

On Stability of Solutions to Equations Describing Incompressible Heat-Conducting Motions Under Navier's Boundary Conditions

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Received: 19 October 2015 / Accepted: 28 June 2017 / Published online: 3 August 2017
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Abstract In this paper we prove existence of global strong-weak two-dimensional solutions to the Navier-Stokes and heat equations coupled by the external force dependent on temperature and the heat dissipation, respectively. The existence is proved in a bounded domain with the Navier boundary conditions for velocity and the Dirichlet boundary condition for temperature. Next, we prove existence of 3d global strong solutions via stability.

Keywords Incompressible heat-conducting fluid · Stability of two-dimensional solution · Navier boundary conditions

1 Introduction

We consider a complete thermodynamical model of unsteady flows of incompressible homogeneous Newtonian fluids in a fixed bounded three-dimensional cylinder. The model comprises evolutionary equations for the velocity, pressure and temperature fields. We restrict our considerations to temperature independent material coefficients, that is, the viscosity and heat conductivity coefficients. We assume that the external force depends on temperature in a very special way (see Theorem 1). The heat equation contains the dissipative heating term which follows from the general theory of incompressible heat-conducting Navier-Stokes motions. We assume that the fluid is Newtonian, the internal energy is proportional to the temperature and the heat flux is proportional to the temperature gradient.

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Let $D = \Omega \times (-a, a)$ be a bounded cylinder, where $\Omega \subset \mathbb{R}^2$ be an open domain, $L = 2a$ is the length of the cylinder which is parallel to the x_3 -axis. The three-dimensional problem under the consideration is as follows

$$\begin{aligned}
 v_t + v \cdot \nabla v - \nu \operatorname{rot}^2 v + \nabla p &= \alpha(\theta) f && \text{in } D_+ = D \times \mathbb{R}_+, \\
 \operatorname{div} v &= 0 && \text{in } D_+, \\
 \theta_t + v \cdot \nabla \theta - \varkappa \Delta \theta &= \nu |\mathbb{D}(v)|^2 && \text{in } D_+, \\
 v \cdot \bar{n} &= 0 && \text{on } S_+ = S \times \mathbb{R}_+, \\
 \bar{n} \times \operatorname{rot} v &= 0 && \text{on } S_+, \\
 \theta &= \theta_* && \text{on } S_+, \\
 v|_{t=0} &= v(0), \quad \theta|_{t=0} = \theta(0) && \text{in } D,
 \end{aligned} \tag{1}$$

where $S = \partial D$, $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the fluid, $x = (x_1, x_2, x_3)$ are given Cartesian coordinates, $p = p(x, t) \in \mathbb{R}$ is the pressure, $\theta = \theta(x, t) \in \mathbb{R}_+$ is the temperature of the fluid, $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ is the external force field. The Cartesian coordinates x are such that x_3 -axis is parallel to the cylinder and it is located inside it. By the dot we denote the scalar product in \mathbb{R}^3 .

By $\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3} = \nabla v + \nabla v^T$ we denote the dilatation tensor, $\operatorname{rot}^2 = \operatorname{rot} \operatorname{rot} = -\Delta$ which holds on any divergence free vector. Moreover, $\nu > 0$ is the constant viscosity coefficient, $\varkappa > 0$ is also the constant heat conductivity coefficient and $\theta_* > 0$ is a constant. Moreover, \bar{n} is the unit outward normal vector to S . The boundary S is split into two parts, i.e. $S = S_1 \cup S_2$, where S_1 is parallel to the x_3 -axis and S_2 is perpendicular to it. Additionally, $S_2 = S_2(-a) \cup S_2(a)$, where $S_2(b)$ meets x_3 -axis at $x_3 = b$.

The function $\alpha = \alpha(\theta)$ from the r.h.s. of (1.1)₁ will be determined later. The term $\alpha(\theta) f$ generates the motion by heating and mechanically by the external force f . Similarly, the term $\nu |\mathbb{D}(v)|^2$ from the r.h.s. of (1.1)₃ heats the fluid by the viscous dissipation.

The aim of this paper is to prove the existence and uniqueness of a two-dimensional solution to problem (1), the global existence of sufficiently small three-dimensional perturbations of the 2d-solution and the existence of a global solution to problem (1) close to the two-dimensional solution assuming that the differences of external forces and initial data are sufficiently small in appropriate norms.

The important assumption is such that the external force f does not decay in time. This implies that the solution does not vanish as time tends to infinity. Then, via stability and periodicity, the existence of stationary 2d and 3d solutions can be proved.

The stability of a two-dimensional solution to the Navier-Stokes system with a nondecaying in time external force has been examined in [21] and [23]. In [21] the system supplemented by the periodic boundary conditions is considered, while the paper [23] is devoted to the system in a cylinder with the Navier boundary conditions.

Problem (1) is much more difficult than the problem considered in [19], where the r.h.s. of (1)₃ is assumed to be zero. Then in [19] it is proved that there exists a global solution under the assumption that $v_{,x_3}(0)$, $\theta_{,x_3}(0)$, $f_{,x_3}$ are sufficiently small in L_2 -norms. Hence the obtained solution also has the same property. This means that the solution in [19] is close to the two-dimensional solution. However, in [19], contrary to this paper, it is not possible to control a behavior of solutions as time tends to infinity.

In paper [22] we proved the existence of a strong global solution to (1)_{1,2,3,6,7} with the slip boundary conditions for velocity which are close for all time to a two-dimensional solution.

The above mentioned result is obtained under much stronger assumptions about α than these supposed in this paper. More precisely, we assumed in [22] that $|\alpha(\theta)| + |\alpha'(\theta)| \leq \alpha_*$ for $\theta \geq \theta_* > 0$, where $\alpha_* > 0$ is a constant.

By a two-dimensional solution to problem (1) we mean functions $w = (w_1(x_1, x_2, t), w_2(x_1, x_2, t), 0) \in \mathbb{R}^2$, $q = q(x_1, x_2, t) \in \mathbb{R}$ and $\vartheta = \vartheta(x_1, x_2, t) \in \mathbb{R}_+$ which satisfy

$$\begin{aligned}
 w_t + w \cdot \nabla w + \nu \tilde{\text{rot}} \text{rot}^{(2)} w + \nabla q &= \alpha(\vartheta)h && \text{in } \Omega_+ \equiv \Omega \times \mathbb{R}_+, \\
 \text{div } w &= 0 && \text{in } \Omega_+, \\
 \vartheta_t + w \cdot \nabla \vartheta - \kappa \Delta \vartheta &= \nu |\mathbb{D}(w)|^2 && \text{in } \Omega_+, \\
 w \cdot \bar{n} &= 0 && \text{on } S_{0+} = S_0 \times \mathbb{R}_+, \\
 \text{rot}^{(2)} w &= 0 && \text{on } S_{0+}, \\
 \vartheta &= \theta_* && \text{on } S_{0+}, \\
 w|_{t=0} = w(0), \quad \vartheta|_{t=0} &= \vartheta(0) && \text{in } \Omega,
 \end{aligned} \tag{2}$$

where S_0 is the boundary of Ω , $S_1 = S_0 \times (-a, a)$, $\text{rot}^{(2)} w = w_{2,x_1} - w_{1,x_2}$, $\tilde{\text{rot}}\varphi = (\varphi_{,x_2}, -\varphi_{,x_1})$ and $h = (h_1(x_1, x_2, t), h_2(x_1, x_2, t), 0) \in \mathbb{R}^2$.

The direct calculation gives

$$\text{rot}^2 w = \tilde{\text{rot}} \text{rot}^{(2)} w.$$

Therefore the main operators in (1)₁ and (2)₁ are compatible. Moreover, it is shown in [23] that

$$\bar{n} \times \text{rot } w|_{S_0} = \bar{\tau} \text{rot}^{(2)} w|_{S_0}$$

and

$$\bar{n} \times \text{rot } w|_{S_2} = 0,$$

where $\bar{n} = (n_1, n_2, 0)$, $\bar{\tau} = (n_2, -n_1, 0)$.

To show the stability of two-dimensional solutions of (2) we introduce the quantities

$$u = v - w, \quad \eta = p - q, \quad \chi = \theta - \vartheta, \quad g = f - h$$

which are solutions to the problem

$$\begin{aligned}
 u_t + u \cdot \nabla u + \nu \text{rot}^2 u + \nabla \eta &= -w \cdot \nabla u - u \cdot \nabla w + [\alpha(\theta) - \alpha(\vartheta)]f + \alpha(\vartheta)g && \text{in } D_+, \\
 \text{div } u &= 0 && \text{in } D_+, \\
 \chi_t + u \cdot \nabla \chi - \kappa \Delta \chi &= -w \cdot \nabla \chi - u \cdot \nabla \vartheta + \nu |\mathbb{D}(u)|^2 + 2\nu \mathbb{D}(u) : \mathbb{D}(w) && \text{in } D_+, \\
 u \cdot \bar{n} &= 0 && \text{on } S_+, \\
 \bar{n} \times \text{rot } u &= 0 && \text{on } S_+, \\
 \chi &= 0 && \text{on } S_+, \\
 u|_{t=0} = v(0) - w(0) \equiv u(0), \quad \chi|_{t=0} &= \theta(0) - \vartheta(0) \equiv \chi(0) && \text{in } \Omega.
 \end{aligned} \tag{3}$$

To formulate main results of the paper we introduce the following notation:

$$\begin{aligned} \bar{A}_1(T) &= \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|h(t)\|_{L_2(\Omega)}^2 dt + \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|h(t)\|_{L_\infty(\Omega)}^{2r} dt, \\ \bar{A}_2 &= \|\operatorname{rot} w(0)\|_{L_2(\Omega)}^2 + \|\vartheta(0)\|_{L_2(\Omega)}^2, \end{aligned}$$

where $T > 0, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The norms introduced above as well as norms occurring in formulations of the main results are defined in Sect. 2.

Theorem 1 *Assume that $S_0 \in C^2, 0 < \sigma < \frac{1}{2}, T > 0, h \in L_2(kT, (k + 1)T; L_2(\Omega)) \cap L_{2r}(kT, (k + 1)T; L_\infty(\Omega))$ for all $k \in \mathbb{N}_0$, where $r = \frac{1}{1-2\sigma}$ and $\bar{A}_1(T) < \infty$. Moreover, assume that $w(0) \in H^1(\Omega), \operatorname{div} w(0) = 0, w(0) \cdot \bar{n}|_{S_0} = 0, \vartheta(0) \in L_2(\Omega), \vartheta(0) \geq \theta_*$ for some constant $\theta_* > 0, \alpha \in C([\theta_*, +\infty))$ and $|\alpha(\vartheta)| \leq a_1 + a_2\vartheta^\sigma$ for $\vartheta \geq \theta_*$, where $a_1 \geq 0, a_2 > 0$ are constants. Then there exists a unique solution*

$$\begin{aligned} (w, \vartheta, q) &\in H^{2,1}(\Omega \times (kT, (k + 1)T)) \times C([kT, (k + 1)T]; L_2(\Omega)) \\ &\cap L_2(kT, (k + 1)T; H^1(\Omega)) \times L_2(kT, (k + 1)T; H^1(\Omega)), \quad k \in \mathbb{N}_0, \end{aligned}$$

to problem (2) with $\int_{\Omega} q dx = 0$ for a.a. $t > 0$. Moreover,

$$\begin{aligned} &\|w\|_{H^{2,1}(\Omega \times (kT, (k+1)T))}^2 + \|\vartheta\|_{C([kT, (k+1)T]; L_2(\Omega))}^2 + \|\vartheta\|_{L_2(kT, (k+1)T; H^1(\Omega))}^2 \\ &+ \|q\|_{L_2(kT, (k+1)T; H^1(\Omega))}^2 \leq c(\bar{A}_1, \bar{A}_2), \end{aligned} \tag{4}$$

where $c(\bar{A}_1, \bar{A}_2)$ does not depend on k .

Theorem 1 is proved in Sect. 3. The proof of Theorem 1 is based on the Faedo-Galerkin approximations combined with a step by step method. Since we need in the proof H^2 -estimate for the velocity and H^1 -estimate for the temperature, we assume that $|\alpha(\vartheta)| \leq a_1 + a_2\vartheta^\sigma$ for $\vartheta \geq \theta_*$, where $0 < \sigma < \frac{1}{2}, a_1, a_2 > 0$. This restriction is generated by the dissipative heating term appearing in the r.h.s. of (1)₃. In the case of the lack of the dissipative term, the L_p -estimate for the temperature follows immediately, so the above restriction on α is not necessary. To obtain estimate (4) we show that the data in time intervals $[kT, (k + 1)T], k \in \mathbb{N}_0$ do not increase with k .

Our next result is the theorem about the stability of the two-dimensional solution obtained in Theorem 1 in the set of three-dimensional solutions.

Let $a > 0$ be given. Then we define

$$\begin{aligned} \mathcal{M} &= \left\{ (h, f) \in (L_{2,loc}(\mathbb{R}_+; L_2(\Omega)) \cap L_{2r,loc}(\mathbb{R}_+; L_\infty(\Omega))) \times L_{2,loc}(\mathbb{R}_+; L_\infty(D)) : \right. \\ &\left. \exists T > 0 \text{ such that } T > a \left(\bar{A}_1(T) + \bar{A}_2 + \sup_{k \in \mathbb{N}_0} \|f\|_{L_2(kT, (k+1)T; L_\infty(D))} \right) \right\}, \end{aligned}$$

where $a = \frac{4\tilde{c}_2\tilde{c}_0}{\tilde{c}_1}, \tilde{c}_0 > 0$ is the constant from (43) and $\tilde{c}_1, \tilde{c}_2 > 0$ are the constants from inequality (48).

Theorem 2 Let $S_0 \in C^2$, $0 < \sigma < \frac{1}{2}$, $h \in L_{2,loc}(\mathbb{R}_+; L_2(\Omega)) \cap L_{2r,loc}(\mathbb{R}_+; L_\infty(\Omega))$, where $r = \frac{1}{1-2\sigma}$ and let $\bar{A}_1(T) < \infty$ for every $T > 0$. Suppose that the assumptions of Theorem 1 concerning $w(0)$, $\vartheta(0)$ and α are satisfied. Let $g \in C(\mathbb{R}_+; L_\infty(D))$, $u(0) \in H^1(D)$, $\text{div } u(0) = 0$, $u(0) \cdot \bar{n}|_S = 0$, $\chi(0) \in L_2(D)$, $\theta(0) \geq \theta_*$, $\alpha \in C^1([\theta_*, +\infty))$, $|\alpha'(\theta)| \leq a_3$ for $\theta \geq \theta_*$, where $a_3 > 0$ is a constant. Assume that $(h, f) \in \mathcal{M}$. Moreover, suppose that

$$\|\text{rot } u(0)\|_{L_2(D)}^2 + \|\chi(0)\|_{L_2(DE)}^2 \leq \gamma \tag{5}$$

and

$$\|g(t)\|_{L_\infty(D)}^2 \leq \delta\gamma \quad \text{for all } t \in \mathbb{R}_+. \tag{6}$$

There exists $T > 0$ such that if $\delta, \gamma > 0$ are sufficiently small and if $(u, \chi, \eta) \in H^{2,1}(D \times (kT, (k + 1)T)) \times L_2(kT, (k + 1)T; H^1(D)) \times L_2(kT, (k + 1)T; H^1(D))$ with $\chi_t \in L_2(kT, (k + 1)T; H^{-1}(D))$ $k \in \mathbb{N}_0$, is a solution to problem (3) then

$$\|\text{rot } u(t)\|_{L_2(D)}^2 + \|\chi(t)\|_{L_2(D)}^2 \leq \bar{c}\gamma \quad \text{for all } t \in [kT, (k + 1)T], \quad k \in \mathbb{N}_0 \tag{7}$$

and

$$\begin{aligned} &\|u\|_{H^{2,1}(D \times (kT, (k+1)T))}^2 + \|\chi\|_{L_2(kT, (k+1)T; H^1(D))}^2 \\ &\quad + \|\chi_t\|_{L_2(kT, (k+1)T; H^{-1}(D))}^2 + \|\nabla \eta\|_{L_2(kT, (k+1)T; L_2(D))}^2 \leq \bar{c}\gamma \\ &\text{for all } k \in \mathbb{N}_0, \end{aligned} \tag{8}$$

where $\bar{c} = \bar{c}(T)$ does not depend on k and (w, ϑ, q) is the two-dimensional solution to problem (1), which exists in virtue of Theorem 1.

To prove Theorem 2 we derive a nonlinear differential inequality (see the proof of Lemma 9) and then apply a contradiction argument.

Theorems 1 and 2 imply the following global existence result.

Theorem 3 Let $S_0 \in C^2$, $0 < \sigma < \frac{1}{2}$, $h \in L_{2,loc}(\mathbb{R}_+; L_2(\Omega)) \cap L_{2r,loc}(\mathbb{R}_+; L_\infty(\Omega))$, where $r = \frac{1}{1-2\sigma}$ and let $\bar{A}_1(T) < \infty$ for every $T > 0$. Assume that the assumptions of Theorem 1 concerning $w(0)$, $\vartheta(0)$ and α are satisfied. Let $v(0) \in H^1(D)$, $\text{div } v(0) = 0$, $v(0) \cdot \bar{n}|_S = 0$, $\theta(0) \in L_2(D)$, $\theta(0) \geq \theta_*$, $\alpha \in C^1([\theta_*, +\infty))$, $|\alpha'(\theta)| \leq a_3$ for $\theta \geq \theta_*$, where $a_3 > 0$ is a constant. Moreover, assume that $f \in \mathcal{M}$, $g \in C(\mathbb{R}_+; L_\infty(0))$, and let conditions (5)–(6) be satisfied. There exists $T > 0$ such that if δ and γ are sufficiently small then there exists a unique global solution $(v, \theta, p) \in H^{2,1}(D \times (kT, (k + 1)T)) \times C([kT, (k + 1)T]; L_2(D)) \cap L_2(kT, (k + 1)T; H^1(D)) \times L_2(kT, (k + 1)T; H^1(D))$, $k \in \mathbb{N}_0$ of problem (1) with $\int_\Omega p dx = 0$.

In view of Theorem 1 we prove Theorem 2 by using estimates (7)–(8) derived for the Faedo-Galerkin approximations.

There is a wide literature concerning equations describing motions of incompressible heat conducting fluids. Papers [6, 8, 10, 16, 19] are concerned with solvability results for the Boussinesq equations, that is, for equations (1)_{1,2,3}, where $v|\mathbb{D}(v)|^2$ disappears. Kagei in [10] and Morimoto in [16] proved the existence of weak solutions to the three-dimensional Boussinesq equations with temperature – independent viscosity, while in [8] the weak solvability of the three-dimensional problem with temperature – dependent viscosity has been

examined. Diaz and Galiano [6] showed existence of weak solutions to the following initial-boundary value problem

$$\begin{aligned}v_t + v \cdot \nabla v - \operatorname{div}(v(\theta)\mathbb{D}(v)) &= F(\theta), \quad \operatorname{div} v = 0 \\ \theta_t + v \cdot \nabla \theta - \Delta \varphi(\theta) &= 0 \\ v|_S &= 0, \quad \varphi(\theta)|_S = \varphi_D,\end{aligned}$$

under some restrictions on the functions $v(\theta)$, $\varphi(\theta)$, $F(\theta)$.

The question of the uniqueness of a solution to the two-dimensional Boussinesq equations with the Dirichlet boundary conditions and with temperature dependent viscosity has been studied in [9].

The recent papers concerning the Boussinesq system are [2] and [15]. In [2] Brandolese and Schonbek examine a large time behavior of both weak and strong solutions to the three-dimensional system. Liu and Li [15] study the stability question for the Cauchy problem to equations of type (1), where the r.h.s. of (1)₃ vanishes and the r.h.s. of (1)₁ equals $\theta \bar{e}_3$, $\bar{e}_3 = (0, 0, 1)$. They consider two solutions (v_1, θ_1) and (v_2, θ_2) , where $\|\theta_1(0)\|_{L_1}$ and $\|\theta_2(0)\|_{L_2} + \|v_2(0)\|_{B_{2,1}^{1/2}}$ are sufficiently small. The following stability is shown: $v_1 - v_2, \theta_1 - \theta_2$ are small if initially they are small. Since the Cauchy problem is considered, the technique of used function spaces is in the spirit of the French School. In contrast to our result solutions in [14] converge to zero as time tends to infinity.

Papers of Kagei [11, 12] are devoted to the two-dimensional system (1)_{1,2,3} where the left-hand side of equation (1)₃ contains also the term $-e_2 \cdot v$, $e_2 = (0, 1)$. Considering both the periodic and Dirichlet boundary conditions the author discusses the existence, uniqueness and large time behavior questions in dependence on assumptions about the right-hand side of (1)₁ and initial data. Kagei [11] touched the very difficult 2d Bérnard convection problem. For a strongly nonhomogeneous boundary condition for the temperature the motion becomes very chaotic, so the concept of the strange attractor appears. The author relaxes the boundary condition on the temperature assuming that it is more homogeneous. Then he is able to derive many qualitative properties for the obtained weak solutions.

The Navier-Stokes system with temperature dependent viscosity coupled with the equation of conservation of internal energy has been studied in [3, 5, 7, 17].

Buliček, Feireisl, Málek [3] and Buliček, Kaplický, Málek [4] proved the existence of suitable weak solutions to the Navier-Stokes-Fourier system in [3] and to the Stokes-Fourier in [4]. Moreover, Feireisl and Málek [7] studied the existence of weak solutions to the Navier-Stokes-Fourier system with temperature dependent viscosity. Naumann [17] proved the existence of weak solutions to system (1)₁–(1)₃ with the Dirichlet boundary conditions and with temperature-dependent viscosity and heat conductivity coefficients. Moreover, it is assumed that the r.h.s. of (1)₁ disappears. Consighlieri, Rodrigues, Shilkin [5] consider problem similar to (1)₁–(1)₃ with temperature-dependent viscosity in 2d-torus, that is, with periodic boundary conditions. The existence of weak solutions is proved. Moreover, the viscosity coefficient $\nu = \nu(\theta)$ satisfies $\nu_1, \nu_2, \nu_1 \leq \nu(\theta) \leq \nu_2$, where ν_1, ν_2 are two positive constants. Kagei, Ružička, Thäter [13] proved the existence, uniqueness and stability of the motionless state for system of type (1), where $v \cdot \nabla v$ and $v \cdot \nabla \theta$ disappear. But in (1)₃ additionally appears the term $\theta v \cdot f$. They proved the existence of an unique solution in the class: $v \in C([0, T]; D(P\Delta))$, $\theta \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$, where P is the Helmholtz projection. Shilkin in [18] studies the classical solvability of the coupled system of two nonlinear parabolic equations with temperature-dependent viscosity which models

a Poiseuille-type flow of an incompressible viscous fluid. He shows the existence of solutions to equations simpler than (1) because the terms $v \cdot \nabla v$ and $v \cdot \nabla \theta$ disappear. Moreover, it is supposed that $\alpha(\theta) = 1$. The considered system is complemented by the Dirichlet conditions for the velocity and temperature. The problem is considered in a bounded domain.

The paper consists of four sections. In Sect. 2 notation and some auxiliary results are introduced. In Sect. 3 the two-dimensional problem is considered. Some useful a priori estimates are derived and Theorem 1 is proved. Section 4 contains the proofs of Theorems 2 and 3.

2 Notation and Auxiliary Results

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\Omega \subset \mathbb{R}^n$ be an open set. By $L_p(\Omega)$, $p \in [1, \infty]$ we denote the Lebesgue space of integrable functions and by $W_p^m(\Omega)$, $m \in \mathbb{N}_0$, $p \in [1, \infty]$ the Sobolev space. In the special case of $p = 2$ we use the notation: $H^m(\Omega) = W_2^m(\Omega)$.

Let $I \subset \mathbb{R}$ be an open interval. Then $H^{2,1}(\Omega \times I)$ denotes the space of functions with the finite norm

$$\|u\|_{H^{2,1}(\Omega \times I)} = \left(\|u_t\|_{L_2(\Omega \times I)}^2 + \sum_{0 \leq |\alpha| \leq 2} \|D_x^\alpha u\|_{L_2(\Omega \times I)}^2 \right)^{1/2},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha_i \in \mathbb{N}_0$, $i = 1, \dots, n$.

Let X be a Banach space. By $L_p(I; X)$ we denote the space of all measurable functions $u : I \rightarrow X$ with the finite norm

$$\|u\|_{L_p(I; X)} = \left(\int_I \|u(t)\|_X^p dt \right)^{1/p} \quad \text{if } 1 < p < \infty$$

and

$$\|u\|_{L_\infty(I; X)} = \text{esssup}_{t \in I} \|u(t)\|_X.$$

By $C(\bar{I}; X)$ we denote the space of all continuous functions $u : \bar{I} \rightarrow X$ with the finite norm

$$\|u\|_{C(\bar{I}; X)} = \sup_{t \in \bar{I}} \|u(t)\|_X.$$

Let $u = u(x_1, \dots, x_n)$ be a scalar-valued function defined on $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. Then by ∇u or u_x we denote the gradient of u . For $u : \Omega \rightarrow \mathbb{R}^n$, $n = 2, 3$, by ∇u or u_x we denote the matrix $[u_{ix_j}]_{i,j=1,\dots,n}$. Moreover, then $|u_x| = \sqrt{\sum_{i,j=1}^n |u_{ix_j}|^2}$.

We need the following lemma.

Lemma 1 *Let $\theta(0) \in L_2(D)$ and assume that $\theta(0) \geq \theta_*$ for some $\theta_* > 0$. Moreover, assume that functions v, θ such that $\nabla v \in L_{\frac{20}{7}}(D \times (kT, (k+1)T))$, $k \in \mathbb{N}_0$, $\theta \in L_{\frac{10}{3}}(D \times (kT, (k+1)T))$, $k \in \mathbb{N}_0$ satisfy (1). Then*

$$\theta \geq \theta_* \quad \text{a.e. in } D \times (kT, (k+1)T), \quad k \in \mathbb{N}_0. \tag{9}$$

Proof Multiply (1)₃ by $(\theta - \theta_*)_- = \min\{\theta - \theta_*, 0\}$. Then

$$\frac{d}{dt} \int_D (\theta - \theta_*)_-^2 dx + 2\kappa \int_D |\nabla(\theta - \theta_*)_-|^2 dx = 2\nu \int_D |\mathbb{D}(v)|^2 (\theta - \theta_*)_- dx \leq 0$$

for all $t \in (kT, (k + 1)T)$, $k \in \mathbb{N}_0$.

Integrating the above equality with respect to time yields

$$\begin{aligned} & \int_D (\theta - \theta_*)_-^2 dx + 2\kappa \int_{D \times (kT, (k+1)T)} |\nabla(\theta - \theta_*)_-|^2 dx dt \\ &= 2\nu \int_{D \times (kT, (k+1)T)} |\mathbb{D}(v)|^2 (\theta - \theta_*)_- dx dt + \int_D (\theta(kT) - \theta_*)_-^2 dx, \quad k \in \mathbb{N}_0. \end{aligned}$$

The l.h.s. of the above equality implies that $\theta - \theta_* \in L_{10/3}(D \times (kT, (k + 1)T))$. Then, by the Hölder inequality, the first integral on the r.h.s. is bounded if $\mathbb{D}(v) \in L_{20/7}(D \times (k + 1)T)$.

Hence

$$\int_D (\theta - \theta_*)_-^2 dx \leq \int_D (\theta(kT) - \theta_*)_-^2 dx = 0,$$

so $(\theta - \theta_*)_- = 0$ a.e. in $D \times (kT, (k + 1)T)$, which implies (9). □

Consider the problem

$$\begin{aligned} w_{2x_1} - w_{1x_2} &= b \quad \text{in } \Omega, \\ w_{1x_1} + w_{2x_2} &= 0 \quad \text{in } \Omega, \\ w \cdot \bar{n} &= 0 \quad \text{on } S_0, \end{aligned} \tag{10}$$

where $\Omega \subset \mathbb{R}^2$ is an open domain, S_0 is the boundary of Ω .

Lemma 2 (See [23]) *Let $S_0 \in C^{s+1}$, $b \in H^s(\Omega)$, $s \in \mathbb{N}_0$. Then there exists a solution $w \in H^{s+1}(\Omega)$ to problem (10) and*

$$\|w\|_{H^{s+1}(\Omega)} \leq c \|b\|_{H^s(\Omega)}.$$

Now, consider the elliptic overdetermined problem

$$\begin{aligned} \text{rot } u &= b \quad \text{in } D, \\ \text{div } u &= 0 \quad \text{in } D, \\ u \cdot \bar{n} &= 0 \quad \text{on } S, \end{aligned} \tag{11}$$

where $D \subset \mathbb{R}^3$ is a cylinder.

Lemma 3 (See [20, 23]) *Let $S_1 \in C^{i+1}$, $b \in H^i(D)$, $i = 1, 2$, $\text{div } b = 0$. Then there exists a solution $u \in H^{i+1}(D)$ to problem (11) and*

$$\|u\|_{H^{i+1}(D)} \leq c \|b\|_{H^i(D)}, \quad i = 0, 1, \quad H^0(D) = L_2(D).$$

In Sect. 3 (see Lemma 5) we need the following lemma.

Lemma 4 *Let w be a sufficiently regular function satisfying*

$$\begin{aligned} \operatorname{div} w &= 0 \quad \text{in } \Omega, \\ \operatorname{rot}^{(2)} w &= 0 \quad \text{on } S_0. \end{aligned} \tag{12}$$

Then

$$\int_{\Omega} w \cdot \nabla w \cdot \tilde{\operatorname{rot}} \operatorname{rot} w^{(2)} dx = 0.$$

Proof Since $\operatorname{rot}^{(2)} w = w_{2x_1} - w_{1x_2}$ and $\tilde{\operatorname{rot}} \phi = (\phi_{x_2}, -\phi_{x_1})$, we have

$$\begin{aligned} & \int_{\Omega} w \cdot \nabla w \cdot \tilde{\operatorname{rot}} \operatorname{rot} w^{(2)} dx \\ &= \int_{\Omega} [w \cdot \nabla w_1 (\operatorname{rot}^{(2)} w)_{x_2} - w \cdot \nabla w_2 (\operatorname{rot}^{(2)} w)_{x_1}] dx \\ &= \int_{\Omega} [(w \cdot \nabla w_1 \operatorname{rot}^{(2)} w)_{x_2} - (w \cdot \nabla w_2 \operatorname{rot}^{(2)} w)_{x_1}] dx \\ &\quad - \int_{\Omega} [(w \cdot \nabla w_1)_{x_2} \operatorname{rot}^{(2)} w - (w \cdot \nabla w_2)_{x_1} \operatorname{rot}^{(2)} w] dx \\ &= - \int_{S_0} (w \cdot \nabla w_2 n_1 - w \cdot \nabla w_1 n_2) \operatorname{rot}^{(2)} w dS_0 \\ &\quad - \int_{\Omega} (w \cdot \nabla w_{1x_2} - w \cdot \nabla w_{2x_1}) \operatorname{rot}^{(2)} w dx \\ &\quad - \int_{\Omega} (w_{1x_2} w_{1x_1} + w_{2x_2} w_{1x_2} - w_{1x_1} w_{2x_1} - w_{2x_1} w_{2x_2}) \operatorname{rot}^{(2)} w dx \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By condition (12)₂ we get that $I_1 = 0$. Next, by (12)₁

$$\begin{aligned} I_2 &= \int_{\Omega} w \cdot \nabla \operatorname{rot}^{(2)} w \operatorname{rot}^{(2)} w dx = \frac{1}{2} \int_{\Omega} w \cdot \nabla (\operatorname{rot}^{(2)} w)^2 dx \\ &= \frac{1}{2} \int_{S_0} w \cdot \tilde{n} (\operatorname{rot}^{(2)} w)^2 dS_0 = 0. \end{aligned}$$

Finally,

$$I_3 = - \int_{\Omega} \operatorname{div} w (\operatorname{rot}^{(2)} w)^2 dx = 0.$$

This concludes the proof. □

3 Existence of a Solution to Two-Dimensional Problem

First, we derive a priori estimates for a two-dimensional solution to problem (2). Introduce the function $\bar{\vartheta} = \vartheta - \theta_*$. Then system (2) takes the form

$$\begin{aligned}
 w_t - w \cdot \nabla w + \nu \operatorname{rot} \operatorname{rot}^{(2)} w + \nabla q &= \alpha (\bar{\vartheta} + \theta_*) h && \text{in } \Omega_+, \\
 \operatorname{div} w &= 0 && \text{in } \Omega_+, \\
 \bar{\vartheta}_t + w \cdot \nabla \bar{\vartheta} - \kappa \Delta \bar{\vartheta} &= \nu |\mathbb{D}(w)|^2 && \text{in } \Omega_+, \\
 w \cdot \bar{n} &= 0 && \text{on } S_{0+}, \\
 \operatorname{rot}^{(2)} w &= 0 && \text{on } S_{0+}, \\
 \bar{\vartheta} &= 0 && \text{on } S_{0+}, \\
 w|_{t=0} = w(0), \quad \bar{\vartheta}|_{t=0} &= \bar{\vartheta}(0) && \text{in } \Omega,
 \end{aligned} \tag{13}$$

where $\bar{\vartheta}(0) = \vartheta(0) - \theta_*$. Assume that

$$\vartheta(0) \geq \theta_*. \tag{14}$$

Then $\vartheta \geq \theta_*$ a.e. in Ω_+ (see Lemma 1). Suppose also that

$$\alpha \in C([\theta_*, +\infty))$$

and

$$|\alpha(\vartheta)| \leq a_1 + a_2 \vartheta^\sigma \quad \text{for } \vartheta \geq \theta_*, \tag{15}$$

where $a_1 \geq 0, a_2 > 0, 0 < \sigma < \frac{1}{2}$ are constants.

Introduce the notation

$$\begin{aligned}
 X(t) &= \|\operatorname{rot} w\|_{L_2(\Omega)}^2 + \|\bar{\vartheta}\|_{L_2(\Omega)}, \\
 Y(t) &= \|\operatorname{rot} w\|_{H^1(\Omega)}^2 + \|\bar{\vartheta}\|_{L_2(\Omega)}.
 \end{aligned}$$

Lemma 5 *Let $S_0 \in C^2$ and $T > 0$ be fixed. Assume that conditions (14)–(15) are satisfied and let $w(0) \in H^1(\Omega), \operatorname{div} w(0) = 0, w(0) \cdot \bar{n}|_{S_0} = 0, \vartheta(0) \in L_2(\Omega)$. Moreover, let*

$$A_1 = \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|h(t)\|_{L_2(\Omega)}^2 dt + \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|h(t)\|_{L_4(\Omega)}^{2r} dt < \infty,$$

where $r = \frac{1}{1-2\sigma}$. Assume that $(w, \bar{\vartheta}, q)$ is a sufficiently regular solution to problem (13). Then

$$X(kT) \leq \frac{\bar{c}_2 A_1}{1 - e^{-\bar{c}_1 T}} + X(0) \equiv A_2, \tag{16}$$

where $\bar{c}_1, \bar{c}_2 > 0$ are constants from inequality (21). Furthermore,

$$X(t) + \int_{kT}^t Y(t') dt' \leq c(A_1 + A_2), \tag{17}$$

where $t \in (kT, (k + 1)T]$, $k \in \mathbb{N}_0$, $c > 0$ is a constant.

Proof Multiplying equation (13)₁ by $\text{rot} \text{rot}^{(2)} w = -\Delta w$, integrating over Ω and using Lemma 2 yields

$$\frac{1}{2} \frac{d}{dt} \|\text{rot } w\|_{L_2(\Omega)}^2 + \nu \|\text{rot } w\|_{H^1(\Omega)}^2 = - \int_{\Omega} \alpha(\bar{\vartheta} + \theta_*) h \Delta w dx.$$

Using (15) we have

$$\begin{aligned} & \frac{d}{dt} \|\text{rot } w\|_{L_2(\Omega)}^2 + 2\nu \|\text{rot } w\|_{H^1(\Omega)}^2 \\ & \leq c(\|h\|_{L_4(\Omega)} \|\bar{\vartheta}\|_{L_{4\sigma}(\Omega)}^\sigma \|\Delta w\|_{L_2(\Omega)} + \|h\|_{L_2(\Omega)} \|\Delta w\|_{L_2(\Omega)}), \end{aligned}$$

where $\|\bar{\vartheta}\|_{L_{4\vartheta}(\Omega)} = (\int_{\Omega} \bar{\vartheta}^{4\sigma} dx)^{\frac{1}{4\sigma}}$ for $0 < \sigma < \frac{1}{2}$.

By the imbedding theorem, Young inequality and Lemma 3 we get

$$\begin{aligned} & \frac{d}{dt} \|\text{rot } w\|_{L_2(\Omega)}^2 + 2\nu \|\text{rot } w\|_{H^1(\Omega)}^2 \\ & \leq \varepsilon \|\text{rot } w\|_{H^1(\Omega)}^2 + c(\varepsilon)(\|h\|_{L_4(\Omega)}^2 \|\bar{\vartheta}\|_{L_{4\sigma}(\Omega)}^{2\sigma} + \|h\|_{L_2(\Omega)}^2). \end{aligned}$$

Assuming that ε is sufficiently small we obtain

$$\frac{d}{dt} \|\text{rot } w\|_{L_2(\Omega)}^2 + c_1 \|\text{rot } w\|_{H^1(\Omega)}^2 \leq c_2(\|h\|_{L_4(\Omega)}^2 \|\bar{\vartheta}\|_{L_2(\Omega)}^{2\sigma} + \|h\|_{L_2(\Omega)}^2). \tag{18}$$

Now, multiplying (13)₃ by $\bar{\vartheta}$ gives

$$\frac{1}{2} \frac{d}{dt} \|\bar{\vartheta}\|_{L_2(\Omega)}^2 + \varkappa \int_{\Omega} |\nabla \bar{\vartheta}|^2 dx = \nu \int_{\Omega} |\mathbb{D}(w)|^2 \bar{\vartheta} dx \leq c \|w\|_{H^2(\Omega)}^2 \|\bar{\vartheta}\|_{L_2(\Omega)}. \tag{19}$$

Using the Poincaré inequality and Lemma 2 we obtain

$$\frac{d}{dt} \|\bar{\vartheta}\|_{L_2(\Omega)} + c_3 \|\bar{\vartheta}\|_{L_2(\Omega)} \leq c \|w\|_{H^2(\Omega)}^2 \leq c_4 \|\text{rot } w\|_{H^1(\Omega)}^2. \tag{20}$$

Multiplying (18) by a constant $c_5 > 0$ so large that $\frac{c_1 c_5}{2} > c_4$ and adding (18) and (20) we get

$$\begin{aligned} & \frac{d}{dt} (\|\text{rot } w\|_{L_2(\Omega)}^2 + \|\bar{\vartheta}\|_{L_2(\Omega)}) + c(\|\text{rot } w\|_{H^1(\Omega)}^2 + \|\bar{\vartheta}\|_{L_2(\Omega)}) \\ & \leq c(\|h\|_{L_4(\Omega)}^2 \|\bar{\vartheta}\|_{L_2(\Omega)}^{2\sigma} + \|h\|_{L_2(\Omega)}^2) \\ & \leq \varepsilon \|\bar{\vartheta}\|_{L_2(\Omega)} + c(\varepsilon)(\|h\|_{L_4(\Omega)}^{2r} + \|h\|_{L_2(\Omega)}^2), \end{aligned}$$

where we used that $2\sigma < 1$ and applied the Young inequality with $r' = \frac{1}{2\sigma} > 1$, $r = \frac{r'}{r'-1} = \frac{1}{1-2\sigma}$. Choosing ε sufficiently small we have

$$\begin{aligned} & \frac{d}{dt} (\|\operatorname{rot} w\|_{L_2(\Omega)}^2 + \|\bar{\vartheta}\|_{L_2(\Omega)}) + \bar{c}_1 (\|\operatorname{rot} w\|_{H^1(\Omega)}^2 + \|\bar{\vartheta}\|_{L_2(\Omega)}) \\ & \leq \bar{c}_2 (\|h\|_{L_4(\Omega)}^{2r} + \|h\|_{L_2(\Omega)}^2). \end{aligned}$$

Hence

$$\frac{d}{dt} X + \bar{c}_1 X \leq \bar{c}_2 (\|h\|_{L_4(\Omega)}^{2r} + \|h\|_{L_2(\Omega)}^2), \tag{21}$$

so

$$\frac{d}{dt} (e^{\bar{c}_1 t} X) \leq \bar{c}_2 e^{\bar{c}_1 t} (\|h\|_{L_4(\Omega)}^{2r} + \|h\|_{L_2(\Omega)}^2).$$

Therefore,

$$X(t) \leq \bar{c}_2 \int_{kT}^t (\|h(t')\|_{L_4(\Omega)}^{2r} + \|h(t')\|_{L_2(\Omega)}^2) dt' + e^{-\bar{c}_1(t-kT)} X(kT) \tag{22}$$

for $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$. Inserting $t = (k+1)T$ to (22) implies

$$X((k+1)T) \leq \bar{c}_1 \int_{kT}^{(k+1)T} (\|h(t')\|_{L_4(\Omega)}^{2r} + \|h(t')\|_{L_2(\Omega)}^2) dt' + e^{-\bar{c}_1 T} X(kT).$$

Therefore, by iteration inequality (16) follows. Moreover, integrating (21) over (kT, t) and using (16), we get (17). □

Lemma 6 *Let the assumptions of Lemma 5 hold. Then*

$$\sup_{kT \leq t \leq (k+1)T} \|w(t)\|_{H^1(\Omega)}^2 \leq c(A_1 + A_2) \quad \text{for } k \in \mathbb{N}_0 \tag{23}$$

and

$$\|w\|_{L_2(kT, t; H^2(\Omega))}^2 + \|\bar{\vartheta}\|_{L_2(kT, t; H^1(\Omega))}^2 \leq c(A_1 + A_2) \quad \text{for } t \in (kT, (k+1)T], \quad k \in \mathbb{N}_0, \tag{24}$$

where $c > 0$ does not depend on k .

Proof Estimate $\|w\|_{H^1(\Omega)}^2 \leq c\|\operatorname{rot} w\|_{L_2(\Omega)}^2$ (see Lemma 2) and (17) imply (23). Next, by the estimate

$$\|w\|_{H^2(\Omega)}^2 \leq c\|\operatorname{rot} w\|_{H^1(\Omega)}^2$$

we get

$$\|w\|_{L_2(kT, t; H^2(\Omega))}^2 \leq c(A_1 + A_2). \tag{25}$$

Now, integrating (19) over (kT, t) , $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$ and using (17) and (25) we obtain

$$\|\bar{\vartheta}\|_{L_2(kT, t; H^1(\Omega))}^2 \leq c(A_1 + A_2).$$

This ends the proof. □

Lemma 7 *Let the assumptions of Lemma 5 be satisfied. Let $(w, \bar{\vartheta}, q)$ be a sufficiently regular solution to problem (13). Then*

$$\|w_t\|_{L_2(\Omega \times (kT, (k+1)T))}^2 + \|\vartheta_t\|_{L_2(kT, (k+1)T; H^{-1}(\Omega))}^2 + \|\nabla q\|_{L_2(\Omega \times (kT, (k+1)T))}^2 \leq c(A_1^2, A_2^2),$$

where c does not depend on k .

Proof Multiply (13)₁ by w_t and integrate over Ω and with respect to time from kT to $(k + 1)T$. We get

$$\begin{aligned} & \|w_t\|_{L_2(\Omega \times (kT, (k+1)T))}^2 + \int_{\Omega} |\text{rot}^{(2)} w((k + 1)T)|^2 dx \\ & \leq c \left(\int_{kT}^{(k+1)T} \int_{\Omega} |\alpha(\bar{\vartheta} + \theta_*) h w_t| dx dt + \int_{kT}^{(k+1)T} \int_{\Omega} |w \cdot \nabla w \cdot w_t| dx dt \right. \\ & \quad \left. + \int_{\Omega} |\text{rot}^{(2)} w(kT)|^2 dx \right) \leq \varepsilon \|w_t\|_{L_2(\Omega \times (kT, (k+1)T))}^2 \\ & \quad + c \left(\int_{kT}^{(k+1)T} \|h\|_{L_4(\Omega)} \|\bar{\vartheta}\|_{L_2(\Omega)}^\sigma \|w_t\|_{L_2(\Omega)} dt + \int_{kT}^{(k+1)T} \|h\|_{L_2(\Omega)} \|w_t\|_{L_2(\Omega)} dt \right) \\ & \quad + c(\varepsilon) \left(\int_{kT}^{(k+1)T} \int_{\Omega} |w|^2 |w_x|^2 dx dt + \int_{\Omega} |\text{rot}^{(2)} w(kT)|^2 dx \right). \end{aligned} \tag{26}$$

Using in (26) the estimate

$$\begin{aligned} & \int_{kT}^{(k+1)T} \|h\|_{L_4(\Omega)} \|\bar{\vartheta}\|_{L_2(\Omega)}^\sigma \|w_t\|_{L_2(\Omega)} dt \\ & \leq \varepsilon \|w_t\|_{L_2(\Omega \times (kT, (k+1)T))}^2 + c(\varepsilon) \sup_{kT \leq t \leq (k+1)T} \|\bar{\vartheta}\|_{L_2(\Omega)}^{2\sigma} \int_{kT}^{(k+1)T} \|h\|_{L_4(\Omega)}^2 dt, \end{aligned}$$

next applying Lemmas 5–6 and assuming that ε is sufficiently small we obtain

$$\|w_t\|_{L_2(\Omega \times (kT, (k+1)T))}^2 \leq c(A_1, A_2).$$

Next, (13)₁ yields

$$\begin{aligned} & \|\nabla q\|_{L_2(\Omega \times (kT, (k+1)T))}^2 \\ & \leq c(\|w_t\|_{L_2(\Omega \times (kT, (k+1)T))}^2 + \|w\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2) \end{aligned}$$

$$\begin{aligned}
 &+ \|w\|_{C((kT, (k+1)T]; H^1(\Omega))}^2 \cdot \|w\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2 + \|h\|_{L_2(kT, (k+1)T; L_2(\Omega))}^2 \\
 &+ \|\bar{\vartheta}\|_{L_\infty((kT, (k+1)T]; L_2(\Omega))}^{2\sigma} \|h\|_{L_2(kT, (k+1)T; L_4(\Omega))}^2 \\
 &\leq c(A_1, A_2).
 \end{aligned}$$

Finally, equation (13)₃ and Lemmas 5–6 imply

$$\begin{aligned}
 &\|\bar{\vartheta}_t\|_{L_2(kT, (k+1)T; H^{-1}(\Omega))}^2 \\
 &\leq c(\|\bar{\vartheta}\|_{L_2(kT, (k+1)T; H^1(\Omega))}^2 + \int_{kT}^{(k+1)T} \|w\|_{L_4(\Omega)}^2 \|\bar{\vartheta}_x\|_{L_4(\Omega)}^2 dt \\
 &\quad + \int_{kT}^{(k+1)T} \|w_x\|_{L_4(\Omega)}^2 \|w_x\|_{L_2(\Omega)}^2 dt) \\
 &\leq c[\|\bar{\vartheta}\|_{L_2(kT, (k+1)T; H^1(\Omega))}^2 + \|w\|_{C((kT, (k+1)T]; H^1(\Omega))}^2 (\|\bar{\vartheta}\|_{L_2(kT, (k+1)T; H^1(\Omega))}^2 \\
 &\quad + \|w\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2)] \\
 &\leq c(A_1, A_2).
 \end{aligned}$$

This ends the proof. □

Now, our goal is to prove the existence of a strong-weak solution to problem (13). Denote

$$V = \text{the closure of } \{v \in C^\infty(\bar{\Omega}) : \operatorname{div} v = 0, v \cdot \bar{n} = 0 \text{ on } S_0\} \text{ in } H^1(\Omega).$$

Definition 1 Let $T > 0$ be given. We call a function $(w, \bar{\vartheta})$ a weak solution to problem (13) if $w \in L_\infty(kT, (k+1)T; L_2(\Omega)) \cap L_2(kT, (k+1)T; V)$, $\bar{\vartheta} \in L_\infty(kT, (k+1)T; L_2(\Omega)) \cap L_2(kT, (k+1)T; H_0^1(\Omega))$, $w_t \in L_2(kT, (k+1)T; V^*)$, $\bar{\vartheta}_t \in L_2(kT, (k+1)T; H^{-1}(\Omega))$ for all $k \in \mathbb{N}_0$ and

$$\begin{aligned}
 &\frac{d}{dt} \int_{\Omega} w \cdot \phi dx + \nu \int_{\Omega} \operatorname{rot}^{(2)} w \cdot \operatorname{rot}^{(2)} \phi dx + \int_{\Omega} w \cdot \nabla w \cdot \phi dx \\
 &= \int_{\Omega} \alpha(\bar{\vartheta} + \theta_*) h \cdot \phi dx \quad \text{for every } \phi \in V,
 \end{aligned}$$

in the sense of distributions on $(kT, (k+1)T)$,

$$\begin{aligned}
 &\frac{d}{dt} \int_{\Omega} \bar{\vartheta} \psi dx + \varkappa \int_{\Omega} \nabla \bar{\vartheta} \cdot \nabla \psi dx + \int_{\Omega} w \cdot \nabla \bar{\vartheta} \psi dx \\
 &= \nu \int_{\Omega} |\mathbb{D}(w)|^2 \psi dx \quad \text{for every } \psi \in H_0^1(\Omega),
 \end{aligned}$$

in the sense of distributions on $(kT, (k+1)T)$,

$$w|_{t=kT} = w(kT),$$

$$\bar{\vartheta}|_{t=kT} = \bar{\vartheta}(kT)$$

for all $k \in \mathbb{N}_0$.

Definition 2 The triple $(w, q, \bar{\vartheta})$ is called a strong-weak solution to problem (13) if $(w, \bar{\vartheta})$ is a weak solution to (13) and if additionally $w \in L_\infty(kT, (k + 1)T; H^1(\Omega)) \cap L_2(kT, (k + 1)T; H^2(\Omega))$, $\nabla q \in L_2(kT, (k + 1)T; L_2(\Omega))$ for all $k \in \mathbb{N}_0$.

Now, we can sketch the proof of Theorem 1.

Proof of Theorem 1 We prove the existence of a strong-weak solution to problem (13) by applying the Faedo-Galerkin approximations. Let $\{\phi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ be bases in V and $H_0^1(\Omega)$, respectively. For each $m \in \mathbb{N}$, we define an approximate solution $w_m(t) = \sum_{i=1}^m c_{im}(t)\phi_i$, $\bar{\vartheta}_m(t) = \sum_{i=1}^m d_{im}(t)\psi_i$, which satisfies the following system of ordinary differential equations

$$\begin{aligned} & \int_{\Omega} w'_m(t) \cdot \phi_j dx + \nu \int_{\Omega} \text{rot}^{(2)} w_m(t) \cdot \text{rot}^{(2)} \phi_j dx + \int_{\Omega} w_m(t) \cdot \nabla w_m(t) \cdot \phi_j dx \\ &= \int_{\Omega} \alpha(\bar{\vartheta}_m(t) + \theta_*) h(t) \cdot \phi_j dx \\ & \text{for } j = 1, \dots, m, \quad t \in [kT, (k + 1)T], \quad k \in \mathbb{N}_0, \end{aligned} \tag{27}$$

$$\begin{aligned} & \int_{\Omega} \bar{\vartheta}'_m(t) \psi_j dx + \varkappa \int_{\Omega} \nabla \bar{\vartheta}_m(t) \cdot \nabla \psi_j dx + \int_{\Omega} w_m(t) \cdot \nabla \bar{\vartheta}_m(t) \psi_j dx \\ &= \nu \int_{\Omega} |\mathbb{D}(w_m(t))|^2 \psi_j dx \quad \text{for } j = 1, \dots, m, \quad t \in [kT, (k + 1)T], \quad k \in \mathbb{N}_0, \end{aligned} \tag{28}$$

$$w_m|_{t=kT} = w_{kTm},$$

$$\bar{\vartheta}_m|_{t=kT} = \bar{\vartheta}_{kTm},$$

where $w_{kTm} \in \text{span}\{\phi_1, \dots, \phi_m\}$, $w_{kTm} \rightarrow w(kT)$ strongly in V as $m \rightarrow \infty$, $\bar{\vartheta}_{kTm} \in \text{span}\{\psi_1, \dots, \psi_m\}$, $\bar{\vartheta}_{kTm} \rightarrow \bar{\vartheta}(kT)$ strongly in $L^2(\Omega)$.

We choose a special basis $\{\phi_j\}_{j=1}^\infty$ which consists of the eigenfunctions of the Stokes system with Navier’s boundary conditions, that is, ϕ_j and some p_j satisfy

$$\begin{aligned} -\nu \Delta \phi_j + \nabla p_j &= \lambda_j \phi_j && \text{in } \Omega, \\ \text{div } \phi_j &= 0 && \text{in } \Omega, \\ \phi_j \cdot \bar{n} &= 0 && \text{on } S_0, \\ \text{rot } \phi_j \times \bar{n} &= 0 && \text{on } S_0, \end{aligned} \tag{29}$$

where $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$. In what follows we will use the weak formulation of (29), that is,

$$\nu \int_{\Omega} \text{rot } \phi_j \cdot \text{rot } v dx = \lambda_j \int_{\Omega} \phi_j \cdot v dx \quad \forall v \in V, \quad j = 1, 2, \dots \tag{30}$$

Now, we multiply the both sides of (27) by λ_j and use (30) to the first and second term on the left-hand side of (27). We also use the equality $\int_{\Omega} \text{rot } w_m(t) \cdot \text{rot}^{(2)} \phi_j dx = \int_{\Omega} \text{rot } w_m(t) \cdot \text{rot } \phi_j dx$. To the other two terms in (27) we apply (30) and the identity

$$\int_{\Omega} \text{rot } \phi_j \cdot \text{rot } v dx = - \int_{\Omega} v \cdot \text{rot}^2 \phi_j dx \quad \forall v \in V.$$

Next, multiplying the resulting system of equations by c_{jm} and summing for $j = 1, \dots, m$ we get

$$\frac{1}{2} \frac{d}{dt} \|\text{rot } w_m\|_{L_2(\Omega)}^2 + \nu \|\text{rot } w_m\|_{H^1(\Omega)}^2 = - \int_{\Omega} \alpha(\bar{\vartheta}_m + \theta_*) h \text{rot}^2 w_m dx,$$

where we used that by Lemma 4

$$\int_{\Omega} w_m(t) \cdot \nabla w_m(t) \cdot \text{rot}^2 w_m(t) dx = 0.$$

Similarly, multiplying (28) by $d_{jm}(t)$ and summing for $j = 1, \dots, m$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|\bar{\vartheta}_m\|_{L_2(\Omega)}^2 + \varkappa \int_{\Omega} |\nabla \bar{\vartheta}_m|^2 dx = \nu \int_{\Omega} |\mathbb{D}(w_m)|^2 \bar{\vartheta}_m dx.$$

Then repeating the considerations from the proofs of Lemmas 5–7 we derive the following estimate

$$\begin{aligned} & \|w_m\|_{L_{\infty}(kT, (k+1)T; V)} + \|w_m\|_{L_2(kT, (k+1)T; H^2(\Omega))} \\ & + \|\bar{\vartheta}_m\|_{L_{\infty}(kT, (k+1)T; L_2(\Omega))} + \|\bar{\vartheta}_m\|_{L_2(kT, (k+1)T; H_0^1(\Omega))} \\ & + \|w_{mt}\|_{L_2(kT, (k+1)T; L_2(\Omega))} + \|\bar{\vartheta}_{mt}\|_{L_2(kT, (k+1)T; H^{-1}(\Omega))} \\ & \leq c(A_1, A_2), \end{aligned} \tag{31}$$

where $c(A_1, A_2)$ does not depend on k . By (31) and the Aubin-Lions lemma (see [14], Theorem 5.1) there exist subsequences of w_m and $\bar{\vartheta}_m$ still denoted by w_m and $\bar{\vartheta}_m$ such that

$$\begin{aligned} w_m & \rightarrow w \text{ weakly in } L_2(kT, (k+1)T; H^2(\Omega)) \\ & \text{and } * \text{-weakly in } L_{\infty}(kT, (k+1)T; V), \\ w_{mt} & \rightarrow w_t \text{ weakly in } L_2(kT, (k+1)T; L_2(\Omega)), \\ w_m & \rightarrow w \text{ strongly in } L_2(kT, (k+1)T; H^1(\Omega)), \\ \bar{\vartheta}_m & \rightarrow \bar{\vartheta} \text{ weakly in } L_2(kT, (k+1)T; H_0^1(\Omega)) \\ & \text{and } * \text{-weakly in } L_{\infty}(kT, (k+1)T; L_2(\Omega)), \\ \bar{\vartheta}_{mt} & \rightarrow \bar{\vartheta}_t \text{ weakly in } L_2(kT, (k+1)T; H^{-1}(\Omega)), \\ \bar{\vartheta}_m & \rightarrow \bar{\vartheta} \text{ strongly in } L_2(kT, (k+1)T; L_2(\Omega)). \end{aligned}$$

The above convergences enable passing to the limit in (27)–(28) in the standard way (after multiplying (27)–(28) by appropriate functions of t and integrating them in $[kT, (k + 1)T]$). In particular passing to the limit in the term on the r.h.s. of (27) is possible thanks to the strong convergence of $\bar{\vartheta}_m$ in $L_2(kT, (k + 1)T; L_2(\Omega))$, which yields that for a subsequence of $\bar{\vartheta}_m$ we have

$$\bar{\vartheta}_m \rightarrow \bar{\vartheta} \quad \text{a.e. in } \Omega \times (kT, (k + 1)T).$$

Therefore,

$$\alpha(\bar{\vartheta}_m + \theta_*) \rightarrow \alpha(\bar{\vartheta} + \theta_*) \quad \text{a.e. in } \Omega \times (kT, (k + 1)T).$$

Moreover,

$$\int_{kT}^{(k+1)T} \int_{\Omega} |\alpha(\bar{\vartheta}_m + \theta_*)|^2 dx dt \leq c(1 + \|\bar{\vartheta}_m\|_{L_2(kT, (k+1)T; L_2(\Omega))}^{2\sigma}) \leq c.$$

Hence (see [14], Lemma 1.3)

$$\alpha(\bar{\vartheta}_m + \theta_*) \rightarrow \alpha(\bar{\vartheta} + \theta_*) \quad \text{weakly in } L_2(\Omega \times (kT, (k + 1)T)).$$

The last convergence suffices to pass to the limit in the r.h.s. of (27). This way we obtain that $(w, q, \bar{\vartheta})$ is the strong-weak solution to problem (13). This ends the proof of Theorem 1. \square

4 Stability of Two-Dimensional Solution

To prove the stability of the two-dimensional solution which exists in virtue of Theorem 1 we need some a priori estimates of solutions to problem (3). Assume that

$$\theta(0) \geq \theta_*, \tag{32}$$

where $\theta_* > 0$ is the constant from assumption (14) and

$$|\alpha'(\theta)| \leq a_3 \quad \text{for } \theta \geq \theta_*, \tag{33}$$

where $a_3 > 0$ is a constant. First, we formulate the lemma

Lemma 8 *Let (32)–(33) and the assumptions of Theorem 1 be satisfied. Assume that $f \in L_{2,loc}(\mathbb{R}_+; L_{\infty}(D))$. Moreover, assume that (w, ϑ, q) is the two-dimensional solutions of problem (1) which exists in view of Theorem 1. Let (u, χ, η) be a sufficiently regular solution to problem (3). Then*

$$\begin{aligned} & \frac{d}{dt} (\|\text{rot } u\|_{L_2(D)}^2 + \|\chi\|_{L_2(D)}^2) + c(\|\text{rot}^2 u\|_{L_2(D)}^2 + \|\chi\|_{H^1(D)}^2) \\ & \leq c[\|\text{rot } u\|_{L_2(D)}^6 + \|w\|_{W^1_3(D)}^2 (\|\text{rot } u\|_{L_2(D)}^2 + \|\chi\|_{L_2(D)}^2) \\ & \quad + \|\bar{\vartheta}_x\|_{L_2(D)}^2 \|u\|_{L_3(D)}^2 + \|\text{rot } u\|_{L_2(D)}^4 \|\chi\|_{L_2(D)}^2 + \|\text{rot } u\|_{L_2(D)}^2 \|w_x\|_{L_3(D)}^2 \\ & \quad + \|\chi\|_{L_2(D)}^2 \|f\|_{L_{\infty}(D)}^2 + \|g\|_{L_{\infty}(D)}^2 + \|g\|_{L_{\infty}(D)}^2 \|\bar{\vartheta}\|_{L_2(D)}^{2\sigma}] \end{aligned} \tag{34}$$

for $t > 0$.

Proof Multiplying equation (3)₁ by $\text{rot}^2 u$ and integrating over D gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\text{rot} u\|_{L_2(D)}^2 + \nu \|\text{rot}^2 u\|_{L_2(D)}^2 \\ & \leq - \int_D u \cdot \nabla u \cdot \text{rot}^2 u \, dx - \int_D w \cdot \nabla u \cdot \text{rot}^2 u \, dx - \int_D u \cdot \nabla w \cdot \text{rot}^2 u \, dx \\ & \quad + \int_D \alpha'(\zeta) \chi f \text{rot}^2 u \, dx + \int_D \alpha(\vartheta) g \text{rot}^2 u \, dx, \end{aligned} \tag{35}$$

where $\zeta = \beta\theta + (1 - \beta)\vartheta$, $\beta \in (0, 1)$. Consider

$$\begin{aligned} - \int_D u \cdot \nabla u \cdot \text{rot}^2 u \, dx &= \int_D \text{rot}(u \cdot \nabla u) \text{rot} u \, dx \\ &= \int_D u \cdot \nabla \text{rot} u \cdot \text{rot} u \, dx + \int_D \text{rot} u \cdot \nabla u \cdot \text{rot} u \equiv I, \end{aligned}$$

where the summation convention over repeated indices is applied. By the boundary conditions

$$\int_D u \cdot \nabla \text{rot} u \cdot \text{rot} u \, dx = 0,$$

so

$$|I| \leq c \|u_x\|_{L_3(D)}^3.$$

Therefore, the Hölder and Young inequalities applied in (35) yield

$$\begin{aligned} & \frac{d}{dt} \|\text{rot} u\|_{L_2(D)}^2 + \nu \|\text{rot}^2 u\|_{L_2(D)}^2 \\ & \leq c (\|u_x\|_{L_3(D)}^3 + \|w \cdot \nabla u\|_{L_2(D)}^2 + \|u \cdot \nabla w\|_{L_2(D)}^2 \\ & \quad + \|\chi\|_{L_2(D)}^2 \|f\|_{L_\infty(D)}^2 + \|g\|_{L_\infty(D)}^2 + \|g\|_{L_\infty(D)}^2 \|\vartheta\|_{L_2(D)}^{2\sigma}). \end{aligned} \tag{36}$$

Consider the terms on the right-hand side of (36). First, using the interpolation inequality (see [1])

$$\|u_x\|_{L_3(D)} \leq c \|u_x\|_{H^1(D)}^{1/2} \|u_x\|_{L_2(D)}^{1/2}$$

together with the estimates

$$\|u_x\|_{L_2(D)} \leq c \|u\|_{H^1(D)} \leq c \|\text{rot} u\|_{L_2(D)} \tag{37}$$

and

$$\|u_x\|_{H^1(D)} \leq c \|\text{rot} u\|_{H^1(D)} \leq c \|\text{rot}^2 u\|_{L_2(D)} \tag{38}$$

(see Lemma 3) we have

$$\|u_x\|_{L_3(D)}^3 \leq \varepsilon \|\text{rot}^2 u\|_{L_2(D)}^2 + c(\varepsilon) \|\text{rot} u\|_{L_2(D)}^6,$$

where the Young inequality has been also used. Next,

$$\|w \cdot \nabla u\|_{L_2(D)}^2 \leq c \|w\|_{L_\infty(D)}^2 \|\nabla u\|_{L_2(D)}^2 \leq c \|w\|_{W_3^1(D)}^2 \|\nabla u\|_{L_2(D)}^2,$$

where we used that w only depends on x_1, x_2 . Finally, the third term on the right-hand side of (36) is bounded by

$$c \|u\|_{L_6(D)}^2 \|\nabla w\|_{L_3(D)}^2 \leq c \|u\|_{H^1(D)}^2 \|\nabla w\|_{L_3(D)}^2.$$

Hence inequality (36) yields

$$\begin{aligned} & \frac{d}{dt} \|\operatorname{rot} u\|_{L_2(D)}^2 + c \|\operatorname{rot}^2 u\|_{L_2(D)}^2 \\ & \leq c (\|\operatorname{rot} u\|_{L_2(D)}^6 + \|w\|_{W_3^1(D)}^2 \|u\|_{H^1(D)}^2 \\ & \quad + \|\chi\|_{L_2(D)}^2 \|f\|_{L_\infty(D)}^2 + \|g\|_{L_\infty(D)}^2 + \|g\|_{L_\infty(D)}^2 \|\bar{\vartheta}\|_{L_2(D)}^{2\sigma}). \end{aligned} \tag{39}$$

Now, multiply (3)₃ by χ and integrate the result over D . We get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\chi\|_{L_2(D)}^2 + \varkappa \|\nabla \chi\|_{L_2(D)}^2 \\ & = - \int_D w \cdot \nabla \chi \chi \, dx - \int_D u \cdot \nabla \bar{\vartheta} \chi \, dx + \nu \int_D |\mathbb{D}(u)|^2 \chi \, dx + 2\nu \int_D \mathbb{D}(u) : \mathbb{D}(w) \chi \, dx. \end{aligned} \tag{40}$$

The first term on the right-hand side of (40) is bounded by

$$\varepsilon \|\nabla \chi\|_{L_2(D)}^2 + c(\varepsilon) \|w\|_{L_\infty(D)}^2 \|\chi\|_{L_2(D)}^2 \leq \varepsilon \|\nabla \chi\|_{L_2(D)}^2 + c(\varepsilon) \|w\|_{W_3^1(D)}^2 \|\chi\|_{L_2(D)}^2.$$

The second term is estimated by

$$\varepsilon \|\chi\|_{L_6(D)}^2 + c(\varepsilon) \|\nabla \bar{\vartheta}\|_{L_2(D)}^2 \|u\|_{L_3(D)}^2.$$

We estimate the third term as follows

$$\begin{aligned} \int_D |\mathbb{D}(u)|^2 \chi \, dx & \leq \|u_x\|_{H^1(D)} \|u_x\|_{L_2(D)} \|\chi\|_{L_3(D)} \\ & \leq \varepsilon \|u_x\|_{H^1(D)}^2 + c(\varepsilon) \|u_x\|_{L_2(D)}^2 \|\chi\|_{L_3(D)}^2 \\ & \leq \varepsilon \|u_x\|_{H^1(D)}^2 + c(\varepsilon) \|u_x\|_{L_2(D)}^2 \|\chi\|_{H^1(D)} \|\chi\|_{L_2(D)} \\ & \leq \varepsilon (\|u_x\|_{H^1(D)}^2 + \|\chi_x\|_{L_2(D)}^2) + c(\varepsilon) \|u_x\|_{L_2(D)}^4 \|\chi\|_{L_2(D)}^2, \end{aligned}$$

where the interpolation inequality

$$\|\chi\|_{L_3(D)} \leq c \|\chi\|_{H^1(D)}^{1/2} \|\chi\|_{L_2(D)}^{1/2}$$

has been used (see [1]). Finally,

$$\int_D \mathbb{D}(u) : \mathbb{D}(w) \chi \, dx \leq \varepsilon \|\chi\|_{L_6(D)}^2 + c(\varepsilon) \|u_x\|_{L_2(D)}^2 \|w_x\|_{L_3(D)}^2.$$

Using the above estimates in (36) we obtain

$$\begin{aligned} \frac{d}{dt} \|\chi\|_{L_2(D)}^2 + c \|\nabla \chi\|_{L_2(D)}^2 &\leq \varepsilon (\|u_x\|_{H^1(D)}^2 + \|\chi_x\|_{L_2(D)}^2 + \|\chi\|_{L_6(D)}^2) \\ &\quad + c(\varepsilon) (\|w\|_{W_3^1(D)}^2 \|\chi\|_{L_2(D)}^2 + \|\nabla \bar{\vartheta}\|_{L_2(D)}^2 \|u\|_{L_3(D)}^2) \\ &\quad + \|u_x\|_{L_2(D)}^4 \|\chi\|_{L_2(D)}^2 + \|u_x\|_{L_2(D)}^2 \|w_x\|_{L_3(D)}^2. \end{aligned} \tag{41}$$

Inequalities (37)–(39) and (41) imply (34). □

Introduce the notation

$$\begin{aligned} X(t) &= \|\operatorname{rot} u(t)\|_{L_2(D)}^2 + \|\chi(t)\|_{L_2(D)}^2, \\ A(t) &= \|w(t)\|_{W_3^1(D)}^2 + \|\vartheta_x(t)\|_{L_2(D)}^2 + \|f(t)\|_{L_\infty(D)}^2, \\ G(t) &= \|g(t)\|_{L_\infty(D)}^2. \end{aligned} \tag{42}$$

By Lemma 6

$$\begin{aligned} \int_{kT}^t A(t') dt' &\leq \tilde{c}_0 \left(A_1 + A_2 + \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|f\|_{L_\infty(D)}^2 dt \right) \equiv A_3(T) \\ &\text{for all } t \in (kT, (k+1)T], \quad k \in \mathbb{N}_0, \end{aligned} \tag{43}$$

where $\tilde{c}_0 > 0$ does not depend on k .

Lemma 9 *Let the assumptions of Lemma 8 hold. Assume that*

$$\frac{\tilde{c}_1 T}{4} > \tilde{c}_2 A_3(T), \tag{44}$$

where $\tilde{c}_1, \tilde{c}_2 > 0$ are the constants from (48). Let γ and δ be so small that

$$\tilde{c}_2 M \gamma^2 + \delta \leq \frac{\tilde{c}_1}{2}, \tag{45}$$

where $M > 0$ is the constant from (49). Moreover, let T be so large and δ be so small that

$$e^{-\frac{\tilde{c}_1}{4} T} + MT\delta \leq 1. \tag{46}$$

If $X(0) \leq \gamma$ and $G(t) \leq \delta \gamma$ for all $t \in \mathbb{R}_+$, then $X(t) \leq \tilde{c} \gamma$ for all $t \in \mathbb{R}_+$, where $\tilde{c} = \tilde{c}(T)$ does not depend on k .

Proof By Lemma 5

$$\sup_{t \in (kT, (k+1)T]} \|\bar{\vartheta}\|_{L_2(D)}^{2\sigma} \leq c(A_1 + A_2) \quad \text{for all } k \in \mathbb{N}_0, \tag{47}$$

where $c > 0$ does not depend on k . Now, by (42) and estimate (47), inequality (34) can be written in the form

$$\frac{dX}{dt} + \tilde{c}_1 X \leq \tilde{c}_2 (X^3 + AX + G) \quad \text{for } t \in (kT, (k+1)T], \quad k \in \mathbb{N}_0. \tag{48}$$

Hence

$$\frac{dZ}{dt} + \tilde{c}_1 Z \leq \tilde{c}_2 \left(e^{2\tilde{c}_2 \int_{kT}^t A(t') dt'} Z^3 + e^{-\tilde{c}_2 \int_{kT}^t A(t') dt'} G \right)$$

for $t \in (kT, (k + 1)T]$, $k \in \mathbb{N}_0$,

where

$$Z(t) = e^{-\tilde{c}_2 \int_{kT}^t A(t') dt'} X(t).$$

Inequality (43) implies

$$e^{2\tilde{c}_2 \int_{kT}^t A(t') dt'} \leq M^2 \quad \text{for all } t \in (kT, (k + 1)T], \quad k \in \mathbb{N}_0,$$

where $M > 0$ is some constant which does not depend on k . Therefore

$$\frac{dZ}{dt} + \tilde{c}_1 Z \leq \tilde{c}_2 (M^2 Z^3 + G) \quad \text{for all } t \in (kT, (k + 1)T], \quad k \in \mathbb{N}_0. \tag{49}$$

Assume that

$$X(kT) \leq \gamma \quad \text{for some } k \in \mathbb{N}_0. \tag{50}$$

We will show that

$$X((k + 1)T) \leq \gamma. \tag{51}$$

Inequality (50) implies that

$$Z(kT) \leq \gamma.$$

We will prove that $Z(t) \leq \gamma$ for all $t \in [kT, (k + 1)T]$. Let $t^* = \inf\{t : Z(t) > \gamma\}$. Then

$$\frac{dZ}{dt}(t^*) + \tilde{c}_1 \gamma \leq \tilde{c}_2 (M^2 \gamma^3 + \delta \gamma).$$

By assumption (45) we have

$$\frac{dZ}{dt}(t^*) < 0,$$

so $Z(t) \leq \gamma$ for all $t \in [kT, (k + 1)T]$. This implies that

$$X(t) \leq \gamma M \quad \text{for all } t \in (kT, (k + 1)T]. \tag{52}$$

Using (52) in (48) we get

$$\frac{dX}{dt} + \tilde{c}_1 X \leq \tilde{c}_2 (\gamma^2 M^2 X + AX + G) \quad \text{for all } t \in (kT, (k + 1)T]$$

and (45) yields

$$\frac{dX}{dt} + \frac{\tilde{c}_1}{2} X \leq \tilde{c}_2 (AX + G) \quad \text{for all } t \in (kT, (k + 1)T].$$

Hence

$$\frac{d}{dt} \left(e^{\frac{\tilde{c}_1}{2} t - \tilde{c}_2 \int_{kT}^t A(t') dt'} X \right) \leq e^{\frac{\tilde{c}_1}{2} t - \tilde{c}_2 \int_{kT}^t A(t') dt'} G.$$

After integrating the above inequality with respect to t from $t = kT$ to $t \in (kT, (k + 1)T]$ we obtain

$$X(t) \leq e^{-\frac{\tilde{c}_1}{2}(t-kT)+\tilde{c}_2 \int_{kT}^t A(t')dt'} X(kT) + e^{-\frac{\tilde{c}_1}{2}t+\tilde{c}_2 \int_{kT}^t A(t')dt'} \int_{kT}^t e^{\frac{\tilde{c}_1}{2}t'-\tilde{c}_2 \int_{kT}^{t'} A(t'')dt''} G(t')dt'$$

for $t \in (kT, (k + 1)T]$. (53)

For $t = (k + 1)T$ we have

$$X((k + 1)T) \leq e^{-\frac{\tilde{c}_1}{2}T+\tilde{c}_2 \int_{kT}^{(k+1)T} A(t)dt} X(kT) + e^{\tilde{c}_2 \int_{kT}^{(k+1)T} A(t')dt'} \int_{kT}^{(k+1)T} G(t')dt'.$$

Now, assumption (44) and estimate (43) imply

$$\frac{\tilde{c}_1 T}{4} > \tilde{c}_2 \int_{kT}^t A(t')dt' \quad \text{for } t \in (kT, (k + 1)T].$$

Hence

$$X((k + 1)T) \leq e^{-\frac{\tilde{c}_1 T}{4}} X(kT) + M \int_{kT}^{(k+1)T} G(t)dt.$$

Now, (50), (46) and the assumption $G(t) \leq \delta\gamma$ for all $t \in \mathbb{R}_+$ yield (51). Therefore, we showed that $X(kT) \leq \gamma$ for all $k \in \mathbb{N}_0$. Next, by (53) we get

$$X(t) \leq M\gamma + M \int_{kT}^{(k+1)T} G(t')dt' \leq M\gamma(1 + \delta T) \equiv \tilde{c}\gamma$$

for all $t \in (kT, (k + 1)T]$, $k \in \mathbb{N}_0$.

This ends the proof. □

Lemma 10 *Let the assumptions of Lemma 9 be satisfied. Moreover, assume that*

$$\sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|f(t)\|_{L^\infty(D)}^2 dt < \infty.$$

Then the following estimate holds

$$\|u\|_{H^{2,1}(D \times (kT, (k+1)T))}^2 + \|\chi\|_{L_2(kT, (k+1)T; H^1(D))}^2 + \|\chi_t\|_{L_2(kT, (k+1)T; H^{-1}(D))}^2 + \|\nabla q\|_{L_2(kT, (k+1)T; L_2(D))}^2 \leq c(T)\gamma$$

Proof Integrating (34) with respect to time from kT to $t \in [kT; (k + 1)T]$ and using estimates (37)–(38) yields

$$\|u\|_{L_2(kT, t; H^2(D))}^2 + \|\chi\|_{L_2(kT, t; H^1(D))}^2 \leq c(T)\gamma \tag{54}$$

for $t \in [kT, (k+1)T]$, $k \in \mathbb{N}_0$. Now, estimate (8) implies the estimates

$$\|u_t\|_{L_2(kT, (k+1)T; L_2(D))}^2 \leq c(T)\gamma,$$

$$\|\nabla\eta\|_{L_2(kT, (k+1)T; L_2(D))}^2 \leq c(T)\gamma$$

and

$$\|\chi_t\|_{L_2(kT, (k+1)T; H^{-1}(D))}^2 \leq c(T)\gamma \quad \text{for all } k \in \mathbb{N}_0.$$

This completes the proof. \square

The proofs of Theorems 2 and 3 The assertion of Theorem 2 is the immediate consequence of Lemmas 8–10. The assertion of Theorem 3 follows by using estimates of Lemmas 9–10 and the Faedo-Galerkin method. \square

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