Periodic L_p Estimates by $\mathscr{R}$-Boundedness: Applications to the Navier-Stokes Equations

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Abstract

General evolution equations in Banach spaces are investigated. Based on an operator-valued version of de Leeuw's transference principle, time-periodic L_p estimates of maximal regularity type are carried over from $\mathscr{R}$-bounds of the family of solution operators ($\mathscr{R}$-solvers) to the corresponding resolvent problems. With this method, existence of time-periodic solutions to the Navier-Stokes equations is shown for two configurations: in a periodically moving bounded domain and in an exterior domain, subject to prescribed time-periodic forcing and boundary data.

Keywords Evolution equations · Time-periodic solutions · L_p estimates · Navier-Stokes equations · Inhomogeneous boundary data

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1 Introduction

The study of time-periodic solutions to evolution equations is the study of oscillations. In this article we investigate time-periodic solutions corresponding to time-periodic data, that is, systems of forced oscillation. A number of different methods, further described below, are traditionally used to carry out a mathematical investigation of such solutions. In the following we introduce a new technique to establish *a priori* estimates of maximal L_p regularity type for linearized equations. Such estimates are essential in the study of nonlinear problems, which we will demonstrate by some examples. For notational simplicity, we consider

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only 2π-periodic problems. By a simple scaling argument, however, all our results extend to $\mathcal{T}$-periodic problems for any $\mathcal{T} > 0$.

The study of 2π-time-periodic solutions to evolution equations can be carried out in a framework where the time axis is replaced with a torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Consider for example an abstract evolution equation

$$\partial_t u + Au = f \quad \text{in } \mathbb{T} \tag{1.1}$$

in a Banach space X, where A is a linear operator on X. Since the time domain is a torus, a solution to (1.1) is intrinsically time-periodic. We refer to estimates of the solution in $\mathrm{L}_p(\mathbb{T}; X)$ norms as *periodic L_p estimates*. Estimates that include all highest-order norms of the solution are said to be of maximal regularity, which in the case (1.1) above means an estimate of type

$$\|\partial_t u\|_{\mathrm{L}_p(\mathbb{T},X)} + \|Au\|_{\mathrm{L}_p(\mathbb{T},X)} + \|u\|_{\mathrm{L}_p(\mathbb{T},X)} \leq C\|f\|_{\mathrm{L}_p(\mathbb{T},X)}. \tag{1.2}$$

Such estimates lead to a characterization of $\partial_t + A$ as a homeomorphism in an $\mathrm{L}_p(\mathbb{T}, X)$ setting, which is critical to the analysis of non-linear problems. In the following, we study the case when X is a UMD space and $p \in (1, \infty)$, and we show how to establish periodic L_p estimates of maximal regularity type for a large class of abstract evolution equations based on their $\mathcal{R}$-solvers, that is, solution operators of the associated resolvent problems that satisfy specific $\mathcal{R}$-bounds. For the notion of UMD spaces and $\mathcal{R}$-boundedness, we refer to Sect. 2.3. In particular, we include cases where 0 lies in the spectrum of A, which constitutes a particular challenge and where traditional methods have shortcomings. As we explain below in more detail, in this case the classical maximal regularity estimate (1.2) is not available and $\partial_t + A$ can only be a realized as a homeomorphism in an adapted framework of function spaces.

Typically, L_p estimates are established via Fourier multipliers, often via the multiplier theorem of Mikhlin, which was extended to operator-valued multipliers by WEIS [77]. He showed that Mikhlin's theorem remains valid in the operator-valued case if boundedness is replaced with $\mathcal{R}$-boundedness in the assumptions. On the strength of this result, it is possible to establish maximal regularity in $\mathrm{L}_p((0, T), X)$ norms for initial-value problems such as

$$\partial_t u + Au = f \quad \text{in } (0, T), \quad u(0) = u_0 \tag{1.3}$$

by establishing $\mathcal{R}$-boundedness on the resolvent family

$$\{\lambda R(\lambda, A) \mid \lambda \in \Sigma_{\theta, \lambda_0}\}, \tag{1.4}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A for a resolvent parameter λ and

$$\Sigma_{\theta, \lambda_0} := \left\{\lambda \in \mathbb{C} \mid \arg(\lambda) \leq \pi - \theta, \ |\lambda| \geq \lambda_0\right\} \qquad (0 \leq \theta < \frac{\pi}{2})$$

is a sector, which excludes a ball around the origin if $\lambda_0 > 0$. Note that for many problems, in particular in unbounded domains, the inclusion of the origin is not possible since 0 does not belong to the resolvent set. However, for the derivation of L_p estimates for the initial-value problem (1.3), the origin can be excluded from the sector unless $T = \infty$ is required. Since the appearance of [77], $\mathcal{R}$-bounds for resolvent families of the form (1.4) have been established for a substantial number of boundary-value problems, which lead to L_p maximal

regularity estimates in UMD spaces X for the associated initial-value problem, see [15, 68] for example, and the references therein.

In this article, we develop a technique to obtain *periodic* L$_p$ estimates of maximal regularity type from these $\mathscr{R}$-bounds. In particular, we focus on problems where the origin belongs to the spectrum $\sigma(A)$ of A, so that $\mathscr{R}$-bounds can at best be established in $\Sigma_{\theta,\lambda_0}$ for some $\lambda_0 > 0$. In the case that 0 is included in the resolvent set as well as in the $\mathscr{R}$-bounds, that is, the operator family (1.4) is $\mathscr{R}$-bounded for $\lambda_0 = 0$, classical maximal L$_p$ regularity can be established, which was shown by ARENDT and BU [2]. If this is not the case, classical periodic L$_p$ estimates of maximal regularity type such as (1.2) cannot be established because invertibility of the linear time-periodic problem would require invertibility of the associated steady-state problem, that is, that 0 belongs to the resolvent set $\rho(A)$. Therefore, we introduce an alternative functional setting in order to characterize the parabolic operator $\partial_t + A$ as a homeomorphism with respect to data in L$_p$ spaces.

The technique developed in the following is based on the transference principle introduced by DE LEEUW in [14] for scalar-valued multipliers and generalized to the operator-valued case by HYTÖNEN, VAN NEERVEN, VERAAR, and WEIS [33]. It states that L$_p$ boundedness of a continuous Fourier multiplier on L$_p(\mathbb{R})$ is retained when the multiplier is restricted to $\mathbb{Z}$ and thus becomes a multiplier in the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ setting. Despite the transference principle seeing little usage outside the field of harmonic analysis, we believe it to be an effective tool in the analysis of periodic solutions to partial differential equations. The promotion of this viewpoint is one of the main purposes of this article since it provides us with an extremely useful tool to derive periodic L$_p$ estimates from known $\mathscr{R}$-bounds on resolvent families. If for example the resolvent family (1.4) is $\mathscr{R}$-bounded in a sector containing the full imaginary axis $i\mathbb{R}$, that is, for $\lambda_0 = 0$, then the operator-valued version of the transference principle combined with the operator-valued version of Mikhlin's multiplier theorem by WEIS [77] immediately yields the periodic L$_p$ estimates (1.2). If, however, $\mathscr{R}$-bounds are only available in a sector excluding the origin, that is, $\lambda_0 > 0$, a decomposition technique has to be introduced. Expanding (1.1) into a Fourier series, we introduce the projection of u into a lower frequency part $u_\ell := \sum_{|k| \le \lambda_0} u_k e^{ikt}$ corresponding to the finite number of modes $k \in \mathbb{Z}$ with $|k| \le \lambda_0$, and a complementary higher frequency part $u_h = u - u_\ell$. Based on the $\mathscr{R}$-bounds of (1.4), we can combine the transference principle with Mikhlin's multiplier theorem to establish periodic L$_p$ estimates for the higher frequency part. Provided that a (possibly different) framework of Banach spaces can be identified that ensures periodic maximal L$_p$ regularity for the lower frequency part, we can combine the two parts to establish periodic maximal L$_p$ regularity for the full problem within the resulting function spaces, where the different Fourier coefficients of the solution and the data might belong to different function spaces. Due to the bespoke framework introduced for the lower frequency part, the resulting type of L$_p$ estimates may not be classical in the sense of (1.2). They are, however, effective in the investigation of time-periodic solutions to corresponding nonlinear problems, which we demonstrate by specific examples. This decomposition technique was successfully applied before, for example, to treat the time-periodic Oseen problem [29]. While for this exterior-domain problem, the non-zero Fourier modes of a solution can be shown to belong to classical Sobolev spaces, the zero-order mode, which corresponds to a time-independent solution, requires a treatment in a framework of homogeneous Sobolev spaces.

Whereas the study of time-periodic solutions to ordinary differential equations goes back to the nineteenth century, one of the first investigations of time-periodic partial differential equations is due to PRODI [55], who examined the (parabolic) 1D heat equation. Although the work of PRODI is predated by a few other articles [3, 36, 69, 79], it seems that [55] is the

first rigorous, by contemporary standards, investigation into the matter. Around the same time, articles also appeared on time-periodic solutions to the (hyperbolic) wave equation [25, 56]. In the following years, the foundation was laid for the methods that have nowadays become standard in the study of time-periodic partial differential equations. We shall give a brief overview of the main ideas. Consider for this purpose a time-periodic abstract evolution equation

$$\partial_t u + Au = F(t, u) \text{ in } \mathbb{R}, \quad u(t + 2\pi) = u(t) \tag{1.5}$$

in the classical setting with the whole of $\mathbb{R}$ as time axis, but still considered as equation in a Banach space X for some operator A and for 2π-time-periodic (nonlinear) data F.

By far the most popular method that emerged is based on the identification of solutions to (1.5) as fixed points of the so-called Poincaré operator.[1] The basic idea goes back to the pioneering work of Poincaré [53, 54] on dynamical systems. The Poincaré operator, sometimes also referred to as *translation operator along trajectories*, is the mapping $\Phi : X \to X$ that maps an initial value u_0 to the value $u(2\pi)$ of the solution u to the associated initial-value problem

$$\partial_t u + Au = F(t, u) \text{ in } (0, \infty), \quad u(0) = u_0. \tag{1.6}$$

In other words, if $t \mapsto S(t, u_0)$ is the solution operator to (1.6) for the initial value u_0, then the Poincaré operator is given by $\Phi(u_0) := S(2\pi, u_0)$. It is obvious that a fixed point $w_0 = \Phi(w_0)$ of Φ is the initial value of a 2π-periodic solution. In this sense a fixed point of the Poincaré operator induces a solution to (1.5). The main challenge in the application of this method is to construct a setting of Banach spaces such that the Poincaré operator is well defined and admits a fixed point. In some cases, this can be carried out directly for the nonlinear problem, but often the method is first employed to obtain suitable *a priori* estimates of maximal regularity type for the linearization of (1.5), which are subsequently used to investigate the nonlinear problem by classical nonlinear functional analysis. In the context of time-periodic partial differential equations, BROWDER introduced the Poincaré operator approach in [10], and around the same time KRASNOSEL'SKIĬ [43] and his student KOLESOV [39–41] advanced the method. The investigation of time-periodic solutions as fixed points of the Poincaré operator depends heavily on the framework in which a solution operator to the initial-value problem (1.6) can be realized. To illustrate this issue, assume that A generates a sufficiently regular semi-group and consider the linear case $F(t, u) = F(t)$. The solution operator S then takes the form

$$S(t, u_0) := e^{-tA} u_0 + \int_0^t e^{-(t-\tau)A} F(\tau) \, d\tau,$$

and a fixed point w_0 of the Poincaré operator is therefore given by

$$w_0 = S(2\pi, w_0) \quad \Longleftrightarrow \quad w_0 = \left(I - e^{-2\pi A}\right)^{-1} \int_0^{2\pi} e^{-(2\pi - \tau)A} F(\tau) \, d\tau$$

provided $0 \in \rho(A)$, so that $1 \in \rho(e^{-2\pi A})$ and $I - e^{-2\pi A}$ is thereby invertible. In this case, *a priori* estimates for the time-periodic solution $u(t) := S(t, w_0)$ can be established in the

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[1]Not to be confused with the *Poincaré mapping*, which is a related but different notion from the theory of dynamical systems.

setting of Banach spaces in which S is realized. If, on the other hand, $0 \in \sigma(A)$, the representation formula for w_0 above is not valid and it becomes much more difficult to establish *a priori* estimates for the corresponding time-periodic solution. For this reason, the Poincaré operator approach is seemingly always carried out in a setting where $0 \in \rho(A)$. General applications of the method can be found in articles going back to [1, 59, 71, 76] for example, but also in more recent work such as [4, 13, 38, 47, 52]. More examples can be found in articles devoted to specific equations; so many that an exhaustive list is beyond the scope of our exposition here. We shall mention only the work of GEISSERT, HIEBER and NGUYEN [31] in which the restriction $0 \in \rho(A)$ is circumvented by introducing interpolation spaces.

Provided one is able to establish suitable energy estimates for the problem under consideration, time-periodic solutions can also be obtained via a Galerkin approximation scheme. The existence of a time-periodic solution then has to be accomplished in a finite-dimensional setting and is thus reduced to finding periodic solutions to an ordinary differential equation. In the finite-dimensional setting, the Poincaré operator is compact, and it is therefore much less critical to establish existence of a fixed point. The time-periodic incompressible Navier-Stokes problem is a good example of a system that can be treated with energy methods; see for example [30, 57]. Also the time-periodic wave equation with suitable damping can be solved in this way [58]. Moreover, time-periodic solutions to the compressible Navier-Stokes equations can be established, as was first shown for the one-dimensional case by MATSUMURA and NISHIDA [50] and then extended by several authors, *cf.* [35, 74] and the references therein. Since energy estimates typically lead to *a priori* estimates in Hilbert space settings, this method is not always suited to establish optimal *a priori* estimates for linear parabolic problems though. Nevertheless, it gives a strong tool in the case of hyperbolic or mixed hyperbolic-parabolic systems.

We also want to mention a very different method, which is due to SEIDMAN [62], who intentionally avoids using the Poincaré operator and shows existence of weak time-periodic solutions in L$_p$($\mathbb{T} \times \Omega$) spaces to a nonlinear evolution equation based on the theory of monotone operators.

A different approach is based on a representation formula that arises from the principle that a solution to the initial-value problem (1.6) (at least in the linear case $F(t, u) = F(t)$) tends to a periodic orbit as $t \to \infty$ regardless of the initial value. Equivalently formulated, a solution to the initial-value problem with time-periodic right-hand side

$$\partial_t u + Au = F(t) \ \text{in} \ (R, \infty), \quad u(R) = u_0,$$

tends to a periodic orbit as $R \to -\infty$. Assuming again that A generates a sufficiently regular semi-group, this principle leads to the formula

$$u(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)A} F(\tau) \, \mathrm{d}\tau \tag{1.7}$$

for the time-periodic solution. Observe that (1.7) constitutes a solution formula for the time-dependent problem $\partial_t u + Au = F$ on the whole real line $\mathbb{R}$. It is easy to verify that this integral expression indeed leads to a periodic solution of the same period as F. As with the Poincaré operator approach, the challenge with the method based on (1.7) is to construct a framework of Banach spaces such that the integral expression is well defined. Since F is time-periodic and therefore non-decaying, this clearly requires suitable decay properties of the semi-group, which again leads to $0 \notin \sigma(A)$ as a sufficient assumption. Under this assumption, however, the representation (1.7) can be very useful, which was demonstrated already in the paper [55] by PRODI. A similar idea was used by FIFE [26] and in subsequent

papers [5, 27, 51, 72] as well as a number of articles on specific equations such as the Navier-Stokes equations [42, 78].

The principle described above gives rise to yet another approach. If namely the solution $u(t)$ to the initial-value problem (1.6) tends to a periodic orbit as $t \to \infty$, then the sequence $u_n(t) := u(t + n2\pi)$, $n \in \mathbb{N}$, will tend to a periodic solution to (1.5) as $n \to \infty$. This idea was employed in the context of partial differential equations by FICKEN and FLEISHMAN [25] as early as 1957, and it was later used to investigate time-periodic solutions to the Navier-Stokes equations in the incompressible case [49, 63] and the compressible case [75].

Finally, we mention the perhaps most natural approach to time-periodic partial differential equations, namely the decomposition of data and solution into a Fourier series with respect to time. In the linear case, the investigation then reduces to an analysis of the individual Fourier coefficients, each of which satisfying a resolvent problem in the Banach space X, that is, a time-independent problem; see for example [9, 12, 32, 56, 60, 61]. This technique, however, has some limitations since it is difficult to obtain satisfactory estimates of the Fourier series based only on estimates of the individual coefficients. Typically, this method only leads to suitable *a priori* estimates when working in a framework of absolutely convergent Fourier series, see the recent articles [16, 17, 20] for examples from fluid dynamics, or when Parseval's identity can be invoked, which requires a Hilbert space setting, that is, an investigation in $L_2((0, 2\pi), H)$ for some Hilbert space H. Our following analysis based on Fourier multipliers on the torus $\mathbb{T}$ offers one way to overcome these limitations and to establish *a priori* estimates in a general $L_p((0, 2\pi), X)$ Banach space setting. While there are well-established sufficient conditions for L_p multipliers on the torus, that is, in the context of Fourier series (see [33, Sect. 5.7.c] for example), in our approach we use a Fourier multiplier theorem in the continuous setting and combine it with a transference principle to obtain results for the time-periodic setting. This allows us to directly make use of the $\mathscr{R}$-solvers established in the context of initial-value problems as shown in our examples in Sect. 4 and Sect. 5.

In order to illustrate another significant novelty of our approach, we return to the requirement $0 \notin \sigma(A)$ that is needed in both the Poincaré operator approach and in the method based on the representation formula (1.7). The root cause of this restriction is the necessity in both techniques that the investigation of (1.5) is carried out in the framework of function spaces of the corresponding initial-value problem (1.6). Specifically, in both methods the time-periodic solution is characterized as a special solution to the initial-value problem, and can therefore only be estimated in the framework in which the initial-value problem is rendered well posed. However, this framework is not suitable for *a priori* estimates of solutions to the corresponding stationary problem $Au = F$ when $0 \in \sigma(A)$. Since a stationary solution is trivially also time periodic, it is clear why the restriction $0 \notin \sigma(A)$ is imposed. In our approach, based solely on Fourier multipliers, both the Poincaré operator and the representation formula (1.7) are avoided, and we are able to construct a bespoke setting of Banach spaces that enables us to also treat cases where $0 \in \sigma(A)$.

As an application of the method, we shall consider the time-periodic incompressible viscous fluid flow inside a bounded container with oscillating boundary. The corresponding Navier-Stokes equations were recently investigated by FARWIG, KOZONO, TSUDA and WEGMANN [24], who established existence of mild solutions by using an adaption of the solution formula (1.7) to non-autonomous systems. These solutions were later shown to have more regularity, in particular, to be strong solutions [23]. Their approach was based on a reformulation of the problem into a time-independent reference domain and on the analysis of the resulting non-autonomous linearized system. In the present paper, we begin similarly by reformulating the problem on a time-independent reference frame, but we then focus on

the autonomous linearized problem, that is, the time-periodic Stokes problem in a bounded domain. To establish suitable maximal regularity estimates for this problem, we employ the abstract method developed before and use $\mathscr{R}$-bounds for the associated resolvent problem, which were derived in [65, 66]. This allows us to treat the non-autonomous linear problem by perturbation arguments. Using the contraction mapping principle, we can further establish the existence of strong solutions to the original nonlinear problem under suitable smallness assumptions on the data, in particular, on the body motion. Note that time-periodic maximal regularity for the Stokes problem in a bounded domain was established before by MAEKAWA and SAUER [48] in the case of homogeneous boundary data. However, for the problem with a moving boundary, homogeneous boundary data mean that fluid particles at the moving boundary would have zero velocity, which would be rather unphysical. Therefore, we establish maximal regularity for the case of inhomogeneous Dirichlet boundary condition.

As another example, we study the time-periodic flow around a (non-moving) rigid body, for which we establish maximal L$_p$ regularity of the associated linearization, which is the Stokes problem in an exterior domain. Although we derive the linear theory in the same way as for the bounded domain, the existence result is different: Existence of time-periodic solutions in a bounded domain can be shown in a framework of classical Sobolev spaces, and in the final results the decomposition technique described above is not visible. This is possible since 0 is not in the spectrum of the underlying linear operator. However, for the exterior-domain problem the situation is different, and the zero-order mode, which is the time mean of the periodic function, has to be treated in a separate functional framework of homogeneous Sobolev spaces. Moreover, in order to treat the full nonlinear problem by a fixed-point argument, we have to consider solutions with additional pointwise spatial decay, and a large part of Sect. 5 is concerned with the asymptotic properties of the time-periodic solutions. Since this analysis of the exterior-domain problem is already quite extensive, the even more involved case of the Navier-Stokes flow inside a periodically moving exterior domain is treated in the separate article [21], which is based on the results and methods presented here.

In previous works, see [18, 29, 44–46] for example, the derivation of *a priori* estimates for strong solutions to time-periodic problems in a domain $\Omega \subset \mathbb{R}^N$ was based on the availability of a suitable theory on the whole space $\mathbb{R}^N$. There, estimates were established by a combination of classical (scalar-valued) Fourier multiplier theorems in the Euclidean space with a subsequent transference to the locally compact Abelian group $\mathbb{T} \times \mathbb{R}^N$. The *a priori* estimates on the domain were then recovered by localization arguments. In contrast, the approach presented here uses the theory of *operator-valued* Fourier multipliers to derive *a priori* estimates for the time-periodic problem from $\mathscr{R}$-bounds for the associated resolvent families. The method can be used even if such $\mathscr{R}$-bounds are not available on the whole imaginary axis, and besides the specific examples presented here, it was recently applied in the study of other problems from fluid mechanics, see [21, 22].

The article is divided into a more theoretical first part (Sect. 3), where abstract linear and nonlinear time-periodic problems are investigated, and a second part devoted to applications (Sects. 4 and 5). The theoretical part focuses on an abstract time-periodic boundary-value problem, and we show statements in general terms of the periodic maximal L$_p$ regularity. It is based on a combination of the decomposition technique described above with the existence of suitable $\mathscr{R}$-solvers; see Theorems 3.1 and 3.2. As examples of the effectiveness of this approach, we subsequently investigate time-periodic solutions to the N-dimensional Navier-Stokes equations in a periodically moving bounded domain in Sect. 4, and to the three-dimensional Navier-Stokes equations in an exterior domain (at rest) in Sect. 5.

2 Notation and Preliminaries

2.1 General Notation

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all natural numbers, integers, and real and complex numbers, respectively, and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. To denote generic constants, we use the symbol C, and $C_{a,b,\dots}$ indicates the dependency of the constant on the quantities a, b, …. Here, the constants C and $C_{a,b,\dots}$ may change from line to line.

For any domain $D \subset \mathbb{R}^N$, $N \in \mathbb{N}$, we denote Lebesgue spaces, Sobolev spaces, and Besov spaces on D by $L_q(D)$, $H_q^m(D)$ and $B_{q,p}^s(D)$, respectively, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{H_q^m(D)}$, and $\|\cdot\|_{B_{q,p}^s(D)}$ denote their norms. For partial derivatives, we write $\partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ for $j = 1, \dots, N$. Let $\nabla f = (\partial_1 f, \dots, \partial_N f)$ and $\nabla^2 f = (\partial_i \partial_j f \mid i, j = 1, \dots, N)$. Let $\hat{H}_q^1(D)$ be the homogeneous space defined by

$$\hat{H}_q^1(D) = \{\varphi \in L_{q,\mathrm{loc}}(D) \mid \nabla \varphi \in L_q(D)^N\}.$$

For a topological vector space V, we let V' denote its dual space. In the following, X and Y will always denote Banach spaces, and $\mathscr{L}(X, Y)$ denotes the space of bounded linear operators from X to Y, and we simply write $\mathscr{L}(X) = \mathscr{L}(X, X)$. Sometimes, we do not distinguish between a space X and its N-fold Cartesian product X^N, and we simply write $\|\cdot\|_X$ for the norm of X^N. If X is a Banach space over $\mathbb{C}$, the set of all X-valued holomorphic functions defined on $U \subset \mathbb{C}$ is denoted by $\mathrm{Hol}\,(U, X)$. For $\varepsilon \in (0, \pi)$ and $\lambda_0 > 0$ we define the sectors

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \varepsilon\}, \quad \Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \Sigma_\varepsilon \mid |\lambda| \geq \lambda_0\}.$$

2.2 Time-Periodic Framework

The study of partial differential equations in a setting where both the data and the corresponding solutions are time periodic can conveniently be carried out in a framework where the time axis is replaced with a torus group. In the following, we consider only the torus

$$\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},$$

which provides us with a framework to study 2π-periodic solutions. We endow $\mathbb{T}$ with the quotient topology inherited from $\mathbb{R}$ via the quotient mapping

$$\pi_Q : \mathbb{R} \to \mathbb{T}, \quad \pi_Q(t) := [t] = \{t + 2n\pi \mid n \in \mathbb{Z}\}.$$

Additionally, the quotient mapping induces a differentiable structure on the torus, and we can therefore investigate equations such as (1.1) as a differential equation on the smooth manifold $\mathbb{T}$. A solution u in this setting corresponds to a classical time-periodic solution $u \circ \pi_Q$ in the Euclidean setting and vice versa. Usually, we tacitly identify functions u on $\mathbb{T}$ with their time-periodic analogue $u \circ \pi_Q$ on $\mathbb{R}$.

The topology on $\mathbb{T}$ turns it into a compact group with a (normalized) Haar measure $d\tau$ such that

$$\forall u \in C(\mathbb{T}): \quad \int_{\mathbb{T}} u(\tau) \, d\tau = \frac{1}{2\pi} \int_0^{2\pi} u \circ \pi_Q(t) \, dt,$$

where $C(\mathbb{T})$ is the class of all continuous functions on $\mathbb{T}$. Bochner-Lebesgue spaces $L_p(\mathbb{T}, X)$ for $p \in [1, \infty]$ are then defined in the usual manner.

The differentiable structure gives rise to the space

$$C^\infty(\mathbb{T}, X) := \{u : \mathbb{T} \to X \mid u \circ \pi_Q \in C^\infty(\mathbb{R}, X)\}$$

of vector-valued smooth functions on $\mathbb{T}$ for any Banach space X. The simple structure of $\mathbb{T}$ implies that the set of Schwartz-Bruhat functions $\mathscr{S}(\mathbb{T}, X)$ (see [11, 18]) coincides with the set of smooth functions, that is,

$$\mathscr{S}(\mathbb{T}, X) = C^\infty(\mathbb{T}, X),$$

which is endowed with the semi-norm topology induced by the family $\rho_\ell(u) :=$ $\sup_{\tau \in \mathbb{T}} \|\partial_t^\ell u(\tau)\|_X$, $\ell \in \mathbb{N}_0$. We refer to the space

$$\mathscr{S}'(\mathbb{T}, X) = \mathscr{L}(\mathscr{S}(\mathbb{T}), X)$$

as the space of X-valued tempered distributions on $\mathbb{T}$. One may observe that the notion of classical distributions on $\mathbb{T}$ (also known as periodic distributions) coincides with the notion of tempered distributions on $\mathbb{T}$. Derivatives of distributions $u \in \mathscr{S}'(\mathbb{T}, X)$ are defined as distributions $\partial_t^\ell u \in \mathscr{S}'(\mathbb{T}, X)$ by duality in the usual way.

As a (locally) compact Abelian group, the torus $\mathbb{T}$ has a Fourier transform $\mathscr{F}_\mathbb{T}$ associated to it, see [18] for example. Obviously, this Fourier transform corresponds to the classic expansion of a function on $\mathbb{T}$ into a Fourier series. In this paper, however, it is essential to treat it as Fourier transform in the same framework as the Fourier transform $\mathscr{F}_\mathbb{R}$ on the real line, which is defined by

$$\mathscr{F}_\mathbb{R} : \mathscr{S}(\mathbb{R}, X) \to \mathscr{S}(\mathbb{R}, X), \qquad \mathscr{F}_\mathbb{R}[u](\xi) := \frac{1}{2\pi} \int_\mathbb{R} u(x)\, e^{-i\xi x}\, dx,$$

$$\mathscr{F}_\mathbb{R}^{-1} : \mathscr{S}(\mathbb{R}, X) \to \mathscr{S}(\mathbb{R}, X), \qquad \mathscr{F}_\mathbb{R}^{-1}[v](x) := \int_\mathbb{R} v(\xi)\, e^{i\xi x}\, d\xi$$

and extended to mappings $\mathscr{F}_\mathbb{R}, \mathscr{F}_\mathbb{R}^{-1} : \mathscr{S}'(\mathbb{R}, X') \to \mathscr{S}'(\mathbb{R}, X')$ by duality. To this end, we recall that $\mathbb{Z}$, endowed with discrete topology and counting measure, can be viewed as the dual group of $\mathbb{T}$. The X-valued Schwartz-Bruhat space on $\mathbb{Z}$ is given by

$$\mathscr{S}(\mathbb{Z}, X) = \{\psi : \mathbb{Z} \to X \mid \forall \ell \in \mathbb{N}_0 : \sup_{k \in \mathbb{Z}} |k|^\ell \|\psi(k)\|_X < \infty\}$$

and is equipped with the locally convex topology induced by the family of semi-norms $\hat{\rho}_\ell$, $\ell \in \mathbb{N}_0$, where $\hat{\rho}_\ell(\psi) := \sup_{k \in \mathbb{Z}} |k|^\ell \|\psi(k)\|_X$.

In the setting of vector-valued Schwartz-Bruhat spaces, the Fourier transform $\mathscr{F}_\mathbb{T}$ is the homeomorphism given by

$$\mathscr{F}_\mathbb{T} : \mathscr{S}(\mathbb{T}, X) \to \mathscr{S}(\mathbb{Z}, X), \quad \mathscr{F}_\mathbb{T}[u](k) := \int_\mathbb{T} u(t) e^{-ikt}\, dt,$$

with inverse mapping

$$\mathscr{F}_\mathbb{T}^{-1} : \mathscr{S}(\mathbb{Z}, X) \to \mathscr{S}(\mathbb{T}, X), \quad \mathscr{F}_\mathbb{T}^{-1}[w](t) := \sum_{k \in \mathbb{Z}} w(k) e^{ikt}.$$

As above, by a duality argument, $\mathscr{F}_{\mathbb{T}}$ extends to a homeomorphism on the space of tempered distributions $\mathscr{F}_{\mathbb{T}} : \mathscr{S}'(\mathbb{T}, X) \to \mathscr{S}'(\mathbb{Z}, X)$ in the usual way.

A standard verification shows that Lebesgue spaces $L_p(\mathbb{T}, X)$ are embedded into $\mathscr{S}'(\mathbb{T}, X)$, which enables us to define vector-valued Sobolev spaces as

$$H_p^m(\mathbb{T}, X) := \left\{ u \in L_p(\mathbb{T}, X) \mid \partial_t^\ell u \in L_p(\mathbb{T}, X) \text{ for } \ell = 0, \dots, m \right\},$$

$$\|u\|_{H_p^m(\mathbb{T}, X)} := \left(\sum_{\ell=0}^m \|\partial_t^\ell u\|_{L_p(\mathbb{T}, X)}^p \right)^{\frac{1}{p}}$$

for $m \in \mathbb{N}$ and $p \in [1, \infty)$. A standard mollification argument shows that $\mathscr{S}(\mathbb{T}, X) = C^\infty(\mathbb{T}, X)$ lies dense in $L_p(\mathbb{T}, X)$ and $H_p^m(\mathbb{T}, X)$. For $\theta \in (0, 1)$ we further define the fractional Sobolev space $H_p^\theta(\mathbb{T}, X)$ via Fourier transform by

$$H_p^\theta(\mathbb{T}, X) := \left\{ u \in L_p(\mathbb{T}, X) \mid \mathscr{F}_{\mathbb{T}}^{-1}\left[|k|^\theta \mathscr{F}_{\mathbb{T}}[u](k) \right] \in L_p(\mathbb{T}, X) \right\},$$

$$\|u\|_{H_p^\theta(\mathbb{T}, X)} := \left\| \mathscr{F}_{\mathbb{T}}^{-1}\left[(1 + |k|)^\theta \mathscr{F}_{\mathbb{T}}[u](k) \right] \right\|_{L_p(\mathbb{T}, X)}.$$

This is an adaption of the vector-valued Bessel potential spaces to the torus; see [33, Sect. 5.6] for the definition in the Euclidean setting. If X is a UMD space and $p \in (1, \infty)$, then they have the same properties as in the scalar-valued case, see also [73, Chap. 9].

2.3 $\mathscr{R}$-Boundedness and Operator-Valued Fourier Multipliers

A family of operators $\mathcal{T} \subset \mathscr{L}(X, Y)$ is called $\mathscr{R}$-bounded in $\mathscr{L}(X, Y)$ if there exists some $C > 0$ such that for all $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \in \mathcal{T}^n$, and $\{f_j\}_{j=1}^n \in X^n$, we have

$$\Big\| \sum_{k=1}^n r_k T_k f_k \Big\|_{L_1((0,1), Y)} \le C \Big\| \sum_{k=1}^n r_k f_k \Big\|_{L_1((0,1), X)}, \tag{2.1}$$

where r_k, $k \in \mathbb{N}$, denote the Rademacher functions given by $r_k : [0, 1] \to \{-1, 1\}$, $t \mapsto \mathrm{sign}(\sin 2^k \pi t)$. The smallest constant C such that (2.1) holds is called the $\mathscr{R}$-bound of $\mathcal{T}$ and is denoted by $\mathscr{R}_{\mathscr{L}(X,Y)}\mathcal{T}$, If $\mathcal{S} \subset \mathscr{L}(X, Y)$ and $\mathcal{U} \subset \mathscr{L}(Y, Z)$ are further operator families, we have

$$\mathscr{R}_{\mathscr{L}(X,Y)}\{S + T \mid S \in \mathcal{S}, \, T \in \mathcal{T}\} \le \mathscr{R}_{\mathscr{L}(X,Y)}\mathcal{S} + \mathscr{R}_{\mathscr{L}(X,Y)}\mathcal{T},$$
$$\mathscr{R}_{\mathscr{L}(X,Z)}\{U T \mid U \in \mathcal{U}, \, T \in \mathcal{T}\} \le \mathscr{R}_{\mathscr{L}(Y,Z)}\mathcal{U} \cdot \mathscr{R}_{\mathscr{L}(X,Y)}\mathcal{T}; \tag{2.2}$$

see [33, Remark 5.3.14] for example. Due to Kahane's inequality, one may further replace the spaces $L_1((0, 1), X)$ and $L_1((0, 1), Y)$ in (2.1) with $L_p((0, 1), X)$ and $L_p((0, 1), Y)$, respectively, for any $p \in [1, \infty)$. In what follows, this choice of p makes no difference.

We introduce the notion of operator-valued Fourier multipliers on $\mathbb{R}$ and $\mathbb{T}$. For $M \in L_\infty(\mathbb{R}, \mathscr{L}(X, Y))$ we define the operator

$$\mathrm{op}_{\mathbb{R}}[M] : \mathscr{S}(\mathbb{R}, X) \to \mathscr{S}'(\mathbb{R}, Y), \quad \mathrm{op}_{\mathbb{R}}[M]f := \mathscr{F}_{\mathbb{R}}^{-1}[M \, \mathscr{F}_{\mathbb{R}}[f]].$$

For $m \in L_\infty(\mathbb{Z}, \mathscr{L}(X, Y))$ we define the operator

$$\mathrm{op}_{\mathbb{T}}[m] : \mathscr{S}(\mathbb{T}, X) \to \mathscr{S}'(\mathbb{T}, Y), \quad \mathrm{op}_{\mathbb{T}}[m]f := \mathscr{F}_{\mathbb{T}}^{-1}[m \, \mathscr{F}_{\mathbb{T}}[f]].$$

If there exists a continuous extension of $\mathrm{op}_{\mathbb{T}}[m]$ to a bounded operator

$$\mathrm{op}_{\mathbb{T}}[m] : \mathrm{L}_p(\mathbb{T}, X) \to \mathrm{L}_p(\mathbb{T}, Y),$$

we call m an $\mathrm{L}_p(\mathbb{T})$-multiplier. To identify such $\mathrm{L}_p(\mathbb{T})$-multipliers, we shall make use of an operator-valued transference principle, which relates $\mathrm{L}_p(\mathbb{T})$-multipliers to $\mathrm{L}_p(\mathbb{R})$-multipliers, and combine it with an operator-valued multiplier theorem due to WEIS [77]. For its formulation, we need the notion of $\mathscr{R}$-boundedness of operator families introduced above as well as the notion of UMD spaces. Recall that a Banach space X is called a UMD space (or a space of class $\mathcal{HT}$) if the Hilbert transform H defined by

$$Hf(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{f(t-s)}{s} \, \mathrm{d}s, \qquad f \in \mathscr{S}(\mathbb{R}, X),$$

extends to a bounded linear operator $\mathscr{L}(\mathrm{L}_p(\mathbb{R}, X))$ for $p \in (1, \infty)$.

Now we can state the multiplier theorem due to WEIS [77, Theorem 3.4], which is an operator-valued version of the classical Mikhlin theorem.

Theorem 2.1 (WEIS) *Let X and Y be UMD-spaces and $p \in (1, \infty)$. Let $M \in \mathrm{L}_\infty(\mathbb{R}, \mathscr{L}(X, Y))$ be differentiable in $\mathbb{R} \setminus \{0\}$ and such that*

$$\mathscr{R}_{\mathscr{L}(X,Y)}\big\{M(t) \mid t \in \mathbb{R} \setminus \{0\}\big\} \leq r_0, \qquad \mathscr{R}_{\mathscr{L}(X,Y)}\big\{tM'(t) \mid t \in \mathbb{R} \setminus \{0\}\big\} \leq r_0, \qquad (2.3)$$

for some $r_0 > 0$. Then $\mathrm{op}_{\mathbb{R}}[M]$ extends to a bounded operator $\mathrm{op}_{\mathbb{R}}[M] : \mathrm{L}_p(\mathbb{R}, X) \to \mathrm{L}_p(\mathbb{R}, Y)$, that is, M is an $\mathrm{L}_p(\mathbb{R})$-multiplier, and

$$\|\mathrm{op}_{\mathbb{R}}[M]\|_{\mathscr{L}(\mathrm{L}_p(\mathbb{R},X),\mathrm{L}_p(\mathbb{R},Y))} \leq C_p \, r_0 \qquad (2.4)$$

for some constant $C_p > 0$ depending only on p but independent of r_0.

In order to investigate L_p-boundedness of operators associated with Fourier multipliers on $\mathbb{T}$, we combine this theorem with the following result that allows an investigation of L_p-boundedness of a Fourier multiplier $m \in \mathrm{L}_\infty(\mathbb{Z}, \mathscr{L}(X, Y))$ in the torus setting by extending it to a multiplier $M \in \mathrm{L}_\infty(\mathbb{R}; \mathscr{L}(X, Y))$ in the Euclidean setting.

Proposition 2.2 (Operator-valued transference principle) *Let X, Y be Banach spaces and $p \in (1, \infty)$. If*

$$M \in \mathrm{L}_\infty(\mathbb{R}, \mathscr{L}(X, Y)) \cap C(\mathbb{R}, \mathscr{L}(X, Y))$$

is an $\mathrm{L}_p(\mathbb{R})$-multiplier, that is, $\mathrm{op}_{\mathbb{T}}[M] \in \mathscr{L}(\mathrm{L}_p(\mathbb{R}, X), \mathrm{L}_p(\mathbb{R}, Y))$, then $M_{|\mathbb{Z}}$ is an $\mathrm{L}_p(\mathbb{T})$-multiplier, that is, $\mathrm{op}_{\mathbb{T}}[M_{|\mathbb{Z}}] \in \mathscr{L}(\mathrm{L}_p(\mathbb{T}, X), \mathrm{L}_p(\mathbb{T}, Y))$, and

$$\|\mathrm{op}_{\mathbb{T}}[M_{|\mathbb{Z}}]\|_{\mathscr{L}(\mathrm{L}_p(\mathbb{T},X),\mathrm{L}_p(\mathbb{T},Y))} \leq \|\mathrm{op}_{\mathbb{R}}[M]\|_{\mathscr{L}(\mathrm{L}_p(\mathbb{R},X),\mathrm{L}_p(\mathbb{T},Y))}. \qquad (2.5)$$

Proof This is a special version of [33, Prop.5.7.1], which is a generalization of the scalar-valued case originally due to DE LEEUW [14]. $\qquad\qquad\square$

Combining the operator-valued transference principle with the Weis multiplier theorem, we directly obtain the following result, which we employ when studying Fourier multipliers on the torus $\mathbb{T}$.

Corollary 2.3 *Let X and Y be UMD spaces, and let*

$$M \in \mathrm{L}_\infty(\mathbb{R}, \mathscr{L}(X, Y)) \cap C(\mathbb{R}, \mathscr{L}(X, Y)) \cap C^1(\mathbb{R} \setminus \{0\}, \mathscr{L}(X, Y))$$

satisfy (2.3) *for some* $r_0 > 0$. *Then* $M_{|\mathbb{Z}}$ *is an* $\mathrm{L}_p(\mathbb{T})$-*multiplier, and*

$$\|\mathrm{op}_{\mathbb{T}}[M_{|\mathbb{Z}}]\|_{\mathscr{L}(\mathrm{L}_p(\mathbb{T}, X), \mathrm{L}_p(\mathbb{T}, Y))} \leq C_p r_0 \tag{2.6}$$

for some constant $C_p > 0$ *only depending on* p.

Proof By Theorem 2.1, M is an $\mathrm{L}_p(\mathbb{R})$-multiplier, and Proposition 2.2 implies that $M_{|\mathbb{Z}}$ is an $\mathrm{L}_p(\mathbb{T})$-multiplier. Finally, estimate (2.6) follows from (2.5) and (2.4). □

3 Maximal $\mathbf{L}_p$ Regularity for Periodic Evolution Equations

We next show how the operator-valued transference principle (Proposition 2.2) can be utilized to establish periodic L_p estimates of the maximal regularity type for a large class of abstract linear evolution equations based on their $\mathscr{R}$-solvers, that is, the $\mathscr{R}$-bounded family of solution operators to the associated resolvent problem. We consider the abstract time-periodic boundary-value problem

$$\begin{cases} \partial_t u + A u = F & \text{in } \mathbb{T}, \\ \qquad B u = T G & \text{in } \mathbb{T}. \end{cases} \tag{3.1}$$

Here, A is an abstract (differential) operator, B is a boundary (differential) operator and T plays the role of a trace operator. Using this notion, we avoid the introduction of suitable trace spaces whose identification strongly relies on the underlying function spaces. We show that the existence of a unique solution to (3.1) can be derived from suitable properties of the associated generalized resolvent problem

$$\begin{cases} i\sigma w + A w = f, \\ \qquad B w = T g, \end{cases} \tag{3.2}$$

for $\sigma \in \mathbb{R}$. More precisely, while for small k the existence of *a priori* bounds in a suitable functional framework is sufficient, for large k we require the existence of $\mathscr{R}$-solvers, that is, a functional framework such that the solution operator satisfies suitable $\mathscr{R}$-bounds. As we shall also see in the examples of the subsequent sections, this lack of $\mathscr{R}$-bounds for the whole line often appears in applications. Moreover, the presented approach allows one to use different function spaces for different modes, which can be a useful and necessary modification; see also Theorem 5.2 below, where the zero-order mode requires separate treatment.

Theorem 3.1 *Let X, Y, Z and W be UMD spaces such that X and W are continuously embedded into Y. Assume that* $A \in \mathscr{L}(X, Y)$, $B \in \mathscr{L}(X, Z)$ *and* $T \in \mathscr{L}(W, Z)$. *Let* $\lambda_0 \in \mathbb{R}$, $\beta \in [0, 1]$, *and let*

$$\mathcal{A} \in C^1(\mathbb{R} \setminus (-\lambda_0, \lambda_0), \mathscr{L}(Y \times Y \times W, X))$$

be an operator-valued function such that for all $\sigma \in \mathbb{R} \setminus (-\lambda_0, \lambda_0)$ and all $(f, g) \in Y \times W$, the function $w = \mathcal{A}(\sigma)(f, \sigma^\beta g, g)$ is the unique solution in X to the generalized resolvent problem (3.2). Assume the validity of the $\mathscr{R}$-bounds

$$\mathscr{R}_{\mathscr{L}(Y \times Y \times W, X)}(\{(\sigma \frac{d}{d\sigma})^\ell \mathcal{A}(\sigma) \mid \sigma \in \mathbb{R} \setminus (-\lambda_0, \lambda_0)\}) \leq r_0, \tag{3.3}$$

$$\mathscr{R}_{\mathscr{L}(Y \times Y \times W, Y)}(\{(\sigma \frac{d}{d\sigma})^\ell (i\sigma \, \mathcal{A}(\sigma)) \mid \sigma \in \mathbb{R} \setminus (-\lambda_0, \lambda_0)\}) \leq r_0 \tag{3.4}$$

for $\ell = 0, 1$ and some constant $r_0 > 0$.

Moreover, for $k \in \mathbb{Z}$ with $|k| \leq k_0 := \max\{k \in \mathbb{Z} \mid k \leq \lambda_0\}$, let $\mathcal{X}_k, \mathcal{Y}_k, \mathcal{Z}_k$ and $\mathcal{W}_k$ be Banach spaces such that $(ik\mathrm{I} + A) \in \mathscr{L}(\mathcal{X}_k, \mathcal{Y}_k)$, $B \in \mathscr{L}(\mathcal{X}_k, \mathcal{Z}_k)$ and $T \in \mathscr{L}(\mathcal{W}_k, \mathcal{Z}_k)$, and such that for all $(f, g) \in \mathcal{Y}_k \times \mathcal{W}_k$ there exists a unique solution $w \in \mathcal{X}_k$ to (3.2) with $\sigma = k$ such that

$$\|w\|_{\mathcal{X}_k} \leq C_k(\|f\|_{\mathcal{Y}_k} + \|g\|_{\mathcal{W}_k}) \tag{3.5}$$

for some constant $C_k > 0$.

Then for any $p \in (1, \infty)$ and (F, G) defined by

$$F(t) = \sum_{k=-k_0}^{k_0} F_k e^{ikt} + F_h(t), \qquad G(t) = \sum_{k=-k_0}^{k_0} G_k e^{ikt} + G_h(t), \tag{3.6}$$

with $(F_k, G_k) \in \mathcal{Y}_k \times \mathcal{W}_k$ for $k \in \mathbb{Z}$, $|k| \leq k_0$, and $(F_h, G_h) \in \mathrm{L}_p(\mathbb{T}; Y) \times (\mathrm{L}_p(\mathbb{T}, W) \cap \mathrm{H}_p^\beta(\mathbb{T}, Y))$ such that $(\mathscr{F}_{\mathbb{T}}[F_h](k), \mathscr{F}_{\mathbb{T}}[G_h](k)) = 0$ for all $|k| \leq k_0$, there exists a unique element

$$(u_{-k_0}, \ldots, u_{k_0}, u_h) \in \mathcal{X}_{-k_0} \times \cdots \times \mathcal{X}_{k_0} \times (\mathrm{L}_p(\mathbb{T}, X) \cap \mathrm{H}_p^1(\mathbb{T}, Y)) \tag{3.7}$$

with $\mathscr{F}_{\mathbb{T}}[u_h](k) = 0$ for $|k| \leq k_0$, such that

$$u(t) := \sum_{k=-k_0}^{k_0} u_k e^{ikt} + u_h(t) \tag{3.8}$$

is the unique solution to the time-periodic problem (3.1), and

$$\|u_k\|_{\mathcal{X}_k} \leq C_k(\|F_k\|_{\mathcal{Y}_k} + \|G_k\|_{\mathcal{W}_k}) \tag{3.9}$$

$$\|u_h\|_{\mathrm{L}_p(\mathbb{T}, X) \cap \mathrm{H}_p^1(\mathbb{T}, Y)} \leq C\, r_0(\|F_h\|_{\mathrm{L}_p(\mathbb{T}, Y)} + \|G_h\|_{\mathrm{H}_p^\beta(\mathbb{T}, Y) \cap \mathrm{L}_p(\mathbb{T}, W)}), \tag{3.10}$$

for some constant $C > 0$. In particular,

$$\|u\|_{\mathbb{X}_p} := \sum_{k=-k_0}^{k_0} \|u_k\|_{\mathcal{X}_k} + \|u_h\|_{\mathrm{L}_p(\mathbb{T}, X) \cap \mathrm{H}_p^1(\mathbb{T}, Y)}$$

$$\leq C\, r_0(\|F_h\|_{\mathrm{L}_p(\mathbb{T}, Y)} + \|G_h\|_{\mathrm{H}_p^\beta(\mathbb{T}, Y) \cap \mathrm{L}_p(\mathbb{T}, W)}) + \sum_{k=-k_0}^{k_0} C_k \|(F_k, G_k)\|_{\mathcal{Y}_k \times \mathcal{W}_k} \tag{3.11}$$

$$=: \|F\|_{\mathbb{Y}_p} + \|G\|_{\mathbb{W}_p}.$$

Proof Let $\varphi = \varphi(\sigma)$ be a $C^\infty(\mathbb{R})$ function which equals 1 for $|\sigma| \geq (\lambda_0 + k_0 + 1)/2$, and equals 0 for $|\sigma| \leq \lambda_0$. We define u as in (3.8), where u_k is the unique solution to (3.2) with $\sigma = k$ and $(f, g) = (F_k, G_k)$ for $|k| \leq k_0$, and

$$u_h = \mathscr{F}_{\mathbb{T}}^{-1}[\varphi(k)\mathcal{A}(k)(\mathscr{F}_{\mathbb{T}}[F](k), k^\beta \mathscr{F}_{\mathbb{T}}[G](k), \mathscr{F}_{\mathbb{T}}[G](k))].$$

One readily verifies that u formally satisfies the time-periodic problem (3.1). Moreover, for $|k| \leq k_0$ we directly conclude $u_k \in \mathcal{X}_k$ and estimate (3.9) since $(F_k, G_k) \in \mathcal{Y}_k \times \mathcal{W}_k$. To show that u_h also belongs to the claimed function space, first observe that $\varphi(\sigma)\mathcal{A}(\sigma) = 0$ for $|\sigma| \leq \lambda_0$, so that $\varphi\mathcal{A} \in C^1(\mathbb{R}, \mathscr{L}(Y \times Y \times W, X))$. In view of [15, Corollary 3.7], from (3.3) and (3.4) it follows that

$$\mathscr{R}_{\mathscr{L}(Y \times Y \times W, X)}(\{(\sigma \frac{d}{d\sigma})^\ell (\varphi(\sigma)\mathcal{A}(\sigma)) \mid \sigma \in \mathbb{R}\}) \leq \|\varphi\|_{\mathrm{H}^1_\infty(\mathbb{R})} r_0,$$

$$\mathscr{R}_{\mathscr{L}(Y \times Y \times W, Y)}(\{(\sigma \frac{d}{d\sigma})^\ell (i\sigma\varphi(\sigma)\mathcal{A}(\sigma)) \mid \sigma \in \mathbb{R}\}) \leq \|\varphi\|_{\mathrm{H}^1_\infty(\mathbb{R})} r_0.$$

We can thus apply Corollary 2.3 to conclude that $k \mapsto \varphi(k)\mathcal{A}(k) \in \mathscr{L}(Y \times Y \times W, X)$ and $k \mapsto ik\varphi(k)\mathcal{A}(k) \in \mathscr{L}(Y \times Y \times W, Y)$ are $\mathrm{L}_p(\mathbb{T})$-multipliers. We thus deduce $u_h \in \mathrm{L}_p(\mathbb{T}, X) \cap \mathrm{H}^1_p(\mathbb{T}, Y)$, and (3.10) follows from the above $\mathscr{R}$-bounds together with (2.6). In total, we have shown existence of a solution in the asserted function class.

To prove the uniqueness statement, let u be of the form (3.8) and satisfy the homogeneous equations, that is, (3.1) with $(f, g) = 0$. For $k \in \mathbb{Z}$ with $|k| > k_0$, we set $u_k = \mathscr{F}_{\mathbb{T}}[u_h](k)$. Then an application of the Fourier transform to (3.1) gives

$$\begin{cases} iku_k + Au_k = 0, \\ \qquad\quad Bu_k = 0. \end{cases}$$

We have $u_k \in X$ for $|k| > k_0$ and $u_k \in \mathcal{X}_k$ for $|k| \leq k_0$, so that $u_k = 0$ follows from the uniqueness assumption for the respective resolvent problem. We thus conclude $u = 0$, which completes the proof. □

The important case of homogeneous boundary conditions can be incorporated in the above setting in several different manners. Of course, the simplest way is to take boundary data $g = 0$, but one may also consider $T = 0$ as an (abstract) trace operator. Another very common way is to simply set $B = 0$ and $T = 0$, that is, to drop the abstract boundary condition in (3.1), and to incorporate the boundary condition in the function space X. Then (3.1) reduces to the time-periodic X-valued ordinary differential equation

$$\partial_t u + Au = F \quad \text{in } \mathbb{T}, \tag{3.12}$$

and (3.2) becomes a proper resolvent problem

$$i\sigma w + Aw = f. \tag{3.13}$$

In this case, the statement of Theorem 3.1 also simplifies significantly, and it can be formulated using the notion of closed linear operators.

Theorem 3.2 *Let Y be a UMD space, and let $A : D(A) \to Y$ be a closed operator with domain $D(A) \subset Y$ and resolvent set $\rho(A)$. Let $\lambda_0 \in \mathbb{R}$ such that $\{i\sigma \mid \sigma \in \mathbb{R} \setminus (-\lambda_0, \lambda_0)\} \subset \rho(A)$, and assume the validity of the $\mathscr{R}$-bounds*

$$\mathscr{R}_{\mathscr{L}(Y)}\big(\{\lambda(\lambda I - A)^{-1} \mid \lambda = i\sigma, \ \sigma \in \mathbb{R} \setminus (-\lambda_0, \lambda_0)\}\big) \leq r_0 \tag{3.14}$$

for some constant $r_0 > 0$.

Moreover, for $k \in \mathbb{Z}$ with $|k| \leq k_0 := \max\{k \in \mathbb{Z} \mid k \leq \lambda_0\}$, let $\mathcal{X}_k$ and $\mathcal{Y}_k$ be Banach spaces such that $(ik I + A) \in \mathscr{L}(\mathcal{X}_k, \mathcal{Y}_k)$ is a linear homeomorphism.

Then for any $p \in (1, \infty)$ and F defined as in (3.6), the function u defined in (3.8) is a solution to the time-periodic problem (3.12), where $u_k = (ik I + A)^{-1} F_k$, and

$$\|u_h\|_{L_p(\mathbb{T}, X) \cap H_p^1(\mathbb{T}, Y)} \leq C\, r_0 \|F_h\|_{L_p(\mathbb{T}, Y)} \tag{3.15}$$

for some constant $C > 0$. The solution u is unique in the sense that the Fourier coefficients $u_k = \mathscr{F}_{\mathbb{T}}[u](k)$ are unique in $\mathcal{X}_k$ if $|k| \leq k_0$, and in $D(A)$ if $|k| > k_0$.

Proof We set $X = D(A) \subset Y$, equipped with the usual graph norm, which is a UMD space since $(i\lambda_0 I - A) : X \to Y$ is a homeomorphism. We further set $\mathcal{A}(\sigma) = (i\sigma I + A)^{-1}$ for $|\sigma| \geq \lambda_0$. Then $\mathcal{A}(\sigma)$ is a solution operator for the resolvent problem (3.13) for $|\sigma| \geq \lambda_0$, and we have $\mathcal{A} \in C^\infty(\mathbb{R} \setminus (-\lambda_0, \lambda_0), \mathscr{L}(Y, X))$ by analyticity of the resolvent mapping. Moreover, due to the identities

$$i\sigma \mathcal{A}(\sigma) = i\sigma(i\sigma I + A)^{-1}, \qquad \sigma \frac{d}{d\sigma}(i\sigma \mathcal{A}(\sigma)) = i\sigma(i\sigma I + A)^{-1} + \sigma^2(i\sigma I + A)^{-2},$$

and the formulas from (2.2), the assumed $\mathscr{R}$-bound (3.14) directly implies (3.4). Since

$$\mathcal{A}(\sigma) = \mathcal{A}(\lambda_0)\mathcal{A}(\lambda_0)^{-1}\mathcal{A}(\sigma) = \frac{\lambda_0 - \sigma}{\sigma}\mathcal{A}(\lambda_0)\, i\sigma(i\sigma I + A)^{-1} + I,$$

$$\sigma \frac{d}{d\sigma}\mathcal{A}(\sigma) = i\sigma(i\sigma I + A)^{-2},$$

and since $A(\lambda_0)$ is a homeomorphism in $\mathscr{L}(Y, X)$, we infer (3.3) from (3.14) in the same way. Now the statement directly follows from Theorem 3.1 with $W = Z = \mathcal{W}_k = \mathcal{Z}_k = \{0\}$ and $B = T = 0$. $\qquad\square$

Next we use the derived linear theory from Theorem 3.1 to obtain a solution to the non-linear problem

$$\begin{cases} \partial_t u + A u = \mathcal{F}(u) & \text{in } \mathbb{T}, \\ Bu = T\mathcal{G}(u) & \text{in } \mathbb{T}, \end{cases} \tag{3.16}$$

where $\mathcal{F}$ and $\mathcal{G}$ are nonlinear functions. To formulate precise assumptions, we introduce the spaces $\mathbb{X}_p$, $\mathbb{Y}_p$ and $\mathbb{W}_p$ for $p \in (1, \infty)$ via

$$\mathbb{X}_p := \Big\{ u = u_\ell + u_h \mid u_\ell(t) = \sum_{k=-k_0}^{k_0} u_k e^{ikt}, \ u_k \in \mathcal{X}_k \text{ for } |k| \leq k_0,$$

$$u_h \in \mathrm{L}_p(\mathbb{T}, X) \cap \mathrm{H}_p^1(\mathbb{T}, Y) \Big\},$$

$$\mathbb{Y}_p := \Big\{ F = F_\ell + F_h \mid F_\ell(t) = \sum_{k=-k_0}^{k_0} F_k e^{ikt}, \ F_k \in \mathcal{Y}_k \text{ for } |k| \leq k_0, \ F_h \in \mathrm{L}_p(\mathbb{T}, Y) \Big\},$$

$$\mathbb{W}_p := \Big\{ G = G_\ell + G_h \mid G_\ell(t) = \sum_{k=-k_0}^{k_0} G_k e^{ikt}, \ G_k \in \mathcal{W}_k \text{ for } |k| \leq k_0,$$

$$G_h \in \mathrm{L}_p(\mathbb{T}, W) \cap \mathrm{H}_p^\beta(\mathbb{T}, Y) \Big\},$$

which are equipped with the norms introduced in (3.11). Note that for each $(F, G) \in \mathbb{Y}_p \times \mathbb{W}_p$, Theorem 3.1 provides existence of a unique solution $u \in \mathbb{X}_p$ to the linear problem (3.1). To obtain an existence result for the nonlinear problem (3.16), we assume have to impose suitable conditions on the nonlinear terms.

Theorem 3.3 *In the situation of Theorem* 3.1, *let* $p \in (1, \infty)$, $\alpha > 0$ *and* $\mathcal{U} \subset \mathbb{X}_p$ *be such that* $\mathcal{F} : \mathcal{U} \to \mathbb{Y}_p$ *and* $\mathcal{G} : \mathcal{U} \to \mathbb{W}_p$ *satisfy*

$$\exists C > 0 \, \forall u, v \in \mathcal{U} : \quad \|\mathcal{F}(u) - \mathcal{F}(v)\|_{\mathbb{Y}_p} + \|\mathcal{G}(u) - \mathcal{G}(v)\|_{\mathbb{W}_p} \leq C \big(\|u\|_{\mathbb{X}_p}^\alpha + \|v\|_{\mathbb{X}_p}^\alpha \big) \|u - v\|_{\mathbb{X}_p}. \tag{3.17}$$

Assume that $0 \in \mathcal{U}$. *Then there exists* $\varepsilon > 0$ *such that if*

$$\|\mathcal{F}(0)\|_{\mathbb{Y}_p} + \|\mathcal{G}(0)\|_{\mathbb{W}_p} \leq \varepsilon^{1+\alpha}, \tag{3.18}$$

then problem (3.17) *possesses a unique solution* $u \in \mathbb{X}_p$ *satisfying* $\|u\|_{\mathbb{X}_p} \leq \varepsilon$.

Proof Due to Theorem 3.1, for any $v \in \mathcal{U}$ there exists a unique solution $u \in \mathbb{X}_p$ to

$$\begin{cases} \partial_t u + Au = \mathcal{F}(v) & \text{in } \mathbb{T}, \\ \quad\quad Bu = T\mathcal{G}(v) & \text{in } \mathbb{T}. \end{cases} \tag{3.19}$$

Let $\Xi : \mathcal{U} \to \mathbb{X}_p$, $v \mapsto u$, be the corresponding solution mapping. From estimates (3.11), (3.17) and (3.18), we obtain

$$\begin{aligned} \|u\|_{\mathbb{X}_p} = \|\Xi(v)\|_{\mathbb{X}_p} &\leq C \big(\|\mathcal{F}(v)\|_{\mathbb{Y}_p} + \|\mathcal{G}(v)\|_{\mathbb{W}_p} \big) \\ &\leq C \big(\|\mathcal{F}(v) - \mathcal{F}(0)\|_{\mathbb{Y}_p} + \|\mathcal{G}(v) - \mathcal{G}(0)\|_{\mathbb{W}_p} + \|\mathcal{F}(0)\|_{\mathbb{Y}_p} + \|\mathcal{G}(0)\|_{\mathbb{W}_p} \big) \\ &\leq C \|v\|_{\mathbb{X}_p}^{\alpha+1} + \varepsilon^{\alpha+1}. \end{aligned}$$

Therefore, if we choose $\varepsilon > 0$ so small that $(C + 1)\varepsilon^\alpha \leq 1$ and such that

$$\mathcal{I}_\varepsilon := \{ v \in \mathbb{X}_p \mid \|v\|_{\mathbb{X}_p} \leq \varepsilon \} \subset \mathcal{U},$$

then the restriction $\Xi \colon \mathcal{I}_\varepsilon \to \mathcal{I}_\varepsilon$ is a self-mapping. Moreover, using (3.11) and (3.17) again, we deduce

$$\|\Xi(v) - \Xi(w)\|_{\mathbb{X}_p} \le C\big(\|\mathcal{F}(v) - \mathcal{F}(w)\|_{\mathbb{Y}_p} + \|\mathcal{G}(v) - \mathcal{G}(w)\|_{\mathbb{W}_p}\big)$$

$$\le C\big(\|v\|_{\mathbb{X}_p}^\alpha + \|w\|_{\mathbb{X}_p}^\alpha\big)\|v - w\|_{\mathbb{X}_p} \le 2C\varepsilon^\alpha \|v - w\|_{\mathbb{X}_p}$$

for $v, w \in \mathcal{I}_\varepsilon$, so that $\Xi \colon \mathcal{I}_\varepsilon \to \mathcal{I}_\varepsilon$ is a contraction if we choose $\varepsilon > 0$ so small that $2C\varepsilon^\alpha < 1$. By Banach's fixed-point theorem, we thus obtain the existence of a unique fixed point $u = \Xi(u) \in \mathcal{I}_\varepsilon$, that is, a solution u to (3.16) with the asserted properties. $\qquad\square$

4 On the Navier-Stokes Equations in a Bounded Periodically Moving Domain

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 2$, whose boundary, Γ, is a compact connected C^2 hypersurface. We assume that for each $t \in \mathbb{R}$ there exists an injective map $\phi(\cdot, t) \colon \Omega \to \mathbb{R}^N$ such that $\phi(y, 0) = 0$ and $\phi(y, t + 2\pi) = \phi(y, t)$ for $t \in \mathbb{R}$ and $y \in \Omega$; possessing the regularity

$$\phi \in C^0(\mathbb{T}; C^3(\Omega)^N) \cap C^1(\mathbb{T}; C^1(\Omega)^N). \tag{4.1}$$

Again, we identify 2π-periodic functions on $\mathbb{R}$ with functions on $\mathbb{T}$. Let Ω_t be a domain in $\mathbb{R}^N$ given by setting

$$\Omega_t = \{x = y + \phi(y, t) \mid y \in \Omega\} \quad (t \in \mathbb{R}), \tag{4.2}$$

that is, Ω_t is the image of the transformation

$$\Phi_t \colon \Omega \to \mathbb{R}^N, \quad \Phi_t(y) = y + \phi(y, t),$$

for $t \in \mathbb{R}$. Notice that Ω_t is a given bounded periodically moving domain in $\mathbb{R}^N$ such that $\Omega_{t+2\pi} = \Omega_t$. Let Γ_t be the boundary of Ω_t, which is given by $\Gamma_t = \{x = y + \phi(y, t) \mid y \in \Gamma\}$. We consider the Navier-Stokes equations in Ω_t:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p} = \mathbf{F}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_t, \quad \mathbf{u}|_{\Gamma_t} = \mathbf{h}|_{\Gamma_t} \tag{4.3}$$

for $t \in (0, 2\pi)$. Here, $\mathbf{u} = (u_1, \ldots, u_N)^\top$ is an unknown velocity field, $M^\top$ being the transposed M, $\mathfrak{p}$ an unknown pressure field, $\mathbf{F} = (F_1, \ldots, F_N)^\top$ a prescribed external force, and $\mathbf{h} = (h_1, \ldots, h_N)^\top$ a velocity field that prescribes the boundary velocity. Assume that $\mathbf{F}(t + 2\pi) = \mathbf{F}(t)$ and $\mathbf{h}(t + 2\pi) = \mathbf{h}(t)$ for any $t \in \mathbb{R}$. Then system (4.3) describes the flow of an incompressible viscous fluid around a periodically moving body, subject to a time-periodic external force and with prescribed time-periodic boundary conditions. Note that a natural choice for $\mathbf{h}$ would be the flow velocity associated with the transformation Φ_t, which means that the fluid particles adhere to the boundary. This choice corresponds to a no-slip condition, which we further discuss in Remark 4.6 below.

We transform (4.3) into a system in Ω by using the change of variables induced by Φ_t, namely $x = y + \phi(y, t)$. For this purpose, we assume that

$$\|\phi\|_{\mathrm{L}_\infty(\mathbb{T}, \mathrm{H}_\infty^3(\Omega))} + \|\partial_t \phi\|_{\mathrm{L}_\infty(\mathbb{T}, \mathrm{H}_\infty^1(\Omega))} < \varepsilon_0 \tag{4.4}$$

with some small number $\varepsilon_0 > 0$. By this smallness assumption, we assure the existence of the inverse transformation: $y = x + \psi(x, t)$. The associated Jacobi matrix $\partial(t, y)/\partial(t, x)$ is given by the formulas:

$$\frac{\partial t}{\partial t} = 1, \quad \frac{\partial t}{\partial x_j} = 0, \quad \frac{\partial y_\ell}{\partial t} = \frac{\partial \psi_\ell}{\partial t}, \quad \frac{\partial y_\ell}{\partial x_j} = \delta_{\ell j} + \frac{\partial \psi_\ell}{\partial x_j}$$

for $j, \ell = 1, \ldots, N$. Set $a_{\ell 0}(y, t) = (\partial \psi_\ell/\partial t)(y + \phi(y, t), t)$, and $a_{\ell j}(y, t) = (\partial \psi_\ell/\partial x_j)(y + \phi(y, t), t)$. Then partial derivatives transform as

$$\frac{\partial f}{\partial t} = \frac{\partial g}{\partial t} + \sum_{\ell=1}^{N} a_{\ell 0}(y, t)\frac{\partial g}{\partial y_\ell}, \quad \frac{\partial f}{\partial x_j} = \frac{\partial g}{\partial y_j} + \sum_{\ell=1}^{N} a_{\ell j}(y, t)\frac{\partial g}{\partial y_\ell} \tag{4.5}$$

for $f(x, t) = g(y, t)$. Let J and J_0 be defined by $J = \det(\partial x/\partial y) = 1 + J_0(y, t)$, which is the Jacobian of Φ_t. By the L_∞-bounds in (4.4) and a Neumann-series argument, we obtain

$$\|\psi\|_{L_\infty(\mathbb{T}, H^2_\infty(\Omega))} + \|\partial_t\psi\|_{L_\infty(\mathbb{T}, L_\infty(\Omega))} < C\varepsilon_0,$$

so that we have

$$\sup_{t\in\mathbb{R}}\|a_{\ell j}(\cdot, t)\|_{H^2_\infty(\Omega)} + \sup_{t\in\mathbb{R}}\|\partial_t a_{\ell j}(\cdot, t)\|_{L_\infty(\Omega)} + \sup_{t\in\mathbb{R}}\|a_{0j}(\cdot, t)\|_{L_\infty(\Omega)}$$
$$+ \sup_{t\in\mathbb{R}}\|J_0(\cdot, t)\|_{H^2_\infty(\Omega)} + \sup_{t\in\mathbb{R}}\|\partial_t J_0(\cdot, t)\|_{L_\infty(\Omega)} \leq C\varepsilon_0 \tag{4.6}$$

with some constant $C > 0$ for $j, \ell = 1, \ldots, N$. For notational simplicity, we set

$$\mathbf{v}(y, t) = (v_1, \ldots, v_N)^\top = \mathbf{u}(x, t), \qquad \mathfrak{q}(y, t) = \mathfrak{p}(x, t).$$

By (4.5) we have

$$\partial_t\mathbf{u} = \partial_t\mathbf{v} + \sum_{\ell=1}^{N} a_{\ell 0}\frac{\partial\mathbf{v}}{\partial y_\ell}, \quad \mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{v}\cdot(\mathrm{I} + \mathrm{A})\nabla\mathbf{v},$$

$$\Delta\mathbf{u} = \Delta\mathbf{v} + \sum_{\ell=1}^{N}(a_{\ell j} + a_{j\ell})\frac{\partial^2\mathbf{v}}{\partial y_\ell\partial y_j} + \sum_{j,\ell,m=1}^{N} a_{\ell j}a_{mj}\frac{\partial^2\mathbf{v}}{\partial y_\ell\partial y_m}$$

$$+ \sum_{\ell,m=1}^{N}\left(\frac{\partial a_{m\ell}}{\partial y_\ell} + \sum_{j=1}^{N} a_{\ell j}\frac{\partial a_{mj}}{\partial y_\ell}\right)\frac{\partial\mathbf{v}}{\partial y_m},$$

$$\mathrm{div}\,\mathbf{u} = J^{-1}\{\mathrm{div}\,\mathbf{v} + \mathrm{div}\,(J_0\mathbf{v}) + \sum_{j,\ell=1}^{N}\frac{\partial}{\partial y_\ell}(a_{\ell j}Jv_j)\}. \quad \nabla\mathfrak{p} = (\mathrm{I} + \mathrm{A})\nabla\mathfrak{q}, \tag{4.7}$$

where A is an $(N \times N)$-matrix whose (j, k)-th component is a_{jk}. Setting $w_\ell = v_\ell + J_0 v_\ell + \sum_{j=1}^{N} a_{\ell j}Jv_j$, we have $J\mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{w}$ with $\mathbf{w} = (w_1, \ldots, w_N)^\top$. Notice that $\mathbf{w} = (\mathrm{I} + J_0\mathrm{I} + \mathrm{A}^\top J)\mathbf{v}$. In view of (4.6), choosing $\varepsilon > 0$, we see that there exists an $(N \times N)$-matrix B_{-1} such that $(\mathrm{I} + J_0\mathrm{I} + \mathrm{A}^\top J)^{-1} = \mathrm{I} + \mathrm{B}_{-1}$ and

$$\sup_{t\in\mathbb{R}}\|\mathrm{B}_{-1}(\cdot, t)\|_{H^2_\infty(\Omega)} \leq C\varepsilon_0, \quad \sup_{t\in\mathbb{R}}\|\partial_t\mathrm{B}_{-1}(\cdot, t)\|_{L_\infty(\Omega)} \leq C\varepsilon_0. \tag{4.8}$$

Summing up, we see that (4.3) is transformed to the following equations:

$$\partial_t \mathbf{w} - \Delta \mathbf{w} + \nabla \mathfrak{q} = \mathbf{G} + \mathscr{L}(\mathbf{w}, \mathfrak{q}) + \mathscr{N}(\mathbf{w}), \quad \operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{w}|_\Gamma = \mathbf{H}|_\Gamma, \quad (4.9)$$

where $\mathbf{G}$ and $\mathbf{H}$ are prescribed data and

$$
\begin{aligned}
\mathscr{L}(\mathbf{w}, \mathfrak{q}) &= -\partial_t(\mathrm{B}_{-1}\mathbf{w}) - \sum_{\ell=1}^N a_{\ell 0} \frac{\partial}{\partial y_\ell}((\mathrm{I} + \mathrm{B}_{-1})\mathbf{w}) + \Delta(\mathrm{B}_{-1}\mathbf{w}) \\
&\quad + \sum_{\ell=1}^N (a_{\ell j} + a_{j\ell}) \frac{\partial^2}{\partial y_\ell \partial y_j}((\mathrm{I} + \mathrm{B}_{-1})\mathbf{w}) + \sum_{j,\ell,m=1}^N a_{\ell j} a_{mj} \frac{\partial^2}{\partial y_\ell \partial y_m}((\mathrm{I} + \mathrm{B}_{-1})\mathbf{w}) \\
&\quad + \sum_{\ell,m=1}^N \left(\frac{\partial a_{m\ell}}{\partial y_\ell} + \sum_{j=1}^N a_{\ell j} \frac{\partial a_{mj}}{\partial y_\ell} \right) \frac{\partial}{\partial y_m}((\mathrm{I} + \mathrm{B}_{-1})\mathbf{w}) - A\nabla\mathfrak{q}, \\
\mathscr{N}(\mathbf{v}) &= ((\mathrm{I} + \mathrm{B}_{-1})\mathbf{w}) \cdot (\mathrm{I} + A)\nabla((\mathrm{I} + \mathrm{B}_{-1})\mathbf{w}).
\end{aligned}
$$

$$(4.10)$$

Observe that, due to $\operatorname{div} \mathbf{w} = 0$ and the boundedness of Ω, for the existence of solutions to (4.9) it is necessary that the boundary data satisfy

$$\int_\Gamma \mathbf{H} \cdot \mathbf{n} \, d\sigma = 0 \tag{4.11}$$

for all $t \in \mathbb{R}$, where $\mathbf{n}$ denotes the unit outer normal vector at Γ. The following theorem is our main result in this section.

Theorem 4.1 *Let $2 < p < \infty$ and $N < q < \infty$. Then, there exists numbers $\varepsilon, \varepsilon_0 > 0$ such that if the condition (4.4) is valid, and if the prescribed terms $\mathbf{G} \in L_p(\mathbb{T}, L_q(\Omega)^N)$ and $\mathbf{H} \in H_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N)$ satisfy the compatibility condition (4.11) as well as the smallness condition*

$$\|\mathbf{G}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{H}\|_{H_p^1(\mathbb{T}, L_q(\Omega))} + \|\mathbf{H}\|_{L_p(\mathbb{T}, H_q^2(\Omega))} \leq \varepsilon^2,$$

then problem (4.9) admits a solution $(\mathbf{w}, \mathfrak{q})$ with

$$\mathbf{w} \in H_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N), \quad \mathfrak{q} \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)),$$

possessing the estimate

$$\|\partial_t \mathbf{w}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{w}\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\nabla\mathfrak{q}\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq \varepsilon.$$

Moreover, $\mathbf{w}$ is the unique velocity field with these properties while the pressure $\mathfrak{q}$ is unique up to addition by a function only depending on time.

To prove Theorem 4.1, we consider the following linearization of equations (4.9):

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p} = \mathbf{F}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}|_\Gamma = \mathbf{H}|_\Gamma. \tag{4.12}$$

For a maximal-regularity theorem for problem (4.12), we consider the associated resolvent problem

$$\lambda \mathbf{w} - \Delta \mathbf{w} + \nabla \mathfrak{p} = \mathbf{f}, \quad \operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega, \quad \mathbf{w}|_\Gamma = \mathbf{h}|_\Gamma. \tag{4.13}$$

At first, we consider the case of homogeneous boundary conditions $\mathbf{h} = 0$, for which we have the following lemma, which holds for bounded and exterior domains simultaneously.

Lemma 4.2 *Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Let Ω be a bounded domain or an exterior domain in $\mathbb{R}^N$ ($N \geq 2$) with C^2 boundary. There exist operator families $(\mathscr{S}(\lambda)) \subset \mathscr{L}(\mathrm{L}_q(\Omega)^N, \mathrm{H}_q^2(\Omega)^N)$ and $(\mathscr{P}(\lambda)) \subset \mathscr{L}(\mathrm{L}_q(\Omega)^N, \hat{\mathrm{H}}_q^1(\Omega)))$ such that for every $\lambda \in \Sigma_\varepsilon \setminus \{0\}$ and every $\mathbf{f} \in \mathrm{L}_q(\Omega)^N$ the pair $(\mathbf{w}, \mathfrak{p}) = (\mathscr{S}(\lambda)\mathbf{f}, \mathscr{P}(\lambda)\mathbf{f})$ is a solution to (4.13) with $\mathbf{h} = 0$ and satisfies the estimate*

$$|\lambda|\,\|\mathscr{S}(\lambda)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} + \|\nabla^2 \mathscr{S}(\lambda)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} + \|\nabla \mathscr{P}(\lambda)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \leq C\|\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \tag{4.14}$$

with some constant $C > 0$ depending on Ω, q and ε. Moreover, there exist constants $\lambda_0, r_0 > 0$, depending on Ω, q and ε, such that

$$\mathscr{S} \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathscr{L}(\mathrm{L}_q(\Omega)^N, \mathrm{H}_q^2(\Omega)^N)), \quad \mathscr{P} \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathscr{L}(\mathrm{L}_q(\Omega)^N, \hat{\mathrm{H}}_q^1(\Omega))),$$

and

$$\mathscr{R}_{\mathscr{L}(\mathrm{L}_q(\Omega)^N, \mathrm{H}_q^{2-j}(\Omega)^N)}(\{(\lambda \partial_\lambda)^\ell (\lambda^{j/2} \mathscr{S}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq r_0,$$

$$\mathscr{R}_{\mathscr{L}(\mathrm{L}_q(\Omega)^N, \mathrm{L}_q(\Omega)^N)}(\{(\lambda \partial_\lambda)^\ell (\nabla \mathscr{P}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq r_0$$

for $\ell = 0, 1$, $j = 0, 1, 2$. Additionally, if Ω is bounded, then there exist $\mathscr{S}(0) \in \mathscr{L}(\mathrm{L}_q(\Omega)^N, \mathrm{H}_q^2(\Omega)^N)$ and $\mathscr{P}(0) \in \mathscr{L}(\mathrm{L}_q(\Omega)^N, \hat{\mathrm{H}}_q^1(\Omega))$ such that $(\mathbf{w}, \mathfrak{p}) = (\mathscr{S}(0)\mathbf{f}, \mathscr{P}(0)\mathbf{f})$ is a solution to (4.13) for $\lambda = 0$ and satisfies

$$\|\mathscr{S}(0)\mathbf{f}\|_{\mathrm{H}_q^2(\Omega)} + \|\nabla \mathscr{P}(0)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \leq C\|\mathbf{f}\|_{\mathrm{L}_q(\Omega)}. \tag{4.15}$$

In all these cases, the velocity field $\mathbf{w}$ is unique while the pressure $\mathfrak{p}$ is unique up to an additive constant.

Proof Existence of unique solutions to (4.13) satisfying (4.14) with a uniform constant C for $\lambda \in \Sigma \setminus \{0\}$ was shown in [8]. The analyticity of the associated family of solution operators in $\Sigma_{\varepsilon,\lambda_0}$ for some $\lambda_0 > 0$ together with the asserted $\mathscr{R}$-bounds was established in [65, Theorem 1.6] and [66, Theorem 9.1.4]. Finally, the statement for $\lambda = 0$ follows from Fredholm's alternative principle since the embedding $\mathrm{H}_q^2(\Omega) \hookrightarrow \mathrm{L}_q(\Omega)$ is compact if Ω is bounded, and the solution to (4.13) is unique. $\qquad \square$

As in the results from Sect. 3, in the case of non-zero boundary values, the $\mathscr{R}$-solvers for the resolvent problem (4.13) are more involved than in the situation of Lemma 4.2. To state the result, we introduce the space

$$\mathcal{X}_q = \{(F_1, F_2, F_3, F_4) \mid F_1, F_2 \in \mathrm{L}_q(\Omega)^N, F_3 \in \mathrm{H}_q^1(\Omega)^N, F_4 \in \mathrm{H}_q^2(\Omega)^N\}.$$

Then we have the following result.

Lemma 4.3 *Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Let Ω be a bounded domain or an exterior domain in $\mathbb{R}^N$ ($N \geq 2$) with C^2 boundary. There exist constants $\lambda_0, r_0 > 0$ and operator families*

$$S \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathscr{L}(\mathcal{X}_q(\Omega), \mathrm{H}_q^2(\Omega)^N)), \qquad P \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathscr{L}(\mathcal{X}_q(\Omega), \hat{\mathrm{H}}_q^1(\Omega)))$$

such that for any $\mathbf{f} \in L_q(\Omega)^N$ and $\mathbf{h} \in H_q^2(\Omega)^N$, satisfying $\int_\Gamma \mathbf{h} \cdot \mathbf{n}\, d\sigma = 0$ if Ω is bounded, the pair $(\mathbf{w}, \mathfrak{p})$ defined by $\mathbf{w} = \mathcal{S}(\lambda)(\mathbf{f}, \lambda\mathbf{h}, \lambda^{1/2}\mathbf{h}, \mathbf{h})$ and $\mathfrak{p} = \mathcal{P}(\lambda)(\mathbf{f}, \lambda\mathbf{h}, \lambda^{1/2}\mathbf{h}, \mathbf{h})$ is a solution to (4.13), and

$$\mathscr{R}_{\mathscr{L}(\mathcal{X}_q(\Omega), H_q^{2-j}(\Omega)^N)}(\{(\lambda\frac{d}{d\lambda})^\ell(\lambda^{j/2}\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_0,$$

$$\mathscr{R}_{\mathscr{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^N)}(\{(\lambda\frac{d}{d\lambda})^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_0$$

for $\ell = 0, 1$, $j = 0, 1, 2$. The pair $(\mathbf{w}, \mathfrak{p})$ is unique in the sense of Lemma 4.2.

Proof See [65, Theorem 1.6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we can prove the following result on the time-periodic linear problem (4.12).

Proposition 4.4 Let $1 < p, q < \infty$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. Then, for any $\mathbf{F} \in L_p(\mathbb{T}, L_q(\Omega)^N)$ and $\mathbf{H} \in H_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N)$ satisfying (4.11), problem (4.12) admits a solution $(\mathbf{u}, \mathfrak{p})$ with

$$\mathbf{u} \in H_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N), \quad \mathfrak{p} \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega))$$

possessing the estimate

$$\|\partial_t\mathbf{u}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\nabla\mathfrak{p}\|_{L_p(\mathbb{T}, L_q(\Omega))}$$
$$\leq C\big(\|\mathbf{F}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\partial_t\mathbf{H}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{H}\|_{L_p(\mathbb{T}, H_q^2(\Omega))}\big). \tag{4.16}$$

Here, $\mathbf{u}$ is unique in the described function class, while $\mathfrak{p}$ is unique up to addition by a function only depending on time.

Proof We first consider the case $\mathbf{H} = 0$, for which proceed analogously to the proof of Theorem 3.1. Let $\varphi = \varphi(\sigma)$ be a $C^\infty(\mathbb{R})$ function that equals 1 for $|\sigma| \geq \lambda_0 + 1/2$ and 0 for $|\sigma| \leq \lambda_0 + 1/4$. Set

$$\mathbf{u}_h = \mathscr{F}_\mathbb{T}^{-1}[\mathcal{S}(ik)\varphi(k)\mathscr{F}_\mathbb{T}[\mathbf{F}](k)], \quad \mathfrak{p}_h = \mathscr{F}_\mathbb{T}^{-1}[\mathcal{P}(ik)\varphi(k)\mathscr{F}_\mathbb{T}[\mathbf{F}](k)].$$

Then $\mathbf{u}_h$ and $\mathfrak{p}_h$ satisfy the equations

$$\partial_t\mathbf{u}_h - \mu\Delta\mathbf{u}_h + \nabla\mathfrak{p}_h = \mathbf{F}_h, \quad \operatorname{div}\mathbf{u}_h = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}_h|_\Gamma = 0,$$

where we have set $\mathbf{F}_h = \mathscr{F}_\mathbb{T}^{-1}[\varphi(k)\mathscr{F}_\mathbb{T}[\mathbf{F}](k)]$. Moreover, arguing as for the proof of Theorem 3.1, we can use the $\mathscr{R}$-bounds from Lemma 4.2 and invoke Corollary 2.3 to deduce

$$\|\partial_t\mathbf{u}_h\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}_h\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\nabla\mathfrak{p}_h\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C\|\mathbf{F}_h\|_{L_p(\mathbb{T}, L_q(\Omega))}$$
$$\leq C\|\mathbf{F}\|_{L_p(\mathbb{T}, L_q(\Omega))}. \tag{4.17}$$

Now, in view of Lemma 4.2, we set

$$\mathbf{u}(t) = \mathbf{u}_h(t) + \sum_{|k|\leq\lambda_0} e^{ikt}\mathcal{S}(ik)\mathscr{F}_\mathbb{T}[\mathbf{f}](k), \quad \mathfrak{p}(t) = \mathfrak{p}_h(t) + \sum_{|k|\leq\lambda_0} e^{ikt}\mathcal{P}(ik)\mathscr{F}_\mathbb{T}[\mathbf{f}](k). \tag{4.18}$$

Then, $\mathbf{u}$ and $\mathfrak{p}$ satisfy (4.12) with $\mathbf{H} = 0$, and from (4.14), (4.15) and (4.17) we conclude the estimate

$$\|\partial_t \mathbf{u}\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{u}\|_{L_p(\mathbb{T},H_q^2(\Omega))} + \|\nabla \mathfrak{p}\|_{L_p(\mathbb{T},L_q(\Omega))} \leq C \|\mathbf{F}\|_{L_p(\mathbb{T},L_q(\Omega))}. \tag{4.19}$$

Thus, we have shown existence for $\mathbf{H} = 0$.

Now consider arbitrary $\mathbf{H} \in H_q^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N)$ satisfying (4.11). Fix $\lambda_1 > \lambda_0$ with λ_0 from Lemma 4.3. We use the $\mathscr{R}$-bounded solution operators $\mathcal{S}$ and $\mathcal{P}$ from Lemma 4.3 to define

$$\mathbf{u}_1 = \mathscr{F}_{\mathbb{T}}^{-1}[\mathcal{S}(ik + \lambda_1)\big(0, (ik + \lambda_1)\tilde{\mathbf{H}}_k, (ik + \lambda_1)^{1/2}\tilde{\mathbf{H}}_k, \tilde{\mathbf{H}}_k\big)],$$

$$\mathfrak{p}_1 = \mathscr{F}_{\mathbb{T}}^{-1}[\mathcal{P}(ik + \lambda_1)\big(0, (ik + \lambda_1)\tilde{\mathbf{H}}_k, (ik + \lambda_1)^{1/2}\tilde{\mathbf{H}}_k, \tilde{\mathbf{H}}_k\big)],$$

where $\tilde{\mathbf{H}}_k = \mathscr{F}[\mathbf{H}](k)$. Then $(\mathbf{u}_1, \mathfrak{p}_1)$ is a solution to the auxiliary problem

$$\partial_t \mathbf{u}_1 + \lambda_1 \mathbf{u}_1 - \mu \Delta \mathbf{u}_1 + \nabla \mathfrak{p}_1 = 0, \quad \operatorname{div} \mathbf{u}_1 = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}_1|_\Gamma = \mathbf{H},$$

and by the multiplier theorem from Corollary 2.3, we have

$$\mathbf{u}_1 \in H_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N), \quad \mathfrak{p}_1 \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega))$$

and the estimate

$$\|\partial_t \mathbf{u}_1\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{u}_1\|_{L_p(\mathbb{T},H_q^2(\Omega))} + \|\nabla \mathfrak{p}_1\|_{L_p(\mathbb{T},L_q(\Omega))}$$

$$\leq C(\|\partial_t \mathbf{H}\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{H}\|_{L_p(\mathbb{T},H_q^2(\Omega))}).$$

Here, we have used the interpolation inequality

$$\|\mathbf{H}\|_{H_p^{1/2}(\mathbb{T},H_q^1(\Omega))} \leq C(\|\partial_t \mathbf{H}\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{H}\|_{L_p(\mathbb{T},H_q^2(\Omega))}).$$

as well as the trivial estimate

$$\|\mathscr{F}_{\mathbb{T}}^{-1}[(ik + \lambda_1)\tilde{\mathbf{H}}_k]\|_{L_p(\mathbb{T},L_q(\Omega))} \leq \|\mathscr{F}_{\mathbb{T}}^{-1}[ik\tilde{\mathbf{H}}_k]\|_{L_p(\mathbb{T},L_q(\Omega))} + \lambda_1 \|\mathscr{F}_{\mathbb{T}}^{-1}[\tilde{\mathbf{H}}_k]\|_{L_p(\mathbb{T},L_q(\Omega))}$$

$$= \|\partial_t \mathbf{H}\|_{L_p(\mathbb{T},L_q(\Omega))} + \lambda_1 \|\mathbf{H}\|_{L_p(\mathbb{T},L_q(\Omega))}.$$

Now consider the problem

$$\partial_t \mathbf{u}_2 - \mu \Delta \mathbf{u}_2 + \nabla \mathfrak{p}_2 = \mathbf{F} + \lambda_1 \mathbf{u}_1, \quad \operatorname{div} \mathbf{u}_2 = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}_2|_\Gamma = 0.$$

As shown in the first part of the proof, there exists a solution $(\mathbf{u}_2, \mathfrak{q}_2)$ in the claimed function class and satisfying (4.19) with $\mathbf{F}$ replaced with $\mathbf{F} + \lambda_1 \mathbf{u}_1$. Then $(\mathbf{u}, \mathfrak{q}) = (\mathbf{u}_1 + \mathbf{u}_2, \mathfrak{q}_1 + \mathfrak{q}_2)$ is a solution to (4.12), belongs to the correct function class, and satisfies estimate (4.16).

For the uniqueness statement, let $(\mathbf{u}, \mathfrak{p})$ be in the considered function class and satisfy (4.12) with $\mathbf{F} = 0$ and $\mathbf{H} = 0$. Then, for each $k \in \mathbb{Z}$, setting $\tilde{\mathbf{u}}_k = \mathscr{F}[\mathbf{u}](k)$ and $\tilde{\mathfrak{p}}_k = \mathscr{F}_{\mathbb{T}}[\mathfrak{p}](k)$, we see that $\tilde{\mathbf{u}}_k \in H_q^2(\Omega)^3$ and $\tilde{\mathfrak{p}}_k \in \hat{H}_q^1(\Omega)$ satisfy the homogeneous equations

$$ik\tilde{\mathbf{u}}_k - \mu \Delta \tilde{\mathbf{u}}_k + \nabla \tilde{\mathfrak{p}}_k = 0, \quad \operatorname{div} \tilde{\mathbf{u}}_k = 0 \quad \text{in } \Omega, \quad \tilde{\mathbf{u}}_k|_\Gamma = 0.$$

Thus, the uniqueness statement from Lemma 4.2 yields that $\tilde{\mathbf{u}}_k = \nabla \tilde{\mathfrak{p}}_k = 0$, which shows that $\mathbf{u} = \nabla \mathfrak{p} = 0$. This completes the proof of Proposition 4.4. $\quad\square$

To conclude the proof of Theorem 4.1, we next derive an estimate of the nonlinear term $\mathscr{N}$ in the spirit of the assumptions of the abstract theory from Theorem 3.3. Moreover, we identify $\mathscr{L}$ as a small perturbation. For this purpose, we introduce the quantity

$$E(\mathbf{u}, \mathfrak{p}) := \|\partial_t \mathbf{u}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\nabla \mathfrak{p}\|_{L_p(\mathbb{T}, L_q(\Omega))}.$$

Lemma 4.5 *Let $p \in (2, \infty)$ and $q \in (N, \infty)$, and let $\varepsilon_0 > 0$ such that (4.4) is satisfied. Then there exists $C > 0$ such that*

$$\|\mathscr{L}(\mathbf{u}_1, \mathfrak{p}_1)\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C\varepsilon_0 E(\mathbf{u}_1, \mathfrak{p}_1), \tag{4.20}$$

$$\|\mathscr{L}(\mathbf{u}_1 - \mathbf{u}_2, \mathfrak{p}_1 - \mathfrak{p}_2)\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C\varepsilon_0 E(\mathbf{u}_1 - \mathbf{u}_2, \mathfrak{p}_1 - \mathfrak{p}_2), \tag{4.21}$$

$$\|\mathscr{N}(\mathbf{u}_1)\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C(1 + \varepsilon_0) E(\mathbf{u}_1, \mathfrak{p}_1)^2, \tag{4.22}$$

$$\|\mathcal{N}(\mathbf{u}_1) - \mathcal{N}(\mathbf{u}_2)\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C(1 + \varepsilon_0)\big(E(\mathbf{u}_1, \mathfrak{p}_1) + E(\mathbf{u}_2, \mathfrak{p}_2)\big) E(\mathbf{u}_1 - \mathbf{u}_2, \mathfrak{p}_1 - \mathfrak{p}_2) \tag{4.23}$$

for all $\mathbf{u}_1, \mathbf{u}_2 \in H_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N)$ and $\mathfrak{p}_1, \mathfrak{p}_2 \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega))$.

Proof Firstly, by (4.4), (4.6) and (4.8) we directly conclude (4.20), and (4.21) follows from (4.20) since $\mathscr{L}$ is a linear operation. In a similar way, noting that $N < q < \infty$, by Sobolev's imbedding theorem we have

$$\|\mathscr{N}(\mathbf{u}_1(\cdot, t))\|_{L_q(\Omega)} \leq C(1 + \varepsilon_0) \|\mathbf{u}_1(\cdot, t)\|_{L_q(\Omega)} \|\mathbf{u}_1(\cdot, t)\|_{H_q^2(\Omega)}. \tag{4.24}$$

By real interpolation theorem, we know that

$$H_p^1(\mathbb{T}, L_q(\Omega)) \cap L_p(\mathbb{T}, H_q^2(\Omega)) \hookrightarrow C^0(\mathbb{T}, B_{q,p}^{2(1-1/p)}(\Omega)),$$
$$\sup_{t \in \mathbb{T}} \|f(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq C(\|f\|_{H_p^1(\mathbb{T}, L_q(\Omega))} + \|f\|_{L_p(\mathbb{T}, H_q^2(\Omega))}) \tag{4.25}$$

since we have $p > 2$, and we obtain

$$\|f\|_{L_\infty(\mathbb{T}, L_q(\Omega))} \leq C\|f\|_{L_\infty(\mathbb{T}, B_{q,p}^{2(1-1/p)}(\Omega))} \leq C(\|f\|_{H_p^1(\mathbb{T}, L_q(\Omega))} + \|f\|_{L_p(\mathbb{T}, H_q^2(\Omega))}). \tag{4.26}$$

Therefore, we conclude (4.22). In a similar way, since

$$\mathcal{N}(\mathbf{u}_1) - \mathcal{N}(\mathbf{u}_2)$$
$$= ((I + B_{-1})(\mathbf{u}_1 - \mathbf{u}_2)) \cdot (I + A)\nabla((I + B_{-1})\mathbf{u}_1) + ((I + B_{-1})\mathbf{u}_2)$$
$$\cdot (I + A)\nabla((I + B_{-1})(\mathbf{u}_1 - \mathbf{u}_2)),$$

by Sobolev's inequality and the assumption $N < q < \infty$, we have

$$\|\mathcal{N}(\mathbf{u}_1) - \mathcal{N}(\mathbf{u}_2)\|_{L_q(\Omega)} \leq C(\|\mathbf{u}_1 - \mathbf{u}_2\|_{L_q(\Omega)} \|\mathbf{u}_1\|_{H_q^2(\Omega)} + \|\mathbf{u}_2\|_{L_q(\Omega)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{H_q^2(\Omega)}).$$

Thus, from (4.26) we finally deduce (4.23). $\qquad\square$

Proof of Theorem 4.1 We conclude the proof using the contraction mapping principle in the very same fashion as in Theorem 3.3. We introduce the underlying space $\mathcal{I}_\varepsilon$ defined by

$$\mathcal{I}_\varepsilon = \{(\mathbf{u}, \mathfrak{p}) \mid \mathbf{u} \in H_p^1(\mathbb{T}, L_q(\Omega)) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N), \ \mathfrak{p} \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)), \ E(\mathbf{u}, \mathfrak{p}) \leq \varepsilon\}.$$

Given $(\mathbf{u}, \mathfrak{p}) \in \mathcal{I}_\varepsilon$, let $\mathbf{v}$ and $\mathfrak{q}$ satisfy

$$\partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla \mathfrak{q} = \mathbf{G} + \mathscr{L}(\mathbf{u}, \mathfrak{p}) + \mathscr{N}(\mathbf{u}), \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}|_\Gamma = \mathbf{H}|_\Gamma. \quad (4.27)$$

The existence of $(\mathbf{v}, \mathfrak{q})$ follows from Proposition 4.4 since by Lemma 4.5, the forcing term in (4.27) belongs to $L_p(\mathbb{T}, L_q(\Omega)^N)$. More precisely, combining estimates (4.20) and (4.22) with the smallness assumption on $\mathbf{G}$ and $\mathbf{H}$, choosing $\varepsilon_0 \le C\varepsilon$ and applying Proposition 4.4 gives the unique existence of a solution $(\mathbf{v}, \mathfrak{q})$ of (4.27) with

$$\mathbf{v} \in H_p^1(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H_q^2(\Omega)^N), \quad \mathfrak{q} \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)),$$

possessing the estimate

$$E(\mathbf{v}, \mathfrak{q}) \le C(\varepsilon^2 + \varepsilon E(\mathbf{u}, \mathfrak{p}) + E(\mathbf{u}, \mathfrak{p})^2). \quad (4.28)$$

Since we assume that $E(\mathbf{u}, \mathfrak{p}) \le \varepsilon$, by (4.28) we have $E(\mathbf{v}, \mathfrak{q}) \le 3C\varepsilon^2$. Therefore, choosing $\varepsilon > 0$ so small that $3C\varepsilon \le 1$, we have $E(\mathbf{v}, \mathfrak{q}) \le \varepsilon$, so that $(\mathbf{v}, \mathfrak{q}) \in \mathcal{I}_\varepsilon$. Thus, the map Ξ acting on $(\mathbf{u}, \mathfrak{p}) \in \mathcal{I}_\varepsilon$ by setting $\Xi(\mathbf{u}, \mathfrak{p}) = (\mathbf{v}, \mathfrak{q})$ is a map from $\mathcal{I}_\varepsilon$ into $\mathcal{I}_\varepsilon$.

We next prove that Ξ is a contraction map. Let $(\mathbf{u}_i, \mathfrak{p}_i)$ $(i = 1.2)$ be any two elements of $\mathcal{I}_\varepsilon$ and set $(\mathbf{v}_i, \mathfrak{q}_i) = \Xi(\mathbf{u}_i, \mathfrak{p}_i)$. Then, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ and $\mathfrak{q} = \mathfrak{q}_1 - \mathfrak{q}_2$ satisfy the equations

$$\partial_t \mathbf{v} - \Delta \mathbf{v} + \nabla \mathfrak{q} = \mathscr{L}(\mathbf{u}_1 - \mathbf{u}_2, \mathfrak{p}_1 - \mathfrak{p}_2) + (\mathscr{N}(\mathbf{u}_1) - \mathscr{N}(\mathbf{u}_2))$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}|_\Gamma = 0.$$

Applying Proposition 4.4 and using (4.21) and (4.23) now gives that

$$E(\mathbf{v}_1 - \mathbf{v}_2, \mathfrak{q}_1 - \mathfrak{q}_2) \le C\varepsilon E(\mathbf{u}_1 - \mathbf{u}_2, \mathfrak{p}_1 - \mathfrak{p}_2).$$

Choosing $\varepsilon > 0$ smaller if necessary, we may assume that $C\varepsilon < 1$, and so Ξ is a contraction map on $\mathcal{I}_\varepsilon$, which yields the unique existence of $(\mathbf{u}, \mathfrak{p}) \in \mathcal{I}_\varepsilon$ such that $(\mathbf{u}, \mathfrak{p}) = \Xi(\mathbf{u}, \mathfrak{p})$. Obviously, $(\mathbf{u}, \mathfrak{p})$ is the required solution of (4.9). This completes the proof of Theorem 4.1. $\qquad \square$

Remark 4.6 In the case of no-slip boundary conditions, the fluid particles at the boundary are attached to the body, so that the fluid velocity coincides with the velocity of boundary particles. Then, the boundary data $\mathbf{h}$ in (4.3) are given by

$$\mathbf{h}(x, t) = \partial_t \Phi_t(y) = \partial_t \phi(t, y),$$

where $x = \Phi_t(y) = y + \phi(y, t)$, and in the formulation (4.9) on a time-independent domain, this corresponds to boundary data

$$\mathbf{H} = (I + J_0 I + A^\top J)\partial_t \phi.$$

Therefore, the assumptions on $\mathbf{H}$ in Theorem 4.1 are additional regularity and smallness assumptions on ϕ in this case. Moreover, the compatibility condition (4.11) is satisfied if and only if Φ_t preserves the volume of Ω.

5 On Periodic Navier-Stokes Flow Around a Body at Rest

5.1 Problem and Main Results

Let Ω be an exterior domain in $\mathbb{R}^3$, that is, a domain that is the complement of a compact set. We assume that its boundary Γ is a connected C^2 hypersurface. Let $b > 0$ be a sufficiently large number such that $\Omega^c \subset B_b$, where $\Omega^c = \mathbb{R}^3 \setminus \Omega$ and $B_b = \{x \in \mathbb{R}^N \mid |x| < b\}$. We further set $S_b = \{x \in \mathbb{R}^N \mid |x| = b\}$. We consider the Navier-Stokes equations in Ω:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla \mathfrak{p} = \mathbf{F}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = \mathbf{h}|_\Gamma. \tag{5.1}$$

Here, $\partial_j = \partial / \partial x_j$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)^\top$ is an unknown velocity field, $\mathfrak{p}$ an unknown pressure field, $\mathbf{F} = (F_1, \ldots, F_N)^\top$ a prescribed external force, and $\mathbf{h} = (h_1, \ldots, h_N)^\top$ are prescribed boundary data. Assume that $\mathbf{F}(t + 2\pi) = \mathbf{F}(t)$ and $\mathbf{h}(t + 2\pi) = \mathbf{h}(t)$ for any $t \in \mathbb{R}$. Then (5.1) describes the fluid flow around a body, subject to time-periodic external forcing $\mathbf{F}$ and with boundary data $\mathbf{h}$.

Given any time-periodic function $f = f(x, t)$, with period 2π, we write

$$f_S(x) = \int_{\mathbb{T}} f(x, t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} f(x, t) \, dt, \qquad f_\perp(x, t) = f(x, t) - f_S(x), \tag{5.2}$$

and f_S and $f_\perp$ are called the stationary part of f and the oscillatory part of f, respectively. By means of this decomposition, we divide the data and the solution into two parts, which have different asymptotic properties as $|x| \to \infty$. To quantify this spatial decay, we set

$$< f_S >_\ell = \sup_{x \in \Omega} |f_S(x)| (1 + |x|)^\ell, \qquad < f_\perp >_{p, \ell} = \sup_{x \in \Omega} \| f_\perp(x, \cdot) \|_{L_p(\mathbb{T})} (1 + |x|)^\ell$$

for $\ell \in \mathbb{R}$ and $p \in (1, \infty)$. We shall prove the following theorem.

Theorem 5.1 *Let $2 < p < \infty$ and $3 < q < \infty$. Assume that $\mathbf{F} = \mathbf{F}_S + \mathbf{F}_\perp$ with $\mathbf{F}_S = \operatorname{div} \mathbf{G}_S$ and $\mathbf{F}_\perp = \operatorname{div} \mathbf{G}_\perp$. Then there exists a small constant $\varepsilon > 0$ depending on p and q such that if $\mathbf{F}$ and $\mathbf{h} \in H_p^1(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, H_q^2(\Omega)^3)$ satisfy the smallness condition*

$$< \mathbf{F}_S >_3 + < \mathbf{G}_S >_2 + < \mathbf{F}_\perp >_{p,2} + < \mathbf{G}_\perp >_{p,1} + \|\mathbf{h}\|_{H_p^1(\mathbb{T}, L_q(\Omega))} + \|\mathbf{h}\|_{L_p(\mathbb{T}, H_q^2(\Omega))} < \varepsilon^2, \tag{5.3}$$

then problem (5.1) admits a solution $(\mathbf{u}, \mathfrak{p})$ such that $\mathbf{u} = \mathbf{u}_S + \mathbf{u}_\perp$ and $\mathfrak{p} = \mathfrak{p}_S + \mathfrak{p}_\perp$ with

$$\mathbf{u}_S \in H_q^2(\Omega)^3, \quad \mathbf{u}_\perp \in H_p^1(\mathbb{T}, L_q(\Omega)) \cap L_p(\mathbb{T}, H_q^2(\Omega)), \quad \mathfrak{p}_S \in H_q^1(\Omega), \quad \mathfrak{p}_\perp \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega))$$

satisfying the estimate

$$< \mathbf{u}_S >_1 + < \nabla \mathbf{u}_S >_2 + \|\mathbf{u}_S\|_{H_q^2(\Omega)} + \|\mathfrak{p}_S\|_{H_q^1(\Omega)}$$

$$+ < \mathbf{u}_\perp >_{p,1} + < \nabla \mathbf{u}_\perp >_{p,2} + \|\mathbf{u}_\perp\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\partial_t \mathbf{u}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\nabla \mathfrak{p}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))}$$

$$\leq \varepsilon.$$

Moreover, $\mathbf{u}$ and $\mathfrak{p}_S$ are the unique functions with these properties while the purely periodic pressure $\mathfrak{p}_\perp$ is unique up to addition by a function only depending on time.

Our proof of Theorem 5.1 is based on the study of the associated linearized system, the time-periodic Stokes problem

$$\partial_t \mathbf{v} - \mu \Delta \mathbf{v} + \nabla \mathfrak{p} = \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}|_\Gamma = \mathbf{h}. \tag{5.4}$$

We shall derive the following theorem, which ensures existence of a unique solution to (5.4) in a framework of spatially weighted spaces. For shorter notation, we set $L_{q,3b}(\Omega) = \{f \in L_q(\Omega) \mid \operatorname{supp} f \subset B_{3b}\}$.

Theorem 5.2 *Let $1 < p < \infty$, $3 < q < \infty$ and $\ell \in (0, 3]$. For all $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp \in L_p(\mathbb{T}, L_q(\Omega)^3)$ such that $\mathbf{f}_S = \operatorname{div} \mathbf{G}_S + \mathbf{g}_S$ and $\mathbf{f}_\perp = \operatorname{div} \mathbf{G}_\perp + \mathbf{g}_\perp$ with $\mathbf{g} = \mathbf{g}_S + \mathbf{g}_\perp \in L_p(\mathbb{T}, L_{q,3b}(\Omega)^3)$ and*

$$< \mathbf{G}_S >_2 + < \operatorname{div} \mathbf{G}_S >_3 + < \mathbf{G}_\perp >_{p,\ell} + < \operatorname{div} \mathbf{G}_\perp >_{p,\ell+1} < \infty,$$

problem (5.4) admits a solution $(\mathbf{v}, \mathfrak{p})$ with

$$\mathbf{v} \in H^1_p(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, H^2_q(\Omega)^3), \quad \mathfrak{p} \in L_p(\mathbb{T}, \hat{H}^1_q(\Omega)),$$

possessing the estimate

$$\|\mathbf{v}_S\|_{H^2_q(\Omega)} + < \mathbf{v}_S >_1 + < \nabla \mathbf{v}_S >_2 + \|\mathfrak{p}_S\|_{H^1_q(\Omega)} + < \mathfrak{p}_S >_2$$

$$+ \|\partial_t \mathbf{v}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{v}_\perp\|_{L_p(\mathbb{T}, H^2_q(\Omega))} + < \mathbf{v}_\perp >_{p,\ell} + < \nabla \mathbf{v}_\perp >_{p,\ell+1} + \|\nabla \mathfrak{p}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))}$$

$$\leq C(< \operatorname{div} \mathbf{G}_S >_3 + < \mathbf{G}_S >_2 + < \operatorname{div} \mathbf{G}_\perp >_{p,1+\ell} + < \mathbf{G}_\perp >_{p,\ell}$$

$$+ \|\mathbf{g}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{h}\|_{H^1_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{h}\|_{L_p(\mathbb{T}, H^2_q(\Omega))}). \tag{5.5}$$

Here, $\mathbf{v}$ and $\mathfrak{p}_S$ are the unique functions with these properties while the purely periodic pressure $\mathfrak{p}_\perp$ is unique up to addition by a function only depending on time.

For the proof of this theorem, we split (5.4) into two separate problems by means of the decomposition (5.2). In the following subsections these problems are analyzed independently of each other in the case of vanishing boundary data. In Sect. 5.4 we return to the original linear and nonlinear problems (5.4) and (5.1) and complete the proofs of Theorems 5.1 and 5.2.

5.2 Stationary Solutions to the Stokes Problem

Here we examine time-independent solutions to (5.4) with vanishing boundary data $\mathbf{h} = 0$, that is, solutions $(\mathbf{u}, \mathfrak{p})$ to the stationary problem

$$-\mu \Delta \mathbf{u} + \nabla \mathfrak{p} = \operatorname{div} \mathbf{F} + \mathbf{g}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = 0. \tag{5.6}$$

We shall derive the following result.

Proposition 5.3 *Let $3 < q < \infty$. If $\mathbf{F}$ satisfies the condition $< \operatorname{div} \mathbf{F} >_3 + < \mathbf{F} >_2 < \infty$ and $\mathbf{g} \in L_{q,3b}(\Omega)^3$, then problem (5.6) admits a unique solution $(\mathbf{u}, \mathfrak{p}) \in H^2_q(\Omega)^3 \times H^1_q(\Omega)$ possessing the estimate*:

$$\|\mathbf{u}\|_{H^2_q(\Omega)} + < \mathbf{u} >_1 + < \nabla \mathbf{u} >_2 + \|\mathfrak{p}\|_{H^1_q(\Omega)} + < \mathfrak{p} >_2 \leq C(< \operatorname{div} \mathbf{F} >_3 + < \mathbf{F} >_2 + \|\mathbf{g}\|_{L_q(\Omega)}) \tag{5.7}$$

with some constant $C > 0$.

For the proof, we first consider the Stokes equations in $\mathbb{R}^3$:

$$-\mu\Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathbb{R}^3. \tag{5.8}$$

As is well-known (cf. Galdi [28, pp.239-240]), there exist fundamental solutions $\mathbf{U} = (U_{ij}(x))$ and $\mathbf{q} = (q_1(x), q_2(x), q_3(x))^\top$ of equations (5.8) with

$$U_{ij}(x) = -\frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3}\right), \quad q_j(x) = \frac{1}{4\pi}\frac{x_j}{|x|^3}. \tag{5.9}$$

If we set

$$\mathbf{u}(x) = \mathbf{U}*\mathbf{f}(x) := \int_{\mathbb{R}^3}\mathbf{U}(y)\mathbf{f}(x-y)\,dy, \quad \mathfrak{p}(x) = \mathbf{q}*\mathbf{f} := \int_{\mathbb{R}^3}\mathbf{q}(y)\cdot\mathbf{f}(x-y)\,dy, \tag{5.10}$$

then, $\mathbf{u}$ and $\mathfrak{p}$ formally satisfy equations (5.8). We prove the following lemma.

Lemma 5.4 *Let $3 < q < \infty$.*

(1) Let $\mathbf{F}$ be a function satisfying $< \operatorname{div}\mathbf{F} >_3 + < \mathbf{F} >_2 < \infty$ and set $\mathbf{u} = \mathbf{U}*(\operatorname{div}\mathbf{F})$ and $\mathfrak{p} = \mathbf{q}*(\operatorname{div}\mathbf{F})$. Then, we have

$$\|\mathbf{u}\|_{H_q^2(\mathbb{R}^3)} + \|\mathfrak{p}\|_{H_q^1(\mathbb{R}^3)} + < \mathbf{u} >_1 + < \nabla\mathbf{u} >_2 + < \mathfrak{p} >_2 \le C(< \mathbf{F} >_2 + < \operatorname{div}\mathbf{F} >_3)$$

with some constant $C > 0$.

(2) Let $\mathbf{g} \in L_q(\mathbb{R}^3)$ such that $\mathbf{g}$ vanishes for $|x| > b$ with some constant $b > 0$. Let $\mathbf{v} = \mathbf{U}*\mathbf{g}$ and $\mathfrak{q} = \mathbf{q}*\mathbf{g}$. Then, we have

$$\|\mathbf{v}\|_{H_q^2(\mathbb{R}^3)} + \|\mathfrak{q}\|_{H_q^1(\mathbb{R}^3)} + < \mathbf{v} >_1 + < \nabla\mathbf{v} >_2 + < \mathfrak{q} >_2 \le C\|\mathbf{g}\|_{L_q(\mathbb{R}^3)}$$

for some constant C.

Proof (1) The theory of singular integrals (see [70, Ch. 2, Thm. 3] for example) yields that

$$\|\nabla^2\mathbf{u}\|_{L_q(\mathbb{R}^3)} + \|\nabla\mathfrak{p}\|_{L_q(\mathbb{R}^3)} \le C\|\operatorname{div}\mathbf{F}\|_{L_q(\mathbb{R}^3)} \le C_q < \operatorname{div}\mathbf{F} >_3;$$

$$\|\nabla\mathbf{u}\|_{L_q(\mathbb{R}^3)} + \|\mathfrak{p}\|_{L_q(\mathbb{R}^3)} \le C\|\mathbf{F}\|_{L_q(\mathbb{R}^3)} \le C_q < \mathbf{F} >_2.$$

For notational simplicity, set $\gamma = < \operatorname{div}\mathbf{F} >_3 + < \mathbf{F} >_2$. By the Gaussian divergence theorem, we write

$$\mathbf{u}(x) = \int_{|y|\le|x|/2}\mathbf{U}(y)(\operatorname{div}\mathbf{F})(x-y)\,dy - \int_{|y|=|x|/2}\mathbf{U}(y)\frac{y}{|y|}\cdot\mathbf{F}(x-y)\,d\omega$$

$$+ \int_{|x|/2\le|y|\le2|x|}\nabla\mathbf{U}(y)\mathbf{F}(x-y)\,dy + \int_{|y|\ge2|x|}\nabla\mathbf{U}(y)\mathbf{F}(x-y)\,dy.$$

Noting that $|x-y| \ge |x|/2$ for $|y| \le |x|/2$, $|x-y| \le 3|x|$ for $|x|/2 \le |y| \le 2|x|$, and $|x-y| \ge |y|/2$ for $|y| \ge 2|x|$, by (5.9) we have

$$|\mathbf{u}(x)| \le C\gamma\left\{(1+|x|)^{-3}\int_{|y|\le|x|/2}|y|^{-1}\,dy + |x|^{-1}(1+|x|)^{-2}\int_{|y|=|x|/2}d\omega\right.$$

$$\left. + |x|^{-2}\int_{|z|\le3|x|}|z|^{-2}\,dz + \int_{|y|\ge2|x|}|y|^{-4}\,dy\right\} \le C\gamma|x|^{-1}$$

for $x \neq 0$. When $|x| \leq 1$, noting that $|(\operatorname{div}\mathbf{F})(x-y)| \leq \gamma$ for $|y| \leq 2$ and $|(\operatorname{div}\mathbf{F})(x-y)| \leq C\gamma|y|^{-3}$ for $|y| \geq 2$, we have

$$|\mathbf{u}(x)| \leq \int_{|y|\leq 2} |\mathbf{U}(y)(\operatorname{div}\mathbf{F})(x-y)|\,dy + \int_{|y|\geq 2} |\mathbf{U}(y)(\operatorname{div}\mathbf{F})(x-y)|\,dy$$

$$\leq C\gamma \left\{ \int_{|y|\leq 2} |y|^{-1}\,dy + \int_{|y|\geq 2} |y|^{-4}\,dy \right\} \leq C\gamma.$$

In total, we thus have $< \mathbf{u} >_1 \leq C\gamma$. In particular, noting that $3 < q < \infty$, we obtain

$$\|\mathbf{u}\|_{L_q(\mathbb{R}^3)} \leq C_q < \mathbf{u} >_1 \leq C_q\gamma.$$

Similarly to before, we proceed with the estimate of $\nabla\mathbf{u}$ and write

$$\nabla\mathbf{u}(x) = \int_{|y|\leq|x|/2} \nabla\mathbf{U}(y)(\operatorname{div}\mathbf{F})(x-y)\,dy - \int_{|y|=|x|/2} \nabla\mathbf{U}(y)\frac{y}{|y|}\cdot\mathbf{F}(x-y)\,d\omega$$

$$+ \int_{|x|/2\leq|y|\leq 2|x|} \nabla^2\mathbf{U}(y)\mathbf{F}(x-y)\,dy + \int_{|y|\geq 2|x|} \nabla^2\mathbf{U}(y)\mathbf{F}(x-y)\,dy.$$

Then, we have

$$|\nabla\mathbf{u}(x)| \leq C\gamma \left\{ (1+|x|)^{-3} \int_{|y|\leq|x|/2} |y|^{-2}\,dy + |x|^{-2}(1+|x|)^{-2} \int_{|y|=|x|/2} d\omega \right.$$

$$\left. + |x|^{-3} \int_{|z|\leq 3|x|} |z|^{-2}\,dz + \int_{|y|\geq 2|x|} |y|^{-5}\,dy \right\} \leq C\gamma|x|^{-2}$$

for $x \neq 0$. When $|x| \leq 1$, arguing as above, we have

$$|\nabla\mathbf{u}(x)| \leq \int_{|y|\leq 2} |\nabla\mathbf{U}(y)(\operatorname{div}\mathbf{F})(x-y)|\,dy + \int_{|y|\geq 2} |\nabla\mathbf{U}(y)(\operatorname{div}\mathbf{F})(x-y)|\,dy$$

$$\leq C\gamma \left\{ \int_{|y|\leq 2} |y|^{-2}\,dy + \int_{|y|\geq 2} |y|^{-5}\,dy \right\} \leq C\gamma.$$

Summing up, we have $< \nabla\mathbf{u} >_2 \leq C\gamma$.

In the very same way, we can use $|\mathbf{q}(y)| \leq C|y|^{-2}$ and $|\nabla\mathbf{q}(y)| \leq C|y|^{-3}$ for $y \neq 0$ to deduce $< \mathfrak{p} >_2 \leq C\gamma$. In particular, $\|\mathfrak{p}\|_{L_q(\mathbb{R}^3)} \leq C_q < \mathfrak{p} >_2 \leq C_q\gamma$, because $3 < q < \infty$. This completes the proof of the first part of Lemma 5.4.

(2) As before, the theory of singular integral operators yields that

$$\|\nabla^2\mathbf{v}\|_{L_q(\mathbb{R}^3)} + \|\nabla\mathfrak{q}\|_{L_q(\mathbb{R}^3)} \leq C\|\mathbf{g}\|_{L_q(\mathbb{R}^3)}. \tag{5.11}$$

By estimates for weak singular integral operators (cf. [28, II.11]), we further have

$$\|\nabla\mathbf{v}\|_{L_q(B_{2b})} + \|\mathbf{v}\|_{L_q(B_{2b})} + \|\mathfrak{q}\|_{L_q(B_{2b})} \leq C_{b,q}\|\mathbf{g}\|_{L_q(\mathbb{R}^3)}. \tag{5.12}$$

Since $|\mathbf{U}(x)| \leq C|x|^{-1}$, $|\nabla\mathbf{U}(x)| \leq C|x|^{-2}$ and $|\mathbf{q}(x)| \leq C|x|^{-2}$, noting that $\mathbf{g}(y) = 0$ for $|y| > b$, for $|x| > 2b$ we have

$$|\mathbf{v}(x)| \leq C\int_{|y|<b}\frac{|\mathbf{g}(y)|}{|x-y|}\,dy \leq C_b|x|^{-1}\int_{|y|<b}|\mathbf{g}(y)|\,dy \leq C_b|x|^{-1}\|\mathbf{g}\|_{\mathrm{L}_q(\mathbb{R}^3)};$$

$$|\nabla\mathbf{v}(x)| \leq C\int_{|y|<b}\frac{|\mathbf{g}(y)|}{|x-y|^2}\,dy \leq C_b|x|^{-2}\int_{|y|<b}|\mathbf{g}(y)|\,dy \leq C_b|x|^{-2}\|\mathbf{g}\|_{\mathrm{L}_q(\mathbb{R}^3)}; \quad (5.13)$$

$$|\mathbf{q}(x)| \leq C\int_{|y|<b}\frac{|\mathbf{g}(y)|}{|x-y|^2}\,dy \leq C_b|x|^{-2}\int_{|y|<b}|\mathbf{g}(y)|\,dy \leq C_b|x|^{-2}\|\mathbf{g}\|_{\mathrm{L}_q(\mathbb{R}^3)}.$$

In particular, setting $(B_{2b})^c = \mathbb{R}^3 \setminus B_{2b}$, we conclude

$$\|\mathbf{v}\|_{\mathrm{L}_q((B_{2b})^c)} + \|\nabla\mathbf{v}\|_{\mathrm{L}_q((B_{2b})^c)} + \|\mathbf{q}\|_{\mathrm{L}_q((B_{2b})^c)} \leq C_{b,q}\|\mathbf{g}\|_{\mathrm{L}_q(\mathbb{R}^3)},$$

because $3 < q < \infty$, which, combined with (5.11) and (5.12), yields that

$$\|\mathbf{v}\|_{\mathrm{H}_q^2(\mathbb{R}^3)} + \|\mathbf{q}\|_{\mathrm{H}_q^1(\mathbb{R}^3)} \leq C_{q,b}\|\mathbf{g}\|_{\mathrm{L}_q(\mathbb{R}^3)}.$$

By Sobolev's inequality, we have

$$\sup_{|x|\leq 2b}(1+|x|)|\mathbf{v}(x)| + \sup_{|x|\leq 2b}(1+|x|)^2|\nabla\mathbf{v}(x)| + \sup_{|x|\leq 2b}(1+|x|)^2|\mathbf{q}(x)|$$

$$\leq C(\|\mathbf{v}\|_{\mathrm{H}_q^2(\mathbb{R}^3)} + \|\mathbf{q}\|_{\mathrm{H}_q^1(\mathbb{R}^3)})$$

$$\leq C_{q,b}\|\mathbf{g}\|_{\mathrm{L}_q(\mathbb{R}^3)},$$

which, combined with (5.13), yields that

$$<\mathbf{v}>_1 + <\nabla\mathbf{v}>_2 + <\mathbf{q}>_2 \leq C_{q,b}\|\mathbf{g}\|_{\mathrm{L}_q(\mathbb{R}^3)}.$$

This completes the proof of the second part of Lemma 5.4.   □

Proof of Proposition 5.3 To construct a solution operator for problem (5.6), we follow the method from [64] and first consider the case where $\mathbf{f} \in \mathrm{L}_{q,3b}(\Omega)^3$. Firstly, let $\mathbf{f}_0$ be the zero extension of $\mathbf{f}$ to the complement of Ω, that is, we set $\mathbf{f}_0(x) = \mathbf{f}(x)$ for $x \in \Omega$ and $\mathbf{f}_0(x) = 0$ for $x \notin \Omega$. Let $\mathcal{T}_0\mathbf{f}_0 = \mathbf{U} * \mathbf{f}_0$ and $P_0\mathbf{f}_0 = \mathbf{q} * \mathbf{f}_0$. Secondly, let $\mathbf{f}_b$ be the restriction of $\mathbf{f} \in \mathrm{L}_{q,3b}(\Omega)^3$ to Ω_{4b}, and let $\mathcal{A}_0$ and $\mathcal{B}_0$ be the operators acting on $\mathbf{f}_b \in \mathrm{L}_q(\Omega_{4b})^3$ such that $\mathcal{A}_0\mathbf{f}_b \in \mathrm{H}_q^2(\Omega_{4b})^3$, $\mathcal{B}_0\mathbf{f}_b \in \hat{\mathrm{H}}_q^1(\Omega_{4b})$ satisfy the equations

$$-\mu\Delta\mathcal{A}_0\mathbf{f}_b + \nabla\mathcal{B}_0\mathbf{f}_b = \mathbf{f}_b, \quad \mathrm{div}\,\mathcal{A}_0\mathbf{f} = 0 \quad \text{in } \Omega_{4b}, \quad \mathcal{A}_0\mathbf{f}_b|_\Gamma = \mathcal{A}_0\mathbf{f}_b|_{S_{4b}} = 0, \quad (5.14)$$

and possess the estimate

$$\|\mathcal{A}_0\mathbf{f}_b\|_{\mathrm{H}_q^2(\Omega_{4b})} + \|\mathcal{B}_0\mathbf{f}_b\|_{\hat{\mathrm{H}}_q^1(\Omega_{4b})} \leq C\|\mathbf{f}_b\|_{\mathrm{L}_q(\Omega_{4b})}. \quad (5.15)$$

Since $\mathcal{B}_0\mathbf{f}_b$ is only defined up to a constant, choosing a constant suitably, we may assume that

$$\int_{\Omega_{4b}}(P_0\mathbf{f}_0 - \mathcal{B}_0\mathbf{f}_b)\,dx = 0. \quad (5.16)$$

In what follows, let φ be a function in $C^\infty(\mathbb{R}^3)$ that equals 1 for $x \in B_{2b}$ and 0 for $x \notin B_{3b}$, and let $\mathbb{B}$ be the Bogovskiĭ operator [6, 7]. For $\mathbf{f} \in L_{q,3b}(\Omega)^3$, let

$$\mathcal{V}_0\mathbf{f} = (1-\varphi)\mathcal{T}_0\mathbf{f}_0 + \varphi\mathcal{A}_0\mathbf{f}_b + \mathbb{B}[(\nabla\varphi)(\mathcal{T}_0\mathbf{f}_0 - \mathcal{A}_0\mathbf{f}_b)], \quad \mathcal{W}_0\mathbf{f} = (1-\varphi)P_0\mathbf{f}_0 + \varphi\mathcal{B}_0\mathbf{f}_b. \quad (5.17)$$

Inserting these formulas into equations (5.6), we have

$$-\mu\Delta\mathcal{V}_0\mathbf{f} + \nabla\mathcal{W}_0\mathbf{f} = \mathbf{f} + \mathcal{R}_1\mathbf{f}, \quad \operatorname{div}\mathcal{V}_0\mathbf{f} = 0 \quad \text{in } \Omega, \quad \mathcal{V}_0\mathbf{f}|_\Gamma = 0, \quad (5.18)$$

where

$$\mathcal{R}_1\mathbf{f} = 2\mu(\nabla\varphi) \cdot (\nabla\mathcal{T}_0\mathbf{f}_0 - \nabla\mathcal{A}_0\mathbf{f}_b) + \mu(\Delta\varphi)(\mathcal{T}_0\mathbf{f}_0 - \mathcal{A}_0\mathbf{f}_b)$$
$$- \mu\Delta\mathbb{B}[(\nabla\varphi) \cdot (\mathcal{T}_0\mathbf{f}_0 - \mathcal{A}_0\mathbf{f}_b)] - (\nabla\varphi)(P_0\mathbf{f}_0 - \mathcal{B}_0\mathbf{f}_b).$$

Employing the same arguments as in [34] and [37], we shall show in Lemma 5.5 below that the inverse operator $(I + \mathcal{R}_1)^{-1} \in \mathscr{L}(L_{q,3b}(\Omega)^3)$ exists and

$$\|(I + \mathcal{R}_1)^{-1}\mathbf{f}\|_{L_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)} \quad (5.19)$$

for any $\mathbf{f} \in L_{q,3b}(\Omega)^3$. Postponing proving (5.19), we continue the proof of Proposition 5.3. Setting

$$\mathcal{U}_0\mathbf{f} := \mathcal{V}_0(I + \mathcal{R}_1)^{-1}\mathbf{f}$$
$$= (1-\varphi)\mathcal{T}_0((I + \mathcal{R}_1)^{-1}\mathbf{f})_0 + \varphi\mathcal{A}_0((I + \mathcal{R}_1)^{-1}\mathbf{f})_b$$
$$+ \mathbb{B}[(\nabla\varphi)(\mathcal{T}_0((I + \mathcal{R}_1)^{-1}\mathbf{f})_0 - \mathcal{A}_0((I + \mathcal{R}_1)^{-1}\mathbf{f}))_b],$$
$$\mathcal{Q}_0\mathbf{f} := \mathcal{W}_0(I + \mathcal{R}_1)^{-1}\mathbf{f} = (1-\varphi)P((I + \mathcal{R}_1)^{-1}\mathbf{f})_0 + \varphi\mathcal{B}_0((I + \mathcal{R}_1)^{-1}\mathbf{f})_b,$$

we see that $(\mathbf{u}, \mathfrak{p}) = (\mathcal{U}_0\mathbf{f}, \mathcal{Q}_0\mathbf{f})$ is a solution to problem (5.6). Moreover, by (5.19), Lemma 5.4 (2) and (5.15), we have

$$\sup_{x\in\Omega}(1 + |x|)|\mathcal{U}_0\mathbf{f}(x)| \le C\|\mathbf{f}\|_{L_q(\Omega)}, \qquad \sup_{x\in\Omega}(1 + |x|)^2|\nabla\mathcal{U}_0\mathbf{f}(x)| \le C\|\mathbf{f}\|_{L_q(\Omega)},$$
$$\sup_{x\in\Omega}(1 + |x|)^2|\mathcal{Q}_0\mathbf{f}(x)| \le C\|\mathbf{f}\|_{L_q(\Omega)}, \qquad \|\mathcal{U}_0\mathbf{f}\|_{H^2_q(\Omega)} + \|\mathcal{Q}_0\mathbf{f}\|_{H^1_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)}. \quad (5.20)$$

We now consider the case where $\mathbf{f} = \operatorname{div}\mathbf{F} + \mathbf{g}$ with $< \operatorname{div}\mathbf{F} >_3 + < \mathbf{F} >_2 < \infty$ and $\mathbf{g} \in L_{q,3b}(\Omega)$. We write $\mathbf{f} = \operatorname{div}((1-\varphi)\mathbf{F}) + \mathbf{h}$ with $\mathbf{h} = \varphi\operatorname{div}\mathbf{F} + (\nabla\varphi) \cdot \mathbf{F} + \mathbf{g}$. Let

$$\mathbf{u} = (1-\varphi)\mathcal{T}_0\mathbf{f}_0 + \mathbb{B}[(\nabla\varphi)\mathcal{T}_0\mathbf{f}_0], \quad \mathfrak{p} = (1-\varphi)P_0\mathbf{f}_0.$$

Notice that $\mathbf{f}_0 = \operatorname{div}((1-\varphi)\mathbf{F}) + \mathbf{h}_0$. We see that $\mathbf{u}$ and $\mathfrak{p}$ satisfy the equations:

$$-\mu\Delta\mathbf{u} + \nabla\mathfrak{p} = (1-\varphi)\mathbf{f}_0 + \mathcal{R}_2\mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = 0,$$

where we have set

$$\mathcal{R}_2\mathbf{f} = 2\mu(\nabla\varphi) \cdot \nabla\mathcal{T}_0\mathbf{f}_0 + \mu(\Delta\varphi)\mathcal{T}_0\mathbf{f}_0 - \mu\Delta\mathbb{B}[(\nabla\varphi) \cdot \mathcal{T}_0\mathbf{f}_0] - (\nabla\varphi)P_0\mathbf{f}_0.$$

Moreover, by Lemma 5.4, we have

$$\sup_{x\in\Omega}(1+|x|)|\mathbf{u}(x)| + \sup_{x\in\Omega}(1+|x|)^2|\nabla\mathbf{u}(x)| + \sup_{x\in\Omega}(1+|x|)^2|\mathfrak{p}(x)| + \|\mathbf{u}\|_{\mathrm{H}^2_q(\Omega)} + \|\mathfrak{p}\|_{\mathrm{H}^1_q(\Omega)}$$

$$\leq C(< \operatorname{div}\mathbf{F} >_3 + < \mathbf{F} >_2 + \|\mathbf{g}\|_{\mathrm{L}_q(\Omega)}).$$

Notice that

$$\|\mathcal{R}_2\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \leq C(< \operatorname{div}\mathbf{F} >_3 + < \mathbf{F} >_2 + \|\mathbf{g}\|_{\mathrm{L}_q(\Omega)})$$

and supp $\mathcal{R}_2\mathbf{f} \subset B_{3b}$. Thus, if we define operators $\mathcal{S}_0$ and $\mathcal{P}_0$ acting on $\mathbf{f}$ by setting

$$\mathcal{S}_0\mathbf{f} = (1-\varphi)\mathcal{T}_0\mathbf{f}_0 + \mathbb{B}[(\nabla\varphi)\cdot\mathcal{T}_0\mathbf{f}_0] + \mathcal{U}_0(\varphi\mathbf{f} + \mathcal{R}_2\mathbf{f}),$$

$$\mathcal{P}_0\mathbf{f} = (1-\varphi)P_0\mathbf{f}_0 + \mathcal{Q}_0(\varphi\mathbf{f} + \mathcal{R}_2\mathbf{f}),$$

then, $\mathbf{u} = \mathcal{S}_0\mathbf{f}$ and $\mathfrak{p} = \mathcal{P}_0\mathbf{f}$ satisfy (5.6), and combining the estimates above gives that

$$\sup_{x\in\Omega}(1+|x|)|(\mathcal{S}_0\mathbf{f})(x)| + \sup_{x\in\Omega}(1+|x|)^2|(\nabla\mathcal{S}_0\mathbf{f})(x)| + \sup_{x\in\Omega}(1+|x|)^2|(\mathcal{P}_0\mathbf{f})(x)|$$

$$+ \|\mathcal{S}_0\mathbf{f}\|_{\mathrm{H}^2_q(\Omega)} + \|\mathcal{P}_0\mathbf{f}\|_{\mathrm{H}^1_q(\Omega)} \leq C(< \operatorname{div}\mathbf{F} >_3 + < \mathbf{F} >_2 + \|\mathbf{g}\|_{\mathrm{L}_q(\Omega)}),$$

which completes the proof of Proposition 5.3.   □

Lemma 5.5 *The operator* $\mathrm{I} + \mathcal{R}_1 : \mathrm{L}_{q,3b}(\Omega)^3 \to \mathrm{L}_{q,3b}(\Omega)^3$ *is invertible and there exists* $C > 0$ *such that* (5.19) *holds for any* $\mathbf{f} \in \mathrm{L}_{q,3b}(\Omega)^3$.

Proof In view of (5.16) and $\nabla(P_0\mathbf{f}_0 - \mathcal{B}_0\mathbf{f}_b) \in \mathrm{L}_q(\Omega_{4b})^3$, we have $(P_0\mathbf{f}_0 - \mathcal{B}_0\mathbf{f}_b) \in \mathrm{H}^1_q(\Omega_{4b})$. Thus, $\mathcal{R}_1\mathbf{f} \in \mathrm{H}^1_q(\Omega)^3$ and supp $\mathcal{R}_1\mathbf{f} \subset D_{2b,3b}$, where $D_{2b,3b} = \{x \in \mathbb{R}^3 \mid 2b \leq |x| \leq 3b\}$. Thus, by Rellich's compactness theorem, $\mathcal{R}_1$ is a compact operator on $\mathrm{L}_{q,3b}(\Omega)^3$. Let Ker$(\mathrm{I} + \mathcal{R}_1) = \{\mathbf{f} \in \mathrm{L}_{q,3b}(\Omega)^3 \mid (\mathrm{I}+\mathcal{R}_1)\mathbf{f} = 0\}$. By Fredholm's alternative principle, if Ker$(\mathrm{I}+\mathcal{R}_1) = \{0\}$, then $\mathrm{I} + \mathcal{R}_1$ is invertible, and therefore we have (5.19). To verify this, we choose $\mathbf{f} \in$ Ker$(\mathrm{I} + \mathcal{R}_1)$ arbitrarily, and we shall show that $\mathbf{f} = 0$. Since $(\mathrm{I} + \mathcal{R}_1)\mathbf{f} = 0$, we have $\mathbf{f} = -\mathcal{R}_1\mathbf{f} \in \mathrm{H}^1_q(\Omega)$ and supp $\mathbf{f} \subset D_{2b,3b}$. Let $\mathbf{u} = \mathcal{V}_0\mathbf{f}$ and $\mathfrak{p} = \mathcal{W}_0\mathbf{f}$. Then by (5.18) we see that

$$-\mu\Delta\mathbf{u} + \nabla\mathfrak{p} = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = 0. \tag{5.21}$$

Since $3 < q < \infty$ and $\mathbf{u} \in \mathrm{H}^2_q(\Omega)^3$ and $\mathfrak{p} \in \hat{\mathrm{H}}^1_q(\Omega)$, we have $\mathbf{u} \in \mathrm{H}^2_{2,\mathrm{loc}}(\Omega)$ and $\mathfrak{p} \in \mathrm{H}^1_{2,\mathrm{loc}}(\Omega)$. Let ψ be a $C^\infty(\mathbb{R}^3)$ function which equals 1 for $|x| < 1$ and 0 for $|x| > 2$ and set $\psi_R(x) = \psi(x/R)$ for $R > 4b$. From (5.21) it follows that

$$0 = (-\mu\Delta\mathbf{u} + \nabla\mathfrak{p}, \psi_R\mathbf{u}) = \mu(\nabla\mathbf{u}, \psi_R\nabla\mathbf{u}) + \mu(\nabla\mathbf{u}, (\nabla\psi_R)\mathbf{u}) - (\mathfrak{p}, (\nabla\psi_R)\cdot\mathbf{u}). \tag{5.22}$$

Using Lemma 5.4 (2), we obtain

$$|\mathbf{u}(x)| \leq C|x|^{-1}, \quad |\nabla\mathbf{u}(x)| \leq C|x|^{-2}, \qquad |\mathfrak{p}| \leq C|x|^{-2}$$

for $|x| > 4b$, and so we have

$$|(\nabla\mathbf{u}, (\nabla\psi_R)\mathbf{u})| \leq C\|\nabla\psi\|_{L_\infty(\mathbb{R}^3)}R^{-1}\int_{2R\leq|x|\leq3R}|x|^{-3}\,dx = O(R^{-1}) \to 0,$$

$$|(\mathfrak{p}, (\nabla \psi_R)\mathbf{u})| \le C \|\nabla \psi\|_{L_\infty(\mathbb{R}^3)} R^{-1} \int_{2R \le |x| \le 3R} |x|^{-3} \, \mathrm{d}x = O(R^{-1}) \to 0$$

as $R \to \infty$, and so taking $R \to \infty$ in (5.22), we have $\|\nabla \mathbf{u}\|_{L_2(\Omega)} = 0$, which implies that $\mathbf{u}$ is a constant vector. But, $\mathbf{u}|_\Gamma = 0$, and so $\mathbf{u} = 0$. Thus, by the first equation of (5.21), $\nabla \mathfrak{p} = 0$, which shows that $\mathfrak{p}$ is a constant. But, $\mathfrak{p}(x) = O(|x|^{-2})$ as $|x| \to \infty$, and so $\mathfrak{p} = 0$. Therefore, by (5.17) we have

$$\begin{aligned} (1-\varphi)\mathcal{T}_0\mathbf{f}_0 + \varphi\mathcal{A}_0\mathbf{f}_b + \mathbb{B}[(\nabla\varphi)\cdot(\mathcal{T}_0\mathbf{f}_0 - \mathcal{A}_0\mathbf{f}_b)] &= 0, \\ (1-\varphi)P_0\mathbf{f}_0 + \varphi\mathcal{B}_0\mathbf{f}_b &= 0 \end{aligned} \tag{5.23}$$

in Ω_{4b}. Since $\mathbb{B}[(\nabla\varphi)\cdot(\mathcal{T}_0\mathbf{f}_0 - \mathcal{A}_0\mathbf{f}_b)]$ vanishes for $x \notin D_{2b,3b}$ and $\varphi(x) = 0$ for $|x| > 3b$ and $1 - \varphi(x) = 0$ for $|x| < 2b$, we have

$$\mathcal{A}_0\mathbf{f}_b = 0, \quad \mathcal{B}_0\mathbf{f}_b = 0 \quad \text{for } |x| < 2b, \quad \mathcal{T}_0\mathbf{f}_0 = 0, \quad P_0\mathbf{f}_0 = 0 \quad \text{for } |x| > 3b. \tag{5.24}$$

Let

$$\mathbf{w}(x) = \begin{cases} (\mathcal{A}_0\mathbf{f}_b)(x) & \text{for } x \in \Omega_{4b}, \\ 0 & \text{for } x \notin \Omega, \end{cases} \qquad \mathfrak{q}(x) = \begin{cases} (\mathcal{B}_0\mathbf{f}_b)(x) & \text{for } x \in \Omega_{4b}, \\ 0 & \text{for } x \notin \Omega, \end{cases}$$

and then, by (5.15) and (5.24) $\mathbf{w} \in H_q^2(B_{4b})^3$ and $\mathfrak{q} \in H_q^1(B_{4b})$, and $\mathbf{w}$ and $\mathfrak{q}$ satisfy equations:

$$-\mu\Delta\mathbf{w} + \nabla\mathfrak{q} = \mathbf{f}_0, \quad \mathrm{div}\,\mathbf{w} = 0 \quad \text{in } B_{4b}, \quad \mathbf{w}|_{S_{4b}} = 0. \tag{5.25}$$

On the other hand, by (5.24), we know that $\mathcal{T}_0\mathbf{f}_0$ and $P_0\mathbf{f}_0$ also satisfy equations (5.25), and so the uniqueness of solutions yields that $\mathbf{w} = \mathcal{T}_0\mathbf{f}_0$ and $\nabla(\mathfrak{q} - P_0\mathbf{f}_0) = 0$ in B_{4b}. Noting that $\mathfrak{q} = \mathcal{B}_0\mathbf{f}_b$, by (5.16) we have $\mathfrak{q} = P_0\mathbf{f}_0$. In particular, $(\nabla\varphi)\cdot(\mathcal{T}_0\mathbf{f}_0 - \mathcal{A}_0\mathbf{f}_b) = 0$. Thus, from (5.23) we even have

$$\begin{aligned} 0 &= \mathcal{T}_0\mathbf{f}_0 - \varphi(\mathcal{T}_0\mathbf{f}_0 - \mathcal{A}_0\mathbf{f}_b) = \mathcal{T}_0\mathbf{f}_0, \\ 0 &= P_0\mathbf{f}_0 - \varphi(P_0\mathbf{f}_0 - \mathcal{B}_0\mathbf{f}_b) = P_0\mathbf{f}_0, \end{aligned}$$

which gives that $\mathbf{f} = -\mu\Delta\mathcal{T}_0\mathbf{f}_0 + \nabla P_0\mathbf{f}_0 = 0$ in Ω_{4b}. Thus, we have $\mathbf{f} = 0$. This completes the proof of existence of $(I + \mathcal{R}_1)^{-1}$. $\square$

5.3 Purely Oscillatory Solutions to the Stokes Problem

In this section we consider the oscillatory part of the linearization (5.4) for $\mathbf{h} = 0$, which is given by

$$\partial_t\mathbf{v}_\perp - \mu\Delta\mathbf{v}_\perp + \nabla\mathfrak{p}_\perp = \mathbf{f}_\perp, \quad \mathrm{div}\,\mathbf{v}_\perp = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}_\perp|_\Gamma = 0, \tag{5.26}$$

where the subscript $\perp$ indicates that all functions have vanishing time mean. We shall prove the following result.

Proposition 5.6 *Let* $1 < p, q < \infty$. *Then, for any* $\mathbf{f}_\perp \in L_p(\mathbb{T}, L_q(\Omega))$ *with* $\int_\mathbb{T} \mathbf{f}_\perp(\cdot, s)\,\mathrm{d}s = 0$, *problem* (5.26) *admits a solution* $(\mathbf{v}_\perp, \mathfrak{p}_\perp)$ *with*

$$\mathbf{v}_\perp \in H_p^1(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, H_q^2(\Omega)^3), \quad \mathfrak{p}_\perp \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)),$$

$$\int_\mathbb{T} \mathbf{v}_\perp(\cdot, s)\,\mathrm{d}s = 0, \quad \int_\mathbb{T} \mathfrak{p}_\perp(\cdot, s)\,\mathrm{d}s = 0,$$

possessing the estimate

$$\|\partial_t \mathbf{v}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{v}_\perp\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\nabla \mathfrak{p}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C \|\mathbf{f}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))}. \qquad (5.27)$$

Here, $\mathbf{v}_\perp$ is unique in the above function class while $\mathfrak{p}_\perp$ is unique up to addition by a function only depending on time.

In order to prove Proposition 5.6, we first consider the corresponding resolvent problem

$$\lambda \mathbf{w} - \mu \Delta \mathbf{w} + \nabla \mathfrak{r} = \mathbf{f}, \quad \operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega, \quad \mathbf{w}|_\Gamma = 0, \qquad (5.28)$$

for which we have the results from Lemma 4.2. Observe that, with regard to the time-periodic problem (5.4), we are mainly interested in the resolvent problem (5.28) with $\lambda = ik$ for $k \in \mathbb{Z}$, and Lemma 4.2 gives a framework where this problem is uniquely solvable, but merely for $k \in \mathbb{Z} \setminus \{0\}$. This is the main reason why we only consider the purely oscillatory problem (5.26) in Proposition 5.6. Apart from this, Proposition 5.6 can be proved in the same way as Proposition 4.4.

Proof of Proposition 5.6 Let λ_0 be as in Lemma 4.2, and let $\varphi = \varphi(\sigma)$ be a $C^\infty(\mathbb{R})$ function that equals 1 for $|\sigma| \geq \lambda_0 + 1/2$ and 0 for $|\sigma| \leq \lambda_0 + 1/4$. Using the operator families $\mathscr{S}$ and $\mathscr{P}$ from Lemma 4.2, we set

$$\mathbf{v}_h = \mathscr{F}_{\mathbb{T}}^{-1}[\mathscr{S}(ik)\varphi(k)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k)], \quad \mathfrak{p}_h = \mathscr{F}_{\mathbb{T}}^{-1}[\mathscr{P}(ik)\varphi(k)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k)].$$

Then $\mathbf{v}_h$ and $\mathfrak{p}_h$ satisfy the equations

$$\partial_t \mathbf{v}_h - \mu \Delta \mathbf{v}_h + \nabla \mathfrak{p}_h = \mathbf{H}_h, \quad \operatorname{div} \mathbf{v}_h = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}_h|_\Gamma = 0,$$

where we have set $\mathbf{H}_h = \mathscr{F}_{\mathbb{T}}^{-1}[\varphi(k)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k)]$. Moreover, arguing as in the proof of Theorem 3.1, we can use the $\mathscr{R}$-bounds from Lemma 4.2 and employ Corollary 2.3 to deduce

$$\|\partial_t \mathbf{v}_h\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{v}_h\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|\nabla \mathfrak{p}_h\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C \|\mathbf{H}_h\|_{L_p(\mathbb{T}, L_q(\Omega))}$$
$$\leq C \|\mathbf{H}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))}. \qquad (5.29)$$

Now, in view of Lemma 4.2, we set

$$\mathbf{v}_\perp(t) = \mathbf{v}_h(t) + \sum_{0 < |k| \leq \lambda_0} e^{ikt} \mathscr{S}(ik)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k),$$

$$\mathfrak{p}_\perp(t) = \mathfrak{p}_h(t) + \sum_{0 < |k| \leq \lambda_0} e^{ikt} \mathscr{P}(ik)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k).$$

Then, $\mathbf{v}_\perp$ and $\mathfrak{p}_\perp$ satisfy equations (5.26), and from (4.14) and (5.29) we conclude estimate (5.27). Thus, we have shown the existence part of Proposition 5.6. The uniqueness statement follows exactly as in the proof of Proposition 4.4 noting that $\mathscr{F}_{\mathbb{T}}[\mathbf{v}_\perp](0) = 0$ and $\mathscr{F}_{\mathbb{T}}[\mathfrak{p}_\perp](0) = 0$ by assumption. □

Next we examine the pointwise decay of the solution $(\mathbf{v}_\perp, \mathfrak{p}_\perp)$. More precisely, we show decay properties of $\|\mathbf{v}_\perp(x, \cdot)\|_{L_p(\mathbb{T})}$ with respect to the x-variable, as stated in the following lemma.

Lemma 5.7 *In the situation of Proposition* 5.6, *let* $3 < q < \infty$ *and* $\ell \in (0, 3]$ *such that* $\mathbf{f}_\perp = \mathrm{div}\,\mathbf{F}_\perp + \mathbf{g}_\perp$ *with*

$$\int_{\mathbb{T}} \mathbf{F}_\perp(x, t)\,\mathrm{d}t = 0, \qquad < \mathbf{F}_\perp >_{p,\ell} + < \mathrm{div}\,\mathbf{F}_\perp >_{p,\ell+1} < \infty,$$
$$\int_{\mathbb{T}} \mathbf{g}_\perp(x, t)\,\mathrm{d}t = 0, \qquad \mathbf{g}_\perp \in \mathrm{L}_p(\mathbb{T}, \mathrm{L}_{q,3b}(\Omega)). \tag{5.30}$$

Then, $\mathbf{v}_\perp$ *has the following asymptotics:*

$$< \mathbf{v}_\perp >_{p,\ell} + < \nabla\mathbf{v}_\perp >_{p,\ell+1} \leq C(< \mathrm{div}\,\mathbf{F}_\perp >_{p,\ell+1} + < \mathbf{F}_\perp >_{p,\ell} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))}) \tag{5.31}$$

with some constant $C > 0$.

Remark 5.8 Since $3 < q < \infty$, we have $\|\mathrm{div}\,\mathbf{F}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))} \leq C_q < \mathrm{div}\,\mathbf{F}_\perp >_{p,\ell+1}$ and so

$$\|\mathbf{f}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))} \leq C_q(< \mathrm{div}\,\mathbf{F}_\perp >_{p,\ell+1} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))}). \tag{5.32}$$

Therefore, Proposition 5.6 really shows existence for $\mathbf{f}_\perp$ as in Lemma 5.7.

To prove (5.31), we use the following theorem due to EITER and KYED [19], which collects properties of the velocity fundamental solution $\Gamma_\perp$ to (5.26), which is a tensor field $\Gamma_\perp$ such that $\mathbf{v}_\perp := \Gamma_\perp * \mathbf{H}_\perp$ is formally a solution to (5.26) for $\Omega = \mathbb{R}^3$.

Theorem 5.9 *Let*

$$\Gamma_\perp = \mathscr{F}_{\mathbb{R}^3 \times \mathbb{T}}^{-1}\left[\frac{1 - \delta_\mathbb{Z}}{\mu|\xi|^2 + ik}\left(\mathrm{I} - \frac{\xi \otimes \xi}{|\xi|^2}\right)\right]. \tag{5.33}$$

Then, it holds $\Gamma_\perp \in \mathrm{L}_q(\mathbb{R}^3 \times \mathbb{T})^{3\times 3}$ *for* $q \in (1, 5/3)$, *and* $\partial_j \Gamma_\perp \in \mathrm{L}_q(\mathbb{R}^3 \times \mathbb{T})$ *for* $q \in (1, 5/4)$, $j = 1, 2, 3$. *Moreover, for any* $\alpha \in \mathbb{N}_0^3$, $\delta > 0$ *and* $r \in [1, \infty)$, *there exists a constant* $C_{\alpha,\delta} > 0$ *such that*

$$\forall |x| \geq \delta: \quad \|\partial_x^\alpha \Gamma_\perp(x, \cdot)\|_{\mathrm{L}_r(\mathbb{T})} \leq \frac{C_{\alpha,\delta}}{|x|^{3+|\alpha|}}.$$

Remark 5.10 This theorem holds for any dimension $N \geq 2$ replacing $5/3$, $5/4$ and $3 + |\alpha|$ with $(N + 2)/N$, $(N + 2)/(N + 1)$ and $N + |\alpha|$, respectively.

Proof of Lemma 5.7 Since we assume that $3 < q < \infty$, by Sobolev's inequality, we have

$$\sup_{|x|\leq 4b} \|\mathbf{v}_\perp(\cdot, x)\|_{\mathrm{L}_p(\mathbb{T})} + \sup_{|x|\leq 4b} \|(\nabla\mathbf{v}_\perp)(\cdot, x)\|_{\mathrm{L}_p(\mathbb{T})} \leq C\|\mathbf{v}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{H}_q^2(\Omega))} \leq C\|\mathbf{H}_\perp\|_{\mathrm{L}_p(\mathbb{T}, \mathrm{L}_q(\Omega))}.$$

It thus remains to estimate $\mathbf{v}_\perp$ for $|x| > 4b$. To this end, recall the operator families $\mathscr{S}$ and $\mathscr{P}$ given in Lemma 4.2. As seen in the proof of Proposition 5.6, we have $\mathbf{v}_\perp = \mathscr{F}_\mathbb{T}^{-1}[\mathscr{S}(ik)\mathscr{F}_\mathbb{T}[\mathbf{H}_\perp](k)]$ and $\mathfrak{p}_\perp = \mathscr{F}_\mathbb{T}^{-1}[\mathscr{P}(ik)\mathscr{F}_\mathbb{T}[\mathbf{H}_\perp](k)]$. We shall first give a representation formula of $\mathscr{S}(ik)$ for $k \in \mathbb{Z} \setminus \{0\}$ for $|x| > 3b$, which will be used to investigate the asymptotic behavior of $\mathbf{v}_\perp$ for $|x| > 3b$. Notice that $\mathscr{S}(ik) \in \mathscr{L}(\mathrm{L}_q(\Omega)^3, \mathrm{H}_q^2(\Omega)^3)$ and $\mathscr{P}(ik) \in \mathscr{L}(\mathrm{L}_q(\Omega)^3, \hat{\mathrm{H}}_q^1(\Omega))$ satisfy the estimate

$$\|\mathscr{S}(ik)\mathbf{f}\|_{\mathrm{H}_q^2(\Omega)} + \|\nabla\mathscr{P}(ik)\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \leq C\|\mathbf{f}\|_{\mathrm{L}_q(\Omega)} \tag{5.34}$$

for $\mathbf{f} \in L_q(\Omega)^3$, where C depends solely on q and Ω. Moreover, $\mathbf{u} = \mathscr{F}_{\mathbb{T}}[\mathbf{v}_\perp](k) = \mathscr{S}(ik)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k)$ and $\mathfrak{q} = \mathscr{F}_{\mathbb{T}}[\mathfrak{p}_\perp](k) = \mathscr{P}(ik)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k)$ satisfy the equations

$$ik\mathbf{u} - \mu\Delta\mathbf{u} + \nabla\mathfrak{q} = \mathbf{f}_k, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = 0, \tag{5.35}$$

where $\mathbf{f}_k = \mathscr{F}_{\mathbb{T}}[\mathbf{f}_\perp](k)$. Let φ be a function in $C_0^\infty(\mathbb{R}^3)$ that equals 1 for $|x| < 2b$ and 0 for $|x| > 3b$. Let

$$\mathbf{w} = (1 - \varphi)\mathscr{S}(ik)\mathbf{f}_k + \mathbb{B}[(\nabla\varphi)\cdot\mathscr{S}(ik)\mathbf{f}_k], \quad \mathfrak{r} = (1-\varphi)\mathscr{P}(ik)\mathbf{f}_k. \tag{5.36}$$

Then $\mathbf{w} \in H_q^2(\mathbb{R}^3)^3$ and $\mathfrak{r} \in \hat{H}_q^1(\mathbb{R}^3)$. Moreover, by (5.35) $\mathbf{w}$ and $\mathfrak{r}$ satisfy the equations

$$ik\mathbf{w} - \mu\Delta\mathbf{w} + \nabla\mathfrak{r} = (1-\varphi)\mathbf{f}_k + \mathcal{R}_3(ik)\mathbf{f}_k, \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \mathbb{R}^3,$$

where we have set

$$\begin{aligned}
\mathcal{R}_3(\lambda)\mathbf{f} = {} & 2\mu(\nabla\varphi)\cdot\nabla\mathscr{S}(\lambda)\mathbf{f} + \mu(\Delta\varphi)\mathscr{S}(\lambda)\mathbf{f} - (\nabla\varphi)\mathscr{P}(\lambda)\mathbf{f} \\
& + (\lambda - \mu\Delta)\mathbb{B}[(\nabla\varphi)\cdot\mathscr{S}(\lambda)\mathbf{f}].
\end{aligned} \tag{5.37}$$

By the uniqueness of solutions to the Stokes resolvent problem in $\mathbb{R}^3$, we have $\mathbf{w} = \mathcal{T}(ik)((1-\varphi)\mathbf{f}_k + \mathcal{R}_3(ik)\mathbf{f}_k)$, where

$$\mathcal{T}(\lambda)\mathbf{f} = \mathscr{F}_{\mathbb{R}^3}^{-1}\left[\frac{1}{\mu|\xi|^2 + \lambda}\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)\mathscr{F}_{\mathbb{R}^3}[\mathbf{f}]\right]. \tag{5.38}$$

Since $1 - \varphi = 1$ and $\mathbb{B}[(\nabla\varphi)\cdot\mathcal{S}(ik)\mathbf{f}_k] = 0$ for $|x| > 4b$, by (5.36) we thus have

$$\mathscr{S}(ik)\mathbf{f}_k = \mathcal{T}(ik)((1-\varphi)\mathbf{f}_k) + \mathcal{T}(ik)(\mathcal{R}_3(ik)\mathbf{f}_k) \quad (|x| > 4b) \tag{5.39}$$

for any $k \in \mathbb{Z} \setminus \{0\}$. Thus, we have

$$\begin{aligned}
\mathbf{v}_\perp &= \mathscr{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k))\mathscr{S}(ik)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k)] \\
&= \mathscr{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k))\mathcal{T}(ik)\mathscr{F}_{\mathbb{T}}[(1-\varphi)\mathbf{H}_\perp](k))] \\
&\quad + \mathscr{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k))\mathcal{T}(ik)(\mathcal{R}_3(ik)\mathscr{F}_{\mathbb{T}}[\mathbf{H}_\perp](k))]
\end{aligned} \tag{5.40}$$

for $|x| > 4b$. Moreover, from Lemma 4.2 we conclude

$$\mathscr{R}_{\mathscr{L}(L_q(\Omega)^3, H_q^1(\mathbb{R}^3)^3)}(\{(\lambda\partial_\lambda)^\ell\mathcal{R}_3(\lambda) \mid \lambda \in \mathbb{R} \setminus [-\lambda_0, \lambda_0]\}) \le r_0 \quad (\ell = 0, 1), \tag{5.41}$$

$$\|\mathcal{R}_3(ik)\mathbf{f}_k\|_{H_q^1(\mathbb{R}^3)} \le r_0\|\mathbf{f}_k\|_{L_q(\Omega)} \tag{5.42}$$

for any $k \in \mathbb{Z} \setminus \{0\}$ with some constant r_0. In particular, we define $\mathcal{R}_4\mathbf{H}_\perp$ by setting $\mathcal{R}_4\mathbf{H}_\perp = \mathscr{F}_{\mathbb{T}}^{-1}[(1 - \delta_{\mathbb{Z}}(k))\mathcal{R}_3(ik)\mathbf{f}_k]$. Then, employing Corollary 2.3 in the same way as in the proof of Proposition 5.6, we see that

$$\begin{aligned}
& \operatorname{supp}\mathcal{R}_4\mathbf{H}_\perp \subset D_{2b,3b} := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} \mid 2b < |x| < 3b\}, \\
& \|\mathcal{R}_4\mathbf{H}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} \le C\|\mathbf{H}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))}.
\end{aligned} \tag{5.43}$$

Recalling that $\mathbf{H}_\perp = \operatorname{div} \mathbf{F}_\perp + \mathbf{g}_\perp$, we set $\mathbf{G} = (1 - \varphi)\mathbf{F}_\perp$ and $\mathbf{h} = (\nabla\varphi)\mathbf{F}_\perp + (1 - \varphi)\mathbf{g}_\perp + \mathcal{R}_4\mathbf{H}_\perp$. In virtue of (5.33), (5.38) and (5.40), we then have

$$
\mathbf{v}_\perp = \Gamma_\perp * (\operatorname{div}\mathbf{G}) + \Gamma_\perp * \mathbf{h}
$$
$$
= \int_\mathbb{T}\int_{\mathbb{R}^3} \Gamma_\perp(y, s)(\operatorname{div}\mathbf{G})(x - y, t - s)\,\mathrm{d}y\mathrm{d}s + \int_\mathbb{T}\int_{\mathbb{R}^3} \Gamma_\perp(y, s)\mathbf{h}(x - y, t - s)\,\mathrm{d}y\mathrm{d}s.
$$
$$(5.44)$$

for $|x| > 4b$. Set $\mathbf{v}_1 = \Gamma_\perp * (\operatorname{div}\mathbf{G})$ and $\mathbf{v}_2 = \Gamma_\perp * \mathbf{h}$. By the divergence theorem of Gauß, we write

$$
\mathbf{v}_1(x, t) = \nabla\Gamma_\perp * \mathbf{G}(x, t)
$$
$$
= \int_\mathbb{T}\int_{|y|\leq 1} \nabla\Gamma_\perp(y, s)\mathbf{G}(x - y, t - s)\,\mathrm{d}y\mathrm{d}s
$$
$$
+ \int_\mathbb{T}\int_{1\leq|y|\leq|x|/2} \nabla\Gamma_\perp(y, s)\mathbf{G}(x - y, t - s)\,\mathrm{d}y\mathrm{d}s
$$
$$
+ \int_\mathbb{T}\int_{|x|/2\leq|y|\leq 2|x|} \nabla\Gamma_\perp(y, s)\mathbf{G}(x - y, t - s)\,\mathrm{d}y\mathrm{d}s
$$
$$
+ \int_\mathbb{T}\int_{|y|\geq 2|x|} \nabla\Gamma_\perp(y, s)\mathbf{G}(x - y, t - s)\,\mathrm{d}y\mathrm{d}s.
$$

Let r_0 and r_1 be exponents such that $p < r_0 < \infty$, $r_1 \in (1, 5/4)$ and $1 + 1/r_0 = 1/r_1 + 1/p$. Then, we have Young's inequality

$$
\|f * g\|_{\mathrm{L}_{r_0}(\mathbb{T})} \leq \|f\|_{\mathrm{L}_{r_1}(\mathbb{T})}\|g\|_{\mathrm{L}_p(\mathbb{T})}
$$
$$(5.45)$$

for $f * g(t) = \int_\mathbb{T} f(s)g(t - s)\,\mathrm{d}s$. Setting $\gamma_\ell = <\mathbf{G}>_{p,\ell}$, from Theorem 5.9 we thus conclude

$$
\|\mathbf{v}_1(x, \cdot)\|_{\mathrm{L}_{r_0}(\mathbb{T})} \leq \gamma_\ell\|\nabla\Gamma_\perp\|_{\mathrm{L}_{r_1}(B_1\times\mathbb{T})}(1 + |x|)^{-\ell} + C_1\gamma_\ell\int_{1\leq|y|\leq|x|/2} |y|^{-4}\,dy(1 + |x|)^{-\ell}
$$
$$
+ C_1\gamma_\ell(|x|/2)^{-4}\int_{|z|\leq 3|x|} (1 + |z|)^{-\ell}\,dz + C_1\gamma_\ell\int_{|y|\geq 2|x|} |y|^{-4-\ell}\,dy.
$$

Noting that $p \leq r_0$ and $\gamma_\ell \leq <\mathbf{F}_\perp>_{p,\ell}$, we infer

$$
\|\mathbf{v}_1(x, \cdot)\|_{\mathrm{L}_p(\mathbb{T})} \leq C_b|x|^{-\min\{\ell, 4\}} <\mathbf{F}_\perp>_{p,\ell} \quad \text{for } |x| \geq 4b.
$$

Analogously, we write

$$
\nabla\mathbf{v}_1(x, t) = \int_\mathbb{T}\int_{|y|\leq 1} \nabla\Gamma_\perp(y, s)(\operatorname{div}\mathbf{G})(x - y, t - s)\,dyds
$$
$$
+ \int_\mathbb{T}\int_{1\leq|y|\leq|x|/2} \nabla\Gamma_\perp(y, s)(\operatorname{div}\mathbf{G})(x - y, t - s)\,dyds
$$
$$
+ \int_\mathbb{T}\int_{|x|/2\leq|y|\leq 2|x|} \nabla\Gamma_\perp(y, s)(\operatorname{div}\mathbf{G})(x - y, t - s)\,dyds
$$
$$
+ \int_\mathbb{T}\int_{|y|\geq 2|x|} \nabla^\ell\Gamma_\perp(y, s)(\operatorname{div}\mathbf{G})(x - y, t - s)\,dyds.
$$

Setting $\gamma_{\ell+1} = <\operatorname{div}\mathbf{G}>_{p,\ell+1}$, by Theorem 5.9 and (5.45) we have

$$\|\nabla\mathbf{v}_1(x,\cdot)\|_{\mathrm{L}_{r_0}(\mathbb{T})} \le \gamma_{\ell+1}\|\nabla\Gamma_\ell\|_{\mathrm{L}_{r_1}(B_1\times\mathbb{T})}(1+|x|)^{-\ell-1}$$

$$+ C_1\gamma_{\ell+1}\int_{1\le|y|\le|x|/2}|y|^{-4}\,dy(1+|x|)^{-\ell-1}$$

$$+ C_1\gamma_{\ell+1}(|x|/2)^{-4}\int_{|z|\le3|x|}(1+|z|)^{-\ell-1}\,dz$$

$$+ C_1\gamma_{\ell+1}\int_{|y|\ge2|x|}|y|^{-5-\ell}\,dy.$$

Since we have

$$<\operatorname{div}\mathbf{G}>_{p,\ell+1} \le <\operatorname{div}\mathbf{F}_\perp>_{p,\ell+1} + <(\nabla\varphi)\mathbf{F}_\perp>_{p,\ell+1}$$

$$\le <\operatorname{div}\mathbf{F}>_{p,\ell+1} + \|\nabla\varphi\|_{\mathrm{L}_\infty(\mathbb{R}^3)}3b<\mathbf{F}>_{p,\ell}$$

and $p\le r_0$, we thus obtain

$$\|\nabla\mathbf{v}_1(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})} \le C_b|x|^{-\min\{\ell+1,4\}}(<\operatorname{div}\mathbf{F}_\perp>_{p,\ell+1} + <\mathbf{F}_\perp>_{p,\ell}) \quad \text{for } |x|\ge 4b.$$

Finally, we use that $\mathbf{h}(y,s)$ vanishes for $|y|\ge 3b$. For $m=0,1$ we thus have

$$\nabla^m\mathbf{v}_2(x,t) = \int_{\mathbb{T}}\int_{|x-y|\le3b}\nabla^m\Gamma_\perp(y,s)\mathbf{h}(x-y,t-s)\,dy\,ds$$

Since $|x|\ge 4b$ and $|x-y|\le 3b$ implies $|y|\ge|x|/4\ge b$, by Theorem 5.9 and (5.45), we deduce

$$\|\nabla^m\mathbf{v}_2(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})} \le \int_{|x-y|\le3b}\|\nabla^m\Gamma_\perp(y,\cdot)\|_{\mathrm{L}_p(\mathbb{T})}\|\mathbf{h}(x-y,\cdot)\|_{\mathrm{L}_1(\mathbb{T})}\,dy$$

$$\le C_m|x|^{-3-m}\|\mathbf{h}\|_{\mathrm{L}_1(B_{3b}\times\mathbb{T})}.$$

Noting (5.43), we can estimate the last term as

$$\|\mathbf{h}\|_{\mathrm{L}_1(B_{3b}\times\mathbb{T})} \le C\|\mathbf{h}\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(B_{3b}))} \le C\big(<\mathbf{F}_\perp>_{p,\ell} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathcal{R}_4\mathbf{H}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\big)$$

$$\le C\big(<\mathbf{F}_\perp>_{p,\ell} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + \|\mathbf{H}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))}\big),$$

For $|x|\ge 4b$ we now conclude

$$\|\nabla^m\mathbf{v}_2(x,\cdot)\|_{\mathrm{L}_p(\mathbb{T})} \le C|x|^{-3-m}\big(<\mathbf{F}_\perp>_{p,\ell} + \|\mathbf{g}_\perp\|_{\mathrm{L}_p(\mathbb{T},\mathrm{L}_q(\Omega))} + <\operatorname{div}\mathbf{F}_\perp>_{p,\ell+1}\big)$$

in virtue of estimate (5.32). Since $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$ for $|x|\ge 4b$, this completes the proof of Lemma 5.7.   □

5.4 Existence of Periodic Solutions

The linear theory from Theorem 5.2 is now a direct consequence of Proposition 5.3 and Lemma 5.7 if $\mathbf{h}=0$. For the case of non-zero boundary data $\mathbf{h}$ we proceed similarly to the proof of Proposition 4.4.

Proof of Theorem 5.2 At first consider the case $\mathbf{h} = 0$. Let $(\mathbf{v}_S, \mathfrak{p}_S) = (\mathbf{u}, \mathfrak{p})$ be the unique solution to (5.6) with $\mathbf{F} = \mathbf{G}_S$ and $\mathbf{g} = \mathbf{g}_S$, which exists due to Proposition 5.3, and let $(\mathbf{v}_\perp, \mathfrak{p}_\perp)$ be the unique solution to (5.26), which exists due to Proposition 5.6 and Lemma 5.7. Then $\mathbf{v} = \mathbf{v}_S + \mathbf{v}_\perp$ and $\mathfrak{p} = \mathfrak{p}_S + \mathfrak{p}_\perp$ defines a solution $(\mathbf{v}, \mathfrak{p})$ to (5.4) with $\mathbf{h} = 0$, and (5.5) follows from (5.7), (5.27), (5.31) and (5.32).

To show existence for arbitrary $\mathbf{h} \in H^1_p(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H^2_q(\Omega)^N)$, we fix $\lambda_1 > \lambda_0$ with λ_0 from Lemma 4.3 and define

$$\mathbf{v}_1 = \mathscr{F}^{-1}_{\mathbb{T}}[\mathcal{S}(ik + \lambda_1)\big(0, (ik + \lambda_1)\tilde{\mathbf{h}}_k, (ik + \lambda_1)^{1/2}\tilde{\mathbf{h}}_k, \tilde{\mathbf{h}}_k\big)],$$

$$\mathfrak{p}_1 = \mathscr{F}^{-1}_{\mathbb{T}}[\mathcal{P}(ik + \lambda_1)\big(0, (ik + \lambda_1)\tilde{\mathbf{h}}_k, (ik + \lambda_1)^{1/2}\tilde{\mathbf{h}}_k, \tilde{\mathbf{h}}_k\big)],$$

where $\mathcal{S}$ and $\mathcal{P}$ are the $\mathscr{R}$-bounded solution operators from Lemma 4.3, and $\tilde{\mathbf{h}}_k = \mathscr{F}[\mathbf{h}](k)$. Then $(\mathbf{v}_1, \mathfrak{p}_1)$ is a solution to the auxiliary problem

$$\partial_t \mathbf{v}_1 + \lambda_1 \mathbf{v}_1 - \mu \Delta \mathbf{v}_1 + \nabla \mathfrak{p}_1 = 0, \quad \operatorname{div} \mathbf{v}_1 = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{v}_1|_\Gamma = \mathbf{h}|_\Gamma.$$

Following the proof of Proposition 4.4 and invoking Corollary 2.3, we further conclude

$$\mathbf{v}_1 \in H^1_p(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, H^2_q(\Omega)^N), \quad \mathfrak{p}_1 \in L_p(\mathbb{T}, \hat{H}^1_q(\Omega))$$

and the estimate

$$\|\partial_t \mathbf{v}_1\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{v}_1\|_{L_p(\mathbb{T}, H^2_q(\Omega))} + \|\nabla \mathfrak{p}_1\|_{L_p(\mathbb{T}, L_q(\Omega))}$$

$$\leq C(\|\partial_t \mathbf{h}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{h}\|_{L_p(\mathbb{T}, H^2_q(\Omega))}).$$

Now let $\varphi \in C^\infty_0(\Omega)$ such that $\varphi \equiv 1$ in B_{2b} and $\varphi \equiv 0$ in $\mathbb{R}^3 \setminus B_{3b}$. Let $D_{2b,3b} = \{x \in \mathbb{R}^N \mid 2b < |x| < 3b\}$ and

$$H^2_{q,0,a}(D_{2b,3b}) = \{f \in H^2_q(D_{2b,3b}) \mid \partial^\alpha_x f|_{S_L} = 0 \text{ for } L = 2b, 3b \text{ and } |\alpha| \leq 1,$$

$$\int_{D_{2b,3b}} f(x)\,dx = 0\}.$$

According to [67, Lemma 5], we know that $(\nabla \varphi) \cdot \mathbf{v}_1(t) \in H^2_{q,0,a}(D_{2b,3b})$ for a.a. $t \in \mathbb{R}$, and setting $\mathbf{w}_1 = \varphi \mathbf{v}_1 - \mathbb{B}[(\nabla \varphi) \cdot \mathbf{v}_1]$, we see that

$$\mathbf{w}_1 \in H^1_p(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, H^2_q(\Omega)^3), \quad \operatorname{supp} \mathbf{w}_1 \subset B_{3b} \cap \Omega, \quad \operatorname{div} \mathbf{w}_1 = 0, \quad \mathbf{w}_1|_\Gamma = \mathbf{h},$$

$$\|\partial_t \mathbf{w}_1\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{w}_1\|_{L_p(\mathbb{T}, H^2_q(\Omega))} \leq C(\|\partial_t \mathbf{h}\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{h}\|_{L_p(\mathbb{T}, H^2_q(\Omega))}).$$

$$(5.46)$$

Now let $(\mathbf{w}_2, \mathfrak{q}_2)$ be the unique solution to

$$\partial_t \mathbf{w}_2 - \Delta \mathbf{w}_2 + \nabla \mathfrak{q}_2 = \mathbf{f} - \partial_t \mathbf{w}_1 - \Delta \mathbf{w}_1, \quad \operatorname{div} \mathbf{w}_2 = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{w}_2|_\Gamma = 0,$$

which exists due to the first part of the proof. Note that $\mathbf{w}_1$ vanishes in $\mathbb{R}^3 \setminus B_{3b}$, so that $(\mathbf{v}, \mathfrak{p}) = (\mathbf{w}_1 + \mathbf{w}_2, \mathfrak{q}_2)$ is a solution to (5.4), and estimate (5.5) follows from the corresponding estimate for $\mathbf{w}_2$ and the properties listed in (5.46).

The uniqueness assertion follows by decomposing a solution $(\mathbf{v}, \mathfrak{p})$ into a stationary and an oscillatory part by means of (5.2) and employing the uniqueness statements from Proposition 5.3 and Proposition 5.6. $\qquad \square$

Finally, we complete the proof of Theorem 5.1 on the existence of time-periodic solutions to the nonlinear problem (5.1) in an exterior domain. Observe that the abstract result from Theorem 3.3 cannot be applied directly since we use spaces of pointwise decaying functions that allow for an appropriate treatment of the nonlinear convection term.

Proof of Theorem 5.1 We employ Banach's contraction mapping principle. Define

$$\mathcal{I}_\varepsilon = \{(\mathbf{v},\mathfrak{q}) \mid \mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_S, \quad \mathfrak{q} = \mathfrak{q}_\perp + \mathfrak{q}_S, \quad \mathbf{v}_\perp \in H_p^1(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, H_q^2(\Omega)^3),$$

$$\mathbf{v}_S \in H_q^2(\Omega)^3, \quad \operatorname{div}\mathbf{v} = 0, \quad \mathfrak{q}_\perp \in L_p(\mathbb{T}, \hat{H}_q^1(\Omega)), \quad \mathfrak{q}_S \in H_q^1(\Omega),$$

$$\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\varepsilon} \le \varepsilon\},$$

where we set

$$\|(\mathbf{v},\mathfrak{q})\|_{\mathcal{I}_\varepsilon} = \|\partial_t \mathbf{v}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{v}_\perp\|_{L_p(\mathbb{T},H_q^2(\Omega))} + \|\mathbf{v}_S\|_{H_q^2(\Omega)} + \|\nabla\mathfrak{q}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))}$$

$$+ \|\mathfrak{q}_S\|_{H_q^1(\Omega)} + <\mathbf{v}_\perp>_{p,1} + <\nabla\mathbf{v}_\perp>_{p,2} + <\mathbf{v}_S>_1 + <\nabla\mathbf{v}_S>_2.$$

For $(\mathbf{v},\mathfrak{q}) \in \mathcal{I}_\varepsilon$, let $(\mathbf{u},\mathfrak{p})$ be the solution of the linear system of equations

$$\partial_t \mathbf{u} - \mu\Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{F} + \mathcal{N}(\mathbf{v}), \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}|_\Gamma = \mathbf{h}|_\Gamma, \tag{5.47}$$

where $\mathcal{N}(\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{v}$. Theorem 5.2 yields that

$$\|(\mathbf{u},\mathfrak{p})\|_{\mathcal{I}_\varepsilon} \le C(<\mathbf{G}_S>_3 + <\mathbf{H}_S>_2 + <\mathcal{N}(\mathbf{v})_S>_3 + <\tilde{\mathcal{N}}(\mathbf{v})_S>_2 + <\mathbf{G}_\perp>_{p,2}$$

$$+ <\mathbf{H}_\perp>_{p,1} + <\mathcal{N}(\mathbf{v})_\perp>_{p,2} + <\tilde{\mathcal{N}}(\mathbf{v})_\perp>_{p,1} \tag{5.48}$$

$$+ \|\mathbf{h}\|_{H_p^1(\mathbb{T},L_q(\Omega))} + \|\mathbf{h}\|_{L_p(\mathbb{T},H_q^2(\Omega))})$$

provided that the right-hand side of (5.48) is finite. Here, we write $\tilde{\mathcal{N}}(\mathbf{v}) = \mathbf{v}\otimes\mathbf{v}$, so that $\operatorname{div}\tilde{\mathcal{N}}(\mathbf{v}) = \mathcal{N}(\mathbf{v})$ since $\operatorname{div}\mathbf{v} = 0$. We further have

$$\mathcal{N}(\mathbf{v})_S = \mathbf{v}_S\cdot\nabla\mathbf{v}_S + \int_\mathbb{T} \mathbf{v}_\perp\cdot\nabla\mathbf{v}_\perp \, dt$$

$$\tilde{\mathcal{N}}(\mathbf{v})_S = \mathbf{v}_S\otimes\mathbf{v}_S + \int_\mathbb{T} \mathbf{v}_\perp\otimes\mathbf{v}_\perp \, dt;$$

$$\tag{5.49}$$

$$\mathcal{N}(\mathbf{v})_\perp = \mathbf{v}_S\cdot\nabla\mathbf{v}_\perp + \mathbf{v}_\perp\cdot\nabla\mathbf{v}_S + \mathbf{v}_\perp\cdot\nabla\mathbf{v}_\perp - \int_\mathbb{T} \mathbf{v}_\perp\cdot\nabla\mathbf{v}_\perp \, dt$$

$$\tilde{\mathcal{N}}(\mathbf{v})_\perp = \mathbf{v}_S\otimes\mathbf{v}_\perp + \mathbf{v}_\perp\otimes\mathbf{v}_S + \mathbf{v}_\perp\otimes\mathbf{v}_\perp - \int_\mathbb{T} \mathbf{v}_\perp\otimes\mathbf{v}_\perp \, dt.$$

Notice that $\operatorname{div}\tilde{\mathcal{N}}(\mathbf{v})_S = \mathcal{N}(\mathbf{v})_S$ and $\operatorname{div}\tilde{\mathcal{N}}(\mathbf{v})_\perp = \mathcal{N}(\mathbf{v})_\perp$. To estimate these nonlinear terms, we choose $\sigma > 0$ so small that $\sigma + 3/q < 2(1 - 1/p)$, which is possible since $2/p + 3/q < 2$ by assumption. By Sobolev inequality and real interpolation, we then have

$$\|\mathbf{v}_\perp\|_{L_\infty(\mathbb{T},L_\infty(\Omega))} \le C\|\mathbf{v}_\perp\|_{L_\infty(\mathbb{T},W_q^{\sigma+3/q}(\Omega))} \le C\|\mathbf{v}_\perp\|_{L_\infty(\mathbb{T},B_{q,p}^{2(1-1/p)}(\Omega))}$$

$$\le C(\|\partial_t\mathbf{v}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{v}_\perp\|_{L_p(\mathbb{T},H_q^2(\Omega))}), \tag{5.50}$$

Using Hölder's inequality with $p > 2$, we further obtain

$$< \mathcal{N}(\mathbf{v})_S >_3 \le C(< \mathbf{v}_S >_1 < \nabla \mathbf{v}_S >_2 + < \mathbf{v}_\perp >_{p,1} < \nabla \mathbf{v}_\perp >_{p,2});$$

$$< \tilde{\mathcal{N}}(\mathbf{v})_S >_2 \le C(< \mathbf{v}_S >_1^2 + < \mathbf{v}_\perp >_{p,1}^2);$$

$$< \mathcal{N}(\mathbf{v})_\perp >_{p,2} \le C(< \mathbf{v}_S >_1 < \nabla \mathbf{v}_\perp >_{p,2} + < \mathbf{v}_\perp >_{p,1} < \nabla \mathbf{v}_S >_2$$
$$+ \|\mathbf{v}_\perp\|_{L_\infty(\mathbb{T}, L_\infty(\Omega))} < \nabla \mathbf{v}_\perp >_{p,2} + < \mathbf{v}_\perp >_{p,1} < \nabla \mathbf{v}_\perp >_{p,2});$$

$$< \tilde{\mathcal{N}}(\mathbf{v})_\perp >_{p,1} \le C(< \mathbf{v}_S >_1 < \mathbf{v}_\perp >_{p,1} + \|\mathbf{v}_\perp\|_{L_\infty(\mathbb{T}, L_\infty(\Omega))} < \mathbf{v}_\perp >_{p,1} + < \mathbf{v}_\perp >_{p,1}^2). \tag{5.51}$$

Combining (5.48) with (5.51) and (5.50) yields that

$$\|(\mathbf{u}, \mathfrak{p})\|_{\mathcal{I}_\varepsilon} \le C(< \mathbf{G}_S >_3 + < \mathbf{H}_S >_2 + < \mathbf{G}_\perp >_{p,2} + < \mathbf{H}_\perp >_{p,1}$$
$$+ \|\mathbf{h}\|_{H_p^1(\mathbb{T}, L_q(\Omega))} + \|\mathbf{h}\|_{L_p(\mathbb{T}, H_q^2(\Omega))} + \|(\mathbf{v}, \mathfrak{q})\|_{\mathcal{I}_\varepsilon}^2).$$

Recalling the smallness assumption (5.3) and that $(\mathbf{v}, \mathfrak{q}) \in \mathcal{I}_\varepsilon$, we have $\|(\mathbf{u}, \mathfrak{p})\|_{\mathcal{I}_\varepsilon} \le C_0 \varepsilon^2$ for some constant C_0. Thus, choosing $\varepsilon > 0$ so small that $C_0 \varepsilon \le 1$, we have $\|(\mathbf{u}, \mathfrak{p})\|_{\mathcal{I}_\varepsilon} \le \varepsilon$, which implies that $(\mathbf{u}, \mathfrak{p}) \in \mathcal{I}_\varepsilon$. Therefore, if we define a map Ξ acting on $(\mathbf{v}, \mathfrak{q}) \in \mathcal{I}_\varepsilon$ by setting $\Xi(\mathbf{v}, \mathfrak{q}) = (\mathbf{u}, \mathfrak{p})$, then Ξ is a map from $\mathcal{I}_\varepsilon$ into itself.

In an analogous way, we see that for any $(\mathbf{v}_i, \mathfrak{q}_i) \in \mathcal{I}_\varepsilon$ $(i = 1, 2)$,

$$\|\Xi(\mathbf{v}_1, \mathfrak{q}_1) - \Xi(\mathbf{v}_2, \mathfrak{q}_2)\|_{\mathcal{I}_\varepsilon} \le C_1 \varepsilon \|(\mathbf{v}_1, \mathfrak{q}_1) - (\mathbf{v}_2, \mathfrak{q}_2)\|_{\mathcal{I}_\varepsilon}$$

for some constant C_1. Thus, choosing $\varepsilon > 0$ smaller if necessary, we have $C_1 \varepsilon < 1$, which shows that Ξ is a contraction map on $\mathcal{I}_\varepsilon$. Therefore, there exists a unique $(\mathbf{u}, \mathfrak{p}) \in \mathcal{I}_\varepsilon$ such that $\Xi(\mathbf{u}, \mathfrak{p}) = (\mathbf{u}, \mathfrak{p})$, which is the required unique solution to (4.9). This completes the proof of Theorem 5.1. $\qquad\square$

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing Interests The authors have no competing interests to declare that are relevant to the content of this article.

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