Identification of Unknown Filter in a Half-Strip

Volodymyr Dilnyi^{1,2} · Khrystyna Huk²

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Abstract We consider the analogue of the classic Wiener filtering theory to a half-strip of complex domain. In this paper all test signal obtained that solving the filter identification problem for the Hardy spaces in a half-strip. This result can be used for the investigations of electrical, optical, acoustical signals.

Keywords Signal processing · Amplitude spectrum · Convolution · Fourier transform · Hardy spaces · Filter

1 Introduction

In Wiener's mathematical filtering theory [1] a signal is a function of the continuous ($t \in \gamma$, $\gamma : R \rightarrow C$) time parameter t and a filter Φ is a device ("a box") transforming an input signal into a certain output signal, $g \rightarrow \Phi g$. The energy of a signal g is proportional to

$$\int_{\gamma} |g(z)|^2 |dz|.$$

Without entering into the physical nature (optical, electrical) of a stationary filter Φ , we consider it as a translation invariant linear operator on the corresponding L^2 space.

The function $G = \mathcal{F}^{-1}g$, where $g \rightarrow \mathcal{F}^{-1}g$ is the inverse Fourier transforms, represents what is called the amplitude spectrum of g .

✉ K. Huk
khrystyna.huk2711@gmail.com

V. Dilnyi
dilnyiiv@gmail.com

¹ Institute of Mathematics, Cracow University of Technology, Warszawska str., 24, 31-155, Cracow, Poland

² Educational and Scientific Institute of Physics, Mathematics, Economics and Innovative Technologies, Drohobych Ivan Franco State Pedagogical University, Stryiska Street, 3, Drohobych, 82100, Ukraine

The following are among the major problem of signal processing:

determine an unknown filter $\Phi : g \rightarrow \psi$ ("black box") from an analysis of g and ψ ; in particular, reconstruct, if possible, a filter knowing the energy densities $|\mathcal{F}^{-1}g|^2$, $|\mathcal{F}^{-1}f|^2$ of an input-output pair.

The last problem is a variation of M. Kac' famous question "Can one hear the shape of a drum?" [2]. Above and similar problems are studied by N. Wiener, D. Newman, B. Nyman, A. Beurling, H. Reinhard, P. Masani [3].

2 The Half-Strip Case

The complex domain provides a natural processing framework for signals with intensity and direction components (see [4–6]).

We consider the above problem for the case of an unknown filter f on the half-strip $D_\sigma = \{z : |\Im z| < \sigma, \Re z < 0\}$, $\sigma > 0$. The aim of this paper is to construct all detecting signals g on $D_\sigma^* = \mathbb{C} \setminus \overline{D_\sigma}$ under some natural conditions.

The circle of ideas surrounding signal processing and Fourier transform has a long history, and has found a number applications. J. Martinez, R. Heusdens and R. Hendriks [7] investigate the characteristics of the signal and the connections of the generalized Fourier transform to analyticity. E. Sejdic, I. Djurovic and L. Stankovic relate the Fractional Fourier transform to other mathematical transforms and discuss various approaches for practical realizations of this transform [8].

Let $E^p[D_\sigma]$ and $E^p[D_\sigma^*]$, $1 \leq p < +\infty$, $\sigma > 0$, be the (Hardy) spaces of holomorphic functions respectively in the domains D_σ and D_σ^* , for which

$$\|f\| := \sup_{\mu} \left\{ \int_{\mu} |f(z)|^p |dz| \right\}^{1/p} < +\infty,$$

where supremum is taken over all segments μ , that are contained in D_σ and D_σ^* respectively. The amplitude spectrum of a signal $g \in E^p[D_\sigma^*]$ is defined by the formula

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{\partial D_\sigma} g(w) e^{zw} dw. \quad (1)$$

Obviously, the above equality is a natural analogue of the Fourier transform for the half-strip.

The *filter identification problem* for the half-strip is to find, if possible, a test signal $g \in E^2[D_\sigma^*]$ whose output

$$f * g(\tau) = \int_{\partial D_\sigma} g(w) f(w + \tau) dw$$

measured at all time moments $\tau \leq 0$ defines uniquely an unknown filter $f \in E^2[D_\sigma]$. More precisely, the question is whether there exists $g \in E^2[D_\sigma^*]$ such that $g * f(\tau) = 0$ for all $\tau \leq 0$ implies $f \equiv 0$?

Let $H^p(\mathbb{C}_+)$, $1 \leq p < +\infty$, be the Hardy space of holomorphic in $\mathbb{C}_+ = \{z : \Re z > 0\}$ functions f , for which

$$\|f\|^p = \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x + iy)|^p dy \right\} < +\infty.$$

Theorem S [9] *The space $H^p(\mathbb{C}_+)$, $1 \leq p < +\infty$, coincides with the space of holomorphic in $\mathbb{C}_+$ functions f , for which*

$$\|f\|_*^p = \sup_{\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p dr \right\} < +\infty.$$

Let $H_\sigma^p(\mathbb{C}_+)$ be a space of holomorphic in $\mathbb{C}_+$ functions, for which

$$\|g\| := \sup_{\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})} \left\{ \int_0^{+\infty} |g(re^{i\varphi})|^p e^{-p\sigma r |\sin \varphi|} dr \right\}^{1/p} < +\infty.$$

Formula (1) establish the bijections of $E^2[D_\sigma^*]$ onto $H_\sigma^2(\mathbb{C}_+)$ and inverse formula

$$g(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} G(x) e^{-xw} dx, \quad \Re w > 0 \quad (2)$$

holds.

3 The Main Result

Theorem 1. Let the amplitude spectrum G of a signal $g \in E^2[D_\sigma^*]$ be continuous and zero-free in $\{z : \Re z \geq 0\}$, $f \in E^2[D_\sigma]$. Then

$$(\forall \tau \leq 0) \quad f * g(\tau) = 0 \quad (3)$$

implies $f \equiv 0$ if and only if one of the following conditions holds:

a) g admits a holomorphic continuation as an entire function and

$$(\forall c \in \mathbb{R}) : g(w) \exp(-ce^{-\frac{w\pi}{2\sigma}}) \notin E^2[D_\sigma]; \quad (4)$$

b) g does not admit an analytic continuation as an entire function.

Proof The integral boundary function $h : \mathbb{R} \rightarrow \mathbb{R}$ of a function $G \in H_\sigma^p(\mathbb{C}_+)$ is defined [10] up to an additive constant at points of continuity by the equality [10]

$$h(t_2) - h(t_1) = \lim_{x \rightarrow 0+} \int_{t_1}^{t_2} \ln |G(x + iy)| dy - \int_{t_1}^{t_2} \ln |G(iy)| dy.$$

Since G is continuous on $\overline{\mathbb{C}_+}$ and $G(z) \neq 0$ for all $z \in \overline{\mathbb{C}_+}$, we have $|\ln |G(z)|| \leq M_R$ for $z \in \{z : \Re z \geq 0, |z| \leq R\}$. Therefore by Fatou's lemma $h \equiv \text{const}$. Since G is zero-free in $\mathbb{C}_+$, by the criterion of solvability [11, 12] equation (3) has only trivial (zero) solution iff

$$\limsup_{x \rightarrow +\infty} \left(\frac{\ln |G(x)|}{x} + \frac{2\sigma}{\pi} \ln x \right) = +\infty \quad (5)$$

We will show that (5) implies condition a) or b). Consider the two alternatives:

$$1) \quad \lim_{x \rightarrow +\infty} \frac{\ln |G(x)|}{x} = -\infty. \quad (6)$$

From the last formula we obtain that the integral on the right hand side of (2) converges uniformly on any compact subset of $\mathbb{C}_+$, hence g is an entire function.

We suppose the contrary to (4). Since g is an analytic function in an each closed rectangle $\overline{M}_k$, $k < 0$, where $M_k = \{z : z \in D_\sigma, \Re z > k\}$. By Cauchy formula we obtain

$$\int_{\partial M_k} g(w) e^{zw} dw = 0, \quad k < 0,$$

then by (1) we have

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{\partial D_\sigma \setminus \overline{M}_k} g(w) e^{zw} dw. \quad (7)$$

Let $q(w) := g(w) \exp(-ce^{-\frac{w\sigma}{2\sigma}}) \in E^2[D_\sigma]$ for some $c > 0$.

Hence for $k < 0$ we obtain

$$\begin{aligned} |G(x)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{\partial(D_\sigma \setminus \overline{M}_k)} q(w) \exp\left(ce^{-\frac{w\pi}{2\sigma}}\right) e^{wx} dw \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\partial(D_\sigma \setminus \overline{M}_k)} |q(w)| \exp\left(ce^{-\frac{u\pi}{2\sigma}} \cos \frac{v\pi}{2\sigma}\right) e^{ux} |dw| \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^k |q(u - i\sigma)| e^{ux} du + \int_{-\sigma}^{\sigma} |q(k + iv)| \exp\left(ce^{-\frac{k\pi}{2\sigma}} \cos \frac{v\pi}{2\sigma}\right) e^{kx} dv \right. \\ &\quad \left. + \int_{-\infty}^k |q(u + i\sigma)| e^{ux} du \right) \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\frac{e^{kx}}{\sqrt{2x}} \left(\int_{-\infty}^0 |q(u - i\sigma)|^2 du \right)^{1/2} + \exp\left(ce^{-\frac{k\pi}{2\sigma}}\right) e^{kx} \sqrt{2\sigma} \right. \\ &\quad \cdot \left. \left(\int_{-\sigma}^{\sigma} |q(k + iv)|^2 dv \right)^{1/2} + \frac{e^{kx}}{\sqrt{2x}} \left(\int_{-\infty}^0 |q(u + i\sigma)|^2 du \right)^{1/2} \right). \end{aligned}$$

The last inequality is a consequence of Schwarz inequality. If $k = -\frac{2\sigma}{\pi} \ln x$, then

$$\begin{aligned} |G(x)| &\leq \frac{c_7}{\sqrt{x}} \exp\left(-\frac{2\sigma}{\pi} x \ln x\right) + c_8 e^{cx} \exp\left(-\frac{2\sigma}{\pi} x \ln x\right) \\ &\leq c_9 e^{cx} \exp\left(-\frac{2\sigma}{\pi} x \ln x\right), \quad x > 1. \end{aligned}$$

Let $\psi(z) := G(z)e^{-cz} \exp(-\frac{2\sigma}{\pi} z \ln z)$. Obviously $\psi \in L^2(\partial\mathbb{C}_+)$ and for all $\varepsilon > 0$ we have $\psi(x)e^{-\varepsilon x} \in L^2(0; +\infty)$ and by equality (1) we obtain $G \in H^2_\sigma(\mathbb{C}_+)$, hence for all $\gamma \in (1; 2]$

$$(\forall \varepsilon > 0) : \sup_{|\varphi| < \frac{\pi}{2}} \left\{ \int_0^{+\infty} |\psi(re^{i\varphi})|^2 \exp(-\varepsilon r e^\gamma) dr \right\} < +\infty.$$

From a Phragmen-Lindelof type theorem for the half-plane [13], we obtain $\psi \in H^2(\mathbb{C}_+)$. It means

$$G(z)e^{cz} \exp\left(\frac{2\sigma}{\pi} z \ln z\right) \in H^2(\mathbb{C}_+),$$

and

$$\left| G(z) \exp\left\{\frac{2\sigma}{\pi} z \ln z\right\} \right| \leq \frac{e^{cx}}{\sqrt{x}}.$$

Therefore $\ln |G(x)| + \frac{2\sigma}{\pi} x \ln x \leq cx$, if $x \geq 1$, since

$$\limsup_{x \rightarrow +\infty} \left(\frac{\ln |G(x)|}{x} + \frac{2\sigma}{\pi} \ln x \right) < +\infty. \quad (8)$$

This is a contradiction.

The second alternative is

$$2) \quad \limsup_{x \rightarrow +\infty} \frac{\ln |G(x)|}{x} > -\infty. \quad (9)$$

Let $g \in E^2_*[D_\sigma]$ be an entire function. Then g is an analytic function in each closed rectangle $\overline{M}_k$, $k < 0$, where $M_k = \{z : z \in D_\sigma, \Re z > k\}$. By the Cauchy formula we obtain

$$\int_{\partial M_k} g(w)e^{zw} dw = 0, \quad k < 0,$$

then by (7) we obtain

$$|G(x)| = \frac{1}{\sqrt{2\pi}} \int_{\partial D_\sigma \setminus \overline{M}_k} |g(w)| e^{xu} |dw| = \frac{1}{\sqrt{2\pi}} (I_1 + I_2 + I_3), \quad z = x + iy, w = u + iv,$$

for $x > 0$. Then, by Schwarz inequality

$$\begin{aligned} I_1 &= \int_{-\infty}^k |g(u - i\sigma)| e^{xu} du \leq \left(\int_{-\infty}^k |g(u - i\sigma)|^2 du \cdot \int_{-\infty}^k e^{2xu} \right)^{1/2} du \\ &\leq \left(\int_{-\infty}^0 |g(u - i\sigma)|^2 du \cdot \frac{e^{2xk}}{2x} \right)^{1/2} \leq c_2 \frac{e^{xk}}{\sqrt{x}}, \end{aligned}$$

analogously

$$I_3 = \int_{-\infty}^k |g(u + i\sigma)| e^{xu} du \leq c_3 \frac{e^{xk}}{\sqrt{x}}.$$

Further

$$I_2 = \int_{-\sigma}^{\sigma} |g(k + iv)| e^{xk} dv \leq \max_{v \in [-\sigma; \sigma]} \{|g(k + iv)|\} e^{xk} \int_{-\sigma}^{\sigma} dv \leq J(k) e^{kx},$$

where $J(k) = 2\sigma \max\{|g(t + iv)| : v \in [-\sigma; \sigma], t \in [k; 0]\}$, $k < 0$.

If $\sup_{k < 0} \{J(k)\} < +\infty$, then the function g belongs to the Hardy spaces in both domains D_{σ} and D_{σ}^* . Then $g \equiv 0$ and by (1) $G \equiv 0$ hence the theorem is proved. In the other way from the nonincreasiness of J we have $\lim_{k \rightarrow -\infty} J(k) = +\infty$. Let J_1 be defined on intervals of decreasing of J as $J_2 = J$. Then the inverse function J_1 of the function $-J$ increase on $(-\infty; 0)$ and $\lim_{s \rightarrow -\infty} J_1(s) = -\infty$. As k in (6) is an arbitrary negative number, we can choose $k = J_1(-x)$, then $I_2 \leq J(J_1(-x)) e^{xJ_1(-x)} = x e^{xJ_1(-x)}$. Hence $|G(x)| \leq c_4 x e^{xJ_1(-x)}$, $x > 1$, and we obtain

$$\lim_{x \rightarrow +\infty} \frac{\ln |G(x)|}{x} = \lim_{x \rightarrow +\infty} J_1(-x) = -\infty.$$

This is a contradiction.

Conversely, b) suppose, condition (9) is not valid. Then by (6) the integral on the right hand side of (2) converges uniformly on any compact subset of $\mathbb{C}_+$, hence g is the entire function. This is a contradiction. Condition (5) is a simple consequence of (9).

Condition a) let $G \in H_{\sigma}^2(\mathbb{C}_+)$ and (4) is valid. Then by (1) equality (2) is valid for all $w \in \mathbb{C}_+$. Let $G_1(z) = G(z) \exp(\frac{2\sigma}{\pi} z \ln z - cz) \in H^2(\mathbb{C}_+)$ for some $c > 0$. Then after a change of the line of integration from $\{x : x > 0\}$ to $\{t \exp(-\frac{(w-c)\pi}{2\sigma}) : t > 0\}$ we obtain

$$\begin{aligned} |g(w)| &= \frac{1}{\sqrt{2\pi}} \left| \int_0^{+\infty} G_1(t e^{-\frac{(w-c)\pi}{2\sigma}}) \exp\left(-\frac{2\sigma}{\pi} t \ln t e^{-\frac{(w-c)\pi}{2\sigma}}\right) e^{-\frac{(w-c)\pi}{2\sigma}} dt \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\int_0^{+\infty} |G_1(t e^{-\frac{(w-c)\pi}{2\sigma}}) e^{-\frac{(w-c)\pi}{2\sigma}}|^2 dt \cdot \int_0^{+\infty} \left| \exp\left(-\frac{4\sigma}{\pi} t \ln t e^{-\frac{(w-c)\pi}{2\sigma}}\right) \right|^2 dt \right)^{1/2} \\ &\leq c_{10} \left(e^{-\frac{(u-c)\pi}{2\sigma}} \exp\left(\frac{1}{2} \ln\left(\frac{4\sigma}{\pi} e^{-\frac{(u-c)\pi}{2\sigma}} \cos \frac{v\pi}{2\sigma}\right)\right) \right. \\ &\quad \cdot \exp\left(\exp\left(\ln\left(\frac{4\sigma}{\pi} e^{-\frac{(u-c)\pi}{2\sigma}} \cos \frac{v\pi}{2\sigma}\right) - 1\right)\right) \Big)^{1/2}. \end{aligned}$$

The last estimate follows from [14]. Then we have

$$|g(w)| \leq c_{11} e^{-\frac{2u\pi}{\sigma}} \exp\left(\frac{1}{e} \frac{2\sigma}{\pi} e^{-\frac{(u-c)\pi}{2\sigma}} \cos \frac{v\pi}{2\sigma}\right).$$

Function $q_1(w) = g(w) \exp(-c_{12} e^{-\frac{w\pi}{2\sigma}})$, where $c_{12} = \frac{2\sigma}{\pi} e^{\frac{c\pi}{2\sigma}}$, satisfy the conditions of the Phragmen–Lindelof type theorem for a half-strip (see [11]) because $g_2 \in L^2(\partial D_{\sigma})$. Then $g_2 \in E^2[D_{\sigma}]$, hence condition (5) is valid for $c = c_{12}$. $\square$

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