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ORIGINAL PAPER

Projected solutions for finite-dimensional quasiequilibrium problems

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Abstract
The concept of projected solution has been introduced in Aussel et al. (J Optim Theory Appl 170:818–837, 2016) for studying quasivariational problems where the constraint map may not be a self-map. Aim of this paper is to establish a new result on the existence of projected solutions for finite-dimensional quasiequilibrium problems without any monotonicity assumptions and without assuming the compactness of the feasible set. These two facts allow us to improve some recent results. Additionally, we deduce the existence of projected solutions for quasivariational inequalities, quasioptimization problems and generalized Nash equilibrium problems. Also, a comparison with similar results is provided.

Keywords Quasiequilibrium problem · Projected solution · Generalized quasivariational inequality · Generalized Nash equilibrium problem · Quasioptimization problem

1 Introduction

A lot of mathematical equilibrium models (as variational inequalities, optimization problems, Nash equilibrium problems, fixed point problems and others) which are apparently different can be viewed as special case of the so called equilibrium problem. This general problem has been studied by various authors in recent years; the interested reader can see Bigi et al. (2019) and the references therein.

More precisely, let $C \subseteq \mathbb{R}^n$ be a nonempty set and $f : C \times C \to \mathbb{R}$ be a function; the equilibrium problem is the following:

$$\text{find } x \in C \text{ s.t. } f(x, y) \geq 0, \quad \forall y \in C$$

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A generalization of the equilibrium problem is the quasiequilibrium problem, that is, an equilibrium problem in which the constraint set depends on the considered point. Let $K : C \rightrightarrows C$ be a set-valued map; the quasiequilibrium problem associated to f and K is the following:

$$\text{find } x \in K(x) \text{ s.t. } f(x, y) \geq 0, \quad \forall y \in K(x)$$

Clearly, an equilibrium problem is a quasiequilibrium problem where $K(x) = C$ for all $x \in C$. In the literature, in most of the existence results for quasiequilibrium problems the constraint map is taken as a self-map, that is, $K(C) \subseteq C$, but sometimes (for example in the model of electricity market described in Aussel et al. (2016)) the constraint map K may not be a self-map, that is, $K(C) \nsubseteq C$, and the existence of fixed point of K may not be verifed. For this reason, Aussel et al. (2016) developed the concept of projected solution for quasivariational inequalities and generalized Nash equilibria, and later Cotrina and Zúñiga (2019) adapted this concept to quasiequilibrium problems.

The aim of this paper is to prove results on the existence of projected solutions of quasiequilibrium problems without requiring any monotonicity assumptions and without assuming the compactness of the feasible set. Subsequently, the last part of the paper is devoted to apply the previous results for having existence of solutions for quasivariational inequality problems, Nash equilibrium problems, and quasioptimization problems. A comparison with analogous results is provided.

2 Preliminaries

We start by recalling some notions which we will use in the rest of this paper. We denote by $\|\cdot\|$ and by $\langle\cdot,\cdot\rangle$ the euclidean norm and the scalar product in $\mathbb{R}^n$, respectively. Given a nonempty subset $C \subseteq \mathbb{R}^n$, we denote by co$C$ the convex hull of C and by cl$C$ the closure of C.

The graph of the set-valued map $\Phi : C \rightrightarrows \mathbb{R}^m$ is

$$\text{gph}\Phi = \{(x, y) \in C \times \mathbb{R}^m : y \in \Phi(x)\}$$

and Φ is said to be closed if gphΦ is a closed subset of $C \times \mathbb{R}^m$. The map Φ is lower semicontinuous at $x \in C$ if for each open set Ω such that $\Phi(x) \cap \Omega \neq \emptyset$ there exists a neighborhood U_x of x such that $\Phi(x') \cap \Omega \neq \emptyset$ for every $x' \in U_x \cap C$; it is upper semicontinuous at x if for each open set Ω such that $\Phi(x) \subseteq \Omega$ there exists a neighborhood U_x of x such that $\Phi(x') \subseteq \Omega$ for every $x' \in U_x \cap C$; and it is continuous at x if it is both upper and lower semicontinuous at x. The closed graph theorem for set-valued maps affirms that a set-valued map with values in a compact set is closed if and only if it is upper semicontinuous with closed values.

If $\Phi : C \rightrightarrows C$, a fixed point of Φ is a point $x \in C$ satisfying $x \in \Phi(x)$, and the set of the fixed points of Φ is denoted by fixΦ.

Given $y \in \mathbb{R}^n$, the metric projection of y onto C is the set

$$P_C(y) = \{x \in C : \|y - x\| \leq \|y - z\|, \ \forall z \in C\}$$

The points of $P_C(y)$ are called best approximations of y in C. It is well-known that if C is closed and convex then for each y there exists a unique best approximation (Deutsch 2001, Theorem 3.5) that will be denoted by $x = p_C(y)$. Moreover, the metric projection map $p_C : \mathbb{R}^n \to C$ is a continuous function (Deutsch 2001, Theorem 5.5).

The Kolmogorov's characterization of the best approximation says that $x = p_C(y)$ if and only if

$$\langle y - x, z - x \rangle \le 0, \quad \forall z \in C$$

This characterization can be equivalently written $y \in x + N_C(x)$ where $N_C(x)$ is the normal cone of C at x defined as

$$N_C(x) = \{x^* \in \mathbb{R}^n : \langle x^*, z - x \rangle \le 0, \ \forall z \in C\}$$

Finally, we recall that a function $f : C \to \mathbb{R}$ is quasiconvex if all its sublevel sets are convex.

3 Existence of projected solutions

In this section we prove our main result which concerns the existence of projected solutions for a quasiequilibrium problem in which the constraint map is not necessarily self-map.

Definition 1 Let $C \subseteq \mathbb{R}^n$, $K : C \rightrightarrows \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given. A point $\bar{x} \in C$ is said to be a projected solution of the quasiequilibrium problem if and only if

$$\exists \bar{y} \in K(\bar{x}) \text{ with } \bar{x} \in P_C(\bar{y}) \text{ s.t. } f(\bar{y}, y) \ge 0, \quad \forall y \in K(\bar{x})$$

Notice that we need to define f on the whole space $\mathbb{R}^n \times \mathbb{R}^n$ for treating the projected quasiequilibrium problems since the range of K is no longer C. Clearly, if $\bar{y} \in C$ then $\bar{x} = \bar{y}$ is the classical solution of the quasiequilibrium problem. This happens when K is a self-map.

Theorem 1 *Let C be a closed, convex set, and assume that $K(C)$ is bounded. Then, the quasiequilibrium problem admits a projected solution if the following properties hold:*

(i) *K is continuous with nonempty, closed, and convex values;*
(ii) *$f(y, y) \ge 0$ for all $y \in K(C)$;*
(iii) *f is upper semicontinuous on $K(C) \times \mathbb{R}^n$;*
(iv) *$f(y, \cdot)$ is quasiconvex on $\mathbb{R}^n$ for all $y \in K(C)$.*

Proof First we recall that the convexity and the closedness of C imply the continuity of the single-valued map p_C. Define $\widehat{C} = \text{clco}K(C)$ which is compact and consider the set-valued maps $\widehat{K} : \widehat{C} \rightrightarrows \widehat{C}$ and $F : \text{fix}\widehat{K} \rightrightarrows \mathbb{R}^n$ defined as

$$\widehat{K}(y) = K(p_C(y)) \quad \text{and} \quad F(y) = \{z \in \mathbb{R}^n : f(y,z) < 0\}$$

respectively. The set-valued map $\widehat{K}$ is continuous (Aliprantis and Border 2006, Theorem 17.23) since composition of two continuous set-valued maps. Moreover, it has nonempty, closed and convex values from *(i)*. From the closed graph theorem, the map $\widehat{K}$ has closed graph, hence $\text{fix}\widehat{K}$ is closed. Moreover, the Kakutani fixed point theorem guarantees that $\text{fix}\widehat{K}$ is not empty. Clearly, $\text{fix}\widehat{K} \subseteq K(C)$ and hence F has convex values from *(iv)*. Since $(y,z) \in \text{gph}F$ is equivalent to affirm that $f(y,z) < 0$, the fact that F has open graph descends from *(iii)*.

By contradiction assume that $F(y) \cap \widehat{K}(y) \neq \emptyset$ for all $y \in \text{fix}\widehat{K}$. From a famous selection result (Michael 1956, Theorem 3.1‴) $\widehat{K}$ admits a continuous selection and hence, thanks to Proposition 1.10.4 and Proposition 1.10.2 in Aubin and Cellina (1984), we deduce that $F \cap \widehat{K}$ has a continuous selection $g : \text{fix}\widehat{K} \to \widehat{C}$. The set-valued map $\Phi : \widehat{C} \rightrightarrows \widehat{C}$ defined as

$$\Phi(y) = \begin{cases} \widehat{K}(y) & \text{if } y \notin \text{fix}\widehat{K} \\ \{g(y)\} & \text{if } y \in \text{fix}\widehat{K} \end{cases}$$

is lower semicontinuous (Castellani and Giuli 2020, Lemma 2.3) and the existence of a continuous selection $\varphi : \widehat{C} \to \widehat{C}$ of Φ is guaranted by Theorem 3.1‴ in Michael (1956) again. Then φ extends g out from $\text{fix}\widehat{K}$. The Brouwer fixed point theorem affirms that φ has a fixed point $y \in \widehat{C}$. Clearly, $y \in \text{fix}\widehat{K}$ and this implies that y is a fixed point of g, i.e., $y = g(y) \in F(y)$. Hence $f(y,y) < 0$ which contradicts *(ii)*.

Therefore, there exists $\bar{y} \in \text{fix}\widehat{K}$ such that $F(\bar{y}) \cap \widehat{K}(\bar{y}) = \emptyset$, i.e.

$$f(\bar{y}, z) \geq 0, \quad \forall z \in \widehat{K}(\bar{y}) = K(\bar{x})$$

which means that $\bar{x} = p_C(\bar{y})$ is a projected solution. □

Now we analyze in detail the assumptions of Theorem 1, in particular we focus mostly on how they are used inside the proof.

Remark 1 The fact that $f(y,y) \geq 0$ for all $y \in K(C)$ is used only to contradict that $F(y) \cap \widehat{K}(y) \neq \emptyset$ for all $y \in \text{fix}\widehat{K}$. For this reason, assumption *(ii)* may be relaxed by

(ii-a) $f(y,y) \geq 0$ for all $y \in \text{fix}\widehat{K}$.

Remark 2 The upper semicontinuity of K and the closedness of its values are used only to prove that fix$\widehat{K}$ is closed. For this reason, assumption *(i)* may be substituted by

(i-a) K is lower semicontinuous with nonempty, convex values;
(i-b) fix$\widehat{K}$ is closed.

We notice that fix$\widehat{K}$ is not empty, even without the hypothesis that K is closed (Castellani and Giuli 2015, Corollary 3.1).

The next example shows that assumptions *(i-a)* and *(i-b)* together are strictly weaker than *(i)*.

Example 1 Consider the closed, convex set $C = [0, +\infty) \times [0, +\infty)$ and the set-valued map $K : C \rightrightarrows \mathbb{R}^2$ as

$$K(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = -1,\ y_1 \leq 0,\ y_2 \leq 0\} & \text{if } x = (0,0) \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 > -1,\ y_1 < 0,\ y_2 < 0\} & \text{if } x \neq (0,0) \end{cases}$$

Clearly K has not closed values and it is not upper semicontinuous at (0, 0). Moreover we observe that $K(C) \cap C = \emptyset$. Anyway the set

$$\text{fix}\widehat{K} = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = -1,\ y_1 \leq 0,\ y_2 \leq 0\}$$

is closed. Notice that K is lower semicontinuous with convex values. Now consider the function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as $f(y, z) = z_1 - y_1$. This function satisfies all the three assumptions of Theorem 1. Therefore, the existence of a projected solution is guaranteed by Remark 2 and Theorem 1. It is an easy calculation to show that $\bar{x} = (0, 0)$ is the unique projected solution of the quasiequilibrium problem with $\bar{y} = (-1, 0)$.

Now we consider the set-valued map $T : C \rightrightarrows \mathbb{R}^n$ defined as $T(x) = K(x) \cap (x + N_C(x))$ to characterize the closedness of fix$\widehat{K}$ in a simple way. In general, there is not a relationship between the closedness of a set-valued map and the closedness of its range. For instance the map $\Phi : [1, +\infty) \rightrightarrows \mathbb{R}$ defined as $\Phi(x) = \left[\frac{1}{x+1}, \frac{1}{x}\right]$ has closed graph, but $\Phi([1, +\infty)) = (0, 1]$. Vice versa the map $\Phi : [1, +\infty) \rightrightarrows \mathbb{R}$ defined as $\Phi(x) = [x, x + 1)$ has not closed values, but $\Phi([1, +\infty)) = [1, +\infty)$. Instead the map T enjoys this property.

Proposition 1 *Let $C \subseteq \mathbb{R}^n$ be a closed, convex set, $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map and $T : C \rightrightarrows \mathbb{R}^n$ defined as $T(x) = K(x) \cap (x + N_C(x))$. Then $T(C)$ is closed if and only if T is closed.*

Proof Assume that T is closed and consider $\{y_k\} \subseteq T(C)$ such that $y_k \to y \in \mathbb{R}^n$. By assumption, for each $k \in \mathbb{N}$ there exists $x_k \in C$ such that $y_k \in T(x_k)$ and this is equivalent to affirm

$$\begin{cases} y_k \in K(x_k) \\ x_k = p_C(y_k) \end{cases}$$

Since p_C is continuous, then

$$x_k = p_C(y_k) \to p_C(y) = x$$

Therefore $(x_k, y_k) \in \mathrm{gph} T$ and $(x_k, y_k) \to (x, y)$, hence $y \in T(x) \subseteq T(C)$ because T is closed.

For the converse, we consider $(x_k, y_k) \in \mathrm{gph} T$ such that $(x_k, y_k) \to (x, y) \in C \times \mathbb{R}^n$. Since $y_k \in T(x_k)$ we have that

$$\begin{cases} y_k \in K(x_k) \\ x_k = p_C(y_k) \end{cases}$$

Since $(x_k, y_k) \to (x, y)$ and p_C is continuous, then $x = p_C(y)$. Furthermore, $y_k \in T(x_k) \subseteq T(C)$ and $T(C)$ is closed, then $y \in T(C)$, i.e., there exists $z \in C$ such that $y \in T(z)$. This implies that $z = p_C(y)$. The uniqueness of the best approximation guarantees that $x = z$ and $y \in T(x)$. □

The set $\mathrm{fix}\widehat{K}$ may be characterized by means of T, indeed $y \in T(x)$ is equivalent to affirm that

$$\begin{cases} y \in K(x) \\ y \in x + N_C(x) = p_C^{-1}(x) \end{cases}$$

that is, $y \in K(p_C(y))$ and $x = p_C(y)$. Thanks to this reformulation and Proposition 1, assumption *(i-b)* may be replaced by

(i-b′) T is closed,

or, equivalently, by

(i-b″) $T(C)$ is closed.

In other words, we have the following equivalence

$$\mathrm{fix}\widehat{K} \text{ closed} \quad \Leftrightarrow \quad T(C) \text{ closed} \quad \Leftrightarrow \quad T \text{ closed}$$

Moreover since the set-valued map $x + N_C(x)$ is closed, the closedness of K is sufficient to ensure the closedness of $\mathrm{fix}\widehat{K}$.

Remark 3 The quasiconvexity of $f(x, \cdot)$ is used only to prove that F has convex values. For this reason the assumption *(iv)* may be replaced by *(iv-a)* the set $F(y) = \{z \in \mathbb{R}^n : f(y,z) < 0\}$ is convex for all $y \in K(C)$.

Alternatively, the upper semicontinuity of f and the quasiconvexity of $f(x,\cdot)$ are used only to show that $F \cap \widehat{K}$ admits a continuous selection in $\mathrm{fix}\widehat{K}$. For this reason, just ask that $F \cap \widehat{K}$ verifies the assumption of Theorem 3.1‴ in Michael (1956) and then the assumptions *(iii)* and *(iv)* may be changed in

(iii-a) $F \cap \widehat{K}$ is lower semicontinuous with convex values on $\mathrm{fix}\widehat{K}$.

Collecting all these remarks, we can state the following result.

Theorem 2 *Let C be a closed, convex set, and assume that $K(C)$ is bounded. Then, the quasiequilibrium problem admits a projected solution if the following properties hold*:

(*i-a*) *K is lower semicontinuous with nonempty convex values*;
(*i-b*) $\mathrm{fix}\widehat{K}$ *is closed*;
(*ii-a*) $f(y,y) \geq 0$ *for all* $y \in \mathrm{fix}\widehat{K}$;
(*iii-a*) *$F \cap \widehat{K}$ is lower semicontinuous with convex values on* $\mathrm{fix}\widehat{K}$.

Recently, Cotrina and Zúñiga (2019) established an existence result for quasiequilibrium problems (Theorem 3) assuming suitable assumptions on two auxiliary set-valued maps defined on the product space $C \times \mathbb{R}^n$:

$$Q(x,y) = p_C(y) \times K(x) \quad \text{and} \quad R(x,y) = F(y) \cap K(x)$$

Unlike in Cotrina and Zúñiga (2019) where the compactness of the feasible region C is required, our approach allows us to achieve the existence of projected solutions assuming the closedness of C only. Now, we show that all the other assumptions of Theorem 2 coincide with the assumptions on Q and R in Theorem 3 in Cotrina and Zúñiga (2019).

- Assumption 1 in Theorem 3 affirms that Q is lower semicontinuous with nonempty convex values. Since p_C is continuous, this fact is equivalent to assume *(i-a)*.

- Assumption 2 in Theorem 3 requires the boundedness of $Q(C \times \mathbb{R}^n)$. It is easy to show that $Q(C \times \mathbb{R}^n) = C \times K(C)$ and therefore Assumption 2 is equivalent to the boundedness of C and the boundedness of $K(C)$. Anyway, if we assume that C is closed, Assumption 2 stresses the fact that C must be compact.

- Assumption 3 in Theorem 3 is the closedness of $\mathrm{fix}Q$ that may be equivalently rewritten

$$\text{fix}Q = \{(x,y) \in C \times \mathbb{R}^n \,:\, y \in \text{fix}\widehat{K} \text{ and } x = p_C(y)\}$$

Since p_C is continuous, the closedness of fix$Q$ is equivalent to the closedness of fix$\widehat{K}$, that is, assumption *(i-b)*.

- Assumption 4 in Theorem 3 affirms that R is lower semicontinuous with convex values on fix$Q$. Since (x, y) must belongs to fix$Q$, thanks to the previous characterization, we have

$$R(x,y) = F(y) \cap K(x) = F(y) \cap K(p_C(y)) = F(y) \cap \widehat{K}(y), \quad \forall y \in \text{fix}\widehat{K};$$

hence Assumption 4 is equivalent to assume *(iii-a)*.

- Finally, Assumption 5 in Theorem 3 requires that $f(z,z) \geq 0$ for every $z \in M$, where

$$M = \{w \in K(C) \,:\, \text{there exists } u \in C \text{ such that } (u,w) \in \text{fix}Q\}$$

Clearly, this fact is equivalent to assume *(iv-a)*.

In conclusion, all the assumptions of Theorem 3 in Cotrina and Zúñiga (2019) coincide with the assumptions of Theorem 2 except the compactness of C that is not required in our result. This fact allows us to consider a large class of quasiequilibrium problems as reported in the following example.

Example 2 Let $C = [0, +\infty)$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as $f(x,y) = -x + y$ and $K : C \rightrightarrows \mathbb{R}$ defined as $K(z) = [-\frac{1}{z+1}, \frac{1}{z+1}]$. Since C is not compact then Theorem 3 in Cotrina and Zúñiga (2019) cannot be applied. Instead, all the assumptions of Theorem 1 are fulfilled: the constraint map K is continuous, with nonempty, closed, and convex values, $K(C) = [-1, 1]$ is compact, f is continuous, quasiconvex, and $f(x,x) = 0$ for all $x \in \mathbb{R}$. Therefore a projected solution exists and it is easy to see that $\bar{x} = 0$ is the projected solution with $\bar{y} = -1$.

4 Three special cases

Let us now consider three different problems where we apply Theorem 1: generalized quasivariational inequality, quasioptimization and generalized Nash equilibrium.

4.1 Generalized quasivariational inequality problem

The generalized quasivariational inequality problem was introduced in Chan and Pang (1982). The concept of projected solution for a generalized quasivariational inequality was first investigated in Aussel et al. (2016). Let $C \subseteq \mathbb{R}^n$ be a nonempty set, $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows \mathbb{R}^n$ be two set-valued maps. A point $\bar{x} \in C$ is said to be a projected solution for the generalized quasivariational inequality if and only if there exists $\bar{y} \in K(\bar{x})$ with $\bar{x} \in P_C(\bar{y})$ such that

$$\exists \bar{y}^* \in \Phi(\bar{y}) \text{ with } \langle \bar{y}^*, y - \bar{y} \rangle \geq 0, \quad \forall y \in K(\bar{x})$$

The existence of projected solutions for the quasivariational inequality problem was proved in Aussel et al. (2016) assuming that the convex values of the set-valued map K have nonempty interior and the operator Φ is pseudomonotone. In the next result we establish the existence avoiding these two restrictive assumptions.

Theorem 3 *Let C be a closed, convex set, and assume that $K(C)$ is bounded. Then, the generalized quasivariational inequality admits a projected solution if the following properties hold*:

(i) *K is lower semicontinuous with nonempty, convex values*;
(ii) $\mathrm{fix}(K \circ p_C)$ *is closed*;
(iii) *Φ is upper semicontinuous on $K(C)$ with nonempty compact, convex values.*

Proof The result descends from Theorem 2 taking

$$f(y,z) = \max_{y^* \in \Phi(y)} \langle y^*, z - y \rangle$$

First we check that f verifies the assumptions of Theorem 1. Clearly $f(y,y) = 0$. Fixed $y \in K(C)$ the set

$$\begin{aligned} \{z \in \mathbb{R}^n : f(y,z) < a\} &= \left\{ z \in \mathbb{R}^n : \max_{y^* \in \Phi(y)} \langle y^*, z - y \rangle < a \right\} \\ &= \bigcap_{y^* \in \Phi(y)} \{z \in \mathbb{R}^n : \langle y^*, z - y \rangle < a\} \end{aligned}$$

is convex for all $a \in \mathbb{R}$ being an intersection of subspaces. The function f is upper semicontinuity on $K(C) \times \mathbb{R}^n$ from *(iii)* and Lemma 17.30 in Aliprantis and Border (2006). Then, thanks to Remark 1 and Remark 3, the assumptions of Theorem 2 are satisfied and there exist $\bar{x} \in C$ and $\bar{y} \in K(\bar{x})$ with $\bar{x} = p_C(\bar{y})$ such that

$$\max_{\bar{y}^* \in \Phi(\bar{y})} \langle \bar{y}^*, y - \bar{y} \rangle \geq 0, \quad \forall y \in K(\bar{x})$$

which is equivalent to affirm that

$$\inf_{y \in K(\bar{x})} \max_{y^* \in \Phi(\bar{y})} \langle y^*, y - \bar{y} \rangle \geq 0$$

Thanks to the Sion's minimax theorem we have that

$$\max_{\bar{y}^* \in \Phi(\bar{y})} \inf_{y \in K(\bar{x})} \langle \bar{y}^*, y - \bar{y} \rangle \geq 0$$

which means that $\bar{x}$ is a projected solution. □

Example 3 Let $C = [-2, 0] \times [0, 2]$, $K : C \rightrightarrows \mathbb{R}^2$ defined as

$$K(x_1, x_2) = \{(z_1, z_2) \in \mathbb{R}^2 : -4 - x_2 \le z_1 \le 0, -1 \le z_2 \le 1\}$$

and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $\Phi(x_1, x_2) = (x_1^2, 1 + x_2^2)$. Clearly, $\text{fix}K = [-2, 0] \times [0, 1]$ but the generalized quasivariational inequality has no classic solution. Instead, all the assumptions of Theorem 3 are satisfied and the existence of a projected solution is guaranteed. In particular, it is easy to see that there are two projected solutions: $\bar{x}^1 = (0, 0)$ with $\bar{y}^1 = (0, -1)$ and $\bar{x}^2 = (-2, 0)$ with $\bar{y}^2 = (-4, -1)$. Notice that Φ is not pseudomonotone, indeed $\langle \Phi(0, 0), (-2, 0) - (0, 0) \rangle = 0$ and $\langle \Phi(-2, 0), (-2, 0) - (0, 0) \rangle = -8$. Therefore Theorem 3.2 in Aussel et al. (2016) can not be applied.

In addition to the already mentioned Theorem 3.2 in Aussel et al. (2016), an existence result for this kind of problems has been achieved in Theorem 8 by Cotrina and Zúñiga (2019). Anyway, Theorem 8 requires (adapting opportunely the notations) the compactness of C and $K(C)$ instead of the closedness of C and the boundedness of $K(C)$.

4.2 Quasioptimization problem

In this subsection, we consider a special optimization problem, known as quasioptimization problem. The term quasioptimization emphasizes the fact that this problem is an optimization problem where the constraint set depends on the solution and, moreover, it also highlights the parallelism to quasivariational inequalities which express the necessary optimality conditions. Let $C \subseteq \mathbb{R}^n$ be a nonempty set, $h : C \to \mathbb{R}$ be an objective function and $K : C \rightrightarrows C$ be a set-valued map; the quasioptimization problem is the following:

$$\text{find } \bar{x} \in C \text{ s.t. } \bar{x} \in K(\bar{x}) \text{ and } h(\bar{x}) \le h(y), \quad \forall y \in K(\bar{x})$$

Now let us consider the related concept of projected solution. Assume that the domain of h is $\mathbb{R}^n$ and $K : C \rightrightarrows \mathbb{R}^n$. A point $\bar{x} \in C$ is said to be a projected solution for a quasioptimization problem if and only if there exists $\bar{y} \in \mathbb{R}^n$ with $\bar{x} \in P_C(\bar{y})$ such that

$$\bar{y} \in K(\bar{x}) \text{ and } h(\bar{y}) \le h(y), \quad \forall y \in K(\bar{x})$$

The concept of projected solution for quasioptimization problems was first developed in Aussel et al. (2016) who proved a result on the existence of projected solutions for the quasioptimization problem using the quasivariational inequality with the normal operator N_h^a associated to the adjusted level sets of h. Using a common approach (see for instance Cotrina and Zúñiga 2019), we establish the following existence result as direct consequence of Theorem 2.

Theorem 4 *Let C be a closed, convex set, and assume that $K(C)$ is bounded. Then, the quasioptimization problem admits a projected solution if the following properties hold*:

(i) *K is lower semicontinuous with nonempty, convex values*;
(ii) $\mathrm{fix}(K \circ p_C)$ *is closed*;
(iii) *h continuous and quasiconvex on $\mathbb{R}^n$*.

Proof It is sufficient to apply Theorem 2 to the auxiliary function $f(x, y) = h(y) - h(x)$. First, we check that f verifies the assumptions of Theorem 1. Clearly $f(x, x) = 0$ and the set

$$\{y \in \mathbb{R}^n : f(x, y) < a\} = \{y \in \mathbb{R}^n : h(y) < h(x) + a\}$$

is convex for all $x \in K(C)$ and $a \in \mathbb{R}$. Moreover f is continuous. Then, thanks to Remark 1 and Remark 3, the assumptions of Theorem 2 are satisfied and there exist $\bar{x} \in C$ and $\bar{y} \in K(\bar{x})$ with $\bar{x} = p_C(\bar{y})$ such that

$$h(y) - h(\bar{y}) = f(\bar{y}, y) \geq 0, \quad \forall y \in K(\bar{x})$$

which means that $\bar{x}$ is a projected solution. □

Our result improves Theorem 4.1 in Aussel et al. (2016) where, by the way, K is closed and lower semicontinuous with convex values with nonempty interior and the normal operator N_h^a must satisfies a suitable continuity assumption.

Clearly, as observed in Remark 3, the assumption *(iii)* on continuity and quasiconvexity of h may be replaced by the assumption *(iii-a)*. This is equivalent to affirm that the set-valued map $H : \mathrm{fix}\widehat{K} \rightrightarrows \mathbb{R}^n$ defined as

$$H(y) = \{z \in \widehat{K}(y) : h(y) > h(z)\}$$

is lower semicontinuous and convex values. Thanks to this fact, our result can be compared with the analogous Theorem 6 in Cotrina and Zúñiga (2019). The only difference is that the proof of Theorem 6 in Cotrina and Zúñiga (2019) requires the compactness of C. Indeed Theorem 6 descends from Theorem 4 in Cotrina and Zúñiga (2019) where the compactness of C is required.

4.3 Generalized Nash equilibrium problem

Now we consider a generalized Nash equilibrium problem that is a noncooperative game in which the strategy set of each player depends on the strategies of all other players. Let $M = \{1, \ldots, m\}$ be a finite set of players. Each player i has a set of possible strategies $C_i \subseteq \mathbb{R}^{n_i}$. We denote by $x = (x_1, \ldots, x_m) \in \prod_{i \in M} C_i = C$ the vector formed by all decision variables and by $x_{-i} \in C_{-i} = \prod_{j \neq i} C_j$ we denote the strategy vector of all the players different from player i. Each player i has an objective loss function

$\theta_i : C \to \mathbb{R}$ that depends on all players' strategies. Furthermore, each player's strategy must belong to a set identified by the set-valued map $K_i : C_{-i} \rightrightarrows C_i$.

The generalized Nash equilibrium problem consists in finding $x \in C$ such that, for each $i \in M$, one has

$$x_i \in K_i(x_{-i}) \text{ and } \theta_i(x_i, x_{-i}) \leq \theta_i(y_i, x_{-i}), \quad \forall y_i \in K_i(x_{-i})$$

The concept of projected Nash equilibrium has been introduced in Aussel et al. (2016) where the existence of such equilibrium was investigated. Assume that the range of each constraint map is the whole space, that is, $K_i : C_{-i} \rightrightarrows \mathbb{R}^{n_i}$ for each $i \in M$, and assume that the domain of each loss function θ_i is the the space $\mathbb{R}^N = \prod_{i \in M} \mathbb{R}^{n_i}$ with $N = \sum_{i \in M} n_i$. A point $\bar{x} \in C$ is said to be a projected solution for the generalized Nash equilibrium problem if and only if there exists $\bar{y} \in \mathbb{R}^N$ with $\bar{x} \in P_C(\bar{y})$ such that, for all $i \in M$, one has

$$\bar{y}_i \in K_i(\bar{x}_{-i}) \text{ and } \theta_i(\bar{y}_i, \bar{y}_{-i}) \leq \theta_i(y_i, \bar{y}_{-i}), \quad \forall y_i \in K_i(\bar{x}_{-i})$$

In order to apply Theorem 2, we need to introduce a "cumulative" constraint set-valued map. First, we denote by $\overline{K}_i$ the set-valued map $\overline{K}_i : C \rightrightarrows C_i$ defined as $\overline{K}_i(x) = K_i(x_{-i})$, hence the set-valued map $\overline{K} : C \rightrightarrows \mathbb{R}^N$ is defined as

$$\overline{K}(x) = \overline{K}_1(x) \times \cdots \times \overline{K}_m(x)$$

The lower semicontinuity of each $\overline{K}_i$ is inherited by $\overline{K}$.

Lemma 5 *If each $\overline{K}_i$ is lower semicontinuous, then $\overline{K}$ is lower semicontinuous.*

Proof Consider the continuous function $\Delta : C \to C^m$ as $\Delta(x) = (x, \ldots, x) \in C^m$ and the product map $S : C^m \rightrightarrows \mathbb{R}^N$ defined as $S(x_1, \ldots, x_m) = \prod_{i \in M} \overline{K}_i(x_i)$. The map $\overline{K}$ is the composition of Δ and S.

Thanks to Theorem 17.28 in Aliprantis and Border (2006), the map S is lower semicontinuous, and moreover composition of lower semicontinuous is lower semicontinuous for Theorem 17.23 in Aliprantis and Border (2006); then $\overline{K}$ is lower semicontinuous. □

When M is finite (as in this case), finding a projected solution for the generalized Nash equilibrium problem amounts to find a projected solution for the quasiequilibrium problem associated to the Nikaido-Isoda function (Nikaido and Isoda 1955)

$$f(x, y) = \sum_{i=1}^{m} [\theta_i(y_i, x_{-i}) - \theta_i(x_i, x_{-i})]$$

and with $\overline{K}(x) = \prod_{i \in M} \overline{K}_i(x)$ defined as above. Indeed, if $\bar{x}$ is a projected solution for the generalized Nash equilibrium problem, there exists $\bar{y} \in \mathbb{R}^N$ with $\bar{x} \in P_C(\bar{y})$ such that $\bar{y}_i \in K_i(\bar{x}_{-i})$ for all $i \in M$. Therefore $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in \prod_{i \in M} \overline{K}_i(\bar{x}) = \overline{K}(\bar{x})$. Furthermore, all the terms of the Nikaido-Isoda function are nonnegative for any

$y_i \in K_i(\bar{x}_{-i})$ and then for any $y \in \overline{K}(\bar{x})$, hence $\bar{x}$ is a projected solution for the quasiequilibrium problem.

Conversely, if $\bar{x}$ is a a projected solution for the quasiequilibrium problem, there exists $\bar{y} \in \mathbb{R}^N$ with $\bar{x} \in P_C(\bar{y})$ such that $\bar{y} \in \overline{K}(\bar{x})$. By contradiction, assume that exists an index $i \in M$ and a strategy $y_i \in \overline{K}_i(\bar{x})$ such that $\theta_i(y_i, \bar{x}_{-i}) > \theta_i(\bar{x}_i, \bar{x}_{-i})$. Since $f(\bar{x}, y) \geq 0$ for all $y \in \overline{K}(\bar{x})$, choosing $y_j = \bar{y}_j$ for all $j \neq i$ leads to the contradiction

$$f(\bar{x}, y) = \theta_i(y_i, \bar{y}_{-i}) - \theta_i(\bar{y}_i, \bar{y}_{-i}) < 0$$

and hence $\bar{x}$ is a projected solution for the generalized Nash equilibrium problem.

Now, we are in position to establish our existence result for projected Nash equilibria.

Theorem 6 *For each $i \in M$ let C_i be a closed, convex set, and assume that $K_i(C_{-i})$ is bounded. Then, the generalized Nash equilibrium problem admits a projected solution if the following properties hold:*

(i) *K_i are lower semicontinuous with nonempty, convex values for all $i \in M$;*

(ii) $\mathrm{fix}(\overline{K} \circ p_C)$ *is closed;*

(iii) *θ_i are upper semicontinuous and $\theta_i(\cdot, x_{-i})$ are convex on $\mathbb{R}^N$ for all $i \in M$.*

Proof We have already shown that finding a projected solution for the generalized Nash equilibrium problem is equivalent to find a projected solution for the quasiequilibrium problem associated to the Nikaido-Isoda function. So, it is sufficient to check that all the conditions of Theorem 2 are fulfilled. The set $C = \prod_{i \in M} C_i$ is closed and convex, the set $\overline{K}(C) = \prod_{i \in M} \overline{K}_i(C)$ is bounded, the set-valued map $\overline{K}$ is lower semicontinuous for Lemma 5, and it has nonempty convex values from *(i)*. Furthermore, f is upper semicontinuous on $\overline{K}(C) \times \mathbb{R}^N$ from *(iii)*. Moreover,

$$f(x, x) = \sum_{i=1}^{n} [\theta_i(x_i, x_{-i}) - \theta_i(x_i, x_{-i})] = 0$$

The quasiconvexity of $f(x, \cdot)$ descends to the fact that the Nikaido-Isoda function is convex since sum of convex functions. Then, thanks to Remarks 1 and 3, all the assumptions of Theorem 2 are satisfied and there exists a projected solution. □

An analogous result was proved in Cotrina and Zúñiga (2019). However, the proof of Theorem 9 in Cotrina and Zúñiga (2019) is based on compactness of the domain C as required in Theorem 3 in Cotrina and Zúñiga (2019).

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Declarations

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