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RESEARCH ARTICLE

On certain representations of pricing functionals

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Abstract
We revisit two classical problems: the determination of the law of the underlying with respect to a risk-neutral measure on the basis of option prices, and the pricing of options with convex payoffs in terms of prices of call options with the same maturity (all options are European). The formulation of both problems is expressed in a language loosely inspired by the theory of inverse problems, and several proofs of the corresponding solutions are provided that do not rely on any special assumptions on the law of the underlying and that may, in some cases, extend results currently available in the literature. Furthermore, we consider a related problem, arising from nonparametric option pricing, on the reconstruction of put option prices in an approximation scheme where a sequence of measures converges to the (image) measure of the underlying's return at fixed maturities.

Keywords Option pricing · Breeden-Litzenberger formula · Convex payoffs · Distributions (generalized functions)

JEL Classification G13 · CO2

1 Introduction

Let $S, \beta \colon \Omega \times [0, T] \to \mathbb{R}_+$ denote the price processes of a risky asset and of a numéraire (that we shall assume to be the money-market account, for simplicity), respectively, in an arbitrage-free market, modeled on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, \mathbb{P})$, where $T > 0$ is a fixed time horizon and $\mathscr{F}_0$ is the trivial σ-algebra. Assuming that pricing takes place with respect to a risk-neutral probability measure $\mathbb{Q}$, the price at time zero of a European option with maturity T and payoff profile $g \colon \mathbb{R}_+ \to \mathbb{R}$ on the asset with price process S is given by

$$\pi(g) = \mathbb{E}_{\mathbb{Q}} \beta_T^{-1} g(S_T).$$

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We shall call the map $g \mapsto \pi(g)$, defined on the set of all measurable functions g such that the right-hand side is finite, the pricing functional.

A rather general and natural question, of clear relevance also for practical purposes, is the following: suppose that the action of π is known on a set of functions G, i.e. that $\pi(g)$ is known for every $g \in G$. Is it possible to enlarge the set of functions G where π is determined, i.e. to compute $\pi(f)$ for some functions f that do not belong to G? We are going to discuss some questions of this type (although not in this generality), through the representation of the pricing functional as a Lebesgue–Stieltjes measure, that is

$$\pi(g) = \int_{\mathbb{R}_+} g(x)\, dF(x),$$

where F is the right-continuous version of the distribution function of S_T with respect to the measure $(d\mathbb{Q}/d\mathbb{P})\beta_T^{-1} \cdot \mathbb{P}$, i.e. the measure with density with respect to $\mathbb{P}$ equal to the stochastic discount factor.

If a collection G of payoff profiles g is such that the prices $\pi(g)$ of the corresponding options are known, the set $M := (g, \pi(g))_{g \in G}$ will be called a measurement set, and a measurement set that determines F will be called a representation. That is to say, if knowing M allows one to reconstruct F, then knowing M is equivalent to knowing the pricing functional itself, which is why we say that it is a representation (of dF, or of π). An interesting and important example of a representation is given by prices of put options: if G is composed of all functions $g_k : x \mapsto (k - x)^+$, $k \in \mathbb{R}_+$, then M as defined above is a representation. More precisely, it is shown in Sect. 4 (see also Sect. 6 for an alternative derivation) that if $P(k) = \pi(g_k)$ denotes the price at time zero of the put option with maturity T and strike $k \geq 0$, then $D^+P(k) = F(k)$ for every $k \in \mathbb{R}_+$, where $D^+P(k)$ denotes the right derivative of P at k. The problem of reconstructing the law of the underlying from option prices was probably considered first in Breeden and Litzenberger (1978), where the authors showed that, denoting the price of a call option with maturity T and strike k by $C(k)$, the second derivative of C is the density of S_T with respect to the measure $(d\mathbb{Q}/d\mathbb{P})\beta_T^{-1} \cdot \mathbb{P}$, i.e. the first derivative of F (the function C is implicitly assumed to be of class C^2 in Breeden and Litzenberger 1978). This result, known as the Breeden–Litzenberger formula, has found many applications, e.g. in static hedging, nonparametric density estimation, and local volatility models (see, e.g., Ait-Sahalia and Lo 1998; Bossu et al. 2021; Itkin 2020; Marinelli and d'Addona 2017 and references therein, as well as Talponen and Viitasaari 2014, where an interesting extension to the multidimensional setting is presented and further references regarding static hedging are given). In general, F is not of class C^1, hence C is not of class C^2, but, if they are, then the Breeden–Litzenberger formula follows immediately from $D^+P = F$ and the put-call parity relation. Note, however, that for pricing purposes it suffices to determine F rather than its derivative, and the relation $D^+P = F$ is obtained here without any a priori assumptions on F.

The reconstruction of F from a set of option prices is interesting in its own right, but sometimes less information is enough for the problem at hand (roughly speaking, this is just the idea behind static hedging). Using the above terminology, if one needs a measurement set M, it may be possible to determine a measurement set M' that contains M, without necessarily recovering F first. The simplest example is the pricing of options with continuous piecewise linear payoff profile (such as straddles, strips, and strangles—see, e.g., Hull 2015, §12.4). Another one is the pricing of options with payoff function equal to the difference of convex functions in terms of call options. Even though, in the latter case, the measurement set of all call options is already a representation, there is an alternative pricing formula that avoids the differentiation of C, which might be preferable for numerical purposes. Such pricing formula for options with convex payoff profiles is not new, but we give nonetheless several proofs: a very concise one, a longer one that highlights the role of convexity, and a third one that is extremely simple if sufficient regularity is present. We also show that similar ideas can be used to "localize" the pricing formula, i.e. to price options with payoff profiles that are piecewise the difference of convex functions.

The main content is organized as follows: we collect in Sect. 2 some auxiliary facts from measure theory, convexity, and the theory of distribution. Definitions, motivations, basic properties, and examples pertaining to pricing functionals, measurement sets, and representations are given in Sect. 3. Qualitative properties of the functions P and C, as defined above, are discussed in Sect. 4, without any assumption on F. Moreover, we show that F is the right derivative of P by two methods, that is, using the integration-by-parts formula for càdlàg functions of finite variation and by a denseness argument, respectively. In Sect. 5 we revisit the fact that prices of options with convex profile are determined by prices of call options for all positive strikes. This is proved in two ways: by an integration argument, that uses essentially only the Fubini theorem, and via the above-mentioned integration-by-parts formula. The results of the previous two sections are derived by yet another approach in Sects. 4–5, that is, using the theory of distributions. An interesting aspect of this method is that it provides a particularly handy way to make computations, also in cases that do not directly follow from the setups of the previous two sections. We conclude in Sect. 7 where, motivated by empirical issues in nonparametric pricing of European options treated in Marinelli and d'Addona (2017), we consider a kind of representation where a sequence of measures converging towards dF intervenes. In particular, assume that the distribution of logarithmic returns admits a density f and that there exists a sequence of functions (f_n) converging to f in $L^2(\mathbb{R})$. Considering f_n as an approximation of f and computing approximating put prices for all strikes accordingly, we show that the knowledge of such approximating prices suffices to uniquely determine the true put option prices for all strikes.

2 Preliminaries

We shall use some elementary facts from measure theory and convexity, that we recall for convenience. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces, and μ a measure on the former. If $\phi\colon X \to Y$ is a measurable function, then the image measure or pushforward of μ through ϕ is the measure on $(Y, \mathscr{B})$ defined by $\phi_*\mu\colon B \mapsto \mu(\phi^{-1}(B))$. If

$g\colon Y \to \overline{\mathbb{R}}$ is a measurable function, then

$$\int_Y g\,d\phi_*\mu = \int_X g \circ \phi\,d\mu, \tag{2.1}$$

in the sense that g is $\phi_*\mu$-integrable if and only if $g \circ \phi$ is μ-integrable, and in this case the integrals coincide (see, e.g., Cohn 2013, §2.6.8). Interpreting precomposition as pullback, hence writing $\phi^* g := g \circ \phi$, and using the notation $m(f) := \langle m, f \rangle := \langle f, m \rangle := \int f\,dm$ for any function f integrable with respect to a measure m, the identity (2.1) can be written in the simple and suggestive form

$$\big\langle \phi^* g, \mu \big\rangle = \big\langle g, \phi_*\mu \big\rangle.$$

We shall extensively use an integration-by-parts formula for Lebesgue–Stieltjes integrals (see, e.g., Dellacherie and Meyer 1980, p. 343). Let the functions $F, G\colon \mathbb{R} \to \mathbb{R}$ be càdlàg (i.e. right-continuous with left limits) and with finite variation (i.e. having bounded variation on every bounded interval). Then, for any two real numbers $a < b$,

$$F(b)G(b) - F(a)G(a) = \int_{]a,b]} G(x-)\,dF(x) + \int_{]a,b]} F(x)\,dG(x). \tag{2.2}$$

If G is continuous, one can obviously replace $G(x-)$ by $G(x)$. Whenever dG is an atomless measure we shall just write $\int_a^b F\,dG$ instead of denoting the interval of integration as a subscript. Moreover, we set $\mathbb{R}_+ := [0, +\infty[$.

Let us recall a few facts about convex functions (see, e.g., Simon 2011, Chapter 1). Let $I \subset \mathbb{R}$ be an open interval and $f\colon I \to \mathbb{R}$ be a convex function. Then f is everywhere left- and right-differentiable, that is, for any $x \in I$ the left and right derivatives

$$D^- f(x) := \lim_{h \to 0-} \frac{f(x+h) - f(x)}{h}, \qquad D^+ f(x) := \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

exists and are finite, and $D^- f(x) \leq D^+ f(x)$. Moreover, both $D^- f$ and $D^+ f$ are increasing functions, $D^- f$ is left-continuous, and $D^+ f$ is right-continuous. It follows that f is differentiable except at the countable set of points where $D^- f$ and $D^+ f$ do not coincide. The subdifferential of f at x is defined as

$$\partial f(x) = \big\{ z \in \mathbb{R} : f(y) - f(x) \geq z(y - x) \quad \forall y \in I \big\}.$$

It can be shown that $\partial f(x) = [D^- f(x), D^+ f(x)]$ and that, for any $x_1, x_2 \in I$, $x_1 < x_2$, it holds $D^+ f(x_1) \leq D^- f(x_2)$, hence $\partial f(x_1) \cap \partial f(x_2)$ is either empty, if the last inequality is strict, or equal to $\{D^- f(x_2)\}$, if the last inequality is an equality. The right derivative $D^+ f$, being increasing, hence of bounded variation, and right-continuous, defines a (Lebesgue–Stieltjes) measure m via the prescription

$$D^+ f(b) - D^+ f(a) =: m(]a, b]), \qquad a, b \in I,\ a \leq b.$$

In this sense, the positive measure m can be interpreted as the second derivative of f.

We shall also use elementary properties of distributions, for which we refer, e.g., to Schwartz (1966) or Duistermaat and Kolk (2010). If E is an open subset of the real line, the space of infinitely differentiable functions from E to $\mathbb{R}$ with compact support will be denoted by $\mathscr{D}(E)$. Elements of the dual $\mathscr{D}'(E)$ of $\mathscr{D}(E)$, i.e. linear continuous functionals on $\mathscr{D}(E)$, are called distributions. The indication of the set E will be omitted if it coincides with $\mathbb{R}$. The pairing between distributions and infinitely differentiable functions with compact support will be denoted by $\langle\cdot,\cdot\rangle$. We recall that, given a distribution $F \in \mathscr{D}'(E)$, its distributional derivative F' is the distribution defined by

$$\left\langle F', \phi\right\rangle = -\left\langle F, \phi'\right\rangle \qquad \forall \phi \in \mathscr{D}(E).$$

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ induces the distribution, denoted by the same symbol, defined by

$$\langle f, \phi\rangle := \int_{\mathbb{R}} f\phi \qquad \forall \phi \in \mathscr{D}.$$

Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise of class C^1, with discontinuity points (x_n). Denoting the Dirac measure at $a \in \mathbb{R}$ by δ_a, and using the standard notations $\Delta f(x) := f(x+) - f(x-)$, one has (see Schwartz 1966, p. 37)

$$f' = \sum_n \Delta f(x_n)\delta_{x_n} + [f'],$$

where f' stands for the derivative of f in the sense of distributions, and $[f']$ for the derivative in the classical sense over the open intervals $]x_n, x_{n+1}[$ where f is continuously differentiable (a corresponding result for higher-order derivatives can be obtained by induction). If f is a function with finite variation, then f' coincides, in the sense of distributions, with the Lebesgue–Stieltjes measure df. We shall need to consider functions f that are piecewise differences of convex functions, i.e. of the form

$$f = \sum_n f_n, \qquad f_n\colon [a_n, a_{n+1}[\to \mathbb{R},$$

where the sum is (at most) countable, and for every n there exist convex functions h^1_n, h^2_n on $\mathbb{R}$ such that $f_n = h^1_n - h^2_n$ on $[a_n, a_{n+1}[$. Then f is càdlàg and has finite variation. We are going to compute the first and second distributional derivatives of f. It is clear that it is enough to consider, without loss of generality, f with support equal to $[a, b]$ and $f = h^1 - h^2$ on $[a, b[$, for h^1 and h^2 convex functions on $\mathbb{R}$.

Lemma 2.1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a càdlàg function with finite variation, and $[a, b] \subset \mathbb{R}$ a compact interval. The distributional derivative of $f_{[a,b]} := f 1_{[a,b]}$ is*

$$f'_{[a,b]} = df\big|_{]a,b[} + f(a)\delta_a - f(b-)\delta_b.$$

Proof Let us denote $f_{[a,b]}$, for the purposes of this proof only, simply by f. The distributional derivative f' is defined by the identity $\langle f', \phi \rangle = -\langle f, \phi' \rangle$ for every $\phi \in \mathscr{D}$, where

$$\langle f, \phi' \rangle = \int_{\mathbb{R}} f \phi' = \int_a^b f\, d\phi = \int_{]a,b]} f\, d\phi$$

and, thanks to the integration-by-parts formula (2.2),

$$f(b)\phi(b) - f(a)\phi(a) = \int_{]a,b]} f\, d\phi + \int_{]a,b]} \phi\, df,$$

hence

$$\int_{]a,b]} f\, d\phi = -\int_{]a,b]} \phi\, df + f(b)\phi(b) - f(a)\phi(a),$$

i.e.

$$\begin{aligned} \langle f', \phi \rangle &= \int_{]a,b]} \phi\, df + f(a)\phi(a) - f(b)\phi(b) \\ &= \int_{]a,b[} \phi\, df + \phi(b)(f(b) - f(b-)) + f(a)\phi(a) - f(b)\phi(b) \\ &= \int_{]a,b[} \phi\, df + f(a)\phi(a) - f(b-)\phi(b). \end{aligned}$$

□

Proposition 2.2 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a difference of convex functions and let* $[a, b] \subset \mathbb{R}$ *be a compact interval. Then the first and second distributional derivatives of* $f_{[a,b]} := f 1_{[a,b]}$ *are*

$$\begin{aligned} f'_{[a,b]} &= df\big|_{]a,b[} + f(a)\delta_a - f(b)\delta_b, \\ f''_{[a,b]} &= dD^+ f\big|_{]a,b[} + f(a)\delta'_a - f(b)\delta'_b + D^+ f(a)\delta_a - D^+ f(b-)\delta_b. \end{aligned}$$

Proof Let us write again f to denote $f_{[a,b]}$ for simplicity, and only for the purposes of this proof. The function f is continuous on $\mathbb{R}$ and, being the difference of convex functions, has finite variation. The previous lemma then yields the expression for f'. To compute f'', let us recall that f is absolutely continuous with classical derivative equal to $D^+ f$ a.e., so that

$$\begin{aligned} \langle f'', \phi \rangle = -\langle f', \phi' \rangle &= -\int_{]a,b[} \phi'\, df - f(a)\phi'(a) + f(b-)\phi'(b) \\ &= -\int_{]a,b[} D^+ f\, d\phi - f(a)\phi'(a) + f(b-)\phi'(b). \end{aligned}$$

Since $D^+ f$ is càdlàg and of finite variation, the integration-by-parts formula (2.2) yields

$$D^+ f(b)\phi(b) - D^+ f(a)\phi(a) = \int_{]a,b]} D^+ f \, d\phi + \int_{]a,b]} \phi \, dD^+ f,$$

hence

$$\begin{aligned} -\int_{]a,b[} D^+ f \, d\phi &= -\int_{]a,b]} D^+ f \, d\phi \\ &= \int_{]a,b]} \phi \, dD^+ f + D^+ f(a)\phi(a) - D^+ f(b)\phi(b) \\ &= \int_{]a,b[} \phi \, dD^+ f + D^+ f(a)\phi(a) - D^+ f(b-)\phi(b). \end{aligned}$$

Collecting terms concludes the proof:

$$\langle f'', \phi \rangle = \int_{]a,b[} \phi \, dD^+ f - f(a)\phi'(a) + f(b-)\phi'(b) + D^+ f(a)\phi(a) - D^+ f(b-)\phi(b). \quad \square$$

Remark 2.3 Let $a < b$ be real numbers and I be any interval with endpoints a and b. Note that $f 1_{[a,b]}$ coincides in $\mathscr{D}'$ with $f 1_I$, for any choice of I. Therefore their distributional derivatives are also the same.

3 Pricing functionals, measurements and representations

3.1 Pricing functionals

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathscr{F}_t)_{t \in [0,T]}$, with $T > 0$ a fixed time horizon, and let $S\colon \Omega \times [0, T] \to \mathbb{R}_+$ be the price process of an asset. We assume also that $\beta\colon \Omega \times [0, T] \to]0, \infty[$ is the price process of a further asset used as numéraire, normalized with $\beta_0 = 1$ and uniformly bounded from below, and that the market where both assets are traded is free of arbitrage, so that the set Q of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted price process $\beta^{-1} S$ is a $\mathbb{Q}$-local martingale is not empty. For any $\mathscr{F}_T$-measurable claim X such that $\beta_T^{-1} X$ is bounded, the value $\mathbb{E}_{\mathbb{Q}} \beta_T^{-1} X$ is an arbitrage-free price at time zero of X for every $\mathbb{Q} \in \mathsf{Q}$. From now on we shall fix a measure $\mathbb{Q} \in \mathsf{Q}$. For any measurable bounded function $g\colon \mathbb{R}_+ \to \mathbb{R}$, the bounded $\mathscr{F}_T$-measurable random variable $g(S_T)$ is the payoff of a European option on S with payoff profile g, the price of which at time zero is $\mathbb{E}_{\mathbb{Q}} \beta_T^{-1} g(S_T)$.

We shall call *pricing functional* the map

$$\pi : g \longmapsto \mathbb{E}_{\mathbb{Q}} \beta_T^{-1} g(S_T) = \mathbb{E} \frac{d\mathbb{Q}}{d\mathbb{P}} \beta_T^{-1} g(S_T),$$

defined first on the set of measurable bounded functions $g\colon \mathbb{R}_+ \to \mathbb{R}$. Let μ be the measure on $\mathscr{F}_T$ defined by

$$\mu(A) := \mathbb{E}_{\mathbb{Q}}\beta_T^{-1}1_A, \tag{3.1}$$

that is, μ is the measure on $\mathscr{F}_T$ the Radon-Nikodym derivative of which with respect to $\mathbb{P}$ is

$$\frac{d\mu}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}}\beta_T^{-1}.$$

Note that μ is (in general) not a probability measure: in fact, $\mu(\Omega) = \mathbb{E}_{\mathbb{Q}}\beta_T^{-1}$ need not be one, and could be interpreted as the price at time zero of a zero-coupon bond maturing at time T with face value equal to one. In this case, μ is a sub-probability measure, i.e. $\mu(\Omega) \leq 1$. The pricing functional can then be written as

$$\pi : g \longmapsto \int_\Omega g(S_T)\,d\mu.$$

Denoting the pushforward of μ through S_T by $S_*\mu$, i.e. the measure on the Borel σ-algebra of $\mathbb{R}$ defined by

$$S_*\mu\colon B \longmapsto \mu(S_T^{-1}(B)),$$

one has

$$\mathbb{E}_{\mathbb{Q}}\beta_T^{-1}g(S_T) = \int_\Omega g(S_T)\,d\mu = \int_{\mathbb{R}} g\,d(S_*\mu).$$

Therefore, denoting the distribution function of the measure $S_*\mu$ by F, i.e.

$$F(x) := \mu(S_T \leq x) = \mathbb{E}_{\mathbb{Q}}\beta_T^{-1}1_{\{S_T\leq x\}},$$

the pricing functional can be written as

$$\pi : g \longmapsto \int_{\mathbb{R}_+} g\,dF.$$

In other words, the pricing functional can be identified with F, or with $S_*\mu$. Note also that the pricing functional can naturally be extended to every $g \in L^1(dF)$. If $g \geq 0$, as is mostly the case for payoff functions, then $\pi(g)$ is simply the norm of g in $L^1(dF)$.

Remark 3.1 If one just assumes that there exists a pricing functional, defined as a positive linear functional on a certain set of functions, then an integral representation of π holds in many cases. This is essentially the content of various forms of the Riesz representation theorem. For instance, if π is continuous on $C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ that are zero at infinity, endowed with the supremum norm, then there exists a unique finite Radon measure m on $\mathbb{R}$ such that $\pi(g) = \int g\,dm$ for every $g \in C_0(\mathbb{R})$. If π is continuous on $C_c(\mathbb{R})$, the space of continuous functions on

$\mathbb{R}$ with compact support with the topology of uniform convergence on compact sets, then there exists a unique Radon measure m on $\mathbb{R}$ (not necessarily finite) such that $\pi(g) = \int g\,dm$ for every $g \in C_c(\mathbb{R})$. On the other hand, if π is just assumed to be continuous on $\mathscr{L}^\infty(\mathbb{R})$, the Banach space of bounded measurable functions with the supremum norm, then an integral representation of π is only possible with respect to a bounded additive set function, not a measure. A completely analogous situation arises if continuity of π is assumed on $L^\infty(\mathbb{R})$. On the other hand, if π is assumed to be weak* continuous on $L^\infty(\mathbb{R})$, then there exists $\phi \in L^1_+(\mathbb{R})$ such that $\pi(g) = \int \phi g$ for every $g \in L^\infty(\mathbb{R})$. However, the weak* convergence of a sequence (g_n) in L^∞, i.e. the existence of $g \in L^\infty$ such that

$$\lim_{n\to\infty} \int f g_n = \int fg \quad \forall f \in L^1(\mathbb{R}),$$

does not seem to have a clear economic interpretation.

3.2 Measurements and representations

Depending on the problem at hand, the pricing functional $\pi : g \mapsto dF(g)$ may or may not be known. If dF is assumed a priori to be known, as, for instance, in the Black-Scholes model with given volatility, then π is trivially known. An analogous situation arises in the case where dF is assumed to belong to a family of finite measures $(dF_\theta)_{\theta\in\Theta}$ parametrized by a finite-dimensional parameter θ: once an estimate $\hat{\theta}$ is obtained, so that $dF_{\hat{\theta}}$ is used in the definition of π, one falls back into the previous (quite tautological) case. On the other hand, in many other situations, for instance when no parametric assumptions on dF are made, the pricing functional π is only known through its action on a set of "test functions" $(g_j)_{j\in J}$, e.g. with g_j the payoff profile of a call or put option with a strike price indexed by $j \in J$. The next definition is hence natural.

Definition 3.2 A *measurement* (of dF) is a pair $(g, \pi(g))$, where $g : \mathbb{R}_+ \to \mathbb{R}$ is a measurable function integrable with respect to dF. A *measurement set* (of dF) is a collection of measurements.

A typical situation of practical relevance is given by (g_j) being a collection of payoff profiles of (European) options. For instance, for any $j \geq 0$, let g_j be the payoff function of a put option with strike price j, that is, $g_j : x \mapsto (j - x)^+$. If the price of the put option with strike j is known for every $j > 0$, then we have a measurement $M = (g_j, \pi_j)_{j\in J}$ setting $J =]0, +\infty[$, $g_j : x \mapsto (j - x)^+$, and $\pi_j = dF(g_j)$.

Remark 3.3 a) A measurement set is just a subset of the graph of π.
b) The term "measurement", by no means standard, somewhat mimics an analogous one used in the theory of inverse problems, where, in a (usually) more rigid functional setting, the expression "measurement operator" is sometimes used.
c) In view of the linearity of integration, if π is known on a set $G \subseteq L^1(dF)$, then it is known also on the vector space generated by G. Similarly, it would seem natural to augment M with its accumulation points, i.e. to take its closure, in $L^1(dF) \times \mathbb{R}$.

However, since we treat dF as unknown, this operation would not be plausible. Some accumulation points can be added nonetheless, as we shall see below, so long as they are constructed without using dF or, more precisely, assuming that all is known about dF is (the vector space generated by) M.

Measurement sets can be ordered by inclusion, hence they can be compared. If M is a measurement set, the vector space generated by M, itself a measurement set, will be denoted by $\hat{M}$.

Definition 3.4 Let M_1 and M_2 be two measurement sets. One says that M_1 is *finer* than M_2 if $\hat{M}_1$ contains M_2, and that M_1 and M_2 are *equivalent* if M_1 is finer than M_2 and M_2 is finer than M_1, i.e. if $\hat{M}_1 = \hat{M}_2$. A *representation* is a measurement set finer than $(1_A, dF(A))_{A\in\mathscr{A}}$, where $\mathscr{A}$ is the Borel σ-algebra of $\mathbb{R}_+$.

Remark 3.5 It would be enough to take as $\mathscr{A}$ any collection of subsets of $\mathbb{R}_+$ generating the Borel σ-algebra $\mathscr{B}(\mathbb{R}_+)$ such that, for any two finite measures m and n on $\mathscr{B}(\mathbb{R}_+)$ with $m(\mathbb{R}_+) = n(\mathbb{R}_+)$, $m(A) = n(A)$ for every $A \in \mathscr{A}$ implies $m = n$. This is the case, for instance, if $\mathscr{A}$ is stable with respect to finite intersection (see, e.g., Cohn 2013, Corollary 1.6.3).

Apart from the natural inverse problem of recovering the measure dF from a sufficiently rich collection of option prices, possibly providing an algorithm to do so, it is interesting also to describe relations between measurement sets. For instance, if one needs F only to price a certain set of options, instead of reconstructing F it could suffice to identify a measurement set that already allows to accomplish the task. In the simplest case, if g is the payoff profile of the option to price and g belongs to the vector space generated by an available measurement set M, there is clearly no need to recover F. In spite of its simplicity, this is precisely how one can proceed to price options with continuous piecewise linear payoff profile. In fact, as is well known, these options can be priced in terms of linear combinations (independent of F!) of prices of put options with strikes at the "juncture" points of the piecewise linear profile. A more sophisticated fact is that call option prices for every positive strike price allow to price option with arbitrary convex payoff. In this case, however, if g is an arbitrary convex function and $\mathrm{pr}_1 M$, the projection on $L^1(dF)$ of the measurement set M, is the vector space generated by $(x \mapsto (x-k)^+)_{k\in\mathbb{R}_+}$, it is not true in general that $g \in \mathrm{pr}_1 M$. It is true, however, that g is an accumulation point of $\mathrm{pr}_1 M$, as discussed in Sect. 5 below.

It was mentioned above that it would not be meaningful to extend a measurement set M taking its closure in $L^1(dF)\times\mathbb{R}$, as F is considered unknown. However, one can indeed add some cluster points, if they are defined by procedures that do not involve F. In particular, at least two possibilities exist:

(a) let $(g_n) \subseteq \mathrm{pr}_1 M$ be a sequence that converges pointwise to g and for which there exists $h \in L^1(dF)$ such that $|g_n(x)| \leq h(x)$ for all $x \in \mathbb{R}_+$. The dominated convergence theorem then implies that $g \in L^1(dF)$ and that $\pi(g) = \lim_{n\to\infty} \pi(g_n)$;
(b) let $(g_n) \subseteq \mathrm{pr}_1 M$ be such that $g_n \uparrow g$, i.e. (g_n) is an increasing sequence that converges pointwise to g, and such that $(\pi(g_n))$ is bounded from above, i.e. $\sup_n \pi(g_n) < \infty$. Then $0 \leq g_n - g_0 \uparrow g - g_0$ and, by the monotone conver-

gence theorem,

$$\pi(g - g_0) = \lim_n \pi(g_n - g_0) = \sup_n \pi(g_n) - \pi(g_0) < \infty,$$

hence $g - g_0 \in L^1(dF)$, i.e. $g \in L^1(dF)$ with $\pi(g) = \sup_n \pi(g_n)$.

The cluster points constructed in (a) and (b) do not depend on knowing F, hence they could reasonably be added to the measurement set M. The measurement sets obtained by adding to $\hat{M}$ the cluster points described in (a) and (b) will be denoted by M^d and M^m, respectively. We shall see that if M_1 is the measurement set of all call options, and M_2 the measurement set of all convex options, then M_1^m is finer than M_2. Since M_2 is finer than M_1^m (the pointwise supremum of a family of convex functions is convex), M_1^m and M_2 are equivalent measurement sets. In other words, one cannot replicate a convex payoff with just call payoffs, but one can approximate a convex payoff by a combination of call payoffs with any pricing accuracy.

Taking suitable limits of sequences of measurements is not the only possible way to enrich a measurement set. In fact, one can also perform several operations on $(\pi_j)_{j\in J} \subseteq \mathbb{R}$, using the structure of $\mathbb{R}$: they can for instance be added, multiplied, and functions $\phi\colon \mathbb{R}^n \to X$ can be applied to n of them, with X suitable sets, and so on. Note that M could also be seen as a linear map from the space of finite measures $\mathscr{M}^1(\mathbb{R}_+)$ to $\mathbb{R}^J$, mapping dF to $(dF(g_j))_{j\in J}$. Viewing elements of $\mathbb{R}^J$ as functions from J to $\mathbb{R}$, the problem at hand may imply that these functions in the codomain have additional properties, for instance they may be monotone, or convex, or continuous, or differentiable, depending on the inputs (g_j). Depending on the range of M in the codomain $\mathbb{R}^J$, different operations may be applied. For instance, taking derivatives on $\mathbb{R}^J$ or on $C(J)$ would not make sense, but it would make sense on $C^1(J)$, or in the a.e. sense if we knew that the range is made of Lipschitz continuous functions. We shall see that this point of view is also fruitful, showing that the right derivative of put prices, seen as a function P of the strike price, is equal to F. We shall also see that the price of an option with arbitrary convex payoff can be written in terms of an integral of C, where $C(k)$ is the price of the call option with strike k.

In some cases one does not observe a measurement directly, but a function of a measurement. This is the case, for instance, of implied volatility. If $g_k\colon x \mapsto (k-x)^+$ is the payoff function of a put option with strike k, there is a one-to-one correspondence between $\pi_k := \pi(g_k)$ and the Black-Scholes implied volatility, given by a function $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\pi_k = \mathsf{BS}(S_0, k, T, v(\pi_k))$. Here $\mathsf{BS}(S_0, k, T, \sigma)$ denotes the Black-Scholes price at time zero of a put option on an underlying with price at time zero equal to S_0, strike k, time to maturity T, volatility σ, and interest rate as well as dividend rate equal to zero. In particular, if the implied volatility is known for every strike $k > 0$, inverting the function v we obtain the measurement set of put prices $M = (g_k)_{k\geq 0}$, which is a representation. In other words, implied volatility for all strikes uniquely determines the pricing functional or, equivalently, the measure dF. We may then say, with a slight abuse of terminology, that implied volatility is a representation.

Let X be a locally compact space and $\phi\colon \mathbb{R} \to X$ be a measurable isomorphism, i.e. a bijection such that both ϕ and ϕ^{-1} are measurable. This is the case, for instance, if ϕ is a homeomorphism. Then, for any $g \in L^1(dF)$, one has

$$dF(g) = \langle g, dF \rangle = \big\langle \phi^*(\phi^{-1})^* g, dF \big\rangle = \big\langle (\phi^{-1})^* g, \phi_* dF \big\rangle.$$

This change of parametrization can also be interpreted in terms of measurement sets, saying that the measurement set $M = (g_j, dF(g_j))$ of dF is in bijective correspondence with the measurement set

$$M' = \big((\phi^{-1})^* g_j, \phi_* dF((\phi^{-1})^* g_j)\big) = \big((\phi^{-1})^* g_j, dF(g_j)\big)$$

of $\phi_* dF$. Even though the two measurements are isomorphic (as sets), they may have quite different properties. Let us consider, for instance, the reparametrization from price to logarithmic return: setting $S_T = S_0 \exp(\sigma X_T + m)$, where $\sigma > 0$ and m are constants, the pricing functional can be written as

$$\pi : g \longmapsto \int_{\mathbb{R}} g(S_0 e^{\sigma x + m}) dF_X(x),$$

where F_X is the distribution function of the measure $(X_T)_* \mu$, the support of which is $\mathbb{R}$. If g is the payoff function of a put option with strike k, then $x \mapsto g(S_0 e^{\sigma x+m}) = (k - S_0 e^{\sigma x+m})^+$ does *not* have compact support. This is clearly in stark contrast to the expression of $\pi(g)$ in terms of dF, where the intersection of the supports of g and dF is compact. As will be seen, several analytic arguments strongly depend on this property, that hence cannot be used with the new parametrization, even though the values of the corresponding integrals are the same.

Finally, we remark that it is sometimes useful to extend the definition of measurement set allowing for an extra collection (dF_j) of (possibly signed) measures for which a relation to dF is known. For instance, let g_k, for any $k \geq 0$, be the payoff function of a put option with strike price k, that is, $g_k : x \mapsto (k - x)^+$. Moreover, let (dF_n) be a sequence of Radon measures converging weakly to dF as $n \to \infty$. Each measure dF_n can be thought of as an approximation to the law dF, and $dF_n(g_k)$ as the price of a put option with strike k under the approximating law dF_n. If all such prices can be observed, then we have an "extended" measurement set $\widetilde{M} := (g_j, \pi_j, dF_j)_{j \in J}$, where $J = \mathbb{R}_+ \times \mathbb{N}$, $F_j = F_{kn}$, $F_{kn} = F_n$ for every k, $g_j = g_{kn} = g_k$ for every n, and $\pi_j = \pi_{kn} = F_n(g_k)$. Note that $g_k \in C_b(\mathbb{R})$ for every k, hence $F_n(g_k) \to F(g_k)$ as $n \to \infty$. In particular, if we define the (standard) measurement set $M = (g_j, \pi_j)_{j \in \mathbb{R}_+}$ as $g_j : x \mapsto (j - x)^+$ and $\pi_j = dF(g_j)$, then we could say that $\widetilde{M}$ "implies" M. That is, for every $g \in \mathrm{pr}_1 M$ there exists a sequence $(dF_n) \subset \mathrm{pr}_3 M$ such that $\pi(g) = dF(g)$ is the limit of $dF_n(g) \subset \mathrm{pr}_2 M$. An example of this type, motivated by empirical nonparametric option pricing, is discussed in Sect. 7 below.

4 Put and call option prices and the pricing functional

Let us define the functions $P,\ C\colon \mathbb{R}_+ \to [0, +\infty]$ by

$$P(k) := \int_{\mathbb{R}_+} (k-x)^+ \, dF(x), \qquad C(k) := \int_{\mathbb{R}_+} (x-k)^+ \, dF(x).$$

Note that $P(k)$ is finite for all k as $\mathbb{R}_+ \ni x \mapsto (k-x)^+$ is bounded (in fact, it is continuous with compact support), but $C(k)$ is finite if and only

$$\int_k^\infty x \, dF(x) < \infty,$$

hence C is everywhere finite if and only dF has a finite mean

$$\overline{dF} := \int_{\mathbb{R}_+} x \, dF(x).$$

The financial interpretation of $\overline{dF} < \infty$ is that $\mathbb{E}_{\mathbb{Q}} \beta_T^{-1} S_T$ is finite. This is clearly not a limitation. Moreover, we recall that $F(\infty) := \lim_{x\to\infty} F(x)$ can be identified with the price at time zero of a zero-coupon bond expiring at time T and with unit face value, therefore $F(\infty) \leq 1$. However, we shall explicitly mention hypotheses on F because some of the considerations to follow may be interesting for general F, independently of the underlying financial interpretation.

The functions P and C will play a central role, so we discuss some of their properties. They are all rather basic, but they are fully justified for completeness.

Proposition 4.1 *The function P is increasing, locally Lipschitz continuous, Lipschitz continuous if dF is finite, convex, and satisfies the inequality $P(k) \leq kF(k)$ for every $k \geq 0$. Moreover, $P(0) = 0$ and*

$$\lim_{k\to\infty} \frac{P(k)}{k} = F(\infty),$$

hence, in particular, $\lim_{k\to\infty} P(k) = \infty$.

Proof Since $k \mapsto (k-x)^+$ is increasing for every $x \in \mathbb{R}_+$ and integration with respect to a positive measure is positivity preserving, P is increasing. Similarly, as $k \mapsto k^+$ is 1-Lipschitz continuous, $k \mapsto (k-x)^+$ is 1-Lipschitz continuous uniformly with respect to x, hence P is locally Lipschitz continuous as well and globally Lipschitz continuous if $F(\infty) < \infty$. To prove convexity, note that, for any $x \in \mathbb{R}_+$, $k \mapsto k - x$ is affine, in particular convex, and $y \mapsto y^+$ is convex increasing, hence the composite function $k \mapsto (k-x)^+$ is convex. Finally, integration with respect to a positive measure preserves convexity, hence P is convex. The identity $P(0) = 0$ follows immediately by the definition of P, as does the estimate $P(k) \leq kF(k)$, where $F(k) \leq F(\infty)$.

Finally,

$$\frac{P(k)}{k} = \frac{1}{k}\int_{[0,k]} (k-x)\,dF(x) = \int_{\mathbb{R}_+} \Big(1-\frac{x}{k}\Big)1_{[0,k]}\,dF(x),$$

where $(1-x/k)1_{[0,k]} \to 1$ for all $x \geq 0$ as $k \to \infty$ and $(1-x/k)1_{[0,k]} \in [0,1]$ for all $x, k \geq 0$, hence the dominated convergence theorem implies

$$\lim_{k\to\infty} \frac{P(k)}{k} = \int_{\mathbb{R}_+} dF = F(\infty). \qquad \square$$

Proposition 4.2 *Assume that* $\overline{dF} < \infty$. *The function* C *is decreasing, Lipschitz continuous, and convex. Moreover,* $C(0) = \overline{dF}$ *and* $\lim_{k\to\infty} C(k) = 0$.

Proof The proof of monotonicity, Lipschitz continuity, and convexity are entirely similar to the corresponding proof for put options, noting that $k \mapsto (x-k)^+$ is decreasing. The definition of C immediately implies that $C(0) = \int_{\mathbb{R}_+} x\,dF(x)$, and also that

$$C(k) = \int_k^\infty (x-k)\,dF(x) \leq \int_{[k,\infty[} x\,dF(x),$$

where the right-hand side converges to zero as $k \to \infty$ because $\int_{\mathbb{R}_+} x\,dF(x)$ is finite by assumption. $\square$

By a direct computation one can obtain estimates for local and global Lipschitz constants. In fact, the 1-Lipschitz continuity of $x \mapsto x^+$, hence also of $k \mapsto (k-x)^+$, yields, for any $k_1, k_2 \geq 0$,

$$\begin{aligned} |P(k_2) - P(k_1)| &\leq \int_{[0,k_1\vee k_2]} \big|(k_2-x)^+ - (k_1-x)^+\big|\,dF(x) \\ &= \int_{[0,k_1\vee k_2[} \big|(k_2-x)^+ - (k_1-x)^+\big|\,dF(x) \\ &\leq \int_{[0,k_1\vee k_2[} |k_2-k_1|\,dF(x) = |k_2-k_1|\,F_-(k_1\vee k_2), \end{aligned}$$

where F_- stands for the left-continuous version of F, defined by $F_-(x) := F(x-) := \lim_{h\to 0+} F(x-h)$. The same estimate holds for P replaced by C. One can actually show, using subdifferentials, that the Lipschitz continuity estimates thus obtained are sharp. In fact, for any $k_1, k_2 \geq 0$, convexity implies

$$P(k_2) \geq P(k_1) + \partial P(k_1)(k_2-k_1)$$

where[1] ∂ stands for the subdifferential in the sense of convex analysis. Hence, if $k_1 \geq k_2$, $P(k_1) \geq P(k_2)$ because P is increasing, which also implies that $\partial P(x) \subset \mathbb{R}_+$

[1] Since $\partial P(k_1)$ is in general a set, one should write $P(k_2) \geq P(k_1) + y(k_2-k_1)$ for every $y \in \partial P(k_1)$. This slight abuse of notation shall not create any harm though.

for every $x > 0$, hence

$$\left|P(k_1) - P(k_2)\right| = P(k_1) - P(k_2) \le \partial P(k_1)(k_1 - k_2) = \partial P(k_1)\,|k_1 - k_2|\,.$$

Similarly, if $k_1 \le k_2$,

$$\left|P(k_1) - P(k_2)\right| = P(k_2) - P(k_1) \le \partial P(k_2)(k_2 - k_1) = \partial P(k_2)\,|k_1 - k_2|\,.$$

Recalling that $\partial P(k) = [D^- P(k), D^+ P(k)]$ for every $k > 0$, it easily follows that

$$\left|P(k_1) - P(k_2)\right| \le D^- P(k_1 \vee k_2)\ |k_1 - k_2|\,.$$

As $D^+ P = F$ and the left-continuous version of $D^+ P$ is $D^- P$, it follows that $D^- P = F_-$.

We are going to show that F is the right derivative of P, and that a similar relation holds between C and F. We give two proofs, one that relies on the integration-by-parts formula for càdlàg functions, and a slightly more indirect one based on a denseness result. Namely, we show that the set of put payoffs are total in $L^1(dF)$, i.e. that for any $g \in L^1(dF)$ there exist a sequence of *finite* linear combinations of put payoffs that converges to g in $L^1(dF)$. Then we show that the two approaches are in fact equivalent. A third approach, using distributions, will be given in Sect. 6 below.

4.1 Reconstruction of *F* via integration by parts

We shall apply the integration-by-parts formula (2.2) to establish a connection between the distribution function F and the price functions for put and call options P and C.

Theorem 4.3 *One has $P' = F$ a.e. in $\mathbb{R}_+$ and $D^+ P(x) = F(x)$ for every $x \in \mathbb{R}_+$. Moreover, if the measure dF has finite mean, then $C' = F - F(\infty)$ a.e. in $\mathbb{R}_+$ and $D^+ C(x) = F(x) - F(\infty)$ for every $x \in \mathbb{R}_+$.*

Proof Let $k \ge 0$ and $G\colon x \mapsto k - x$. The integration-by-parts formula

$$G(k)F(k) - G(0)F(0) = \int_{]0,k]} G(x)\,dF(x) + \int_{]0,k]} F(x)\,dG(x)$$

yields

$$\begin{aligned}\int_0^k F(x)\,dx &= kF(0) + \int_{]0,k]} (k - x)\,dF(x)\\ &= \int_{[0,k]} (k - x)\,dF(x)\\ &= \int_{\mathbb{R}_+} (k - x)^+\,dF(x) = P(k).\end{aligned}$$

The Lebesgue differentiation theorem then implies that $P' = F$ a.e. in $\mathbb{R}_+$. Moreover, since F is right-continuous by definition, and P is convex, hence right-differentiable, we also have $D^+P(x) = F(x)$ for every $x \in \mathbb{R}_+$.

Obtaining a relation between C and F along the same lines is a bit more involved: if $k > 0$ and $G\colon x \mapsto x - k$, one has, for any $a > k$,

$$G(a)F(a) - G(k)F(k) = \int_{]k,a]} G(x)\,dF(x) + \int_{]k,a]} F(x)\,dG(x),$$

i.e.

$$(a-k)F(a) = \int_{]k,a]} (x-k)\,dF(x) + \int_k^a F(x)\,dx,$$

which is equivalent to

$$\int_{[k,a]} (x-k)\,dF(x) = \int_k^a (F(a) - F(x))\,dx.$$

Therefore, by the monotone convergence theorem,

$$\begin{aligned} \lim_{a\to\infty} \int_{[k,a]} (x-k)\,dF(x) &= \lim_{a\to\infty} \int_{\mathbb{R}_+} 1_{[k,a]}(x-k)\,dF(x) \\ &= \int_{[k,\infty[} (x-k)\,dF(x) \\ &= \int_{\mathbb{R}_+} (x-k)^+\,dF(x) = C(k), \end{aligned}$$

as well as

$$\begin{aligned} \lim_{a\to\infty} \int_k^a (F(a) - F(x))\,dx &= \lim_{a\to\infty} \int_{\mathbb{R}_+} 1_{[k,a]}(F(a) - F(x))\,dx \\ &= \int_k^\infty (F(\infty) - F(x))\,dx, \end{aligned}$$

hence

$$C(k) = \int_k^\infty (F(\infty) - F(x))\,dx. \tag{4.1}$$

This implies $C' = F - F(\infty)$ a.e. as well as, by right continuity of F and convexity of C, $D^+C(x) = F(x) - F(\infty)$ for every $x \in \mathbb{R}_+$. □

The finiteness of the integral on the right-hand side of (4.1) is implied by the finiteness of $C(k)$, which in turn follows by the assumption that dF has finite mean. One may also easily see directly that the last assumption implies that the integral is finite. In fact, this produces another proof of the identity (4.1): by Tonelli's theorem,

$$\begin{aligned}\int_k^\infty (F(\infty) - F(x))\,dx &= \int_k^\infty \int_{]x,\infty[} dF(y)\,dx = \int_{[k,\infty[} \int_k^y dx\,dF(y)\\ &= \int_{[k,\infty[} (k-y)\,dF(y) = \int_{\mathbb{R}_+} (k-y)^+\,dF(y)\\ &= C(k).\end{aligned}$$

The relation between C and F can of course be obtained also from put-call parity, once the relation between P and F has been obtained: if follows from the identity $x - k = (x-k)^+ - (k-x)^+$, upon integrating with respect to dF, that

$$\int_{\mathbb{R}_+} x\,dF(x) - k\int_{\mathbb{R}_+} dF = C(k) - P(k),$$

hence, by Lebesgue's differentiation theorem, $-F(\infty) = C'(k) - P'(k) = C'(k) - F(k)$ for a.a. $k \in \mathbb{R}_+$, as well as $D^+C(k) = F(k) - F(\infty)$ for every $k \in \mathbb{R}_+$ by the same argument used above.

4.2 Reconstruction of F by approximation in $L^1(dF)$

Let V be the vector space generated by put payoff profiles, i.e. by the family of functions $\mathbb{R}_+ \ni x \mapsto (k-x)^+, k \geq 0$. We are going to show the following approximation result.

Lemma 4.4 *Let $a > 0$. For any $\varepsilon > 0$ there exists $\phi \in V$ such that*

$$\left\|\phi - 1_{[0,a]}\right\|_{L^1(dF)} < \varepsilon.$$

Proof Since F is right-continuous, there exists $b > a$ such that $F(b) - F(a) < \varepsilon$. Set $\phi_a(x) := (a-x)^+$, $\phi_b(x) := (b-x)^+$, $\alpha = 1/(b-a)$, and $\phi := \alpha\phi_b - \alpha\phi_a$. Then easy computations show that $\phi\colon \mathbb{R}_+ \to [0,1]$ is a continuous function with support $[0,b]$, equal to one on $[0,a]$. More precisely,

$$\phi(x) = \begin{cases} \alpha(b-a) = 1, & 0 \leq x \leq a,\\ \alpha b - \alpha x, & a \leq x \leq b,\\ 0, & x \geq b.\end{cases}$$

Since $\phi = 1_{[0,a]} + \phi 1_{]a,b]}$, we have

$$\left|\phi - 1_{[0,a]}\right| = \phi 1_{]a,b]} \leq 1_{]a,b]},$$

hence

$$\left\|\phi - 1_{[0,a]}\right\|_{L^1(dF)} \leq \int_{[0,b]} 1_{]a,b]}\,dF = F(b) - F(a) < \varepsilon. \qquad \square$$

This shows that we can explicitly approximate F by P. The proof of the lemma also shows that $D^+P = F$: for any $a > 0$, take a sequence (b_n) converging to a from the right, and call ϕ_n the corresponding approximating sequence converging to $1_{[0,a]}$ in $L^1(dF)$, for which

$$\int \phi_n \, dF = \int \frac{1}{b_n - a}\big((b_n - x)^+ - (a - x)^+\big)\, dF(x) = \frac{P(b_n) - P(a)}{b_n - a},$$

hence

$$F(a) = \lim_{n\to\infty} \int \phi_n \, dF = \lim_{n\to\infty} \frac{P(b_n) - P(a)}{b_n - a} = D^+P(a).$$

This approach to proving that $D^+P = F$ (that, by the way, does not require any further condition on F) is probably the most elementary. Note that the approach via integration by parts of Sect. 4.1 also implies

$$F(a) = D^+P(a) = \lim_{n\to\infty} \frac{P(b_n) - P(a)}{b_n - a} = \lim_{n\to\infty} \int \phi_n \, dF,$$

while here we prove the seemingly more precise limiting relation $\phi_n \to 1_{[0,a]}$ in $L^1(dF)$. This, however, can be deduced from F. Riesz's lemma[2] since both $1_{[0,a]}$ and ϕ_n are positive, $\phi_n \to 1_{[0,a]}$ a.e. and $\int \phi_n \, dF \to \int 1_{[0,a]} \, dF$, it follows that $\phi_n \to 1_{[0,a]}$ in $L^1(dF)$. Therefore also the integration-by-parts proof of $D^+P = F$, together with F. Riesz's lemma, implies that indicator functions of intervals can be obtained as limits in $L^1(dF)$ of linear combinations of put payoff profiles, that are explicitly determined.

Even though the previous lemma is enough to obtain F from P, a more general denseness result holds.

Proposition 4.5 *The vector space V generated by put payoff profiles is dense in $L^1(dF)$.*

Proof Let $g \in L^1(dF)$ and $\varepsilon > 0$. Since simple functions are dense in $L^1(dF)$, there exists $n \in \mathbb{N}$ and $A_i :=]a_i, b_i]$, $0 \le a_i \le b_i$, and $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that

$$\Big\| g - \sum_{i=1}^n c_i 1_{A_i} \Big\|_{L^1(dF)} \le \varepsilon/2.$$

By Lemma 4.4, the indicator function of any interval of $\mathbb{R}_+$ open to the left and closed to the right can be approximated by an element of V. Therefore, for every $i = 1, \ldots, n$ there exists $\phi_i \in V$ such that (all norms until the end of the proof are meant to be in $L^1(dF)$)

$$\big\| \phi_i - 1_{A_i} \big\| \le \frac{1}{n\,|c_i|} \frac{\varepsilon}{2},$$

[2] This result is often called Scheffé's lemma: in a general measure space with measure μ, if $f_n \to f$ μ-a.e. and $\int |f_n| \, d\mu \to \int |f| \, d\mu$, then $f_n \to f$ in $L^1(\mu)$.

hence, setting $\phi := \sum c_i \phi_i$,

$$\begin{aligned}\|g - \phi\| &\leq \Big\| g - \sum_{i=1}^{n} c_i 1_{A_i} \Big\| + \sum_{i=1}^{n} |c_i| \, \big\| 1_{A_i} - \phi_i \big\| \\ &\leq \varepsilon/2 + \sum_{i=1}^{n} |c_i| \frac{1}{n\,|c_i|} \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Since ϕ clearly belongs to V, the proof is completed. □

5 Convex payoffs

We are going to show that prices of call options for all strikes determine the price of any option with arbitrary convex payoff function (the result is not new—see, e.g., Karatzas and Shreve 1998, pp. 51–52, with a different proof), thus also for options with payoff function that can be written as the difference of two convex functions.

Using the language of Sect. 3, let $M_1 = (g, dF(g))_{g \in G}$ be the measurement set with G the set of convex functions on $\mathbb{R}_+$ (satisfying the assumption below), and $M_2 = (g_k, dF(g_k))_{k \in \mathbb{R}_+}$, $g_k : x \mapsto (x-k)^+$, the measurement set of call options (for all strikes). It is evident that M_1 is finer than M_2. We shall show that M_2^m is finer than M_1, hence that M_1 and M_2^m are equivalent (in particular, M_1 is a representation). The proof will actually establish that, for any $g \in G$, $\pi(g)$ can be written in terms of an integral of the function $C : k \mapsto \pi(g_k)$. This will then be shown to belong to M_2^m.

Throughout this section we assume that $g : \mathbb{R}_+ \to \mathbb{R}$ is the restriction to $\mathbb{R}_+$ of a convex function h on an open set $I \supset \mathbb{R}_+$. In particular, $D^+ g(0) > -\infty$. In order to avoid trivialities, we also assume that $g \in L^1(dF)$. We recall that g is continuous, differentiable almost everywhere, right-differentiable on $[0, \infty[$, and that $D^+ g$ is increasing and càdlàg. In particular, $D^+ g$ has finite variation, thus generates a Lebesgue–Stieltjes measure that we shall denote by m, or also by dg'. The positive measure m can also be identified with the second derivative of g in the sense of distributions.

Proposition 5.1 *Assume that $\overline{dF} < \infty$ and let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be the call option price function. Then*

$$\int_{\mathbb{R}_+} g \, dF = g(0) F(\infty) + D^+ g(0) \overline{dF} + \int_{]0,\infty[} C \, dm. \tag{5.1}$$

Proof One has

$$g(x) = g(0) + \int_0^x D^+ g(y) \, dy,$$

where $D^+g(y) - D^+g(0) = m(]0, y])$ for every $y > 0$, hence, by Tonelli's theorem,

$$\begin{aligned} g(x) &= g(0) + D^+g(0)x + \int_0^x \int_{]0,y]} dm(k)\,dy \\ &= g(0) + D^+g(0)x + \int_{]0,\infty[} \int_{[k,x]} dy\,dm(k) \\ &= g(0) + D^+g(0)x + \int_{]0,\infty[} (x-k)^+\,dm(k). \end{aligned}$$

Integrating both sides with respect to dF and appealing again to Tonelli's theorem completes the proof. □

Note that

$$\int_{[0,\infty[} C\,dm = C(0)m(\{0\}) + \int_{]0,\infty[} C\,dm = D^+g(0)\overline{dF} + \int_{]0,\infty[} C\,dm,$$

i.e. (5.1) could be written in the more symmetric form

$$\int_{\mathbb{R}_+} g\,dF = g(0)F(\infty) + \int_{\mathbb{R}_+} C\,dm.$$

Analogously, since

$$\int_{[0,\infty[} g\,dF = g(0)F(0) + \int_{]0,\infty[} g\,dF,$$

(5.1) could also be written as

$$\int_{]0,\infty[} g\,dF = g(0)(F(\infty) - F(0)) + D^+g(0)\overline{dF} + \int_{]0,\infty[} C\,dm.$$

Corollary 5.2 *Let $I \subseteq \mathbb{R}$ be an open set containing $\mathbb{R}_+$ and $h_1, h_2 \colon I \to \mathbb{R}_+$ convex functions belonging to $L^1(dF)$. If $g = h_1 - h_2$ and ν is the Lebesgue–Stieltjes (signed) measure induced by $D^+h_1 - D^+h_2$, i.e. $\nu([0,x]) := D^+h_1(x) - D^+h_2(x)$, then*

$$\int_{\mathbb{R}_+} g\,dF = g(0)F(\infty) + D^+g(0)\overline{dF} + \int_{]0,\infty[} C\,d\nu.$$

Slightly more generally, one can also write

$$\beta_T^{-1} g(S_T) = g(0)\beta_T^{-1} + D^+g(0)\beta_T^{-1}S_T + \int_{]0,\infty[} \beta_T^{-1}(S_T - k)^+\,d\nu(k),$$

hence, taking conditional expectation with respect to $\mathscr{F}_t$, for any $t \in [0, T]$, and multiplying by β_t,

$$\begin{aligned}\beta_t \mathbb{E}_{\mathbb{Q}}\big[\beta_T^{-1} g(S_T) \big| \mathscr{F}_t\big] &= g(0)\beta_t \mathbb{E}_{\mathbb{Q}}\big[\beta_T^{-1} \big| \mathscr{F}_t\big] \\ &\quad + D^+ g(0)\beta_t \mathbb{E}_{\mathbb{Q}}\big[\beta_T^{-1} S_T \big| \mathscr{F}_t\big] \\ &\quad + \int_{]0,\infty[} \beta_t \mathbb{E}_{\mathbb{Q}}\big[\beta_T^{-1}(S_T - k)^+ \big| \mathscr{F}_t\big]\, d\nu(k) \\ &= g(0)B(t, T) + D^+ g(0) S_t + \int_{]0,\infty[} C_t(k)\, d\nu(k),\end{aligned}$$

where $C_t(k) := \beta_t \mathbb{E}_{\mathbb{Q}}\big[\beta_T^{-1}(S_T - k)^+ \big| \mathscr{F}_t\big]$ is the price at time t of the call option with strike k.

It is actually possible to prove Proposition 5.1 using only the integration-by-parts formula (2.2). Even though the proof is longer than the previous one, some of its ingredients may be interesting in their own right. We begin with a useful reduction step.

Lemma 5.3 *Assume that $\overline{dF} < \infty$. The claim of Proposition 5.1 holds if and only if it does under the additional assumptions that $g(0) = D^+ g(0) = 0$ and m has compact support.*

Proof Clearly only sufficiency needs a proof. The extra assumption $g(0) = D^+ g(0) = 0$ comes at no loss of generality as one can reduce to this situation simply replacing the function g by the function $x \mapsto g(x) - g(0) - D^+ g(0)x$, which is still convex, being the sum of a convex function and an affine function, as well as in $L^1(dF)$, because $\overline{dF}$ is finite by assumption. Let us then assume that $g(0) = D^+ g(0) = 0$. Let (χ_n) be a sequence of smooth cutoff functions such that $\chi_n : \mathbb{R}_+ \to [0, 1]$ has support equal to $[0, n + 1]$ and is equal to one on $[0, n]$. Setting, for every $n \in \mathbb{N}$, $m_n := \chi_n \cdot m$ and

$$g_n^{(1)}(x) := m_n([0, x]) = \int_{[0,x]} \chi_n\, dm, \qquad g_n(x) := \int_0^x g_n^{(1)}(y)\, dy,$$

it is immediately seen that $g_n^{(1)}$ is positive, $g_n' = g_n^{(1)}$ a.e., $D^+ g_n = g_n^{(1)}$, and g_n is convex. Therefore, by hypothesis,

$$\int_{\mathbb{R}_+} g_n\, dF = \int_{]0,\infty[} C\, dm_n = \int_{]0,\infty[} C \chi_n\, dm.$$

Several applications of the monotone convergence theorem imply that (g_n) converges pointwise from below to g, hence, finally, that

$$\int_{\mathbb{R}_+} g\, dF = \int_{]0,\infty[} C\, dm. \qquad \square$$

Note that the "normalizing" assumptions $g(0) = 0$ and $D^+ g(0) = 0$ imply that

$$\int_{\mathbb{R}_+} g\, dF = \int_{]0,\infty[} g\, dF$$

and that

$$\int_{]0,\infty[} C\,dm = \int_{\mathbb{R}_+} C\,dm,$$

respectively. The former is evident, and the latter follows from $m(\{0\}) = D^+g(0) = 0$. Therefore

$$\int_0^\infty g\,dF = \int_{]0,\infty[} g\,dF = \int_{]0,\infty[} C\,dm = \int_{\mathbb{R}_+} C\,dm.$$

An alternative proof of Proposition 5.1 We shall assume, as allowed by Lemma 5.3, that $g(0) = D^+g(0) = 0$ and that m has compact support, which implies that, for all x sufficiently large, g is differentiable at x and $g'(x)$ is constant. For the rest of the proof, we shall write, with a harmless abuse of notation, g' to denote D^+g. Since g is continuous and F is càdlàg, the integration-by-parts formula (2.2) yields, for any $a \in \mathbb{R}_+$,

$$g(a)F(a) - g(0)F(0) = \int_{]0,a]} g(x)\,dF(x) + \int_{]0,a]} F(x)\,dg(x).$$

Therefore, as $g(0) = 0$ and the Lebesgue–Stieltjes measure dg is absolutely continuous with respect to Lebesgue measure with density g',

$$\int_{[0,a]} g(x)\,dF(x) = g(a)F(a) - \int_0^a g'(x)F(x)\,dx,$$

hence

$$\int_0^\infty g(x)\,dF(x) = \lim_{a\to+\infty}\Big(g(a)F(a) - \int_0^a g'(x)F(x)\,dx\Big).$$

Since g' is increasing and càdlàg, and C is continuous, another application of the integration-by-parts formula (2.2) yields, for any $a \in \mathbb{R}_+$,

$$g'(a)C(a) - g'(0)C(0) = \int_0^a g'(x)\,dC(x) + \int_{]0,a]} C(x)\,dg'(x),$$

hence, recalling that $g'(0) = 0$,

$$\int_0^a g'(x)\,dC(x) = g'(a)C(a) - \int_{]0,a]} C\,dm.$$

Moreover, the identity $C' = F - F(\infty)$ a.e. implies

$$\int_0^a g'(x)\,dC(x) = -F(\infty)g(a) + \int_0^a g'(x)F(x)\,dx,$$

hence

$$-\int_0^a g'(x)F(x)\,dx = -F(\infty)g(a) - \int_0^a g'(x)\,dC(x)$$
$$= -F(\infty)g(a) - g'(a)C(a) + \int_{]0,a]} C\,dm,$$

thus also

$$\int_{\mathbb{R}_+} g(x)\,dF(x) = \lim_{a\to+\infty}\left(g(a)(F(a)-F(\infty)) - g'(a)C(a) + \int_{]0,a]} C\,dm\right).$$

Note that g' is increasing by convexity of g and $g'(0) = 0$, hence g' is positive, therefore g is increasing and positive because $g(0) = 0$. Therefore

$$\left|g(a)(F(a)-F(\infty))\right| = g(a)(F(\infty)-F(a)) = \int_{]a,+\infty[} g(a)\,dF \le \int_{]a,+\infty[} g(x)\,dF(x),$$

where the last term converges to zero as $a \to +\infty$ because $g \in L^1(dF)$ by assumption. In particular,

$$\lim_{a\to+\infty} g(a)(F(a)-F(\infty)) = 0.$$

Moreover, as g' is constant at infinity and C tends to zero at infinity, we also have

$$\lim_{a\to+\infty} g'(a)C(a) = 0,$$

which allows to conclude that

$$\int_{\mathbb{R}_+} g\,dF = \int_{]0,\infty[} C\,dm. \qquad \square$$

Let us show that $\int_{]0,\infty[} C\,dm \in M_2^m$. By Tonelli's theorem,

$$\int_{]0,\infty[} C\,dm = \int_{]0,\infty[}\int_{\mathbb{R}_+} (x-k)^+\,dF(x)\,dm(k)$$
$$= \int_{\mathbb{R}_+}\int_{]0,\infty[} (x-k)^+\,dm(k)\,dF(x).$$

For any $n \in \mathbb{N}$, let $(k_i)_{i=0,\ldots,2^n}$ be a dyadic partition of $]0,n]$. Then

$$\sum_{i=1}^{2^n} (x-k_{i+1})^+ 1_{]k_i,k_{i+1}]}(k) \uparrow (x-k)^+ \qquad \forall x,\ k \in \mathbb{R}_+$$

as $n \to \infty$, hence, again by Tonelli's theorem,

$$\int_{]0,\infty[} \sum_{i=1}^{2^n} (x - k_{i+1})^+ 1_{]k_i,k_{i+1}]}(k)\, dm(k)$$
$$= \sum_{i=1}^{2^n} m\big(]k_i, k_{i+1}]\big)(x - k_{i+1})^+ \uparrow \int_{]0,\infty[} (x - k)^+\, dm(k) \qquad \forall x \in \mathbb{R}_+.$$

Then

$$g_n := \sum_{i=1}^{2^n} m\big(]k_i, k_{i+1}]\big)(x - k_{i+1})^+$$

defines a sequence of elements in the vector space generated by M_2 that monotonically converges pointwise to the function $x \mapsto \int_{]0,\infty[}(x-k)^+\, dm(k)$, which belongs to $L^1(dF)$ by assumption, therefore also to M_2^m.

Remark 5.4 It is more convenient to work with the call price function C, rather than with the put price function P, because C vanishes at infinity, while P grows linearly at infinity (see Propositions 4.1 and 4.2). However, the identity

$$x - k = (x - k)^+ - (x - k)^- = (x - k)^+ - (k - x)^+$$

yields, upon integrating both sides with respect to dF,

$$\int_{\mathbb{R}_+} x\, dF(x) - k \int_{\mathbb{R}_+} dF(x) = \overline{dF} - kF(\infty) = C(k) - P(k),$$

i.e.

$$C(k) = P(k) - kF(\infty) + \overline{dF}, \tag{5.2}$$

hence $k \mapsto P(k) - kF(\infty) + \overline{dF} \in L^1(m)$, even though, in general, P need not belong to $L^1(dF)$. A formula relating the integral of g with respect to dF with the integral of P with respect to m for a special class of functions g will be discussed in the next section.

Remark 5.5 A small variation of the argument used in the proof of Lemma 5.3 shows that every C^2 function g is the difference of two convex functions h_1 and h_2 (taking the positive and negative part of g''). A simple sufficient condition ensuring that the functions h_1 and h_2 can be chosen in $L^1(dF)$ is that there exists a function $h \in L^1(dF)$ with $h'' = |g''|$.

6 A distributional approach

We are going to show that most properties of the functions F, P and C discussed in the previous sections can also be obtained using Schwartz's distributions. The main advantage of this approach is that several results reduce, in their formal aspect, to simple calculus for distributions. Some work is needed, however, to remove the regularity assumptions on test functions typical of this approach.

Throughout this section, the functions F, P, and C (the last one if $\overline{dF}$ is finite) are assumed to be extended to $\mathbb{R}$ setting them equal to zero on $]-\infty, 0[$. All of them obviously belong to $L^1_{\mathrm{loc}}(\mathbb{R})$, hence they can be considered as distributions in $\mathscr{D}'$.

Let us start with a simple but useful observation.

Lemma 6.1 *Let F' be the distributional derivative of F. Then $dF = F'$ in $\mathscr{D}'$.*

Proof In fact, for any $\phi \in \mathscr{D}$, one has

$$\langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_{\mathbb{R}} F(x)\phi'(x)\,dx,$$

where, thanks to the integration-by-parts formula (2.2),

$$-\int_{\mathbb{R}} F(x)\phi'(x)\,dx = \int_{[0,\infty[} \phi(x)\,dF(x).$$ □

In the next subsection we derive the main results of Sect. 4 using distributions.

6.1 On the functions *P*, *C*, and *F*

The price function for put options P can be written in terms of convolutions of distributions. We shall denote the function $x \mapsto x^+$ by $(\cdot)^+$ and the right-continuous Heaviside function $1_{\mathbb{R}_+}$ by H.

Proposition 6.2 *The following identities hold in $\mathscr{D}'$:*

(a) $P = (\cdot)^+ * F'$;
(b) $P = H * F$;
(c) $P' = F$.

Proof The function P is the convolution of $(\cdot)^+$ with the measure dF, therefore, since $dF = F'$ in $\mathscr{D}'$ and both F' and $(\cdot)^+$ are distributions supported on $\mathbb{R}_+$, the convolution of $(\cdot)^+$ and F' is well-defined in the sense of distributions, and $P = (\cdot)^+ * F'$ in $\mathscr{D}'$. Standard calculus in $\mathscr{D}'$ then yields

$$P = (\cdot)^+ * F' = \big((\cdot)^+\big)' * F = H * F,$$

thus also, denoting the Dirac measure at the origin by δ,

$$P' = H' * F = \delta * F = F.$$ □

Corollary 6.3 *The function P is convex and satisfies the identity $D^+ P = F$.*

Proof It follows by $P' = F$ that $P'' = F'$, both as identities in $\mathscr{D}'$. Since F' coincides in $\mathscr{D}'$ with the measure dF, it is a positive distribution. Then P, as a distribution with positive second derivative, is a convex function. This and the identity $P' = F$ in $\mathscr{D}'$ imply that $P' = F$ holds also in the a.e. sense in $\mathbb{R}$, and that one can choose a right-continuous version of P', so that $D^+P = F$. □

The properties of P have thus been obtained starting from the properties of its second distributional derivative, that is reversing the path followed in the previous section, where convexity of P was proved first, which implied first-order differentiability outside a countable set of points first, hence second-order differentiability in the sense of measures.

It seems not possible, on the other hand, to treat the call option price function C by similar arguments, because one would formally have $C = (\cdot)^- * F'$, where the convolution is not well-defined in the sense of distributions. In fact, $(\cdot)^-$ and F' do not have their support "on the same side" of $\mathbb{R}$, and none of them has compact support. Nonetheless, properties of C can be deduced from those of P taking (5.2) into account.

Proposition 6.4 *One has, as an identity in* $\mathscr{D}'$,

$$C = P - F(\infty)(\cdot)^+ + \overline{dF}H, \tag{6.1}$$

and, as identities in $\mathscr{D}'(]0, \infty[)$,

$$C' = P' - F(\infty), \quad C'' = P''. \tag{6.2}$$

In particular, C is convex on $]0, \infty[$. *Moreover,* $D^+C(x) = F(x) - F(\infty)$ *for every* $x \in \mathbb{R}_+$.

Proof Identity (6.1) is just a rewriting of (5.2), considering C and P as distributions on $\mathbb{R}$. Differentiating in $\mathscr{D}'$ yields

$$\begin{aligned} C' &= P' - F(\infty)H + \overline{dF}\delta, \\ C'' &= P'' - F(\infty)\delta + \overline{dF}\delta', \end{aligned} \tag{6.3}$$

that, by restriction of the support, yield (6.2). In particular, $C \in \mathscr{D}'(]0, \infty[)$ has a positive second derivative, hence it is a convex function on $]0, \infty[$, then also on $\mathbb{R}_+$, and one can choose a right-continuous version of C' on $\mathbb{R}_+$ such that $C'(x) = F(x) - F(\infty)$ for a.a. $x \in \mathbb{R}_+$, with $D^+C(x) = F(x) - F(\infty)$ for every $x \in \mathbb{R}_+$. □

Remark 6.5 The identity (6.1) can be interpreted also as an identity of càdlàg functions on $\mathbb{R}$. Moreover, since δ' is not a measure, it follows by (6.3) that the function C is *not* convex on $\mathbb{R}$ (this also clearly follows from $C(0) = \overline{dF}$ and $C(k) = 0$ for all $k < 0$).

6.2 Convex payoffs

We are going to prove (5.1) using distributions. Note that, assuming that $g(0) = Dg^+(0) = 0$, (5.1) could heuristically be written as $\langle C'', g\rangle = \langle C, g''\rangle$, which seems very natural indeed. It is clear, however, that it makes no immediate sense if g is just a convex function in $L^1(dF)$. However, the identity has a meaning if $\langle\cdot,\cdot\rangle$ is interpreted as the duality between measures and continuous functions, rather than between distributions and test functions.

Let us start proving (5.1) under an extra regularity assumption on g.

Lemma 6.6 *Assume that* $g \in C^2_c(\mathbb{R})$. *Then*

$$\int_{\mathbb{R}_+} g\,dF = \langle C'', g\rangle + F(\infty)g(0) + \overline{dF}g'(0).$$

Proof By Lemma 6.1 and Proposition 6.2, the identity $P'' = F' = dF$ holds true in $\mathscr{D}'$. This immediately implies $\langle P'', g\rangle = \langle P, g''\rangle$ for every $g \in \mathscr{D}$, hence also, since P'' is a distribution of order at most two,

$$\langle P'', g\rangle = \int_{\mathbb{R}_+} g\,dF = \langle P, g''\rangle \qquad \forall g \in C^2_c(\mathbb{R}).$$

Therefore, using identity (6.3),

$$\begin{aligned}\langle P'', g\rangle = \int_{\mathbb{R}_+} g\,dF &= \langle C'', g\rangle + F(\infty)\,\langle \delta, g\rangle - \overline{dF}\,\langle \delta', g\rangle \\ &= \langle C'', g\rangle + F(\infty)g(0) + \overline{dF}g'(0).\end{aligned}$$

□

As a next step, we consider the case where $g \in C^2(\mathbb{R})$, i.e. we remove the assumption of compactnes of the support. This is the hardest part of the argument in this subsection.

Lemma 6.7 *Assume that* $g \in C^2(\mathbb{R})$. *Then, for any* $a \in \mathbb{R}_+$,

$$\int_{[0,a]} g\,dF = \int_0^a Cg'' - C(a)g'(a) + D^+C(a)g(a)$$

Proof As already seen, we can and shall assume, without loss of generality, that $g(0) = g'(0) = 0$. Let $a \in \mathbb{R}_+$, $a \neq 0$, (χ_n) be a sequence of smooth cutoff function taking values in $[0, 1]$, equal to one on $[-a, a]$, and equal to zero on $[a + 1/n, \infty[$. Then $g\chi_n \in C^2_c(\mathbb{R})$ and

$$(g\chi_n)'' = g''\chi_n + 2g'\chi_n' + g\chi_n'',$$

hence

$$\int_{\mathbb{R}_+} g\chi_n \, dF = \langle C'', g\chi_n \rangle = \langle C, (g\chi_n)'' \rangle$$
$$= \langle C, g''\chi_n \rangle + 2\langle C, g'\chi_n' \rangle + \langle C, g\chi_n'' \rangle.$$

We are going to pass to the limit as $n \to \infty$. One has

$$\int_{\mathbb{R}_+} g\chi_n \, dF = \int_{[0,a]} g \, dF + \int g\chi_n 1_{]a,a+1/n[} \, dF,$$

where $g\chi_n 1_{]a,a+1/n[} \to 0$ pointwise, hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} g\chi_n \, dF = \int_{[0,a]} g \, dF.$$

An entirely similar, slightly simpler reasoning shows that

$$\lim_{n\to\infty} \langle C, g''\chi_n \rangle = \lim_{n\to\infty} \int_{\mathbb{R}_+} Cg''\chi_n = \int_0^a Cg''.$$

Moreover,

$$\langle C, g'\chi_n' \rangle = \int_{\mathbb{R}} C(x)g'(x)\chi_n'(x) \, dx = \int_a^{a+1/n} C(x)g'(x)\chi_n'(x) \, dx,$$

where $-\chi_n'$ converges to $\delta_a + R$ in the sense of distributions, with R a distribution with support contained in $]-\infty, -a]$, hence

$$\lim_{n\to\infty} \langle C, g'\chi_n' \rangle = -C(a)g'(a).$$

The term $\langle C, g\chi_n'' \rangle$ is more difficult to treat because χ_n'' converges to $-\delta_a'$ in the sense of distributions (modulo terms with support in the strictly negative reals, that we are going to ignore), but C is just right-differentiable, not of class C^1. We can nonetheless argue as follows: let (ρ_m) be a sequence of mollifiers with support contained in $[-1/m, 0]$ and set $C_m := C * \rho_m$. Then $C_m \in C^\infty(\mathbb{R})$ and

$$\langle C_m, g\chi_n'' \rangle = \langle C_m g, \chi_n'' \rangle = -\langle (C_m g)', \chi_n' \rangle,$$

hence

$$\lim_{n\to\infty} \langle C_m, g\chi_n'' \rangle = C_m'(a)g(a) + C_m(a)g'(a).$$

Thus one has

$$\int_0^a C_m'' g = \int_0^a C_m g'' + C_m'(a)g(a) - C_m(a)g'(a).$$

We can now pass to the limit as $m \to \infty$: the continuity of C implies that C_m converges to C uniformly on $[0, a]$, hence

$$\lim_{m\to\infty} \int_0^a C_m g'' = \int_0^a C g'', \qquad \lim_{m\to\infty} C_m(a) = C(a).$$

Setting $dF_m := C_m'' = dF * \rho_m$ and $\widetilde{\rho}_m : x \mapsto \rho_m(-x)$, so that the support of $\widetilde{\rho}_m$ is contained in $[0, 1/m]$, one has

$$\int_0^a g C_m'' = \int g 1_{[0,a]} \, dF_m = \int g 1_{[0,a]} * \widetilde{\rho}_m \, dF,$$

where

$$\begin{aligned} &\lim_{m\to\infty} g 1_{[0,a]} * \widetilde{\rho}_m(x) = g(x) \qquad \forall x \in]0, a[, \\ &\lim_{m\to\infty} g 1_{[0,a]} * \widetilde{\rho}_m(0) = 0, \\ &\lim_{m\to\infty} g 1_{[0,a]} * \widetilde{\rho}_m(a) = g(a-) = g(a), \end{aligned}$$

i.e.

$$\lim_{m\to\infty} g 1_{[0,a]} * \widetilde{\rho}_m(x) = g 1_{]0,a]}(x) \qquad \forall x \in \mathbb{R},$$

or, in other words, $g 1_{[0,a]} * \widetilde{\rho}_m$ converges to the càglàd version of $g 1_{[0,a]}$ as $m \to \infty$. Therefore, by the dominated convergence theorem,

$$\lim_{m\to\infty} \int_0^a g C_m'' = \lim_{m\to\infty} \int g 1_{[0,a]} * \widetilde{\rho}_m \, dF = \int_{]0,a]} g \, dF = \int_{[0,a]} g \, dF,$$

where the last equality follows from $g(0) = 0$.

Since $C_m \in C^\infty(\mathbb{R})$ and C is right-differentiable with increasing incremental quotients on $\mathbb{R}_+$ (because of convexity thereon), the dominated convergence theorem yields

$$\begin{aligned} C_m'(a) = D^+ C_m(a) &= \lim_{h\to 0+} \frac{C_m(a+h) - C_m(a)}{h} \\ &= \lim_{h\to 0+} \int_{\mathbb{R}} \frac{C(a-y+h) - C(a-y)}{h} \rho_m(y) \, dy \\ &= \int_{\mathbb{R}} D^+ C(a-y) \rho_m(y) \, dy, \end{aligned}$$

hence also, recalling that the support of ρ_m is contained in $\mathbb{R}_-$ and that $D^+ C$ is right-continuous,

$$\lim_{m\to\infty} C_m'(a) - D^+ C(a) = \lim_{m\to\infty} \int_{\mathbb{R}} \big(D^+ C(a-y) - D^+ C(a) \big) \rho_m(y) \, dy = 0.$$

We have thus shown that

$$\int_{[0,a]} g\,dF = \int_0^a Cg'' - C(a)g'(a) + D^+C(a)g(a)$$

for every $g \in C^2(\mathbb{R})$. □

To remove the assumption that $g \in C^2$, assuming instead that it is convex, we can apply the same regularization by convolution: let g be convex and set $g_n := g * \rho_n$, with the sequence of mollifiers (ρ_n) chosen as before. Then $g_n \in C^\infty$ and

$$\int_{[0,a]} g_n\,dF = \int_0^a Cg_n'' - C(a)g_n'(a) + D^+C(a)g_n(a),$$

where $g_n \to g$ uniformly on $[0,a]$ and $\lim_{n\to\infty} g_n'(a) = D^+g(a)$. Moreover, using the same argument as before,

$$\lim_{n\to\infty} \int_0^a Cg_n'' = \lim_{n\to\infty} \int C1_{[0,a]} * \widetilde{\rho}_n\,dm = \int_{]0,a]} C\,dm.$$

We infer that the following claim is true.

Proposition 6.8 *Assume that g is convex. Then, for any $a \in \mathbb{R}_+$, $a > 0$,*

$$\int_{[0,a]} g\,dF = \int_{]0,a]} g\,dF = \int_{]0,a]} C\,dm - C(a)D^+g(a) + D^+C(a)g(a). \quad (6.4)$$

Note that until here we have not used the assumption that $g \in L^1(dF)$. To complete the proof of (5.1), we let a tend to infinity using two lemmas proved next, according to which the last two terms on the right-hand side of (6.4) tend to zero. It is precisely at this point that we use the assumption that $g \in L^1(dF)$.

Lemma 6.9 *Assume that dF is a finite measure and let $g \in L^1(dF)$ be increasing. Then*

$$\lim_{a\to\infty} g(a)(F(\infty) - F(a)) = 0.$$

In particular, if $\overline{dF} < \infty$ then $\lim_{a\to\infty} D^+C(a)g(a) = 0$.

Proof Assume first that $g(0) = 0$, so that g is positive. Then, as g is increasing,

$$g(a)(F(\infty) - F(a)) = \int_{]a,\infty[} g(a)dF(x) \leq \int_{]a,\infty[} g(x)dF(x),$$

and

$$\lim_{a\to\infty} \int_{]a,\infty[} g(x)dF(x) = 0$$

because $g \in L^1(dF)$. If $g(0) < 0$, then consider the function $\tilde{g} := |g(0)| + g$, which is increasing and belongs to $L^1(dF)$. The identity

$$g(a)(F(\infty) - F(a)) = \tilde{g}(a)(F(\infty) - F(a)) - |g(0)|\,(F(\infty) - F(a)) = 0$$

immediately implies the claim. □

Lemma 6.10 *Assume that* $\overline{dF} < \infty$. *Let* $g \in L^1(dF)$ *be absolutely continuous and such that* g' *is increasing (possibly after a suitable modification on a set of Lebesgue measure zero). Then*

$$\lim_{a\to\infty} g'(a) \int_a^\infty (F(\infty) - F(y))\,dy = 0,$$

or, equivalently, $\lim_{a\to\infty} C(a)g'(a) = 0$.

Proof The assumption $\overline{dF} < \infty$ guarantees that the function C is well defined and

$$C(a) = \int_a^\infty (F(\infty) - F(y))\,dy \qquad \forall a \in \mathbb{R}_+.$$

Then we can write

$$\begin{aligned}
g'(a)C(a) &= \int_a^\infty g'(a)(F(\infty) - F(y))\,dy \\
&\le \int_a^\infty g'(y)(F(\infty) - F(y))\,dy \\
&= \int_a^\infty g'(y) \int_{]y,\infty[} dF(x)\,dy \\
&= \int_{]a,\infty[} \int_a^x g'(y)\,dy\,dF(x) = \int_{]a,\infty[} g(x)\,dF(x) - \int_{]a,\infty[} g(a)\,dF(x),
\end{aligned}$$

where

$$\lim_{a\to\infty} \int_{]a,\infty[} g(x)\,dF(x) = 0$$

because $g \in L^1(dF)$. Moreover,

$$\left| \int_{]a,\infty[} g(a)\,dF(x) \right| \le \int_{]a,\infty[} |g(a)|\,dF(x).$$

Let us first consider the case that $g'(0) \ge 0$, so that g' is positive and g is increasing. If there exists $a_0 \in \mathbb{R}_+$ such that $g(a_0) \ge 0$, then

$$\lim_{a\to\infty} \int_{]a,\infty[} g(a)\,dF(x) \le \lim_{a\to\infty} \int_{]a,\infty[} g(x)\,dF(x) = 0.$$

Otherwise, if $g(x) \leq 0$ for all $x \in \mathbb{R}_+$, then $|g| = -g$ is decreasing, therefore

$$\lim_{a\to\infty} \int_{]a,\infty[} |g(a)| \, dF(x) \leq \lim_{a\to\infty} |g(1)| \, (F(\infty) - F(a)) = 0.$$

Let us now consider the case that $g'(0) < 0$: introduce the function $\tilde{g}(x) := g(x) + \left|g'(0)\right| x$, for which $\tilde{g}'(0) = g'(0) + \left|g'(0)\right| \geq 0$, and note that $\tilde{g}'$ is increasing. The assumption $\overline{dF} < \infty$ implies that $\tilde{g} \in L^1(dF)$, hence the previous part of the proof shows that $\lim_{a\to\infty} C(a)\tilde{g}'(a) = 0$. Writing $C(a)g'(a) = C(a)\tilde{g}'(a) - C(a)\left|g'(0)\right|$ and recalling that $\lim_{a\to\infty} C(a) = 0$, the proof is completed. □

6.3 Payoffs as piecewise difference of convex functions

Even though the set of functions that can be written as the difference of two convex functions is quite rich (see, e.g., Bačák and Borwein 2011), it does not contain any discontinuous function. So, for instance, for digital options we cannot produce a pricing formula such as (5.1). However, the distributional approach allows to obtain in a quite efficient way pricing formulas for options the payoff of which can be written piecewise as the difference of convex functions. The formulas involve, apart from integrals of C, also pointwise evaluations of C and F. We shall consider an option with payoff equal to the càdlàg restriction of a convex function to a compact interval.

Proposition 6.11 *Let $g_0 \colon \mathbb{R} \to \mathbb{R}$ be a convex function, $[a, b] \subset \mathbb{R}_+$ a compact interval, and $g := g_0 1_{[a,b[}$. Then*

$$\begin{aligned}\int_{]a,b[} g \, dF = \int_{]a,b[} C \, dm + C(a)D^+g(a) - D^+C(a)g(a) \\ - C(b)D^+g(b-) + D^+C(b-)g(b-).\end{aligned}$$

Proof One has

$$\int_{\mathbb{R}_+} g \, dF = \int_{[a,b[} g \, dF = \int_{]a,b[} g \, dF + g(a)\Delta F(a),$$

where $\Delta F(a) = D^+C(a) - D^+C(a-)$ and, by the dominated convergence theorem,

$$\int_{]a,b[} g \, dF = \lim_{x\to b-} \int_{]a,x]} g \, dF.$$

Since

$$\int_{]a,x]} g \, dF = \int_{[0,x]} g \, dF - \int_{[0,a]} g \, dF,$$

it follows by (6.4) that

$$\int_{]a,x]} g\,dF = \int_{]a,x]} C\,dm + C(a)D^+g(a) - D^+C(a)g(a) \\ - C(x)D^+g(x) + D^+C(x)g(x).$$

Taking the limit for x going to b from the left, one obtains, recalling that C is continuous and both g and D^+C are càdlàg,

$$\int_{]a,b[} g\,dF = \int_{]a,b[} C\,dm + C(a)D^+g(a) - D^+C(a)g(a) \\ - C(b)D^+g(b-) + D^+C(b-)g(b-).$$

□

We are going to present an alternative way to obtain the same formula, using Proposition 2.2, that is slightly longer but that starts from very basic principles and shows how the distributional approach allows to quickly compute the price of an option under the mild assumption that F admits a continuous density.

Alternative proof of Proposition 6.11 Let us consider $g := g_0 1_{[a,b[}$ as a distribution, and assume first that F is of class C^1, which implies that C is of class C^2. Then

$$\int_{\mathbb{R}_+} g\,dF = \int_a^b g\,dF = \langle g, C''\rangle = \langle g'', C\rangle,$$

where, thanks to Proposition 2.2,

$$\langle g'', C\rangle = \int_{]a,b[} C\,dm + C(a)D^+g(a) - C'(a)g(a) \\ - C(b)D^+g(b-) + C'(b)g(b-).$$

If C is not twice continuously differentiable, setting $C_n := C * \rho_n$, with (ρ_n) a sequence of mollifiers chosen as before, then C_n and $dF_n := dF * \rho_n$ are both in C^∞ and

$$\int_a^b g\,dF_n = \langle g'', C_n\rangle = \int_{]a,b[} C_n\,dm + C_n(a)D^+g(a) - C_n'(a)g(a) \\ - C_n(b)D^+g(b-) + C_n'(b)g(b-).$$

We are now going to pass to the limit as $n \to \infty$: C_n converges to C uniformly on compact sets, hence $C_n(a)$ and $C_n(b)$ converge to $C(a)$ and $C(b)$, respectively, and

$$\lim_{n\to\infty} \int_{]a,b[} C_n\,dm = \int_{]a,b[} C\,dm.$$

As before, the choice of (ρ_n) and the right continuity of D^+C imply that $C_n'(a)$ and $C_n'(b)$ converge to $D^+C(a)$ and $D^+C(b)$, respectively, hence

$$\lim_{n\to\infty}\int_a^b g\,dF_n = \int_{]a,b[} C\,dm + C(a)D^+g(a) - D^+C(a)g(a) \\ - C(b)D^+g(b-) + D^+C(b)g(b-).$$

Writing

$$\int_a^b g\,dF_n = \int g_0 1_{[a,b[} * \widetilde{\rho}_n\,dF$$

we can use again an argument already met before, which shows that

$$\lim_{n\to\infty} g_0 1_{]a,b[} * \widetilde{\rho}_n(x) = g_-(x) \qquad \forall x \in \mathbb{R},$$

where g_- denotes the càglàd version of g. Therefore, by the dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty}\int_a^b g\,dF_n = \int_{]a,b]} g_-\,dF &= \int_{]a,b[} g_-\,dF + g(b-)\Delta F(b)\\ &= \int_{]a,b[} g\,dF + g(b-)(D^+C(b) - D^+C(b-)).\end{aligned}$$

Rearranging terms we are left with

$$\int_{]a,b[} g\,dF = \int_{]a,b[} C\,dm + C(a)D^+g(a) \\ -D^+C(a)g(a) - C(b)D^+g(b-) + D^+C(b-)g(b-),$$

as claimed. □

7 Pricing via approximated laws of logarithmic returns

In the standard Black-Scholes (BS) model one assumes that $S_T = \exp(\varsigma\sqrt{T}Z - \varsigma^2 T/2)$ in law, where the volatility ς is constant and Z is a standard Gaussian random variable. This family of random variables (indexed by ς, with time to maturity T fixed as before) can be embedded in the larger class defined by $S_T = \exp(\sigma X + m)$, where σ and m are constants, and X is a random variable with density $f \in L^2 := L^2(\mathbb{R})$. This rather general family of laws can be used as setup for empirical nonparametric option pricing, essentially by projecting the density f on radial basis functions (see Marinelli and d'Addona 2023). More precisely, one considers expansions of f in terms of Hermite functions, so that the lognormal distribution of returns corresponds exactly to the zeroth-order expansion of f. The approach can thus be thought of as a perturbation of the BS model at fixed time to maturity. The following problem then arises: let $(f_n) \subset L^1 \cap L^2$ be a sequence of functions converging to f in L^2, and let $P_n(k)$ be the "fictitious" price of a put option with strike k, obtained replacing the density f with its approximation f_n. Suppose that the $P_n(k)$ are known for all $k \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Is this information enough to determine the function P, i.e. to obtain the put option prices in the "true" model?

Assuming, for simplicity, that $S_0 = 1$, $\sigma = 1$ and $m = 0$, and denoting the distribution function of $X = \log S_T$ with respect to the measure μ (see (3.1)) by F_X, it is immediately seen that $F_X(x) = F(e^x)$ for every $x \in \mathbb{R}$ and that

$$P(k) = \int_{\mathbb{R}} (k - e^x)^+ \, dF_X(x) = \int_{\mathbb{R}} (k - e^x)^+ f(x) \, dx,$$

hence

$$P_n(k) = \int_{\mathbb{R}} (k - e^x)^+ f_n(x) \, dx.$$

Note that F_X and f are supported on the whole real line and that F and F_X are in bijective correspondence, hence F_X is in bijective correspondence also with P.

The sequence $P_n(k)$ does not necessarily converge to $P(k)$ as $n \to \infty$, because the function $x \mapsto (k - e^x)^+$ belongs to L^∞ but not to L^2, hence it induces a continuous linear form on L^1, but not on L^2. Moreover, convergence in $L^2(\mathbb{R})$ does not imply convergence in $L^1(\mathbb{R})$.

We are going to show that the function P can be reconstructed from approximation to option prices with payoff of the type

$$\theta_{k_1,k_2}(x) := (k_2 - e^x)^+ - \frac{k_2}{k_1}(k_1 - e^x)^+, \qquad k_1, k_2 > 0.$$

More precisely, to identify the pricing functional P, it suffices to know, for any sequence (f_n) converging to f in L^2, the values $\langle \theta_{k_1,k_2}, f_n \rangle$ for all $k_1, k_2 > 0$ and all $n \geq 0$, where $\langle \cdot, \cdot \rangle$ stands for the scalar product of L^2. In fact, for any $k_1, k_2 > 0$, the function θ_{k_1,k_2} is in L^2, hence, for any sequence $(f_n) \subset L^1 \cap L^2$ converging to f in L^2 (weak convergence in L^2 would also suffice), one has

$$P_n(k_2) - \frac{k_2}{k_1} P_n(k_1) = \langle \theta_{k_1,k_2}, f_n \rangle \longrightarrow \langle \theta_{k_1,k_2}, f \rangle = P(k_2) - \frac{k_2}{k_1} P(k_1).$$

Moreover,

$$\frac{k_2}{k_1} P(k_1) = \int_{\mathbb{R}} \frac{k_2}{k_1} (k_1 - e^x)^+ f(x) \, dx,$$

where $(k_1 - e^x)^+ \in]0, k_1]$ for all $x \in \mathbb{R}$, hence $\frac{k_2}{k_1}(k_1 - e^x)^+ \in]0, k_2]$ for all $x \in \mathbb{R}$, and

$$\frac{k_2}{k_1}(k_1 - e^x)^+ = \begin{cases} k_2 - \dfrac{k_2}{k_1} e^x, & \text{if } x \leq \log k_1, \\ 0, & \text{if } x \geq \log k_1, \end{cases}$$

so that

$$\lim_{k_1\to 0} \frac{k_2}{k_1}(k_1 - e^x)^+ = 0 \qquad \forall x \in \mathbb{R}.$$

Therefore the function $x \mapsto \frac{k_2}{k_1}(k_1 - e^x)^+$ converges to zero as $k_1 \to 0$ in L^p for every $p \in [1, \infty[$ by the dominated convergence theorem. In particular, since $f \in L^2$,

$$\lim_{k_1\to 0} \frac{k_2}{k_1} P(k_1) = \lim_{k_1\to 0} \int_{\mathbb{R}} \frac{k_2}{k_1}(k_1 - e^x)^+ f(x)\,dx = 0. \tag{7.1}$$

We have thus shown that

$$\lim_{k_1\to 0} \lim_{n\to\infty} \langle \theta_{k_1,k_2}, f_n \rangle = P(k_2) \qquad \forall k_2 > 0,$$

thus also the following statement.

Proposition 7.1 *Let $(f_n) \subset L^1 \cap L^2$ be a sequence converging to f in L^2. There is a bijection between*

$$\Big(\langle \theta_{k_1,k_2}, f_n \rangle\Big)_{\substack{k_1,\,k_2 > 0\\ n \geq 0}}$$

and P.

Completely analogously, if $P(k_1)$ is known, then

$$P(k_2) = \frac{k_2}{k_1} P(k_1) + \lim_{n\to\infty} \langle \theta_{k_1,k_2}, f_n \rangle = \frac{k_2}{k_1} P(k_1) + \lim_{n\to\infty} \Big(P_n(k_2) - \frac{k_2}{k_1} P_n(k_1) \Big).$$

Remark 7.2 The function $x \mapsto \frac{k_2}{k_1}(k_1 - e^x)^+$ does not converge to zero in L^∞ as $k_1 \to 0$, as

$$\sup_{x\in\mathbb{R}} \frac{k_2}{k_1}(k_1 - e^x)^+ = k_2.$$

However, the convergence in (7.1) also holds with $f \in L^1$, i.e. without any extra integrability assumption on f, because $\frac{k_2}{k_1}(k_1 - e^x)^+ f(x) \leq k_2 f(x)$ for every $x \in \mathbb{R}$, hence the result follows by the dominated convergence theorem.

Note that Proposition 7.1 can be interpreted as providing a sort of generalized representation of dF, but, as discussed at the end of Sect. 3, it cannot be formulated in the language introduced there. Even the extended measurements of the type (g_j, π_j, dF_j), with dF_j a family of measures weakly converging to dF, is not enough. In fact, it is not difficult to check that the pushforward of $f_n\,dx$ through $x \mapsto e^x$, denoted by dF_n, does not converge weakly to dF, in general. However, setting $M_1 = (g_k, dF_n(g_k), dF_n)_{k>0,\, n\in\mathbb{N}}$, where $g_k \colon x \mapsto (k - x)^+$, we have shown that M_1 “implies” $M_2 = (\theta_{k_1,k_2}, \pi_{k_1,k_2})_{k_1,k_2>0}$, where implication is meant as in the last paragraph of Sect. 3, and that M_2^m is finer than M, the measurement set composed of put prices, which is a representation.

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