INVITED SURVEY

Preference learning and multiple criteria decision aiding: differences, commonalities, and synergies–part I

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Abstract

Multiple criteria decision aiding (MCDA) and preference learning (PL) are established research fields, which have different roots, developed in different communities – the former in the decision sciences and operations research, the latter in AI and machine learning – and have their own agendas in terms of problem setting, assumptions, and criteria of success. In spite of this, they share the major goal of constructing practically useful decision models that either support humans in the task of choosing the best, classifying, or ranking alternatives from a given set, or even automate decision-making by acting autonomously on behalf of the human. Therefore, MCDA and PL can complement and mutually benefit from each other, a potential that has been exhausted only to some extent so far. By elaborating on the connection between MCDA and PL in more depth, our goal is to stimulate further research at the junction of these two fields. To this end, we first review both methodologies, MCDA in this part of the paper and PL in the second part, with the intention of highlighting their most common elements. In the second part, we then compare both methodologies in a systematic way and give an overview of existing work on combining PL and MCDA.

Keywords Preference learning · Preference modelling · Multiple criteria decision aiding · Multiple criteria decision making · Machine Learning

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1 Introduction

The notion of "preference" has a long tradition in various scientific disciplines, including economics and the social sciences, operations research and the decision sciences, psychology, and philosophy. Over the past decades, it has also been studied intensively in artificial intelligence (AI), where preferences provide a means for specifying desires in a declarative and intelligible way, a point of critical importance for effective knowledge representation and reasoning. To this end, suitable representation formalisms and modeling languages have been developed (Rossi et al. 2011).

Before preferences can be used for any sort of reasoning or inference, they have to be acquired and formalized in one way or the other. Preference acquisition is not an easy task, however, especially in domains with many and possibly complex sets of alternatives. This may easily lead to a knowledge acquisition bottleneck. As an alternative to the manual specification of preferences, *preference learning* (PL) builds on machine learning methodology to support and partly automate the design of preference models in a data-driven way. Roughly speaking, preference learning is concerned with the automated acquisition of preference models from observed (stated or revealed) preference information, that is data from which (possibly uncertain) preference representations can be inferred in a direct or indirect way (Fürnkranz and Hüllermeier 2010). Meanwhile, preference learning has established itself as an independent branch of machine learning research. It attracted renewed attention with the recent advent of generative AI systems such as ChatGPT OpenAI (2022), which seek to create artifacts of high quality, complying as much as possible with the preferences of the user.

Methods for eliciting preferences and constructing preference models from explicit or implicit preference information and feedback have also been studied in other fields, notably in multiple criteria decision aiding[1] (MCDA). The latter aims at helping a decision maker to choose the best alternative, or to rank or classify alternatives on the basis of their performances on multiple criteria (Greco et al. 2016a). To this end, MCDA has developed a wide variety of decision models, most of which aggregate the evaluations on individual criteria into an overall assessment of an alternative. MCDA differs from PL in various ways. For example, while MCDA mostly focuses on a single decision maker (an individual or a group of well-identified stakeholders), a model in PL typically refers to an entire population of individuals and seeks to optimize performance "on average". Moreover, the interaction between the decision maker and the decision analyst (facilitator) is at the core of MCDA, while being less emphasized in PL. Correspondingly, preference data is sparser but also less corrupted in MCDA, compared to PL, where training data is commonly assumed to be more massive but also afflicted with noise and various sorts of inaccuracies. Besides, with the goal of learning a presumably existing "ground truth" (the data-generating process), PL is *inductive* in nature, whereas MCDA does not assume such a ground truth and should rather be thought of as a *constructive* process (cf. Sect. 2.3). The two frameworks are sketched in Fig. 1.

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[1] An alternative name to MCDA is multiple criteria decision making (MCDM), however, in this paper, we will use the name MCDA as it better underlines the supporting role of the considered methodology.
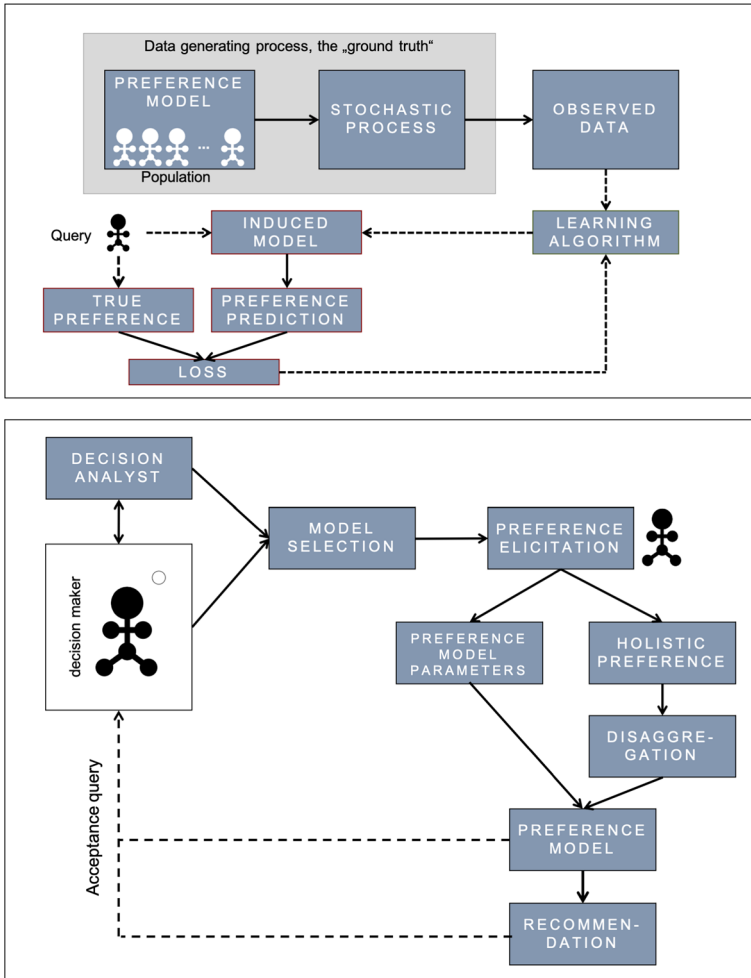
Fig. 1 Simplified illustration of the frameworks of preference learning (above) and multiple criteria decision aiding (below)

This being said, both MCDA and PL do share the major goal of constructing practically useful decision models, i.e., mathematical models that either support humans in the task of choosing the best among a set of alternatives, classifying them into predefined and preferentially ordered decision classes, or ranking these alternatives from the best to the worst, or even automating decision making by acting autonomously on behalf of the human. Therefore, in spite of differences in their objectives and research agenda, or rather because of them, we argue that PL and MCDA can cross-fertilize and mutually benefit from each other, and indeed, several interesting contributions at the junction of PL and MCDA have been made in the recent past. While this is a welcome development, we believe that the potential is far from exhausted. This is the main motivation for the current paper, which, by elaborating on the connection

between PL and MCDA in more depth, will hopefully stimulate further research in that direction.

Our paper comes in two parts. This first part continues with a brief survey of the state-of-the-art in MCDA. The survey focuses on MCDA based on constructive preference modeling from holistic preference information, also referred to as decision examples. This is because this methodology of MCDA is the closest to PL. A similar survey of PL comes in the second part. In the latter, both methodologies are also compared with each other, and their differences but also commonalities are presented and discussed in a systematic way. Moreover, we elaborate on existing work on combining PL and MCDA.

2 Multiple criteria decision aiding

2.1 Decision aiding

Before we pass to Multiple Criteria Decision Aiding (MCDA), we should stress the Decision Aiding (DA) aspect of the methodology. The aim of DA is to analyze the decision-making context by identifying the problem with its actors, possible alternatives and their consequences, as well as the stakes (Roy 2000). Among the actors, there are stakeholders (actors concerned by the decision), single or multiple decision makers (users), and an analyst (facilitator). For DA it is important how the decision-making process will unfold, increasing consistency between the values underlying the objectives, and the quality of recommendations based on results from models and computational procedures designed with respect to some working hypotheses. Decision problems considered within DA involve a set $\mathcal{A}$ of alternatives (actions, solutions, objects, etc.) evaluated from multiple points of view. The following three main problem statements are considered in DA:

- *Choice* of the best alternative, with multiobjective optimization as a particular case.
- *Classification* (called also *sorting*) of alternatives into pre-defined and preference-ordered classes.
- *Ranking* of alternatives from the best to the worst.

The above decision problems involve, in general, multiple evaluation criteria (multiple objectives). A decision-maker (DM) seldom has a single clear criterion in mind. Seldom is there a common unit for all scales of criteria, which are rather heterogeneous. That is why it may be very difficult to define *a priori* a unique criterion able to take into account all relevant points of view. Formally, the *criterion* is a real-valued function g_i defined on $\mathcal{A}$, reflecting a value of each alternative from a particular point of view, such that in order to compare any two alternatives $a, b \in \mathcal{A}$ from this point of view, it is sufficient to compare two values: $g_i(a)$ and $g_i(b)$, called *performances*. It is worth stressing that criteria may have *ordinal* evaluation scales, where a difference of performance does not have the meaning of intensity, or *cardinal* evaluation scales, where a difference of performance has the meaning of intensity, of either interval or ratio type. The type of criterion scale determines the allowed arithmetic operations on the

performances. For example, additions of (weighted) performances on ordinal scales have no meaning. In the rest of this section, we assume by default that criteria are of the *gain-type*, i.e., the greater the performance, the better, unless explicitly specified that they are of the *cost-type*.

Building a family of criteria $\mathcal{G} = \{g_1, \ldots, g_n\}$ for a decision problem at hand is the preliminary step of MCDA. According to Roy (1996), the family $\mathcal{G}$ should be exhaustive, monotonic, and non-redundant. With the exception of trivial cases, the evaluation criteria are in conflict in the sense that, improving an alternative on one criterion causes deterioration on other criteria, thus, in set $\mathcal{A}$, there is no alternative being the best on all criteria. The only objective information that stems from a multiple criteria evaluation of alternatives is the dominance relation in set $\mathcal{A}$. The meaning of the dominance relation is the following: alternative a dominates alternative b if a is not worse than b on all criteria and on at least one criterion, it is strictly better. It is clear that such a binary relation is a weak partial order. As such, the dominance relation leaves many alternatives incomparable. Incomparability prevents the formulation of an unambiguous recommendation in the best choice, classification, or ranking. This is why the main preoccupation of MCDA consists of aggregating the multiple criteria into a *preference model* making the alternatives more comparable in light of users' preferences.

2.2 The three-step methodology of decision aiding

In this way, we arrived at the point where the MCDA methodology should try to "enrich" the dominance relation using preference information elicited from the decision makers (DMs). Thus, the steps of MCDA that go beyond this point are the following:

- Get preference information from DMs revealing their value system confronted with the multiple criteria evaluation of alternatives from set $\mathcal{A}$.
- Learn or build a preference model that aggregates the vector evaluations of alternatives. The preference model induces a preference relation in set $\mathcal{A}$, making comparable a larger number of alternatives than the dominance relation.
- Exploit the preference relation in set $\mathcal{A}$ to work out a recommendation in terms of choice, classification, or ranking.

This three-step methodology loops in the case of *interactive methods*.

2.3 Preference handling in MCDA and MCDM

Dealing with DM's preferences is at the core of MCDA. However, they are rarely clearly structured and known *a priori*. Hence, the questions of an analyst and the use of dedicated methods should be oriented toward shaping these preferences. In fact, MCDA proceeds by progressively forming a conviction and communicating about the foundations of these convictions. The models, procedures, and provided results constitute a communication and reflection tool. Indeed, they allow the participants of the decision process to carry forward their process of thinking, discover what is

important to them, and learn about their values. Such elements of responses obtained by the DMs contribute to recommending and justifying a decision, that increases the consistency between the evolution of the process and the objectives and value systems of the stakeholders. Thus, Decision Aiding should be perceived as a *constructive learning process*, as opposed to machine discovery of an optimum, of a pre-existing preference system, or a truth that exists outside the involved stakeholders.

The above remark highlights the subtle difference between the two schools known as MCDA (Multiple Criteria Decision Aiding) and MCDM (Multiple Criteria Decision Making). As the research area of both schools is the same, they often meet under the label Multiple Criteria Decision Analysis (Ehrgott et al. 2010).

MCDA underlines the "aiding" in a process, involving the DMs in the co-construction of their preferences (Roy 2005). It assumes that preferences of the DM with respect to considered alternatives do not pre-exist in the DM's mind. This implies that the concepts, models and methods proposed by MCDA must not be considered as a means of discovering a pre-existing truth. They have to be seen as keys to doors giving access to elements of knowledge contributing to the acceptance of a final recommendation.

MCDM relies on normative and prescriptive Decision Analysis, including tools for identifying, representing, and formally assessing important aspects of a decision, for prescribing a recommended course of action maximizing the utility (Bell et al. 1988). Decision Analysis assumes an ideal rationality and aims at giving an "objectively" best recommendation.

It is important to note that the methodology of MCDA/MCDM has evolved over the past 50 years with the advancement of computing technology. The most affected part of this methodology has been the modeling of preferences. Recently, due to an abundance of data about human choices a "knowledge-driven" approach is prevailing, i.e., holistic preference information is known first and then the model is built explaining the past decisions and predicting future ones. The knowledge-driven trend makes the preference modeling in MCDA similar to Preference Learning according to the Machine Learning paradigm. This is why in MCDA the term Preference Learning is now often used in the sense of model building from holistic preference information. This observation clearly motivates the present paper.

The issue of preference handling in MCDA and MCDM is considered in the first two steps of the three-step DA methodology described in Sect. 2.2. Below, we first characterize the preference models, and then the ways of obtaining preference information used to build these models.

2.4 Preference models

The preference model aggregates the multiple criteria evaluations of alternatives. This aggregation should take into account the preferences of the DM so that the resulting preference model would induce a preference relation in set $\mathcal{A}$ richer than the dominance relation.

One can distinguish three families of preference models based on specific ways of aggregation:

- Multiple Attribute Utility Theory (MAUT) (KeeneyandRaiffa [1976]) using a value function assigning to each alternative $a \in \mathcal{A}$ a real value, e.g., the weighted sum of performances $U(a) = \sum_{i=1}^{n} k_i \times g_i(a)$, where k_i are weights interpreted as substitution rates among criteria, or a more general additive function $U(a) = \sum_{i=1}^{n} u_i[g_i(a)]$, where u_i are marginal value functions, or non-additive integrals handling interactions among criteria, like Choquet integral for cardinal criteria, and Sugeno integral for ordinal criteria (Grabisch [1996]).
- Outranking methods (Roy [2005]) using a system of binary relations, e.g., the outranking relation $S = \{\sim, \succ^w, \succ^s\}$, where $\sim$ means indifference, $\succ^w$ weak preference, and $\succ^s$ strong preference; relation aSb reads: "alternative a is at least as good as alternative b".
- Decision rule models (Greco et al. [2001]) composed of logical statements relating conditions on particular criteria and a decision, e.g., "if $g_i(a) \succeq r_i$ & $g_j(a) \succeq r_j$ & $\ldots g_k(a) \succeq r_k$, then $a \rightarrow$ Class t or better" for classification, and "if $g_i(a) \succeq^{\geq h(i)} g_i(b)$ & $g_j(a) \succeq^{\geq h(j)} g_j(b)$ & $\ldots g_p(a) \succeq^{\geq h(p)} g_p(b)$, then aSb" for choice or ranking, where $\succeq$ is a weak preference relation, $r_i, r_j, \ldots, r_k$ are threshold values on selected criteria $\{g_i, g_j, \ldots, g_k\} \subseteq \mathcal{G}$ induced from data, $\succeq^{\geq h(\cdot)}$ is a weak preference relation with intensity in degree at least $h(\cdot)$, and $h(i), h(j), \ldots, h(p)$ are degrees of intensity of preference on cardinal criteria $\{g_i, g_j, \ldots, g_p\} \subseteq \mathcal{G}$, also induced from data.

The above preference models have different capacities for representing DM's preferences. Comparison of these models at the axiomatic level shows that the decision rule model requires the weakest axioms of all three, which means that the value function model or the outranking model is able to represent particular preferences if and only if the rule model is able to (Słowiński et al. [2002]). Another thorough comparison of these models, acknowledging the above claim was performed by Greco et al. ([2004]).

The classification rules having the syntax exemplified above are called *dominance-based rules* because, in the condition part, they involve a simple dominance relation between a partial performance profile of an alternative and a vector of threshold values, usually inferred by induction from data (Greco et al. [2016b]). The rules are the aggregation operators distinguished by the following features:

- they account for the most complex interactions among criteria,
- they are non-compensatory because the conditions on particular criteria are considered individually,
- they accept ordinal evaluation scales and do not convert ordinal evaluations into cardinal ones, because the dominance relation only uses order.

Moreover, rules identify values that drive DM's decisions–each rule is an intelligible scenario of a causal relationship between performances on a subset of criteria and a comprehensive judgment.

2.5 Elicitation of preference information

Preference elicitation can be either direct or indirect. In the case of direct elicitation, the DM is expected to provide parameters related to a supposed form of the preference model. Experience indicates that direct elicitation of numerical values of model

parameters by DMs demands much of their cognitive effort. For example, defining weights k_i, $i = 1, \ldots, n$, of a weighted sum model requires specification of substitution rates k_i/k_j between all pairs of criteria $\{g_i, g_j\} \subset \mathcal{G}$. These substitution rates must be, moreover, constant in the whole range of criteria scales. In the case of heterogeneous criteria, it is not easy to say how much gain in performance on one criterion compensates for a loss of performance on another criterion. The same difficulty concerns the determination of relative importance weights of criteria having different semantics than the weights of the weighted sum. Other preference model parameters, like indifference, preference, and veto thresholds on particular criteria are not easier to elicit directly. A long time ago, Fishburn (1967) listed and classified twenty-four methods of estimating additive utilities. Such a large number of methods for estimating parameters of the additive value function proves the difficulty of direct elicitation.

For this reason, indirect preference elicitation through holistic judgments, i.e., *decision examples*, gained importance in MCDA. Decision examples are relatively "easy" preference information. Decisions can also be observed without the active participation of DMs. For example, the choices made by people on the Internet provide rich information about their preferences. Finally, psychologists confirm that DMs are more confident exercising their decisions than explaining them (Slovic et al. 1977; March 1978).

Decision examples may either be provided by the DM on a set of real or hypothetical alternatives, or may come from observation of DM's past decisions. Methods based on indirect preference information are considered more user-friendly than approaches based on direct elicitation of preferences. Moreover, the preference model constructed using decision examples aims to be *compatible* with the preference information, i.e., to reproduce the holistic judgments observed in decision examples. For this, the DMs can verify if the preference model represents faithfully enough their preferences and what is the impact of the provided preference information on the recommendation produced by the method. This makes the whole decision-aiding process more transparent, increasing the DM's confidence in the final recommendation. As noted by Stewart (2005), MCDA includes a comprehensive process involving a rich interplay between human judgment, data analysis, and computational processes. Indeed, transparency and explainability of the decision-aiding process are necessary for an efficient interplay.

Decision examples elicited in the indirect approach may have different forms depending on the problem statement and the wishes of stakeholders. These may be: (i) pairwise comparisons of some real or fictitious alternatives, (ii) assignments of some alternatives to decision classes, as their typical examples, (iii) intensity of preference expressed on some quadruples of alternatives, e.g., a is preferred to z more than c is preferred to d, (iv) rank related preferences, e.g., alternative f should be among 5% of the best ones.

The MCDA methods based on the indirect preference information having the form of decision examples are called *ordinal regression methods*. They were first proposed for the additive value function preference model in the method called UTA (Jacquet-Lagrèze and Siskos 1982). In UTA, the preference information consists of a set of pairwise comparisons of so-called reference alternatives $A^R = \{a^*, b^*, \ldots\}$. Usually, $A^R \subseteq \mathcal{A}$, and these alternatives are relatively well-known to the DM, so that they are

able to say for some $a^*, b^* \in A^R$, that $a^* \succeq b^*$ or $b^* \succeq a^*$ or $a^* \sim b^*$. A *compatible* additive value function, reproducing as far as possible the pairwise comparisons of reference alternatives, is found as a result of solving a linear programming (LP) problem where variables are marginal values in the breakpoints of piecewise-linear marginal value functions. While the UTA method and its many variants gained a high popularity witnessed by many applications (Siskos et al. 2005), it suffers from the problem of non-uniqueness of LP solutions. In other words, the LP solution identifies a single compatible value function, while there may exist many instances of compatible value functions that reproduce the pairwise comparisons of reference alternatives equally well. Different compatible instances may, however, compare non-reference alternatives differently, making the recommendation ambiguous.

The issue of ambiguity has been resolved by Greco et al. (2008) using the approach called Robust Ordinal Regression (ROR) (Greco et al. 2010d). In ROR, the preference data are the same as in UTA, however, the ordinal regression finds the whole set of compatible instances of the value function. The reconstruction of the DM's judgments by the ordinal regression in terms of compatible instances of the value function is a preference learning step. As ROR works in a loop with incremental elicitation of preferences, it is called *constructive learning* (Corrente et al. 2013a). In constructive learning the model aims to reconstruct as faithfully as possible the preference information of the DM, while the DM learns from the consequences of applying all compatible instances of the model on the whole set of alternatives. Constructive learning is 'robust' because it tells the truth about preferences based on partial, not exhaustive, preference information. For example, it does not require information about all possible pairwise comparisons, like the AHP method (Saaty 2005).

As ROR has the most in common with PL, we will characterize this approach in more detail in the next subsection.

2.6 Robust ordinal regression (ROR)

A typical scenario of the ROR approach is the following (Greco et al. 2010d). Suppose a DM wants to rank countries with respect to their trend for innovation. Three criteria are used to evaluate the countries: innovation performance, direct innovation inputs, and innovation environment. The preference model chosen to rank the countries is a value function. In order to build a value function representing the preferences of the DM, one needs to know some preference information elicited by the DM. Typically, the DM is sure of some pairwise comparisons between countries that do not dominate each other. These countries constitute a set of reference alternatives $A^R = \{a^*, b^*, \ldots\}$. The pairwise comparisons have the form $a^* \succeq b^*$, $b^* \succeq e^*$, $c^* \sim d^*$, and so on, but they do not need to be exhaustive. Moreover, the DM may want to distinguish between intensities of preference between some pairs of reference alternatives, telling, for example, that the intensity of preference of a^* over b^* is not smaller than that of b^* over e^*. This preference information is a starting point for the search for a preference model, i.e., the value function. If there is no instance of the preference model reproducing the provided holistic judgments expressed on reference alternatives, the DM could either decide to work with the inconsistency or revise some pieces of preference

information impeding the incompatibility. Then, using ROR, i.e., solving a series of special optimization problems, one obtains two preference relations in the set of all countries, called *necessary* and *possible*. While the first is true for all instances of the value function compatible with the DM's preference information, the other is true for at least one such compatible instance of the value function. Moreover, ROR can provide extreme ranks of particular countries and a representative compatible instance of the value function. Analyzing the two outcomes of ROR, as well as the form of the displayed representative preference model, the DM gains insights into their preferences at the current stage of interaction. This stimulates a reaction of the DM who may add a new or revise the old preference information. Such an interactive process ends when the necessary preference relation, extreme ranks, and/or a representative ranking yields a recommendation which, according to the DM, is decisive and convincing.

Before moving to the survey of ROR applied to MCDA with various preference models, let us mention the interesting adaptation of ROR to decisions under risk and uncertainty. This is possible after reformulation of decision under risk and uncertainty in terms of MCDA where criteria are some meaningful quantiles of gain or loss (Corrente et al. 2016b).

Moreover, ROR has been adapted to multiple criteria group decision (Greco et al. 2012). In this case, several DMs cooperate in a decision problem to make a collective decision. DMs share the same "description" of the decision problem (the same set of alternatives, family of criteria, and performance matrix). Each DM provides their own preference information, composed of pairwise comparisons of some reference alternatives. The collective preference model accounts for the preference expressed by each DM. Two approaches for handling preferences of multiple DMs are considered:

- DMs have similar expertise and compass the whole, rather small set $\mathcal{A}$. Then, the preferences of different DMs are interrelated, and the examination of necessary relations obtained for each DM permits the identification of the spaces of agreement and disagreement between the DMs, which helps in negotiations.
- DMs have different expertise and express their opinions on different parts of rather big set $\mathcal{A}$. Then, knowledge of all DMs is combined into preference information of a single fictitious DM, and examination of necessary relations obtained for coalitions of DMs permits the identification of the spaces of agreement and disagreement between the coalitions.

2.7 ROR with value function preference model

Let us formulate the ordinal regression problem based on the above-mentioned preference information. In order to model the DM's preference information, we are using in this subsection a general additive value function:

$$U(a) = \sum_{i=1}^{n} u_i(g_i(a)) \tag{1}$$

where indices of criteria, $i = 1, \ldots n$, form set $\mathcal{J}$, the marginal value functions u_i, $i \in \mathcal{J}$, are monotone, non-decreasing and normalized so that the additive value

([1](#)) is bounded within the interval $[0, 1]$. Note that for simplicity of notation, we write $u_i(a)$, $i \in \mathcal{J}$, instead of $u_i(g_i(a))$. Consequently, the basic set of constraints defining general additive value functions compatible with the preference information composed of pairwise comparisons and comparisons of intensities of preferences, has the following form:

$$\left.\begin{array}{l} U(a^*) \geq U(b^*) + \varepsilon, \quad \text{if } a^* \succ b^*, \text{ for } \{a^*, b^*\} \in A^R \\ U(a^*) = U(b^*), \quad \text{if } a^* \sim b^*, \text{ for } \{a^*, b^*\} \in A^R \\ U(a^*) - U(b^*) \geq U(c^*) - U(d^*) + \varepsilon, \quad \text{if } (a^*, b^*) \succ^* (c^*, d^*), \text{ for } \{a^*, b^*\, c^*, d^*\} \in A^R \\ U(a^*) - U(b^*) = U(c^*) - U(d^*), \quad \text{if } (a^*, b^*) \sim^* (c^*, d^*), \text{ for } \{a^*, b^*\, c^*, d^*\} \in A^R \\ u_i(x_i^h) - u_i(x_i^{(h-1)}) \geq 0, \quad h = 2, \ldots, m_i(A^R), \\ u_i(x_i^1) = 0, \quad \sum_{i=1}^{n} u_i(x_i^{m_i(A^R)}) = 1, \end{array}\right\} E_{BASE}^{A^R}$$

where $(a^*, b^*) \succ^* (c^*, d^*)$ means a^* is preferred to b^* more than c^* is preferred to d^*, $(a^*, b^*) \sim^* (c^*, d^*)$ means a^* is preferred to b^* as much as c^* is preferred to d^*, $m_i(A^R)$ is the number of different performances of reference alternatives on criterion $i \in \mathcal{J}$, $x_i^1, \ldots, x_i^{m_i(A^R)}$ are the ordered performances of reference alternatives on criterion $i \in \mathcal{J}$, $x_i^h < x_i^{h+1}, h = 1, \ldots, m_i(A^R) - 1$. The first four constraints represent pairwise comparisons of reference alternatives, as well as comparisons between pairs of reference alternatives, and the last two are monotonicity and normalization constraints.

General monotonic marginal value functions defined in this way do not involve any arbitrary or restrictive parametrization (Greco et al. [2010d](#)). On the contrary, the majority of existing methods employ marginal value functions that are linear or piecewise linear (Siskos et al. [2005](#)). Note that piecewise linear functions require specification of the number of characteristic points which is not easy for most DMs.

In order to verify that the set of value functions $\mathcal{U}_{A^R}$ compatible with preference information provided by the DM is not empty, the following LP problem has to be solved:

$$Maximize : \varepsilon, \text{ s.t. } E_{BASE}^{A^R}. \tag{2}$$

Let us denote by ε^* the maximal value of ε obtained from the solution of the above LP problem, i.e., $\varepsilon^* = max\ \varepsilon$, s.t. $E_{BASE}^{A^R}$. It corresponds to a margin of the representation error. One concludes that $\mathcal{U}_{A^R}$ is not empty, if $E_{BASE}^{A^R}$ is feasible and $\varepsilon^* > 0$. In such a case, all pieces of preference information can be properly reproduced by at least one instance of the value function. On the contrary, when $E_{BASE}^{A^R}$ is infeasible or the margin of the representation error is not greater than zero, some pieces of DM's preference on the set of reference alternatives A^R are conflicting, and thus, some pairwise comparisons or comparisons between pairs cannot be reproduced by any additive value function. In the latter case, the DM can choose one of three options:

- revise some pairwise comparisons or comparisons between pairs that make the reproduction by an additive value function impossible; the "troublesome" items of preference information can be identified by solving an integer LP problem (Mousseau et al. [2003](#)),

- continue to work with a set of non-perfectly compatible value functions $\mathcal{U}_{A^R}$ that reproduce the preference information with an error not greater than ε^*,
- replace the additive value function by a more complex model, e.g., the Choquet integral (Angilella et al. 2010), or an additive value function augmented by components modeling positive and negative interactions between some criteria without requiring that all criteria are expressed on the same scale (Greco et al. 2014).

The consequence of considering many instances of the preference model is a univocal recommendation. In the constructive learning perspective, where the aim is not to predict, but rather to construct the preferences from scratch, the user has an interest in investigating what are the consequences of using all instances from $\mathcal{U}_{A^R}$ on the set of alternatives $\mathcal{A}$. Obviously, the final recommendation in terms of ranking, best choice, or classification may vary substantially depending on which compatible instance is selected. Remark also that ROR does not consider a probability distribution on the set of all compatible instances of the value function, assigning to all of them the same credibility.

When comparing a pair of alternatives $(a, b) \in \mathcal{A} \times \mathcal{A}$ in terms of the recommendation that is provided by any compatible (or non-perfectly compatible) value function from set $\mathcal{U}_{A^R}$, it is reasonable to verify whether a is weakly preferred to b or vice versa, for all or at least one compatible instance of the value function. Answering these questions, ROR methods (Greco et al. 2008) produce two preference relations in the set of alternatives $\mathcal{A}$:

- a *necessary* weak preference relation $\succsim^N$ holds for a pair of alternatives $(a, b) \in \mathcal{A} \times \mathcal{A}$, in case $U(a) \geq U(b)$ for all compatible instances of the value function,
- a *possible* weak preference relation $\succsim^P$ holds for a pair of alternatives $(a, b) \in \mathcal{A} \times \mathcal{A}$, in case $U(a) \geq U(b)$ for at least one compatible instance of the value function.

Thus defined, the necessary relations specify the most certain recommendation worked out on the basis of all compatible instances of the value function, while the possible relations identify a recommendation provided by at least one compatible instance of the value function. Consequently, the necessary outcomes can be considered robust regarding the available preference information, as they guarantee that a definite relation is the same whichever compatible instance of the preference model would be used. To verify the truth of the necessary and possible weak preference relations the following linear programs need to be solved (Corrente et al. 2013a):

$$Maximize : \varepsilon$$

s.t.

$$\left. \begin{array}{l} U(b) - U(a) \geq \varepsilon \\ E_{BASE}^{A^R} \end{array} \right\} E^N(a, b),$$

and

$$Maximize : \varepsilon$$

s.t.

$$\left.\begin{array}{l} U(a) - U(b) \geq 0 \\ E^{A^R}_{BASE} \end{array}\right\} E^P(a, b).$$

We conclude that $a \succsim^N b$, if $E^N(a, b)$ is not feasible or $\varepsilon_* = max \ \varepsilon$, s.t. $E^N(a, b)$, is not greater than 0, and that $a \succsim^P b$, if $E^P(a, b)$ if feasible and $\varepsilon^* = max \ \varepsilon$, s.t. $E^P(a, b)$, is greater than 0.

Let us remark that preference relations $\succsim^N$ and $\succsim^P$ are meaningful only if there exists at least one compatible instance of the value function. Observe also that in this case, for any $a^*, b^* \in A^R$, $a^* \succsim b^* \Rightarrow a^* \succsim^N b^*$ and $a^* \succ b^* \Rightarrow \neg(b^* \succsim^P a^*)$. In fact, if the DM says for $a^*, b^* \in A^R$ that $a^* \succsim b^*$, then for any compatible instance of the value function $U(a^*) \geq U(b^*)$, and, therefore, $a^* \succsim^N b^*$. Moreover, if $a^* \succ b^*$, then for any compatible instance of the value function $U(a^*) > U(b^*)$, and, consequently, there is no compatible instance of the value function such that $U(b^*) \geq U(a^*)$, which means $\neg(b^* \succsim^P a^*)$.

The necessary weak preference relation $\succsim^N$ is a partial preorder (i.e., it is reflexive ($a \succsim^N a$, since for all $a \in \mathcal{A}$, $U(a) = U(a)$), and transitive (for all $a, b, c \in \mathcal{A}$, if $a \succsim^N b$ and $b \succsim^N c$, then $a \succsim^N c$). Possible weak preference relation $\succsim^P$ is a strongly complete binary relation (i.e., for all $a, b \in \mathcal{A}$, $a \succsim^P b$ or $b \succsim^P a$), and negatively transitive (i.e., $\forall a, b, c \in \mathcal{A}$, if $\neg(a \succsim^P b)$ and $\neg(b \succsim^P c)$, then $\neg(a \succsim^P c)$).

Moreover, for any pair $(a, b) \in \mathcal{A} \times \mathcal{A}$, $a \succsim^N b \Rightarrow a \succsim^P b$, i.e., $\succsim^N \subseteq \succsim^P$, and for all $(a, b) \in \mathcal{A} \times \mathcal{A}$, $a \succsim^N b$ or $b \succsim^P a$.

ROR with a general additive value function preference model has also been applied to multiple criteria classification problems (Greco et al. 2010c; Köksalan and Ozpeynirci 2009). Within ROR for multiple criteria classification one can also handle additional preference information about the intensity of preference, in a way similar to (Figueira et al. 2009).

Since different compatible instances of the value function may order a given alternative from set $\mathcal{A}$ at different positions in the ranking, or classify this alternative to different classes, Kadziński et al. (2012) proposed to identify the extreme ranks, or classes, for each alternative, as well as the interval of scores it can attain. This procedure, based on mixed-integer LP, is known in ROR as Extreme Ranking Analysis.

Although richer than the dominance relation, the necessary preference relation $\succsim^N$ may still leave some pairs of alternatives incomparable, i.e., $a \succsim^P b$ and $b \succsim^P a$. To get an estimate of the probability that a randomly selected compatible instance of the value function ranks alternative a higher than b, or vice versa, one can perform the Stochastic Multiobjective Acceptability Analysis (SMAA) (Kadziński and Tervonen 2013). It consists of using a *Hit & Run* sampling of compatible instances of the value function within constraints $E^{A^R}_{BASE}$, and checking the frequency with which:

- $a \succ b$, which gives *pairwise winning index* $p(a, b)$,
- a gets position h in the ranking, which gives *rank acceptability index* r_a^h.

In SMAA, all compatible instances of the value function are equiprobable. In (Corrente et al. 2016a), it has been proposed to induce the probability distribution on

compatible instances of the value function from additional *uncertain* preference information of the type: $a \succ b$ is more credible than $b \succ a$. The procedure is composed of three steps:

- sampling a number (s_v) of instances of the value function $U_t \in \mathcal{U}_{A^R}$, $t = 1, \ldots, s_v$, compatible with the certain part of preference information provided by the DM,
- inducing a probability distribution $\left\{ w_t(U_t) \in [0, 1] : \sum_{t=1}^{s_v} w_t(U_t) = 1 \right\}$ on the sample of instances generated in the previous step, taking into account the uncertain part of preference information via LP,
- sampling probability distributions compatible with uncertain preference information, and taking a barycenter distribution as a representative one for the final ranking.

Yet another way of dealing with multiple compatible value functions in the UTA-like approach has been presented by Bous et al. (2010) as ACUTA method. It is based on the computation of the analytic center of a polyhedron formed by compatible additive value functions for the selection of the representative value function.

2.8 ROR with non-additive value functions

When *interactions* among criteria appear in the user's preferences, the general additive model is not able to represent them. Interactions may be either negative (*negative synergy*) if the importance weight of a subset of criteria is smaller than the sum of importance weights of all criteria belonging to the subset, or positive (*positive synergy*) if, conversely, the importance weight of a subset of criteria is greater than the sum of importance weights of all criteria belonging to the subset.

To cope with interactions among criteria, one can use *non-additive integrals* as preference models, such as Choquet integral for cardinal criteria with an interval scale, and Sugeno integral for ordinal criteria (Grabisch 1996). It should be noted, however, that these non-additive integrals require that the evaluations on all criteria are expressed on the same scale, which is hard to satisfy in real-world MCDA (Roy 2009). Another popular nonlinear aggregation function able to handle interactions among criteria expressed on ratio scales is the multilinear model characterized recently by Pelegrina et al. (2020). Differently from the Choquet integral, the multilinear model does not need the commensurability assumption and is obtained as the linear interpolation using all vertices of the hypercube $[0, 1]^n$ (Grabisch 2016).

The commensurability assumption has also been abandoned in the UTA^{GMS}-INT method (Greco et al. 2014). The authors proposed to aggregate interacting criteria with heterogeneous scales using a value function composed not only of the sum of marginal non-decreasing value functions $u_i(a)$, $i = 1, \ldots, n$, but also of sums of functions:

$$syn_{i_1,i_2}^+ : [x_{i_1}^{min}, x_{i_1}^{max}] \times [x_{i_2}^{min}, x_{i_2}^{max}] \to [0, 1],$$
$$syn_{i_1,i_2}^- : [x_{i_1}^{min}, x_{i_1}^{max}] \times [x_{i_2}^{min}, x_{i_2}^{max}] \to [0, 1], \quad \text{for all } (i_1, i_2) \in \mathcal{J} \times \mathcal{J}, \ i_1 > i_2,$$

where $x_{i_1}^{min}$, $x_{i_1}^{max}$ and $x_{i_2}^{min}$, $x_{i_2}^{max}$ are the worst (min) and the best (max) performances on criteria $i_1, i_2 \in \mathcal{J}$, respectively. Functions $syn_{i_1,i_2}^{+}(x_{i_1}, x_{i_2})$ and $syn_{i_1,i_2}^{-}(x_{i_1}, x_{i_2})$, are non-decreasing in both their two arguments, for all pairs of (possibly) interacting criteria $(i_1, i_2) \in \mathcal{J} \times \mathcal{J}$, such that $i_1 > i_2$. They represent positive and negative interactions, respectively, and augment or decrease the additive component of the value function. In other words, they are *bonus* and *penalty* functions with respect to the main additive component. Obviously, a pair of interacting criteria can either be in positive or negative synergy, which means that $syn_{i_1,i_2}^{+}(\cdot, \cdot)$ and $syn_{i_1,i_2}^{-}(\cdot, \cdot)$ are mutually exclusive.

For an alternative $a \in \mathcal{A}$, the value function is defined as:

$$U^{int}(a) = \sum_{i=1}^{n} u_i(a) + \sum_{(i_1,i_2)\in\mathcal{J}\times\mathcal{J}, i_1>i_2} syn_{i_1,i_2}^{+}(g_{i_1}(a), g_{i_2}(a))$$
$$- \sum_{(i_1,i_2)\in\mathcal{J}\times\mathcal{J}, i_1>i_2} syn_{i_1,i_2}^{-}(g_{i_1}(a), g_{i_2}(a)).$$

U^{int} takes into account all possible positive and negative interactions between pairs of criteria, which makes it similar to the 2-additive Choquet integral, however, the capacity of representation of preferences is greater for the former than for the latter (Greco et al. 2014). In practical applications, the U^{int} model is even too general, thus a limited number of pairs of interacting criteria is advised for consideration. One can easily identify a minimal set of pairs of criteria for which there is either positive or negative interaction using a procedure presented by Greco et al. (2014).

2.9 ROR with outranking relation preference model

The value function model is a complete and transitive preference relation with a compensatory logic. In many real-life decision situations, when modeling users' preferences it is reasonable to consider: (i) incomparability between alternatives, (ii) intransitive indifference and intransitive preference, and (iii) non-compensatory multiple criteria aggregation.

MCDA methods based on a preference model in the form of a valued binary relation called *outranking relation*, such as ELECTRE and PROMETHEE, answer these needs (Greco et al. 2016a). Outranking relation S groups three basic preference relations: indifference ($\sim$), weak preference ($\succ^w$) and strong preference ($\sim^s$). aSb reads: "alternative a is at least as good as alternative b". As this relation is constructed for all ordered pairs of alternatives $\{a, b\} \in \mathcal{A} \times \mathcal{A}$, one can conclude the following relations between a and b:

- $aSb \ \wedge \ bSa \ \Leftrightarrow \ a \sim b$ (indifference),
- $aSb \ \wedge \ \neg(bSa) \ \Leftrightarrow \ a \succ^w b \ \vee \ a \succ^s b$ (large preference),
- $\neg(aSb) \ \wedge \ \neg(bSa) \ \Leftrightarrow \ a \ ? \ b$ (incomparability).

S is an incomplete and intransitive relation on set $\mathcal{A}$, constructed via concordance and discordance tests (Figueira et al. 2013). Outranking methods assume that evaluation criteria from family $\mathcal{G}$ are *pseudo-criteria* equipped with comparison thresholds.

A pseudo-criterion is a real-valued function $g_i \in \mathcal{G}$ associated with two threshold functions, $q_i(\cdot)$ and $p_i(\cdot)$, whose arguments are performances of the alternatives. By convention, the thresholds are functions of the worst performance of the two alternatives being compared. They are called indifference and preference thresholds, respectively. The thresholds satisfy the following condition: for all ordered pairs of alternatives $(a, b) \in \mathcal{A} \times \mathcal{A}$, such that $g_i(a) \geq g_i(b)$, $g_i(a) + p_i(g_j(b))$ and $g_i(a) + q_i(g_i(b))$ are non-decreasing monotone functions of $g_i(b)$, such that $p_i(g_i(b)) \geq q_i(g_i(b)) \geq 0$ for all $a, b \in \mathcal{A}$. In this definition, the thresholds are functions of the performance, but they can also be defined as constant values $p_i \geq q_i \geq 0, i \in \mathcal{J}$. Their meaning is the following: indifference threshold q_i is the maximum difference of performances of the compared alternatives allowing them to be considered equivalent with respect to criterion g_i; preference threshold p_i is the minimum difference of performances of the compared alternatives switching the DM's preference with respect to criterion g_i from weak to strict.

The concordance test checks if the coalition of criteria concordant with the hypothesis aSb is strong enough. This strength is measured by the concordance index:

$$C(a, b) = \frac{\sum_{i=1}^{n} w_i \times C_i(a, b)}{\sum_{i=1}^{n} w_i},$$

where $w_i, i \in \mathcal{J}$, are weights of criteria which do not have the meaning of substitution rates, like weights $k_i, i \in \mathcal{J}$, of the weighted sum, because the former weights are not multiplied by performances on the corresponding criteria, but by 0–1 individual concordance coefficients $C_i(a, b)$ defined as:

$$C_i(a, b) = \begin{cases} 1 & \text{if } g_i(a) - g_i(b) \geq -q_i, \\ \frac{g_i(a) - g_i(b) + p_i}{p_i - q_i} & \text{if } -p_i < g_i(a) - g_i(b) < -q_i, \\ 0 & \text{if } g_i(a) - g_i(b) \leq -p_i. \end{cases}$$

Thus, the aggregation is not compensatory and weights $w_i, i \in \mathcal{J}$, indicate a voting power of $g_i \in \mathcal{G}$. The concordance test is positive if $C(a, b) \geq \lambda$, where $\lambda \in [0.5, 1]$ is a cutting level (concordance threshold).

The discordance test checks if among criteria discordant with the hypothesis aSb there is a strong opposition against aSb. Criterion $g_i \in \mathcal{G}$ opposes strongly to the assertion aSb if $g_i(b) - g_i(a) \geq v_i$, where $v_i \geq p_i$ is called the veto threshold of g_i. In this situation, criterion g_i makes a veto to the assertion aSb.

In conclusion, outranking relation aSb is true if and only if $C(a, b) \geq \lambda$ and there is no criterion strongly opposed (making veto) to the hypothesis. In this way, for each ordered pair $(a, b) \in \mathcal{A} \times \mathcal{A}$ one obtains relation S being true (1) or false (0), presented as a directed outranking graph where the nodes correspond to alternatives and there is an arc from node a to node b if aSb. To choose the best alternative of $\mathcal{A}$, one can exploit the outranking relation by searching for a kernel of the outranking graph. When the graph has no cycles, its kernel is unique and includes alternatives that do not mutually outrank; each alternative from outside the kernel is outranked by at least one alternative from the kernel. Thus, the kernel is composed of the best incomparable alternatives.

The many parameters to be fixed by the DM in the outranking method can be avoided in the ROR approach. Then, the DM provides preference information in terms of holistic pairwise comparisons of some reference alternatives: a^*Sb^* or $a^*S^cb^*$ for some $a^*, b^* \in A^R \subseteq \mathcal{A}$, where S^c is a negation of the outranking relation. Moreover, according to Greco et al. (2011), where ROR was adapted to ELECTRE methods, the DM has a possibility to specify $[q_{i*}, q_i^*]$ and $[p_{i*}, p_i^*]$, being the ranges of indifference and preference thresholds, respectively, allowed by the DM.

Assuming $\sum_{i=1}^n w_i = 1$, one can substitute $C(a, b) = \sum_{i=1}^n w_i C_i(a, b)$ by $C(a, b) = \sum_{i=1}^n \Psi_i(a, b)$, where marginal concordance functions $\Psi_i(a, b), i \in \mathcal{J}$, are non-decreasing functions of $g_i(a) - g_i(b)$:

$$\Psi_i(a, b) = \begin{cases} \Psi_i(\beta_i, \alpha_i) & \text{if } g_i(a) - g_i(b) \geq -q_i, \\ \frac{g_i(a) - g_i(b) + p_i}{p_i - q_i} & \text{if } -p_i < g_i(a) - g_i(b) < -q_i, \\ 0 & \text{if } g_i(a) - g_i(b) \leq -p_i, \end{cases}$$

and α_i, β_i are, respectively, the worst and the best-observed performances on criterion $g_i, i = 1, \ldots, n$.

The outranking model compatible with preference information provided by the DM is a set of marginal concordance functions $\Psi_i(a, b)$, cutting levels λ, indifference q_i, preference p_i, and veto thresholds $v_i, i = 1, \ldots, n$, reproducing the DM's preference information concerning pairs $(a^*, b^*) \in A^R \times A^R$.

In this case, the ordinal regression compatibility constraints E^{A^R} are:

$$\left.\begin{array}{l} \text{If } a^*Sb^*, \text{ for } (a^*, b^*) \in A^R \times A^R: \\ \text{(i) } C(a^*, b^*) = \sum_{i=1}^n \Psi_i(a^*, b^*) \geq \lambda, \\ \text{(ii) } g_i(b^*) - g_i(a^*) + \varepsilon \leq v_i, \ i = 1, \ldots, n. \\ \text{If } a^*S^cb^*, \text{ for } (a^*, b^*) \in A^R \times A^R: \\ \text{(iii) } C(a^*, b^*) = \sum_{i=1}^n \Psi_i(a^*, b^*) + \varepsilon \leq \lambda + M_0(a^*, b^*), \\ \text{(iv) } g_i(b^*) - g_i(a^*) \geq v_i - \delta M_i(a^*, b^*), \ i = 1, \ldots, n, \\ \text{(v) } M_i(a^*, b^*) \in \{0, 1\}, \ i = 0, 1, \ldots, n, \\ \text{(vi) } \sum_{i=0}^n M_i(a^*, b^*) \leq n. \\ \text{(vii) } 0.5 \leq \lambda \leq 1, \\ \text{(viii) } v_i \geq p_i^* + \varepsilon, \ i = 1, \ldots, n, \\ \text{(ix) } \varepsilon \geq 0. \end{array}\right\} E_{BASE}^{A^R}$$

where δ is a big constant value. Here, constraints (i) and (ii) ensure positive concordance and non-discordance tests, respectively. For $a^*S^cb^*$, either the concordance test or the non-discordance test should be negative, which corresponds to constraints (iii) or (iv)–(vi), respectively. Remark that for $M_i = 0$, the corresponding constraint (iii) or (iv) is satisfied, which means that either concordance $C(a^*, b^*)$ is below cutting level λ or difference $g_i(b^*) - g_i(a^*)$ is exceeding veto threshold v_i.

Given a pair of alternatives $(a, b) \in \mathcal{A}$, alternative a *possibly outranks* alternative b (written as aS^Pb) if and only if $\varepsilon^* > 0$, where (Greco et al. 2011):

$$\varepsilon^* = \max \varepsilon$$

s.t.

$$\left.\begin{array}{l} C(a, b) = \sum_{i=1}^{n} \Psi_i(a, b) \geq \lambda, \\ g_i(b) - g_i(a) + \varepsilon \leq v_i, \ i = 1, \ldots, n, \\ E_{BASE}^{A^R}. \end{array}\right\} E^P(a, b).$$

This means that if $\varepsilon^* > 0$ and constraints $E^P(a, b)$ are feasible (for assertion aSb, concordance test does not fail and no criterion makes veto), then a outranks b for at least one compatible instance of the outranking model.

Moreover, alternative a *necessarily outranks* alternative b (written as $aS^N b$) if and only if $\varepsilon^* \leq 0$, where:

$$\varepsilon^* = \max \varepsilon$$

s.t.

$$\left.\begin{array}{l} C(a, b) = \sum_{i=1}^{n} \Psi_i(a, b) + \varepsilon \leq \lambda + M_0(a, b), \\ g_i(b) - g_i(a) \geq v_i - \delta M_i(a, b), \ i = 1, \ldots, n, \\ M_i(a, b) \in \{0, 1\}, \ i = 0, \ldots, n, \ \sum_{i=0}^{n} M_i(a, b) \leq n, \\ E_{BASE}^{A^R}. \end{array}\right\} E^N(a, b).$$

This means, in turn, that if $\varepsilon^* \leq 0$ or constraints $E^N(a, b)$ are infeasible (for assertion $aS^c b$ either concordance test fails or one of the criteria makes veto), then a outranks b for all compatible instances of the outranking model. In other words, $aS^N b$ is true because $aS^c b$ is not true for all compatible instances of the outranking model.

For any pair of alternatives $(a, b) \in \mathcal{A}$:

$$aS^N b \Leftrightarrow \neg(aS^{cP} b), \ \text{ as well as, } \ aS^{cP} b \Leftrightarrow \neg(aS^N b),$$
$$aS^P b \Leftrightarrow \neg(aS^{cN} b), \ \text{ as well as, } \ aS^{cN} b \Leftrightarrow \neg(aS^P b),$$

thus, there are two sources of information (S^N, S^P) about four relations $(S^N, S^{cN}, S^P, S^{cP})$ in set $\mathcal{A}$. They have the following properties:

$$aS^N b \Rightarrow aS^P b$$
$$aS^N b \Rightarrow \neg(aS^{cN} b) \ \text{ as well as } \ aS^{cN} b \Rightarrow \neg(aS^N b).$$

Moreover, for given preference information $a^* Sb^*$:

$$a^* Sb^* \Rightarrow a^* S^N b^* \ \text{ and } \ a^* Sb^* \Rightarrow \neg(b^* S^P a^*).$$

To work out a recommendation in the choice problem, it is advised to find a kernel of the necessary outranking graph S^N. In case of the ranking problem, one can exploit the necessary outranking graph including S^N and S^{cN} relations, using the Net Flow Score procedure. For each alternative $a \in \mathcal{A}$, the score $NFS(a) = strength(a) - weakness(a)$, where $strength(a)$ counts all relations (arcs) with other alternatives z,

such that $aS^N z$ or $zS^{cN}a$, while $weakness(a)$ counts all relations (arcs) with other alternatives v, such that $aS^{cN}v$ or $vS^N a$. The resulting ranking is a complete preorder determined by $NFS(a) \in \mathcal{A}$.

2.10 ROR handling hierarchy of criteria

In many real-world decision problems, family of criteria $\mathcal{G}$ has a *hierarchical* structure, which means that the performances on some criteria are aggregates of performances on sub-criteria. In (Keeney and Raiffa 1976), one can read: "Almost everyone who has seriously thought about the objectives in a complex problem has come up with some sort of hierarchy of objectives", A similar conviction is expressed by Belton and Stewart (2002): "In the process of structuring the problem, it is possible (even likely) that the criteria may have been constructed hierarchically in terms of a value tree". For example, in the land use problem considered by Belton and Stewart (2002), the economic benefit criterion has three sub-criteria which are the outputs of agriculture, forestry, and secondary industry. Consideration of a hierarchy of criteria is also a way of decreasing the problem's complexity by reasoning about the problem at different levels of generality. Once the hierarchy is built, the DMs may first judge the alternatives on the elementary criteria (leaves of the hierarchy tree) and then receive feedback about their impact on criteria performances at higher levels of the hierarchy tree.

ROR has been adapted to handle the hierarchy of criteria in, the so-called, Multiple Criteria Hierarchy Process (MCHP), first, for a general additive value function as a preference model within the ranking problem (Corrente et al. 2012). In this case, the marginal value functions refer to all levels of the hierarchy tree, representing values of particular scores of the alternatives on criteria, sub-criteria, sub-sub-criteria, etc.

In MCHP, the DM is asked to provide preference information concerning a particular criterion or sub-criterion $G_{\mathbf{r}}$. It has the form of a partial preorder $\succsim_{\mathbf{r}}$ on A^R or a partial preorder $\succsim_{\mathbf{r}}^*$ on $A^R \times A^R$, which are interpreted as pairwise comparisons of reference alternatives, or statements regarding comparisons of the intensity of preference between pairs of reference alternatives.

The value function assessing alternative $a \in \mathcal{A}$ with respect to criterion or sub-criterion $G_{\mathbf{r}}$ is:

$$U_{\mathbf{r}}(a) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(g_{\mathbf{t}}(a))$$

where $E(G_{\mathbf{r}})$ is the set of indices of lowest-level sub-criteria descending from $G_{\mathbf{r}}$ located at a higher level.

It is worth noting that in the case of MCHP, the DM can express their preferences for reference alternatives either comprehensively, or with respect to particular sub-criteria $G_{\mathbf{r}}$, which allows a more precise expression, and then representation, of preferences.

Since the first proposal of ROR-MCHP with the general additive value function for the ranking problem, this methodology has been extended to:

- preference model in terms of an outranking relation used in ELECTRE and PROMETHEE methods for multiple criteria ranking (Corrente et al. 2013b), and for multiple criteria classification (Corrente et al. 2016c, 2017a),
- preference model in terms of an outranking relation used in ELECTRE III method for multiple criteria ranking with imprecise weights of criteria and stochastic multiobjective acceptability analysis (Corrente et al. 2017b),
- preference model in terms of Choquet integral for multiple criteria ranking with stochastic multiobjective acceptability analysis (Angilella et al. 2016).

2.11 ROR with rule preference model

As mentioned before, a preference model built of "*if . . . , then . . .*" decision rules induced from examples of decisions has a double advantage in MCDA: it combines the relatively easy elicitation of indirect preference information with the transparency and explainability of decision rules.

The decision examples come either from observations of DM's past decisions made in the same decision context or from conscious elicitation by the DM on demand of the analyst. In the latter case, they concern some reference alternatives, as in other ROR approaches. In both cases, decision examples may, however, be *inconsistent with the dominance principle* commonly accepted for multiple criteria decision problems. Decisions are inconsistent with the dominance principle if:

- In case of classification: alternative a has been assigned to a worse decision class than alternative b, although a is at least as good as b on all the considered criteria, i.e., a dominates b.
- In case of choice and ranking: a pair of alternatives (a, b) has been assigned a degree of intensity of preference smaller than pair (c, d), although differences in evaluations between a and b on all the considered criteria are at least as large as the respective differences of evaluations between c and d, i.e., pair (a, b) dominates pair (c, d).

The inconsistency may come from many sources. Examples include:

- Missing attributes (attributes without preference scale or criteria) in the description of alternatives.
- Unstable preferences or hesitation of DMs.

Handling these inconsistencies is important for preference learning. They cannot be simply considered as noise or error to be eliminated from data, or amalgamated with consistent data by some averaging operators. They should be identified and presented as uncertain patterns.

The concept of *rough set* introduced by Pawlak (1982) appeared to be useful for handling inconsistency in data, however, originally it was limited to inconsistency with respect to the indiscernibility principle. Objects (alternatives) that have the same description are said to be *indiscernible*. The indiscernibility relation thus generated in the universe of discourse U constitutes the mathematical basis of rough set theory. It induces a partition of the universe into blocks of indiscernible objects, called *granules*, which can be used to build knowledge about a real or abstract world. Any subset X of

the universe may be expressed in terms of these blocks either precisely (as a union of granules) or approximately. In the latter case, the subset X may be characterized by two ordinary sets, called *lower approximation* and *upper approximation* of set X. A rough set is defined by means of these two approximations, which coincide in the case of an ordinary set. The lower approximation of X is composed of all the granules included in X (whose elements, therefore, certainly belong to X), while the upper approximation of X consists of all the granules which have a non-empty intersection with X (whose elements, therefore, may belong to X). The difference between the upper and lower approximations constitutes the boundary region of the rough set, whose elements cannot be characterized with certainty as belonging or not to X by using the available information. The information about objects from the boundary region is, therefore, inconsistent or ambiguous. The cardinality of the boundary region states, moreover, the extent to which it is possible to express X in exact terms, on the basis of the available information. For this reason, this cardinality is used as a measure of the vagueness of information about X. The lower and upper approximations of a partition of U into decision classes prepare the ground for inducing, respectively, *certain* and *possible* knowledge patterns in the form of "$if \ldots, then \ldots$" decision rules.

To handle inconsistencies with respect to the dominance principle, which is typical for preference data, Greco et al. (2001) generalized the original rough set concept substituting the indiscernibility relation with a dominance relation in the rough approximation of decision classes. The resulting methodology was called *Dominance-based Rough Set Approach* (DRSA). An important consequence of DRSA is the possibility of inferring the DM's preference model in terms of decision rules induced either from lower approximations of decision classes (certain rules), or from upper approximations of decision classes (possible rules), or from the difference between upper and lower approximations (approximate rules supported by inconsistent examples).

In DRSA, the granules used for building approximations of decision classes are dominance cones in the evaluation space. DRSA accepts all scales of criteria because it exploits the order only which is a primitive feature of ordinal, interval, and ratio evaluation scales.

Let us explain **DRSA to multiple criteria classification** problems first.

The finite universe of discourse U is composed of classification examples concerning reference alternatives from set $A^R \in \mathcal{A}$, such that, without loss of generality, $g_i : U \rightarrow \mathbb{R}$ for each $i = 1, \ldots, n$, and, for all alternatives $a^*, b^* \in U$, $g_i(a^*) \geq g_i(b^*)$ means that "a^* is at least as good as b^* with respect to criterion g_i", which is denoted as $a^* \succeq_i b^*$. Therefore, $\succeq_i$ is a complete preorder, i.e., a strongly complete and transitive binary relation, defined on U on the basis of evaluations $g_i(\cdot)$. Furthermore, there is a decision attribute d which makes a partition of U into a finite number of decision classes, called *classification*, **Cl**=$\{Cl_1, \ldots, Cl_p\}$, such that each $a^* \in U$ belongs to one and only one class Cl_t, $t = 1, \ldots, p$. We suppose that the classes are preference ordered, i.e., for all $r, s = 1, \ldots, p$, such that $r > s$, the alternatives from Cl_r are preferred to the alternatives from Cl_s. More formally, if $\succeq$ is a *comprehensive weak preference relation* on U, i.e., if for all $a^*, b^* \in U, a^* \succeq b^*$ reads "a^* is at least as good as b^*", then we suppose

$$[a^* \in Cl_r, \ b^* \in Cl_s, \ r > s] \Rightarrow a^* \succ b^*,$$

where $a^* \succ b^*$ means $a^* \succeq b^*$ and *not* $b^* \succeq a^*$. The above assumptions are typical for consideration of a multiple criteria classification problem (also called *ordinal classification with monotonicity constraints*).

In DRSA, the explanation of the assignment of alternatives to preference-ordered decision classes is made on the base of their evaluation with respect to a subset of criteria $P \subseteq \mathcal{J}$. This explanation is called *approximation* of decision classes with respect to P. Of course, most commonly, $P = \mathcal{J}$. In order to take into account the order of decision classes, in DRSA the classes are not considered one by one but, instead, unions of classes are approximated: *upward union* from class Cl_t to class Cl_p, denoted by $Cl_t^{\geq}$, and *downward union* from class Cl_t to class Cl_1, denoted by $Cl_t^{\leq}$, i.e.:

$$Cl_t^{\geq} = \bigcup_{s \geq t} Cl_s, \quad Cl_t^{\leq} = \bigcup_{s \leq t} Cl_s, \quad t = 1, \ldots, p.$$

The statement $a^* \in Cl_t^{\geq}$ reads "a^* belongs to at least class Cl_t", while $a^* \in Cl_t^{\leq}$ reads "a^* belongs to at most class Cl_t". Let us remark that $Cl_1^{\geq} = Cl_p^{\leq} = U, Cl_p^{\geq} = Cl_p$ and $Cl_1^{\leq} = Cl_1$. Furthermore, for $t = 2, \ldots, p$, we have:

$$Cl_{t-1}^{\leq} = U \setminus Cl_t^{\geq} \quad \text{and} \quad Cl_t^{\geq} = U \setminus Cl_{t-1}^{\leq}.$$

The key idea of the rough set approach is an explanation (approximation) of knowledge generated by the decision attribute, using *granules of knowledge* generated by condition attributes. In DRSA, where condition attributes are criteria and decision classes are preference ordered, the knowledge to be explained is the assignment of alternatives to *upward* and *downward unions of classes*, and the granules of knowledge are sets of alternatives contained in *dominance cones* defined in the space of evaluation criteria.

Alternative a^* *dominates* alternative b^* with respect to $P \subseteq \mathcal{J}$ (shortly, a^* *P-dominates* b^*), denoted by $a^* D_P b^*$, if for every criterion $i \in P$, $g_i(a^*) \geq g_i(b^*)$. The relation of P-dominance is reflexive and transitive, i.e., it is a partial preorder.

Given a set of criteria $P \subseteq \mathcal{J}$ and $a^* \in U$, the granules of knowledge used for approximation in DRSA are:

- a set of alternatives dominating a^*, called *P-dominating set*,

$$D_P^+ (a^*) = \{b^* \in U : b^* D_P a^*\},$$

- a set of alternatives dominated by a^*, called *P-dominated set*,

$$D_P^- (a^*) = \{b^* \in U : a^* D_P b^*\}.$$

In terms of multiple criteria evaluations, the above definitions correspond to:

$$D_P^+ \left(a^*\right) = \{b^* \in U : g_i(b^*) \geq g_i(a^*), \text{ for all } i \in P\},$$
$$D_P^- \left(a^*\right) = \{b^* \in U : g_i(b^*) \leq g_i(a^*), \text{ for all } i \in P\}.$$

Given $P \subseteq \mathcal{J}$, the inclusion of an alternative $a^* \in U$ to the upward union of classes $Cl_t^{\geq}$, $t = 2, \ldots, p$, is *inconsistent with the dominance principle* if one of the following conditions holds:

- a^* belongs to class Cl_t or better but it is P-dominated by an alternative b^* belonging to a class worse than Cl_t, i.e., $a^* \in Cl_t^{\geq}$ but $D_P^+(a^*) \cap Cl_{t-1}^{\leq} \neq \emptyset$,
- a^* belongs to a worse class than Cl_t but it P-dominates an object b^* belonging to class Cl_t or better, i.e., $a^* \notin Cl_t^{\geq}$ but $D_P^-(a^*) \cap Cl_t^{\geq} \neq \emptyset$.

The above considerations bring us closer to the definition of rough approximations of the upward and downward unions of decision classes.

The *P-lower approximation* of $Cl_t^{\geq}$, denoted by $\underline{P}(Cl_t^{\geq})$, and the *P-upper approximation* of $Cl_t^{\geq}$, denoted by $\overline{P}(Cl_t^{\geq})$, are defined as follows ($t = 2, \ldots, p$):

$$\underline{P}(Cl_t^{\geq}) = \{a^* \in U : D_P^+(a^*) \subseteq Cl_t^{\geq}\},$$
$$\overline{P}(Cl_t^{\geq}) = \{a^* \in U : D_P^-(a^*) \cap Cl_t^{\geq} \neq \emptyset\}.$$

The definition of *P-lower approximation* and the *P-upper approximation* of $Cl_t^{\leq}$, is analogous ($t = 1, \ldots, p - 1$):

$$\underline{P}(Cl_t^{\leq}) = \{a^* \in U : D_P^-(a^*) \subseteq Cl_t^{\leq}\},$$
$$\overline{P}(Cl_t^{\leq}) = \{a^* \in U : D_P^+(a^*) \cap Cl_t^{\leq} \neq \emptyset\}.$$

The *P-boundaries* of $Cl_t^{\geq}$ and $Cl_t^{\leq}$, denoted by $Bn_P(Cl_t^{\geq})$ ($t = 2, \ldots, p$) and $Bn_P(Cl_t^{\leq})$ ($t = 1, \ldots, p - 1$), respectively, are defined as follows:

$$Bn_P(Cl_t^{\geq}) = \overline{P}(Cl_t^{\geq}) \setminus \underline{P}(Cl_t^{\geq}), \quad Bn_P(Cl_t^{\leq}) = \overline{P}(Cl_t^{\leq}) \setminus \underline{P}(Cl_t^{\leq}).$$

For every $P \subseteq \mathcal{J}$, the alternatives that are consistent in the sense of the dominance principle with all upward and downward unions of classes are *P-correctly classified*. For every $P \subseteq \mathcal{J}$, the *quality of classification* **Cl** by set of criteria P is defined as the ratio of the number of alternatives P-consistent with the dominance principle and the number of all alternatives in U. Since the P-consistent alternatives are those which do not belong to any P-boundary $Bn_P(Cl_t^{\geq})$ or $Bn_P(Cl_t^{\leq})$, and $Bn_P(Cl_t^{\geq}) = Bn_P(Cl_{t-1}^{\leq})$ for $t = 2, \ldots, p$, the quality of classification **Cl** by a set of criteria $P \in \mathcal{J}$ can be written as

$$\gamma_P(\mathbf{Cl}) = \frac{\left| U \setminus \left(\bigcup_{t \in \{2, \ldots, p\}} Bn_P(Cl_t^{\geq}) \right) \right|}{|U|} = \frac{\left| U \setminus \left(\bigcup_{t \in \{1, \ldots, p-1\}} Bn_P(Cl_t^{\leq}) \right) \right|}{|U|}.$$

$\gamma_P(\mathbf{Cl})$ can be seen as a degree of consistency of the classification examples, where P is the set of criteria and **Cl** is the considered classification.

Each minimal (in the sense of inclusion) subset $P \subseteq \mathcal{J}$, such that $\gamma_P(\mathbf{Cl}) = \gamma_{\mathcal{J}}(\mathbf{Cl})$, is called a *reduct* of classification **Cl**, and is denoted by $RED_{\mathbf{Cl}}$. Let us remark that for a given set of classification examples, one can have more than one

reduct. The intersection of all reducts is called the *core* and is denoted by $CORE_{\mathbf{Cl}}$. Criteria in $CORE_{\mathbf{Cl}}$ cannot be removed from consideration without deteriorating the quality of classification **Cl**. This means that, in set $\mathcal{J}$, there are three categories of criteria:

- *indispensable* criteria included in the core,
- *exchangeable* criteria included in some reducts, but not in the core,
- *redundant* criteria, neither indispensable nor exchangeable, and thus not included in any reduct.

The dominance-based rough approximations of upward and downward unions of decision classes can serve to induce a model of classification decisions (preference model) in terms of "*if . . . , then . . .*" decision rules. Given the preference information in terms of classification examples, it is meaningful to consider the following five types of decision rules intending to classify alternative $x \in \mathcal{A}$:

1) *certain* $D_{\geq}$-*decision rules*, providing lower profiles of alternatives belonging to $\underline{P}(Cl_t^{\geq})$:
   if $g_{i_1}(x) \geq r_{i_1}$ *and* . . . *and* $g_{i_h}(x) \geq r_{i_h}$, *then* $x \in Cl_t^{\geq}$,
   $t = 2, \ldots, p$, $\{i_1, ..., i_h\} \in P$, $r_{i_1}, \ldots, r_{i_h} \in \mathbb{R}$;
2) *possible* $D_{\geq}$-*decision rules*, providing lower profiles of alternatives belonging to $\overline{P}(Cl_t^{\geq})$:
   if $g_{i_1}(x) \geq r_{i_1}$ *and* . . . *and* $g_{i_h}(x) \geq r_{i_h}$, *then* x *possibly belongs to* $Cl_t^{\geq}$,
   $t = 2, \ldots, p$, $\{i_1, ..., i_h\} \in P$, $r_{i_1}, \ldots, r_{i_h} \in \mathbb{R}$;
3) *certain* $D_{\leq}$-*decision rules*, providing upper profiles of alternatives belonging to $\underline{P}(Cl_t^{\leq})$:
   if $g_{i_1}(x) \leq r_{i_1}$ *and* . . . *and* $g_{i_h}(x) \leq r_{i_h}$, *then* $x \in Cl_t^{\leq}$,
   $t = 1, \ldots, p - 1$, $\{i_1, ..., i_h\} \in P$, $r_{i_1}, \ldots, r_{i_h} \in \mathbb{R}$;
4) *possible* $D_{\leq}$-*decision rules*, providing upper profiles of alternatives belonging to $\overline{P}(Cl_t^{\leq})$:
   if $g_{i_1}(x) \leq r_{i_1}$ *and* . . . *and* $g_{i_h}(x) \leq r_{i_h}$, *then* x *possibly belongs to* $Cl_t^{\leq}$,
   $t = 1, \ldots, p - 1$, $\{i_1, ..., i_h\} \in P$, $r_{i_1}, \ldots, r_{i_h} \in \mathbb{R}$;
5) *approximate* $D_{\geq\leq}$-*decision rules*, providing simultaneously lower and upper profiles of alternatives belonging to $Cl_s \cup Cl_{s+1} \cup \ldots \cup Cl_t$, without the possibility of discerning to which class: *if* $g_{i_1}(x) \geq r_{i_1}$ *and* . . . *and* $g_{i_k}(x) \geq r_{i_k}$ *and* $g_{i_{k+1}}(x) \leq r_{i_{k+1}}$ *and* . . . *and* $g_{i_h}(x) \leq r_{i_h}$, *then* $x \in Cl_s \cup Cl_{s+1} \cup \ldots \cup Cl_t$, $\{i_1, \ldots, i_h\} \subseteq P$, $s, t \in \{1, \ldots, p\}$, $s < t$, $r_{i_1}, \ldots, r_{i_h} \in \mathbb{R}$.

The rules of type 1) and 3) represent certain preference patterns extracted from decision examples, while the rules of types 2) and 4) represent possible preference patterns. Rules of type 5) represent doubtful preference patterns. Depending on the classification examples supporting the induced rules, they are characterized by different values of adopted interestingness measures. In (Greco et al. 2016c), some rule interestingness measures with desirable properties are characterized in four perspectives of Bayesian and likelihoodist confirmation. The interestingness measures are used to classify new alternatives by decision rules in situations where these alternatives are matched by (i) no rule, (ii) exactly one rule, (iii) several rules (even contradictory). Such a classification scheme has been proposed by Błaszczyński et al. (2007).

The above definitions of rough approximations are based on a strict application of the dominance principle. In practice, however, it is beneficial to accept a limited proportion of inconsistent examples in lower approximations, particularly for large data sets. Two parametric versions of DRSA have been proposed, relaxing in different ways the definition of rough approximations: Variable Precision DRSA (Inuiguchi et al. 2009) and Variable Consistency DRSA (Błaszczyński et al. 2009). Statistical interpretation of these two parametric DRSA from the viewpoint of empirical risk minimization typical for machine learning has been given by Kusunoki et al. (2021). It is also worth mentioning the stochastic DRSA model presented in (Kotłowski and Słowiński 2008; Kotłowski et al. 2008).

Algorithms for induction of decision rules from rough approximations of upward and downward unions of decision classes perform various strategies. The most popular induction strategy, called minimal-cover, offers a minimal set of rules representing the decision examples in the most concise way (i.e., without any redundancy) (Błaszczyński et al. 2012). It is usually performed by greedy heuristics of sequential covering type, giving an approximately minimal-cover set of rules. Other strategies aim at discovering a set of decision rules satisfying some pre-defined user's requirements, e.g., with respect to minimal support, maximal length, or maximal number of rules. Yet other strategies propose to induce an exhaustive set of rules, i.e., all rules compatible with the provided preference information, which form the most comprehensive knowledge base with respect to the analyzed data set. The recent trend consists of integrating several base classifiers into ensembles or committees of classifiers (Kotłowski and Słowiński 2009). Several methods have been proposed to get diverse base classifiers to be integrated within an ensemble of classifiers. The best known are bagging (Błaszczyński et al. 2010) and boosting (Dembczyński et al. 2010a), which modify the set of objects by sampling or weighting particular examples and use the same learning algorithm to create base classifiers.

ROR has been applied to the rule preference model by Kadziński et al. (2015). It consists of taking into account all minimal-cover (MC) by sets of decision rules compatible with the provided decision examples when computing recommendations for non-reference alternatives. An MC set of rules is a single base instance of the preference model. In order to generate all MC sets of rules, first an exhaustive set of rules is induced and then, all compatible MC sets of rules are obtained by solving a series of integer LP problems which guarantee that each reference alternative is covered by at least one rule from each MC set. Such an ensemble of MC classifiers is applied to the set of all considered alternatives, producing two types of classification results. Alternative a is necessarily assigned to class Cl_h if and only if a is assigned to Cl_h for all MC sets of rules, and a is possibly assigned to Cl_h, if and only if a is assigned to Cl_h for at least one MC set of rules. Since all compatible MC sets of rules are considered, it is also possible to compute class acceptability indices defined as the share of MC sets of rules assigning an alternative to a single class or a set of contiguous classes. The robustness analysis is continued in a twofold way. First, one selects a representative MC set of rules that builds on the class acceptability indices and highlights the most stable part of the robust classification. Then, for each alternative, one can identify rules that are decisive in terms of recommendation in the greatest number of compatible MC sets. These rules directly influence the assignment

suggested by different instances, being of particular interest to the DM willing to understand the obtained recommendation.

Let us move to the explanation of **DRSA to multiple criteria choice and ranking** problems. While the classification of alternatives is based on their absolute evaluation on the considered criteria, choice and ranking refer to relative evaluation by means of pairwise comparisons of alternatives. Indeed, the position of an alternative in the ranking depends on the quality of other competing alternatives. This required an adaptation of DRSA to pairwise comparisons of alternatives instead of absolute multiple criteria evaluations considered in classification problems. Thus, in the case of choice and ranking problems, the rough approximation concerns a comprehensive preference relation in the set of reference alternatives, and the approximation is done using marginal preference relations on particular criteria for pairs of reference alternatives (Greco et al. 2001).

Technically, the modifications of DRSA necessary to approach the problems of choice and ranking are twofold:

1) A *pairwise comparison table* (PCT) is considered instead of the simple classification table: PCT is a classification table whose rows represent pairs of reference alternatives for which multiple criteria evaluations and a comprehensive preference relation are known.
2) A *dominance principle* is considered for comparisons of pairs of alternatives instead of simple alternatives: if alternative a is preferred to b at least as strongly as c is preferred to d on all the considered criteria, then a must be comprehensively preferred to b at least as strongly as c is comprehensively preferred to d.

The application of DRSA to the choice or ranking problems proceeds as follows. First, the DM gives examples of pairwise comparisons of reference alternatives from set $A^R \in \mathcal{A}$. This can be a partial or complete ranking from the best to the worst of these alternatives. From this set of decision examples, a PCT is constructed with an ordered pair of reference alternatives in each row, pairs of evaluations of these alternatives on particular criteria in the columns corresponding to criteria, and comprehensive preference relation in the last column corresponding to the decision. The pairs of evaluations on particular criteria for a pair of reference alternatives $(a^*, b^*) \in A^R \times A^R$ can be converted to graded differences of evaluations (intensities of preference) in case of criteria with interval or ratio scale. Remark that such a PCT with comprehensive preference relations in the decision is analogous to a classification table where instead of single alternatives there are pairs of alternatives and instead of decision classes there are comprehensive preference relations, e.g., outranking relation a^*Sb^* and non-outranking relation $a^*S^cb^*$ in the simplest case. Thus, all the DRSA methodology developed for the multiple criteria classification can be applied to PCT and yields lower and upper approximations of the comprehensive preference relations, e.g., S and S^c, composed of pairs of reference objects. These approximations prepare the ground for induction of certain, possible, and approximate "*if* …, *then*…" decision rules. VC-DRSA methodology applies perfectly to this scheme (Szeląg et al. 2014a). For example, in the case of comprehensive preference relations S and S^c, decision rules induced from VC-DRSA lower approximations have the following form:

- *if* $g_{i_1}(x) - g_{i_1}(y) \geq \delta_{i_1}$ *and* ... *and* $g_{i_h}(x) - g_{i_h}(y) \geq \delta_{i_h}$ *and* $(g_{i_{h+1}}(x) \geq r_{i_{h+1}}$ *and* $g_{i_{h+1}}(y) \leq s_{i_{h+1}})$ *and* ... *and* $(g_{i_z}(x) \geq r_{i_z}$ *and* $g_{i_z}(y) \leq s_{i_z})$, *then* xSy, where $\{i_1, \ldots, i_h\} \subseteq P$ indicate criteria with an interval or ratio scale, $\{i_{p+1}, \ldots, i_z\} \subseteq P$ indicate criteria with an ordinal scale, and $\delta_{i_1}, \ldots, \delta_{i_h} \in \mathbb{R}$, as well as $r_{i_{p+1}}, \ldots, r_{i_z}, s_{i_{h+1}}, \ldots, s_{i_z} \in \mathbb{R}$;
- *if* $g_{i_1}(x) - g_{i_1}(y) \leq \gamma_{i_1}$ *and* ... *and* $g_{i_k}(x) - g_{i_k}(y) \leq \gamma_{i_k}$ *and* $(g_{i_{k+1}}(x) \leq r_{i_{k+1}}$ *and* $g_{i_{k+1}}(y) \geq s_{i_{k+1}})$ *and* ... *and* $(g_{i_v}(x) \leq r_{i_v}$ *and* $g_{i_v}(y) \geq s_{i_v})$, *then* $xS^c y$, where $\{i_1, \ldots, i_k\} \subseteq P$ indicate criteria with an interval or ratio scale, $\{i_{k+1}, \ldots, i_v\} \subseteq P$ indicate criteria with an ordinal scale, and $\gamma_{i_1}, \ldots, \gamma_{i_k} \in \mathbb{R}$, as well as $r_{i_{k+1}}, \ldots, r_{i_v}, s_{i_{k+1}}, \ldots, s_{i_v} \in \mathbb{R}$.

These rules constitute a preference model of the DM. They are applied to the set of alternatives $\mathcal{A}$ with the intention of ranking them. Any pair of alternatives $(a, b) \in \mathcal{A} \times \mathcal{A}$ can match these rules in one of four ways:

- aSb and $\neg(aS^c b)$, which is *true outranking* $aS^T b$,
- $aS^c b$ and $\neg(aSb)$, which is *false outranking* $aS^F b$,
- aSb and $aS^c b$, which is *contradictory outranking* $aS^K b$,
- $\neg(aSb)$ and $\neg(aS^c b)$, which is *unknown outranking* $aS^U b$.

This 4-valued outranking relation underlines the presence or the absence of positive or negative arguments for outranking. For any $(a, b) \in \mathcal{A} \times \mathcal{A}$ it can be faithfully represented by 3-valued fuzzy relation R_{3v}:

$$R_{3v}(a, b) = \frac{[aSb] + (1 - [aS^c b])}{2},$$

where $[\,\cdot\,]$ denotes the indicator function taking value 0 or 1. In order to obtain a final recommendation, relation R_{3v} is exploited using a ranking method. In (Szeląg et al. 2014a), several ranking methods were compared with respect to a set of desirable properties. It appeared that the best ranking method is the Net Flow Rule which builds a ranking of alternatives using a scoring function $NFS(a)$ for all $a \in \mathcal{A}$:

$$\begin{aligned}
NFS(a) &= \sum_{b \in \mathcal{A} \setminus \{a\}} (R_{3v}(a, b) - R_{3v}(b, a)) \\
&= \sum_{b \in \mathcal{A} \setminus \{a\}} \left([aSb] - [bSa] - [aS^c b] + [bS^c a]\right),
\end{aligned}$$

where $[\,\cdot\,]$ denotes again a (0–1) indicator function. The recommendation for ranking is a complete preorder on set $\mathcal{A}$ determined by $NFS(\cdot)$, and the best choice recommendation is alternative $\hat{a} \in \mathcal{A}$, such that $NFS(\hat{a}) = \max_{a \in \mathcal{A}} \{NFS(a)\}$.

In (Dembczyński et al. 2010b), the DRSA for choice and ranking has been compared with statistical learning of a utility function for single objects, based on the boosting technique. Similar to the Net Flow Rule, the utility function also gives a linear ranking of objects. In conclusion of a computational experiment, the authors state that both analyzed approaches to preference learning share good properties of the decision rule preference model and have good performance in massive-data learning problems.

DRSA has been adapted to a large variety of multi-attribute decision problems (Słowiński et al. 2015). Let's list a few of them:

- decision under uncertainty and time preference (Greco et al. 2010b),
- robustness analysis for decision under uncertainty (Kadziński et al. 2016),
- classification with missing data (Szeląg et al. 2017),
- classification with a hierarchical structure of evaluation criteria (Dembczyński et al. 2002),
- classification with imprecise evaluations and assignments (Dembczyński et al. 2009),
- interactive evolutionary multiobjective optimization (Corrente et al. 2024).

The DRSA methodology gained popularity as a useful tool for structuring preference data prior to the induction of the preference model in terms of decision rules. It is based on elementary concepts and mathematical tools (sets and set operations, binary relations), without recourse to any complex algebraic or analytical structures (Greco et al 2010a); the decision rules have very natural interpretation and the key concept of dominance is rational and objective.

3 Conclusion

This paper is the first part of our systematic comparison of multiple criteria decision aiding (MCDA) and preference learning (PL), in which we provided a review of the MCDA methodology highlighting its recent trend in preference modeling from decision examples. This review will be continued with a similar overview of PL in the second part of the paper, which furthermore compares both methodologies in a systematic way, and gives an overview of existing work on combining PL and MCDA.

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