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C^s-smooth isogeometric spline spaces over planar bilinear multi-patch parameterizations

Mario Kapl[1,2] · **Vito Vitrih**[3]

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Abstract

The design of globally C^s-smooth ($s \geq 1$) isogeometric spline spaces over multi-patch geometries with possibly extraordinary vertices, i.e. vertices with valencies different from four, is a current and challenging topic of research in the framework of isogeometric analysis. In this work, we extend the recent methods Kapl et al. Comput. Aided Geom. Des. **52–53**:75–89, 2017, Kapl et al. Comput. Aided Geom. Des. **69**:55–75, 2019 and Kapl and Vitrih J. Comput. Appl. Math. **335**:289–311, 2018, Kapl and Vitrih J. Comput. Appl. Math. **358**:385–404, 2019 and Kapl and Vitrih Comput. Methods Appl. Mech. Engrg. **360**:112684, 2020 for the construction of C^1-smooth and C^2-smooth isogeometric spline spaces over particular planar multi-patch geometries to the case of C^s-smooth isogeometric multi-patch spline spaces of degree p, inner regularity r and of a smoothness $s \geq 1$, with $p \geq 2s+1$ and $s \leq r \leq p-s-1$. More precisely, we study for $s \geq 1$ the space of C^s-smooth isogeometric spline functions defined on planar, bilinearly parameterized multi-patch domains, and generate a particular C^s-smooth subspace of the entire C^s-smooth isogeometric multi-patch spline space. We further present the construction of a basis for this C^s-smooth subspace, which consists of simple and locally supported functions. Moreover, we use the C^s-smooth spline functions to perform L^2 approximation on bilinearly parameterized multi-patch domains, where the obtained numerical results indicate an optimal approximation power of the constructed C^s-smooth subspace.

Keywords Isogeometric analysis · Geometric continuity · Multi-patch domain · Bilinear-like · C^s-smooth functions

Mathematics Subject Classification (2010) 65D07 · 65D17 · 65N30

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✉ Mario Kapl
m.kapl@fh-kaernten.at

Extended author information available on the last page of the article.

1 Introduction

Multi-patch spline geometries with possibly extraordinary vertices, i.e. vertices with valencies different from four, are a useful tool in computer-aided design [15, 24] for modeling complex objects, which usually cannot be described just by single-patch geometries. The concept of isogeometric analysis [6, 14, 25] allows the construction of globally C^s-smooth ($s \geq 1$) isogeometric spline spaces over these multi-patch geometries. The smooth spline spaces can then be used to solve high-order partial differential equations (PDEs) on the multi-patch domains directly via the weak form and a standard Galerkin discretization. While in case of fourth-order PDEs such as the biharmonic equation, e.g. [5, 13, 26, 49, 57], the Kirchhoff-Love shell problem, e.g. [3, 7, 41–43], the Cahn-Hilliard equation, e.g. [17, 18, 45], and problems of strain gradient elasticity, e.g. [16, 46, 51], C^1-smooth isogeometric spline functions are needed, even C^2-smooth functions are required in case of sixth order PDEs such as the triharmonic equation, e.g. [5, 34, 57], the phase-field crystal equation, e.g. [5, 20], the Kirchhoff plate model based on the Mindlin's gradient elasticity theory, e.g. [40, 52], and the gradient-enhanced continuum damage model, e.g. [60]. Isogeometric collocation, see, e.g. [1, 4, 19, 35, 47], is another possible application of globally smooth isogeometric spline spaces. Solving the strong form of the PDE requires now in case of a second order PDE C^2-smooth isogeometric spline functions and in case of a fourth-order PDE already C^4-smooth functions.

Beside solving high-order PDEs directly via their Galerkin discretization or via isogeometric collocation by employing exactly C^s-smooth isogeometric spline functions as described in the previous paragraph, there exist further possible strategies in isogeometric analysis to deal with high order PDEs by using function spaces of lower regularity. We will briefly discuss two of them. The first approach is to couple the neighboring patches instead of in a strong sense as for the case of exactly C^s-smooth functions just in a weak sense. Then, the isogeometric spline space for solving the PDE is in general not exactly C^s-smooth but the solution of the PDE is enforced to be approximately C^s-smooth, e.g. by adding penalty terms and jump terms along the interfaces to the weak form of the PDE, see, e.g. [2, 22], or by using Lagrange multipliers, see, e.g. [2, 56]. The second strategy is based on the application of a mixed variational formulation for the corresponding high order PDE, which requires a reformulation of the problem and the solving of a system of lower order PDEs but allows the usage of isogeometric spline spaces of lower regularity, see, e.g. [54, 55].

The construction of globally, exactly C^s-smooth isogeometric spline spaces over multi-patch geometries with possibly extraordinary vertices is mainly based on the observation that an isogeometric function is C^s-smooth over the given multi-patch domain if and only if its associated multi-patch graph surface is G^s-smooth (i.e. geometrically continuous of order s). In this work, we will focus on the design of smooth isogeometric spline spaces over planar multi-patch geometries, which may have extraordinary vertices. So far, most existing techniques are limited to a global smoothness of $s = 1$ and $s = 2$. In case of $s = 1$, these methods can be roughly classified into three approaches depending on the used multi-patch parameterization. While the first strategy employs a multi-patch parameterization, which is C^1-smooth everywhere and therefore possesses a singularity at the extraordinary vertices, see, e.g. [50, 59], the second approach uses a multi-patch parameterization, which is C^1-smooth everywhere except in the neighborhood of an extraordinary vertex, where a special construction of the parameterization is needed, see, e.g [37–39, 49]. In contrast to the first two approaches, where the multi-patch geometry is C^1-smooth at most parts of the multi-patch domain, the used multi-patch parameterization in the third strategy is in general just C^0-smooth at the interfaces. Examples of such parameterizations are (mapped) piecewise bilinear parameterizations, e.g. [8, 26, 36], general analysis-suitable parameterizations, e.g. [13, 27, 28, 30], non-analysis-suitable parameterizations, e.g. [11, 12], and general quadrilateral meshes of arbitrary topology [9, 10, 48]. The recent survey article [29] provides more details about the single methods of the three approaches and also includes further possible constructions.

In case of $s = 2$, there exist only a small number of possible constructions, which mainly follow the third strategy for $s = 1$, see, e.g. [31–35]. All these methods can be applied to the case of (mapped) bilinear multi-patch parameterizations, but the techniques [33–35] work also for a more general class of multi-patch parameterizations, called bilinear-like G^2 multi-patch geometries, cf. [33]. The design of C^s-smooth isogeometric spline spaces for planar multi-patch geometries with possibly extraordinary vertices has not been considered so far for a global smoothness of $s > 2$, and is the topic of this paper. A related approach, which is based on a polar configuration and enables the construction of C^s-smooth isogeometric spline functions with a smoothness of $s \geq 3$, is the technique [58].

In this paper, we study and generate C^s-smooth isogeometric spline functions, which are defined over planar, multi-patch parameterizations. We will restrict ourselves to a smoothness s with $1 \leq s < 20$, which should cover all cases of practical interest. This limitation is due to the fact that one step in the proof of Theorem 2 requires the use of a computer algebra system and has been verified for $1 \leq s < 20$. However, it is worth to mention that numerical tests (not shown in the paper) have indicated the validity of Theorem 2 for a smoothness $s \geq 20$, too, and that then all other results of the paper would be directly applicable to an arbitrary smoothness $s \geq 1$.

The construction of the C^s-smooth spline space is mainly described for the case of bilinearly parameterized multi-patch domains, but can be extended to the wider class of bilinear-like G^s multi-patch geometries, which has been already introduced for the case $s = 2$ in [33], and which allows the modeling of planar multi-patch geometries with curved interfaces and boundaries. The presented study and construction of the globally C^s-smooth isogeometric spline functions can be seen as an extension of the techniques [27, 30] and [33–35] for the design of C^s-smooth isogeometric multi-patch spline spaces for the case of $s = 1$ and $s = 2$, respectively. More precisely, we develop for the case of a planar bilinear multi-patch parameterization a theoretical framework to study the C^s-smoothness condition of an isogeometric function and to characterize the resulting C^s-smooth function. We also use this framework to generate a particular C^s-smooth isogeometric spline space for a given planar, bilinearly parameterized multi-patch domain and to construct a simple and locally supported basis for the C^s-smooth space. Several numerical tests by performing L^2 approximation using the C^s-smooth isogeometric spline space for different s indicate an optimal approximation power of the constructed C^s-smooth space and demonstrate the potential of the space for the use in isogeometric analysis.

The remainder of this paper is organized as follows. In Section 2, we introduce the particular class of planar multi-patch geometries, which consists of bilinearly parameterized quadrilateral patches, and will be used throughout the paper. Moreover, we present the concept of C^s-smooth isogeometric spline spaces over this class of multi-patch geometries. Section 3 studies the C^s-smoothness condition of an isogeometric function across two neighboring patches and describes first the construction of a particular C^s-smooth isogeometric spline space for the case of a bilinearly parameterized two-patch domain. This requires the introduction of auxiliary functions, where some concrete examples of these functions are presented in the Appendix. In Section 4, we then extend the particular construction to the case of bilinearly parameterized multi-patch domains with more than two patches and with possibly extraordinary vertices. For both cases, we also explain the design of a simple basis, which consists of locally supported functions. A first possible generalization of our approach beyond bilinear parameterizations is briefly discussed in Section 5. Numerical experiments in Section 6 indicate optimal approximation properties of the presented C^s-smooth isogeometric multi-patch spline spaces. Finally, we conclude the paper in Section 7.

2 The multi-patch setting and C^s-smooth isogeometric spline spaces

In this section, we will first describe the multi-patch setting, which will be used throughout the paper. Then, we will give a short overview of the concept of C^s-smooth ($s \geq 1$) isogeometric spline spaces over the considered class of multi-patch domains.

Let Ω and $\Omega^{(i)}$, $i \in \mathcal{I}_\Omega$, be open and connected domains in $\mathbb{R}^2$, such that $\overline{\Omega} = \cup_{i \in \mathcal{I}_\Omega} \overline{\Omega^{(i)}}$, where $\mathcal{I}_\Omega$ is the index set of the indices of the patches $\Omega^{(i)}$. Furthermore, let $\Omega^{(i)}$, $i \in \mathcal{I}_\Omega$, be quadrangular patches, which are mutually disjoint, and the closures of any two of them have either an empty intersection, possess exactly one common vertex or share the whole common edge. We will further assume that each patch $\overline{\Omega^{(i)}}$ is parameterized by a bilinear, bijective and regular geometry mapping $\boldsymbol{F}^{(i)}$,

$$\boldsymbol{F}^{(i)} : [0,1]^2 \to \mathbb{R}^2, \quad \boldsymbol{\xi} = (\xi_1, \xi_2) \mapsto \boldsymbol{F}^{(i)}(\boldsymbol{\xi}) = \boldsymbol{F}^{(i)}(\xi_1, \xi_2), \quad i \in \mathcal{I}_\Omega,$$

such that $\overline{\Omega^{(i)}} = \boldsymbol{F}^{(i)}\left([0,1]^2\right)$, see Fig. 1. In addition, we denote by $\boldsymbol{F}$ the multi-patch parameterization consisting of all geometry mappings $\boldsymbol{F}^{(i)}$, $i \in \mathcal{I}_\Omega$. We will also use the splitting of the multi-patch domain $\overline{\Omega}$ into the single patches $\Omega^{(i)}$, $i \in \mathcal{I}_\Omega$, edges $\Gamma^{(i)}$, $i \in \mathcal{I}_\Gamma$, and vertices $\Xi^{(i)}$, $i \in \mathcal{I}_\Xi$, i.e.

$$\overline{\Omega} = \bigcup_{i \in \mathcal{I}_\Omega} \Omega^{(i)} \,\dot{\cup} \bigcup_{i \in \mathcal{I}_\Gamma} \Gamma^{(i)} \,\dot{\cup} \bigcup_{i \in \mathcal{I}_\Xi} \Xi^{(i)},$$

where $\dot{\cup}$ denotes the disjoint union of sets and $\mathcal{I}_\Gamma$ and $\mathcal{I}_\Xi$ are the index sets of the indices of the edges $\Gamma^{(i)}$ and vertices $\Xi^{(i)}$, respectively.

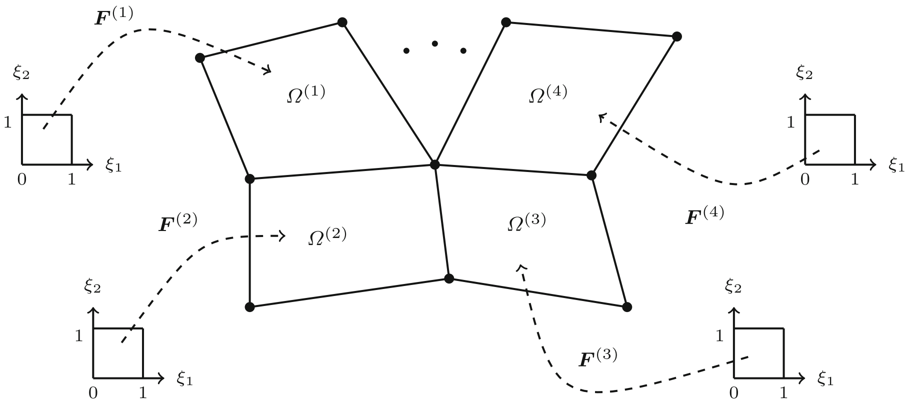

Fig. 1 The multi-patch domain $\overline{\Omega} = \cup_{i \in \mathcal{I}_\Omega} \overline{\Omega^{(i)}}$ with the corresponding geometry mappings $\boldsymbol{F}^{(i)}$, $i \in \mathcal{I}_\Omega$

Let us describe now the isogeometric spline spaces that will be considered in this work. We denote by $\mathcal{S}_h^{p,r}([0,1])$ the univariate spline space of degree p, regularity r and mesh size $h = \frac{1}{k+1}$, which is defined on the unit interval $[0, 1]$, and which is constructed from the uniform open knot vector

$$\left(\underbrace{0, \ldots, 0}_{(p+1)-\text{times}}, \underbrace{\tfrac{1}{k+1}, \ldots, \tfrac{1}{k+1}}_{(p-r)-\text{times}}, \ldots, \underbrace{\tfrac{k}{k+1}, \ldots, \tfrac{k}{k+1}}_{(p-r)-\text{times}}, \underbrace{1, \ldots, 1}_{(p+1)-\text{times}} \right),$$

where k is the number of different inner knots. Furthermore, let $\mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right)$ be the tensor-product spline space $\mathcal{S}_h^{p,r}([0,1]) \otimes \mathcal{S}_h^{p,r}([0,1])$ on the unit-square $[0,1]^2$. We denote the B-splines of the spaces $\mathcal{S}_h^{p,r}([0,1])$ and $\mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right)$ by $N_j^{p,r}$ and $N_{j_1,j_2}^{\boldsymbol{p},\boldsymbol{r}} = N_{j_1}^{p,r} N_{j_2}^{p,r}$, respectively, with $j, j_1, j_2 = 0, 1, \ldots, n-1$, where $n = p+1+k(p-r)$. We assume that $p \geq 2s+1$ and $s \leq r \leq p-(s+1)$. Since the geometry mappings $\boldsymbol{F}^{(i)}$, $i \in \mathcal{I}_\Omega$, are bilinearly parameterized, we trivially have that

$$\boldsymbol{F}^{(i)} \in \mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right) \times \mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right).$$

The space of isogeometric functions on Ω is given as

$$\mathcal{V} = \left\{ \phi \in L^2\left(\overline{\Omega}\right) \mid \phi|_{\overline{\Omega^{(i)}}} \in \mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right) \circ \left(\boldsymbol{F}^{(i)}\right)^{-1},\ i \in \mathcal{I}_\Omega \right\}.$$

In addition, let

$$\mathcal{V}^s = \mathcal{V} \cap \mathcal{C}^s\left(\overline{\Omega}\right)$$

be the space of C^s-smooth isogeometric functions on Ω. For an isogeometric function $\phi \in \mathcal{V}$, we denote the spline functions $\phi \circ \boldsymbol{F}^{(i)}$, $i \in \mathcal{I}_\Omega$, by $f^{(i)}$, and specify their spline representations by

$$f^{(i)}(\xi_1, \xi_2) = \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} d_{j_1,j_2}^{(i)} N_{j_1,j_2}^{\boldsymbol{p},\boldsymbol{r}}(\xi_1, \xi_2), \quad d_{j_1,j_2}^{(i)} \in \mathbb{R}.$$

Moreover, we define the graph $\boldsymbol{\Sigma} \subseteq \Omega \times \mathbb{R}$ of an isogeometric function $\phi \in \mathcal{V}$ as the collection of the graph surface patches $\boldsymbol{\Sigma}^{(i)} : [0, 1]^2 \to \Omega^{(i)} \times \mathbb{R}$, $i \in \mathcal{I}_\Omega$, given by

$$\boldsymbol{\Sigma}^{(i)}(\xi_1, \xi_2) = \left(\boldsymbol{F}^{(i)}(\xi_1, \xi_2), f^{(i)}(\xi_1, \xi_2)\right)^T.$$

The space $\mathcal{V}^s$ can be characterized by means of the concept of geometric continuity of multi-patch surfaces, cf. [24, 53]. An isogeometric function $\phi \in \mathcal{V}$ belongs to the space $\mathcal{V}^s$ if and only if for any two neighboring patches $\Omega^{(i_0)}$ and $\Omega^{(i_1)}$, $i_0, i_1 \in \mathcal{I}_\Omega$, with the common edge $\overline{\Gamma^{(i)}} = \overline{\Omega^{(i_0)}} \cap \overline{\Omega^{(i_1)}}$, $i \in \mathcal{I}_\Gamma$, the associated graph surface patches $\boldsymbol{\Sigma}^{(i_0)}$ and $\boldsymbol{\Sigma}^{(i_1)}$ are G^s-smooth, see, e.g. [21, 36], i.e. there exists a regular, orientation-preserving reparameterization $\Phi^{(i)} = \left(\Phi_1^{(i)}, \Phi_2^{(i)}\right)$, $\Phi_j^{(i)} : [0, 1]^2 \to [0, 1]$, $j = 1, 2$, such that

$$\partial_1^{j_1} \partial_2^{j_2} \boldsymbol{\Sigma}^{(i_1)}\big|_{\overline{\Gamma^{(i)}}} = \partial_1^{j_1} \partial_2^{j_2} \left(\boldsymbol{\Sigma}^{(i_0)} \circ \Phi^{(i)}\right)\big|_{\overline{\Gamma^{(i)}}}, \quad 0 \le j_1+j_2 \le s,\ i_0, i_1 \in \mathcal{I}_\Omega. \tag{1}$$

Here and throughout the paper, we will denote by ∂_ℓ^j the jth partial derivative with respect to the ℓth argument of a multivariate function, while we will denote by ∂^j the jth derivative with respect to the argument of a univariate function.

In the next section, first, the case of a two-patch domain will be analyzed. For this purpose but also for the remainder of the paper, the smoothness s will be restricted to the case $1 \le s < 20$ as explained in the introduction.

3 C^s-smooth isogeometric spline spaces over two-patch domains

In this section, we will consider the case of bilinearly parameterized two-patch domains Ω. In order to simplify the notation, we will denote the patches of the two-patch domain as $\overline{\Omega} = \overline{\Omega^{(i_0)}} \cup \overline{\Omega^{(i_1)}}$, their intersection by $\overline{\Gamma} = \overline{\Omega^{(i_0)}} \cap \overline{\Omega^{(i_1)}}$ and the corresponding reparameterization just as Φ. We will first study the G^s-smoothness condition of the graph surface of a C^s-smooth isogeometric spline function defined on a bilinear two-patch domain, and will then use it to construct a particular C^s-smooth isogeometric spline space. The presented work in this section can be seen as an extension of [27] and [33] for $s = 1$ and $s = 2$, respectively, to a higher smoothness s in case of bilinear two-patch parameterizations. A possible strategy beyond bilinear parameterizations is briefly explained in Section 5.

3.1 G^s-smoothness of graph surfaces

Let $\phi \in \mathcal{V}$, and let $f^{(\tau)} = \phi \circ \boldsymbol{F}^{(\tau)}$, $\tau \in \{i_0, i_1\}$, be the two associated spline functions. To ensure that the graph surface $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{(i_0)} \cup \boldsymbol{\Sigma}^{(i_1)}$ of the isogeometric function ϕ is G^s-smooth, $\boldsymbol{\Sigma}^{(i_0)}$ and $\boldsymbol{\Sigma}^{(i_1)}$ have to be joint above their common edge Γ with G^s-continuity. Without loss of generality, we can assume that $\Phi(0, \xi_2) = (0, \xi_2)$, i.e. the G^0 smoothness across the common interface can be written as

$$\boldsymbol{\Sigma}^{(i_0)}(0, \xi_2) = \boldsymbol{\Sigma}^{(i_1)}(0, \xi_2), \quad \xi_2 \in [0, 1]. \tag{2}$$

In this way, the patches $\Omega^{(i_0)}$ and $\Omega^{(i_1)}$ are parameterized as shown in Fig. 2. Furthermore, the G^1-smoothness can be expressed as

$$\det\left(\partial_1 \boldsymbol{\Sigma}^{(i_1)}(0, \xi_2), \partial_1 \boldsymbol{\Sigma}^{(i_0)}(0, \xi_2), \partial_2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi_2)\right) = 0, \quad \xi_2 \in [0, 1], \tag{3}$$

which is equivalent to

$$\alpha^{(i_0)}(\xi_2)\partial_1 f^{(i_1)}(0, \xi_2) - \alpha^{(i_1)}(\xi_2)\partial_1 f^{(i_0)}(0, \xi_2) - \beta(\xi_2)\partial_2 f^{(i_0)}(0, \xi_2) = 0, \tag{4}$$

for $\xi_2 \in [0, 1]$, with $\alpha^{(\tau)}, \beta : \mathbb{R} \to \mathbb{R}$,

$$\alpha^{(\tau)}(\xi) = \lambda_1 \det\left(\partial_1 \boldsymbol{F}^{(\tau)}(0, \xi), \partial_2 \boldsymbol{F}^{(\tau)}(0, \xi)\right), \quad \tau \in \{i_0, i_1\}, \tag{5}$$

and

$$\beta(\xi) = \lambda_1 \det\left(\partial_1 \boldsymbol{F}^{(i_0)}(0, \xi), \partial_1 \boldsymbol{F}^{(i_1)}(0, \xi)\right).$$

We can select $\lambda_1 \in \mathbb{R}$ in such a way, that

$$||\alpha^{(i_0)} + 1||^2_{L^2} + ||\alpha^{(i_1)} - 1||^2_{L^2} \tag{6}$$

is minimized, cf. [34].

Remark 1 The proposed choice of λ_1 for the functions $\alpha^{(i_0)}$, $\alpha^{(i_1)}$ and β will ensure later together with additional scaling factors that the constructed basis functions across the interfaces in Section 3.2 will be uniformly scaled, see also Remark 3.

Note that $\alpha^{(i_0)} < 0$ and $\alpha^{(i_1)} > 0$, since the geometry mappings $\boldsymbol{F}^{(i_0)}$ and $\boldsymbol{F}^{(i_1)}$ are regular. In addition, we can write β as

$$\beta(\xi) = \alpha^{(i_0)}(\xi)\beta^{(i_1)}(\xi) - \alpha^{(i_1)}(\xi)\beta^{(i_0)}(\xi) \tag{7}$$

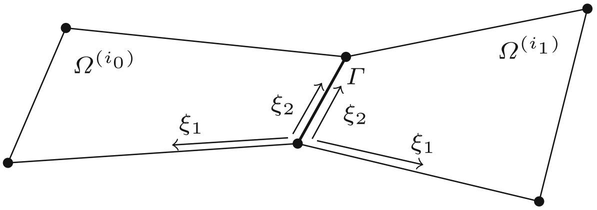

Fig. 2 The parameterization of the two-patch domain $\Omega^{(i_0)} \cup \Omega^{(i_1)}$ with the common edge Γ

with $\beta^{(\tau)} : \mathbb{R} \to \mathbb{R}$,

$$\beta^{(\tau)}(\xi) = \frac{\partial_1 \boldsymbol{F}^{(\tau)}(0,\xi) \cdot \partial_2 \boldsymbol{F}^{(\tau)}(0,\xi)}{||\partial_2 \boldsymbol{F}^{(\tau)}(0,\xi)||^2}, \quad \tau \in \{i_0, i_1\}, \tag{8}$$

where $\alpha^{(i_0)}$, $\alpha^{(i_1)}$, $\beta^{(i_0)}$ and $\beta^{(i_1)}$ are linear polynomials, and β is a quadratic one, cf. [13, 33].

Recall (1), and let $a_{i,j}, b_{i,j} : \mathbb{R} \to \mathbb{R}$,

$$a_{i,j}(\xi) = \left(\partial_1^i \partial_2^j \Phi_1\right)(0,\xi), \quad b_{i,j}(\xi) = \left(\partial_1^i \partial_2^j \Phi_2\right)(0,\xi). \tag{9}$$

By (1), (4) and (9), we observe that

$$a_{1,0}(\xi) = \frac{\alpha^{(i_1)}(\xi)}{\alpha^{(i_0)}(\xi)}, \quad b_{1,0}(\xi) = \frac{\beta(\xi)}{\alpha^{(i_0)}(\xi)}. \tag{10}$$

In a similar way as for the G^1 smoothness, we can derive conditions for the G^ℓ-smoothness, $2 \le \ell \le s$ (see, e.g. [24]). For each particular ℓ, $1 \le \ell \le s$, one only needs to consider the equation

$$\partial_1^\ell \boldsymbol{\Sigma}^{(i_1)}(0,\xi_2) = \partial_1^\ell \left(\boldsymbol{\Sigma}^{(i_0)} \circ \boldsymbol{\Phi}\right)(0,\xi_2), \quad \xi_2 \in [0,1], \tag{11}$$

since the continuity of all mixed derivatives of total order ℓ follows directly from (11) for $1 \le \ell' < \ell$. Now, we would like to express G^ℓ-smoothness conditions in a similar way as in (3). By introducing $\boldsymbol{\Xi}_\ell = \left(\widetilde{\boldsymbol{\Xi}}_\ell, \omega_\ell\right) : \mathbb{R} \to \mathbb{R}^3$ with components $\widetilde{\boldsymbol{\Xi}}_\ell : \mathbb{R} \to \mathbb{R}^2$ and $\omega_\ell : \mathbb{R} \to \mathbb{R}$, defined as

$$\begin{aligned}\boldsymbol{\Xi}_\ell(\xi) &= \left(\widetilde{\boldsymbol{\Xi}}_\ell(\xi), \omega_\ell(\xi)\right) = \partial_1^\ell \boldsymbol{\Sigma}^{(i_1)}(0,\xi) - \partial_1^\ell \left(\boldsymbol{\Sigma}^{(i_0)} \circ \boldsymbol{\Phi}\right)(0,\xi) + a_{\ell,0}\, \partial_1 \boldsymbol{\Sigma}^{(i_0)}(0,\xi) \\ &\quad + b_{\ell,0}\, \partial_2 \boldsymbol{\Sigma}^{(i_0)}(0,\xi),\end{aligned} \tag{12}$$

relations (11) imply

$$\det\left(\boldsymbol{\Xi}_\ell(\xi_2), \partial_1 \boldsymbol{\Sigma}^{(i_0)}(0,\xi_2), \partial_2 \boldsymbol{\Sigma}^{(i_0)}(0,\xi_2)\right) = 0, \quad \xi_2 \in [0,1]. \tag{13}$$

Expanding (13) and multiplying with λ_1 leads to

$$\lambda_\ell\, \alpha^{(i_0)}(\xi_2)\, \omega_\ell(\xi_2) + \eta_\ell(\xi_2)\, \partial_1 f^{(i_0)}(0,\xi_2) + \theta_\ell(\xi_2)\, \partial_2 f^{(i_0)}(0,\xi_2) = 0, \tag{14}$$

for $\xi_2 \in [0,1]$, with $\eta_\ell, \theta_\ell : \mathbb{R} \to \mathbb{R}$,

$$\eta_\ell(\xi) = \lambda_1 \lambda_\ell \det\left(\partial_2 \boldsymbol{F}^{(i_0)}(0,\xi), \widetilde{\boldsymbol{\Xi}}_\ell(\xi)\right), \theta_\ell(\xi) = \lambda_1 \lambda_\ell \det\left(\widetilde{\boldsymbol{\Xi}}_\ell(\xi), \partial_1 \boldsymbol{F}^{(i_0)}(0,\xi)\right). \tag{15}$$

Since $\boldsymbol{F}^{(i_0)}$ is a bilinear mapping, (9), (11) and (13) imply

$$\widetilde{\boldsymbol{\Xi}}_\ell(\xi) = -c_\ell(\xi)\, \partial_1 \partial_2 \boldsymbol{F}^{(i_0)}(0,\xi),$$

with $c_\ell : \mathbb{R} \to \mathbb{R}$, defined as

$$c_\ell(\xi) := \sum_{i=1}^{\ell-1} \binom{\ell}{i} a_{i,0}(\xi)\, b_{\ell-i,0}(\xi), \quad \ell \ge 2. \tag{16}$$

Then by (15), it follows that

$$\eta_\ell(\xi) = -\lambda_1\lambda_\ell\, c_\ell(\xi)\, \det\left(\partial_2 \boldsymbol{F}^{(i_0)}(0,\xi), \partial_1\partial_2 \boldsymbol{F}^{(i_0)}(0,\xi)\right)$$
$$= \lambda_\ell \left(\alpha^{(i_0)}(\xi)\right)' c_\ell(\xi), \tag{17}$$

and

$$\theta_\ell(\xi) = -\lambda_1\lambda_\ell\, c_\ell(\xi)\, \det\left(\partial_1\partial_2 \boldsymbol{F}^{(i_0)}(0,\xi), \partial_1 \boldsymbol{F}^{(i_0)}(0,\xi)\right)$$
$$= \lambda_\ell \left(\alpha^{(i_0)}(\xi)\left(\beta^{(i_0)}(\xi)\right)' - \left(\alpha^{(i_0)}(\xi)\right)' \beta^{(i_0)}(\xi)\right) c_\ell(\xi). \tag{18}$$

The last equalities in (17) and (18) follow directly by computing $\det\left(\partial_2 \boldsymbol{F}^{(i_0)}(0,\xi), \partial_1\partial_2 \boldsymbol{F}^{(i_0)}(0,\xi)\right)$ and $\det\left(\partial_1\partial_2 \boldsymbol{F}^{(i_0)}(0,\xi), \partial_1 \boldsymbol{F}^{(i_0)}(0,\xi)\right)$ for a bilinear patch $\boldsymbol{F}^{(i_0)}$, and using (5) and (8). Inserting ω_ℓ, defined in (12), into (14) and considering coefficient functions at $\partial_1 f^{(i_0)}(0,\xi_2)$ and $\partial_2 f^{(i_0)}(0,\xi_2)$ gives

$$\eta_\ell(\xi) = -\lambda_\ell\, \alpha^{(i_0)}(\xi) a_{\ell,0}(\xi), \theta_\ell(\xi) = -\lambda_\ell\, \alpha^{(i_0)}(\xi) b_{\ell,0}(\xi), \quad 1 \le \ell \le s. \tag{19}$$

Remark 2 For the sake of simplicity, we will choose in the following $\lambda_\ell = 1$ for $\ell = 2, 3, \ldots, s$.

Comparing Eqs. (17) and (18) with (19) directly leads to the following lemma.

Lemma 1 *The functions $a_{\ell,0}$ and $b_{\ell,0}$, $1 \le \ell \le s$, can be expressed by $\alpha^{(\tau)}$, $\beta^{(\tau)}$, $\tau \in \{i_0, i_1\}$, via the recursion*

$$a_{1,0}(\xi) = \frac{\alpha^{(i_1)}(\xi)}{\alpha^{(i_0)}(\xi)}, \quad b_{1,0}(\xi) = \frac{\beta(\xi)}{\alpha^{(i_0)}(\xi)},$$
$$a_{\ell,0}(\xi) = \vartheta(\xi) c_\ell(\xi), \quad b_{\ell,0}(\xi) = \mu(\xi) c_\ell(\xi), \quad 2 \le \ell \le s,$$

where the function c_ℓ is given in (16), and the functions ϑ, $\mu : \mathbb{R} \to \mathbb{R}$ are defined as

$$\vartheta(\xi) = -\frac{\left(\alpha^{(i_0)}(\xi)\right)'}{\alpha^{(i_0)}(\xi)}, \quad \mu(\xi) = -\frac{\left(\alpha^{(i_0)}(\xi)\left(\beta^{(i_0)}(\xi)\right)' - \left(\alpha^{(i_0)}(\xi)\right)' \beta^{(i_0)}(\xi)\right)}{\alpha^{(i_0)}(\xi)}.$$

Explicit expressions for the functions $\boldsymbol{\Xi}_\ell$, η_ℓ and θ_ℓ for the cases $\ell \in \{1, 2, 3\}$ are given in Example 1 in the Appendix. Lemma 1 provides us now with closed form formulae for the functions $a_{\ell,0}(\xi)$ and $b_{\ell,0}(\xi)$, $\ell \ge 2$, which only depend on $\alpha^{(i_0)}(\xi)$, $\alpha^{(i_1)}(\xi)$, $\beta^{(i_0)}(\xi)$ and $\beta^{(i_1)}(\xi)$, and which are equal to

$$a_{\ell,0}(\xi) = \ell!\, \frac{\alpha^{(i_1)}(\xi)\,\beta(\xi)}{\left(\alpha^{(i_0)}(\xi)\right)^\ell}\, \vartheta(\xi) \sum_{j=0}^{\ell-2} N(\ell-1, j+1) \left(\mu(\xi)\, \alpha^{(i_1)}(\xi)\right)^j \left(\vartheta(\xi)\, \beta(\xi)\right)^{\ell-2-j},$$

$$b_{\ell,0}(\xi) = \ell!\, \frac{\alpha^{(i_1)}(\xi)\,\beta(\xi)}{\left(\alpha^{(i_0)}(\xi)\right)^\ell}\, \mu(\xi) \sum_{j=0}^{\ell-2} N(\ell-1, j+1) \left(\mu(\xi)\, \alpha^{(i_1)}(\xi)\right)^j \left(\vartheta(\xi)\, \beta(\xi)\right)^{\ell-2-j},$$

where $\beta(\xi)$ is given in (7) and

$$N(m_1, m_2) = \frac{1}{m_1}\binom{m_1}{m_2}\binom{m_1}{m_2-1}, \quad 1 \le m_2 \le m_1,\ m_1, m_2 \in \mathbb{N},$$

are the well-known Narayana numbers. Summarizing the results of this section implies the following theorem.

Theorem 1 *Let Ω be a bilinearly parameterized two-patch domain, i.e. $\overline{\Omega} = \overline{\Omega^{(i_0)}} \cup \overline{\Omega^{(i_1)}}$. An isogeometric function $\phi \in \mathcal{V}$ belongs to the space $\mathcal{V}^s$ if and only if the two associated spline functions $f^{(\tau)} = \phi \circ \boldsymbol{F}^{(\tau)}$, $\tau \in \{i_0, i_1\}$, fulfill*

$$\partial_1^\ell f^{(i_1)}(0,\xi_2) = \partial_1^\ell \left(f^{(i_0)} \circ \boldsymbol{\Phi} \right)(0,\xi_2), \quad \xi_2 \in [0,1], \quad \ell = 0,1,\ldots,s,$$

or equivalently

$$f^{(i_1)}(0,\xi_2) = f^{(i_0)}(0,\xi_2),$$
$$\alpha^{(i_0)}(\xi_2)\,\omega_\ell(\xi_2) + \eta_\ell(\xi_2)\,\partial_1 f^{(i_0)}(0,\xi_2) + \theta_\ell(\xi_2)\,\partial_2 f^{(i_0)}(0,\xi_2) = 0,$$

for $\xi_2 \in [0,1]$, and $\ell = 1,2,\ldots,s$, where $\alpha^{(i_0)}$ is defined via (5) and (6), and ω_ℓ, η_ℓ and θ_ℓ are expressed by means of (12), (19) and Lemma 1.

3.2 Construction of C^s-smooth isogeometric spline spaces

Theorem 1 describes the C^s-smoothness condition for an isogeometric function $\phi \in \mathcal{V}$. Theorem 1 will provide now an equivalent but simplified condition, which will be the key step for the construction of C^s-smooth isogeometric functions. Before stating the theorem, we need the closed-form expression for $\partial_1^\ell \left(\boldsymbol{\Sigma}^{(i_0)} \circ \boldsymbol{\Phi} \right)(0,\xi)$, which requires the use of the generalized Faà di Bruno's formula [23], i.e.

$$\partial_1^\ell \left(\boldsymbol{\Sigma}^{(i_0)} \circ \boldsymbol{\Phi} \right)(0,\xi) = \sum_{|\boldsymbol{\sigma}|=1}^{\ell} A_{\boldsymbol{\sigma};\ell}(\xi)\, \partial_1^{\sigma_1} \partial_2^{\sigma_2} \boldsymbol{\Sigma}^{(i_0)}(0,\xi), \tag{20}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ is a multi-index with the indices $\sigma_1, \sigma_2 \in \{0,1,\ldots,\ell\}$, $1 \leq \sigma_1 + \sigma_2 \leq \ell$, $|\boldsymbol{\sigma}| = \sigma_1 + \sigma_2$ is the length of the multi-index, and $A_{\boldsymbol{\sigma};\ell} : \mathbb{R} \to \mathbb{R}$,

$$A_{\boldsymbol{\sigma};\ell}(\xi) = \ell\,! \sum_{(\boldsymbol{i},\boldsymbol{j}) \in \mathcal{I}_{\boldsymbol{\sigma};\ell}} \prod_{\rho=1}^{\ell} a_{\rho,0}^{i_\rho}(\xi)\, b_{\rho,0}^{j_\rho}(\xi) \frac{1}{\rho!^{\,i_\rho + j_\rho}\, i_\rho!\, j_\rho!}, \tag{21}$$

with $a_{\rho,0}(\xi)$ and $b_{\rho,0}(\xi)$ given in (9), and with

$$\mathcal{I}_{\boldsymbol{\sigma};\ell} = \left\{ (\boldsymbol{i},\boldsymbol{j}) = ((i_1,i_2,\ldots,i_\ell),(j_1,j_2,\ldots,j_\ell)) \,\middle|\, |\boldsymbol{i}| = \sigma_1, |\boldsymbol{j}| = \sigma_2, \sum_{\rho=1}^{\ell} \rho(i_\rho + j_\rho) = \ell \right\}. \tag{22}$$

Example 2 in the Appendix states the expressions for the functions $A_{\boldsymbol{\sigma};\ell}$ for the case $\ell = 3$ and $|\boldsymbol{\sigma}| \leq 3$.

Theorem 2 *Let Ω be a bilinearly parameterized two-patch domain, i.e. $\overline{\Omega} = \overline{\Omega^{(i_0)}} \cup \overline{\Omega^{(i_1)}}$. An isogeometric function $\phi \in \mathcal{V}$ belongs to the space $\mathcal{V}^s$ if and only if the corresponding spline functions $f^{(i_0)} = \phi \circ \boldsymbol{F}^{(i_0)}$ and $f^{(i_1)} = \phi \circ \boldsymbol{F}^{(i_1)}$ satisfy*

$$f_\ell^{(i_0)}(\xi) = f_\ell^{(i_1)}(\xi) =: f_\ell(\xi), \quad \ell = 0,1,\ldots,s, \tag{23}$$

with

$$f_\ell^{(\tau)}(\xi) = \left(\alpha^{(\tau)}(\xi) \right)^{-\ell} \partial_1^\ell f^{(\tau)}(0,\xi) - \sum_{i=0}^{\ell-1} \binom{\ell}{i} \left(\frac{\beta^{(\tau)}(\xi)}{\alpha^{(\tau)}(\xi)} \right)^{\ell-i} \partial^{\ell-i} f_i(\xi), \tag{24}$$

for $\tau \in \{i_0, i_1\}$.

Proof It directly follows from Theorem 1 that an isogeometric function $\phi \in \mathcal{V}$ belongs to the space $\mathcal{V}^s$ if and only if the associated spline functions $f^{(i_0)}$ and $f^{(i_1)}$ fulfill the equation

$$\left(\alpha^{(i_1)}(\xi)\right)^{-\ell} \left(\partial_1^\ell f^{(i_1)}(0,\xi) - \partial_1^\ell \left(f^{(i_0)} \circ \Phi\right)(0,\xi)\right) = 0, \quad \xi \in [0,1], \tag{25}$$

for $\ell = 0, 1, \ldots, s$. We will prove the equivalence of Eqs. (23) and (25) for any $\ell = 0, 1, \ldots, s$, by means of induction on ℓ. The equivalence of both equations trivially holds for $\ell = 0$ and can be directly obtained for $\ell = 1$ by applying (10) in Eq. (25). We will assume now that the equivalence of the two Eqs. (23) and (25) holds for all $\ell \le s-1$, and we will show it for $\ell = s$. Using the induction assumption, multiplying Eq. (24) for $\tau = i_0$ by $(\alpha^{(i_0)}(\xi))^\ell$ and expressing $\partial_1^\ell f^{(i_0)}(0,\xi)$ gives

$$\partial_1^\ell f^{(i_0)}(0,\xi) = \sum_{i=0}^{\ell} \binom{\ell}{i} \left(\beta^{(i_0)}(\xi)\right)^{\ell-i} \left(\alpha^{(i_0)}(\xi)\right)^{i} \partial^{\ell-i} f_i(\xi), \quad 1 \le \ell \le s-1. \tag{26}$$

Furthermore, differentiating $\partial_1^\ell f^{(i_0)}(0,\xi)$ with respect to the second argument yields

$$\partial_2^j\, \partial_1^\ell f^{(i_0)}(0,\xi) = \sum_{i=0}^{\ell} \sum_{\rho=0}^{j} \binom{\ell}{i}\binom{j}{\rho} \partial^{j-\rho} \left(\left(\beta^{(i_0)}(\xi)\right)^{\ell-i} \left(\alpha^{(i_0)}(\xi)\right)^{i}\right) \partial^{\ell-i+\rho} f_i(\xi), \tag{27}$$

for $1 \le \ell \le s-1$ and $j \ge 0$. In the following, we will skip the arguments in order to simplify the expressions. Using (20) and (27), (25) is equivalent to

$$\begin{aligned}
0 &= \frac{\partial_1^s f^{(i_1)}}{\left(\alpha^{(i_1)}\right)^s} - \frac{\partial_1^s f^{(i_0)}}{\left(\alpha^{(i_0)}\right)^s} \\
&\quad - \sum_{\substack{|\boldsymbol{\sigma}|=1 \\ \sigma_1<s}}^{s} \frac{A_{\boldsymbol{\sigma};s}}{\left(\alpha^{(i_1)}\right)^s} \sum_{i=0}^{\sigma_1}\sum_{\rho=0}^{\sigma_2} \binom{\sigma_1}{i}\binom{\sigma_2}{\rho} \partial^{\sigma_2-\rho}\left(\left(\beta^{(i_0)}\right)^{\sigma_1-i}\left(\alpha^{(i_0)}\right)^{i}\right) \partial^{\sigma_1-i+\rho} f_i \\
&= \frac{\partial_1^s f^{(i_1)}}{\left(\alpha^{(i_1)}\right)^s} - \frac{\partial_1^s f^{(i_0)}}{\left(\alpha^{(i_0)}\right)^s} \\
&\quad - \sum_{\sigma_1=0}^{s-1}\sum_{\sigma_2=0}^{s-\sigma_1} \frac{A_{\boldsymbol{\sigma};s}}{\left(\alpha^{(i_1)}\right)^s} \sum_{i=0}^{\sigma_1}\sum_{\rho=0}^{\sigma_2} \binom{\sigma_1}{i}\binom{\sigma_2}{\rho} \partial^{\sigma_2-\rho}\left(\left(\beta^{(i_0)}\right)^{\sigma_1-i}\left(\alpha^{(i_0)}\right)^{i}\right) \partial^{\sigma_1-i+\rho} f_i \\
&= \frac{\partial_1^s f^{(i_1)}}{\left(\alpha^{(i_1)}\right)^s} - \frac{\partial_1^s f^{(i_0)}}{\left(\alpha^{(i_0)}\right)^s} \\
&\quad - \sum_{i=0}^{s-1}\sum_{\sigma_1=i}^{s-1}\sum_{\sigma_2=0}^{s-\sigma_1}\sum_{\rho=0}^{\sigma_2} \frac{A_{\boldsymbol{\sigma};s}}{\left(\alpha^{(i_1)}\right)^s} \binom{\sigma_1}{i}\binom{\sigma_2}{\rho} \partial^{\sigma_2-\rho}\left(\left(\beta^{(i_0)}\right)^{\sigma_1-i}\left(\alpha^{(i_0)}\right)^{i}\right) \partial^{\sigma_1-i+\rho} f_i.
\end{aligned} \tag{28}$$

It is straightforward to see that $\sigma_1 - i + \rho \leq s - i$. By writing $j = \sigma_1 - i + \rho$ and applying that $\rho \geq 0$ implies $\sigma_1 \leq i + j$, we can express Eq. (28) also as

$$0 = \frac{\partial_1^s f^{(i_1)}}{\left(\alpha^{(i_1)}\right)^s} - \frac{\partial_1^s f^{(i_0)}}{\left(\alpha^{(i_0)}\right)^s} - \sum_{i=0}^{s-1} \sum_{j=0}^{s-i} B_{i,j}^s \, \partial^j f_i, \tag{29}$$

where $B_{i,j}^s : \mathbb{R} \to \mathbb{R}$,

$$B_{i,j}^s = \sum_{\sigma_1=i}^{\min\{s-1,i+j\}} \sum_{\sigma_2=i+j-\sigma_1}^{s-\sigma_1} \frac{A_{\boldsymbol{\sigma};s}}{\left(\alpha^{(i_1)}\right)^s} \binom{\sigma_1}{i} \binom{\sigma_2}{i+j-\sigma_1} \partial^{\sigma_2-(i+j-\sigma_1)} \left(\left(\beta^{(i_0)}\right)^{\sigma_1-i} \left(\alpha^{(i_0)}\right)^i \right). \tag{30}$$

In order to prove the theorem, which means now to demonstrate the equivalence of the Eqs. (23) and (29), it remains to show that functions (30) simplify to

$$B_{i,s-i}^s = \binom{s}{i} \left(\left(\frac{\beta^{(i_1)}}{\alpha^{(i_1)}} \right)^{s-i} - \left(\frac{\beta^{(i_0)}}{\alpha^{(i_0)}} \right)^{s-i} \right), \quad i = 0, 1, \ldots, s-1, \tag{31}$$

and

$$B_{i,j}^s = 0, \quad j = 0, 1, \ldots, s-i-1, \quad i = 0, 1, \ldots, s-1. \tag{32}$$

We will first consider the case $i + j = s$, and will hence prove formula (31). Now, $\sigma_2 = s - \sigma_1$, thus (30) equals

$$B_{i,s-i}^s = \sum_{\sigma_1=i}^{s-1} \frac{A_{(\sigma_1,s-\sigma_1);s}}{\left(\alpha^{(i_1)}\right)^s} \binom{\sigma_1}{i} \left(\beta^{(i_0)}\right)^{\sigma_1-i} \left(\alpha^{(i_0)}\right)^i.$$

Applying (21), we obtain

$$B_{i,s-i}^s = \frac{1}{\left(\alpha^{(i_1)}\right)^s} \sum_{\sigma_1=i}^{s-1} \binom{s}{\sigma_1} \binom{\sigma_1}{i} a_{1,0}^{\sigma_1} b_{1,0}^{s-\sigma_1} \left(\beta^{(i_0)}\right)^{\sigma_1-i} \left(\alpha^{(i_0)}\right)^i.$$

Using $\mu = \sigma_1 - i$ gives

$$\begin{aligned} B_{i,s-i}^s &= \frac{1}{\left(\alpha^{(i_1)}\right)^s} \left(\frac{\alpha^{(i_0)}}{\beta^{(i_0)}} \right)^i \sum_{\mu=0}^{s-i-1} \binom{s}{i+\mu} \binom{i+\mu}{i} \left(a_{1,0}\beta^{(i_0)}\right)^{i+\mu} b_{1,0}^{s-(i+\mu)} \\ &= \frac{1}{\left(\alpha^{(i_1)}\right)^s} \left(\frac{\alpha^{(i_0)}}{\beta^{(i_0)}} \right)^i \sum_{\mu=0}^{s-i-1} \binom{s-i}{\mu} \binom{s}{i} \left(a_{1,0}\beta^{(i_0)}\right)^{i+\mu} b_{1,0}^{s-(i+\mu)}, \end{aligned}$$

where the binomial identity $\binom{s}{i+\mu}\binom{i+\mu}{i} = \binom{s-i}{\mu}\binom{s}{i}$ has been used. Expanding the sum by one additional term and then subtracting it and further using the binomial theorem implies

$$\begin{aligned} B_{i,s-i}^s &= \frac{1}{\left(\alpha^{(i_1)}\right)^s} \left(\frac{\alpha^{(i_0)}}{\beta^{(i_0)}} \right)^i \left(a_{1,0}\beta^{(i_0)}\right)^i \binom{s}{i} \left(\sum_{\mu=0}^{s-i} \binom{s-i}{\mu} \left(a_{1,0}\beta^{(i_0)}\right)^\mu b_{1,0}^{(s-i)-\mu} - \left(a_{1,0}\beta^{(i_0)}\right)^{s-i} \right) \\ &= \frac{1}{\left(\alpha^{(i_1)}\right)^s} \left(\frac{\alpha^{(i_0)}}{\beta^{(i_0)}} \right)^i \left(a_{1,0}\beta^{(i_0)}\right)^i \binom{s}{i} \left(\left(a_{1,0}\beta^{(i_0)} + b_{1,0}\right)^{s-i} - \left(a_{1,0}\beta^{(i_0)}\right)^{s-i} \right). \end{aligned}$$

By (10), it follows that $a_{1,0}\beta^{(i_0)} = \alpha^{(i_1)}\beta^{(i_0)}\left(\alpha^{(i_0)}\right)^{-1}$ and $a_{1,0}\beta^{(i_0)}+b_{1,0} = \beta^{(i_1)}$. Thus, $B^s_{i,s-i}$ finally simplifies to

$$B^s_{i,s-i} = \binom{s}{i}\frac{1}{\left(\alpha^{(i_1)}\right)^{s-i}}\left(\left(\beta^{(i_1)}\right)^{s-i} - \left(\frac{\alpha^{(i_1)}\beta^{(i_0)}}{\alpha^{(i_0)}}\right)^{s-i}\right) = \binom{s}{i}\left(\left(\frac{\beta^{(i_1)}}{\alpha^{(i_1)}}\right)^{s-i} - \left(\frac{\beta^{(i_0)}}{\alpha^{(i_0)}}\right)^{s-i}\right).$$

Let us now consider (32), i.e. $i + j \leq s - 1$. We can show with the help of a computer algebra system that expression (30) is equivalent[1] to

$$\begin{aligned} B^s_{i,j} \;=\; & \frac{(-1)^{s-(i+j)}s(s-1-i)!\binom{s-1}{i}\left(\beta^{(i_1)}\right)^{j-1}\left(\alpha^{(i_1)}\right)^{i-1}}{j!\left(\alpha^{(i_1)}\right)^{s}\left(\alpha^{(i_0)}\right)^{2(s-(i+j))}}\left(i\,\beta^{(i_1)}\left(\alpha^{(i_0)}\right)' + j\,\alpha^{(i_1)}\left(\beta^{(i_0)}\right)'\right) \\ & \cdot\left(\alpha^{(i_0)}\beta^{(i_1)}\left(\alpha^{(i_0)}\right)' + \alpha^{(i_1)}\left(\alpha^{(i_0)}\left(\beta^{(i_0)}\right)' - 2\beta^{(i_0)}\left(\alpha^{(i_0)}\right)'\right)\right)^{s-1-(i+j)} \\ & \cdot\left(\left(\alpha^{(i_1)}\right)^2\beta^{(i_0)} - \alpha^{(i_0)}\alpha^{(i_1)}\beta^{(i_1)} + \left(\alpha^{(i_0)}\right)^2 a_{1,0}\,b_{1,0}\right). \end{aligned} \tag{33}$$

Since the last factor of (33) is equal to zero, i.e.

$$\left(\alpha^{(i_1)}\right)^2\beta^{(i_0)} - \alpha^{(i_0)}\alpha^{(i_1)}\beta^{(i_1)} + \left(\alpha^{(i_0)}\right)^2 a_{1,0}\,b_{1,0} = -\alpha^{(i_1)}\left(\alpha^{(i_0)}\beta^{(i_1)} - \alpha^{(i_1)}\beta^{(i_0)}\right) + \alpha^{(i_1)}\beta = 0,$$

relation (32) holds. Employing (31) and (32), we can now simplify Eq. (29) to

$$\frac{\partial_1^s f^{(i_1)}}{\left(\alpha^{(i_1)}\right)^s} - \frac{\partial_1^s f^{(i_0)}}{\left(\alpha^{(i_0)}\right)^s} - \sum_{i=0}^{s-1}\binom{s}{i}\left(\frac{\beta^{(i_1)}}{\alpha^{(i_1)}}\right)^{s-i}\partial^{s-i} f_i + \sum_{i=0}^{s-1}\binom{s}{i}\left(\frac{\beta^{(i_0)}}{\alpha^{(i_0)}}\right)^{s-i}\partial^{s-i} f_i = 0,$$

which is equivalent to Eq. (23), and which finally concludes the proof. □

The C^s-smooth isogeometric spline space $\mathcal{V}^s$ over a bilinearly parameterized two-patch domain $\overline{\Omega} = \overline{\Omega^{(i_0)}} \cup \overline{\Omega^{(i_1)}}$ can be decomposed into the direct sum of three subspaces, namely

$$\mathcal{V}^s = \mathcal{V}^s_{\Omega^{(i_0)}} \oplus \mathcal{V}^s_{\Omega^{(i_1)}} \oplus \mathcal{V}^s_{\Gamma},$$

where the subspaces $\mathcal{V}^s_{\Omega^{(\tau)}}$, $\tau \in \{i_0, i_1\}$, and $\mathcal{V}^s_{\Gamma}$ are given by

$$\mathcal{V}^s_{\Omega^{(\tau)}} = \left\{\phi \in \mathcal{V}^s \mid f^{(\tau)}(\xi_1,\xi_2) = \sum_{j_1=s+1}^{n-1}\sum_{j_2=0}^{n-1} d^{(\tau)}_{j_1,j_2}N^{\boldsymbol{p},\boldsymbol{r}}_{j_1,j_2}(\xi_1,\xi_2),\ f^{(\widetilde{\tau})}(\xi_1,\xi_2) = 0, \widetilde{\tau} \neq \tau,\ d^{(\tau)}_{j_1,j_2} \in \mathbb{R}\right\}$$

and

$$\mathcal{V}^s_{\Gamma} = \left\{\phi \in \mathcal{V}^s \mid f^{(\tau)}(\xi_1,\xi_2) = \sum_{j_1=0}^{s}\sum_{j_2=0}^{n-1} d^{(\tau)}_{j_1,j_2}N^{\boldsymbol{p},\boldsymbol{r}}_{j_1,j_2}(\xi_1,\xi_2),\ \tau \in \{i_0, i_1\},\ d^{(\tau)}_{j_1,j_2} \in \mathbb{R}\right\},$$

[1]This equivalence has been verified for $1 \leq s < 20$ using the Mathematica file available on https://osebje.famnit.upr.si/~vito.vitrih/notebook.html

respectively. The subspaces $\mathcal{V}^s_{\Omega^{(\tau)}}$, $\tau \in \{i_0, i_1\}$, can be simply described as

$$\mathcal{V}^s_{\Omega^{(\tau)}} = \mathrm{span}\left\{\phi_{\Omega^{(\tau)};j_1,j_2} \mid j_1 = s+1, \ldots, n-1,\ j_2 = 0, 1, \ldots, n-1\right\}, \quad \tau \in \{i_0, i_1\},$$

with the functions $\phi_{\Omega^{(\tau)};j_1,j_2} : \overline{\Omega} \to \mathbb{R}$,

$$\phi_{\Omega^{(\tau)};j_1,j_2}(\boldsymbol{x}) = \begin{cases} \left(N^{\boldsymbol{p},\boldsymbol{r}}_{j_1,j_2} \circ \left(\boldsymbol{F}^{(\tau)}\right)^{-1}\right)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega^{(\tau)}}, \\ 0 & \text{if } \boldsymbol{x} \in \overline{\Omega} \backslash \overline{\Omega^{(\tau)}}. \end{cases} \tag{34}$$

Since the functions $\phi_{\Omega^{(\tau)};j_1,j_2}$, $j_1 = s+1, \ldots, n-1$, $j_2 = 0, 1, \ldots, n-1$, are just "standard" isogeometric spline functions of at least C^s-continuity, they are linearly independent and therefore form a basis of the space $\mathcal{V}^s_{\Omega^{(\tau)}}$. The following theorem specifies now an explicit representation of an isogeometric function $\phi \in \mathcal{V}^s_{\Gamma}$.

Theorem 3 *Let $\phi \in \mathcal{V}^s_{\Gamma}$, then the two associated spline functions $f^{(\tau)} = \phi \circ \boldsymbol{F}^{(\tau)}$, $\tau \in \{i_0, i_1\}$, can be represented as*

$$f^{(\tau)}(\xi_1, \xi_2) = \sum_{i=0}^{s} \left(\sum_{j=0}^{i} \binom{i}{j} \left(\beta^{(\tau)}(\xi_2)\right)^{i-j} \left(\alpha^{(\tau)}(\xi_2)\right)^{j} \partial^{i-j} f_j(\xi_2) \right) M^{p,r}_i(\xi_1), \tag{35}$$

where the functions f_j are defined in (23) and (24), and the functions $M^{p,r}_i : \mathbb{R} \to \mathbb{R}$, which fulfill $\partial^\ell M^{p,r}_i(0) = \delta_{i,\ell}$, $\ell = 0, 1, \ldots, s$, with $\delta_{i,\ell}$ being the Kronecker delta, are given as

$$M^{p,r}_i(\xi) = \sum_{j=i}^{s} \frac{\binom{j}{i} h^i}{\prod_{\ell=0}^{i-1}(p-\ell)} N^{p,r}_j(\xi), \quad i = 0, 1, \ldots, s.$$

Proof By means of the Taylor expansion of $f^{(\tau)}(\xi_1, \xi_2)$ at $(\xi_1, \xi_2) = (0, \xi_2)$, and due to Theorem 2, we obtain that

$$\begin{aligned} f^{(\tau)}(\xi_1, \xi_2) &= f^{(\tau)}(0, \xi_2) + \partial_1 f^{(\tau)}(0, \xi_2)\xi_1 + \partial_1^2 f^{(\tau)}(0, \xi_2)\frac{\xi_1^2}{2} + \ldots + \partial_1^s f^{(\tau)}(0, \xi_2)\frac{\xi_1^s}{s!} + \mathcal{O}\left(\xi_1^{s+1}\right) \\ &= f_0(\xi_2) + \left(\alpha^{(\tau)}(\xi_2) f_1(\xi_2) + \beta^{(\tau)}(\xi_2) f_0'(\xi_2)\right)\xi_1 \\ &\quad + \left(\left(\alpha^{(\tau)}\right)^2(\xi_2) f_2(\xi_2) + 2\beta^{(\tau)}\alpha^{(\tau)}(\xi_2) f_1'(\xi_2) + \left(\beta^{(\tau)}\right)^2(\xi_2) f_0''(\xi_2)\right)\frac{\xi_1^2}{2} + \ldots \\ &\quad + \left(\sum_{j=0}^{s} \binom{s}{j} \left(\beta^{(\tau)}(\xi_2)\right)^{s-j} \left(\alpha^{(\tau)}(\xi_2)\right)^{j} \partial^{s-j} f_j(\xi_2)\right)\frac{\xi_1^s}{s!} + \mathcal{O}\left(\xi_1^{s+1}\right). \end{aligned}$$

Using the fact that the functions $f^{(\tau)}(\xi_1, \xi_2)$, $\tau \in \{i_0, i_1\}$, possess just a spline representation of the form

$$f^{(\tau)}(\xi_1, \xi_2) = \sum_{j_1=0}^{s} \sum_{j_2=0}^{n-1} d^{(\tau)}_{j_1,j_2} N^{\boldsymbol{p},\boldsymbol{r}}_{j_1,j_2}(\xi_1, \xi_2),$$

we further get

$$\begin{aligned} f^{(\tau)}(\xi_1,\xi_2) \; = \;\; & f_0(\xi_2) M_0^{p,r}(\xi_1) + \left(\alpha^{(\tau)}(\xi_2) f_1(\xi_2) + \beta^{(\tau)}(\xi_2) f_0'(\xi_2)\right) M_1^{p,r}(\xi_1) \\ & + \left(\left(\alpha^{(\tau)}\right)^2(\xi_2) f_2(\xi_2) + 2\beta^{(\tau)}\alpha^{(\tau)}(\xi_2) f_1'(\xi_2) + \left(\beta^{(\tau)}\right)^2(\xi_2) f_0''(\xi_2)\right) M_2^{p,r}(\xi_1) + \ldots \\ & + \left(\sum_{j=0}^{s} \binom{s}{j} \left(\beta^{(\tau)}(\xi_2)\right)^{s-j} \left(\alpha^{(\tau)}(\xi_2)\right)^{j} \partial^{s-j} f_j(\xi_2)\right) M_s^{p,r}(\xi_1), \end{aligned}$$

with

$$M_i^{p,r}(\xi_1) = \sum_{j=0}^{s} \lambda_{i,j} N_j^{p,r}(\xi_1), \quad i = 0, 1, \ldots, s.$$

The unknown parameters $\lambda_{i,j}$, $i, j = 0, 1, \ldots, s$, are then determined by the conditions $\partial^\ell M_i^{p,r}(0) = \delta_{i,\ell}$. By the properties of the B-splines $N_j^{p,r}$, we obtain for the unknowns $\lambda_{i,j}$ the following system of $2(s+1)$ equations

$$\partial^\ell M_i^{p,r}(0) = h^{-\ell}\left(\prod_{\rho=0}^{\ell-1}(p-\rho)\right)\sum_{\rho=0}^{\ell}(-1)^{\ell-\rho}\binom{\ell}{\rho}\lambda_{i,\rho}, \quad i, \ell = 0, 1, \ldots, s,$$

which possesses the solution

$$\lambda_{i,0} = \lambda_{i,1} = \cdots = \lambda_{i,i-1} = 0, \quad \lambda_{i,j} = \frac{\binom{j}{i} h^i}{\prod_{\rho=0}^{i-1}(p-\rho)}, \;\; j \geq i.$$

□

The construction of a basis for the space $\mathcal{V}_\Gamma^s$, and hence for the space $\mathcal{V}^s$, is a very challenging task, which requires the study of a lot of different possible cases, cf. [27] and [33] for $s = 1$ and $s = 2$, respectively. This is a direct consequence of the fact that the dimension of the space $\mathcal{V}_\Gamma^s$ heavily depends on the configuration of the two underlying bilinear patches $\boldsymbol{F}^{(i_0)}$ and $\boldsymbol{F}^{(i_1)}$. Therefore, we consider instead of the entire space $\mathcal{V}^s = \mathcal{V}^s_{\Omega^{(i_0)}} \oplus \mathcal{V}^s_{\Omega^{(i_1)}} \oplus \mathcal{V}^s_\Gamma$ a subspace $\mathcal{W}^s \subseteq \mathcal{V}^s$, given as

$$\mathcal{W}^s = \mathcal{V}^s_{\Omega^{(i_0)}} \oplus \mathcal{V}^s_{\Omega^{(i_1)}} \oplus \widetilde{\mathcal{W}}^s_\Gamma, \tag{36}$$

with $\widetilde{\mathcal{W}}^s_\Gamma \subseteq \mathcal{V}^s_\Gamma$. The subspace $\widetilde{\mathcal{W}}^s_\Gamma$ is defined as

$$\widetilde{\mathcal{W}}^s_\Gamma = \operatorname{span}\left\{\phi_{\Gamma;j_1,j_2} \mid j_1 = 0, 1, \ldots, s, \;\; j_2 = 0, 1, \ldots, n_{j_1} - 1\right\},$$

where the functions $\phi_{\Gamma;j_1,j_2} : \overline{\Omega} \to \mathbb{R}$ possess the form

$$\phi_{\Gamma;j_1,j_2}(\boldsymbol{x}) = \begin{cases} \left(f^{(i_0)}_{\Gamma;j_1,j_2} \circ \left(\boldsymbol{F}^{(i_0)}\right)^{-1}\right)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega^{(i_0)}}, \\ \left(f^{(i_1)}_{\Gamma;j_1,j_2} \circ \left(\boldsymbol{F}^{(i_1)}\right)^{-1}\right)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega^{(i_1)}}, \end{cases} \quad j_1 = 0, 1, \ldots, s, \;\; j_2 = 0, 1, \ldots, n_{j_1} - 1, \tag{37}$$

and $n_{j_1} = \dim\left(\mathcal{S}_h^{p-j_1,r+s-j_1}([0,1])\right) = p+1-j_1+k(p-r-s+j_1)$. The functions $f_{\Gamma;j_1,j_2}^{(\tau)} : [0,1]^2 \to \mathbb{R}$, $\tau \in \{i_0, i_1\}$, are further given by

$$f_{\Gamma;j_1,j_2}^{(\tau)}(\xi_1,\xi_2) = \gamma_{j_1}\sum_{i=j_1}^{s}\binom{i}{j_1}\left(\beta^{(\tau)}(\xi_2)\right)^{i-j_1}\left(\alpha^{(\tau)}(\xi_2)\right)^{j_1}\partial^{i-j_1}\left(N_{j_2}^{p-j_1,r+s-j_1}(\xi_2)\right)M_i^{p,r}(\xi_1), \tag{38}$$

where γ_{j_1}, $j_1 = 0, 1, \ldots, s$, are scaling factors of the form $\gamma_{j_1} = h^{-j_1}\prod_{j_2=0}^{j_1-1}(p-j_2)$. The selection of the subspace $\mathcal{W}^s$ is motivated by the numerical results in [33] for $s = 2$, and by our numerical experiments in Section 6 for $s = 1, \ldots, 4$, which indicate that the subspace $\mathcal{W}^s$ (and consequently also the entire space $\mathcal{V}^s$) possesses optimal approximation properties. It remains to show that $\widetilde{\mathcal{W}}_\Gamma^s \subseteq \mathcal{V}_\Gamma^s$, which is covered amongst others by the following theorem.

Theorem 4 *It holds that*

$$\widetilde{\mathcal{W}}_\Gamma^s \subseteq \mathcal{V}_\Gamma^s.$$

Moreover, the functions $\phi_{\Gamma;j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, n_{j_1} - 1$, *form a basis of the space* $\widetilde{\mathcal{W}}_\Gamma^s$.

Proof In order to prove $\widetilde{\mathcal{W}}_\Gamma^s \subseteq \mathcal{V}_\Gamma^s$, we show that $\phi_{\Gamma;j_1,j_2} \in \mathcal{V}_\Gamma^s$ for all $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, n_{j_1} - 1$. Note that any $(s+1)$-tuple of functions $(f_0, f_1, \ldots, f_s)$, which assures after inserting in (35), that the two functions $f^{(\tau)} = \phi \circ \boldsymbol{F}^{(\tau)}$, $\tau \in \{i_0, i_1\}$, belong to the spline space $\mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right)$, defines an isogeometric function $\phi \in \mathcal{V}_\Gamma^s$. One can easily verify by means of representation (35), that the $(s+1)$-tuples formed by

$$(f_0, f_1, \ldots, f_s) = \left(\underbrace{0, 0, \ldots, 0}_{j_1}, \gamma_{j_1} N_{j_2}^{p-j_1,r+s-j_1}, \underbrace{0, \ldots, 0}_{s-j_1}\right), \tag{39}$$

for $j_2 = 0, 1, \ldots, n_{j_1} - 1$, $j_1 = 0, 1, \ldots, s$, yield isogeometric functions $\phi \in \mathcal{V}_\Gamma^s$, since we get $f^{(\tau)} \in \mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right)$, $\tau \in \{i_0, i_1\}$. However, the resulting functions $\phi \in \mathcal{V}_\Gamma^s$ obtained by the $(s+1)$-tuples in (39) are now exactly the isogeometric functions $\phi_{\Gamma;j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, n_{j_1} - 1$, which hence implies that $\phi_{\Gamma;j_1,j_2} \in \mathcal{V}_\Gamma^s$ for all $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, n_{j_1} - 1$. Since the functions $\phi_{\Gamma;j_1,j_2}$ are linearly independent by construction, they form a basis of the space $\widetilde{\mathcal{W}}_\Gamma^s$ □

Remark 3 The scaling factors γ_{j_1} in (38) guarantee together with the proposed choice of λ_1 for the functions $\alpha^{(i_0)}$, $\alpha^{(i_1)}$ and β in Section 3.1 that the basis functions $\phi_{\Gamma;j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, n_{j_1} - 1$, are uniformly scaled. This scaling has been already used for the construction of C^s-smooth isogeometric spline functions for the case of $s = 1$ and $s = 2$ in [29, 30] and [34, 35], respectively.

Finally, we obtain the following theorem as a direct consequence of the results from this subsection.

Theorem 5 *The space $\mathcal{W}^s$, given in (36), is a subspace of the C^s-smooth space $\mathcal{V}^s$. In addition, the functions $\phi_{\Omega^{(\tau)};j_1,j_2}$, $j_1 = s+1, \ldots, n-1$, $j_2 = 0, 1, \ldots, n-1$, $\tau \in \{i_0, i_1\}$, together with the functions $\phi_{\Gamma;j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, n_{j_1} - 1$, form a basis of the space $\mathcal{W}^s$, which further implies that the dimension of the subspace $\mathcal{W}^s$ is independent of the configuration of the two underlying bilinear patches $\boldsymbol{F}^{(i_0)}$ and $\boldsymbol{F}^{(i_1)}$, and is equal to*

$$\dim \mathcal{W}^s = \dim \mathcal{V}^s_{\Omega^{(i_0)}} + \dim \mathcal{V}^s_{\Omega^{(i_1)}} + \dim \widetilde{\mathcal{W}}^s_{\Gamma},$$

with

$$\dim \mathcal{V}^s_{\Omega^{(i_0)}} = \dim \mathcal{V}^s_{\Omega^{(i_1)}} = n(n-(s+1)) \quad \textit{and} \quad \dim \widetilde{\mathcal{W}}^s_{\Gamma} = \sum_{j_1=0}^{s} (n_{j_1} + 1).$$

Remark 4 While for $s = 1$ we have $\widetilde{\mathcal{W}}^1_{\Gamma} = \mathcal{V}^1_{\Gamma}$ except for special configurations of the two patch geometry $\boldsymbol{F}$, see [27], for $s \geq 2$ it always holds $\widetilde{\mathcal{W}}^s_{\Gamma} \subsetneq \mathcal{V}^s_{\Gamma}$, since the linear combinations of the functions $\phi_{\Gamma;j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, n_{j_1} - 1$, are not the only functions in $\mathcal{V}^s_{\Gamma}$, see, e.g. [33] for $s = 2$. An example of a special configuration of the two-patch geometry $\boldsymbol{F}$ with $\widetilde{\mathcal{W}}^s_{\Gamma} \subsetneq \mathcal{V}^s_{\Gamma}$ for $s = 1$, too, is to choose the two geometry mappings $\boldsymbol{F}^{(i_0)}$ and $\boldsymbol{F}^{(i_1)}$ as bilinear mappings $\boldsymbol{F}^{(i_0)} : [0,1]^2 \to [-1,0] \times [0,1]$ and $\boldsymbol{F}^{(i_1)} : [0,1]^2 \to [0,1]^2$. In this case, the dimension of $\mathcal{V}^s_{\Gamma}$ is trivially equal to

$$\dim \mathcal{V}^s_{\Gamma} = (s+1)n,$$

but which is always larger than $\dim \widetilde{\mathcal{W}}^s_{\Gamma}$.

4 C^s-smooth isogeometric spaces over multi-patch domains

In this section, we will extend the construction of the C^s-smooth isogeometric subspace $\mathcal{W}^s \subseteq \mathcal{V}^s$ for bilinearly parameterized two-patch domains, which has been described in the previous section, to the case of bilinear multi-patch domains $\overline{\Omega}$ with more than two patches and with possibly extraordinary vertices. The proposed construction will work uniformly for all possible multi-patch configurations and is much simpler as for the entire C^s-smooth space $\mathcal{V}^s$. Thereby, the design of the subspace $\mathcal{W}^s$ will be based on the results of the two-patch case, and is motivated by the methods [29, 30] and [34, 35], where similar subspaces have been generated for a global smoothness of $s = 1$ and $s = 2$, respectively. There, it has been numerically shown that the corresponding subspaces possess optimal approximation properties. This will be also numerically verified in Section 6 on the basis of an example for the subspace $\mathcal{W}^s$ for $s = 1, \ldots, 4$.

The subspace $\mathcal{W}^s$ will be generated as the direct sum of smaller subspaces corresponding to the single patches $\Omega^{(i)}$, $i \in \mathcal{I}_\Omega$, edges $\Gamma^{(i)}$, $i \in \mathcal{I}_\Gamma$ and vertices $\Xi^{(i)}$, $i \in \mathcal{I}_\Xi$, i.e.

$$\mathcal{W}^s = \left(\bigoplus_{i \in \mathcal{I}_\Omega} \mathcal{W}^s_{\Omega^{(i)}} \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_\Gamma} \mathcal{W}^s_{\Gamma^{(i)}} \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_\Xi} \mathcal{W}^s_{\Xi^{(i)}} \right). \tag{40}$$

In order to ensure h-refinable and well-defined subspaces, we have to assume additionally that the number of inner knots satisfies $k \geq \frac{4s+1-p}{p-r-s}$, which implies $h \leq \frac{p-r-s}{3s-r+1}$. The construction of the particular subspaces in (40) will be based on functions from the subspaces $\mathcal{V}^s_{\Omega^{(\tau)}}$, $\tau \in \{i_0, i_1\}$, and $\mathcal{W}^s_\Gamma$, which have been both defined in Section 3 for the case of a bilinearly parameterized two-patch domain $\overline{\Omega} = \overline{\Omega^{(i_0)}} \cup \overline{\Omega^{(i_1)}}$ with the common edge $\overline{\Gamma} = \overline{\Omega^{(i_0)}} \cap \overline{\Omega^{(i_1)}}$, and will be described in detail below.

4.1 The patch and edge subspaces

We will first describe the construction of the subspaces $\mathcal{W}^s_{\Omega^{(i)}}$, $i \in \mathcal{I}_\Omega$, and $\mathcal{W}^s_{\Gamma^{(i)}}$, $i \in \mathcal{I}_\Gamma$. Analogous to (34) in case of a two-patch domain, we define the functions $\phi_{\Omega^{(i)};j_1,j_2} : \overline{\Omega} \to \mathbb{R}$ in case of a multi-patch domain $\overline{\Omega} = \cup_{i \in \mathcal{I}_\Omega} \overline{\Omega^{(i)}}$ as

$$\phi_{\Omega^{(i)};j_1,j_2}(\boldsymbol{x}) = \begin{cases} \left(N^{\boldsymbol{p},\boldsymbol{r}}_{j_1,j_2} \circ \left(\boldsymbol{F}^{(i)}\right)^{-1} \right)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega^{(i)}}, \\ 0 \quad \text{if } \boldsymbol{x} \in \overline{\Omega} \backslash \overline{\Omega^{(i)}}, \end{cases} \tag{41}$$

and then define the patch subspace $\mathcal{W}^s_{\Omega^{(i)}}$ as

$$\mathcal{W}^s_{\Omega^{(i)}} = \text{span} \left\{ \phi_{\Omega^{(i)};j_1,j_2} | \; j_1, j_2 = s+1, s+2, \ldots, n-1-(s+1) \right\}.$$

We clearly have $\mathcal{W}^s_{\Omega^{(i)}} \subseteq \mathcal{V}^s$, since the functions $\phi_{\Omega^{(i)};j_1,j_2}$, $j_1, j_2 = s+1, \ldots, n-1-(s+1)$, have a support entirely inside $\Omega^{(i)}$, are clearly C^s-smooth on $\Omega^{(i)}$ and have vanishing values and derivatives up to order s on $\partial\Omega^{(i)}$.

Let us construct now the edge subspaces $\mathcal{W}^s_{\Gamma^{(i)}}$, $i \in \mathcal{I}_\Gamma$, where we have to distinguish between boundary and inner edges $\Gamma^{(i)}$. In case of a boundary edge $\Gamma^{(i)} \subseteq \overline{\Omega^{(i_0)}}$, $i_0 \in \mathcal{I}_\Omega$, we can assume without loss of generality that the boundary edge $\Gamma^{(i)}$ is given by $\boldsymbol{F}^{(i_0)}(\{0\} \times (0,1))$. Then, we generate the edge subspace $\mathcal{W}^s_{\Gamma^{(i)}}$ as

$$\mathcal{W}^s_{\Gamma^{(i)}} = \text{span} \left\{ \phi_{\Omega^{(i_0)};j_1,j_2} | \; j_2 = 2s+1-j_1, \ldots, n+j_1-(2s+2), \;\; j_1 = 0, 1, \ldots, s \right\},$$

where the functions $\phi_{\Omega^{(i_0)};j_1,j_2}$ are defined as in (41). Similar to the patch subspace $\mathcal{W}^s_{\Omega^{(i)}}$, the functions $\phi_{\Omega^{(i_0)};j_1,j_2}$, $j_2 = 2s+1-j_1, \ldots, n+j_1-(2s+2)$, $j_1 = 0, 1, \ldots, s$, are trivially C^s-smooth on $\overline{\Omega}$, which implies that $\mathcal{W}^s_{\Gamma^{(i)}} \subseteq \mathcal{V}^s$.

Let us consider now the case of an inner edge $\Gamma^{(i)} \subseteq \overline{\Omega^{(i_0)}} \cap \overline{\Omega^{(i_1)}}$, $i_0, i_1 \in \mathcal{I}_\Omega$. Without loss of generality, we can assume that the two associated geometry mappings $\boldsymbol{F}^{(i_0)}$ and $\boldsymbol{F}^{(i_1)}$ are parameterized as shown in Fig. 2. The edge subspace

$\mathcal{W}^s_{\Gamma^{(i)}}$ is now defined as

$$\mathcal{W}^s_{\Gamma^{(i)}} = \text{span}\left\{\phi_{\Gamma^{(i)};j_1,j_2} |\ \ j_2 = 2s+1-j_1, \ldots, n_{j_1}+j_1-(2s+2),\ \ j_1 = 0, 1, \ldots, s\right\},$$

where the functions $\phi_{\Gamma^{(i)};j_1,j_2} : \overline{\Omega} \to \mathbb{R}$ possess the form

$$\phi_{\Gamma^{(i)};j_1,j_2}(\boldsymbol{x}) = \begin{cases} \left(f^{(i_0)}_{\Gamma^{(i)};j_1,j_2} \circ \left(\boldsymbol{F}^{(i_0)}\right)^{-1}\right)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega^{(i_0)}}, \\ \left(f^{(i_1)}_{\Gamma^{(i)};j_1,j_2} \circ \left(\boldsymbol{F}^{(i_1)}\right)^{-1}\right)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega^{(i_1)}}, \\ 0 & \text{otherwise,} \end{cases} \tag{42}$$

similar to the two-patch case (37), and where the functions $f^{(\tau)}_{\Gamma^{(i)};j_1,j_2}$, $\tau \in \{i_0, i_1\}$, are specified in (38). In contrast to the two-patch case (37), the indices j_2 of the functions $f^{(\tau)}_{\Gamma^{(i)};j_1,j_2}$ are restricted to the choice of $j_2 = 2s+1-j_1, \ldots, n_{j_1}+j_1-(2s+2)$, $j_1 = 0, 1, \ldots, s$, which implies that the functions $f^{(\tau)}_{\Gamma^{(i)};j_1,j_2}$ have vanishing values and derivatives up to order s at all edges $\overline{\Gamma^{(\ell)}}$, $\ell \in \mathcal{I}_\Gamma$, except the edge $\Gamma^{(i)}$, that is for all edges $\overline{\Gamma^{(\ell)}}$, $\ell \in \mathcal{I}_\Gamma \setminus \{i\}$. This property of the functions $\phi_{\Gamma^{(i)};j_1,j_2}$, $j_2 = 2s+1-j_1, \ldots, n_{j_1}+j_1-(2s+2)$, $j_1 = 0, 1, \ldots, s$, together with the fact that the functions $\phi_{\Gamma^{(i)};j_1,j_2}$ possess a support contained in $\overline{\Omega^{(i_0)}} \cup \overline{\Omega^{(i_1)}}$ and are C^s-smooth at the edge $\Gamma^{(i)}$ by construction directly leads to $\mathcal{W}^s_{\Gamma^{(i)}} \subseteq \mathcal{V}^s$.

4.2 The vertex subspaces

We will denote by v_i the patch valency of a vertex $\Xi^{(i)}$, $i \in \mathcal{I}_\Xi$. To generate the vertex subspaces $\mathcal{W}^s_{\Xi^{(i)}}$, we will distinguish between inner and boundary vertices. We will follow a similar approach as used in [29, 30] and [34, 35] for the construction of C^1 and C^2-smooth isogeometric spline functions in the vicinity of the vertex $\Xi^{(i)}$.

Let us start with the case of an inner vertex $\Xi^{(i)}$, $i \in \mathcal{I}_\Xi$. We can assume without loss of generality that all patches $\Omega^{(i_\rho)}$, $\rho = 0, 1, \ldots, v_i - 1$, around the vertex $\Xi^{(i)}$, i.e. $\Xi^{(i)} = \cap_{\rho=0}^{v_i-1} \overline{\Omega^{(i_\rho)}}$, are parameterized and labeled as shown in Fig. 3. In addition, we relabel the common edges $\overline{\Omega^{(i_\rho)}} \cap \overline{\Omega^{(i_{\rho+1})}}$, $\rho = 0, 1, \ldots, v_i - 1$, by $\Gamma^{(i_{\rho+1})}$, where we take the lower index ρ of the indices i_ρ modulo v_i.

The design of the subspace $\mathcal{W}^s_{\Xi^{(i)}}$ is based on the construction of C^s-smooth functions in the vicinity of the vertex $\Xi^{(i)}$, which will be formed by the linear combination of functions $\phi_{\Gamma^{(i_\rho)};j_1,j_2}$, $j_2 = 0, 1, \ldots, 2s - j_1$, $j_1 = 0, 1, \ldots, s$, $\rho = 0, 1, \ldots, v_i - 1$, coinciding at their common supports in the vicinity of the vertex $\Xi^{(i)}$, and by subtracting those "standard" isogeometric spline functions $\phi_{\Omega^{(i_\rho)};j_1,j_2}$, $j_1, j_2 = 0, 1, \ldots, s$, $\rho = 0, 1, \ldots, v_i - 1$, which have been added twice. For this purpose, let us consider the isogeometric spline function $\phi_{\Xi^{(i)}} : \overline{\Omega} \to \mathbb{R}$,

$$\phi_{\Xi^{(i)}}(\boldsymbol{x}) = \begin{cases} \left(f^{\Xi^{(i)}}_{i_\rho} \circ \left(\boldsymbol{F}^{(i_\rho)}\right)^{-1}\right)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{\Omega^{(i_\rho)}},\ \ \rho = 0, 1, \ldots, v_i - 1, \\ 0 & \text{otherwise,} \end{cases} \tag{43}$$

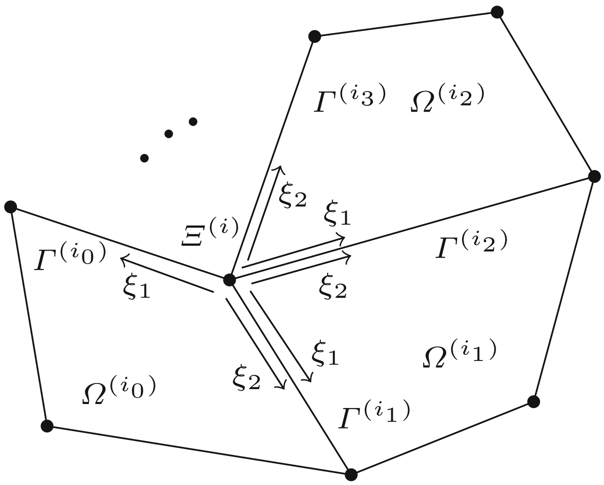

Fig. 3 The parameterization of the patches $\Omega^{(i_0)}, \Omega^{(i_1)}, \ldots, \Omega^{(i_{v_i-1})}$ with the edges $\Gamma^{(i_0)}, \Gamma^{(i_1)}, \ldots, \Gamma^{(i_{v_i-1})}$ around the inner vertex $\Xi^{(i)}$

where the functions $f_{i_\rho}^{\Xi^{(i)}} : [0,1]^2 \to \mathbb{R}$ are given as

$$f_{i_\rho}^{\Xi^{(i)}}(\xi_1, \xi_2) = f_{i_\rho}^{\Gamma^{(i_\rho)}}(\xi_1, \xi_2) + f_{i_\rho}^{\Gamma^{(i_{\rho+1})}}(\xi_1, \xi_2) - f_{i_\rho}^{\Omega^{(i_\rho)}}(\xi_1, \xi_2), \tag{44}$$

with $f_{i_\rho}^{\Gamma^{(i_{\rho+\tau})}}, f_{i_\rho}^{\Omega^{(i_\rho)}} : [0,1]^2 \to \mathbb{R}$,

$$\begin{aligned} f_{i_\rho}^{\Gamma^{(i_{\rho+\tau})}}(\xi_1, \xi_2) &= \sum_{j_1=0}^{s} \sum_{j_2=0}^{2s-j_1} a_{j_1,j_2}^{\Gamma^{(i_{\rho+\tau})}} f_{\Gamma^{(i_{\rho+\tau})};j_1,j_2}^{(i_\rho)}(\xi_{2-\tau}, \xi_{1+\tau}), \quad a_{j_1,j_2}^{\Gamma^{(i_{\rho+\tau})}} \in \mathbb{R}, \quad \tau = 0, 1, \\ f_{i_\rho}^{\Omega^{(i_\rho)}}(\xi_1, \xi_2) &= \sum_{j_1=0}^{s} \sum_{j_2=0}^{s} a_{j_1,j_2}^{(i_\rho)} N_{j_1,j_2}^{\boldsymbol{p},\boldsymbol{r}}(\xi_1, \xi_2), \quad a_{j_1,j_2}^{(i_\rho)} \in \mathbb{R}, \end{aligned} \tag{45}$$

and with the functions $f_{\Gamma^{(i_{\rho+\tau})};j_1,j_2}^{(i_\rho)}$, $\tau = 0, 1$, given in (38). The function $\phi_{\Xi^{(i)}}$ is now C^s-smooth on $\overline{\Omega}$, i.e. $\phi_{\Xi^{(i)}} \in \mathcal{V}^s$, if the coefficients $a_{j_1,j_2}^{\Gamma^{(i_{\rho+\tau})}}$, $a_{j_1,j_2}^{(i_\rho)}$ satisfy the equations

$$\partial_1^{\ell_1} \partial_2^{\ell_2} \left(f_{i_\rho}^{\Gamma^{(i_{\rho+1})}} - f_{i_\rho}^{\Gamma^{(i_\rho)}} \right)(\mathbf{0}) = 0 \quad \text{and} \tag{46a}$$

$$\partial_1^{\ell_1} \partial_2^{\ell_2} \left(f_{i_\rho}^{\Gamma^{(i_{\rho+1})}} - f_{i_\rho}^{\Omega^{(i_\rho)}} \right)(\mathbf{0}) = 0, \tag{46b}$$

for $0 \le \ell_1, \ell_2 \le s$ and $\rho = 0, 1, \ldots, v_i - 1$. The Eqs. (46) form a homogeneous system of linear equations

$$T^{(i)} \boldsymbol{a} = \mathbf{0}, \tag{47}$$

where the vector $\boldsymbol{a}$ consists of all coefficients $a_{j_1,j_2}^{\Gamma^{(i_{\rho+\tau})}}$, $a_{j_1,j_2}^{(i_\rho)}$. Any choice of the vector $\boldsymbol{a}$, which fulfills the linear system (47), yields an isogeometric function (43) belonging to the spline space $\mathcal{V}^s$. Each basis of the kernel $\ker\left(T^{(i)}\right)$ determines $\dim\ker\left(T^{(i)}\right)$ linearly independent C^s-smooth isogeometric spline functions, which are denoted by $\phi_{\Xi^{(i)},j}$, $j \in \{0, 1, \ldots, \dim\ker\left(T^{(i)}\right) - 1\}$, and which can be used to define the vertex subspace $\mathcal{W}^s_{\Xi^{(i)}}$ via

$$\mathcal{W}^s_{\Xi^{(i)}} = \operatorname{span}\left\{\phi_{\Xi^{(i)},j} \mid j \in \{0, 1, \ldots, \dim\ker\left(T^{(i)}\right) - 1\}\right\} \subseteq \mathcal{V}^s.$$

As in the case of the three-patch domain in the numerical examples in Section 6, see Fig. 5 (bottom row), we can employ the algorithm developed in [32] for computing a basis of $\ker\left(T^{(i)}\right)$, which is based on the concept of minimal determining sets (cf. [44]) for the coefficients $\boldsymbol{a}$.

Note that $\dim\ker\left(T^{(i)}\right)$, and hence $\dim\mathcal{W}^s_{\Xi^{(i)}}$, does not depend just on the valency ν_i of the vertex $\Xi^{(i)}$ but also on the configuration of the bilinear patches around the corresponding vertex. The computation of $\dim\ker\left(T^{(i)}\right)$ would require the study of various different possible cases, see, e.g. [8] and [31] for $s = 1$ and $s = 2$, respectively, and is beyond the scope of this paper. However, the following lemma provides us with a first lower and upper bound for $\dim\mathcal{W}^s_{\Xi^{(i)}}$.

Lemma 2 *Let $\Xi^{(i)}$, $i \in \mathcal{I}_{\Xi}$, be an inner vertex of patch valency ν_i. Then,* $\dim\mathcal{W}^s_{\Xi^{(i)}}$ *can be bounded by*

$$\frac{1}{2}\nu_i\, s\,(s+1) \leq \dim\mathcal{W}^s_{\Xi^{(i)}} \leq \frac{1}{2}(s+1)((2+\nu_i)s+2).$$

Proof Clearly, $\dim\ker\left(T^{(i)}\right)$, and hence $\dim\mathcal{W}^s_{\Xi^{(i)}}$, is given by the the number of unknowns in the homogeneous linear system (47) minus the number of linearly independent equations in this linear system. The number of unknowns is now just the number of the possible involved coefficients $a^{\Gamma^{(i_{\rho+\tau})}}_{j_1,j_2}$, $a^{(i_\rho)}_{j_1,j_2}$, which is equal to $\nu_i\,((s+1)^2 + \frac{1}{2}(s+1)(3s+2))$. To estimate the number of linearly independent equations, we will study the Eqs. (46) in more detail, since they form the linear system (47). In doing so, we aim to give a lower and upper bound for this number.

Let us start with the $\nu_i\,(s+1)^2$ linear Eqs. (46b). All these equations are not only linearly independent from each other but also from the linear Eqs. (46a). This is a direct consequence of their construction and of the fact that for each ρ, $\rho = 0, 1, \ldots, \nu_i - 1$, the $(s+1)^2$ coefficients $a^{(i_\rho)}_{j_1,j_2}$, $0 \leq j_1, j_2 \leq s$, arise just in Eqs. (46b), and there only in the equations for the corresponding patch $\Omega^{(i_\rho)}$.

Let us continue with the $\nu_i\,(s+1)^2$ linear Eqs. (46a). For each patch $\Omega^{(i_\rho)}$, $\rho = 0, 1, \ldots, \nu_i - 1$, the $(s+1)^2$ equations contain at most the coefficients $a^{\Gamma^{(i_\rho)}}_{j_1,j_2}$, $a^{\Gamma^{(i_{\rho+1})}}_{j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, 2s - j_1$, and are clearly linearly independent by construction. Therefore, starting with the $(s+1)^2$ equations for patch $\Omega^{(i_0)}$, and adding step by step the $(s+1)^2$ equations for the patches $\Omega^{(i_\rho)}$, $\rho = 1, 2, \ldots, \nu_i - 2$, one can easily verify that the resulting $(\nu_i - 1)\,(s+1)^2$ equations are linearly independent, since in each step for the patches $\Omega^{(i_\rho)}$, $\rho = 1, 2, \ldots, \nu_i - 2$, the coefficients $a^{\Gamma^{(i_\rho)}}_{j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, 2s - j_1$, just arise for the first time. Adding the last $(s+1)^2$ equations, namely the equations for the patch $\Omega^{(i_{\nu_i-1})}$, we cannot assume as before that the $(s+1)^2$ additional equations are linearly independent with all the remaining ones, since now the coefficients $a^{\Gamma^{(i_{v_i-1})}}_{j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, 2s - j_1$, could have already appeared in the equations for patch $\Omega^{(i_0)}$, and therefore some of these equations could be linearly dependent with the previous equations. By assuming the two extreme cases that all or none of these $(s+1)^2$ equations are linearly dependent with the previous equations we get in total $(2v_i - 1)\,(s+1)^2$ as a lower and $2v_i\,(s+1)^2$ as an upper bound for the number of linearly independent equations in (46). Using these bounds to estimate the dimension of $\mathcal{W}^s_{\Xi^{(i)}}$, we finally obtain for $\dim \mathcal{W}^s_{\Xi^{(i)}}$ as a lower and upper bound

$$v_i\,((s+1)^2 + \frac{1}{2}(s+1)(3s+2)) - 2v_i\,(s+1)^2 = \frac{1}{2}\,v_i\,s\,(s+1)$$

and

$$v_i\,\left((s+1)^2 + \frac{1}{2}(s+1)(3s+2)\right) - (2v_i - 1)\,(s+1)^2 = \frac{1}{2}\,(s+1)((2+v_i)s+2),$$

respectively. □

Let us continue with the case of a boundary vertex. This can be handled similarly as an inner vertex by assuming that the two boundary edges are labeled as $\Gamma^{(i_0)}$ and $\Gamma^{(i_{v_i})}$. Then, the only difference in the construction of the C^s-smooth functions $\phi_{\Xi^{(i)},j}$ and of the C^s-smooth space $\mathcal{W}^s_{\Xi^{(i)}} \subseteq \mathcal{V}^s$ is that for the patches $\Omega^{(i_0)}$ and $\Omega^{(i_{v_i-1})}$ the functions $f^{\Gamma^{(i_0)}}_{i_0;j_1,j_2}$ and $f^{\Gamma^{(i_{v_i})}}_{i_{v_i-1};j_1,j_2}$ in (45) are just the standard B-splines. Following the steps in the proof of Lemma 2 and counting the number of unknowns in the adapted homogeneous linear system (47) minus the number of equations (which are now clearly all linearly independent) in this linear system, we further obtain that for any boundary vertex $\Xi^{(i)}$ of valency v_i it holds

$$\dim \mathcal{W}^s_{\Xi^{(i)}} = \left(((s+1)^2 v_i + \frac{1}{2}(s+1)(3s+2))(v_i+1)\right) - 2(s+1)^2 v_i = \frac{1}{2}(s+1)((3+v_i)s+2).$$

For boundary vertices $\Xi^{(i)}$ of patch valency $v_i \in \{1, 2\}$, the vertex subspaces $\mathcal{W}^s_{\Xi^{(i)}}$ can be also directly constructed without solving a homogeneous linear system (47). In case of a boundary vertex $\Xi^{(i)}$ of patch valency $v_i = 2$, we can assume without loss of generality that the two neighboring patches $\Omega^{(i_0)}$ and $\Omega^{(i_1)}$, $i_0, i_1 \in \mathcal{I}_\Omega$, which contain the vertex $\Xi^{(i)}$ and possess the common edge $\overline{\Gamma^{(j_0)}} = \overline{\Omega^{(i_0)}} \cap \overline{\Omega^{(i_1)}}$, $j_0 \in \mathcal{I}_\Gamma$, are parameterized as shown in Fig. 2 and that the vertex $\Xi^{(i)}$ is further given as $\Xi^{(i)} = \boldsymbol{F}^{(i_0)}(\mathbf{0}) = \boldsymbol{F}^{(i_1)}(\mathbf{0})$. Then, the vertex subspace $\mathcal{W}^s_{\Xi^{(i)}}$ can be also generated as

$$\mathcal{W}^s_{\Xi^{(i)}} = \operatorname{span}\left\{\widetilde{\phi}_{\Xi^{(i)};j_1,j_2} \mid j_1 = 0, 1, \ldots, 3s,\ j_2 = \begin{cases} 0, 1, \ldots, 2s - j_1 & \text{if } j_1 \le 2s \\ 0, 1, \ldots, 3s - j_1 & \text{if } j_1 > 2s \end{cases}\right\},$$

with the functions $\widetilde{\phi}_{\Xi^{(i)};j_1,j_2} : \overline{\Omega} \to \mathbb{R}$,

$$\widetilde{\phi}_{\Xi^{(i)};j_1,j_2}(\boldsymbol{x}) = \begin{cases} \phi_{\Gamma^{(j_0)};j_1,j_2}(\boldsymbol{x}) & \text{if } j_1 = 0, 1, \ldots, s \\ \phi_{\Omega^{(i_0)};j_1,j_2}(\boldsymbol{x}) & \text{if } j_1 = s+1, s+2, \ldots, 2s \\ \phi_{\Omega^{(i_1)};j_1-s,j_2}(\boldsymbol{x}) & \text{if } j_1 = 2s+1, 2s+2, \ldots, 3s, \end{cases}$$

where the functions $\phi_{\Omega^{(i_\ell)};j_1,j_2}$, $\ell = 0, 1$, and $\phi_{\Gamma^{(j_0)};j_1,j_2}$ are defined as in (41) and (42), respectively. Clearly, all functions $\widetilde{\phi}_{\Xi^{(i)};j_1,j_2}$ are C^s-smooth on $\overline{\Omega}$, since the functions $\phi_{\Gamma^{(j_0)};j_1,j_2}$, $j_1 = 0, 1, \ldots, s$, $j_2 = 0, 1, \ldots, 2s - j_1$, are C^s-smooth by construction on $\overline{\Omega}$, and the functions $\phi_{\Omega^{(i_\ell)};j_1,j_2}$, $\ell = 0, 1$, $j_1 = s+1, s+2, \ldots, 3s$, $j_2 = 0, 1, \ldots, 2s - j_1$ if $j_1 \leq 2s$ and $j_2 = 0, 1, \ldots, 3s - j_1$ if $j_1 > 2s$, which possess a support in $\overline{\Omega^{(i_\ell)}}$, are C^s-smooth on $\overline{\Omega^{(i_\ell)}}$, and have vanishing values and derivatives of order $\leq s$ along all inner edges $\overline{\Gamma^{(j)}}$, $j \in \mathcal{I}_\Gamma$.

In case of a boundary vertex $\Xi^{(i)}$ of patch valency $v_i = 1$, we can assume without loss of generality that the boundary vertex $\Xi^{(i)}$ is given by $\Xi^{(i)} = \boldsymbol{F}^{(i_0)}(\mathbf{0})$, $i_0 \in \mathcal{I}_\Omega$. Then, the vertex subspace $\mathcal{W}^s_{\Xi^{(i)}}$ can be simply constructed as

$$\mathcal{W}^s_{\Xi^{(i)}} = \operatorname{span}\left\{\phi_{\Omega^{(i_0)};j_1,j_2} |\ j_1, j_2 = 0, 1, \ldots, 2s,\ \ j_1 + j_2 \leq 2s\right\},$$

where the functions $\phi_{\Omega^{(i_0)};j_1,j_2}$ are given as in (41). Again, the functions $\phi_{\Omega^{(i_0)};j_1,j_2}$, $j_1, j_2 = 0, 1, \ldots, 2s$, $j_1 + j_2 \leq 2s$, are entirely contained in $\overline{\Omega^{(i_0)}}$, are C^s-smooth on $\overline{\Omega^{(i_0)}}$, and have vanishing values and derivatives of order $\leq s$ along all inner edges $\overline{\Gamma^{(j)}}$, $j \in \mathcal{I}_\Gamma$, which implies that the functions are C^s-smooth on $\overline{\Omega}$.

Summarizing the results from Section 4, we obtain:

Theorem 6 *The space $\mathcal{W}^s$, given by the direct sum (40), is a subspace of the C^s-smooth space $\mathcal{V}^s$. Moreover, the functions which have been used to generate the spaces $\mathcal{W}^s_{\Omega^{(i)}}$, $i \in \mathcal{I}_\Omega$, $\mathcal{W}^s_{\Gamma^{(i)}}$, $i \in \mathcal{I}_\Gamma$, and $\mathcal{W}^s_{\Xi^{(i)}}$, $i \in \mathcal{I}_\Xi$, form a basis of the space $\mathcal{W}^s$, and the dimension of the space $\mathcal{W}^s$ is equal to*

$$\dim \mathcal{W}^s = \sum_{i \in \mathcal{I}_\Omega} \dim \mathcal{W}^s_{\Omega^{(i)}} + \sum_{i \in \mathcal{I}_\Gamma} \dim \mathcal{W}^s_{\Gamma^{(i)}} + \sum_{i \in \mathcal{I}_\Xi} \dim \mathcal{W}^s_{\Xi^{(i)}},$$

where

$$\dim \mathcal{W}^s_{\Omega^{(i)}} = (n - 2(s+1))^2,$$

$$\dim \mathcal{W}^s_{\Gamma^{(i)}} = \begin{cases} (s+1)\left(n - ks - \left(\frac{7s}{2} + 2\right)\right) & \text{if } \Gamma^{(i)} \text{ is an inner edge}, \\ (s+1)(n - 3s - 2) & \text{if } \Gamma^{(i)} \text{ is a boundary edge}, \end{cases}$$

and

$$\dim \mathcal{W}^s_{\Xi^{(i)}} = \dim\ker\left(T^{(i)}\right),$$

with

$$\dim\ker\left(T^{(i)}\right) = \frac{1}{2}(s+1)((3+v_i)s + 2),$$

if $\Xi^{(i)}$ is a boundary vertex of valency v_i, and with

$$\frac{1}{2}\, v_i\, s\,(s+1) \leq \dim\ker\left(T^{(i)}\right) \leq \frac{1}{2}\,(s+1)((2+v_i)s + 2),$$

if $\Xi^{(i)}$ is an inner vertex of valency v_i.

Proof $\mathcal{W}^s \subseteq \mathcal{V}^s$ follows from the fact that $\mathcal{W}^s_{\Omega^{(i)}} \subseteq \mathcal{V}^s$, $i \in \mathcal{I}_\Omega$, $\mathcal{W}^s_{\Gamma^{(i)}} \subseteq \mathcal{V}^s$, $i \in \mathcal{I}_\Gamma$, and $\mathcal{W}^s_{\Xi^{(i)}} \subseteq \mathcal{V}^s$, $i \in \mathcal{I}_\Xi$, as already shown before. Since the functions which have been used to generate the spaces $\mathcal{W}^s_{\Omega^{(i)}}$, $i \in \mathcal{I}_\Omega$, $\mathcal{W}^s_{\Gamma^{(i)}}$, $i \in \mathcal{I}_\Gamma$, and $\mathcal{W}^s_{\Xi^{(i)}}$, $i \in \mathcal{I}_\Xi$, are linearly independent by definition and/or by construction, they form a basis of the individual spaces $\mathcal{W}^s_{\Omega^{(i)}}$, $\mathcal{W}^s_{\Gamma^{(i)}}$ and $\mathcal{W}^s_{\Xi^{(i)}}$, and therefore, they build a basis of the space $\mathcal{W}^s$. Finally, the dimension of $\mathcal{W}^s$ results from the direct sum (40), from the dimensions of the spaces $\mathcal{W}^s_{\Omega^{(i)}}$, $\mathcal{W}^s_{\Gamma^{(i)}}$ and $\mathcal{W}^s_{\Xi^{(i)}}$ and from Lemma 2. □

Remark 5 In case of a bilinearly parameterized two-patch domain, the two slightly different constructions described in this and in the previous section lead in both cases to the same subspace $\mathcal{W}^s$ with the same basis, which can be easily verified by comparing the two differently generated bases.

An alternative approach for the construction of a vertex subspace $\mathcal{W}^s_{\Xi^{(i)}}$, which leads also for inner vertices to a vertex subspace whose dimension is independent of the configuration of the bilinear patches (and even independent of the valency v_i), is to enforce additionally C^{2s}-smoothness of the functions at the vertex $\Xi^{(i)}$, see, e.g. [29, 30] and [35] for $s = 1$ and $s = 2$, respectively. Thereby, we will compute a subspace $\widehat{\mathcal{W}}^s_{\Xi^{(i)}}$ of the vertex subspace $\mathcal{W}^s_{\Xi^{(i)}}$, i.e. $\widehat{\mathcal{W}}^s_{\Xi^{(i)}} \subseteq \mathcal{W}^s_{\Xi^{(i)}}$. For this purpose, let $\psi_{j_1,j_2} : \overline{\Omega} \to \mathbb{R}$, $j_1, j_2 = 0, 1, \ldots, 2s$ with $j_1 + j_2 \leq 2s$, be functions which are C^s-smooth on $\overline{\Omega}$ and additionally C^{2s}-smooth at the vertex $\Xi^{(i)}$ such that

$$\partial_1^{\ell_1} \partial_2^{\ell_2} \psi_{j_1,j_2} \left(\Xi^{(i)} \right) = \sigma^{\ell_1+\ell_2} \, \delta_{j_1}^{\ell_1} \delta_{j_2}^{\ell_2}, \quad \ell_1, \ell_2 = 0, 1, \ldots, 2s, \; \ell_1 + \ell_2 \leq 2s,$$

where σ is a scaling factor (cf. [30]) given by

$$\sigma = \left(\frac{h}{p \, v_i} \sum_{\rho=0}^{v_i - 1} ||\mathrm{J}\boldsymbol{F}^{(i_\rho)}(\mathbf{0})|| \right)^{-1},$$

with $\mathrm{J}\boldsymbol{F}^{(i_\rho)}$ being the Jacobian of $\boldsymbol{F}^{(i_\rho)}$. Then, isogeometric functions $\widehat{\phi}_{\Xi^{(i)};j_1,j_2} : \overline{\Omega} \to \mathbb{R}$, $j_1, j_2 = 0, 1, \ldots, 2s$, $j_1 + j_2 \leq 2s$, can be defined via $\widehat{\phi}_{\Xi^{(i)};j_1,j_2} = \phi_{\Xi^{(i)}}$, with the functions $\phi_{\Xi^{(i)}}$ given in (43), by means of the interpolation problem

$$\partial_1^{\ell_1} \partial_2^{\ell_2} \phi_{\Xi^{(i)}} \left(\Xi^{(i)} \right) = \partial_1^{\ell_1} \partial_2^{\ell_2} \psi_{j_1,j_2} \left(\Xi^{(i)} \right), \quad \ell_1, \ell_2 = 0, 1, \ldots, 2s, \; \ell_1 + \ell_2 \leq 2s. \tag{48}$$

The isogeometric functions $\phi_{\Xi^{(i)}}$, and therefore the isogeometric functions $\widehat{\phi}_{\Xi^{(i)};j_1,j_2}$, are uniquely determined by (48) and can be computed via the coefficients $a^{\Gamma^{(i_{\rho+\tau})}}_{j_1,j_2}$ and $a^{(i_\rho)}_{j_1,j_2}$ of the spline functions $f^{\Xi^{(i)}}_{i_\rho}$ in (44) with the help of the following equivalent interpolation conditions

$$\begin{aligned}
\partial_1^{\ell_1} \partial_2^{\ell_2} f^{\Gamma^{(i_\rho)}}_{i_\rho}(\mathbf{0}) &= \partial_1^{\ell_1} \partial_2^{\ell_2} \left(\psi_{j_1,j_2} \circ \boldsymbol{F}^{(i_\rho)} \right)(\mathbf{0}), \quad 0 \leq \ell_1 \leq 2s, \; 0 \leq \ell_2 \leq s, \; \ell_1 + \ell_2 \leq 2s, \\
\partial_1^{\ell_1} \partial_2^{\ell_2} f^{\Omega^{(i_\rho)}}_{i_\rho}(\mathbf{0}) &= \partial_1^{\ell_1} \partial_2^{\ell_2} \left(\psi_{j_1,j_2} \circ \boldsymbol{F}^{(i_\rho)} \right)(\mathbf{0}), \quad 0 \leq \ell_1, \ell_2 \leq s,
\end{aligned}$$

for $\rho = 0, 1, \ldots, v_i - 1$. The resulting isogeometric spline functions $\widehat{\phi}_{\Xi^{(i)};j_1,j_2}$, $j_1, j_2 = 0, 1, \ldots, 2s$, $j_1 + j_2 \leq 2s$ are well-defined, C^s-smooth on $\overline{\Omega}$ and even C^{2s}-continuous at the vertex $\Xi^{(i)}$, and determine by

$$\widehat{\mathcal{W}}^s_{\Xi^{(i)}} = \operatorname{span} \left\{ \widehat{\phi}_{\Xi^{(i)};j_1,j_2} \,|\; j_1, j_2 = 0, 1, \ldots, 2s, \; j_1 + j_2 \leq 2s \right\} \subseteq \mathcal{V}^s,$$

a vertex subspace $\widehat{\mathcal{W}}^s_{\Xi^{(i)}}$ with a dimension which is independent of the valency v_i of the vertex $\Xi^{(i)}$ and of the configuration of the bilinear patches around the vertex, since it just equals

$$\dim \widehat{\mathcal{W}}^s_{\Xi^{(i)}} = (s+1)(2s+1). \tag{49}$$

The resulting dimension of the alternative vertex subspace $\widehat{\mathcal{W}}^s_{\Xi^{(i)}}$ can be used now to give an improved lower bound in Lemma 2 for the dimension of $\mathcal{W}^s_{\Xi^{(i)}}$ in case of an inner vertex $\Xi^{(i)}$ when the valency v_i of the corresponding vertex $\Xi^{(i)}$ is small. Since $\widehat{\mathcal{W}}^s_{\Xi^{(i)}} \subseteq \mathcal{W}^s_{\Xi^{(i)}}$, (49) is a further lower bound for the dimension of $\mathcal{W}^s_{\Xi^{(i)}}$, which directly leads with Lemma 2 to the following corollary.

Corollary 1 *Let $\Xi^{(i)}$, $i \in \mathcal{I}_\Xi$, be an inner vertex of patch valency v_i. Then,*

$$\max\Big((s+1)(2s+1), \frac{1}{2}\, v_i\, s\,(s+1)\Big) \le \dim \mathcal{W}^s_{\Xi^{(i)}} \le \frac{1}{2}(s+1)\,((2+v_i)s+2).$$

5 Beyond bilinear parameterization

In this section, we will briefly discuss a first possible generalization of the presented construction to a wider class of multi-patch parameterizations than the considered bilinear one. Motivated by [13] for $s = 1$ and by [33] for $s = 2$, we are interested in multi-patch parameterizations which possess similar connectivity functions as in the bilinear case, in particular linear functions $\alpha^{(i_0)}$, $\alpha^{(i_1)}$, $\beta^{(i_0)}$, $\beta^{(i_1)}$, along the interfaces $\Gamma^{(i)}$, $i \in \mathcal{I}_\Gamma$. There, but also in further publications for the case $s = 1$ or $s = 2$, see, e.g. [29, 30, 34], it was numerically shown that such multi-patch parameterizations can allow the construction of globally C^s-smooth isogeometric spline spaces with optimal approximation properties, similar to the bilinear case. Inspired by [33], we call these particular multi-patch parameterizations bilinear-like G^s and define them as follows.

Definition 1 A multi-patch parameterization $\boldsymbol{F}$ consisting of the geometry mappings

$$\boldsymbol{F}^{(i)} \in \mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right) \times \mathcal{S}_h^{\boldsymbol{p},\boldsymbol{r}}\left([0,1]^2\right), \quad i \in \mathcal{I}_\Omega,$$

is called bilinear-like G^s if for any two neighboring patches $\boldsymbol{F}^{(i_0)}$ and $\boldsymbol{F}^{(i_1)}$, $i_0, i_1 \in \mathcal{I}_\Omega$, assuming without loss of generality that

$$\boldsymbol{F}^{(i_0)}(0,\xi_2) = \boldsymbol{F}^{(i_1)}(0,\xi_2),$$

there exist linear functions $\alpha^{(i_0)}, \alpha^{(i_1)}, \beta^{(i_0)}, \beta^{(i_1)} : \mathbb{R} \to \mathbb{R}$, such that

$$\boldsymbol{F}_\ell^{(i_0)}(\xi) = \boldsymbol{F}_\ell^{(i_1)}(\xi) =: \boldsymbol{F}_\ell(\xi), \quad \ell = 0,1,\ldots,s,$$

with

$$\boldsymbol{F}_\ell^{(\tau)}(\xi) = \left(\alpha^{(\tau)}(\xi)\right)^{-\ell} \partial_1^\ell \boldsymbol{F}^{(\tau)}(0,\xi) - \sum_{i=0}^{\ell-1}\binom{\ell}{i}\left(\frac{\beta^{(\tau)}(\xi)}{\alpha^{(\tau)}(\xi)}\right)^{\ell-i} \partial^{\ell-i}\boldsymbol{F}_i(\xi), \quad \tau \in \{i_0, i_1\}.$$

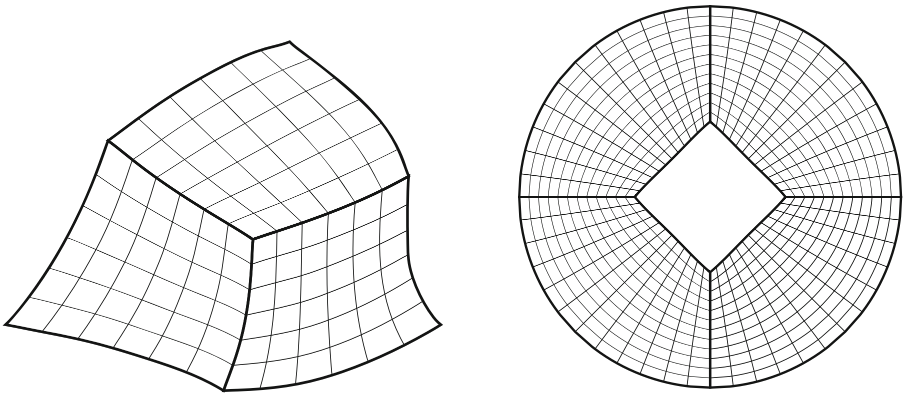

Fig. 4 Two examples of bilinear-like G^3 multi-patch geometries beyond bilinear multi-patch parameterizations, which can have curved interfaces and boundaries

For example, Theorems 2 and 3 can be directly applied by employing bilinear-like G^s multi-patch parameterizations. The advantage of using bilinear-like G^s multi-patch parameterizations instead of bilinear multi-patch parameterizations is the possibility to deal with multi-patch domains with curved interfaces and curved boundaries, see, e.g. [28–30] and [33] for $s = 1$ and for $s = 2$, respectively. Two instances of possible bilinear-like G^3 multi-patch geometries beyond bilinear multi-patch parameterizations are visualized in Fig. 4. Both domains consist of polynomial patches of bi-degree $(p, p) = (7, 7)$, and have been constructed by following the two strategies presented in [29, Section 3.3] for the case of $s = 1$, which have been adapted to the case of $s = 3$. While the left multi-patch geometry is a mapped piecewise bilinear domain generated from the bilinearly parameterized three-patch domain in Section 6, the right multi-patch geometry is a first simple example of a bilinear-like G^3 multi-patch parameterization, which extends the class of mapped bilinear multi-patch domains, and possesses a C^3-smooth outer boundary and an inner boundary with sharp corners. However, a detailed study about the construction of bilinear-like G^s multi-patch geometries as well as the generalization of our method to the design of C^s-smooth isogeometric spline spaces over bilinear-like G^s multi-patch parameterizations are beyond the scope of the paper and will be part of our planned future research.

6 Examples

The goal of this section is to numerically study the approximation power of the isogeometric spline space $\mathcal{W}^s$ by performing L^2 approximation over the two bilinearly parameterized multi-patch domains Ω given in Fig. 5 (left column). More precisely, we will approximate the smooth function $z : \overline{\Omega} \to \mathbb{R}$,

$$z(\boldsymbol{x}) = z(x_1, x_2) = 4\cos(2x_1)\sin(2x_2), \tag{50}$$

visualized in Fig. 5 (middle column) on these two multi-patch domains, by employing isogeometric spline spaces $\mathcal{W}^s$ for a global smoothness $s = 1, \ldots, 4$, and a mesh size $h = \frac{1}{2^L}$, with $L = 0, 1, \ldots, 5$ or $L = 0, 1, \ldots, 6$, for a spline degree $p = 2s + 1$ and for an inner patch regularity $r = s$. Let $\{\phi_j\}_{j=0}^{\dim \mathcal{W}^s - 1}$ be a basis of such an isogeometric spline space $\mathcal{W}^s$, then we compute an approximation $z_h : \overline{\Omega} \to \mathbb{R}$,

$$z_h(\boldsymbol{x}) = \sum_{j=0}^{\dim \mathcal{W}^s - 1} c_j \phi_j(\boldsymbol{x}), \quad c_j \in \mathbb{R},$$

of the function z, by minimizing the objective function

$$\int_\Omega (z_h(\boldsymbol{x}) - z(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x}.$$

Finding a solution of this minimization problem is equivalent to solving the linear system

$$\boldsymbol{M}\boldsymbol{c} = \boldsymbol{z}, \quad \boldsymbol{c} = (c_j)_{j=0}^{\dim \mathcal{W}^s - 1},$$

where M is the mass matrix with the single entries

$$m_{j_1, j_2} = \int_\Omega \phi_{j_1}(\boldsymbol{x}) \phi_{j_2}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{51}$$

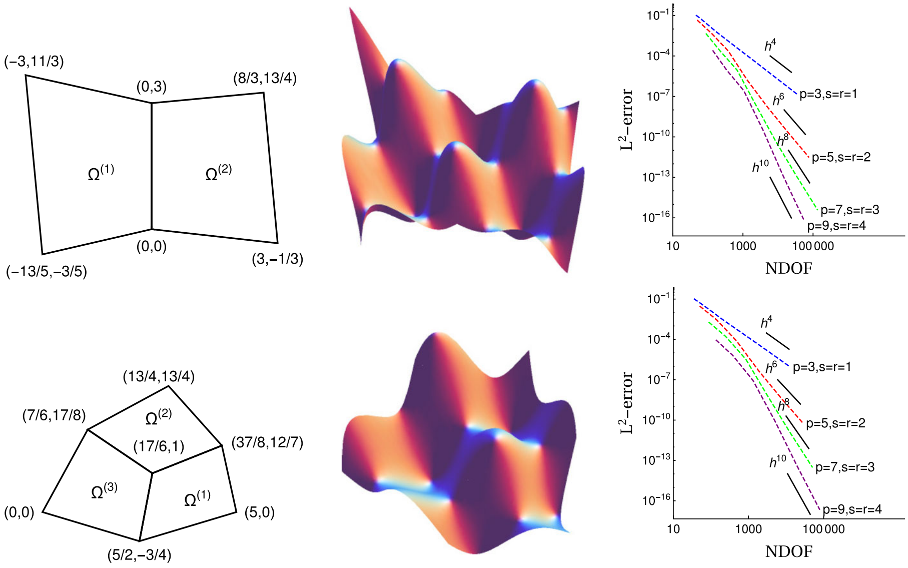

Fig. 5 L^2 approximation of the smooth function (50) (middle column) on two different bilinearly parameterized multi-patch domains Ω (left column) using spline spaces $\mathcal{W}^s$ for a global smoothness $s = 1, \ldots, 4$, and a mesh size $h = \frac{1}{2^L}$, with $L = 0, 1, \ldots, 5$ or $L = 0, 1, \ldots, 6$, for a spline degree $p = 2s + 1$ and for an inner patch regularity $r = s$. The resulting relative L^2 errors (right column) are visualized with respect to the number of degrees of freedom (NDOF)

and $\boldsymbol{z}$ is the right side vector with the single entries

$$z_j = \int_{\Omega} z(\boldsymbol{x})\phi_j(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{52}$$

Using the relation $f_j^{(i)} = \phi_j \circ \boldsymbol{F}^{(i)}$, $i \in \mathcal{I}_{\Omega}$, the entries (51) and (52) can be computed via

$$m_{j_1,j_2} = \sum_{i\in\mathcal{I}_{\Omega}} \int_{[0,1]^2} f_{j_1}^{(i)}(\xi_1,\xi_2) f_{j_2}^{(i)}(\xi_1,\xi_2)|\det(\mathrm{J}\boldsymbol{F}^{(i)}(\xi_1,\xi_2))|\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

and

$$z_j = \sum_{i\in\mathcal{I}_{\Omega}} \int_{[0,1]^2} z(\boldsymbol{F}^{(i)}(\xi_1,\xi_2)) f_j^{(i)}(\xi_1,\xi_2)|\det(\mathrm{J}\boldsymbol{F}^{(i)}(\xi_1,\xi_2))|\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

respectively.

While in case of the bilinearly parameterized two-patch domain, see Fig. 5 (top row and left column), the basis for the space $\mathcal{W}^s$ is generated as described in Section 3.2; in case of the bilinearly parameterized three-patch domain, see Fig. 5 (bottom row and left column), the basis is constructed as explained in Section 4. In the latter case, the design of the vertex subspaces $\mathcal{W}^s_{\Xi^{(i)}}$ has to be slightly modified in case of a mesh size $\frac{p-r-s}{3s-r+1} \leq h \leq 1$ for boundary vertices. Namely, the vertex subspace $\mathcal{W}^s_{\Xi^{(i)}}$ for a boundary vertex $\Xi^{(i)}$ is then just generated by those corresponding functions $\phi_{\Gamma^{(i)};j_1,j_2}$ and/or $\phi_{\Omega^{(i)};j_1,j_2}$, which have not been already used to construct the vertex subspace for another vertex especially for the inner vertex.

Figure 5 (right column) displays the resulting relative L^2 errors with respect to the number of degrees of freedom (NDOF) by performing L^2 approximation on the two different bilinearly parameterized multi-patch domains. In all cases, the numerical results indicate a convergence rate of optimal order of $\mathcal{O}(h^{p+1})$ in the L^2 norm. In case of the three-patch domain, the shown results have been obtained by employing the minimal determining set approach for the construction of the vertex subspaces for the inner vertex. However, the use of the alternative interpolation strategy instead would lead to a nearly indistinguishable result but which is not presented here. The number of degrees of freedom, i.e. the dimensions of the obtained isogeometric spline spaces $\mathcal{W}^s$ for the two different multi-patch domains are reported in Table 1. Again, the dimensions for the spaces based on the alternative interpolation strategy for the three-patch domain are not presented in the table. However, they are very similar to the numbers presented in the table, namely they are the same for the first column, reduced by one for the second column, by two for the third one and by four for the last column.

Table 1 The number of degrees of freedom, i.e. the dimensions of the generated isogeometric spline spaces $\mathcal{W}^s$ in Section 6 for a mesh size $h = \frac{1}{2^L}$, $L = 0, 1, \ldots, 5$, for the two bilinearly parameterized multi-patch domains shown in Fig. 5 (left column)

h	$p = 3, s = r = 1$	$p = 5, s = r = 2$	$p = 7, s = r = 3$	$p = 9, s = r = 4$
Two-patch domain				
1	23	51	90	140
$\frac{1}{2}$	57	126	222	345
$\frac{1}{4}$	173	384	678	1055
$\frac{1}{8}$	597	1332	2358	3675
$\frac{1}{16}$	2213	4956	8790	13715
$\frac{1}{32}$	8517	19116	33942	52995
Three-patch domain				
1	24	52	90	139
$\frac{1}{2}$	66	142	246	379
$\frac{1}{4}$	222	484	846	1309
$\frac{1}{8}$	822	1816	3198	4969
$\frac{1}{16}$	3174	7072	12510	19489
$\frac{1}{32}$	12486	27952	49566	77329

7 Conclusion

We have studied the space of C^s-smooth ($s \geq 1$) isogeometric spline functions on planar, bilinearly parameterized multi-patch domains and have presented the construction of a particular subspace of this C^s-smooth isogeometric spline space. The use of the C^s-smooth subspace is advantageous compared to the use of the entire C^s-smooth space, since the design of the subspace is simple and works uniformly for all possible multi-patch configurations, and furthermore, the numerical experiments by performing L^2 approximation indicate that the subspace already possesses optimal approximation properties.

The construction of the C^s-smooth subspace and of an associated simple and locally supported basis is first described for the case of two-patch domains, and is then extended to the case of multi-patch domains with more than two patches and with possibly extraordinary vertices. In the latter case, the C^s-smooth subspace is generated as the direct sum of spaces corresponding to the individual patches, edges and vertices.

Moreover, a possible generalization of our approach to a more general class of planar multi-patch parameterizations, called bilinear-like G^s multi-patch geometries, is briefly explained. This class of multi-patch parameterizations provides the possibility to model multi-patch domains with curved interfaces and boundaries. A detailed study of this class of geometries is beyond the scope of the paper and is a topic of our future research. Further open problems which are worth to study are, e.g. the theoretical investigation of the approximation properties of the constructed C^s-smooth isogeometric spline space, the use of the C^s-smooth isogeometric spline functions for applications which require functions of high continuity such as solving fourth order PDEs via isogeometric collocation, and the extension of our approach to multi-patch shells and volumes.

Appendix

We present concrete examples of functions which have been introduced in Section 3.

Example 1 We give for the cases $\ell \in \{1, 2, 3\}$ explicit expressions for the functions $\boldsymbol{\Xi}_\ell$, η_ℓ and θ_ℓ, which have been firstly considered in (12) and (15). Based on (12), (19) and Lemma 1, we get for $\ell = 1$

$$\boldsymbol{\Xi}_1(\xi) = \partial_1 \boldsymbol{\Sigma}^{(i_1)}(0, \xi), \quad \eta_1(\xi) = -\lambda_1\, \alpha^{(i_1)}(\xi), \quad \theta_1(\xi) = -\lambda_1\, \beta(\xi),$$

for $\ell = 2$

$$\begin{aligned}\boldsymbol{\Xi}_2(\xi) &= \partial_1^2 \boldsymbol{\Sigma}^{(i_1)}(0, \xi) - \Big(a_{1,0}^2(\xi)\partial_1^2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) \\ &\quad +2a_{1,0}(\xi)b_{1,0}(\xi)\partial_1\partial_2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) + b_{1,0}^2(\xi)\partial_2^2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi)\Big), \\ \eta_2(\xi) &= -\frac{2\alpha^{(i_1)}(\xi)\beta(\xi)}{(\alpha^{(i_0)}(\xi))}\vartheta(\xi), \quad \theta_2(\xi) = -\frac{2\alpha^{(i_1)}(\xi)\beta(\xi)}{(\alpha^{(i_0)}(\xi))}\mu(\xi),\end{aligned}$$

and for $\ell = 3$

$$\begin{aligned}\boldsymbol{\Xi}_3(\xi) &= \partial_1^3 \boldsymbol{\Sigma}^{(i_1)}(0, \xi) - \Big(a_{1,0}^3(\xi)\partial_1^3 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) + 3a_{1,0}^2(\xi)b_{1,0}(\xi)\partial_1^2\partial_2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) \\ &\quad + 3a_{1,0}(\xi)b_{1,0}^2(\xi)\partial_1\partial_2^2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) + b_{1,0}^3(\xi)\partial_2^3 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) + 3a_{1,0}(\xi)a_{2,0}(\xi)\partial_1^2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) \\ &\quad + 3(b_{1,0}(\xi)a_{2,0}(\xi) + a_{1,0}(\xi)b_{2,0}(\xi))\partial_1\partial_2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi) + 3b_{1,0}(\xi)b_{2,0}(\xi)\partial_2^2 \boldsymbol{\Sigma}^{(i_0)}(0, \xi)\Big), \\ \eta_3(\xi) &= \frac{6\alpha^{(i_1)}(\xi)\beta(\xi)\left(\alpha^{(i_0)}(\xi)\alpha^{(i_1)}(\xi)(\beta^{(i_0)}(\xi))' + (\alpha^{(i_0)}(\xi))'(\beta(\xi) - \alpha^{(i_1)}(\xi)\beta^{(i_0)}(\xi))\right)}{(\alpha^{(i_0)}(\xi))^3}\vartheta(\xi), \\ \theta_3(\xi) &= \frac{6\alpha^{(i_1)}(\xi)\beta(\xi)\left(\alpha^{(i_0)}(\xi)\alpha^{(i_1)}(\xi)(\beta^{(i_0)}(\xi))' + (\alpha^{(i_0)}(\xi))'(\beta(\xi) - \alpha^{(i_1)}(\xi)\beta^{(i_0)}(\xi))\right)}{(\alpha^{(i_0)}(\xi))^3}\mu(\xi).\end{aligned}$$

Example 2 We consider particular examples of the functions $A_{\boldsymbol{\sigma};\ell}$ and of the sets $\mathcal{I}_{\boldsymbol{\sigma};\ell}$ introduced in (21) and (22), respectively. Let $|\boldsymbol{\sigma}| \leq 3$. Then, the sets $\mathcal{I}_{\boldsymbol{\sigma};3}$ as well as the functions $A_{\boldsymbol{\sigma};3}$ are equal to

$$\begin{aligned}\mathcal{I}_{(1,0);3} &= \{((0,0,1),(0,0,0))\}, \quad \mathcal{I}_{(0,1);3} = \{((0,0,0),(0,0,1))\}, \\ \mathcal{I}_{(2,0);3} &= \{((1,1,0),(0,0,0))\}, \quad \mathcal{I}_{(0,2);3} = \{((0,0,0),(1,1,0))\}, \\ \mathcal{I}_{(1,1);3} &= \{((1,0,0),(0,1,0)), ((0,1,0),(1,0,0))\}, \\ \mathcal{I}_{(3,0);3} &= \{((3,0,0),(0,0,0))\}, \quad \mathcal{I}_{(2,1);3} = \{((2,0,0),(1,0,0))\}, \\ \mathcal{I}_{(1,2);3} &= \{((1,0,0),(2,0,0))\}, \quad \mathcal{I}_{(0,3);3} = \{((0,0,0),(3,0,0))\},\end{aligned}$$

and

$$\begin{aligned}
A_{(1,0);3}(\xi) &= a_{3,0}(\xi), \quad A_{(0,1);3}(\xi) = b_{3,0}(\xi), \quad A_{(2,0);3}(\xi) = 3a_{1,0}(\xi)a_{2,0}(\xi), \\
A_{(1,1);3}(\xi) &= 3\left(a_{1,0}(\xi)b_{2,0}(\xi) + b_{1,0}(\xi)a_{2,0}(\xi)\right), \quad A_{(0,2);3}(\xi) = 3b_{1,0}(\xi)b_{2,0}(\xi), \\
A_{(3,0);3}(\xi) &= a_{1,0}^3(\xi), \quad A_{(2,1);3}(\xi) = 3a_{1,0}^2(\xi)b_{1,0}(\xi), \\
A_{(1,2);3}(\xi) &= 3a_{1,0}(\xi)b_{1,0}^2(\xi), \quad A_{(0,3);3}(\xi) = b_{1,0}^3(\xi).
\end{aligned}$$

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Declarations

Conflict of interest The authors declare no competing interests.

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Affiliations

Mario Kapl[1,2] · **Vito Vitrih**[3]

Vito Vitrih
vito.vitrih@upr.si

1 Department of Engineering & IT, Carinthia University of Applied Sciences, Europastraße 4, 9524 Villach, Austria

2 Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstraße 69, 4040 Linz, Austria

3 IAM and FAMNIT, University of Primorska, Muzejski trg 2, 6000 Koper, Slovenia