physical sysdof, that is, each* $\mathring{W}^k(T)$ *has the form* (3.1). *Then the associated interpolator commutes with the exterior derivative if and only if Constraint 2 is satisfied.*

Proof Let T be a cell in $\mathcal{T}$ and let $\phi \in \mathring{W}^{k+1}(T)$ for $k \in \mathbb{Z}_{\geq 0}$ be defined as

$$\phi(\omega) = \int_s \mathrm{tr}_s \omega, \quad s \in \mathring{\mathcal{F}}^{k+1}(T),$$

for $\omega \in X^{k+1}(T)$. Stokes theorem implies that, for $\omega \in X^k(T)$, it holds

$$\phi(d\omega) = \int_s \mathrm{tr}_s d\omega = \int_s d\mathrm{tr}_s \omega = \int_{\partial s} \mathrm{tr}_{\partial s} \omega.$$

Now, noticing that

$$\bigoplus_{S \leq T} \mathring{W}^k(S) = \mathrm{span} \left\{ \omega \mapsto \int_s \mathrm{tr}_s \omega \mid s \in \bigcup_{S \leq T} \mathring{\mathcal{F}}^k(S) \right\}$$

it is easy to see that condition (2.20) is satisfied if and only if ∂s is a union of cells in $\bigcup_{S \leq T} \mathring{\mathcal{F}}^k(S)$, i.e., if and only if Constraint 2 is satisfied. □

We have thus proved that if a physical sysdof W is unisolvent or overdetermining, it satisfies necessarily Constraint 1 and 2. If it is unisolvent, then it satisfies the following additional constraint.

Constraint 3 *For each $T \in \mathcal{T}$, the union of all* $\dim T$*-dimensional cells $\mathring{\mathcal{F}}^{(\dim T)}(T)$ covers T.*

Recall that $C^k(\mathcal{F}^\bullet(T))$ denotes the space of k-cochains over $\mathcal{F}^\bullet(T)$. If $\mathcal{T}'$ is a subtriangulation of $\mathcal{T}$, then, Constraint 2 implies that

$$\mathcal{F}^\bullet(\mathcal{T}') := \bigcup_{T\in\mathcal{T}'} \bigcup_{k=0}^{\dim T} \mathring{\mathcal{F}}^k(T)$$

is a cellular complex, then it make sense to consider chains and cochains over $\mathcal{F}^\bullet(\mathcal{T}')$.

Theorem 3.1 *Let W be a unisolvent physical sysdof and assume that the associated interpolator commutes with the exterior derivative. Then Constraint 3 holds.*

Proof If each $\mathring{W}^k(T)$ is of the form (3.1), then the map (2.12) is an isomorphism for each k if and only if the de Rham map $\mathfrak{R}^\bullet = \{\mathfrak{R}^k\}_{k\in\mathbb{Z}_{\geq 0}}$ defined as

$$\begin{aligned}\mathfrak{R}^k : X^k(T) &\to C^k(\mathcal{F}^\bullet(T)),\\ \omega &\mapsto \left\{c \mapsto \int_c \mathrm{tr}_c\omega\right\},\end{aligned}$$

is an isomorphism. If this is the case then $\mathfrak{R}^\bullet$ induces isomorphisms in cohomology, and therefore

$$\dim H^k C^\bullet(\mathcal{F}^\bullet(T)) = \begin{cases}1 \text{ if } k = 0,\\ 0 \text{ otherwise.}\end{cases}$$

Since this is true for each $T \in \mathcal{T}$, then Constraint 3 must hold. □

We have proved that if a physical sysdof is unisolvent, it satisfies necessarily Constraint 1 (with equality), 2, and 3. The reversed implication, whether the Constraint 1 (with equality), 2, and 3 yield in particular unisolvence, is an open question.

4 Physical degrees of freedom for $\mathcal{P}_r^- \Lambda^k$

In this section, we consider the compatible finite element system $X^k = \mathcal{P}_r^- \Lambda^k$ and the overdetermining physical degrees of freedom introduced in [19]. We recall which are the usual spanning families for the spaces $\mathcal{P}_r^- \Lambda^k(T)$ and we will assign a k-cell to each element of the family. The resulting physical degrees of freedom are overdetermining, mirroring the fact that the spanning families have linear dependencies. Removing redundant elements from the spanning families in order to obtain bases yields unisolvent degrees of freedom, see an example for $k = 1$ in [18]. It is also possible to work with spanning families provided that their linear dependencies are taken into account in the construction of the bases as explained in [3].

4.1 Small simplices

We start by recalling some notation introduced in *Example 2.2*. Let T be an n-simplex, with barycentric coordinates $\lambda_0, \ldots, \lambda_n$. For s, n nonnegative integers, we recall that $\mathcal{I}(s, n)$ is the set of multi-indices $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n)$ such that $\alpha_i \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=0}^{n} \alpha_i = s$. We will use the notation

$$\lambda^{\boldsymbol{\alpha}} := \lambda_0^{\alpha_0} \ldots \lambda_n^{\alpha_n}.$$

Definition 4.1 Le T be an n-simplex and let $\boldsymbol{\alpha} \in \mathcal{I}(r-1, n)$ for $r \in \mathbb{Z}_{>0}$. Then $s^{\boldsymbol{\alpha}}$ is the *small* n-simplex contained in T defined as $s^{\boldsymbol{\alpha}} = \{z_{\boldsymbol{\alpha}}(x),\ x \in T\}$, where

$$z_{\boldsymbol{\alpha}} \quad : \quad x \longmapsto z_{\boldsymbol{\alpha}}(x) = \frac{1}{r} \sum_{i=0}^{n} [\lambda_i(x) + \alpha_i]\, x_i\,.$$

Let $\boldsymbol{x}_0^{\boldsymbol{\alpha}}, \ldots, \boldsymbol{x}_n^{\boldsymbol{\alpha}}$ be the vertices of $s^{\boldsymbol{\alpha}}$ and let $\sigma \in \Sigma_0(k, n)$ for $0 \leq k \leq n$. Define $s_\sigma^{\boldsymbol{\alpha}}$ as the *small* k-simplex spanned by $\boldsymbol{x}_{\sigma(0)}^{\boldsymbol{\alpha}}, \ldots, \boldsymbol{x}_{\sigma(k)}^{\boldsymbol{\alpha}}$.

See Fig. 1 for an example with $r = 3$.

4.2 Bases and spanning families

In this section, we summarize which are the canonical spanning families for the spaces $\mathcal{P}_r^- \Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ for an n-simplex T. In particular, we show how one can extract a basis from them. These results are stated clearly in [14] but they were already known in [4]. Let T be an n-simplex. The sets

$$\begin{aligned} \mathcal{SP}_r^- \Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} \omega^\sigma \mid \boldsymbol{\alpha} \in \mathcal{I}(r-1, n),\ \sigma \in \Sigma_0(k, n)\}, \\ \mathcal{S}\mathring{\mathcal{P}}_r^- \Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} \omega^\sigma \in \mathcal{SP}_r^- \Lambda^k(T) \mid \mathcal{S}(\boldsymbol{\alpha}) \cup \mathcal{R}(\sigma) = \{0, \ldots, n\}\} \end{aligned}$$

are spanning families for the spaces $\mathcal{P}_r^- \Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ respectively. In general, they are not linearly independent sets. For example, if T is a 2-simplex, then

$$\mathcal{S}\mathring{\mathcal{P}}_2^- \Lambda^1(T) := \{\lambda_0 \omega^{(12)}, \lambda_1 \omega^{(02)}, \lambda_2 \omega^{(01)}\} \tag{4.1}$$

and a direct computation shows that $\lambda_0 \omega^{(12)} - \lambda_1 \omega^{(02)} + \lambda_2 \omega^{(01)} = 0$. Therefore, one has to introduce a constraint to remove some elements and obtain a linearly

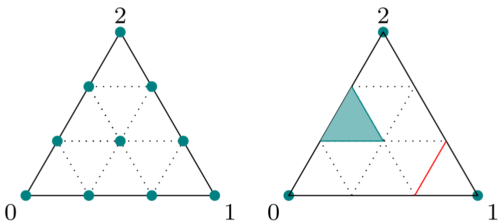


Fig. 1 Example of small k-simplices, $k = 0$ (left), and $k = 1, 2$ (right), for $r = 3$. The nodes of the principal lattice of degree r in T coincide with the small nodes. In blue, the small triangle $s^{(1,0,1)}$; in red, the small edge $s_{(02)}^{(0,2,0)}$

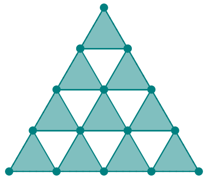


Fig. 2 Overdetermining degrees of freedom for $\mathcal{P}_4^- \Lambda^k(T)$, where T is a 2-simplex

independent set. The sets

$$\begin{aligned}\mathcal{BP}_r^- \Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} \omega^\sigma \in \mathcal{SP}_r^- \Lambda^k(T) \mid \min \mathcal{S}(\boldsymbol{\alpha}) \geq \min \mathcal{R}(\sigma)\}, \\ \mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} \omega^\sigma \in \mathcal{S}\mathring{\mathcal{P}}_r^- \Lambda^k(T) \mid \min \mathcal{R}(\sigma) = 0, \}\end{aligned}$$

are bases of the spaces $\mathcal{P}_r^- \Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ respectively (see [14]). Returning to example (4.1), we have that

$$\mathcal{B}\mathring{\mathcal{P}}_2^- \Lambda^1(T) = \{\lambda_1 \omega^{(02)}, \lambda_2 \omega^{(01)}\} \subsetneq \mathcal{S}\mathring{P}_2^- \Lambda^1(T).$$

In other words, the constraint removes the element $\lambda_0 \omega^{(12)}$ from the set $\mathcal{S}\mathring{P}_2^- \Lambda^1(T)$.

4.3 Small *k*-cells

In this section, we will denote by $\mathring{\mathcal{F}}_r^{-,k}(T)$ the set $\mathring{\mathcal{F}}^k(T)$ to emphasize that we are constructing a system of degrees of freedom for $\mathring{\mathcal{P}}_r^- \Lambda^k(T)$. For each simplex T, $n := \dim T = 0, 1, 2$, we will define a bijection between $\mathcal{S}\mathring{\mathcal{P}}_r^- \Lambda^k(T)$ (or $\mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^k(T)$) and $\mathring{\mathcal{F}}_r^{-,k}(T)$, $k = 0, \ldots, n$. In the first case, we will obtain an overdetermining system of degrees of freedom, that is, in general the de Rham map

$$\mathfrak{R}^k : \mathcal{P}_r^- \Lambda^k(T) \to C^k(\mathcal{F}^\bullet(T)) \tag{4.2}$$

will be only injective. In the second case, Eq. 4.2 will be an isomorphism.

Overdetermining degrees of freedom Let T be a n-simplex. For each $S \leq T$, we define $\mathring{\mathcal{F}}_r^{-,k}(T)$ in the following way. To each k-form $\lambda^{\boldsymbol{\alpha}} \omega^\sigma \in \mathcal{S}\mathring{\mathcal{P}}_r^- \Lambda^k(S)$, assign the small k-simplex $s_\sigma^{\boldsymbol{\alpha}}$. An example for $n = 2$ and $r = 4$ is given in Fig. 2. Note that, with this choices, if $r > 1$, we have

$$|\mathring{\mathcal{F}}_r^{-,1}(T)| = |\mathcal{S}\mathring{\mathcal{P}}_r^- \Lambda^1(T)| > |\mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^1(T)| = \dim \mathring{\mathcal{P}}_r^- \Lambda^1(T),$$

therefore Constraint 1 is satisfied, but not with equality. It is easy to check that Constraint 2 is satisfied, while Constraint 3 is not because of the "holes". Nevertheless, the resulting system of degrees of freedom is overdetermining (for a proof see [9]). On the other hand, one is left with the problem of extracting a unisolvent subspace from $\mathring{W}^1(T)$.

Minimal degrees of freedom We describe now a system of physical degrees of freedom which satisfies Constraints 1' and 3. We limit ourselves to the two dimensional case.

- Case $k = 0$, $n = 0, 1, 2$. To each $\lambda^{\boldsymbol{\alpha}}\omega^\sigma \in \mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^0(T)$, assign the point $s_\sigma^{\boldsymbol{\alpha}}$. For $n = 0, 1, 2$, $\mathring{\mathcal{F}}_r^{-,0}(T)$ is the set of points on the principal lattice of order r of T which lay *on the interior* of T.
- Case $k = n = 1$. Assign to each $\lambda^{\boldsymbol{\alpha}}\omega^\sigma \in \mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^1(T)$ the small simplex $s^{\boldsymbol{\alpha}}$.
- Case $k = 1$, $n = 2$. Here to each $\lambda^{\boldsymbol{\alpha}}\omega^\sigma \in \mathcal{B}\mathring{\mathcal{P}}_r^- \lambda^1(T)$ assign the (small) edge $s_\sigma^{\boldsymbol{\alpha}}$. In other words, the set $\mathring{\mathcal{F}}_r^1(T)$ is obtained by the overdetermining one by removing the small edges parallel to the line $\lambda_0 = 0$.
- Case $k = 2$, $n = 2$. The set $\mathring{\mathcal{F}}_r^{-,2}(T)$ is then uniquely determined by Constraint 2. We assign to each $\lambda^{\boldsymbol{\alpha}}\omega^\sigma \in \mathcal{B}\mathring{\mathcal{P}}_r^- \Lambda^2(T)$ the unique cell in $\mathring{\mathcal{F}}_r^{-,2}(T)$ containing the small 2-simplex $s^{\boldsymbol{\alpha}}$.

Figure 3 gives an example of this construction for $r = 4$.

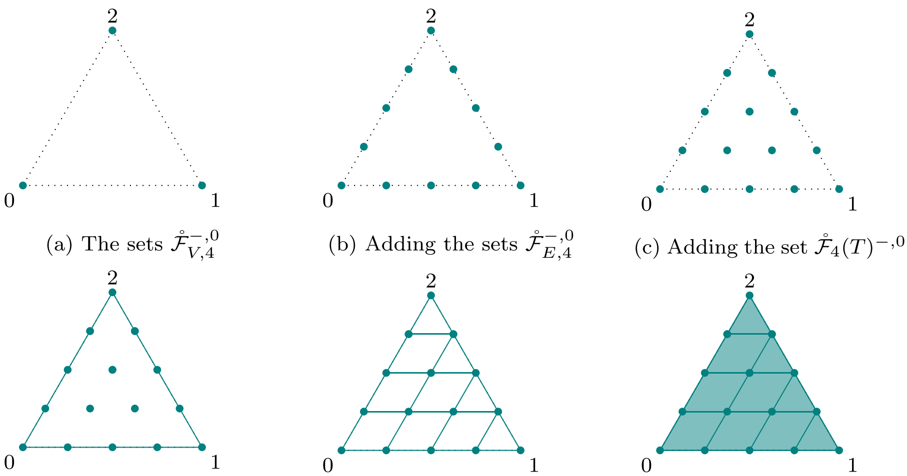
Now, for a 2-simplex T, we set $\mathcal{F}_r^{-,k}(T) := \bigcup_{S \le T} \mathring{\mathcal{F}}_r^{-,k}(S)$ for $k \le \dim S$, $k = 0, 1, 2$. These k-cells coincide with those obtained by chopping those defined in [1] as explained in [2]. The associated spaces

$$W_r^{-,k}(T) := \operatorname{span}\left\{\phi : \omega \mapsto \int_s \operatorname{tr}_{T,s}\omega \mid s \in \mathcal{F}_r^{-,k}(T)\right\}$$

are indeed dual spaces of $\mathcal{P}_r^- \Lambda^k(T)$ for $k = 0, 1, 2$.

5 Physical degrees of freedom for $\mathcal{P}_{r-k}\Lambda^k$

In this section, we will consider the compatible finite element system $X^k = \mathcal{P}_{r-k}\Lambda^k$. We will discuss only the two-dimensional case. As in the previous section, for each simplex T, we will associate a k-cell topologically contained in T to each element of



(a) The sets $\mathring{\mathcal{F}}_{V,4}^{-,0}$ (b) Adding the sets $\mathring{\mathcal{F}}_{E,4}^{-,0}$ (c) Adding the set $\mathring{\mathcal{F}}_4(T)^{-,0}$ (d) Adding the sets $\mathring{\mathcal{F}}_{E,4}^{-,1}$ (e) Adding the set $\mathring{\mathcal{F}}_4^{-,1}(T)$ (f) Adding the set $\mathring{\mathcal{F}}_4^{-,2}(T)$

Fig. 3 Construction of the cellular complex $\mathcal{F}_4^{-,\bullet}(T)$

a spanning family of $\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$ obtaining in this way a system of physical degrees of freedom which is not minimal. Removing redundant elements from the spanning families in order to obtain bases will result in minimal degrees of freedom. We will motivate this choice and discuss why we conjecture the unisolvence of this system of degrees of freedom. Numerical evidence of this fact is provided in the next section.

We start with an example consisting in a cellular complex (constraint 2 is thus verified) which induces a system of physical degrees of freedom which satisfies Constraint 1 with equality but not Constraint 3.

Consider the simplicial complex in Fig. 4 in a triangle T, for $r = 3$.

This complex satisfies Constraint 1 with equality but not Constraint 3. Indeed, there are, respectively, 10 ($= \dim\mathcal{P}_{3-0}\Lambda^0(T)$) small nodes, $3 \times 3 + 3$ ($= \dim\mathcal{P}_{3-1}\Lambda^1(T)$) small edges, and 3 ($= \dim\mathcal{P}_{3-2}\Lambda^2(T)$) small triangles. On the other hand, the complex of small simplices in Fig. 4a is topologically equivalent to the union of a circle $\mathbb{S}^1$ and a point c, the center of $\mathbb{S}^1$. Thus, T is contractible contrarily to $\mathbb{S}^1 \cup \{c\}$ which is not. In other words, we have the right number of degrees of freedom, but the cohomology of the complex $C^\bullet(\mathcal{F}^\bullet(T))$ is different from the cohomology of $X^\bullet(T)$. It follows that the de Rham map

$$\mathfrak{R}^k : \mathcal{P}_{3-k}\Lambda^k(T) \to C^k(\mathcal{F}^\bullet(T))$$

cannot be an isomorphism for all $k = 0, 1, 2$. This example shows that having a correct number of small simplices for the spaces $\mathcal{P}_{r-k}\Lambda^k(T)$ is not the only condition to fulfill. Intuitively, these cells have to be chosen in order to give a connected graph that *covers* T completely (for example, see Fig. 4b), as we are going to explain in the following.

5.1 Spanning families and bases

The discussion which follows is analogous to the one in Section 4.2. The sets

$$\begin{aligned} \mathcal{SP}_{r-k}\Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma \mid \boldsymbol{\alpha} \in \mathcal{I}(r-k, n),\ \sigma \in \Sigma_0(k-1, n)\}, \\ \mathcal{S}\mathring{\mathcal{P}}_{r-k}\Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma \in \mathcal{SP}_{r-k}\Lambda^k(T) \mid \mathcal{S}(\boldsymbol{\alpha}) \cup \mathcal{R}(\sigma) = \{0, \dots, n\}\} \end{aligned}$$

are spanning families for the spaces $\mathcal{P}_{r-k}\Lambda^k(T)$ and $\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$ respectively. For example, if T is a 2-simplex, the elements

$$\lambda_0\lambda_1\lambda_2 d\lambda_0,\ \lambda_0\lambda_1\lambda_2 d\lambda_1,\ \lambda_0\lambda_1\lambda_2 d\lambda_2 \tag{5.1}$$

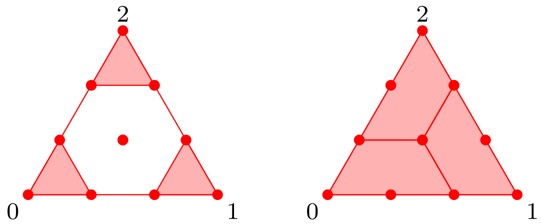


Fig. 4 Two cellular complexes: a non admissible configuration (left), and an admissible configuration (right)

belong to $\mathcal{S}\mathring{\mathcal{P}}_3\Lambda^1(T)$, but they satisfy

$$\lambda_0\lambda_1\lambda_2 d\lambda_0 + \lambda_0\lambda_1\lambda_2 d\lambda_1 + \lambda_0\lambda_1\lambda_2 d\lambda_2 = 0.$$

Again, the introduction of constraints allows to remove the "redundant," elements. For example, the above constraint removes $\lambda_0\lambda_1\lambda_2 d\lambda_2$ from the list (5.1). The sets

$$\begin{aligned}\mathcal{B}\mathcal{P}_{r-k}\Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma \in \mathcal{S}\mathcal{P}_{r-k}\Lambda^k(T) \mid \min \mathcal{S}(\boldsymbol{\alpha}) \notin \mathcal{R}(\sigma)\},\\ \mathcal{B}\mathring{\mathcal{P}}_{r-k}\Lambda^k(T) &:= \{\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma \in \mathcal{S}\mathring{\mathcal{P}}_{r-k}\Lambda^k(T) \mid \min \mathcal{S}(\boldsymbol{\alpha}) \notin \mathcal{R}(\sigma)\}\end{aligned}$$

are bases of the spaces $\mathcal{P}_{r-k}\Lambda^k(T)$ and $\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$ respectively (see [14]).

5.2 Small *k*-cells

In this section, we will denote by $\mathring{\mathcal{F}}^k_{r-k}(T)$ the set $\mathring{\mathcal{F}}^k(T)$ to emphasize that we are constructing a system of degrees of freedom for $\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$.

To introduce physical degrees of freedom, we exploit the isomorphism (2.17). Assume that we have a cellular complex $\mathcal{F}^\bullet_r(T)$ such that the relative de Rham map induces an isomorphism

$$\mathring{\mathcal{P}}_{r-k}\Lambda^k(T) \simeq C^k(\mathcal{F}^\bullet_r(T), \mathcal{F}^\bullet_r(\partial T)).$$

Then we combine this isomorphism with Eq. 2.17 and the unisolvent physical degrees of freedom for the space $\mathcal{P}^-_r\Lambda^k(T)$, obtaining

$$C^k(\mathcal{F}^\bullet_r(T), \mathcal{F}^\bullet_r(\partial T)) \simeq \mathring{\mathcal{P}}_{r-k}\Lambda^k(T) \simeq \mathcal{P}^-_{r-\dim T}\Lambda^{\dim T-k}(T)^* \simeq C_k(\mathcal{F}^{-,\bullet}(T)). \tag{5.2}$$

The isomorphisms (5.2) suggest how these physical degrees of freedom can be constructed: the cells in $\mathring{\mathcal{F}}^k_{r-k}(T)$ must be "dual," with respect to those in $\mathcal{F}^{-,\dim T-k}_{r-\dim T}(T)$. More precisely, consider the map

$$\begin{aligned}\Gamma : \mathring{\mathcal{P}}_{r-k}\Lambda^k(T) &\to \mathcal{P}^-_{r-\dim T}\Lambda^{\dim T-k}(T),\\ \lambda^{\boldsymbol{\alpha}} d\lambda_\sigma &\mapsto \lambda^{\boldsymbol{\alpha}-\sigma^c}\omega^{(\sigma^c)},\end{aligned} \tag{5.3}$$

where $\boldsymbol{\alpha} - \sigma^c$ is the multi-index with components

$$(\boldsymbol{\alpha} - \sigma^c)_i = \begin{cases} \alpha_i - 1 \text{ if } i = \sigma(j) \text{ for some } j = 0, \ldots, \dim T - k,\\ \alpha_i \text{ otherwise,}\end{cases}$$

for $i = 0, \ldots, \dim T$. Since $\boldsymbol{\alpha} \in \mathcal{I}(r-k, \dim T)$ and $\sigma^c \in \Sigma_0(\dim T - k, \dim T)$, then

$$\sum_{i=0}^{\dim T} (\boldsymbol{\alpha} - \sigma^c)_i = r - k - \dim T + k - 1 = r - \dim T - 1,$$

that is, $\boldsymbol{\alpha} - \sigma^c \in \mathcal{I}(r - \dim T - 1, \dim T)$ and therefore Γ maps indeed $\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$ into $\mathcal{P}^-_{r-\dim T}\Lambda^{\dim T-k}(T)$. We show also that the map Γ sends $\mathcal{B}\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$ to

$\mathcal{BP}^-_{r-\dim T}\Lambda^{\dim T-k}(T)$. In fact, let $\boldsymbol{\alpha} \in \mathcal{I}(r-k, \dim T)$ and $\sigma \in \Sigma_0(k, \dim T)$ with $\min \mathcal{S}(\boldsymbol{\alpha}) \notin \mathcal{R}(\sigma)$. Then $\min \mathcal{S}(\boldsymbol{\alpha} - \sigma^c) \geq \min \mathcal{S}(\boldsymbol{\alpha})$, but $\min \mathcal{S}(\boldsymbol{\alpha})$ belongs to $\mathcal{R}(\sigma^c)$ and therefore $\min \mathcal{S}(\boldsymbol{\alpha}) \geq \min \mathcal{R}(\sigma^c)$. Combining the two inequalities, we get that

$$\min \mathcal{S}(\boldsymbol{\alpha} - \sigma^c) \geq \min \mathcal{R}(\sigma^c),$$

that is, $\lambda^{\boldsymbol{\alpha}-\sigma^c}\omega^{\sigma^c}$ belongs to $\mathcal{BP}^-_{r-\dim T}\Lambda^{\dim T-k}(T)$. Moreover, it is shown in [14] that the map Γ is an isomorphism. Our construction is as follows:

1. Start from an element $\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma$ belonging to $\mathcal{S}\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$ (resp. $\mathcal{B}\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$);
2. Apply the map Γ to obtain an element $\Gamma(\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma)$ in $\mathcal{SP}^-_{r-\dim T}\Lambda^{\dim T-k}(T)$ (resp. $\mathcal{BP}^-_{r-\dim T}\Lambda^{\dim T-k}(T)$).
3. Associate to $\Gamma(\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma)$ a small $(\dim T - k)$-cell s^* as in Section 4.
4. Find a k-cell s topologically contained in T such that $s \cap s^*$ is a single point.

We give now an example of this construction.

Example 5.1 In this example, we set $n = 1$, $k = 1$, and $r = 3$. Let us consider an oriented segment E of extreme points x_0, x_1 and associated barycentric functions λ_0^E, λ_1^E (namely, λ_0^E, λ_1^E are affine functions over E such that $\lambda_i^E(x_j) = \delta_{ij}$, being δ_{ij} the Kronecker symbol). So, $x = \lambda_0^E(x)\, x_0 + \lambda_1^E(x)\, x_1$ for any $x \in E = [x_0, x_1]$. Let us consider the space $X^1(E) = \mathcal{P}_2\Lambda^1(E)$. The degrees of freedom for $\omega \in X^1(E)$ as defined in Eq. 2.15 are

$$\mathbf{w} \quad \longmapsto \quad \left\{ \int_E \mathbf{w} \cdot \mathbf{t}_E\, (\lambda_0^E)^2, \qquad \int_E \mathbf{w} \cdot \mathbf{t}_E\, \lambda_0^E \lambda_1^E\,, \qquad \int_E \mathbf{w} \cdot \mathbf{t}_E\, (\lambda_1^E)^2 \right\}$$

where $\mathbf{w}$ is the proxy of ω and $\mathbf{t}_E$ is the unit vector associated with E. Indeed, $\eta \in \mathcal{P}_2^-\Lambda^0(E)$ namely η is a polynomial function of degree 2 over E; thus, a basis of these polynomials is the set $\{\, (\lambda_0^E)^2,\ \lambda_0^E\lambda_1^E\,,\ (\lambda_1^E)^2 \,\}$.

Now, let us start from $\mathcal{P}_2^-\Lambda^0(E) = \mathcal{P}_2(E)$. A well-known choice of degrees of freedom for this space is the value of a function at the end points x_0, x_1 and at one internal point, say $\hat{x}$ (e.g., the midpoint of E). Let x' be a point in $(x_0, \hat{x})$ and x'' be a point in $(\hat{x}, x_1)$. So, we have thus divided E into three segments

$$E = [\, x_0\,,\ x'\,] \cup [\, x'\,,\ x''\,] \cup [\, x''\,,\ x_1\,],$$

which are, in some sense, *dual* to the points $x_0, \hat{x}, x_1$ (note that, in this example, $n = 1$), and we can define new degrees of freedom as follows

$$\mathbf{w} \quad \longmapsto \quad \left\{ \int_{[\,x_0\,,\,x'\,]} \mathbf{w} \cdot \mathbf{t}_E\,, \qquad \int_{[\,x'\,,\,x''\,]} \mathbf{w} \cdot \mathbf{t}_E\,, \qquad \int_{[\,x''\,,\,x_1\,]} \mathbf{w} \cdot \mathbf{t}_E \right\}.$$

More precisely, a basis of $\mathring{\mathcal{P}}_2\Lambda^1(E)$ is given by

$$\mathcal{B}\mathring{\mathcal{P}}_2\Lambda^1(E) = \{(\lambda_0^E)^2 d\lambda_1, \lambda_0^E\lambda_1^E d\lambda_1, (\lambda_1^E)^2 d\lambda_0\}.$$

In this case, the map Γ reads as follows:

$$\begin{aligned}\Gamma : \mathcal{B}\mathring{\mathcal{P}}_2\Lambda^1(E) &\rightarrow \mathcal{B}\mathcal{P}_r^-\Lambda^0(E),\\ (\lambda_0^E)^2 d\lambda_1 &\mapsto (\lambda_0^E)^2,\\ \lambda_0^E\lambda_1^E d\lambda_1 &\mapsto \lambda_0^E\lambda_1^E,\\ (\lambda_1^E)^2 d\lambda_0 &\mapsto (\lambda_1^E)^2.\end{aligned}$$

To the three elements of $\mathcal{B}\mathcal{P}_2^-\Lambda^0(E)$, we associate the points x_0, $\hat{x}$, and x_1 respectively. Finally, we write E as a union of three intervals

$$E = [\,x_0\,,\ x'\,] \cup [\,x'\,,\ x''\,] \cup [\,x''\,,\ x_1\,],$$

such that each interval contains only one point.

Isomorphism (5.2) suggests a geometrical interpretation of the isomorphism (2.17): indeed, (5.2) induces the well-known Lefschetz duality (see, e.g., [13] or [12])

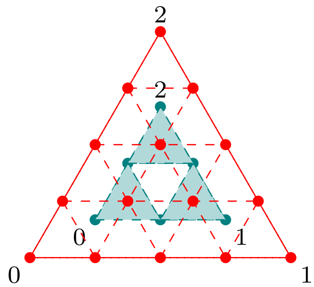
$$H^k(T, \partial T) \cong H_{n-k}(T).$$

Not minimal degrees of freedom Since $\mathring{\mathcal{P}}_r\Lambda^0(T) = \mathring{\mathcal{P}}_r^-\Lambda^0(T) = \mathring{\mathcal{P}}_r(T)$ when $\dim T > 0$, and, for a 1-simplex T, $\mathring{\mathcal{P}}_{r-1}\Lambda^1(T) = \mathring{\mathcal{P}}_r^-\Lambda^1(T)$, we will set $\mathring{\mathcal{F}}_r^0(T) := \mathring{\mathcal{F}}_r^{-,0}(T)$ for any simplex T and $\mathring{\mathcal{F}}_{r-1}^1(T) := \mathring{\mathcal{F}}_r^{-,1}(T)$ for a 1-simplex T. The only cases that remain to be addressed are when T is a 2-simplex and $k = 1$ or 2. We describe now in detail our construction. Consider the principal lattice L of order r on T and the associated small simplices. Note that we can construct a principal lattice L^* of order $r - 2$ on the barycenters of the reversed triangles (see Fig. 5 for an example).

Starting from $\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma \in \mathcal{S}\mathring{\mathcal{P}}_{r-k}\Lambda^k(T)$, apply the Γ map defined in Eq. 5.3 to obtain $\Gamma(\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma) \in \mathcal{B}\mathcal{P}_{r-2}^-\Lambda^{2-k}(T)$. Then, we associate to $\Gamma(\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma)$ a small $(2-k)$-simplex s^* in L^*. Finally, we select a k-cell s which is dual to s^*, that is $s \cap s^*$ is a single point.

- For $k = 1$, note that for each small edge s^* in L^*, there are two small edges of L which share on point with s^*. For the sake of definiteness, if s^* is parallel to the

Fig. 5 Principal lattice L^* of order 3 constructed on the barycenters of the reversed triangles of the principal lattice L of order 5 on T



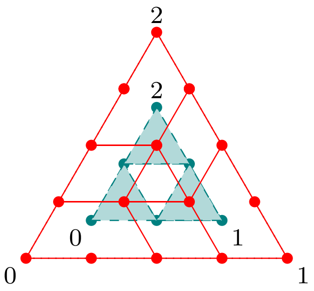


Fig. 6 Internal small edges chosen using the map γ

edge E_σ, we choose the small edge parallel to $E_{\gamma(\sigma)}$ where we have defined the map γ as

$$\begin{aligned}\gamma : \Sigma_0(1,2) &\to \Sigma_0(1,2),\\ (12) &\mapsto (02),\\ (02) &\mapsto (01),\\ (01) &\mapsto (12).\end{aligned}$$

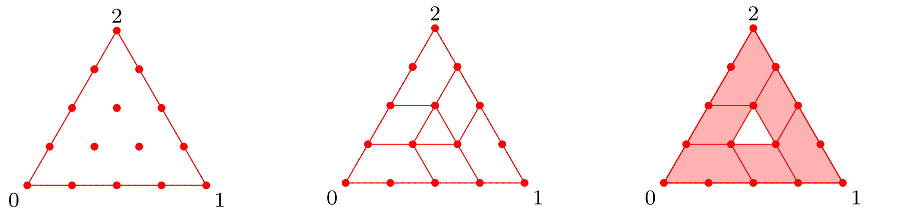
Note that we could also have chosen the small edge parallel to $E_{\gamma^{-1}(\sigma)}$. A more direct way which produces the same result is to assign to each $\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma \in \mathcal{S}\mathring{\mathcal{P}}_{r-1}\Lambda^1(T)$, the small edge $s^{\boldsymbol{\alpha}}_{\gamma(\sigma^c)}$ (see Fig. 6 for an example).

– If $k = 2$, the set $\mathring{\mathcal{F}}^2_{r-2}(T)$ is then uniquely determined by Constraint 2 and from the requirement that each 2-cell contains a single point of the dual principal lattice L^*. More precisely, we assign to each $\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma \in \mathcal{S}\mathring{\mathcal{P}}_{r-2}\Lambda^2(T)$ the unique cell in $\mathring{\mathcal{F}}^2_{r-2}(T)$ containing the point s^*, where s^* in the point in L^* relative to $\Gamma(\lambda^{\boldsymbol{\alpha}} d\lambda_\sigma)$.

Define now

$$\mathcal{F}^\bullet_r(T) := \bigcup_{S \leq T} \bigcup_{k=0}^{\dim S} \mathring{\mathcal{F}}^k_{r-k}(S).$$

An example of this construction for $r = 4$ is given in Fig. 7. As for the spaces $\mathcal{P}^-_r\Lambda^k$, there are holes, which in this case are triangles homothetic to T.



(a) The sets $\mathcal{F}^0_4(T)$ and $\mathring{\mathcal{F}}^1_{E,3}$ (b) Adding the set $\mathring{\mathcal{F}}^1_3(T)$ (c) Adding the sets $\mathring{\mathcal{F}}^2_2(T)$

Fig. 7 Construction of not minimal degrees of freedom for $\mathcal{P}_{4-k}\Lambda^k(T)$, where T is a 2-simplex

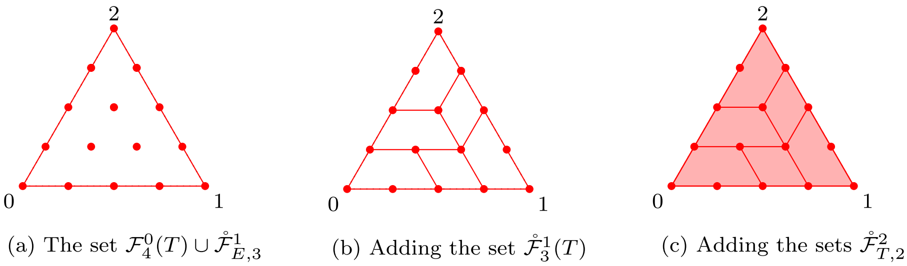


Fig. 8 Minimal degrees of freedom for $\mathcal{P}_{4-k}\Lambda^k(T)$, where T is a 2-simplex

Remark: Due to the presence of holes, as the white one in Fig. 7c, one could argue that, from a homological point of view, T and the supports of degrees of freedom in $\mathcal{F}_r^\bullet(T)$ shown in the same figure are not equivalent. In presence of holes, we have indeed to think in terms of relative homology, as it was done in [19] for the spaces $\mathcal{P}_r^-\Lambda^k$, that is considering homology groups of k-chains modulus the holes of dimension k. Thus, equivalence holds.

Remark: We recall that the presence of holes is related with redundancies in $\mathcal{S}\mathring{\mathcal{P}}_{r-1}\Lambda^1(T)$, namely, the set of possible 1-cochains has bigger cardinality than that of degrees of freedom for $\mathring{\mathcal{P}}_{r-1}\Lambda^1(T)$. As soon as redundancies are eliminated, holes disappear too, as it occurs when we work with a minimal set of degrees of freedom.

Minimal degrees of freedom The same procedure with $\mathcal{B}\mathcal{P}_{r-k}\Lambda^k$ in place of $\mathcal{S}\mathcal{P}_{r-k}\Lambda^k$ leads to a minimal system of degrees of freedom (see Fig. 8). Informally this new set is obtained by the overdetermining one by removing the internal edges parallel to the line $\lambda_1 = 0$ and such that both their vertices belong to the interior of the triangle (see Fig. 9).

5.3 Unisolvence

For a simplex T, let us consider the maps

$$\begin{aligned} \mathring{\mathcal{P}}_{r-k}\Lambda^k(T) &\to \mathring{W}^k(T)^*, \\ \omega &\mapsto \{\phi \mapsto \phi(\omega)\}, \end{aligned} \tag{5.4}$$

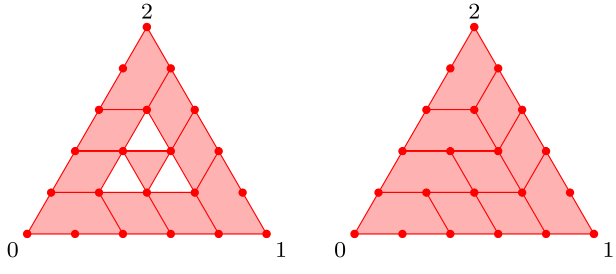


Fig. 9 Not minimal (left) and minimal (right) degrees of freedom for $\mathcal{P}_{5-k}\Lambda^k(T)$, where T is a 2-simplex

for $k = 0, \ldots, \dim T$, where $\mathring{W}^k(T)$ is defined as in (3.1) with $\mathring{\mathcal{F}}^k_{r-k}(T)$ instead of $\mathring{\mathcal{F}}^k(T)$. Condition 2 of Lemma 5 states that the sysdof is unisolvent if and only if these maps are isomorphisms. In two dimensions, this is nontrivial only when T is a 2-simplex and $k = 1, 2$. Moreover, the following result holds true.

Theorem 5.1 *The de Rham map* $\mathfrak{R}^2 : \mathring{\mathcal{P}}_r\Lambda^2(T) \to C^2(\mathcal{F}^\bullet_r(T), \mathcal{F}^\bullet_r(\partial T))$ *is an isomorphism if and only if the de Rham map* $\mathfrak{R}^1 : \mathring{\mathcal{P}}_r\Lambda^1(T) \to C^1(\mathcal{F}^\bullet_r(T), \mathcal{F}^\bullet_r(\partial T))$ *is an isomorphism.*

Proof The proof is similar to the one of the Five Lemma along the commuting diagram, where the starting (resp., ending) horizontal map from (resp., to) 0 in the top and bottom chains are omitted,

$$\begin{array}{ccccccc}
\mathring{\mathcal{P}}_r\Lambda^0(T) & \xrightarrow{d^0} & \mathring{\mathcal{P}}_{r-1}\Lambda^1(T) & \xrightarrow{d^1} & \mathring{\mathcal{P}}_{r-2}\Lambda^2(T) & \xrightarrow{\int_T} & \mathbb{R} \\
\downarrow \mathfrak{R}^0 & & \downarrow \mathfrak{R}^1 & & \downarrow \mathfrak{R}^2 & & \downarrow \iota \\
C^0(\mathcal{F}^\bullet_r(T), \mathcal{F}^\bullet_r(\partial T)) & \xrightarrow{\delta^0} & C^1(\mathcal{F}^\bullet_r(T), \mathcal{F}^\bullet_r(\partial T)) & \xrightarrow{\delta^1} & C^2(\mathcal{F}^\bullet_r(T), \mathcal{F}^\bullet_r(\partial T)) & \longrightarrow & \mathbb{R}
\end{array}$$

since the two rows of the diagram are exact sequences and the vertical operators $\mathfrak{R}^0$ and ι (which stands for the identity) are isomorphisms. If we assume that $\mathfrak{R}^2$ is an isomorphism, we can prove that $\mathfrak{R}^1$ is surjective and injective. The other way around, starting from $\mathfrak{R}^1$ being an isomorphism and proving for $\mathfrak{R}^2$ is analogous. The top and bottom sequences involve finite dimensional spaces; it is hence sufficient to prove that $\mathfrak{R}^1$ is injective.

We thus show that $\mathfrak{R}^1\,\omega = 0$ yields $\omega = 0$. Let $\omega \in \mathring{\mathcal{P}}_{r-1}\Lambda^1(T)$ such that $\mathfrak{R}^1\,\omega = 0$, then $\delta^1(\mathfrak{R}^1\,\omega) = 0$, and by the commutativity of the diagram, we have that $\mathfrak{R}^2(d^1\omega) = 0$. We have assumed that $\mathfrak{R}^2$ is an isomorphism, thus injective, that yields $d^1\omega = 0$. The top sequence is exact, so it exists $\eta \in \mathring{\mathcal{P}}_r\Lambda^0(T)$ such that $d^0\eta = w$. The diagram commutes; we thus have $\delta^0(\mathfrak{R}^0\eta) = \mathfrak{R}^1(d^0\eta) = \mathfrak{R}^1\omega = 0$. From $\delta^0(\mathfrak{R}^0\eta) = 0$, it results $\mathfrak{R}^0\eta = 0$ thus $\eta = 0$, since $\mathfrak{R}^0$ is injective. Consequently, $\omega = 0$. □

The content of Theorem 2 is that the map (5.4) is an isomorphism for $k = 2$ if and only if it is an isomorphism for $k = 1$. Proving that the map $\mathfrak{R}^2$ is an isomorphism is equivalent to showing that any polynomial in $n = 2$ variables of degree $\leq r$, such that its integrals are zero on $\binom{r+2}{2}$ non-overlapping polygons that cover T, is identically zero on T. A proof of this type of result can be found in [1] for a particular case. To the best of the authors knowledge, the general case is an open problem. In the next section, we provide numerical evidence of the unisolvence of the proposed degrees of freedom for $k = 1$, by stating that, varying r, the Vandermonde matrices are invertible. Motivated by these facts and by Theorem 2, we conjecture the following: "The map (5.4) is an isomorphism for $k = 1, 2$."

5.4 Computational examples

In this section, let T be the standard 2-simplex and let E_0, E_1, and E_2 be its edges. The monomial basis of $\mathcal{BP}_{r-1}\Lambda^1(T)$ with elements $\lambda^{\boldsymbol{\alpha}}$ is here replaced by the Bernstein basis with elements

$$\frac{r!}{\alpha_0!\alpha_1!\ldots\alpha_n!}\lambda^{\boldsymbol{\alpha}}.$$

Let us denote by $\Psi : \mathcal{P}_{r-1}\Lambda^1(T) \to \mathcal{F}^1_{r-1}(T)$ the map that associates to each form ω of the basis the corresponding cell. Assume now that we have ordered the elements of $\mathcal{BP}_{r-1}\Lambda^1(T) = \{\omega^1, \omega^2, \ldots\}$ in such a way that the set

$$\{\omega^{1+(r+1)i}, \ldots, \omega^{r+1+(r+1)i}\}$$

is a basis of $\mathrm{ext}_{E_i,T}\mathring{\mathcal{P}}_{r-1}\Lambda^1(E_i)$, for $i = 0, 1, 2$, and, the set

$$\{\omega^{3(r+1)+1}, \ldots, \omega^{\dim \mathcal{P}_{r-1}\Lambda^1(T)}\}$$

is a basis of $\mathring{\mathcal{P}}_{r-1}\Lambda^1(T)$. Define the Vandermonde matrix $\mathcal{V}^{r-1} = (\mathcal{V}^{r-1}_{ij})$ as

$$\mathcal{V}^{r-1}_{ij} = \int_{\Psi(\omega^i)} \omega^j.$$

For $r = 2, 3$, the entries of $\mathcal{V}^{r-1}$ computed on the Bernstein basis are shown below:

$$\mathcal{V}^1 = \frac{1}{8}\begin{pmatrix} 3 & 1 & & & & \\ 1 & 3 & & & & \\ & & 3 & 1 & & \\ & & 1 & 3 & & \\ & & & & 3 & 1 \\ & & & & 1 & 3 \end{pmatrix},$$

$$\mathcal{V}^2 = \frac{1}{81}\begin{pmatrix} 19 & 7 & 1 & & & & & & & & & \\ 7 & 13 & 7 & & & & & & & & & \\ 1 & 7 & 19 & & & & & & & & & \\ & & & 19 & 7 & 1 & & & & & & \\ & & & 7 & 13 & 7 & & & & & & \\ & & & 1 & 7 & 19 & & & & & & \\ & & & & & & 19 & 7 & 1 & & & \\ & & & & & & 7 & 13 & 7 & & & \\ & & & & & & 1 & 7 & 19 & & & \\ 3 & & & & 9 & 3 & 7 & 4 & 1 & 9 & & 3 \\ 1 & 4 & 7 & -3 & & & 3 & & & 3 & 9 & \\ & -9 & -3 & 1 & 4 & 7 & & 3 & 3 & & -3 & 9 \end{pmatrix}.$$

Note that, as we could expect, the matrix $\mathcal{V}^{r-1}$ has the following block structure:

$$\mathcal{V}^{r-1} = \begin{pmatrix} \mathcal{V}^{r-1}_{\mathrm{ext}} & O & O & O \\ O & \mathcal{V}^{r-1}_{\mathrm{ext}} & O & O \\ O & O & \mathcal{V}^{r-1}_{\mathrm{ext}} & O \\ * & * & * & \mathcal{V}^{r-1}_{\mathrm{int}} \end{pmatrix},$$

where O denotes a square matrix of zeroes, and the $*$ stands for a rectangular matrix that is not necessarily filled in completely with zeroes. On the diagonal, we have square blocks with entries, respectively,

$$(\mathcal{V}_{\text{ext}}^{r-1})_{ij} = \int_{\Psi(\omega^i)} \omega^j, \quad \omega^i, \omega^j \in \mathcal{B}\mathring{\mathcal{P}}_{r-1}\Lambda^1(E),$$

and

$$(\mathcal{V}_{\text{int}}^{r-1})_{ij} = \int_{\Psi(\omega^i)} \omega^j, \quad \omega^i, \omega^j \in \mathcal{B}\mathring{P}_{r-1}\Lambda^1(T).$$

On the diagonal, the blocks $\mathcal{V}_{\text{ext}}^{r-1}$ in $\mathcal{V}^{r-1}$ are invertible: this is due to the fact that the degrees of freedom on each edge of T describe completely the space associated with this geometrical entity. To prove unisolvence is thus equivalent to prove that the block $\mathcal{V}^{r-1}$int is invertible. This is true for the values $r = 2, 3, 4, ..., 11$ we have tested. For $r = 4$, we have, for example, that on the Berstein basis

$$\mathcal{V}_{\text{ext}}^3 = \frac{1}{1024}\begin{pmatrix} 175 & 67 & 13 & 1 \\ 65 & 109 & 67 & 15 \\ 15 & 67 & 109 & 65 \\ 1 & 13 & 67 & 175 \end{pmatrix},$$

$$\mathcal{V}_{\text{int}}^3 = \frac{1}{1024}\begin{pmatrix} 76 & 30 & & & & 28 & 6 & 4 \\ 56 & 72 & & & & 32 & 24 & 8 \\ 24 & 8 & 72 & 32 & 56 & & & \\ 18 & 28 & 18 & 52 & 28 & & & \\ 6 & 4 & 30 & 28 & 76 & & & \\ & & -28 & -52 & -18 & 52 & 28 & 18 \\ & & -4 & -28 & -6 & 28 & 76 & 30 \\ & & -8 & -32 & -24 & 32 & 56 & 72 \end{pmatrix}.$$

In a finite element approach, the Vandermonde matrix $\mathcal{V}^{r-1}$ has to be inverted to construct the basis $\{\mathtt{w}_j\}$ of the local discrete space, here $\mathcal{P}_{r-k}\Lambda^k(T)$, in duality with the selected set of dofs. The invertibility of the Vandermonde matrix V^{r-1} does not depend on the basis $\{\omega^j\}$ of $\mathcal{P}_{r-k}\Lambda^k(T)$ which is used to compute the matrix, but the condition number cond $(\mathcal{V}^{r-1}))$ does. To guarantee the computation of the dual basis for any choice of r, it is important to work with a basis $\{\omega^j\}$ such that the condition number of $\mathcal{V}^{r-1}$ does not increase too fast with r. Bernstein basis fulfils this latter requirement. Table 1 reports, for $r = 2, \dots, 11$, the condition number of $\mathcal{V}^{r-1}$ computed on the Bernstein basis and on the monomial one.

Concerning the non-admissible configuration at the beginning of Section 5, another way to see that the set of small edges drawn in Fig. 4a cannot support an unisolvent set of degrees of freedom for fields in $\mathcal{P}_2\Lambda^1(T)$ is to consider the internal block V_{int}^2 of the corresponding Vandermonde matrix, namely

$$V_{\text{int}}^2 = \frac{1}{81}\begin{pmatrix} 6 & -6 & \\ -6 & & 6 \\ & -6 & 6 \end{pmatrix}.$$

The determinant of this block is zero, as the third line is the sum of the first two, thus yielding a singular global Vandermonde matrix.

Table 1 Condition number of the matrix $\mathcal{V}^{r-1}$ for $r = 2, \ldots, 11$

$r-1$	cond $(\mathcal{V}^{r-1})$ with Bernstein basis	cond $(\mathcal{V}^{r-1})$ with monomial basis
1	2.0000	2.0000
2	7.4105	11.8276
3	21.1319	75.6558
4	59.5527	393.0213
5	169.2325	2.3819×10^3
6	474.8238	1.5598×10^4
7	1.3168×10^3	1.0006×10^5
8	3.6197×10^3	6.6383×10^5
9	9.8874×10^3	4.6183×10^6
10	2.6888×10^4	3.1695×10^7

6 Conclusions and future investigations

Many classical finite element spaces are just instances of the two families $\mathcal{P}_r^- \Lambda^k$, often referred to as *the first* or *the trimmed* one, and $\mathcal{P}_{r-k}\Lambda^k$, known as *the second* or *the complete* one. These two families are strictly related the one with the other since we need the first family to construct degrees of freedom for the second and vice versa. In this work, we have put the accent on the fact that the duality relations (2.16) and (2.17) have a geometrical interpretation. Thanks to this geometrical point of view, we have obtained new physical degrees of freedom for the second family $\mathcal{P}_r \Lambda^k$ in the two-dimensional case. Numerical results on the Vandermonde matrices corresponding with the case $k = 1$ and different choices of $r \geq 1$ are presented for $n = 2$. They provide numerical evidence of the fact that the proposed new degrees of freedom, for the second family, are unisolvent. The case $n = 3$ is under study.

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Declarations

Competing interests The authors declare no competing interests.

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