Correction to “A Classification of Spectrum-determined Circulant Digraphs”

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We find an error in the proof of Lemma 4 in [3]. This makes that Theorem 2 and Corollary 3 in [3] are also wrong, and the next counterexample shows this.

Suppose that $p = 3$, $q = 5$, $w = 7$, then $|Z_n^*| = (p-1)p^{a-1}(q-1)q^{b-1}(w-1)w^{c-1} = 48$, $\frac{1}{2}|Z_n| = 52.5$, but $|Z_n^*| < \frac{1}{2}|Z_n|$. This contradicts Lemma 4, which leads that Theorem 2 and Corollary 3 are wrong.

In this note we add the condition $\varphi(n) > \frac{n}{2}$ (That is $|Z_n^*| > \frac{1}{2}|Z_n|$) and give the correct result.

Define $C_n(S)$ and $C_n(T)$ to be two circulant digraphs. In [1], we know that $\text{Spec}(C_n(S)) = \{\alpha_r = \sum_{i \in S} u^{ri} | r = 0, 1, 2, \dots, n-1\}$ and $\text{Spec}(C_n(T)) = \{\beta_r = \sum_{j \in T} u^{rj} | r = 0, 1, 2, \dots, n-1\}$ are the spectra of $C_n(S)$ and $C_n(T)$, respectively. Suppose that $C_n(S) \cong C_n(T)$, then $\text{Spec}(C_n(S)) = \text{Spec}(C_n(T))$. Hence, for any $\alpha_r \in \text{Spec}(C_n(S))$, there exists some $\beta_{\pi(r)} \in \text{Spec}(C_n(T))$ such that

$$\alpha_r = \beta_{\pi(r)}, \quad \forall r \in Z_n. \quad (0.1)$$

Since a set may be labeled as ordered elements, in regular time, we will regard π as a spectrum projection from $\text{Spec}(C_n(S))$ to $\text{Spec}(C_n(T))$. A property of π will be illustrated as follows:

Lemma 1^[2]. Let $\text{Spec}(C_n(S)) = \text{Spec}(C_n(T))$, $\text{Spec}(C_n(S)) = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ and $\text{Spec}(C_n(T)) = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ and π is defined in (0.1). Then for each $\lambda \in Z_n^*$, we have $\alpha_{\lambda r} = \beta_{\lambda \pi(r)}$, $\forall r \in Z_n$.

Lemma 2. Let $n = p^a q^b w^c$, where p, q and w are three distinct odd primes. If $\varphi(n) > \frac{n}{2}$ and $\text{Spec}(C_n(S)) = \text{Spec}(C_n(T))$, then there is $\lambda \in Z_n^*$ such that $\alpha_1 = \beta_\lambda$.

Proof. Let π be the spectrum projection defined by (0.1). Since $\varphi(n) > \frac{n}{2}$, we obtain

$$Z_n^* \cap \pi(Z_n^*) \neq \emptyset, \quad \text{where } \pi(Z_n^*) = \{\pi(r) | r \in Z_n^*\}.$$

Hence, there exists $r \in Z_n^*$ such that $r' = \pi(r) \in Z_n^*$. Thus by (1), $\alpha_r = \beta_{r'}$.

By Lemma 1, we obtain that for each $\mu \in Z_n^*$, $\alpha_{\mu r} = \beta_{\mu r'}$. Let $\mu = r^{-1}$ and $\lambda = r^{-1}r'$, we have $\alpha_1 = \beta_\lambda$. $\square$

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For a polynomial $g(x) = a_0 + a_1x^1 + \cdots + a_mx^m$, let $\partial(g(x))$ denote its degree, i.e., $\partial(g(x)) = m$. And we denote $\partial^+(g(x))$ and $\partial^-(g(x))$ the maximum power indices of positive and negative coefficient terms of $g(x)$, respectively, then $\partial(g(x)) = \max\{\partial^+g(x), \partial^-g(x)\}$. For example, let $g(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$, then $\partial^+(g(x)) = 6$ and $\partial^-(g(x)) = 5$.

Lemma 3^[3]. Let $n = p^aq^bw^c$, where $p < q < w$ are three distinct primes. Suppose that $f(x)$ is an integer coefficient polynomial with $\partial(f(x)) < n$, then $f(x) \equiv 0$ if the following conditions are satisfied

- (1) $f(u) = f(1) = 0$, $f(0) \leq 0$,
 - (2) $\partial^+(f(x)) \leq \varphi(n) = (p-1)p^{a-1}(q-1)q^{b-1}(w-1)w^{c-1}$,
- where $\varphi(\cdot)$ is the Euler function.

Theorem 1. Let $n = p^aq^bw^c$, where p, q and w are three distinct odd primes. For $S \subset Z_n$, if $\varphi(n) > \frac{n}{2}$ and $\max\{s \in S\} \leq \varphi(n)$, then $C_n(S)$ is spectrum-determined.

Proof. Suppose that $\text{Spec}(C_n(T)) = \text{Spec}(C_n(S))$. By Lemma 2, there is $\lambda \in Z_n^*$ such that $\sum_{i \in S} \omega^i = \alpha_1 = \beta_\lambda = \sum_{j \in T} \omega^{\lambda j}$. Now we take $f(x) = S(x) - T(x^\lambda)$ as the spectrum-polynomial of $C_n(S)$ and $C_n(T)$. Then, we can get a polynomial $\overline{f(x)}$ from $f(x)$ by taking the power index of each term of $f(x)$ modulo n , i.e., $\overline{f(x)} = \sum_{i \in S} x^i - \sum_{j \in T} x^{\lambda j}$. Clearly, $\overline{f(u)} = f(u) = \alpha_1 - \beta_\lambda = 0$,

and $\overline{f(1)} = f(1) = |S| - |T| = 0 = f(0)$. Further, since $\max\{s \in S\} \leq \varphi(n)$, $\partial^+(\overline{f(x)}) \leq \varphi(n)$, by Lemma 3, $\overline{f(x)} \equiv 0$, we obtain $S = \lambda T \pmod{n}$, which implies that $C_n(S) \cong C_n(T)$. $\square$

Corollary 1. Let $n = p^aq^bw^c$, where p, q and w are three distinct odd primes. For $S \subset Z_n$, if $\varphi(n) > \frac{n}{2}$ and $D(S) < \varphi(n)$, then $C_n(S)$ is spectrum-determined.

Proof. Let $\text{Spec}(C_n(S)) = \text{Spec}(C_n(T))$, by Lemma 2, there exists $\lambda \in Z_n^*$ such that $\alpha_1 = \beta_\lambda$, then

$$\sum_{i \in S} u^i = \sum_{j \in T} u^{\lambda j} \quad (0.2)$$

Let $s_0 = \min\{s - 1 | s \in S\}$, and define $S' = S - s_0 = \{s - s_0 | s \in S\}$, $T' = \lambda T - s_0 = \{\lambda t - s_0 | t \in T\}$. Then by (0.2), we obtain that

$$\sum_{i \in S} u^{i-s_0} = \sum_{j \in T} u^{\lambda j-s_0} \implies \sum_{i' \in S'} u^{i'} = \sum_{j' \in T'} u^{j'}.$$

Define $f(x) = \sum_{i' \in S'} x^{i'} - \sum_{j' \in T'} x^{j'}$. Clearly, $f(u) = f(1) = 0$ and $\partial^+(f(x)) \leq \max\{i' \in S'\} = \max\{i - s_0 | i \in S\} = D(S) + 1 \leq \varphi(n)$. Further, $0 \notin S'$ and hence $f(0) \leq 0$. By Lemma 3, $f(x) \equiv 0$, $S' = T' \implies S = \lambda T$, and $C_n(S) \cong C_n(T)$. $\square$

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