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Unbounded Solutions for a Fractional Boundary Value Problems on the Infinite Interval

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Abstract In this paper, we consider the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0, \infty),\ \alpha \in (1, 2), \\ u(0) = 0, \quad \lim_{t\to\infty} D_{0+}^{\alpha-1}u(t) = \beta u(\xi), \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. By means of fixed point theorems, sufficient conditions are obtained that guarantee the existence of solutions to the above boundary value problem. The fractional modeling is a generalization of the classical integer-order differential equations and it is a very important tool for modeling the anomalous dynamics of numerous processes involving complex systems found in many diverse fields of science and engineering.

Keywords Boundary value problems · Unbounded solution · Riemann-Liouville fractional derivative · Infinite interval

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1 Introduction

Fractional operators mentioned by Leibnitz in a letter to L'Hospital in 1695, have a long history. Early mathematicians who contributed to fractional differential operators include

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Liouville, Riemann and Holmgrem. However, for a quite long period, the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics. The situation has changed recently, many authors pointed out that fractional calculus are very suitable for the description of memory and hereditary properties of various materials and processes, such effects are in fact neglected in classical models. Nowadays, fractional differential equations are increasingly used to model problems in acoustics and thermal systems, materials and mechanical systems, control and robotics, and other areas of application.

For example, nonlocal epidemics can be modeled with fractional derivatives [1]. The model is given by

$$D_i^\alpha = \lambda_i y_i(1-y_i) + \mu_i(1-y_i)\sum_{j\neq i} y_j - \gamma_i y_i, \quad i = 1, 2, \ldots, n,$$

where y_i is the number of infected individuals in the ith patch. The first term represents the infection within the patch with a rate λ_i. The second term represents the effect of other patches both nearby and far away at a rate μ_i. The recovery rate is represented by γ_i. The results are relevant to foot-and-mouth disease, SARS and avian flu.

The constitutive equation of viscoelastic fluid is given by

$$\tau = \eta_0 \frac{d^\alpha \gamma}{dt^\alpha},$$

in [2], where $\frac{d^\alpha \gamma}{dt^\alpha}$ is the fractional derivative, τ is stress, γ is variable force.

The fractional differential equations for nonlinear oscillation of earthquake are presented in [3]. The fractional differential equations for seepage flow in porous media are suggested in [4].

There are a large number of papers dealing with the solvability of fractional differential equation. Paper [5] investigated the existence and uniqueness of the fractional evolution equation

$$\begin{cases} D_{0+}^a u(t) = Au(t) + B(t)u(t), \\ u(0) = u_0. \end{cases}$$

In paper [6], the author applied the homotopy analysis method (HAM) to solve the following fractional Volterras model for population growth of a species in a closed system.

$$\begin{cases} \dfrac{d^\mu u(t)}{dt^\mu} = au(t) + bu^2(t) - cu(t)\displaystyle\int_0^t u(x)dx, \\ u(0) = \alpha. \end{cases}$$

Papers [7–9] considered boundary value problems for fractional differential equations. In paper [9], the authors investigated the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear fractional differential equation

$$\begin{cases} D_{0+}^a u(t) + f(t, u(t)) = 0, \quad t \in (0,1),\ \alpha \in (1,2], \\ u(0) = u(1) = 0. \end{cases}$$

Much of the theory of integer-order boundary value problems on infinite intervals have been presented in [10–12] and references therein. However, to the best knowledge of the authors, there is no paper concerned with the existence of solutions to fractional boundary value problems on infinite intervals. It is important to bridge this gap between known integer-order boundary value problems studies and unknown fractional boundary value problems theory. Such investigations will provide an important platform for gaining a deeper understanding of our environment.

Motivated by the constitutive equation of viscoelastic fluid come from rheology [2]. In this paper, we will consider the fractional boundary value problems

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0, \infty),\ \alpha \in (1, 2), \\ u(0) = 0, \qquad \lim_{t \to \infty} D_{0+}^{\alpha - 1}u(t) = \beta u(\xi), \end{cases} \tag{1.1}$$

where $0 < \xi < \infty$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative.

The method chosen in this paper is the Leray-Schauder Nonlinear Alternative theorem [13], as following.

Theorem 1.1 *Let C be a convex subset of a Banach space, U be an open subset of C with $0 \in U$. Then every completely continuous map $N : \overline{U} \to C$ has at least one of the following two properties*:

(A1) *N has a fixed point in $\overline{U}$; or*
(A2) *There is an $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = \lambda N x$.*

As we know, $[0, \infty)$ is noncompact. So some vital important inequalities used in [5–9] do not apply to such a case. In order to overcome these difficulties, a special Banach space is introduced so that we can establish some similar inequalities, which guarantee that the functionals defined on $[0, \infty)$ have better properties and then we can proceed with the Leray-Schauder Nonlinear Alternative theorem.

The work presented in this paper has the following new features. Firstly, the Green function for fractional boundary value problems on the infinite interval is given. Secondly, the main tool used in this paper is the Leray-Schauder Nonlinear Alternative theorem and the result obtained is for the existence of the unbounded solution.

2 Preliminary

For the convenience of readers, we provide some background material in this section.

Definition 2.1 The Riemann-Liouville fractional integral of order α for function f is defined as

$$I_{0+}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds, \quad \alpha > 0,$$

provided that the right side is point-wise defined on $(0, \infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ for function f is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha - n + 1}} ds, \quad \alpha > 0,$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0, \infty)$.

Lemma 2.1 *Let $\alpha > 0$. Assume that $u \in C(0,\infty) \cap L(0,\infty)$, then the fractional differential equation*

$$D^{\alpha}_{0+}u(t) = 0$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$, $c_i \in R$, $i = 1, 2, \ldots, N$, as the unique solution.

Lemma 2.2 *Assume that $u \in C(0,\infty) \cap L(0,\infty)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,\infty) \cap L(0,\infty)$, then*

$$D^{\alpha}_{0+} I^{\alpha}_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$$

for some $c_i \in R$, $i = 1, 2, \ldots, N$.

3 Related Lemmas

Now we list some conditions in this section for convenience:

(H1) $\beta > 0$, $\beta\xi^{\alpha-1} < \Gamma(\alpha)$;
(H2) $f : [0,\infty) \times R \to [0,\infty)$ is continuous;
(H3) Let $F(t,u) = f(t,(1+t^{\alpha-1})u)$, $|F(t,u)| \le \varphi(t)\omega(|u|)$ on $[0,\infty) \times R$ with $\omega \in C([0,\infty),[0,\infty))$ nondecreasing and $\varphi \in L^1[0,+\infty)$.

Consider the space $C_\infty([0,\infty), R)$ defined by

$$C_\infty([0,\infty),R) = \left\{ u \in C([0,\infty),R) \;\middle|\; \lim_{t\to+\infty} \frac{u(t)}{1+t^{\alpha-1}} \text{ exists} \right\},$$

with the norm

$$\|u\|_\infty = \sup_{t\in[0,\infty)} \left| \frac{u(t)}{1+t^{\alpha-1}} \right|.$$

Lemma 3.1 ([14]) *C_∞ is a Banach space.*

Lemma 3.2 *If $\beta\xi^{\alpha-1} \neq \Gamma(\alpha)$, then for $y(t) \in L^1[0,\infty)$ the boundary value problem*

$$\begin{cases} D^{a}_{0+}u(t) + y(t) = 0, & t \in (0,\infty),\ \alpha \in (1,2), \\ u(0) = 0, & \lim\limits_{t\to\infty} D^{\alpha-1}_{0+}u(t) = \beta u(\xi), \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} \int_0^\infty y(s)ds - \frac{\beta t^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1} y(s)ds.$$

Proof We may apply Lemma 2.2 to reduce $D^{a}_{0+}u(t) + y(t) = 0$ to an equivalent integral equation

$$u(t) = -I^{\alpha}_{0+} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

for some $c_1, c_2 \in R$. Consequently, the general solution of $D_{0+}^{a}u(t) + y(t) = 0$ is

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

By $u(0) = 0$, $\lim_{t\to\infty} D_{0+}^{a-1}u(t) = \beta u(\xi)$, we have

$$c_1 = \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}\int_0^\infty y(s)ds - \frac{\beta}{(\Gamma(\alpha) - \beta\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi - s)^{\alpha-1}y(s)ds, \qquad c_2 = 0.$$

Therefore, the unique solution of problem (3.1) is

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}\int_0^\infty y(s)ds \\ &- \frac{\beta t^{\alpha-1}}{(\Gamma(\alpha) - \beta\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi - s)^{\alpha-1}y(s)ds. \end{aligned}$$

The proof is complete. □

Lemma 3.3 *If $\beta\xi^{\alpha-1} \neq \Gamma(\alpha)$, then the Green function for the boundary value problem*

$$\begin{cases} D_{0+}^{a}u(t) = 0, & t \in (0,\infty),\ \alpha \in (1,2), \\ u(0) = 0, & \lim\limits_{t\to\infty} D_{0+}^{a-1}u(t) = \beta u(\xi), \end{cases} \tag{3.2}$$

is given by

$$G(t,s) = \begin{cases} \frac{-(t-s)^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}-\beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}, & s \le t,\ s \le \xi, \\ \frac{-(t-s)^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}, & \xi \le s \le t, \\ \frac{\Gamma(\alpha)t^{\alpha-1}-\beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}, & t \le s \le \xi, \\ \frac{t^{\alpha-1}}{\Gamma(\alpha)-\beta\xi^{\alpha-1}}, & t \le s,\ s \ge \xi. \end{cases} \tag{3.3}$$

Proof For $t \le \xi$, the unique solution of (3.1) can be expressed as

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}\left[\int_0^t y(s)ds + \int_t^\xi y(s)ds + \int_\xi^\infty y(s)ds\right] \\ &\quad - \frac{\beta t^{\alpha-1}}{(\Gamma(\alpha) - \beta\xi^{\alpha-1})\Gamma(\alpha)}\left[\int_0^t (\xi - s)^{\alpha-1}y(s)ds + \int_t^\xi (\xi - s)^{\alpha-1}y(s)ds\right] \\ &= \int_0^t \frac{-(t-s)^{\alpha-1}(\Gamma(\alpha) - \beta\xi^{\alpha-1}) + \Gamma(\alpha)t^{\alpha-1} - \beta t^{\alpha-1}(\xi - s)^{\alpha-1}}{(\Gamma(\alpha) - \beta\xi^{\alpha-1})\Gamma(\alpha)}y(s)ds \\ &\quad + \int_t^\xi \frac{\Gamma(\alpha)t^{\alpha-1} - \beta t^{\alpha-1}(\xi - s)^{\alpha-1}}{(\Gamma(\alpha) - \beta\xi^{\alpha-1})\Gamma(\alpha)}y(s)ds \end{aligned}$$

$$+\int_{\xi}^{\infty}\frac{t^{\alpha-1}}{\Gamma(\alpha)-\beta\xi^{\alpha-1}}y(s)ds$$
$$=\int_{0}^{\infty}G(t,s)y(s)ds.$$

For $t\geq\xi$, the unique of (3.1) can be expressed as

$$\begin{aligned}u(t)&=-\frac{1}{\Gamma(\alpha)}\left[\int_0^{\xi}(t-s)^{\alpha-1}y(s)ds+\int_{\xi}^{t}(t-s)^{\alpha-1}y(s)ds\right]\\&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)-\beta\xi^{\alpha-1}}\left[\int_0^{\xi}y(s)ds+\int_{\xi}^{t}y(s)ds+\int_t^{\infty}y(s)ds\right]\\&\quad-\frac{\beta t^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}y(s)ds\\&=\int_0^{\xi}\frac{-(t-s)^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}-\beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}y(s)ds\\&\quad+\int_{\xi}^{t}\frac{-(t-s)^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}y(s)ds\\&\quad+\int_t^{\infty}\frac{t^{\alpha-1}}{\Gamma(\alpha)-\beta\xi^{\alpha-1}}y(s)ds\\&=\int_0^{1}G(t,s)y(s)ds.\end{aligned}$$

The proof is complete. □

Lemma 3.4 *If* (H1) *holds, then the Green function of* (3.2) *satisfies* $G(t,s)\geq 0$, *for* $t,s\in(0,\infty)$.

Proof *Case 1.* $0\leq s\leq t, s\leq\xi$.

$$\begin{aligned}G(t,s)&=\frac{-(t-s)^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}-\beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\\&\geq\frac{-t^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}-\beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\\&=\frac{\beta t^{\alpha-1}\xi^{\alpha-1}-\beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\geq 0.\end{aligned}$$

Case 2. $0\leq\xi\leq s\leq t$.

$$\begin{aligned}G(t,s)&=\frac{-(t-s)^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\\&\geq\frac{-t^{\alpha-1}(\Gamma(\alpha)-\beta\xi^{\alpha-1})+\Gamma(\alpha)t^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\\&=\frac{\beta t^{\alpha-1}\xi^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\geq 0.\end{aligned}$$

Case 3. $0 \le t \le s \le \xi$.

$$G(t,s) = \frac{\Gamma(\alpha)t^{\alpha-1} - \beta t^{\alpha-1}(\xi - s)^{\alpha-1}}{(\Gamma(\alpha) - \beta\xi^{\alpha-1})\Gamma(\alpha)} \ge \frac{\Gamma(\alpha)t^{\alpha-1} - \beta t^{\alpha-1}\xi^{\alpha-1}}{(\Gamma(\alpha) - \beta\xi^{\alpha-1})\Gamma(\alpha)} \ge \frac{t^{\alpha-1}}{\Gamma(\alpha)} \ge 0.$$

Case 4. $0 \le t \le s \le \xi$.

$$G(t,s) = \frac{t^{\alpha-1}}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} \ge 0.$$

The proof is complete. □

For $u \in C_\infty$, define the operator T by

$$(Tu)(t) = \int_0^\infty G(t,s) f(s, u(s)) ds. \tag{3.4}$$

Boundary value problem (1.1) has a solution u if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies Theorem 1.1, which will prove the existence of the fixed points of T. Since the Arzela-Ascoli theorem fails to work in the space C_∞, we need a modified compactness criterion to prove T is compact.

Lemma 3.5 ([14]) *Let* $V = \{u \in C_\infty \mid \|u\|_\infty < l\}$ $(l > 0)$, $V_1 = \{\frac{u(t)}{1+t^{\alpha-1}} \mid u \in V\}$. *If* V_1 *is equicontinuous on any compact intervals of* $[0, +\infty)$ *and equiconvergent at infinity, then* V *is relatively compact on* C_∞.

Remark 3.1 V_1 is called equiconvergent at infinity if and only if for all $\epsilon > 0$, there exists $\upsilon = \upsilon(\epsilon) > 0$ such that $for\ all\ u \in V_1,\ t_1,\ t_2 \ge \upsilon$, it holds,

$$\left| \frac{u(t_1)}{1 + t_1^{\alpha-1}} - \frac{u(t_2)}{1 + t_2^{\alpha-1}} \right| < \epsilon.$$

Lemma 3.6 *If* (H1)–(H3) *hold, then* $T : C_\infty \to C_\infty$ *is completely continuous.*

Proof We divided the proof into three steps.

Step 1: We show that $T : C_\infty \to C_\infty$ is continuous.

Let $u_n \to u$ as $n \to +\infty$ in C_∞, then there exists r_0 such that

$$\max\left\{\|u\|_\infty, \sup_{n \in N\setminus\{0\}} \|u_n\|_\infty\right\} < r_0.$$

We have

$$
\begin{aligned}
&\int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}}\big|f(s,u_n(s))-f(s,u(s))\big|ds\\
&\quad\le \frac{1}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^{+\infty}\big(|f(s,u_n(s))|+|f(s,u(s))|\big)ds\\
&\quad= \frac{1}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^{+\infty}\left(\left|f\left(s,\frac{u_n(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right)\right|+\left|f\left(s,\frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right)\right|\right)ds\\
&\quad= \frac{1}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^{+\infty}\left(\left|F\left(s,\frac{u_n(s)}{1+s^{\alpha-1}}\right)\right|+\left|F\left(s,\frac{u(s)}{1+s^{\alpha-1}}\right)\right)\right|ds\\
&\quad= \frac{2\omega(r_0)}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^{+\infty}\varphi(s)ds<\infty.
\end{aligned}
$$

Hence

$$
\|Tu_n-Tu\|_\infty=\sup_{t\in[0,\infty)}\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}|f(s,u_n(s))-f(s,u(s))|ds\to 0,\quad \text{as } n\to+\infty.
$$

So, T is continuous.

Step 2: We show that $T:C_\infty\to C_\infty$ is relatively compact.

Let Ω be any bounded subset of C_∞, then there exists $K>0$ such that $\|u\|_\infty\le K$. We have

$$
\begin{aligned}
\|Tu\|_\infty&=\sup_{t\in[0,\infty)}\int_0^\infty\left|\frac{G(t,s)}{1+t^{\alpha-1}}f(s,u(s))\right|ds\\
&\le\frac{1}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^\infty|f(s,u(s)|ds\\
&=\frac{1}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^\infty\left|f\left(s,\frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right)\right|ds\\
&=\frac{1}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^\infty\left|F\left(s,\frac{u(s)}{1+s^{\alpha-1}}\right)\right|ds\\
&\le\frac{1}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^\infty\varphi(s)\omega\left(\frac{|u(s)|}{1+s^{\alpha-1}}\right)ds\\
&\le\frac{\omega(K)}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\int_0^\infty\varphi(s)ds<\infty,\quad \text{for } u\in\Omega.
\end{aligned}
$$

Hence $T\Omega$ is uniformly bounded.

Now we show that $T\Omega$ is equicontinuous on any compact interval of $[0,\infty)$.

For any $T>0$, $t_1,\ t_2\in[0,T]$, and $u\in\Omega$, without loss of generality, we may assume that $t_2>t_1$. In fact,

$$
\begin{aligned}
&\left|\frac{(Tu)(t_2)}{1+t_2^{\alpha-1}}-\frac{(Tu)(t_1)}{1+t_1^{\alpha-1}}\right|\\
&\quad=\left|\int_0^\infty\frac{G(t_2,s)}{1+t_2^{\alpha-1}}f(s,u(s))ds-\int_0^\infty\frac{G(t_1,s)}{1+t_1^{\alpha-1}}f(s,u(s))ds\right|\\
&\quad=\int_0^\infty\left[\frac{G(t_2,s)}{1+t_2^{\alpha-1}}ds-\frac{G(t_1,s)}{1+t_2^{\alpha-1}}+\frac{G(t_1,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right]|f(s,u(s))|ds
\end{aligned}
$$

$$\le \omega(K)\int_0^\infty \left[\frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\beta(T+\xi)^{\alpha-1}+\Gamma(\alpha))(t_2^{\alpha-1}-t_1^{\alpha-1})}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)} + \frac{T^{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})}\right]\varphi(s)ds$$
$$\to 0, \quad \text{uniformly as } t_1 \to t_2 \text{ for all } u \in \Omega.$$

Hence $T\Omega$ is locally equicontinuous on $[0,\infty)$.

Step 3: We show that $T : C_\infty \to C_\infty$ is equiconvergent at ∞.

For any $u \in \Omega$,

$$\int_0^\infty f(s,u(s))ds \le \omega(K)\int_0^\infty \varphi(s)ds < \infty,$$

$$\lim_{t\to+\infty}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| = \frac{\beta\xi^{\alpha-1}}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\infty f(s,u(s))ds - \frac{\beta}{(\Gamma(\alpha)-\beta\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1} f(s,u(s))ds.$$

Hence $T\Omega$ is equiconvergent at infinity.

By using Lemma 3.5, we obtain that $T : C_\infty \to C_\infty$ is completely continuous. □

4 Existence of Unbounded Solutions

Theorem 4.1 *Assume that* (H1)–(H3) *hold, Let ω, φ satisfies the following condition*:

(B) $\exists \varrho > 0$ *such that*

$$\frac{\varrho(\Gamma(\alpha)-\beta\xi^{\alpha-1})}{\omega(\varrho)\int_0^\infty \varphi(s)ds} > 1. \tag{4.1}$$

Then boundary value problem (1.1) *has an unbounded solution $u = u(t)$ such that*

$$0 \le \frac{u(t)}{1+t^{\alpha-1}} \le \varrho, \quad \textit{for } t \in [0,\infty).$$

Proof We consider the fractional boundary value problem

$$\begin{cases} D_{0+}^{a}u(t) + \lambda f(t,u(t)) = 0, & t \in (0,\infty),\ \alpha \in (1,2), \\ u(0) = 0, & \lim_{t\to\infty} D_{0+}^{a-1}u(t) = \beta u(\xi), \end{cases} \tag{4.2}$$

for $0 < \lambda < 1$. Solving (4.2) is equivalent to solving the fixed point problem $u = \lambda Tu$.

Let

$$U = \{u \in C_\infty \mid \|u\|_\infty < \varrho\}.$$

We claim that $u \ne \lambda Tu$ for $u \in \partial U$ and $\lambda \in (0,1)$. The claim is immediate, since if there exists $u \in \partial U$ with $u = \lambda Tu$, then for $\lambda \in (0,1)$ we have

$$\begin{aligned}\|u\|_\infty &= \sup_{t\in[0,\infty)}\left|\frac{(\lambda Tu)(t)}{1+t^{\alpha-1}}\right| \\ &\le \sup_{t\in[0,\infty)}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| \\ &= \sup_{t\in[0,\infty)}\left|\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right)ds\right| \\ &= \sup_{t\in[0,\infty)}\left|\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right)ds\right| \\ &\le \int_0^\infty \frac{1}{\Gamma(\alpha)-\beta\xi^{\alpha-1}}\varphi(s)\omega\left(\frac{|u(s)|}{1+s^{\alpha-1}}\right)ds \\ &\le \frac{1}{\Gamma(\alpha)-\beta\xi^{\alpha-1}}\omega(\varrho)\int_0^\infty \varphi(s)ds.\end{aligned}$$

So

$$\varrho \le \frac{1}{\Gamma(\alpha)-\beta\xi^{\alpha-1}}\omega(\varrho)\int_0^\infty \varphi(s)ds.$$

Hence

$$\frac{\varrho(\Gamma(\alpha)-\beta\xi^{\alpha-1})}{\omega(\varrho)\int_0^\infty \varphi(s)ds} \le 1,$$

which contradicts with (4.1). By Theorem 1.1 and Lemma 3.4, boundary value problem (1.1) has an unbounded solution $u=u(t)$ such that

$$0 \le \frac{u(t)}{1+t^{\alpha-1}} \le \varrho, \quad \text{for } t\in[0,\infty).$$

□

Example 4.1 Let $\alpha=\frac{3}{2},\ \xi=1,\ 0<\beta<\Gamma(\frac{3}{2}),\ f(t,u)=\sqrt{|\frac{u}{1+t^{\frac{1}{2}}}|e^{-t}},\ F(t,u)=\sqrt{|u|}e^{-t}$ in problem (1.1). Now we consider the following fractional boundary value problem

$$\begin{cases} D_{0+}^{\frac{3}{2}}u(t)+f(t,u)=0, & t\in(0,\infty), \\ u(0)=0, & \lim_{t\to\infty} D_{0+}^{a-1}u(t)=\beta u(1). \end{cases} \tag{4.3}$$

Choose $\omega(u)=\sqrt{u}$, $\varphi(t)=e^{-t}$, $\varrho>(\frac{1}{\Gamma(\frac{3}{2})-\beta})^2$, we have

1. $\beta>0,\ \beta\xi^{\alpha-1}<\Gamma(\alpha)$;
2. $f:[0,\infty)\times R\to[0,\infty)$ is continuous;
3. $|F(t,u)|=\varphi(t)\omega(|u|)$ on $[0,\infty)\times R$ with $\omega\in C([0,\infty),[0,\infty))$ nondecreasing and $\varphi\in L^1[0,+\infty)$;
4. $\frac{\varrho(\Gamma(\alpha)-\beta\xi^{\alpha-1})}{\omega(\varrho)\int_0^\infty\varphi(s)ds}=\sqrt{\varrho}(\Gamma(\frac{3}{2})-\beta)>1$.

Hence all conditions of Theorem 4.1 hold. Thus with Theorem 4.1, the problem (4.3) has at least a positive solution u such that

$$0 \leq \frac{u(t)}{1+\sqrt{t}} \leq \varrho, \quad \text{for } t \in [0, \infty).$$

5 Conclusion

We consider a fractional differential equation on a infinite interval. Conditions (H1), (H2), (H3) and (4.1) are given to guarantee the solution $u(t)$ (variable force) to be positive. The result obtained in this paper might find some potential applications in the processes associated with complex systems involving long-memory in time, which improves the results of [7–9].

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