THE EQUIVALENCE OF HEEGAARD FLOER HOMOLOGY AND EMBEDDED CONTACT HOMOLOGY VIA OPEN BOOK DECOMPOSITIONS II

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ABSTRACT

This paper is the sequel to (Colin et al. in Publ. Math. Inst. Hautes Études Sci., 2024), and is devoted to proving some of the technical parts of the HF=ECH isomorphism.

Notation specific to [II]

$\sigma^{\tau}_{\infty}, \sigma^{-\infty,i}_{\infty}$ Sections at infinity of $\overline{W}_\tau$ and $\overline{W}_{-\infty,i}$
$\pi_{B_\tau}: \overline{W}_\tau \to B_\tau$ Symplectic fibration with fiber S used in the definition of the chain homotopy
$\overline{\mathbf{b}} = \{\overline{b}_1, \dots, \overline{b}_{2g}\}$ Pushoff of $\overline{\mathbf{a}} = \{\overline{a}_1, \dots, \overline{a}_{2g}\}$
B_τ Base of the projection $\pi_{B_\tau}: \overline{W}_\tau \to B_\tau$
$B_{-\infty,i}$, $i = 1, 2$ Base of $\overline{W}_{-\infty,i}$
$\overline{\mathcal{I}}$ Space of all admissible $\{\overline{J}_\tau \in \mathcal{J}_{\overline{W}_\tau}\}_{\tau\in\mathbf{R}}$
$\mathcal{J}_{\overline{W}_\tau}$ Space of all admissible $\overline{J}_\tau$
$\mathcal{J}_{\overline{W}_{-\infty}}$ Space of admissible $\overline{J}_{-\infty}$
$\overline{J}_\tau$ Almost complex structure on $\overline{W}_\tau$
$\overline{J}_{-\infty} = \overline{J}_{-\infty,1} \cup \overline{J}_{-\infty,2}$ Almost complex structure on $\overline{W}_{-\infty} = \overline{W}_{-\infty,1} \cup \overline{W}_{-\infty,2}$
$\mathcal{L}_{t_0}$ The locus $\{t = t_0\}$, viewed as a subset of B_+, B_-, B_τ, as appropriate
$L^{-\infty,1}_{\overline{\mathbf{a}},1}, L^{-\infty,1}_{\overline{\mathbf{b}},2}, L^{-\infty,1}_{\overline{h}(\overline{\mathbf{b}}),3}, L^{-\infty,1}_{\overline{h}(\overline{\mathbf{a}}),4}$ Lagrangian submanifolds on $\overline{W}_{-\infty,1}$
$L^{-\infty,2}_{\overline{\mathbf{a}},+}, L^{-\infty,2}_{\overline{\mathbf{b}},-}$ Lagrangian submanifolds on $\overline{W}_{-\infty,2}$
$L^{\tau,+}_{\widehat{\mathbf{a}}}, L^{\tau,-}_{\widehat{\mathbf{b}}}$ Lagrangian submanifolds on $\overline{W}_\tau$
$\overline{\mathfrak{m}}(+\infty), \overline{\mathfrak{m}}(-\infty)$ Marked points on $\overline{W}_-$ and $\overline{W}_{-\infty,2}$, respectively
$\overline{\mathfrak{m}}(\tau) = (\overline{\mathfrak{m}}^b(\tau), \overline{\mathfrak{m}}^f(\tau))$ 1-parameter family of marked points on $\overline{W}_\tau$
$n^*(\overline{u}), n^{*,alt}(\overline{u})$ Intersection numbers defined in Equation (3.2.3)
$\overline{W}_\tau$ Total space of the projection $\pi_{B_\tau}: \overline{W}_\tau \to B_\tau$
$\overline{W}_{-\infty} = \overline{W}_{-\infty,1} \cup \overline{W}_{-\infty,2}$ Limit of $\overline{W}_\tau$ as $\tau \to -\infty$
$x^\#_{i1}, x^\#_{i2}, x^\#_{i3}$ Intersection points of $\overline{a}_i$ and $\overline{b}_i$ besides z_∞

* VC supported by the Institut Universitaire de France, ANR Symplexe, ANR Floer Power, and ERC Geodycon.
** PG supported by ANR Floer Power and ANR TCGD.
*** KH supported by NSF Grants DMS-0805352, DMS-1105432, and DMS-1406564.

https://doi.org/10.1007/s10240-024-00146-w

Published online: 02 April 2024

1. Introduction

This paper is the sequel to [I] and is devoted to proving some of the technical parts of the isomorphism between $\widehat{HF}(-M)$ and $\widehat{ECH}(M)$. References from [I] will be written as "Section I.x" to mean "Section x" of [I], for example. The notation from [I] carries over to this paper and is summarized in the "Notation common to [I]–[III]" that appears at the beginning of [I].

In [I] we defined the chain maps

$$\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N),$$

$$\Psi : PFC_{2g}(N) \to \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})).$$

In this paper we prove the following two results:

Theorem **1.0.1** *(Quasi-isomorphism). — The chain maps Φ and Ψ are quasi-isomorphisms, provided the monodromy map h does not have any elliptic periodic points of period $\leq 2g$ and and rotation number $\theta \notin [-\frac{1}{2g}, \frac{1}{2g}]$.*

Theorem 1.0.1 is quite involved and takes up most of this paper. The condition on h is a technical assumption which simplifies gluing and one should be able to remove this condition with more work.

Theorem **1.0.2** *(Stabilization). —* $\widehat{ECH}(M) \simeq PFH_{2g}(N)$.

By successively stabilizing the open book (S, h) and applying Theorem 1.0.1 to the stabilizations, we obtain a sequence of stable Hamiltonian structures (α_0, ω^{2k}) with contact perturbations α^{2k}, for $k \geq g$, such that

$$ECH_{2k}(N, \alpha^{2k}) \cong PFH_{2k}(N, \alpha_0, \omega^{2k})$$

and the stabilization maps

$$ECC_{2k}(N, \alpha^{2k}) \to ECC_{2k+2}(N, \alpha^{2k+2}), \quad \boldsymbol{\gamma} \mapsto e^2 \mathfrak{K}_{2k}(\gamma)$$

are quasi-isomorphisms, where

$$\mathfrak{K}_{2k} : ECC_{2k}(N, \alpha^{2k}) \to ECC_{2k}(N, \alpha^{2k+2})$$

is a chain map inducing the continuation map. Then Theorem 1.0.2 follows from Theorem I.2.5.6.

Theorem 1.0.2 is proved in Section 5.

Notation **1.0.3** *(Sub/superscripts $*$). —* In this paper, as in [I], $*$ is often used to denote a variety of possible subscripts/superscripts (e.g., intersection numbers $n^*(\overline{u})$ in

Equation (3.2.3), curves $\overline{v}_*$ and their domains $\dot{F}_*$ as in Notation 3.4.1, and moduli spaces $\mathcal{M}^*(\star)$ as in Notation 3.2.12).

Organization of this paper. In Sections 3 and 4 we prove the chain homotopy between the chain maps $\Psi \circ \Phi$ and a quasi-isomorphism, as well as the chain homotopy between the chain maps $\Phi \circ \Psi$ and id. The necessary Gromov-Witten type calculations are carried out in Section 2. Finally, Section 5 is devoted to proving Theorem 1.0.2.

2. Gromov-Witten type computations

This section is devoted to Gromov-Witten type calculations which are used in the proof of Theorem 1.0.1. After a brief review of relative Gromov-Witten invariants in Section 2.1, we treat a slightly simpler model situation in Section 2.2. We then tackle the specific situations of interest in this paper in Sections 2.3 and 2.4.

2.1. *Relative Gromov-Witten invariants.* — Relative Gromov-Witten invariants were introduced independently by Ionel and Parker in [IP1] and Li and Ruan in [LR]. The reader can see also McDuff [M] for a review of the topic. Although we were heavily inspired by their work, our definition is different from the original one.

Let X be a four-dimensional symplectic manifold and $V \subset X$ a codimension two symplectic submanifold with trivial self-intersection.[1] Throughout this section we will fix an almost complex structure which is integrable (and therefore locally split) on a neighborhood of V and generic elsewhere. We fix a primitive homology class $A \in H_2(X; \mathbf{Z})$ such that $A \cdot V = d > 0$ and denote by $\mathcal{M}^J_{g,n}(A)$ the moduli space of J-holomorphic maps $u : (F, j, \mathbf{x}) \to (X, J)$, where (F, j) is a genus g closed Riemann surface and $\mathbf{x} = (x_1, \dots, x_n)$ is an ordered n-tuple of marked points. In practice, we will also allow F to have boundary or constrain the evaluation at the marked points on some fixed cycles in X which are not contained in V; straightforward variations on the theme are left to the reader. Whenever there is no risk of confusion, the almost complex structure will be dropped from the notation.

Since A is a primitive class, every J-holomorphic map in the class A is simple, and moreover its image is not contained in V because $A \cdot V > 0$. Therefore the standard Fredholm theory for J-holomorphic curves applies and the moduli space $\mathcal{M}_{g,n}(A)$ is a smooth manifold of dimension

$$\dim \mathcal{M}_{g,n}(A) = 2g - 2 + 2\langle c_1(TX), A\rangle + 2n. \tag{2.1.1}$$

Definition **2.1.1.** — *We define a map* $ev_V : \mathcal{M}_{g,n}(A) \to \mathrm{Sym}^d(V)$ *such that* $ev_V(u)$ *is the set of intersection points between* $u(F)$ *and* V *counted with multiplicity: that is,* $ev_V(u) =$

[1] This simplifying condition will be satisfied in all our examples but is not strictly necessary.

$\{(z_1, d_1), \ldots, (z_l, d_l)\}$ *such that* $\{z_1, \ldots, z_l\} = u(\mathrm{F}) \cap \mathrm{V}$ *and* d_i *is the multiplicity of the intersection point* z_i.

This definition makes sense because of the local properties of J-holomorphic maps. Note that ev_{V} is not sensitive to the cardinality of the preimage, e.g., it does not distinguish between a unique point of F mapped to V with order two and two points of F mapped to V with order one. This is undesirable, but in the examples the possibility of multiple points of F mapped to the same point of V will be excluded by topological considerations.

In order to show that ev_{V} is a smooth map, we need the following weaker version of the Weierstrass preparation theorem. Let D^n_ε denote the ball of radius ε in $\mathbf{R}^n$ centered at 0.

Lemma **2.1.2.** — *If for some* $\varepsilon, \delta > 0$ *a smooth map* $\mathrm{F} : \mathrm{D}^n_\varepsilon \times \mathrm{D}^2_\delta \to \mathbf{C}$ *satisfies the following properties:*

– *for every* $t \in \mathrm{D}^n_\varepsilon$ *the map* $f_t(z) = \mathrm{F}(t, z)$ *is holomorphic, and*
– $f_0(0) = 0$ *is a zero of order k,*

then there exist $0 < \varepsilon' < \varepsilon$ *and* $0 < \delta' < \delta$ *such that on* $\mathrm{D}^n_{\varepsilon'} \times \mathrm{D}^2_{\delta'}$ *we can decompose* $\mathrm{F}(t, z) = \mathrm{P}(t, z)\mathrm{Q}(t, z)$*, where* P *and* Q *are smooth maps which satisfy*

– $p_t(z) = \mathrm{P}(t, z)$ *is a monic polynomial of degree k in z for every* $t \in \mathrm{D}^n_{\varepsilon'}$*, and*
– $\mathrm{Q}(t, z) \neq 0$ *for every* $(t, z) \in \mathrm{D}^n_{\varepsilon'} \times \mathrm{D}^2_{\delta'}$.

In particular f_t and p_t have the same zeros with the same multiplicities for every t and the coefficients of p_t depend smoothly on t. For the proof of this lemma see the proof of [Hör, Theorem 7.51].

Lemma **2.1.3.** — *If every* J*-holomorphic map in* $\mathcal{M}_{g,n}(\mathrm{A})$ *is graphical over* V *near its intersection points with* V*, then the map* $ev_{\mathrm{V}} : \mathcal{M}_{g,n}(\mathrm{A}) \to \mathrm{Sym}^d(\mathrm{V})$ *is smooth.*

The hypothesis that every J-holomorphic be graphical is strong and probably unnecessary, but it will hold for topological reasons in all our applications.

Proof. — Take $u_0 \in \mathcal{M}_{g,n}(\mathrm{A})$ with $ev_{\mathrm{V}}(u_0) = \{(z_1, d_1), \ldots, (z_l, d_l)\}$ and fix a smooth parametrization $\mathrm{D}^m_\varepsilon \to \mathcal{M}_{g,n}(\mathrm{A})$, $t \mapsto u_t$, of a neighborhood of u_0. We will show that the composition

$$\mathrm{D}^m_\varepsilon \to \mathrm{Sym}^d(\mathrm{V}), \qquad t \mapsto ev_{\mathrm{V}}(u_t)$$

is smooth. Choose pairwise disjoint neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_l \subset \mathrm{V}$ around the points $z_1, \ldots, z_l$ and identify each of them holomorphically with a disk $\mathrm{D}^2_\delta \subset \mathbf{C}$ in such a way that z_i is identified with the origin. Since the almost complex structure J is locally split and the J-holomorphic maps are graphical near V, after possibly making ε smaller, in a neighborhood of V we can identify the image of the J-holomorphic maps u_t for $t \in \mathrm{D}^m_\varepsilon$

with graphs of holomorphic functions $f_t^i : \mathcal{U}_i \cong D_\delta^2 \to \mathbf{C}$ for $i = 1, \dots, l$ such that z_i is a zero of order d_i for f_0^i and the zeros of f_t^i correspond to the intersections of the image of u_t with V near z_i. Then we apply Lemma 2.1.2 to obtain monic polynomials p_t^i of degree d_i with the same zeros as f_t^i and whose coefficients are smooth functions of t, after possibly shrinking the neighborhoods $\mathcal{U}_i$.

The zeros of the functions f_t^i (and the polynomials p_t^i) give a map

$$t \mapsto \mathrm{Sym}^{d_1}(\mathcal{U}_1) \times \cdots \times \mathrm{Sym}^{d_l}(\mathcal{U}_l) \subset \mathrm{Sym}^d(V).$$

The holomorphic identifications $\mathcal{U}_i \cong D_\delta^2$ give embeddings

$$\mathrm{Sym}^d(\mathcal{U}_i) \hookrightarrow \mathrm{Sym}^{d_i}(\mathbf{C}) \cong \mathbf{C}^{d_i}.$$

The identification $\mathrm{Sym}^{d_i}(\mathbf{C}) \cong \mathbf{C}^{d_i}$ comes from the correspondence between roots and coefficients of degree d_i monic polynomials and induces the smooth structure on the symmetric product. Since the coefficients of p_t^i define a smooth map $D_\varepsilon^m \to \mathbf{C}^{d_i}$, the map $D_\varepsilon^m \to \mathrm{Sym}^d(\mathbf{C})$ defined by the zeros is also smooth. □

Let $ev_X : \mathcal{M}_{g,n}(A) \to X^n$ be the evaluation at the marked points. The two maps combine into an evaluation map

$$ev = (ev_X, ev_V) : \mathcal{M}_{g,n}(A) \to X^n \times \mathrm{Sym}^d(V).$$

Definition **2.1.4.** — *If $\mathcal{M}_{g,n}(A)$ is compact, the* relative Gromov-Witten invariant $GW^V_{X,A,g,n}$ *is the class*

$$ev_*[\mathcal{M}_{g,n}(A)] \in H_*(X^n \times \mathrm{Sym}^d(V)).$$

In order to obtain numerical invariants, we will intersect $ev_*[\mathcal{M}_{g,n}(A)]$ with homology classes of the appropriate codimension. In practice $\mathcal{M}_{g,n}(A)$ is rarely compact, so we will need to either replace it by a moduli space with point constraints, or compactify it by adding nodal J-holomorphic curves. Instead of developing a general theory, we prefer to deal with this issue case by case in the applications.

2.2. *First relative Gromov-Witten calculation.* — This first calculation is a warm-up for the following more involved ones, and will never be used in the paper. Let Σ be a surface of genus $g \geq 1$. We consider $(X, \omega) = \Sigma \times \mathbf{CP}^1$ with a product symplectic form. Let π_1 and π_2 be the projections of X onto Σ and $\mathbf{CP}^1$, respectively. Let $V = \Sigma \times \{\infty\}$, $A = [\Sigma] + 2g[\mathbf{CP}^1]$, $g = g(\Sigma)$ and $n = g + 1$. The moduli space $\mathcal{M}_{g,g+1}(A)$ then has virtual dimension $8g + 4$ by Lemma 2.2.2 below; this is equal to the dimension of $X^{g+1} \times \mathrm{Sym}^{2g}(V)$. Hence, given generic $\mathbf{x} = ((z_1, y_1), \dots, (z_{g+1}, y_{g+1})) \in X^{g+1}$ and $\mathfrak{z} = \{(z_1', \infty), \dots, (z_{2g}', \infty)\} \in \mathrm{Sym}^{2g}(V)$ we define

$$G_1(\Sigma, \mathbf{x}, \mathfrak{z}) = \# ev^{-1}(\mathbf{x}, \mathfrak{z}),$$

where # denotes the cardinality modulo 2.

Theorem **2.2.1.** — *For generic* $(\mathbf{x}, \mathfrak{z})$, $G_1(\Sigma, \mathbf{x}, \mathfrak{z}) = 1$.

In the proof of Theorem 2.2.1 we will use a product complex structure $J = j_\Sigma \times j_{\mathbf{CP}^1}$ to carry out our calculations. This is legitimate because of certain automatic transversality results, as explained in the next lemma.

First recall that $\mathcal{M}_g(A) = \mathcal{M}_{g,0}(A)$ is the moduli space of J-holomorphic maps $u : (F, j) \to (X, J)$ in the class A, modulo automorphisms of the domain. Here F is a closed surface of genus g and j ranges over all complex structures on F.

Lemma **2.2.2.** — *The moduli space* $\mathcal{M}_{g,g+1}(A)$ *is regular and*

$$\dim \mathcal{M}_{g,g+1}(A) = 8g + 4.$$

Proof. — Every curve $u \in \mathcal{M}_g(A)$ is embedded, since the projection $\pi_1 \circ u$ is a holomorphic map of degree one, and therefore a biholomorphism. Then, by the adjunction formula,

$$\langle c_1(TX), A\rangle = \chi(\Sigma) + A \cdot A = 2g + 2. \tag{2.2.1}$$

Hence the expected dimension of $\mathcal{M}_{g,g+1}(A)$ is $8g + 4$ by Equation (2.1.1). Since every $u \in \mathcal{M}_g(A)$ is embedded and $\langle c_1(TX), A\rangle > 0$, Hofer, Lizan and Sikorav's automatic transversality theorem [HLS, Theorem 1] (first suggested by Gromov [Gr]) implies that $\mathcal{M}_g(A)$ is regular. Since adding free marked points has no effect on regularity, $\mathcal{M}_{g,g+1}(A)$ is also regular and is a manifold of the dimension $8g + 4$. □

Let $u \in \mathcal{M}_g(A)$. Since $\pi_1 \circ u : F \to \Sigma$ has degree one, it follows that (F, j) is biholomorphic to our chosen (Σ, j_Σ). Hence there is a one-to-one correspondence between maps $u \in \mathcal{M}_g(A)$ and meromorphic functions (i.e., branched covers)

$$v = \pi_2 \circ u : \Sigma \to \mathbf{CP}^1$$

of degree $2g$. Meromorphic functions can be studied using the classical Riemann-Roch theorem, which states that:

$$l(D) - l(K - D) = \deg(D) + 1 - g,$$

where D is a divisor on Σ, K is the canonical divisor, and $l(D)$ is the complex dimension of the space of meromorphic functions on Σ with poles at most at D.

Lemma **2.2.3.** — *If* $\deg(D) = 2g - 1 + m$ *with* $m \geq 0$, *then* $l(D) = g + m$.

Proof. — Since $\deg(K) = 2g - 2$, we have $\deg(K - D) < 0$ and $l(K - D) = 0$. Hence $l(D) = \deg(D) + 1 - g = (2g - 1 + m) + 1 - g = g + m$. □

Given $(\mathbf{x}, \mathfrak{z}) \in X^{g+1} \times \mathrm{Sym}^{2g}(V)$ we write $\mathcal{M}_{g,g+1}(A, \mathbf{x}, \mathfrak{z}) = ev^{-1}(\mathbf{x}, \mathfrak{z})$.

Lemma **2.2.4.** — *If $(\mathbf{x}, \mathfrak{z})$ is generic, then $\mathcal{M}_{g,g+1}(A, \mathbf{x}, \mathfrak{z})$ is compact.*

Proof. — Arguing by contradiction, let $u_i \in \mathcal{M}_{g,g+1}(A, \mathbf{x}, \mathfrak{z})$ be a sequence that converges to a limit nodal curve $u : F \to X$. Since u is J-holomorphic, $\pi_1 \circ u$ is holomorphic and one component of F must be biholomorphic to Σ. Hence F is obtained from Σ by attaching spheres which are mapped by u to $\{z_i''\} \times \mathbf{CP}^1$, $1 \leq i \leq k$. Moreover, $k \leq 2g$ by the positivity of intersections, since $u(F)$ represents the homology class $A = [\Sigma] + 2g[\mathbf{CP}^1]$ and $A \cdot [\Sigma] = 2g$.

(1) Suppose that $u(\Sigma) \neq V = \Sigma \times \{\infty\}$. Since $A \cdot [V] = 2g$ and the intersection points of $u(F)$ and V are $(z_1', \infty), \dots, (z_{2g}', \infty)$, we must have

$$\{z_1'', \dots, z_k''\} \subset \{z_1', \dots, z_{2g}'\}.$$

We claim that the map $u|_\Sigma$ cannot exist. Let us write $v = \pi_2 \circ u|_\Sigma : \Sigma \to \mathbf{CP}^1$. Then v is a meromorphic function with poles at $\{z_1', \dots, z_{2g}'\} \setminus \{z_1'', \dots, z_k''\}$, subject to $v(z_i) = y_i$ for all $i = 1, \dots, g+1$. Hence $v \in l(D' - D'')$, where $D' = z_1' + \dots + z_{2g}'$ and $D'' = z_1'' + \dots + z_k''$. By Lemma 2.2.3, $l(D' - D'') \leq g$. [Apply the lemma to $D' - z_1''$, which has degree $2g - 1$. This gives us $l(D' - z_1'') = g$. Then observe that $l(D' - z_1'') \geq l(D' - D'')$, more or less by definition.] On the other hand, there are more constraints $v(z_i) = y_i$ than there are linearly independent functions. Hence v cannot exist provided $\{z_1, \dots, z_{g+1}\}$ and $\{y_1, \dots, y_{g+1}\}$ are generic, a contradiction.

(2) Now suppose that $u(\Sigma) = V$. Then u consists of $u|_\Sigma$, together with $2g$ bubbles $\{z_i''\} \times \mathbf{CP}^1$, $i = 1, \dots, 2g$. The set $\{z_1'', \dots, z_{2g}''\}$ contains $\{z_1, \dots, z_{g+1}\}$ and may also contain $k \leq g - 1$ elements of $\{z_1', \dots, z_{2g}'\}$. (Recall that the points z_i, z_i' and y_i are in generic position.) We then apply the renormalization procedure of [IP1, Proposition 6.6] to the sequence u_i restricted to a neighborhood of V. After choosing suitable restrictions and rescalings, the sequence u_i converges to a nonconstant meromorphic function $\xi : \Sigma \to \mathbf{CP}^1$ which encodes how the curves $u_i(\Sigma)$ approach V. The function ξ has poles at most at $D'' = z_1'' + \dots + z_{2g}''$ and zeros at $2g - k \geq g + 1$ points of $\{z_1', \dots, z_{2g}'\}$. The details of this argument are left to the reader. Since $l(D'') = g + 1$ by Lemma 2.2.3, we have at least as many constraints as linearly independent functions. This implies that $\xi = 0$, which is again a contradiction. □

Proof of Theorem 2.2.1. — By Lemma 2.2.2 and Lemma 2.2.4, $G_1(\mathbf{x}, \mathfrak{z})$ is the number of meromorphic functions $v : \Sigma \to \mathbf{CP}^1$ with poles at $z_1', \dots, z_{2g}'$ such that $v(z_i) = y_i$ for all $i = 1, \dots, g+1$. By Lemma 2.2.3, $l(z_1' + \dots + z_{2g}') = g + 1$. On the other hand, since there are $g + 1$ constraints $v(z_i) = y_i$, there is a unique solution for generic $(\mathbf{x}, \mathfrak{z})$. This shows that $G_1(\mathbf{x}, \mathfrak{z}) = 1$. □

2.3. *Second relative Gromov-Witten calculation.*

2.3.1. *Definition of the invariant.* — Let $\Sigma = \overline{S}$, where $\overline{S} = S \cup D^2$ is obtained by capping off a page S of an open book decomposition with connected binding as in Section I.5.1.2. Also recall the "point at infinity" $z_\infty = \{\rho = 0\} \in \overline{S}$.

Consider (X, ω), where $X = \Sigma \times \mathbf{CP}^1$ and ω is a product symplectic form. Let $V = V_1 \cup V_\infty$, where $V_* = \Sigma \times \{*\}$ and $* = 1, \infty$. We take $A = [\Sigma] + 2g[\mathbf{CP}^1]$, $g = g(\Sigma) = g(S)$ and $n = 1$. We say that piecewise smooth simple closed curves $\overline{a}'_i$, $i = 1, \dots, 2g$, in Σ homotopic to $\overline{a}_i$, $i = 1, \dots, 2g$ are in *good position* if $\Sigma \setminus (\overline{a}'_1 \cup \dots \cup \overline{a}'_{2g})$ is connected and either of the following conditions is satisfied:

- $(\mathfrak{P}_1)$ each $\overline{a}'_i$ is smoothly embedded, $z_\infty \notin \overline{a}'_1 \cup \dots \cup \overline{a}'_{2g}$, each pair $\overline{a}'_i, \overline{a}'_j$, $i \neq j$, intersects transversely in at most one point, and no point in Σ belongs to more than two curves; or
- $(\mathfrak{P}_2)$ the curves $\overline{a}'_i$, $i = 1, \dots, 2g$, all intersect at z_∞ and at no other point, and are smoothly embedded away from z_∞, where they are allowed to have a corner.

We will need Condition $(\mathfrak{P}_2)$ in the proof of Lemma 4.3.4 and Condition $(\mathfrak{P}_1)$ in the computation. Throughout this section we will assume that the curves $\overline{a}'_1, \dots, \overline{a}'_{2g}$ are in good position and write $\overline{\mathbf{a}}' = (\overline{a}'_1, \dots, \overline{a}'_{2g})$. We will denote by $[\overline{a}'_1 \times \dots \times \overline{a}'_{2g}]$ the image of $\overline{a}'_1 \times \dots \times \overline{a}'_{2g}$ in $\mathrm{Sym}^{2g}(V_\infty)$ using the identification $\Sigma \cong V_\infty$ given by the projection $\Sigma \times \mathbf{CP}^1 \to \Sigma$.

Lemma **2.3.1.** — *If $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_1)$, then $[\overline{a}'_1 \times \dots \times \overline{a}'_{2g}]$ is a smooth submanifold of* $\mathrm{Sym}^{2g}(V_\infty)$.

Proof. — By Condition $(\mathfrak{P}_1)$ and after taking complex charts near the double points, it is enough to consider the case of two smooth simple curves $\gamma_i : (-\varepsilon, \varepsilon) \to \mathbf{C}$, $i = 1, 2$, such that $\gamma_1(0) = \gamma_2(0) = 0$ and $(\dot\gamma_1(0), \dot\gamma_2(0))$ is a basis of $\mathbf{C}$. We define

$$\Gamma : (-\varepsilon, \varepsilon)^2 \to \mathrm{Sym}^2(\mathbf{C}), \quad \Gamma(x, y) = \{\gamma_1(x), \gamma_2(y)\}.$$

Recall the identification

$$\Phi : \mathrm{Sym}^2(\mathbf{C}) \to \mathbf{C}^2, \quad \Phi(\{z, w\}) = (z + w, zw),$$

which defines the smooth structure on $\mathrm{Sym}^2(\mathbf{C})$. We have

$$\Phi \circ \Gamma(x, y) = (\gamma_1(x) + \gamma_2(y), \gamma_1(x)\gamma_2(y)).$$

Computing the differential of this map and checking that it is injective at $(0, 0)$ is straightforward and therefore $\Gamma : (-\varepsilon, \varepsilon)^2 \to \mathrm{Sym}^2(\mathbf{C})$ parametrizes a regular surface in $\mathrm{Sym}^2(\mathbf{C})$. □

We will use a product complex structure $J = j_\Sigma \times j_{\mathbf{CP}^1}$ on $\Sigma \times \mathbf{CP}^1$. We recall the moduli space $\mathcal{M}_{g,1}(A)$ and the evaluation map

$$ev = (ev_X, ev_{V_1}, ev_{V_\infty}) : \mathcal{M}_{g,1}(A) \to X \times \mathrm{Sym}^{2g}(V_1) \times \mathrm{Sym}^{2g}(V_\infty)$$

defined in the previous section.

Definition **2.3.2.** — *For every* $\mathfrak{z} = \{(z_1, d_1), \dots, (z_l, d_l)\} \in \mathrm{Sym}^{2g}(V_1)$ *and generic* $\overline{\mathbf{a}}'$ *in good position, we define the relative Gromov-Witten invariant*

$$G_2(\Sigma, \mathfrak{z}) = \# ev^{-1}(\{(z_\infty, 0)\} \times \{\mathfrak{z}\} \times [\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]),$$

where # *denotes the cardinality modulo* 2.

In the definition, the genericity condition on $\overline{\mathbf{a}}'$ is in the space of curves in good position and depends on $\mathfrak{z}$. Before proceeding, a couple of remarks are necessary. First, neither the complex structure nor the placement of the point $(z_\infty, 0)$ is generic. The meaningfulness of this definition will follow from automatic transversality results for the moduli spaces and the evaluation maps. Second, when Condition $(\mathfrak{P}_2)$ holds, the constraint $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]$ may be singular when one or more coordinates are equal to z_∞. While, strictly speaking, a cycle would be enough to define the relative Gromov-Witten invariant (and the set in question surely is a cycle), a number of arguments are simpler if we are allowed to use the machinery of smooth topology. The key observation here is that we can still reason as if $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]$ were smooth because topological constraints prevent the J-holomorphic curves in $\mathcal{M}_{g,1}(A)$ from passing both through $(z_\infty, 0)$ and the singular locus of $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]$. The main result of the section is the following.

Theorem **2.3.3.** — *For every* $\mathfrak{z} \in \mathrm{Sym}^{2g}(V_1 \setminus \{z_\infty\})$ *we have* $G_2(\Sigma, \mathfrak{z}) = 1$.

Recall from the previous section that the moduli space $\mathcal{M}_g(A)$ is regular and can be identified with the space of meromorphic functions $\Sigma \to \mathbf{CP}^1$ with poles of total degree $2g$.

Lemma **2.3.4.** — *The evaluation map*

$$(ev_X, ev_{V_1}) : \mathcal{M}_{g,1}(A) \to X \times \mathrm{Sym}^{2g}(V_1)$$

is a submersion.

Proof. — The first step of the proof is to show that ev_{V_1} is a submersion. Each fiber $\pi^{-1}(\mathfrak{z})$, where $\mathfrak{z} = \{(z_1, d_1), \dots, (z_l, d_l)\} \in \mathrm{Sym}^{2g}(V_1)$, can be viewed, after identifying V_1 with Σ, as the space of meromorphic functions $\Sigma \to \mathbf{CP}^1$ with a pole of order d_i at z_i for $i = 1, \dots, l$. This space is an open dense subset of $H^0(\Sigma, \mathcal{O}(\mathfrak{z}))$, where we identify V_1 with Σ and regard $\mathfrak{z}$ as the divisor $d_1 z_1 + \cdots + d_l z_l$. By Lemma 2.2.3, $H^0(\Sigma, \mathcal{O}(\mathfrak{z}))$ is a complex vector space of dimension $l(\mathfrak{z}) = g + 1$ for each $\mathfrak{z} \in \mathrm{Sym}^{2g}(\Sigma)$; the denseness of $\pi^{-1}(\mathfrak{z})$ is a consequence of the fact that $l(\mathfrak{z}') = g$ if $\mathfrak{z}' \in \mathrm{Sym}^{2g-1}(\Sigma)$ and $\mathfrak{z}' \leq \mathfrak{z}$ (i.e., $\mathfrak{z} - \mathfrak{z}'$ is effective). Hence ev_{V_1} is the restriction of a holomorphic vector bundle

$$\pi : \mathcal{E} \to \mathrm{Sym}^{2g}(\Sigma), \quad \pi^{-1}(\mathfrak{z}) = \mathrm{H}^0(\Sigma, \mathcal{O}(\mathfrak{z}))$$

to an dense open subset of $\mathcal{E}$. In particular, π is a submersion.

For $(u, x) \in \mathcal{M}_{g,1}(A)$ (here x is the marked point on the domain of u, which we identify with Σ), let N_u be the normal bundle of u and D_u the linearized normal Cauchy-Riemann operator on N_u. Since u is an embedding, there is an identification $T_{(u,x)}\mathcal{M}_{g,1}(A) = \ker D_u \oplus T_x\Sigma$ and $d_{(u,x)}ev_X(\xi, v) = \xi(x) + d_x u(v)$, while $d_{(u,x)}ev_{V_1}|_{T_x\Sigma} = 0$.

Since ev_{V_1} is a submersion, then by the discussion above, the pair (ev_X, ev_{V_1}) is a submersion if for every (u, x) there exists $\xi \in \ker D_u$ such that $\xi(x) \neq 0$ and $d_{(u,x)}ev_{V_1}(\xi, 0) = 0$.

We define an action of $\mathbf{C}$ on $\mathcal{M}_{g,1}(A)$ as follows. For every $(u, x) \in \mathcal{M}_{g,1}(A)$, we identify u with a meromorphic function $\Sigma \to \mathbf{CP}^1$ with poles at $ev_{V_1}(u)$, and we define

$$\mathbf{C} \times \mathcal{M}_{g,1}(A) \ni (a, (u, x)) \mapsto (u + a, x).$$

Let $\xi \in \ker D_u$ be the infinitesimal generator of this action. It is clear that $\xi(x) \neq 0$ (and in fact ξ has no zeros) and that $d_{(u,x)}ev_{V_1}(\xi, 0) = 0$ because $ev_{V_1}(u) = ev_{V_1}(u + a)$ for every $a \in \mathbf{C}$. □

For $\mathfrak{z} \in \mathrm{Sym}^{2g}(V_1)$ let us write

$$\mathcal{M}_{g,1}(A, z_\infty, \mathfrak{z}) = ev^{-1}(\{(z_\infty, 0)\} \times \{\mathfrak{z}\} \times [\overline{a}_1' \times \cdots \times \overline{a}_{2g}']).$$

The following statement is a corollary of the automatic transversality for the moduli space $\mathcal{M}_{g,1}(A)$ (see Lemma 2.2.2), Lemma 2.3.4, and the dimension formula (2.1.1).

Lemma **2.3.5.** — *For every $\mathfrak{z} \in \mathrm{Sym}^{2g}(V_1)$ and generic $\overline{\mathbf{a}}'$ in good position, the moduli space $\mathcal{M}_{g,1}(A, z_\infty, \mathfrak{z})$ is regular and zero-dimensional.*

Proof. — For every $\mathfrak{z} \in \mathrm{Sym}^{2g}(V_1)$ the moduli space

$$\widetilde{\mathcal{M}}_{g,1}(A, z_\infty, \mathfrak{z}) = (ev_X, ev_{V_1})^{-1}(\{(z_\infty, 0)\} \times \{\mathfrak{z}\})$$

is regular by Lemma 2.3.4. We consider the restriction of the evaluation map

$$ev_{V_\infty} : \widetilde{\mathcal{M}}_{g,1}(A, z_\infty, \mathfrak{z}) \to \mathrm{Sym}^{2g}(V_\infty).$$

If $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_1)$, generically ev_{V_∞} is transverse to $[\overline{a}_1' \times \cdots \times \overline{a}_{2g}]$ by the standard transversality theorem since $[\overline{a}_1' \times \cdots \times \overline{a}_{2g}]$ is a smooth manifold.

The only issue which remains to be considered is the nonsmoothness of $[\overline{a}_1' \times \cdots \times \overline{a}_{2g}']$ when $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_2)$. In that case, the nonsmoothness is concentrated in the locus where at least one of the coordinates is equal to (z_∞, ∞). However, it is not possible that $u \in \mathcal{M}_{g,1}(\mathrm{A})$ passes through $(z_\infty, 0)$ and (z_∞, ∞) at the same time because $\mathrm{A} \cdot [\{z_\infty\} \times \mathbf{CP}^1] = 1$. Therefore, for every $u \in \mathcal{M}_{g,1}(\mathrm{A})$, $ev_{\mathrm{V}_\infty}(u)$ is disjoint from the singular part of $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]$, and therefore we can still apply the standard transversality theorem. □

Given $\overline{\mathbf{a}}'$ in good position, we define subsets $\mathfrak{G}_1(\overline{\mathbf{a}}'), \mathfrak{G}_2(\overline{\mathbf{a}}') \subset \mathrm{Sym}^{2g}(\Sigma)$ as follows: Let $\mathfrak{G}_1(\overline{\mathbf{a}}')$ be the set of elements $\mathfrak{z} = \{(z_1, d_1), \dots, (z_l, d_l)\} \in \mathrm{Sym}^{2g}(\Sigma)$ such that $z_i \in \overline{a}'_1 \cup \cdots \cup \overline{a}'_{2g}$ or $z_i = z_\infty$ for some $i \in \{1, \dots, l\}$.

If $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_1)$, then we set $\mathfrak{G}_2(\overline{\mathbf{a}}') = \varnothing$. In order to define $\mathfrak{G}_2(\overline{\mathbf{a}}')$ when $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_2)$, we first introduce the set $\mathrm{D}(\overline{\mathbf{a}}')$ of divisors $w_1 + \cdots + w_{2g}$ such that $w_i \in \overline{a}'_i$ for $i = 1, \dots, 2g$ and $w_i = z_\infty$ for at least one i. Then we define $\mathfrak{G}_2(\overline{\mathbf{a}}')$ as the set of divisors $\mathfrak{z}$ of degree $2g$ which are linearly equivalent to a divisor of $\mathrm{D}(\overline{\mathbf{a}}')$, which means that there exist a divisor $\mathfrak{w} \in \mathrm{D}(\overline{\mathbf{a}}')$ and a meromorphic function $f : \Sigma \to \mathbf{CP}^1$ whose associated divisor is $\mathfrak{w} - \mathfrak{z}$. Finally, we set

$$\mathfrak{G}(\overline{\mathbf{a}}') = \mathfrak{G}_1(\overline{\mathbf{a}}') \cup \mathfrak{G}_2(\overline{\mathbf{a}}').$$

Lemma **2.3.6.** — *For every* $\mathfrak{z} \in \mathrm{Sym}^{2g}(\Sigma \setminus \{z_\infty\})$ *we have* $\mathfrak{z} \notin \mathfrak{G}(\overline{\mathbf{a}}')$ *for a generic* $\overline{\mathbf{a}}'$.

Proof. — If $\mathfrak{z} \in \mathrm{Sym}^{2g}(\Sigma \setminus \{z_\infty\})$, then $\mathfrak{z} \notin \mathfrak{G}_1(\overline{\mathbf{a}}')$ is immediate for generic $\overline{\mathbf{a}}'$. Next we show that $\mathfrak{z} \notin \mathfrak{G}_2(\overline{\mathbf{a}}')$ for generic $\overline{\mathbf{a}}'$: Let $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]_\infty$ be the subset of $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]$ where at least one of the coordinates is equal to z_∞. Then $\mathfrak{z} \notin \mathfrak{G}_2(\overline{\mathbf{a}}')$ if and only if the linear equivalence class of $\mathfrak{z}$, seen as a divisor, is disjoint from $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]_\infty$. The linear equivalence class of $\mathfrak{z}$ can be identified with the quotient of the space of nonzero meromorphic functions on Σ with poles at most at $\mathfrak{z}$ by the action of $\mathbf{C}^*$ by multiplication. By Lemma 2.2.3 this quotient is $\mathbf{CP}^g$, so it has real dimension $2g$. On the other hand, $[\overline{a}'_1 \times \cdots \times \overline{a}'_{2g}]_\infty$ is a union of submanifolds of dimension at most $2g - 1$, and therefore generically these two subsets do not intersect in $\mathrm{Sym}^{2g}(\Sigma)$. □

Lemma **2.3.7.** — *If* $\mathfrak{z} \in \mathrm{Sym}^{2g}(\mathrm{V}_1) \setminus \mathfrak{G}(\overline{\mathbf{a}}')$, *then* $\mathcal{M}_{g,1}(\mathrm{A}, z_\infty, \mathfrak{z})$ *is compact.*

Proof. — Arguing by contradiction, let $u_i \in \mathcal{M}_{g,1}(\mathrm{A}, z_\infty, \mathfrak{z})$ be a sequence that converges to a nodal J-holomorphic curve $u : \mathrm{F}' \to \mathrm{X}$. Recall the holomorphic projections $\pi_1 : \mathrm{X} \to \Sigma$ and $\pi_2 : \mathrm{X} \to \mathbf{CP}^1$. Since $u(\mathrm{F})$ represents the homology class $\mathrm{A} = [\Sigma] + 2g[\mathbf{CP}^1]$ and $\pi_1 \circ u$ is holomorphic, one of the irreducible components of F' must be biholomorphic to Σ and all other irreducible components must be spheres which u maps biholomorphically onto $\{w_i\} \times \mathbf{CP}^1$ for $i = 1, \dots, k$. Moreover, since $\mathrm{A} \cdot \mathrm{V}_1 = \mathrm{A} \cdot \mathrm{V}_\infty = 2g$, the positivity of intersections implies that $k \leq 2g$.

(1) Suppose that $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_1)$.

If $w_i \notin \{z_1, \dots, z_l\}$ for some i, then the image of u must contain V_1; otherwise, by the positivity of intersections, the total multiplicity of the intersections between $u(F')$ and V_1 would be at least equal to $2g+1$, contradicting $A \cdot V_1 = 2g$. Hence the image of u consists of V_1 and $2g$ bubbles $\{w_i\} \times \mathbf{CP}^1$ for $i = 1, \dots, 2g$. Moreover one bubble must pass through $(z_\infty, 0)$ and the total number of bubbles intersecting $\overline{a}'_1 \cup \cdots \cup \overline{a}'_{2g}$, counted with multiplicity, is $2g$, which is impossible.

If $w_i \in \{z_1, \dots, z_l\}$ for some i, then there is an intersection between $u(F')$ and V_∞ which does not belong to any of $\overline{a}'_1, \dots, \overline{a}'_{2g}$ since $\mathfrak{z} \notin \mathfrak{G}_1(\overline{\mathbf{a}}')$ (i.e., no z_i lies on $\overline{a}'_1 \cup \cdots \cup \overline{a}'_{2g}$). Hence the image of u consists of V_∞ with some bubbles attached that map to $\{z_i\} \times \mathbf{CP}^1$ with multiplicity d_i for $i = 1, \dots, l$. It is impossible by intersection reasons that such a curve passes through $(z_\infty, 0)$.

(2) Suppose that $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_2)$.

If $w_i \in \{z_1, \dots, z_l\}$ for some i, then again there is an intersection between $u(F')$ and V_∞ which does not belong to any of $\overline{a}'_1, \dots, \overline{a}'_{2g}$, and this gives a contradiction as in (1).

If $w_i \notin \{z_1, \dots, z_l\}$ for some i, then, as in (1), the image of u consists of V_1 and $2g$ bubbles $\{w_1\} \times \mathbf{CP}^1$ with $w_i \in \overline{a}_1$. Moreover $w_i = z_\infty$ for at least one i because the image of u must contain $(z_\infty, 0)$. Define the divisor $\mathfrak{w} = w_1 + \cdots + w_{2g}$ on Σ. Applying the renormalization procedure of [IP1, Proposition 6.6] as in Lemma 2.2.4, we obtain a meromorphic function $\xi : \Sigma \to \mathbf{CP}^1$ with divisor $D(\xi) = \mathfrak{w} - \mathfrak{z}$. Clearly $\mathfrak{w} \in D(\overline{\mathbf{a}}')$, and therefore the existence of ξ contradicts $\mathfrak{z} \notin \mathfrak{G}_2(\overline{\mathbf{a}}')$. □

Proposition **2.3.8.** — *For every $\mathfrak{z} \in \mathrm{Sym}^{2g}(\Sigma \setminus \{z_\infty\})$ and generic $\overline{\mathbf{a}}'$ in good position, the relative Gromov-Witten invariant $G_2(\Sigma, \mathfrak{z})$ is well-defined and its value does not depend on the choice of $\overline{\mathbf{a}}'$ in good position, complex structure on Σ and $\mathfrak{z}$.*

Proof. — Fix $\mathfrak{z} \in \mathrm{Sym}^{2g}(\Sigma \setminus \{z_\infty\})$. By Lemma 2.3.6, $\mathfrak{z} \notin \mathfrak{G}(\overline{\mathbf{a}}')$ for a generic $\overline{\mathbf{a}}'$ in good position and therefore by Lemmas 2.3.5 and 2.3.7 the moduli spaces $\mathcal{M}_{g,1}(A, z_\infty, \mathfrak{z})$ are regular, zero-dimensional, and compact. This proves that $G_2(\Sigma, \mathfrak{z})$ is well-defined.

At first we prove invariance when Condition $(\mathfrak{P}_1)$ holds: in this case $\mathfrak{G}(\overline{\mathbf{a}}') = \mathfrak{G}_1(\overline{\mathbf{a}}')$ and therefore $\mathrm{Sym}^{2g}(V_1) \setminus \mathfrak{G}(\overline{\mathbf{a}}')$ is connected because

$$\mathrm{Sym}^{2g}(V_1) \setminus \mathfrak{G}_1(\overline{\mathbf{a}}') \cong \mathrm{Sym}^{2g}(\Sigma \setminus (\overline{a}'_1 \cup \cdots \cup \overline{a}'_{2g} \cup \{z_\infty\})).$$

Then, if we have pairs

(i) $\overline{\mathbf{a}}'_0$ and $\overline{\mathbf{a}}'_1$ of generic $2g$-tuples of curves in good position satisfying Condition $(\mathfrak{P}_1)$,
(ii) j_0 and j_1 of complex structures on Σ, and
(iii) $\mathfrak{z}_0$ and $\mathfrak{z}_1$ of points in $\mathrm{Sym}^{2g}(V_1)$ such that $\mathfrak{z}_i \notin \mathfrak{G}(\overline{\mathbf{a}}'_i)$, $i = 1, 2$,

then we can find generic paths[2] $\overline{\mathbf{a}}'_t$, j_t and $\mathfrak{z}_t$ interpolating between them, where $\overline{\mathbf{a}}'_t$ is in good position and satisfies Condition $(\mathfrak{P}_1)$, and $\mathfrak{z}_t \in \mathrm{Sym}^{2g}(\mathrm{V}_1) \setminus \mathfrak{G}(\overline{\mathbf{a}}'_t)$.

Since we can repeat the proofs of the Lemmas 2.3.5 and 2.3.7 for paths, the invariance follows in a standard manner.

If, instead, $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_2)$ and $\mathfrak{z} \in \mathrm{Sym}^{2g}(\mathrm{V}_1) \setminus \mathfrak{G}(\overline{\mathbf{a}}')$, then we can find an open neighborhood U of $\mathfrak{z}$ and a small perturbation $\overline{\mathbf{a}}'_t$ of $\overline{\mathbf{a}}'$ such that the good position condition is preserved all along the deformation, the final $2g$-tuple of curves satisfies Condition $(\mathfrak{P}_1)$, and $\mathrm{U} \cap \mathfrak{G}(\overline{\mathbf{a}}'_t) = \varnothing$ all along the deformation. Then we can prove the analogues of Lemmas 2.3.5 and 2.3.7 for paths where $\mathfrak{z}$ remains in U, and again the invariance follows in a standard manner. This concludes the proof of the proposition. □

2.3.2. *Case of* T^2. — We consider the situation where $\Sigma = \mathrm{T}^2$. We suppose that $\overline{\mathbf{a}}'$ satisfies Condition $(\mathfrak{P}_1)$. By the correspondence between J-holomorphic curves in $\mathrm{T}^2 \times \mathbf{CP}^1$ representing the homology class $\mathrm{A} = [\mathrm{T}^2] + 2[\mathbf{CP}^1]$ and degree 2 meromorphic functions on T^2 for a generic $\mathfrak{z} = \{z_1, z_2\} \subset \mathrm{Sym}^2(\mathrm{V}_1) \setminus \mathfrak{G}(\overline{\mathbf{a}}')$ with $z_1 \neq z_2$, the relative Gromov-Witten invariant $\mathrm{G}_2(\mathrm{T}^2, \mathfrak{z})$ is equal to the number of meromorphic functions $v : \mathrm{T}^2 \to \mathbf{CP}^1$ of degree 2 such that $v(z_\infty) = 0$, $v(z_1) = v(z_2) = 1$ and the poles are on $\overline{\mathbf{a}}'$.

Lemma **2.3.9.** — *If $z_1 \neq z_2$, then every meromorphic function on T^2 with poles at z_1 and z_2 of order 1 can be written as $\alpha_1 + \alpha_2 f$, where $\alpha_1, \alpha_2 \in \mathbf{C}$, $\alpha_2 \neq 0$, and f is a fixed function on T^2 with poles at z_1 and z_2 of order 1.*

In other words, f is unique up to a postcomposition by a fractional linear transformation $\mathbf{CP}^1 \to \mathbf{CP}^1$ which fixes ∞.

Proof. — By Lemma 2.2.3, $l(z_i) = 1$ and is given by the constants, and $l(z_1 + z_2) = 2$ and one dimension is taken by the constants. The lemma follows. □

Lemma **2.3.10.** — $\mathrm{G}_2(\mathrm{T}^2, \mathfrak{z}) = 1$.

Proof. — Degree two meromorphic functions on T^2 are branched double covers $\mathrm{T}^2 \to \mathbf{CP}^1$. According to Lemma 2.3.9, a branched double cover $v : \mathrm{T}^2 \to \mathbf{CP}^1$ satisfying $v(z_0) = 0$ and $v(z_1) = v(z_2) = 1$ is determined up to postcomposition by $\eta \in \mathrm{PSL}(2, \mathbf{C})$ which fixes 0 and 1; see Figure 1.

[2] A complication here is that the smooth structure on $\mathrm{Sym}^{2g}(\Sigma)$ depends on the complex structure on Σ, and therefore for different values of t the symmetric product $\mathrm{Sym}^{2g}(\Sigma)$ could have different smooth structures, which are all diffeomorphic but not through the identity. However they can all be put in a smooth fiber bundle over $[0, 1]$ such that the fiber over t has the smooth structure defined by j_t. The necessary modifications to the proof are standard but notationally heavy, so we prefer to leave them to the reader.

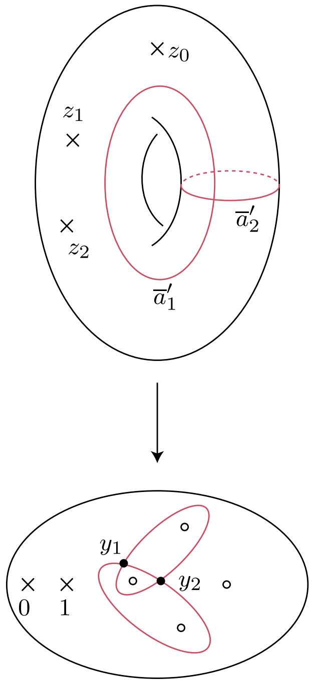


FIG. 1. — The branched double cover v of the torus T^2. The branch points $w_1, \dots, w_4$ of v are indicated by $\circ$. (Color figure online)

Let $w_1, \dots, w_4 \in \mathbf{CP}^1$ be the branch points of v. Since z_1 and z_2 can be chosen generic, we may assume that $w_i \notin \{0, 1, \infty\}$ for $i = 1, \dots, 4$ and are all distinct. Let C_{ij} be a simple closed curve which bounds a small neighborhood of an arc from w_i to w_j. We may assume that C_{12} and C_{23} intersect only at 2 points y_1, y_2. Let $\overline{a}'_1, \overline{a}'_2$ be lifts of C_{12}, C_{23} to T^2 such that $\#(\overline{a}'_1 \cap \overline{a}'_2) = 1$ and $v(\overline{a}'_1 \cap \overline{a}'_2) = y_1$. We now impose the condition $\eta \circ v(z_3) = \eta \circ v(z_4) = \infty$, where $z_3 \in \overline{a}'_1$, $z_4 \in \overline{a}'_2$, and η is a fractional linear transformation which fixes 0, 1. This is possible only if $z_3, z_4 \in v^{-1}(y_i)$ for $i = 1$ or 2. On the other hand, since y_1, y_2 are not branch points, z_3 and z_4 must be distinct. Hence the only possibility is $\{z_3, z_4\} = v^{-1}(y_2)$ and $\eta \in PSL(2, \mathbf{C})$ is the uniquely determined by $\eta(0) = 0$, $\eta(1) = 1$, $\eta(y_2) = \infty$. This completes the proof of the lemma. □

2.3.3. *Reduction to a torus.* — Now we reduce the computation of $G_2(\Sigma, \mathfrak{z})$ to the case $\Sigma = T^2$ by a degeneration argument. In this section we assume again that the curves $\overline{\mathbf{a}}'$ satisfy Condition $(\mathfrak{P}_1)$.

Let $\Sigma_\tau = (\overline{S}, j_{\Sigma_\tau})$, $\tau \in [0, \infty)$, be a 1-parameter family of Riemann surfaces which degenerate into a nodal Riemann surface

$$(\Sigma_\infty, \mathfrak{n}) = ((\Sigma_-, \mathfrak{n}_-) \sqcup (\Sigma_+, \mathfrak{n}_+)) / \mathfrak{n}_- \sim \mathfrak{n}_+,$$

where Σ_- has genus one, Σ_+ has genus $g-1$ and $\mathfrak{n} = \{\mathfrak{n}_-, \mathfrak{n}_+\}/\sim$. Let j_{Σ_+} and j_{Σ_-} be the complex structures on Σ_+ and Σ_- respectively. Topologically Σ_∞ is obtained as follows: Let γ be a separating curve in $\overline{S}$ and let $\sim_0$ be an equivalence relation on $\overline{S}$ such that $x \sim_0 y$ if $x, y \in \gamma$. Then $\Sigma_\infty = \overline{S}/\sim_0$. We assume that z_∞ is disjoint from γ and is mapped to Σ_-. We also assume that the curves in $\overline{\mathbf{a}}'$ are disjoint from γ and that $\overline{\alpha}'_1, \overline{\alpha}'_2$ are mapped to Σ_-, while $\overline{\alpha}'_3, \dots, \overline{\alpha}'_{2g}$ are mapped to Σ_+. Finally we choose $\mathfrak{z} = \{z_1, \dots, z_{2g}\} \in \mathrm{Sym}^{2g}(V_1) \setminus \mathfrak{G}(\overline{\mathbf{a}}')$ such that $z_i \notin \gamma$ for $i = 1, \dots, 2g$, $\{z_1, z_2\}$ are mapped to Σ_- and $\{z_3, \dots, z_{2g}\}$ are mapped to Σ_+. We denote $\mathfrak{z}_- = \{z_1, z_2\}$ and $\mathfrak{z}_+ = \{z_3, \dots, z_{2g}\}$.

Define product complex structures $J_\tau = j_{\Sigma_\tau} \times j_{\mathbf{CP}^1}$ on $\Sigma_\tau \times \mathbf{CP}^1$ and $J_\pm = j_{\Sigma_\pm} \times j_{\mathbf{CP}^1}$ on $\Sigma_\pm \times \mathbf{CP}^1$. We denote $P_\mathfrak{n} = \{\mathfrak{n}\} \times \mathbf{CP}^1$, which we see as a submanifold of both $\Sigma_+ \times \mathbf{CP}^1$ and $\Sigma_- \times \mathbf{CP}^1$. We define the parametric moduli space[3]

$$\mathcal{M}^*_{g,1}(A, z_\infty, \mathfrak{z}) = \left\{ (u, \tau) \mid \tau \in [0, \infty), u \in \mathcal{M}^{J_\tau}_{g,1}(A, z_\infty, \mathfrak{z}) \right\}.$$

Proposition **2.3.11.** — *Let* $\overline{\mathcal{M}}^*_{g,1}(A, z_\infty, \mathfrak{z})$ *be the compactification of* $\mathcal{M}^*_{g,1}(A, z_\infty, \mathfrak{z})$. *Then*

$$\overline{\mathcal{M}}^*_{g,1}(A, z_\infty, \mathfrak{z}) - \mathcal{M}^*_{g,1}(A, z_\infty, \mathfrak{z}) = \mathcal{M}^{J_-}_{1,1}(A_-, z_\infty, \mathfrak{z}_-) \times_{P_\mathfrak{n}} \mathcal{M}^{J_+}_{g-1}(A_+, \mathfrak{z}_+),$$

where $A_+ = [\Sigma_+] + 2(g-1)[\mathbf{CP}^1]$ *and* $A_- = [\Sigma_-] + 2[\mathbf{CP}^1]$.

Proof. — By Gromov compactness, up to passing to a subsequence, (u_n, τ_n) converges to (u_+, u_0, u_-), where $u_\pm$ maps to $\Sigma_\pm \times \mathbf{CP}^1$ and u_0 maps to $\{\mathfrak{n}\} \times \mathbf{CP}^1$. Let $A_\pm$ be the homology class in $H_2(\Sigma_\pm \times \mathbf{CP}^1)$ represented by $u_\pm$, and let $k_0[\mathbf{CP}^1]$ be the homology class represented by u_0. A simple computation shows that $A_\pm = [\Sigma_\pm] + k_\pm[\mathbf{CP}^1]$, where $k_+ + k_- + k_0 = 2g$. The point constraints for $u_\pm$ imply that $k_+ \geq 2g-2$ and $k_- \geq 2$ and, on the other hand, $k_0 \geq 0$ and $k_0 = 0$ if and only if u_0 is constant. This implies that $k_+ = 2g-2$, $k_- = 2$, and u_0 is constant. By Lemma 2.3.7, u_- is irreducible, and therefore $u_- \in \mathcal{M}^{J_-}_{1,1}(A_-, z_\infty, \mathfrak{z}_-)$.

By the positivity of intersections, the image of u_- must intersect $P_\mathfrak{n}$ at a single point w. Without loss of generality we can assume that $w \notin \overline{a}'_3 \cup \dots \cup \overline{a}'_{2g}$, since u_- is the unique element of $\mathcal{M}^{J_-}_{1,1}(A_-, z_\infty, \mathfrak{z}_-)$ by Lemma 2.3.10. Since the Gromov limit of a sequence of connected holomorphic curves is connected, u_+ must also pass through w, once we have identified $\{\mathfrak{n}_+\} \times \mathbf{CP}^1$ with $\{\mathfrak{n}_-\} \times \mathbf{CP}^1$. Then we can repeat the proof of Lemma 2.3.7 and show that u_+ is also irreducible, and therefore $u_+ \in \mathcal{M}^{J_+}_{g-1,1}(A_+, w, \mathfrak{z}_+)$. This shows that

$$\overline{\mathcal{M}}^*_{g,1}(A, z_\infty, \mathfrak{z}) - \mathcal{M}^*_{g,1}(A, z_\infty, \mathfrak{z}) \subseteq \mathcal{M}^{J_-}_{1,1}(A_-, z_\infty, \mathfrak{z}_-) \times_{P_\mathfrak{n}} \mathcal{M}^{J_+}_{g-1}(A_+, \mathfrak{z}_+).$$

[3] We do not know if $\overline{\mathbf{a}}'$ is generic for every τ. The proper way to proceed would be to choose generic paths $\tau \mapsto \mathfrak{z}_\tau$ and $\tau \mapsto \overline{\mathbf{a}}'_\tau$ such that $\mathfrak{z}_\tau \in \mathrm{Sym}^{2g}(V_1) \setminus \mathfrak{G}(\overline{\mathbf{a}}'_\tau)$ for all τ. The caveat about the smooth structure of the symmetric product explained in Footnote 2 also applies here.

The reverse inclusion follows from what Seidel in [Se, Proposition 2.7] calls an average specimen of the "gluing theorem" type. □

Proof of Theorem 2.3.3. — We prove the theorem by induction on the genus of Σ. The case $g=1$ follows from Proposition 2.3.8 and Lemma 2.3.10. If Σ has genus g and we assume that the statement is true for genus $g-1$, we use Proposition 2.3.8 to modify the curves $\overline{\mathbf{a}}'$ and the points $\mathfrak{z}$ so that we can apply Proposition 2.3.11. Then

$$G_2(\Sigma,\mathfrak{z}) = G_2(\Sigma_-,\mathfrak{z}_-)G_2(\Sigma_+,\mathfrak{z}_+)$$

and $G_2(\Sigma,\mathfrak{z})=1$ by Lemma 2.3.10 and the inductive hypothesis. □

2.4. *Third relative Gromov-Witten calculation.*

2.4.1. *Definitions.* — Let D^2 be the closed unit disk $\{|z|\leq 1\}\subset \mathbf{C}$ and let $\Sigma=\overline{S}$. We consider (X,ω), where $X=\Sigma\times D^2$ and ω is a product symplectic form.

Recall $\widehat{\mathbf{a}}\subset\overline{S}$ as well as its components $\widehat{a}_i$, $i=1,\dots,2g$, from Section I.5.2.2; here $\widehat{a}_i$ is an open arc, $\{z_\infty\}=\overline{a}_i-\widehat{a}_i$, and z_∞ is the point at infinity. Then $L_{\widehat{\mathbf{a}}}=\widehat{\mathbf{a}}\times\partial D^2\subset\partial X$ is a Lagrangian submanifold of (X,ω).[4] We write π_i, $i=1,2$, for the projection of X onto the ith factor. Let A be the relative homology class

$$A=[\Sigma]+2g\cdot[D^2]\in H_2(X,L_{\widehat{\mathbf{a}}}),$$

and let $(F,j,x_0,\mathbf{x})$ be a compact Riemann surface (F,j) of genus g with $2g$ boundary components, together with an interior marked point x_0 and boundary marked points $\mathbf{x}=(x_1,\dots,x_{4g})$, where x_i, x_{2g+i} are on the ith boundary component $\partial_i F$.

Again, we work with a product almost complex structure $J=j_\Sigma\times i_{D^2}$ on X. Let $\mathcal{M}_{g,1,4g}(A)$ (resp. $\mathcal{M}_{g,0,0}(A)$) be the moduli space of holomorphic maps

$$u:(F,j,x_0,\mathbf{x})\to(X,J)\quad(\text{resp. } u:(F,j)\to(X,J))$$

such that $u(\partial_i F)\subset L_{\widehat{a}_i}$ and $u_*([F])=A$, modulo automorphisms of the domain. We consider the evaluation maps

$$ev_X:\mathcal{M}_{g,1,4g}(A)\to X,\quad (u,x_0,\mathbf{x})\mapsto u(x_0); \tag{2.4.1}$$

$$ev_\partial:\mathcal{M}_{g,1,4g}(A)\to L_{\widehat{a}_1}\times\cdots\times L_{\widehat{a}_{2g}}\times L_{\widehat{a}_1}\times\cdots\times L_{\widehat{a}_{2g}}, \tag{2.4.2}$$
$$(u,x_0,\mathbf{x})\mapsto(u(x_1),\dots,u(x_{4g})).$$

and $ev=(ev_X,ev_\partial)$. Let $\mathbf{z}=(z_1,\dots,z_{4g})$ be a $4g$-tuple of points in Σ such that $z_i,z_{2g+i}\in\widehat{a}_i$, $i=1,\dots,2g$. Then we write

$$\mathcal{M}_{g,1,4g}(A,z_\infty)=ev_X^{-1}(z_\infty,0)\quad\text{and}$$

[4] We also could have used $L_{\overline{\mathbf{a}}}=\overline{\mathbf{a}}\times\partial D^2\subset\partial X$, but we write $L_{\widehat{\mathbf{a}}}$ to emphasize that the holomorphic maps never intersect $\{z_\infty\}\times\partial D^2$.

$$\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z}) \\ = ev^{-1}((z_\infty, 0), (z_1, 1), \dots, (z_{2g}, 1), (z_{2g+1}, -1), \dots, (z_{4g}, -1)).$$

Definition **2.4.1.** — *The relative Gromov-Witten type invariant* $G_3(\Sigma, \mathbf{z})$ *is the cardinality modulo two of* $\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$ *for* $\mathbf{z}$ *generic.*

Theorem **2.4.2.** — *If* $\mathbf{z}$ *is generic, then* $G_3(\Sigma, \mathbf{z}) = 1$.

Proof. — The regularity and compactness for the moduli space $\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$ are proved in Corollary 2.4.10 and Lemma 2.4.12, respectively. The independence of $G_3(\Sigma, \mathbf{z})$ of the almost complex structure and $\mathbf{z}$ is proved in Lemma 2.4.13, and finally $G_3(\Sigma, \mathbf{z})$ is computed in Sections 2.4.4–2.4.6. □

2.4.2. *Transversality.* — We recall the following automatic transversality result from [HLS]; see Theorem 2 and Remark (1) following the statement of Theorem 2 in [HLS]:

Theorem **2.4.3** *(Hofer-Lizan-Sikorav).* — *Let* (F, j) *be a compact Riemann surface with nonempty boundary,* (M, J) *be an almost complex 4-manifold, and* $L \subset (M, J)$ *be a totally real surface. Let*

$$u_0 : (F, \partial F, j) \to (M, L, J)$$

be a holomorphic map which sends ∂F *to* L. *If* u_0 *is immersed and the sum of the Maslov indices of* $u_0|_{\partial F}$ *with respect to any unitary trivialization of* $u_0^* TM$ *is positive, then the set of holomorphic maps* $u : (F, \partial F, j) \to (M, L, J)$ *near* u_0 *is regular.*

Remark **2.4.4.** — Strictly speaking, Theorem 2.4.3 applies to the case where (M, J) has no boundary. However, we note that Theorem 2.4.3 is a local result, i.e., only uses an open neighborhood of the image of u_0. The complex manifold $(X = \Sigma \times D^2, J)$ can then be slightly enlarged to a product complex manifold $\Sigma \times D^2_{1+\varepsilon}$, where $D^2_{1+\varepsilon}$ is a disk of radius $1 + \varepsilon$, so that the theorem applies. We then note that any nearby curve in the enlargement with boundary on $L_{\widehat{\mathbf{a}}}$ has image in X by considering the projection to $D^2_{1+\varepsilon}$.

Using Theorem 2.4.3, we prove the following key result:

Lemma **2.4.5.** — *The moduli space* $\mathcal{M}_{g,0,0}(A)$ *is regular and has dimension*

$$\dim \mathcal{M}_{g,0,0}(A) = 4g + 2.$$

Proof. — Let $u \in \mathcal{M}_{g,0,0}(A)$, where $u : (F, j) \to (X, J)$. Since u maps each $\partial_i F$ to a distinct $L_{\widehat{a}_i}$, there are no branch points of $\pi_2 \circ u$ along ∂F and the curve u is immersed near ∂F. On the other hand, the restriction of $\pi_1 \circ u$ to $int(F)$ is a biholomorphism onto its image. Hence u is an immersion on all of F.

The total Maslov index $\mu(u)$ relative to the Lagrangian $L_{\widehat{\mathbf{a}}}$, with respect to any unitary trivialization of u^*TX is:

$$\mu(u) = 2g \cdot \mu(\{pt\} \times D^2) + 2c_1(T\Sigma),$$

where $\{pt\} \times D^2$ has boundary on $L_{\widehat{\mathbf{a}}}$ and $\mu(\{pt\} \times D^2)$ is computed with respect to a trivialization of $TX|_{\{pt\}\times D^2}$. An easy calculation gives $\mu(\{pt\} \times D^2) = 2$ and $c_1(T\Sigma) = \chi(\Sigma) = 2 - 2g$. Hence

$$\mu(u) = 2g(2) + 2(2 - 2g) = 4 > 0. \tag{\textbf{2.4.3}}$$

The regularity of $\mathcal{M}_{g,0,0}(A)$ then follows from Theorem 2.4.3, in view of Remark 2.4.4. Using the usual index formula for holomorphic curves with Lagrangian boundary, we obtain:

$$\dim \mathcal{M}_{g,0,0}(A) = -\chi(F) + \mu(u) = -(2 - 4g) + 4 = 4g + 2,$$

This completes the proof of Lemma 2.4.5. □

Corollary **2.4.6.** — *The moduli space $\mathcal{M}_{g,1,4g}(A)$ is regular and has dimension*

$$\dim \mathcal{M}_{g,1,4g}(A) = 8g + 4.$$

Since the point z_∞ is not generic, we also need an automatic transversality result for the evaluation map.

Lemma **2.4.7.** — *The evaluation map $ev_X : \mathcal{M}_{g,1,4g}(A) \to X$ is a submersion.*

Proof. — For every $(u, x_0, \mathbf{x}) \in \mathcal{M}_{g,1,4g}(A)$, the differential

$$d_{(u,x_0,\mathbf{x})} ev_X : T_{(u,x_0,\mathbf{x})}\mathcal{M}_{g,1,4g}(A) \to T_{u(x_0)}X$$

is surjective thank to the following two local deformations of $(u, x_0, \mathbf{x})$: (i) postcomposition of u by a fractional linear transformation and (ii) variation of the point x. □

Corollary **2.4.8.** — *The moduli space $\mathcal{M}_{g,1,4g}(A, z_\infty)$ is regular and has dimension* $\dim \mathcal{M}_{g,1,4g}(A, z_\infty) = 8g$.

Next we consider the evaluation map at the boundary points.

Lemma **2.4.9.** — *For a generic $\mathbf{z} = (z_1, \dots, z_{4g})$, the point*

$$\mathbf{z}_* = ((z_1, 1), \dots, (z_{2g}, 1), (z_{2g+1}, -1), \dots, (z_{4g}, -1))$$

is a regular value of the evaluation map

$$ev_\partial : \mathcal{M}_{g,1,4g}(A, z_\infty) \to L_{\widehat{a}_1} \times \cdots \times L_{\widehat{a}_{2g}} \times L_{\widehat{a}_1} \times \cdots \times L_{\widehat{a}_{2g}},$$

defined as restriction of the map from Equation (2.4.2).

Proof. — We take the generic $\mathbf{z}$ to be a regular value of the composition of ev_∂ with the projection

$$L_{\widehat{a}_1} \times \cdots \times L_{\widehat{a}_{2g}} \times L_{\widehat{a}_1} \times \cdots \times L_{\widehat{a}_{2g}} \xrightarrow{\pi_1} \widehat{a}_1 \times \cdots \times \widehat{a}_{2g} \times \widehat{a}_1 \times \cdots \times \widehat{a}_{2g};$$

by Sard's theorem the set of such $\mathbf{z}$ is open and dense.

We claim that if $\mathbf{z}$ is generic, then a regular value of $\pi_1 \circ ev_\partial$ is a regular value of ev_∂: In fact we decompose

$$T_{(z_j,\pm 1)} L_{\widehat{a}_j} = T_{z_j} \widehat{a}_j \oplus T_{\pm 1} \partial D^2$$

and observe that the directions tangent to ∂D^2 are covered by moving the boundary marked points (since u is an embedding) and the directions tangent to the arcs $\widehat{a}_j$ are covered by moving the J-holomorphic curve in the moduli space as long as $\mathbf{z}$ is a regular value of $\pi_1 \circ ev_\partial$. □

Corollary **2.4.10.** — *The moduli space* $\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$ *is regular and has dimension* $\dim \mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z}) = 0$.

We consider now a slightly different moduli space. Let

$$A' = [\Sigma] + (2g-1)[D^2] \in H_2(X, L_{\widehat{\mathbf{a}}}),$$

and let $\mathbf{z}' = (z_1, \ldots, z_{2g-1}, z_{2g+1}, \ldots, z_{4g-1})$ be a $(4g-2)$-tuple of points such that $z_i, z_{2g+i} \in \widehat{a}_i$, $i = 1, \ldots, 2g-1$. We define the moduli space $\mathcal{M}_{g,1,4g-2}(A', z_\infty, \mathbf{z}')$ in a manner analogous to $\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$.

Lemma **2.4.11.** — *For generic* $\mathbf{z}$ *the moduli space* $\mathcal{M}_{g,1,4g-2}(A', z_\infty, \mathbf{z}')$ *is empty.*

Proof. — For generic $\mathbf{z}'$ the moduli space $\mathcal{M}_{g,1,4g-2}(A', z_\infty, \mathbf{z}')$ is regular; the proof is the same as the proof of Corollary 2.4.10. One can compute from the dimension formula that the virtual dimension of $\mathcal{M}_{g,1,4g-2}(A', z_\infty, \mathbf{z}')$ is -1. This proves the lemma. □

Observe that the lemma is still valid if the unoccupied arc is any $\widehat{a}_i$ instead of $\widehat{a}_{2g}$.

2.4.3. *Compactness.*

Lemma **2.4.12.** — *The moduli space* $\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$ *is compact for generic* $\mathbf{z}$.

Proof. — Let $u_i : F_i \to X$ be a sequence of curves in $\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$ that limits to $u : F \to X$. The compactness of holomorphic curves with this kind of singular Lagrangian boundary condition has already been discussed in detail in Section I.7.3 in the SFT setting, but here the existence of the limit can be justified more easily as follows. The Lagrangian boundary condition is the product of a Lagrangian submanifold in D^2 (i.e., ∂D^2) and an immersed Lagrangian submanifold in $\Sigma = \overline{S}$ (i.e., the arcs $\overline{\mathbf{a}}$). Since we work with split almost complex structures, we can apply the Gromov compactness theorem on the two components separately. On the component where the Lagrangian boundary condition is smooth, this is of course classical, and on the component where the Lagrangian boundary condition is immersed we apply [EES, Theorem 9.2]. It is true that the reference only considers disks, but the only new ingredient in the higher genus case are domain degenerations, which happen at the source, and the necessary adjustments are straightforward.

We will eliminate all possible degenerations in the limit to show that $u \in \mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$. Note in particular that we are assuming that all the points of $\mathbf{z}$ are distinct.

(i) We first claim that the domain of u can have no irreducible component which is a disk. Arguing by contradiction, suppose that u has a disk component u'. Since $\pi_1 \circ u_i|_{int(F_i)}$ is a biholomorphism onto its image for all i, $\mathrm{Im}(u') = \{z\} \times D^2$ for some z. Also there is no fiber component of u of the form $\Sigma \times \{pt\}$ where $pt \in \partial D^2$, since otherwise u cannot pass through $(z_\infty, 0)$ and be in the class A. The point z must then be an element of $\{z_1, \dots, z_{2g}\}$ by considering the intersections of u with $\Sigma \times \{1\}$. Similarly $z \in \{z_{2g+1}, \dots, z_{4g}\}$. This is contradicts the assumption that all points in $\mathbf{z}$ are distinct.

(ii) Next we claim that $u(\partial F)$ does not intersect $\{z_\infty\} \times \partial D^2$. Arguing by contradiction, suppose that $u(\partial F) \cap (\{z_\infty\} \times \partial D^2) \neq \varnothing$. Since u cannot have a disk component $\{z_\infty\} \times D^2$ by (i), there exists a point $z \in \Sigma - \overline{\mathbf{a}}$ close to z_∞, such that u intersects $\{z\} \times D^2$ at two points: once near $(z_\infty, 0)$ and once near $\{z_\infty\} \times \partial D^2$. Hence $\langle u, \{z\} \times D^2 \rangle \geq 2$, which is a contradiction. This also implies that u has no nodes along the boundary: since each u_i maps each component of ∂F_i to a different $L_{\widehat{a}_i}$, such a node could only be at a point of $\{z_\infty\} \times \partial D^2$.

(iii) Finally we eliminate interior nodes. Since u has no disk components and sphere components (the latter easily follows from $\pi_2(X) = 0$), if u has an interior node, it must have a component $u' : F' \to X$ of genus $< g$. Since $\pi_1 \circ u'|_{int(F')}$ cannot map onto the genus g surface $\Sigma - (\widehat{\mathbf{a}} - N(z_\infty))$, we have a contradiction. □

Corollary 2.4.10 and Lemma 2.4.12 imply that $\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$ is a finite set, and therefore the count modulo two

$$G_3(\Sigma, \mathbf{z}) = \#\mathcal{M}_{g,1,4g}(A, z_\infty, \mathbf{z})$$

is well-defined, when $\mathbf{z}$ is generic.

Lemma **2.4.13.** — *The count* $G_3(\Sigma, \mathbf{z})$ *is independent of the choice of product complex structure* J *and* $\mathbf{z}$ *generic.*

Proof. — Given J_i and $\mathbf{z}_i$ generic, $i = 0, 1$, we can connect them by generic paths J_t and $\mathbf{z}_t$, $t \in [0, 1]$. We further assume that the path $\mathbf{z}_t$ satisfies the following (generic) properties:

(1) the number of parameters $t \in [0, 1]$ for which two points in $\mathbf{z}_t$ coincide is finite,
(2) at a given $t \in [0, 1]$ at most two points coincide, and
(3) if for some $t \in [0, 1]$ two points of $\mathbf{z}_t$ coincide, the remaining $4g - 2$ are generic in the sense of Lemma 2.4.11.

We define the parametric moduli space

$$\mathcal{M}^{\{J_t\}}_{g,1,4g}(A, z_\infty, \{\mathbf{z}_t\}) := \{(t, u) \mid t \in [0, 1],\ u \in \mathcal{M}^{J_t}_{g,1,4g}(A, z_\infty, \mathbf{z}_t)\}.$$

We prove the compactness of $\mathcal{M}^{\{J_t\}}_{g,1,4g}(A, z_\infty, \{\mathbf{z}_t\})$. The proof is similar to that of Lemma 2.4.12 and the only difference is that, when two points of $\mathbf{z}_t$ (say, without loss of generality, $z_{2g,t}$ and $z_{4g,t}$) come together for some isolated $t_0 \in (0, 1)$, a family of J_t-holomorphic curves u_t could degenerate, as $t \to t_0$, into a nodal curve with a disk component $\{z_{2g,t_0}\} \times D^2$ and one irreducible component $u' : F' \to X$ with $2g - 1$ boundary components which passes through $(0, z_\infty)$ and $(z_1, 1), \dots, (z_{2g-1}, 1), (z_{2g+1}, -1), \dots, (z_{4g-1}, -1)$. The component u' however cannot exist by Lemma 2.4.11 and Property (3) of the path $\{\mathbf{z}_t\}$.

The regularity of $\mathcal{M}^{\{J_t\}}_{g,1,4g}(A, z_\infty, \{\mathbf{z}_t\})$ follows from Corollary 2.4.10 adapted to a generic 1-parameter family. The compactness and regularity then imply the lemma. ☐

2.4.4. *Reduction to a torus.* — We now explain how to reduce to the case of a torus. As in Section 2.3.3, we degenerate $\Sigma_\tau = (\Sigma, j_{\Sigma_\tau})$, $\tau \in [0, \infty)$, into a nodal Riemann surface

$$(\Sigma_\infty, \mathfrak{n}) = ((\Sigma_-, \mathfrak{n}_-) \sqcup (\Sigma_+, \mathfrak{n}_+)) / \mathfrak{n}_- \sim \mathfrak{n}_+,$$

where Σ_- has genus one and Σ_+ has genus $g - 1$ by pinching along a separating curve γ. We denote $D_\mathfrak{n} = \{\mathfrak{n}\} \times D^2$. Let j_{Σ_+} and j_{Σ_-} be the complex structures on Σ_+ and Σ_-, and define product complex structures $J_\tau = j_{\Sigma_\tau} \times i_{D^2}$ and $J_\pm = j_{\Sigma_\pm} \times i_{D^2}$. Here we assume the following:

(i) z_∞, $\mathbf{z}$, and $\widehat{\mathbf{a}}$ are fixed for all $\tau \in [0,\infty)$;
(ii) γ is disjoint from $\mathbf{z}, \widehat{a}_1, \widehat{a}_2$, but intersects $\widehat{a}_3, \dots, \widehat{a}_{2g}$ at two points each;
(iii) in the limit $z_\infty, z_1, z_2, z_{2g+1}, z_{2g+2} \in \Sigma_-$ and $z_3, \dots, z_{2g}, z_{2g+3}, \dots, z_{4g} \in \Sigma_+$.

Let $\widetilde{\mathbf{a}}$ be the quotient of $\widehat{\mathbf{a}}$ at $\tau = \infty$. Let $v_\tau = \pi_2 \circ u_\tau$, where π_2 is the projection to D^2. We write $\mathbf{z}_- = (z_1, z_2, z_{2g+1}, z_{2g+2})$ and $\mathbf{z}_+ = (z_3, \dots, z_{2g}, z_{2g+3}, \dots, z_{4g})$.

Proposition **2.4.14.** — *Let* $\overline{\mathcal{M}}^{\{J_\tau\}}_{g,1,4g}(A, z_\infty, \mathbf{z})$ *be the compactification of* $\mathcal{M}^{\{J_\tau\}}_{g,1,4g}(A, z_\infty, \mathbf{z})$. *Then*

$$\begin{aligned}\overline{\mathcal{M}}^{\{J_\tau\}}_{g,1,4g}&(A, z_\infty, \mathbf{z}) - \mathcal{M}^{\{J_\tau\}}_{g,1,4g}(A, z_\infty, \mathbf{z})\\ &= \mathcal{M}^{J_-}_{1,1,4}(A_-, z_\infty, \mathbf{z}_-) \times_{D_\mathfrak{n}} \mathcal{M}^{J_+}_{g-1,0,4g-4}(A_+, \mathbf{z}_+).\end{aligned}$$

Proof. — Consider a sequence $(u^{\tau_i}, x_{0,\tau_i}, \mathbf{x}_{\tau_i}) \in \mathcal{M}^{J_{\tau_i}}_{g,1,4g}(A, z_\infty, \mathbf{z})$. We first note that u^{τ_i} cannot degenerate as τ_i approaches a finite τ_*. This is as argued in Lemma 2.4.12. Hence we may assume that $\tau_i \to \infty$.

Let $u = u_- \cup u_0 \cup u_+$ be the limit of u^{τ_i} as $\tau_i \to \infty$, where $\mathrm{Im}(u_\pm) \subset \Sigma_\pm \times D^2$, $u_\pm$ has no components in $D_\mathfrak{n}$, and u_0 maps to $D_\mathfrak{n}$, and let $v_\pm = \pi_2 \circ u_\pm$. By the argument of Part (i) of the proof of Lemma 2.4.12, u cannot have disk components $\{z\} \times D^2$, since otherwise they would introduce extra intersection points with

$$\Sigma_\infty \times \{w\} = ((\Sigma_+ \sqcup \Sigma_-)/\sim) \times \{w\},$$

where $w \in int(D^2)$. In particular, this implies that $u_0 = \varnothing$.

Let $F_\pm$ be the domain of $u_\pm$. By Part (ii) of the proof of Lemma 2.4.12, $u_-(\partial F_-)$ does not intersect $\{z_\infty\} \times \partial D^2$. Let $\#(\partial F_\pm)$ be the number of boundary components of $F_\pm$.

We claim that $\#(\partial F_-) = 2$ and $\#(\partial F_+) = 2g-2$. First observe that $\#(\partial F_-) \geq 2$, since two boundary components are needed to map to $\widetilde{a}_i$, $i = 1, 2$; similarly $\#(\partial F_+) \geq 2g - 2$ since $z_3, \dots, z_{2g}, z_{2g+3}, \dots, z_{4g} \in \Sigma_+$.

The restriction of v_- to each component C of ∂F_- is either a positive degree map $C \to \partial D^2$ or is a constant map to a point $w \in \partial D^2$. If v_- maps C to a point $w \in \partial D^2$, then u_- maps an irreducible component of F_- to a fiber $\Sigma_- \times \{w\}$. This in turn implies that u_- has disk components, a contradiction. Hence the restriction of v_- to each C is a positive degree map. If $\#(\partial F_-) > 2$, then $\deg(v_-|_{\partial F_-}) > 2$. Similarly we obtain that $\deg(v_+|_{\partial F_+}) \geq 2g-2$. This implies that $\deg((v_+ \cup v_-)|_{\partial F}) > 2g$, which is a contradiction. Hence $\#(\partial F_0) = 2$ and $\#(\partial F_-) = 2g - 2$.

The above claim implies that

$$u_- \in \mathcal{M}^{J_-}_{1,1,4}(A_-, z_\infty, \mathbf{z}_-), \quad u_+ \in \mathcal{M}^{J_+}_{g-1,0,4g-4}(A_+, \mathbf{z}_+).$$

Moreover, the images of u_- and u_+ intersect $D_{\mathfrak{n}}$ at the same point because the Gromov limit is compact. This proves the inclusion

$$\begin{aligned}&\overline{\mathcal{M}}^{\{J_\tau\}}_{g,1,4g}(A, z_\infty, \mathbf{z}) - \mathcal{M}^{\{J_\tau\}}_{g,1,4g}(A, z_\infty, \mathbf{z})\\ &\quad\subseteq \mathcal{M}^{J_-}_{1,1,4}(A_-, z_\infty, \mathbf{z}_-) \times_{D_{\mathfrak{n}}} \mathcal{M}^{J_+}_{g-1,0,4g-4}(A_+, \mathbf{z}_+).\end{aligned}$$

The opposite inclusion follows from the usual gluing argument. □

By the usual gluing argument we obtain the following:

Corollary **2.4.15.**

$$G_3(\Sigma, \mathbf{z}) = G_3(\Sigma_-, \mathbf{z}_1) \times G_3(\Sigma_+, \mathbf{z}_+).$$

Proof. — By Proposition 2.4.14 we have

$$G_3(\Sigma, \mathbf{z}) = \#\left(\mathcal{M}^{J_-}_{1,1,4}(A_-, z_\infty, \mathbf{z}_-) \times_{D_{\mathfrak{n}}} \mathcal{M}^{J_+}_{g-1,0,4g-4}(A_+, \mathbf{z}_+)\right).$$

We have, by definition, $\#\mathcal{M}^{J_-}_{1,1,4}(A_-, z_\infty, \mathbf{z}_-) = G_3(T^2, \mathbf{z}_-)$. Moreover, every curve $u \in \mathcal{M}^{J_-}_{1,1,4}(A_-, z_\infty, \mathbf{z}_-)$ intersects $D_{\mathfrak{n}}$ at a unique point $(\mathfrak{n}, w) \in \Sigma_- \times D^2$. Now we observe that $\mathfrak{n}$ plays the role of z_∞ for the arcs $\widetilde{\mathbf{a}}$ in Σ_+ (i.e., it is the unique intersection point of all the arcs). Moreover, we can assume $w = 0$ after an automorphism of $\mathbf{CP}^1$ that fixes $+1$ and -1. This means that, for every curve $u_- \in \#\mathcal{M}^{J_-}_{1,1,4}(A_-, z_\infty, \mathbf{z}_-)$, there are $G_3(\Sigma_+, \mathbf{z}_+)$ curves in $\mathcal{M}^{J_+}_{g-1,0,4g-4}(A_+, \mathbf{z}_+)$ which intersect $D_{\mathfrak{n}}$ at the same point as u_-. □

2.4.5. *Two calculations on* $\mathbf{CP}^1 \times D^2$. — We now calculate two model situations which are key ingredients in the proof of Lemma 2.4.19 below. Note that all the holomorphic curves on $\mathbf{CP}^1 \times D^2$ that are considered below satisfy automatic transversality; see Hofer-Lizan-Sikorav [HLS, Theorem 2′].

Fix real numbers $a > b > 0$. Let $\mathcal{S}_1$ be the set of pairs (v_1, w), where v_1 is a degree 1 holomorphic map $D^2 \to \mathbf{CP}^1$ (more precisely, is a biholomorphism onto its image when restricted to *int*(D^2)) such that $v_1(\partial D^2) \subset \mathbf{R}^+$, $v_1(1) = a$, $v_1(-1) = b$, $v_1(0) = \infty$, and w is a point in D^2 such that $v_1(w) = 0$, and let C_1 be the set of points w for which there is some v_1 with $(v_1, w) \in \mathcal{S}_1$. Similarly, let $\mathcal{S}_2$ be the set of pairs (v_2, w), where v_2 is a degree 1 holomorphic map $D^2 \to \mathbf{CP}^1$ such that $v_2(\partial D^2) \subset \mathbf{R}$, $v_2(1) = a$, $v_2(-1) = b$, $v_2(0) = \infty$, and w is a point on D^2 such that $v_2(w) = -i$, and let C_2 be the set of points w for which there is some v_2 with $(v_2, w) \in \mathcal{S}_2$.

Let $\mathcal{R}_\theta$ be the restriction to D^2 of the radial ray which passes through $0 \in D^2$ and makes an angle of θ with the positive real axis.

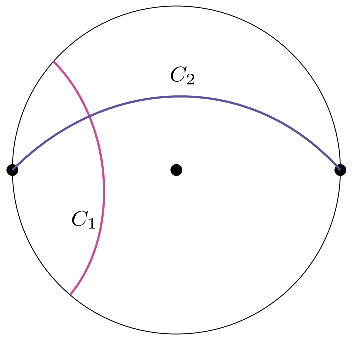


FIG. 2. — Rough pictures of curves C_1 and C_1 appearing in Lemmas 2.4.16 and 2.4.17. (Color figure online)

Lemma **2.4.16.** — *There exists* $0 < \theta_0 < \frac{\pi}{2}$ *such that* $C_1 \subset D^2$ *can be written as the image of a curve which is parametrized by* $\theta \in (\pi - \theta_0, \pi + \theta_0)$, $C_1(\theta) = C_1 \cap int(\mathcal{R}_\theta)$ *(here we are abusing notation and using* C_1 *for both the curve and its image), and*

$$\lim_{\theta \to (\pi+\theta_0)^-} C_1(\theta) = e^{i(\pi+\theta_0)}, \qquad \lim_{\theta \to (\pi-\theta_0)^+} C_1(\theta) = e^{i(\pi-\theta_0)}.$$

See Figure 2 for a rough picture of the curve C_1.

Proof. — By the Schwarz reflection principle, a map v_1 with $(v_1, w) \in \mathcal{S}_1$ extends to a degree 2 branched cover $\mathbf{CP}^1 \to \mathbf{CP}^1$ with two branch points which lie on $\mathbf{R}^+ \subset \mathbf{CP}^1$. Hence v_1 admits a factorization

$$D^2 \xrightarrow{f} \mathbf{H} \xrightarrow{\tilde{v}_1} \mathbf{CP}^1 \xrightarrow{g} \mathbf{CP}^1,$$

where $\mathbf{H} = \{\mathrm{Im}(z) \geq 0\}$ is the upper half plane, $f : D^2 \xrightarrow{\sim} \mathbf{H}$ is a fractional linear transformation, $\tilde{v}_1(z) = z^2$, and $g \in PSL(2, \mathbf{R})$. In order for $v_1(0) = \infty$ and $v_1(w) = 0$ to hold, f must map the ray $\mathcal{R}_\theta$ through 0 and w to the line $\{\mathrm{Re}(z) = 0\} \subset \mathbf{H}$. We may set $f(e^{i\theta}) = 0$ and $f(0) = i$; these conditions uniquely determine $f = f_\theta$. We leave it to the reader to verify that for each θ, there is a one-to-one correspondence between $(v_1, w) \in \mathcal{S}_1$ with $w \in \mathcal{R}_\theta$ and $w \in \mathcal{R}_\theta$ satisfying the following equality of cross ratios:

$$(\tilde{v}_1 \circ f_\theta(w), \tilde{v}_1 \circ f_\theta(1); \tilde{v}_1 \circ f_\theta(-1), -1) = (0, b; a, \infty). \tag{2.4.4}$$

Note that $\tilde{v}_1 \circ f_\theta(\mathcal{R}_\theta) = [-1, 0]$ and that there is at most one $w \in \mathcal{R}_\theta$ such that Equation (2.4.4) holds.

When $\theta = \pi$, $\tilde{v}_1 \circ f_\pi(1) = \infty$ and $\tilde{v}_1 \circ f_\pi(-1) = 0$. Hence there is a unique $w \in \mathcal{R}_\pi$ satisfying Equation (2.4.4). As θ moves from π to $\frac{3\pi}{2}$, the points $\tilde{v}_1 \circ f_\theta(1)$ and $\tilde{v}_1 \circ f_\theta(-1)$ approach each other and become equal when $\theta = \frac{3\pi}{2}$. Hence there exists $0 < \theta_0 < \frac{\pi}{2}$ such that there is no $w \in \mathcal{R}_\theta$ for $\pi + \theta_0 < \theta < \frac{3\pi}{2}$ and there is a unique $w \in \mathcal{R}_\theta$ for $\pi \leq \theta < \pi + \theta_0$. The situation of $\theta \in (\frac{\pi}{2}, \pi]$ is symmetric. The lemma then follows. $\square$

Lemma **2.4.17.** — *There exists a parametrization of* $C_2 \subset D^2$ *by* $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ *such that*

$$\lim_{\theta \to \frac{\pi}{2}+} C_2(\theta) = 1, \qquad \lim_{\theta \to \frac{3\pi}{2}-} C_2(\theta) = -1.$$

See Figure 2 for a rough picture of the curve C_2.

Proof. — As in the proof of Lemma 2.4.16, v_2 can be factored as

$$D^2 \xrightarrow{f_\theta} \mathbf{H} \xrightarrow{\tilde{v}_2} \mathbf{CP}^1 \xrightarrow{g} \mathbf{CP}^1,$$

where $\tilde{v}_2(z) = z^2$ and f_θ maps 0 to i and $\mathcal{R}_\theta$ to $\{\mathrm{Re}(z) = 0, 0 \leq \mathrm{Im}(z) \leq 1\} \subset \mathbf{H}$. Note that there is a unique fractional linear transformation g such that $g(-1) = \infty$, $g(0) = b$, and $g(\infty) = a$. For each $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, there exists $(v_2, C_2(\theta)) \in \mathcal{S}_2$ such that $C_2(\theta)$ is in one of the half-disks of D^2 divided by $\mathcal{R}_\theta \cup \mathcal{R}_{\theta+\pi}$. As a reference point, $C_2(\pi)$ is in the upper half-disk. As θ approaches $\frac{\pi}{2}$ from above, the corresponding v_2 sends -1 and 1 to arbitrarily close points. Hence $\lim_{\theta \to \frac{\pi}{2}+} C_2(\theta) = 1$. Similarly, $\lim_{\theta \to \frac{3\pi}{2}-} C_2(\theta) = -1$. □

The following lemma is immediate from Lemmas 2.4.16 and 2.4.17; the key ingredient is that the endpoints of C_1 and C_2 alternate on ∂D^2:

Lemma **2.4.18.** — *The total count of intersection points between* C_1 *and* C_2 *is* 1 *modulo* 2.

2.4.6. *Reduction to* $\mathbf{CP}^1 \times D^2$. — We now explain how to further reduce from T^2 to S^2. We pinch T^2 along three parallel, disjoint, essential closed curves $\gamma_1, \gamma_2, \gamma_3$ to obtain a "sausage"

$$\big((\Sigma_1, w_1, w_2') \sqcup (\Sigma_2, w_2, w_3') \sqcup (\Sigma_3, w_3, w_1')\big) / \sim, \tag{2.4.5}$$

where $\Sigma_i \simeq S^2$, $i = 1, 2, 3$, and $w_i \sim w_i'$, $i = 1, 2, 3$. More precisely, pick an oriented identification $T^2 \simeq \mathbf{R}^2/\mathbf{Z}^2$ with coordinates (x, y) so that $\overline{a}_1 = \{y = 0\}$ and $\overline{a}_2 = \{x = 0\}$. Then $\gamma_1 = \{x = \frac{1}{4}\}$, $\gamma_2 = \{x = \frac{1}{2}\}$, and γ_3 is obtained from $\{x = \frac{3}{4}\}$ by applying a finger move along the arc $[\frac{3}{4}, 1+\varepsilon] \times \{\frac{1}{2}\}$ so that γ_3 has two intersections $y = \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon$ with $\widehat{a}_2$. We also assume that $z_1, z_3 \in \widehat{a}_1$ lie on $\{\frac{1}{4} < x < \frac{1}{2}\}$ and $z_2, z_4 \in \widehat{a}_2$ lie on $\{\frac{1}{2} - \varepsilon < y < \frac{1}{2} + \varepsilon\}$. Then Σ_1 is obtained from the closure of the connected component of $T^2 - \cup_{i=1}^3 \gamma_i$ which is bounded by (copies of) γ_1 and γ_2, by identifying all of γ_1 to w_1 and all of γ_2 to w_2'. The other components Σ_2 and Σ_3 are defined similarly. In particular, $z_\infty \in \Sigma_3$. See Figure 3.

Lemma **2.4.19.** — $G_3(T^2, \mathbf{z}) = 1$*, where* $\mathbf{z} = \{z_1, \dots, z_4\}$.

Proof. — We degenerate the Riemann surfaces $\Sigma^\tau = (T^2, i_\tau)$, $\tau \to \infty$, by pinching along $\gamma_1 \cup \gamma_2 \cup \gamma_3$. Then a sequence of holomorphic maps

$$u^\tau : (F, j_\tau) \to (\Sigma^\tau \times D^2, J_\tau)$$

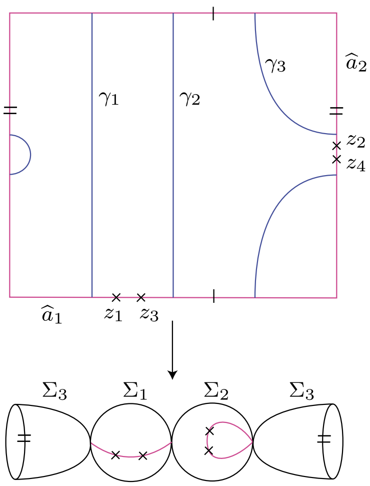


FIG. 3. — The top diagram is the torus T^2, where the sides are identified and the top and the bottom are identified. The arrow indicates the projection of T^2 onto $\Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3/\sim$, given in (2.4.5). The components of $\widetilde{a}_i = \overline{a}_i/\sim$ which do not contain points in the set $\{z_1, \dots, z_4\}$ are not drawn in Σ_i. Also the two disks on the left and the right of the bottom diagram are glued into Σ_3. (Color figure online)

in $\mathcal{M}^{J_\tau}_{1,4}(A, z_\infty, \mathbf{z})$ converges to a nodal curve (u_1, u_2, u_3), where

$$u_i : F_i \to \Sigma_i \times D^2, \quad i = 1, 2, 3,$$

and $F_1 = F_1 = D^2$, $F_3 = \mathbf{CP}^1$. This is because $z_1, z_3 \in \Sigma_1$ and $z_2, z_4 \in \Sigma_2$, and the total number of boundary components $\sum_{i=1}^3 \#(\partial F_i)$ is equal to two by the argument in Section 2.4.4. Now, u_3 must have image $\Sigma_3 \times \{0\}$ since $z_\infty \in \Sigma_3$. The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ for $v_i = \pi_2 \circ u_i$, $i = 1, 2$, were determined in Section 2.4.5. The gluing of intersecting curves u_1 and u_2 is given by the signed intersection number of C_1 and C_2, which is 1 modulo 2 by Lemma 2.4.18. □

Proof of Theorem 2.4.2. — By Lemma 2.4.13 we can write $G_3(\Sigma) = G_3(\Sigma, \mathbf{z})$ for generic $\mathbf{z}$. By Corollary 2.4.15, if Σ has genus g, then $G(\Sigma) = G(T^2)^g$. Then Lemma 2.4.19 implies that $G_3(\Sigma) = 1$. □

3. Homotopy of cobordisms I

In this section and the next we prove Theorem 1.0.1. The chain homotopies that appear in the proof of Theorem 1.0.1 are induced by homotopies of cobordisms $\overline{W}^{\pm}_\tau$ and $\overline{W}^{\mp}_\tau$ which are parametrized by $\tau \in \mathbf{R}$. In this section we treat $\overline{W}^{\pm}_\tau$, leaving $\overline{W}^{\mp}_\tau$ for Section 4. If $\pm$ is understood (as it will be in the rest of this section), then it will be omitted.

We now give a brief informal description of $\overline{W}_\tau$, leaving precise definitions for later. The base B_τ of $\overline{W}_\tau$ is biholomorphic to an annulus with one puncture on each boundary component; the neighborhoods of the punctures are viewed as strip-like ends. As $\tau \to +\infty$, $\overline{W}_\tau$ degenerates to the stacking of $\overline{W}_+$ "on top of" $\overline{W}_-$, where $\overline{W}_+$ and $\overline{W}_-$ are used in the definitions of the chain maps Φ and Ψ. (This is the reason why there is a $\pm$ in $\overline{W}_\tau^{\pm}$.) On the other hand, as $\tau \to -\infty$, $\overline{W}_\tau$ degenerates to $\overline{W}_{-\infty}$, whose base $B_{-\infty}$ is (more or less) given by:

$$(\mathbf{3.0.1}) \qquad B_{-\infty} = ((\mathbf{R} \times [0,1]) \sqcup D)/\sim,$$

where $D = \{|z| \leq 1\} \subset \mathbf{C}$ and $\sim$ identifies $(0,1) \in \mathbf{R} \times [0,1]$ with $1 \in D$ and $(0,0) \in \mathbf{R} \times [0,1]$ with $-1 \in D$.

3.1. *Construction of the homotopy of cobordisms for* $\Psi \circ \Phi$.

3.1.1. *Recollections.* — In this subsection we recall some notation from [I].

Recall that S is a compact oriented surface of genus g with connected boundary (a page of an open book (S, h)), $\overline{S} = S \cup D^2$ is a closed surface obtained by capping off S, $\overline{h} = \overline{h}_m : \overline{S} \xrightarrow{\sim} \overline{S}$ is an extension of h which is dependent on the integer $m \gg 0$ as in Section I.5.1.2, and $\overline{\omega}$ is the area form on $\overline{S}$ from Section I.5.1.2 which is invariant under $\overline{h}$. Also z_∞ is the origin $\rho = 0$ of $D^2 = \{\rho \leq 1\}$ with polar coordinates (ρ, ϕ).

The mapping tori

$$N = (S \times [0,2])/(x,2) \sim (h(x),0), \quad \overline{N} = (\overline{S} \times [0,2])/(x,2) \sim (\overline{h}(x),0)$$

were defined in Section I.5.1. Let $W = \mathbf{R} \times [0,1] \times S$, $\overline{W} = \mathbf{R} \times [0,1] \times \overline{S}$, $W' = \mathbf{R} \times N$, and $\overline{W'} = \mathbf{R} \times \overline{N}$; they admit symplectic fibrations with fibers diffeomorphic to S, $\overline{S}$, S, and $\overline{S}$, respectively. We also have the symplectic fibrations $\pi_{B_+} : W_+ \to B_+$, $\overline{\pi}_{B_+} : \overline{W}_+ \to B_+$, and $\overline{\pi}_{B_-} : \overline{W}_- \to B_-$ from Sections I.5.1.1 and I.5.1.2, with fibers diffeomorphic to S, $\overline{S}$, and $\overline{S}$, respectively.

The fibration W (or $\overline{W}$) was used in the definition of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ and the fibration W' (or $\overline{W'}$) in the definition of $PFC_{2g}(N)$. The positive end of $\overline{W}_+$ and the negative end of $\overline{W}_-$ agree with those of $\overline{W}$ and the negative end of $\overline{W}_+$ and the positive end of $\overline{W}_-$ agree with those of $\overline{W}'$. The fibrations π_{B_+} and $\overline{\pi}_{B_-}$ were used in the definitions of the chain maps

$$\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N),$$
$$\Psi : PFC_{2g}(N) \to \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})).$$

3.1.2. *Definition of the family* $(\overline{W}_\tau, \overline{\mathfrak{m}}(\tau))$. — For each $r \in [2,\infty)$, consider the fibration

$$\pi_r : \mathbf{R} \times \overline{N}_r \to \mathbf{R} \times (\mathbf{R}/r\mathbf{Z}),$$

where

$$\overline{\mathrm{N}}_r = (\overline{\mathrm{S}} \times [0,r])/(x,r) \sim (\overline{h}(x),0)$$

and (s,t) are coordinates on $\mathbf{R} \times (\mathbf{R}/r\mathbf{Z})$. For each $l,r \in [2,\infty)$, define $\overline{\mathrm{W}}_{l,r} = \pi_r^{-1}(\mathrm{B}_{l,r})$, where the base $\mathrm{B}_{l,r}$ is obtained by smoothing the corners of

$$\{-l \le s \le l\} \cup \{0 \le t \le 1\} \subset \mathbf{R} \times (\mathbf{R}/r\mathbf{Z}).$$

Next choose a function

$$\eta = (l,r) : \mathbf{R} \to [2,\infty) \times [2,\infty), \tag{3.1.1}$$

which is obtained by smoothing

$$\eta_0(\tau) = \begin{cases} (\tau+2,2), & \text{for } \tau \ge 0; \\ (2,2-\tau), & \text{for } \tau \le 0; \end{cases}$$

near $\tau = 0$. We then let $\overline{\mathrm{W}}_\tau = \overline{\mathrm{W}}_{\eta(\tau)}$ and $\mathrm{B}_\tau = \mathrm{B}_{\eta(\tau)}$. Let $\pi_{\mathrm{B}_\tau} : \overline{\mathrm{W}}_\tau \to \mathrm{B}_\tau$ be the projection along $\{(s,t)\} \times \overline{\mathrm{S}}$.

3.1.3. *Neck-stretching.* — As $\tau \to +\infty$, the cobordism $\overline{\mathrm{W}}_\tau$ approaches the concatenation of $\overline{\mathrm{W}}_+$ and $\overline{\mathrm{W}}_-$; see Figure 4. On the other hand, for $\tau \ll 0$ we can view the rectangle $[-2,2] \times [\frac{3}{2}, \frac{3}{2} - \tau] \subset \mathrm{B}_\tau$ as a neck and by taking $\tau \to -\infty$ we are stretching along this neck and degenerating the cobordism $\overline{\mathrm{W}}_\tau$ into a 2-component manifold $\overline{\mathrm{W}}_{-\infty} = \overline{\mathrm{W}}_{-\infty,1} \cup \overline{\mathrm{W}}_{-\infty,2}$, which we describe now; see Figure 5.

The base of $\overline{\mathrm{W}}_{-\infty} = \overline{\mathrm{W}}_{-\infty,1} \cup \overline{\mathrm{W}}_{-\infty,2}$ is $\mathrm{B}_{-\infty} = \mathrm{B}_{-\infty,1} \cup \mathrm{B}_{-\infty,2}$, where $\mathrm{B}_{-\infty,1}$ is obtained from $\{-2 \le s \le 2\} \cup \{0 \le t \le 1\} \subset \mathbf{R}^2$ by smoothing the corners and $\mathrm{B}_{-\infty,2} = [-2,2] \times \mathbf{R}$. Here both $\mathbf{R}^2$ and $[-2,2] \times \mathbf{R}$ have coordinates (s,t). The component $\mathrm{B}_{-\infty,1}$ has four strip-like ends: the ends $s \to +\infty$, $s \to -\infty$, $t \to +\infty$, $t \to -\infty$ will be referred to as the top, bottom, left, and right ends. The component $\mathrm{B}_{-\infty,2}$ has two strip-like ends: the ends $t \to +\infty$ and $t \to -\infty$ will be referred to as the left and right ends. As usual, $\mathrm{B}_{-\infty}$ is endowed with identifications of the compactifications of the strip-like ends. More precisely, if the compactification $\check{\mathrm{B}}_{-\infty,1}$ is obtained from $\mathrm{B}_{-\infty,1}$ by attaching $\{\pm\infty\} \times [0,1]$ and $[-2,2] \times \{\pm\infty\}$ and the compactification $\check{\mathrm{B}}_{-\infty,2}$ is obtained from $\mathrm{B}_{-\infty,2}$ by attaching $[-2,2] \times \{\pm\infty\}$, then we identify $(s,\pm\infty) \in \check{\mathrm{B}}_{-\infty,1}$ with $(s,\mp\infty) \in \check{\mathrm{B}}_{-\infty,2}$.

Let $\overline{\mathrm{W}}_{-\infty,i} = \mathrm{B}_{-\infty,i} \times \overline{\mathrm{S}}$ for $i = 1,2$. The 2-component building $\overline{\mathrm{W}}_{-\infty} = \overline{\mathrm{W}}_{-\infty,1} \cup \overline{\mathrm{W}}_{-\infty,2}$ is endowed with identifications of compactifications of the ends. The compactification $\check{\overline{\mathrm{W}}}_{-\infty,1}$ is obtained from $\overline{\mathrm{W}}_{-\infty,1}$ by attaching $\{\pm\infty\} \times [0,1] \times \overline{\mathrm{S}}$ and $[-2,2] \times \{\pm\infty\} \times \overline{\mathrm{S}}$, the compactification $\check{\overline{\mathrm{W}}}_{-\infty,2}$ is obtained from $\overline{\mathrm{W}}_{-\infty,2}$ by attaching $[-2,2] \times \{\pm\infty\} \times \overline{\mathrm{S}}$, and we identify $(s,+\infty,x) \in \check{\overline{\mathrm{W}}}_{-\infty,1}$ with $(s,-\infty,x) \in \check{\overline{\mathrm{W}}}_{-\infty,2}$

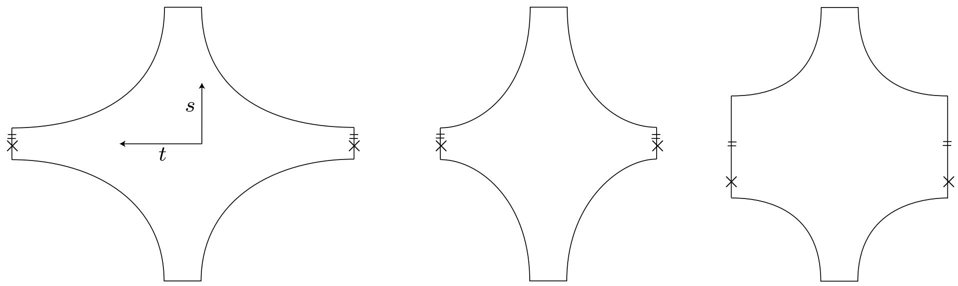


FIG. 4. — The bases of the family $\overline{W}_\tau$. The parameter τ increases as we go to the right. The sides are identified in this picture, as indicated. The location of $\overline{\mathfrak{m}}^b(\tau)$ is indicated by $\times$

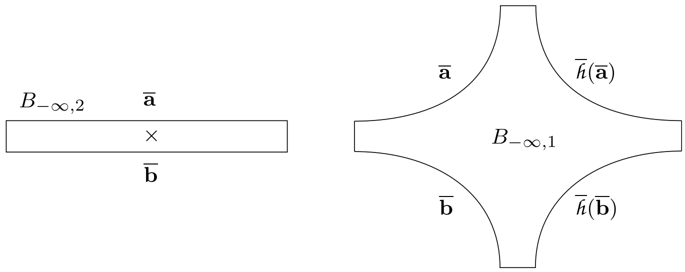


FIG. 5. — The bases of $\overline{W}_{-\infty,2}$ to the left and $\overline{W}_{-\infty,1}$ to the right

and $(s, +\infty, x) \in \check{\overline{W}}_{-\infty,2}$ with $(s, -\infty, \overline{h}(x)) \in \check{\overline{W}}_{-\infty,1}$. We write $\pi_{B_{-\infty,i}} : \overline{W}_{-\infty,i} \to B_{-\infty,i}$ for the projection along $\overline{S}$.

We write $cl(B_\tau)$, $cl(B_+)$, $cl(B_-)$ to denote the compactifications of B_τ, B_+, B_-, obtained by adjoining a point at infinity for each end $s = \pm\infty$. Similarly, we write $cl(B_{-\infty,1})$ for the compactification of $B_{-\infty,1}$, obtained by adding 4 points $s = \pm\infty$ and $t = \pm\infty$, and $cl(B_{-\infty,2})$ for the compactification of $B_{-\infty,2}$, obtained by adding 2 points $t = \pm\infty$.

3.1.4. *Marked points.* — We choose a 1-parameter family of marked points

$$\overline{\mathfrak{m}}(\tau) = (\overline{\mathfrak{m}}^b(\tau), \overline{\mathfrak{m}}^f(\tau)) = ((-l(\tau) + 2, (r(\tau) + 1)/2), z_\infty) \in \overline{W}_\tau.$$

Observe that the map $\tau \mapsto \overline{\mathfrak{m}}^b(\tau)$ is a smoothing of the function

$$\tau \mapsto \begin{cases} \left(-\tau, \frac{3}{2}\right), & \text{for } \tau \geq 0; \\ \left(0, \frac{3-\tau}{2}\right), & \text{for } \tau \leq 0, \end{cases}$$

and therefore $\overline{\mathfrak{m}}^b(\tau)$ is always on the axis $\{t = (r(\tau) + 1)/2\}$ of involution for the involution $(s, t) \mapsto (s, 1 - t)$ of B_τ and always at a fixed distance from the lower component of ∂B_τ. See Figure 4 for a schematic representation of B_τ and the placement of $\overline{\mathfrak{m}}^b(\tau)$.

The following also hold:

(i) as $\tau \to +\infty$, $\overline{\mathfrak{m}}(\tau)$ limits to $\overline{\mathfrak{m}}(+\infty) = (\overline{\mathfrak{m}}^b(+\infty), \overline{\mathfrak{m}}^f(+\infty))$, where $\overline{\mathfrak{m}}^b(+\infty) = (0, \frac{3}{2}) \in B_-$ and $\overline{\mathfrak{m}}^f(+\infty) = z_\infty$;
(ii) as $\tau \to -\infty$, $\overline{\mathfrak{m}}(\tau)$ limits to $\overline{\mathfrak{m}}(-\infty) = (\overline{\mathfrak{m}}^b(-\infty), \overline{\mathfrak{m}}^f(-\infty))$, where $\overline{\mathfrak{m}}^b(-\infty) = (0, 0) \in B_{-\infty,2}$ and $\overline{\mathfrak{m}}^f(-\infty) = z_\infty$.

Convention **3.1.1.** — In this section and the next, $\overline{\mathfrak{m}}$ will denote a 1-parameter family (as opposed to a single marked point).

Let us write $\mathcal{L}_{t_0}$ for the locus $\{t = t_0\}$, viewed as a subset of B_+, B_-, B_τ, as appropriate. Of particular interest is $\mathcal{L}_{(r(\tau)+1)/2}$, which passes through $\overline{\mathfrak{m}}^b(\tau) \in B_\tau$ or $\overline{\mathfrak{m}}^b(+\infty) \in B_-$.

3.1.5. *Stable Hamiltonian structures and symplectic forms.* — We first consider $\overline{W}_\tau$. The stable Hamiltonian structure on $\overline{N}_{r(\tau)} = (\overline{S} \times [0, r(\tau)])/\sim$ is obtained from $(dt, \overline{\omega})$ on $\overline{S} \times [0, r(\tau)]$ by passing to the quotient, where $\overline{\omega}$ is the area form on $\overline{S}$ from Section I.5.1.2. The 2-plane field is $\xi_\tau = \ker dt = T\overline{S}$ and the Hamiltonian vector field is $\overline{R}_\tau = \partial_t$. The symplectic form $\overline{\Omega}_\tau$ is obtained from the symplectic form $ds \wedge dt + \overline{\omega}$ on $\mathbf{R} \times \overline{S} \times [0, r(\tau)]$ by passing to the quotient $\mathbf{R} \times \overline{N}_{r(\tau)}$ and then restricting to $\overline{W}_\tau$.

Next we consider $\overline{W}_{-\infty} = \overline{W}_{-\infty,1} \cup \overline{W}_{-\infty,2}$. Let $\omega_{-\infty,1}$ be the restriction of the area form $ds \wedge dt$ on $\mathbf{R}^2$ to $B_{-\infty,1}$ and let $\omega_{-\infty,2} = ds \wedge dt$ on $B_{-\infty,2} = [-2, 2] \times \mathbf{R}$. Then we set $\overline{\Omega}_{-\infty,i} = \omega_{-\infty,i} + \overline{\omega}$. The stable Hamiltonian structure at the $s \to \pm\infty$ ends of $\overline{W}_{-\infty,1}$ are given by $(dt, \overline{\omega})$ and the stable Hamiltonian structure at the $t \to \pm\infty$ ends of $\overline{W}_{-\infty,1}$ are given by $(ds, \overline{\omega})$.

3.2. *Holomorphic curves and moduli spaces.*

3.2.1. *Lagrangian boundary conditions.* — Recall that the monodromy map $\overline{h} = \overline{h}_m : \overline{S} \to \overline{S}$ depends on the integer $m \gg 0$. Also $\overline{\mathbf{a}} = \{\overline{a}_1, \dots, \overline{a}_{2g}\}$ is the extension of the basis $\mathbf{a} = \{a_1, \dots, a_{2g}\}$ to $\overline{S}$ so that $\overline{a}_i = a_i \cup \overline{a}_{i,0,} \cup \overline{a}_{i,1}$, as described in Section I.5.2.2. We also note that $\overline{\mathbf{a}}$ depends on m.

We first describe the pushoff $\overline{\mathbf{b}}$ of $\overline{\mathbf{a}}$ which also depends on m: Let

$$\varepsilon_0 = \varepsilon_0(m) = \tfrac{2\pi}{mK(m)} > 0,$$

where $K(m)$ is a positive integer such that $\lim_{m\to\infty} K(m) = \infty$, and let $\overline{b}_i$ be a ε_0-close transverse pushoff of $\overline{a}_i$ which satisfies the following:

- in a neighborhood of z_∞, $\overline{b}_i$ is obtained from $\overline{a}_i$ by a $-\varepsilon_0$-rotation; and
- $\overline{a}_i$ and $\overline{b}_i$ intersect at three points $x^\#_{i1}$, $x^\#_{i2}$ and $x^\#_{i3}$ (besides at z_∞); $x^\#_{i2} \in int(S)$ and $x^\#_{i1}, x^\#_{i3} \in \overline{S} - S$.

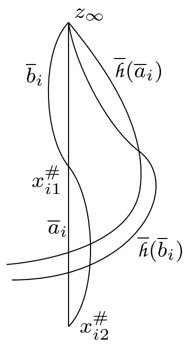


FIG. 6. — The arcs $\overline{a}_i$, $\overline{b}_i$, $\overline{h}(\overline{a}_i)$, and $\overline{h}(\overline{b}_i)$ near z_∞

See Figure 6. We also write $\overline{b}_{i,j}$ for the portion of $\overline{b}_i$ analogous to $\overline{a}_{i,j}$. When we want to signal that z_∞ is an intersection point of $\overline{a}_i$ and $\overline{b}_i$, then we write it as $z_\infty^{\#}$.

Let us write $\phi(\overline{a}_{i,j})$ for the ϕ-coordinate of the portion of $\overline{a}_{i,j}$ near z_∞, subject to the condition $0 \leq \phi(\overline{a}_{i,j}) < 2\pi$; similarly define $\phi(\overline{b}_{i,j})$, etc. Then, near z_∞,

$$(\mathbf{3.2.1}) \qquad 0 < \phi(\overline{b}_{i,j}) < \phi(\overline{a}_{i,j}) < \phi(\overline{h}(\overline{b}_{i,j})) < \phi(\overline{h}(\overline{a}_{i,j})) < c(m),$$

where $c(m) \to 0$ as $m \to \infty$; cf. Section I.5.2.2.

Remark **3.2.1.** — In view of the choices of $\overline{a}_{i,j}$ and $\overline{h}(\overline{a}_{i,j})$ from Section I.5.2.2 and the choices of $\overline{b}_{i,j}$ above, as $m \to \infty$:

– $\phi(\overline{a}_{i,j}) - \phi(\overline{b}_{i,j}) \to 0$ the fastest;
– $\phi(\overline{h}(\overline{a}_{i,j})) - \phi(\overline{a}_{i,j}) \to 0$ the next fastest;
– $\phi(\overline{a}_{i,j}) - \phi(\overline{a}_{i',j'}) \to 0$ the slowest if $(i,j) \neq (i',j')$.

The symplectic fibration

$$\pi_{B_\tau} : (\overline{W}_\tau, \overline{\Omega}_\tau) \to (B_\tau, ds \wedge dt)$$

induces a symplectic connection, defined as the $\overline{\Omega}_\tau$-orthogonal of the tangent plane to the fibers. We place a copy of $\overline{\mathbf{a}}$ on the fiber $\pi_{B_\tau}^{-1}(s,1)$ with $s > l(\tau)$ and use the symplectic connection to parallel transport $\overline{\mathbf{a}}$ along $\partial\overline{W}_\tau$. (Note that $\partial\overline{W}_\tau$ is equal to the *vertical boundary* $\partial_v\overline{W}_\tau := \pi_{B_\tau}^{-1}(\partial B_\tau)$.) This gives us a singular Lagrangian submanifold $L_{\overline{\mathbf{a}}}^{\tau,+}$. (Note that $\overline{\mathbf{a}}$ is a singular Lagrangian submanifold of $\overline{S}$ with singularity z_∞ and is a union of radial rays in a neighborhood of z_∞. Hence the singular set of $L_{\overline{\mathbf{a}}}^{\tau,+}$ and its neighborhood in $L_{\overline{\mathbf{a}}}^{\tau,+}$ are obtained from those of $\overline{\mathbf{a}}$ by multiplying with the upper boundary of B_τ.) Similarly, we place a copy of $\overline{\mathbf{b}}$ on the fiber $\pi_{B_\tau}^{-1}(s,1)$ with $s < -l(\tau)$ and use the symplectic connection to parallel transport $\overline{\mathbf{b}}$ along $\partial\overline{W}_\tau$ to construct the singular Lagrangian submanifold $L_{\overline{\mathbf{b}}}^{\tau,-}$. The Lagrangian submanifolds $L_{\widehat{\mathbf{a}}}^{\tau,+}$, $L_{\widehat{a}_i}^{\tau,+}$, $L_{\widehat{\mathbf{b}}}^{\tau,-}$, $L_{\widehat{b}_i}^{\tau,-}$, etc. are defined similarly, where $\widehat{\mathbf{a}} = \{\widehat{a}_1, \dots, \widehat{a}_{2g}\}$ and $\widehat{a}_i = \overline{a}_i - \{z_i\}$ were defined in Section I.5.2.2 and $\widehat{\mathbf{b}}$ and $\widehat{b}_i$ are defined analogously.

On $(\overline{W}_{-\infty}, \overline{\Omega}_{-\infty})$, we define the singular Lagrangian submanifolds as follows: Let us write

$$\partial B_{-\infty,1} = \sqcup_{i=1}^4 \partial_i B_{-\infty,1},$$

where the boundary components, in order from $i = 1$ to $i = 4$, satisfy

$$s > 0, t > 1/2; \quad s < 0, t > 1/2; \quad s < 0, t < 1/2; \quad \text{and } s > 0, t < 1/2.$$

Also let $\partial_i \overline{W}_{-\infty,1}$ be the component of $\partial \overline{W}_{-\infty,1}$ corresponding to $\partial_i B_{-\infty,1}$. Then we define:

– $L_{\overline{\mathbf{a}},1}^{-\infty,1} = \partial_1 B_{-\infty,1} \times \overline{\mathbf{a}}$ on $\partial_1 \overline{W}_{-\infty,1}$;
– $L_{\overline{\mathbf{b}},2}^{-\infty,1} = \partial_2 B_{-\infty,1} \times \overline{\mathbf{b}}$ on $\partial_2 \overline{W}_{-\infty,2}$;
– $L_{\overline{h}(\overline{\mathbf{b}}),3}^{-\infty,1} = \partial_3 B_{-\infty,1} \times \overline{h}(\overline{\mathbf{b}})$ on $\partial_3 \overline{W}_{-\infty,1}$;
– $L_{\overline{h}(\overline{\mathbf{a}}),4}^{-\infty,1} = \partial_4 B_{-\infty,1} \times \overline{h}(\overline{\mathbf{a}})$ on $\partial_4 \overline{W}_{-\infty,1}$;
– $L_{\overline{\mathbf{a}},+}^{-\infty,2} = \{2\} \times \mathbf{R} \times \overline{\mathbf{a}}$ on $\partial \overline{W}_{-\infty,2}$;
– $L_{\overline{\mathbf{b}},-}^{-\infty,2} = \{-2\} \times \mathbf{R} \times \overline{\mathbf{b}}$ on $\partial \overline{W}_{-\infty,2}$.

3.2.2. *Almost complex structures.* — Recall the space $\mathcal{J}_{\overline{W}}$ of admissible almost complex structures $\overline{J}$ on $\overline{W}$ from Definition I.5.3.2, the space $\mathcal{J}_{\overline{W}'}$ of adapted almost complex structures $\overline{J}'$ on $\overline{W}'$ from Definition I.5.3.14, and the spaces $\mathcal{J}_{\overline{W}_+}$ and $\mathcal{J}_{\overline{W}_-}$ of admissible almost complex structures $\overline{J}_+$ and $\overline{J}_-$ on $\overline{W}_+$ and $\overline{W}_-$ from Definition I.5.4.1.

Definition **3.2.2.** — *An almost complex structure* $\overline{J}_{-\infty,2}$ *on* $\overline{W}_{-\infty,2}$ *is* admissible *if the following hold:*

(1) $\overline{J}_{-\infty,2}$ *is t-invariant,* $\overline{J}_{-\infty,2}(\partial_t) = -\partial_s$*, and* $\overline{J}_{-\infty,2}(T\overline{S}) = T\overline{S}$*; and*
(2) *there exists* $\varepsilon > 0$ *such that* $\overline{J}_{-\infty,2}$ *restricts to the standard complex structure on the subsurface* $D_\varepsilon^2 = \{\rho \leq \varepsilon\} \subset \overline{S}$ *of each fiber.*

The space of all admissible $\overline{J}_{-\infty,2}$ *will be denoted by* $\mathcal{J}_{\overline{W}_{-\infty,2}}$.

Definition **3.2.3.** — *An almost complex structure* $\overline{J}_{-\infty,1}$ *on* $\overline{W}_{-\infty,1}$ *is* admissible *if the following hold:*

(1) *the projection* $\pi_{B_{-\infty,1}}$ *is* $(\overline{J}_{-\infty,1}, j_{-\infty,1})$*-holomorphic with respect to the standard complex structure* $j_{-\infty,1}$ *on* $B_{-\infty,1}$*;*
(2) *there exists* $\varepsilon > 0$ *such that* $\overline{J}_{-\infty,1}$ *restricts to the standard complex structure on the subsurface* $D_\varepsilon^2 = \{\rho \leq \varepsilon\} \subset \overline{S}$ *of each fiber;*

(3) *there exist* $\overline{J} \in \mathcal{J}_{\overline{W}}$ *and* $\overline{J}_{-\infty,2} \in \mathcal{J}_{\overline{W}_{-\infty,2}}$ *such that* $\overline{J}_{-\infty,1}$ *agrees with* $\overline{J}$ *on* $\overline{W} = \mathbf{R} \times [0,1] \times \overline{S}$, *with* $\overline{J}_{-\infty,2}$ *on* $[-2,2] \times [3,+\infty) \times \overline{S}$, *and with* $(id \times \overline{h})_*(\overline{J}_{-\infty,2})$ *on* $[-2,2] \times [-2,-\infty) \times \overline{S}$.

If (3) holds, we say that $\overline{J}_{-\infty,1}$ *is* compatible *with* $\overline{J} \in \mathcal{J}_{\overline{W}}$ *and* $\overline{J}_{-\infty,2} \in \mathcal{J}_{\overline{W}_{-\infty,2}}$. *The space of admissible* $\overline{J}_{-\infty,1}$ *will be denoted by* $\mathcal{J}_{\overline{W}_{-\infty,1}}$.

Definition **3.2.4.** — *An almost complex structure* $\overline{J}_{-\infty} = \overline{J}_{-\infty,1} \cup \overline{J}_{-\infty,2}$ *on* $\overline{W}_{-\infty} = \overline{W}_{-\infty,1} \cup \overline{W}_{-\infty,2}$ *is* admissible *if* $\overline{J}_{-\infty,i} \in \mathcal{J}_{\overline{W}_{-\infty,i}}$ *for* $i = 1, 2$ *and* $\overline{J}_{-\infty,1}$ *is compatible with* $\overline{J}_{-\infty,2}$. *The space of admissible* $\overline{J}_{-\infty}$ *will be denoted by* $\mathcal{J}_{\overline{W}_{-\infty}}$.

Definition **3.2.5.** — *An almost complex structure* $\overline{J}_\tau$ *on* $\overline{W}_\tau$ *is* admissible *if the following hold:*

(1) *the projection* π_{B_τ} *is* $(\overline{J}_\tau, j_\tau)$*-holomorphic with respect to the standard complex structure* j_τ *on* B_τ*;*
(2) *there exists* $\varepsilon > 0$ *such that* $\overline{J}_\tau$ *restricts to the standard complex structure on the subsurface* $D^2_\varepsilon = \{\rho \le \varepsilon\} \subset \overline{S}$ *of each fiber;*
(3) *if* $\tau \ge 0$, *then* $\overline{J}_\tau$ *is the restriction of some* $\overline{J}' \in \mathcal{J}_{\overline{W}'}$*;*
(4) *if* $\tau \le 0$, *then* $\overline{J}_\tau$ *agrees with some* $\overline{J} \in \mathcal{J}_{\overline{W}}$ *on* $\overline{W} = \mathbf{R} \times [0,1] \times \overline{S}$ *and with some* $\overline{J}_{-\infty,2} \in \mathcal{J}_{\overline{W}_{-\infty,2}}$ *on* $[-2,2] \times [3, r(\tau) - 2] \times \overline{S}$ *(provided* $r(\tau) \ge 5$*).*

If (3) holds, we say that $\overline{J}_\tau$ *is* compatible *with* $\overline{J}' \in \mathcal{J}_{\overline{W}'}$ *and if (4) holds, we say that* $\overline{J}_\tau$ *is* compatible *with* $\overline{J} \in \mathcal{J}_{\overline{W}}$ *and* $\overline{J}_{-\infty,2} \in \mathcal{J}_{\overline{W}_{-\infty,2}}$. *The space of all admissible* $\overline{J}_\tau$ *on* $\overline{W}_\tau$ *will be denoted by* $\mathcal{J}_{\overline{W}_\tau}$.

Definition **3.2.6.** — *A family* $\{\overline{J}_\tau \in \mathcal{J}_{\overline{W}_\tau}\}_{\tau \in \mathbf{R}}$ *of almost complex structures is* admissible *if there exist* $\overline{J}' \in \mathcal{J}_{\overline{W}'}$, $\overline{J} \in \mathcal{J}_{\overline{W}}$, $\overline{J}_+ \in \mathcal{J}_{\overline{W}_+}$, $\overline{J}_- \in \mathcal{J}_{\overline{W}_-}$ *and* $\overline{J}_{-\infty} = \overline{J}_{-\infty,1} \cup \overline{J}_{-\infty,2} \in \mathcal{J}_{\overline{W}_{-\infty}}$ *such that the following hold:*

(1) $\overline{J}_\tau$ *converges to* $\overline{J}_{-\infty}$ *as* $\tau \to -\infty$*;*
(2) $\overline{J}_\tau$ *converges to* $\overline{J}_+$ *and* $\overline{J}_-$ *as* $\tau \to +\infty$*;*
(3) $\overline{J}_+$ *and* $\overline{J}_-$ *are compatible with* $\overline{J}'$ *and* $\overline{J}$*; and*
(4) $\overline{J}_\tau$ *is compatible with* $\overline{J}$ *and* $\overline{J}_{-\infty,2}$ *for* $\tau \le 0$ *and with* $\overline{J}'$ *for* $\tau \ge 0$.

The convergence of almost complex structures is to be understood in the sense of neck-stretching as in [BEHWZ, Section 3.4]. The space of all admissible $\{\overline{J}_\tau \in \mathcal{J}_{\overline{W}_\tau}\}_{\tau \in \mathbf{R}}$ *will be denoted by* $\overline{\mathcal{I}}$.

3.2.3. *Some notation and conventions.* — We now collect some notation and conventions.

Notation **3.2.7** *(Tuples and orbit sets).* — When we write a tuple of $\overline{\mathbf{a}} \cap \overline{h}(\overline{\mathbf{a}})$ as $\mathbf{y}$ or an orbit set of $\overline{N}$ as $\boldsymbol{\gamma}$ (with possible superscripts, subscripts and other decorations), it is assumed that $\mathbf{y} \subset S$ and $\boldsymbol{\gamma} \subset N$. In particular, $\mathbf{y}$ and $\boldsymbol{\gamma}$ do not contain any multiples of z_∞ or δ_0.

Notation **3.2.8** *(Sections at* ∞*).* — The sections $\{\rho = 0\}$ of $\overline{W}$, $\overline{W'} = \mathbf{R} \times \overline{N}$, $\overline{W}_\tau$, $\overline{W}_+$, $\overline{W}_-$ and $\overline{W}_{-\infty,i}$ are holomorphic with respect to almost complex structures in $\mathcal{J}_{\overline{W}}$, $\mathcal{J}_{\overline{W'}}$, $\mathcal{J}_{\overline{W}_\tau}$, etc. They are called *sections at* ∞ and are denoted by σ_∞, σ'_∞, σ^τ_∞, σ^+_∞, σ^-_∞ and $\sigma^{-\infty,i}_\infty$.

Notation **3.2.9** *(The intersection numbers* $n^*(\overline{u})$ *and* $n^{*,alt}(\overline{u})$*).* — Let δ_{ρ_0,ϕ_0} be a closed orbit of the Hamiltonian vector field ∂_t which lies on the torus

$$\{\rho = \rho_0\} \subset \overline{N}_r = (\overline{S} \times [0, r])/(x, 1) \sim (\overline{h}(x), 0)$$

for appropriate r and $\rho_0 > 0$ sufficiently small and which passes through the point $(t, \rho, \phi) = (0, \rho_0, \phi_0)$. Since $\overline{h} = \overline{h}_m$ is a $\frac{2\pi}{m}$-rotation on $D^2_{1/2} = \{\rho \le 1/2\} \subset D^2$, the orbit δ_{ρ_0,ϕ_0} winds m times in the longitudinal direction and once in the meridian direction. The point $(0, \rho_0, \phi_0)$ is with respect to balanced coordinates on $\overline{N}_r$; see Section I.5.1.2. We assume additionally that δ_{ρ_0,ϕ_0} does not intersect the projections of the Lagrangians of $\overline{W}_\tau$, $\overline{W}_+$ and $\overline{W}_-$ to $\overline{N}_r$.

Recall from Section 3.2.1 that $\phi(\overline{a}_{i,j})$ is the ϕ-coordinate of $\overline{a}_{i,j}$ near z_∞ such that $0 \le \phi(\overline{a}_{i,j}) < 2\pi$. Also let $\varepsilon_0 = \frac{2\pi}{n K(m)}$ be the constant appearing in the definition of $\overline{\mathbf{b}}$ and let ε_1 be a constant satisfying $0 < \varepsilon_1 < \varepsilon_0$. We consider two possibilities for ϕ_0:

$$\phi_0^\pm = \phi(\overline{a}_{i,j}) \pm \varepsilon_1.$$

Comparing with Equation (3.2.1), we obtain:

$$0 < \phi(\overline{b}_{i,j}) < \phi_0^- < \phi(\overline{a}_{i,j}) < \phi_0^+ < \phi(\overline{h}(\overline{b}_{i,j})) < \phi(\overline{h}(\overline{a}_{i,j})) < c(m). \tag{3.2.2}$$

We write $(\sigma^*_\infty)^{\dagger,\pm}$ for the restriction of $\mathbf{R} \times \delta_{\rho_0,\phi_0^\pm}$ to $\overline{W}_*$, where $* = \varnothing, ', \tau, +,$ or $-$. For $\overline{W}_{-\infty,i}$, we write

$$(\sigma^{-\infty,i}_\infty)^{\dagger,\pm} = B_{-\infty,i} \times \{\rho = \rho_0, \phi = \phi_0^\pm + 2\pi k/m, k \in \mathbf{Z}\}.$$

Finally we define:

$$n^*(\overline{u}) = \langle \overline{u}, (\sigma^*_\infty)^{\dagger,+}\rangle, \qquad n^{*,alt}(\overline{u}) = \langle \overline{u}, (\sigma^*_\infty)^{\dagger,-}\rangle, \tag{3.2.3}$$

where $* = \varnothing, ', \tau, +, -,$ or $(-\infty, i)$. The two quantities n^* and $n^{*,alt}$ can be used interchangeably, except when $\tau = -\infty$; for the most part we will use n^*.

Notation **3.2.10** *(Components of a holomorphic curve* $\overline{u}$*).* — Given a holomorphic curve $\overline{u}$ in $\overline{W}$, $\overline{W'}$, etc., we write

$$\overline{u} = \overline{u}' \cup \overline{u}'' = \overline{u}' \cup \overline{u}^\sharp \cup \overline{u}^\flat \cup \overline{u}^f,$$

where

- $\overline{u}'$ is a possibly disconnected branched cover of σ_∞^*;
- $\overline{u}''$ is the union of irreducible components which do not branch cover σ_∞^*;
- $\overline{u}^\sharp$ is the union of components of $\overline{u}''$ which are asymptotic to a multiple of δ_0 or z_∞ at one or more ends;
- $\overline{u}^\flat$ is the union of the remaining non-fiber components of $\overline{u}''$; and
- $\overline{u}^f$ is the union of fiber components of $\overline{u}$, including ghosts.

If $\overline{u}$ is a multisection, then $\deg \overline{u}$ is the degree of $\overline{u}$ as a multisection.

The choice of hyperbolic orbit. By the definition of the monodromy map h, ∂N is a negative Morse-Bott torus; we denote the negative Morse-Bott family of simple orbits on ∂N by $\mathcal{N}$. Let ϕ_γ be the ϕ-coordinate of $\gamma \in \mathcal{N}$. Also recall the orbit $\delta_0 = \{z_\infty\} \times [0,2]/\sim$ of $\overline{N}$.

Let $\overline{J}' \in \mathcal{J}_{\overline{W}'}$. Without loss of generality, we may assume that there is only one holomorphic cylinder Z_γ in $(\overline{W}' = \mathbf{R} \times \overline{N}, \overline{J}')$ from δ_0 to any orbit $\gamma \in \mathcal{N}$, modulo $\mathbf{R}$-translation. Each Z_γ corresponds to a radial ray $\mathcal{R}_{\phi_\gamma} = \{\phi = \phi_\gamma, \rho \geq 0\} \subset D^2$, which is the asymptotic direction of $\pi_{D^2}(Z_\gamma)$ at the positive end. Here $\pi_{D^2} : \overline{N} - int(N) \to D^2$ is the projection with respect to the balanced coordinates; see Section I.5.1.2.

We now choose a hyperbolic orbit h and an elliptic orbit e in $\mathcal{N}$. The choice of $h = \gamma_{\phi_h}$ is the same as that of Convention I.6.6.4: h is generic and ϕ_h is close to $-\frac{2\pi}{m}$, where the integer m which appears in the definition of $\overline{h}_m$ additionally satisfies the conditions of Section I.5.2.2. In particular, the radial ray $\mathcal{R}_{\phi_h}$ does not lie on the thin wedges from $\overline{a}_i$ to $\overline{h}(\overline{a}_i)$ for all i. There are no restrictions on e except that $e \neq h$.

3.2.4. *Holomorphic maps to* $\overline{W}_\tau$. — Let (F, j) be a compact Riemann surface, possibly disconnected, with two k-tuples of boundary punctures $\mathbf{q}^+ = \{q_1^+, \dots, q_k^+\}$ and $\mathbf{q}^- = \{q_1^-, \dots, q_k^-\}$ on $\partial F = \partial_+ F \sqcup \partial_- F$, such that:

(i) each component of F nontrivially intersects $\partial_+ F$ and $\partial_- F$;
(ii) each of $\partial_+ F$ and $\partial_- F$ is a union of connected components of ∂F; and
(iii) on each component of $\partial_+ F$ (resp. $\partial_- F$) there is at least one puncture from $\mathbf{q}^+$ (resp. $\mathbf{q}^-$) and none from $\mathbf{q}^-$ (resp. $\mathbf{q}^+$).

We write $\dot{F} = F - \mathbf{q}^+ - \mathbf{q}^-$, $\partial_+ \dot{F} = \partial_+ F - \mathbf{q}^+$ and $\partial_- \dot{F} = \partial_- F - \mathbf{q}^-$.

Let $\mathbf{z} = \{z_\infty^p(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$, $p \geq 0$, be a k-tuple of points of $\overline{\mathbf{a}} \cap \overline{h}(\overline{\mathbf{a}})$, where $k \leq 2g$, z_∞ has multiplicity p, and $\overrightarrow{\mathcal{D}}$ is the data at z_∞^p with respect to $\overline{\mathbf{a}} \cap \overline{h}(\overline{\mathbf{a}})$. The definition of $\mathbf{z}$ and the notion of data at z_∞^p are given in Section I.5.7. In particular, by definition, each arc of $\{\overline{a}_i, \overline{h}(\overline{a}_i)\}_{i=1}^{2g}$ is used at most once. Also let $\mathbf{z}' = \{z_\infty^q(\overrightarrow{\mathcal{D}}')\} \cup \mathbf{y}'$, $q \geq 0$, be a k-tuple of points of $\overline{\mathbf{b}} \cap \overline{h}(\overline{\mathbf{b}})$, where z_∞ has multiplicity q and $\overrightarrow{\mathcal{D}}'$ is the data at z_∞^q with respect to $\overline{\mathbf{b}} \cap \overline{h}(\overline{\mathbf{b}})$.

Let $\overline{J}_\tau \in \mathcal{J}_{\overline{W}_\tau}$. If $\overline{u}' : \dot{F} \to (\overline{W}_\tau, \overline{J}_\tau)$ is a branched cover of σ^τ_∞, then $\overline{u}'$ comes equipped with data $\mathcal{C}$ (cf. Definition I.5.7.1), which is a map from $\pi_0(\partial_+\dot{F})$ (resp. $\pi_0(\partial_-\dot{F})$) to the set of arcs $\overline{a}_{i,j}$ (resp. $\overline{b}_{i,j}$). In words, $\overline{u}'$ is viewed as mapping each component of $\partial_+\dot{F}$ (resp. $\partial_-\dot{F}$) to some $L^{\tau,+}_{\overline{a}_{i,j}}$ (resp. $L^{\tau,-}_{\overline{b}_{i,j}}$). Then $\mathcal{C}$ determines the data $\overrightarrow{\mathcal{D}}$ and $\overrightarrow{\mathcal{D}}'$ at the positive and negative ends.

We then make the following definition:

Definition **3.2.11.** — *Let* $\overline{J}_\tau \in \mathcal{J}_{\overline{W}_\tau}$, $\mathbf{z} = \{z^p_\infty(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$ *be a k-tuple of* $\overline{\mathbf{a}} \cap \overline{h}(\overline{\mathbf{a}})$ *and* $\mathbf{z}' = \{z^q_\infty(\overrightarrow{\mathcal{D}}')\} \cup \mathbf{y}'$ *be a k-tuple of* $\overline{\mathbf{b}} \cap \overline{h}(\overline{\mathbf{b}})$.

A degree k multisection of $(\overline{W}_\tau, \overline{J}_\tau)$ from $\mathbf{z}$ to $\mathbf{z}'$ *is a pair* $(\overline{u}, \mathcal{C})$ *consisting of a holomorphic map*

$$\overline{u} = \overline{u}' \cup \overline{u}'' : (\dot{F} = \dot{F}' \sqcup \dot{F}'', j) \to (\overline{W}_\tau, \overline{J}_\tau)$$

which is a degree k multisection of $\pi_{B_\tau} : \overline{W}_\tau \to B_\tau$ *and data* $\mathcal{C}$ *for* $\overline{u}'$*, and which additionally satisfies the following:*

(1) $\overline{u}''(\partial_+\dot{F}'') \subset L^{\tau,+}_{\widehat{\mathbf{a}}}$ *and* $\overline{u}''(\partial_-\dot{F}'') \subset L^{\tau,-}_{\widehat{\mathbf{b}}}$;
(2) $\overline{u}$ *maps each connected component of* $\partial_+\dot{F}$ *to a different* $L^{\tau,+}_{\overline{a}_i}$ *and each connected component of* $\partial_-\dot{F}$ *to a different* $L^{\tau,-}_{\overline{b}_i}$ *(here we are using* $\mathcal{C}$ *to assign some* $L^{\tau,+}_{\overline{a}_i}$ *or* $L^{\tau,-}_{\overline{b}_i}$ *to each component of* $\partial_\pm\dot{F}'$*);*
(3) $\lim_{w\to q_i^+} \pi_{\mathbf{R}} \circ \overline{u}(w) = +\infty$ *and* $\lim_{w\to q_i^-} \pi_{\mathbf{R}} \circ \overline{u}(w) = -\infty$;
(4) $\overline{u}$ *converges to a strip over* $[0,1] \times \mathbf{z}$ *near* $\mathbf{q}^+$ *and to a strip over* $[0,1] \times \mathbf{z}'$ *near* $\mathbf{q}^-$;
(5) *the positive and negative ends of* $\overline{u}$ *which limit to* z_∞ *are described by* $\overrightarrow{\mathcal{D}}$ *and* $\overrightarrow{\mathcal{D}}'$.

Here $\pi_{\mathbf{R}} : \overline{W}_\tau \to \mathbf{R}$ *is the projection to the s-coordinate.*

A $(\overline{W}_\tau, \overline{J}_\tau)$-curve from $\mathbf{y}$ to $\mathbf{y}'$ *is a degree 2g multisection of* $(\overline{W}_\tau, \overline{J}_\tau)$ *satisfying* $n^*(\overline{u}) = m$*. (Recall that the integer m is the integer on which the monodromy map* $\overline{h} = \overline{h}_m$ *depends.)*

Let $\mathcal{M}_{\overline{J}_\tau}(\mathbf{z}, \mathbf{z}')$ be the moduli space of degree k multisections of $(\overline{W}_\tau, \overline{J}_\tau)$ from $\mathbf{z}$ to $\mathbf{z}'$ and let $\mathcal{M}_{\overline{J}_\tau}(\mathbf{z}, \mathbf{z}'; \overline{\mathfrak{m}}(\tau))$ be the moduli space of degree k multisections of $(\overline{W}_\tau, \overline{J}_\tau)$ from $\mathbf{z}$ to $\mathbf{z}'$ and with a marked point mapped to $\overline{\mathfrak{m}}(\tau)$. By a slight abuse of terminology we call this the moduli space of multisections *passing through* $\overline{\mathfrak{m}}(\tau)$.

We write

$$\mathcal{M}_{\{\overline{J}_\tau\}}(\mathbf{z}, \mathbf{z}') := \{(\tau, u) \mid \tau \in \mathbf{R},\ u \in \mathcal{M}_{\overline{J}_\tau}(\mathbf{z}, \mathbf{z}')\},$$

$$\mathcal{M}_{\{\overline{J}_\tau\}}(\mathbf{z}, \mathbf{z}'; \overline{\mathfrak{m}}) := \{(\tau, u) \mid \tau \in \mathbf{R},\ u \in \mathcal{M}_{\overline{J}_\tau}(\mathbf{z}, \mathbf{z}'; \overline{\mathfrak{m}}(\tau))\}.$$

Notation **3.2.12** *(Modifiers).* — For any moduli space $\mathcal{M}_{\star_1}(\star_2)$, we may place modifiers $*$ as in $\mathcal{M}^*_{\star_1}(\star_2)$ to denote the subset of $\mathcal{M}_{\star_1}(\star_2)$ satisfying $*$. Typical self-explanatory modifiers are $I = i$, $n^* = m$, and $\deg = k$. Note that the degree can be inferred from $\star_2$.

The following is a list of non-self-explanatory modifiers:

$\dagger$ = no component of $\overline{u}$ branch covers σ_∞^* with possibly empty branch locus.
s = all the components of $\overline{u}$ are simply covered.
irr = the curve $\overline{u}$ is irreducible.

3.2.5. *Holomorphic maps to* $\overline{W}_{-\infty}$. — We first discuss holomorphic curves without ends at z_∞.

Definition **3.2.13.** — *Let* $\overline{J}_{-\infty,1} \in \mathcal{J}_{\overline{W}_{-\infty,1}}$, $\mathbf{y}_1 \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$, $\mathbf{y}_2 \in \mathcal{S}_{\mathbf{b},\mathbf{a}}$, $\mathbf{y}_3 \in \mathcal{S}_{\mathbf{b},h(\mathbf{b})}$ *and* $\mathbf{y}_4 \in \mathcal{S}_{h(\mathbf{a}),h(\mathbf{b})}$. *(Recall that an element of* $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ *is a tuple of intersection points* $a_i \cap h(a_j)$, *where each* a_i *and each* $h(a_j)$ *is used at most once; the other* $\mathcal{S}_{*,*}$ *are defined analogously. In this definition all the tuples are* k*-tuples.)*

A degree $k \leq 2g$ multisection $\overline{u}$ of $(\overline{W}_{-\infty,1}, \overline{J}_{-\infty,1})$ with ends $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ *is a holomorphic map*

$$\overline{u} : (\dot{F}, j) \to (\overline{W}_{-\infty,1}, \overline{J}_{-\infty,1})$$

which is degree k *multisection of* $\pi_{B_{-\infty,1}} : \overline{W}_{-\infty,1} \to B_{-\infty,1}$ *and which additionally satisfies the following:*

(1) $\overline{u}(\partial\dot{F}) \subset L^{-\infty,1}_{\overline{\mathbf{a}},1} \cup L^{-\infty,1}_{\overline{\mathbf{b}},2} \cup L^{-\infty,1}_{h(\overline{\mathbf{b}}),3} \cup L^{-\infty,1}_{h(\overline{\mathbf{a}}),4}$ *and* $\overline{u}$ *maps each component of* $\partial\dot{F}$ *to a different* $L^{-\infty,1}_{\overline{a}_i,1}$, $L^{-\infty,1}_{\overline{b}_i,2}$, $L^{-\infty,1}_{h(\overline{b}_i),3}$, *or* $L^{-\infty,1}_{h(\overline{a}_i),4}$;
(2) $\overline{u}$ *converges to a strip over* $[0, 1] \times \mathbf{y}_1$ *as* $s \to +\infty$; $[-2, 2] \times \mathbf{y}_2$ *as* $t \to +\infty$; $[0, 1] \times \mathbf{y}_3$ *as* $s \to -\infty$; *and* $[-2, 2] \times \mathbf{y}_4$ *as* $t \to -\infty$.

A $(\overline{W}_{-\infty,1}, \overline{J}_{-\infty,1})$-curve with ends $\mathbf{y}_1, \dots, \mathbf{y}_4$ *is a degree* $2g$ *multisection of* $(\overline{W}_{-\infty,1}, \overline{J}_{-\infty,1})$ *satisfying* $n^*(\overline{u}) = 0$.

We will use the convention to list the ends of a multisection $\overline{u}$ of $\overline{W}_{-\infty,1}$ in counterclockwise order, starting with the top end.

Definition **3.2.14.** — *Let* $\overline{J}_{-\infty,2} \in \mathcal{J}_{\overline{W}_{-\infty,2}}$ *and* $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\mathbf{b},\mathbf{a}}$. *A* degree $k \leq 2g$ multisection $\overline{u}$ of $(\overline{W}_{-\infty,2}, \overline{J}_{-\infty,2})$ with ends $\mathbf{y}$ and $\mathbf{y}'$ *is a holomorphic map*

$$\overline{u} : (\dot{F}, j) \to (\overline{W}_{-\infty,2}, \overline{J}_{-\infty,2})$$

which is degree k *multisection of* $\pi_{B_{-\infty,2}} : \overline{W}_{-\infty,2} \to B_{-\infty,2}$ *and which additionally satisfies the following:*

(1) $\overline{u}(\partial\dot{F}) \subset L^{-\infty,2}_{\overline{\mathbf{a}},+} \cup L^{-\infty,2}_{\overline{\mathbf{b}},-}$ *and* $\overline{u}$ *maps each component of* $\partial\dot{F}$ *to a different* $L^{-\infty,2}_{\overline{a}_i,+}$ *or a different* $L^{-\infty,2}_{\overline{b}_i,-}$;
(2) $\overline{u}$ *converges to a cylinder over* $[-2, 2] \times \mathbf{y}$ *as* $t \to +\infty$ *and to a cylinder over* $[-2, 2] \times \mathbf{y}'$ *as* $t \to -\infty$.

A $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$-curve with ends $\mathbf{y}$ *and* $\mathbf{y}'$ *is a degree $2g$ multisection of* $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$ *satisfying $n^*(\overline{u})=m$.*

Note that the definition is not symmetric in $\mathbf{y}$ and $\mathbf{y}'$; the $t=+\infty$ end is always written first.

Next we discuss holomorphic curves with ends at z_∞. For $s=1,\dots,4$, the data $\overrightarrow{\mathcal{D}}_s$ at $z_\infty^{p_s}$ (cf. Section I.5.7) corresponding to the s^{th} end is given by

$$\overrightarrow{\mathcal{D}}_s=\{(i'_{s,\ell},j'_{s,\ell})\to(i_{s,\ell},j_{s,\ell})\}_{\ell=1}^{p_s}.$$

When $s=1$, $\mathcal{D}^{from}_{s=1}=\{(i'_{1,\ell},j'_{1,\ell})\}_{\ell=1}^{p_1}$ and $\mathcal{D}^{to}_{s=1}=\{(i_{1,\ell},j_{1,\ell})\}_{\ell=1}^{p_1}$ specify the initial points on $\overline{h}(\overline{a}_{i'_{1,\ell}j'_{1,\ell}})$ and terminal points on $\overline{a}_{i_{1,\ell},j_{1,\ell}}$, respectively; the cases $s=2,3,4$ are analogous.

We then extend the definition of a degree k multisection $\overline{u}$ of $\overline{W}_{-\infty,1}$ to include those with ends $\mathbf{z}_s=\{z_\infty^{p_s}(\overrightarrow{\mathcal{D}}_s)\}\cup\mathbf{y}_s$, $s=1,\dots,4$, by attaching data $\mathcal{C}$ to $\overline{u}'$ (cf. Definition I.5.7.1) and modifying Definition 3.2.13 in the same way Definition I.5.7.2 modifies Definition I.4.3.1. Degree k multisections of $\overline{W}_{-\infty,2}$ with ends $\mathbf{z}=\{z_\infty^{p}(\overrightarrow{\mathcal{D}})\}\cup\mathbf{y}$ and $\mathbf{z}'=\{z_\infty^{q}(\overrightarrow{\mathcal{D}}')\}\cup\mathbf{y}'$ are defined similarly.

Let $\mathcal{M}_{\overline{J}_{-\infty,1}}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_4)$ be the moduli space of degree k multisections of $(\overline{W}_{-\infty,1},\overline{J}_{-\infty,1})$ with ends $\mathbf{z}_s$, $s=1,\dots,4$, and let $\mathcal{M}_{\overline{J}_{-\infty,2}}(\mathbf{z},\mathbf{z}')$ be the moduli space of degree k multisections of $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$ with ends $\mathbf{z}$ and $\mathbf{z}'$. Also let $\mathcal{M}_{\overline{J}_{-\infty,2}}(\mathbf{z},\mathbf{z}';\overline{\mathfrak{m}}(-\infty))$ be the moduli space of multisections as above passing through $\overline{\mathfrak{m}}(-\infty))$.

We define the extended moduli spaces $\mathcal{M}^{\dagger,ext}_{\overline{J}_{-\infty,1}}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_4)$ and $\mathcal{M}^{\dagger,ext}_{\overline{J}_{-\infty,2}}(\mathbf{z},\mathbf{z}')$ in a manner similar to that of Section I.5.7.11. The precise definitions will be omitted.

3.2.6. *Indices.* — We now discuss the Fredholm index $\operatorname{ind}(\overline{u})$ and the ECH index $I(\overline{u})$ of a $\overline{W}_\tau$-curve $\overline{u}:\dot{F}\to\overline{W}_\tau$ from $\mathbf{y}$ to $\mathbf{y}'$ (i.e., when $\overline{u}'=\varnothing$). The discussion will be brief since all the key ingredients have already been discussed in Sections I.5.5 and I.5.6.

We remark that, once again, $\operatorname{ind}(\overline{u})$ and $I(\overline{u})$ do not take into account the point constraint $\overline{\mathfrak{m}}(\tau)$ and that the condition "passing through $\overline{\mathfrak{m}}(\tau)$" is a codimension 2 condition; more precisely, adding a marked point increases the dimension of the moduli space by 2 and constraining the marked point to $\overline{\mathfrak{m}}(\tau)$ reduces the dimension by 4.

Let $\check{\overline{W}}_\tau$ be the compactification of $\overline{W}_\tau$ that we identify with $\overline{W}_\tau-\{s>l(\tau)+1\}-\{s<-l(\tau)-1\}$, where $l(\tau)$ is given in Equation (3.1.1). Let $\check{\overline{u}}:\check{F}\to\check{\overline{W}}_\tau$ be the compactification of $\overline{u}$, where $\check{F}$ is obtained by performing a real blow-up of F at its boundary punctures. We also define

$$\begin{aligned}Z_{\mathbf{y},\mathbf{y}'}=&(\{l(\tau)+1\}\times[0,1]\times\mathbf{y})\cup(\{-l(\tau)-1\}\times[0,1]\times\mathbf{y}')\\&\cup((L^{\tau,+}_{\hat{\mathbf{a}}}\cup L^{\tau,-}_{\hat{\mathbf{b}}})\cap\check{\overline{W}}_\tau).\end{aligned}$$

The trivialization $\tau^\star$ of $T\overline{S}$ along $Z_{\mathbf{y},\mathbf{y}'}$ is defined in a manner similar to that of Section I.4.4.2:[5] First we define the trivialization $\tau^\star$ of $T\overline{S}|_{\pi_{B_\tau}^{-1}(l(\tau)+1,1)}$ (resp. $T\overline{S}|_{\pi_{B_\tau}^{-1}(-l(\tau)-1,1)}$) along the $\widehat{a}_i$ (resp. $\widehat{b}_i$) by choosing a nonsingular tangent vector field along $\widehat{a}_i$ (resp. $\widehat{b}_i$). We then parallel transport $\tau^\star$ along $\partial\overline{W}_\tau$ and extend $\tau^\star$ arbitrarily to $\{l(\tau)+1\}\times[0,1]\times\mathbf{y}$ and $\{-l(\tau)-1\}\times[0,1]\times\mathbf{y}'$.

Let $Q_{\tau^\star}(\check{\overline{u}})$ be the relative intersection form given by intersecting $\check{\overline{u}}$ and a pushoff of $\check{\overline{u}}$ in the direction of $J\tau^\star$ along $\partial\check{\overline{W}}_\tau$. Then

$$(\mathbf{3.2.4})\qquad I(\overline{u}) = c_1(\check{\overline{u}}^* T\overline{S},\tau^\star) + Q_{\tau^\star}(\check{\overline{u}}) + \mu_{\tau^\star}(\mathbf{y}) - \mu_{\tau^\star}(\mathbf{y}') - 2g,$$

$$(\mathbf{3.2.5})\qquad \operatorname{ind}(\overline{u}) = -\chi(\dot{F}) + 2c_1(\check{\overline{u}}^* T\overline{S},\tau^\star) + \mu_{\tau^\star}(\mathbf{y}) - \mu_{\tau^\star}(\mathbf{y}') - 2g,$$

These index formulas are obtained by adding the index formulas for holomorphic curves in $\overline{W}_\pm$ of [I]. The index inequality holds as usual:

$$(\mathbf{3.2.6})\qquad \operatorname{ind}(\overline{u}) + 2\delta(\overline{u}) \le I(\overline{u}),$$

where $\delta(\overline{u}) \ge 0$ and equals zero if and only if $\overline{u}$ is an embedding.

In the general case when $\overline{u}' \ne \varnothing$, modifications can be made as in Section I.5.7.5 and one can easily verify the following:

$$(\mathbf{3.2.7})\qquad I(\sigma_\infty^+) = \operatorname{ind}(\sigma_\infty^+) = -1;$$

$$(\mathbf{3.2.8})\qquad I(\sigma_\infty^-) = \operatorname{ind}(\sigma_\infty^-) = 0;$$

$$(\mathbf{3.2.9})\qquad I(\sigma_\infty^\tau) = \operatorname{ind}(\sigma_\infty^\tau) = -1.$$

The Fredholm and ECH indices for $\overline{W}_{-\infty,i}$-curves can be defined and computed similarly. We now prove the following:

Lemma **3.2.15.** — *If there exists a $\overline{W}_{-\infty,2}$-curve $\overline{u}$ with $I=2$ and ends $\mathbf{y}$ and $\mathbf{y}'$, then $\mathbf{y}=\{x^\#_{ij(i)}\}_{i=1}^{2g}$ and $\mathbf{y}'=\{x^\#_{ik(i)}\}_{i=1}^{2g}$, where $j(i)$ is odd and $k(i)$ is even for all i.*

In other words, $\mathbf{y}$ is a summand of the top generator $\Theta_{\overline{\mathbf{a}},\overline{\mathbf{b}}} \in \widehat{HF}(\overline{\mathbf{a}},\overline{\mathbf{b}})$ and $\mathbf{y}'$ is a summand of the top generator $\Theta_{\overline{\mathbf{b}},\overline{\mathbf{a}}} \in \widehat{HF}(\overline{\mathbf{b}},\overline{\mathbf{a}})$. We remind the reader that our convention is that $\mathbf{y}$ and $\mathbf{y}'$ do not contain z_∞.

Proof. — The proof is similar to the index calculation of Lemma 2.2.2. Let $\overline{u}$ be a $\overline{W}_{-\infty,2}$-curve; note that $n^*(\overline{u})=m$ by definition. First consider the situation where $\overline{u}$ is in the homology class consisting of a copy $\{pt\}\times\overline{S}$ of the fiber and $2g$ trivial strips. Then

$$\operatorname{ind}(\{pt\}\times\overline{S}) = -\chi(\overline{S}) + 2c_1(T\overline{S}) = -(2-2g) + 2(2-2g) = 2-2g.$$

[5] Here we are writing $\tau^\star$ instead of τ due to the notational conflict with $\tau\in\mathbf{R}$.

The strips contribute 0 to ind and there are $2g$ intersection points. Hence

$$\mathrm{I}(\overline{u}) = (2-2g) + 0 + 2(2g) = 2g+2$$

when $\mathbf{y}' = \mathbf{y}$. The only way to lower $\mathrm{I}(\overline{u})$ to 2 is to take $\mathbf{y} = \{x^{\#}_{ij(i)}\}_{i=1}^{2g}$ and $\mathbf{y}' = \{x^{\#}_{ik(i)}\}_{i=1}^{2g}$ so that all the $j(i)$ are odd and all the $k(i)$ are even. □

3.2.7. *Regularity.* — We now discuss the regularity of the family $\{\overline{\mathrm{J}}_\tau\}_{\tau\in\mathbf{R}}$.

Definition **3.2.16.**

(1) $\overline{\mathrm{J}}_{-\infty,2} \in \mathcal{J}_{\overline{\mathrm{W}}_{-\infty,2}}$ *is* regular *if all the moduli spaces* $\mathcal{M}^{\dagger,ext}_{\overline{\mathrm{J}}_{-\infty,2}}(\mathbf{z},\mathbf{z}')$ *are transversely cut out.*
(2) $\overline{\mathrm{J}}_{-\infty,1} \in \mathcal{J}_{\overline{\mathrm{W}}_{-\infty,1}}$ *is* regular *if all the moduli spaces* $\mathcal{M}^{\dagger,ext}_{\overline{\mathrm{J}}_{-\infty,1}}(\mathbf{z}_1,\dots,\mathbf{z}_4)$ *are transversely cut out and the restrictions* $\overline{\mathrm{J}}_{-\infty,2}$ *and* $\overline{\mathrm{J}}$ *of* $\overline{\mathrm{J}}_{-\infty,1}$ *to the ends are regular.*
(3) $\overline{\mathrm{J}}_{-\infty} = \overline{\mathrm{J}}_{-\infty,1} \cup \overline{\mathrm{J}}_{-\infty,2} \in \mathcal{J}_{\overline{\mathrm{W}}_{-\infty}}$ *is* regular *if* $\overline{\mathrm{J}}_{-\infty,i}$ *are regular for* $i=1,2$.

Definition **3.2.17.** — *The family* $\{\overline{\mathrm{J}}_\tau\}_{\tau\in\mathbf{R}} \in \overline{\mathcal{I}}$ *is* regular *if:*

(1) *all the moduli spaces* $\mathcal{M}^{\dagger,ext}_{\{\overline{\mathrm{J}}_\tau\}}(\mathbf{z},\mathbf{z}')$ *are transversely cut out;*
(2) *the restriction* $\overline{\mathrm{J}}$ *of* $\overline{\mathrm{J}}_\tau$ *to the positive and negative ends is regular;*
(3) $\overline{\mathrm{J}}_+$ *and* $\overline{\mathrm{J}}_-$ *in the limit* $\tau\to+\infty$ *are regular; and*
(4) $\overline{\mathrm{J}}_{-\infty}$ *in the limit* $\tau\to-\infty$ *is regular.*

Let $\overline{\mathcal{I}}^{reg}$ be the space of regular $\{\overline{\mathrm{J}}_\tau\}\in\overline{\mathcal{I}}$. As usual, we have:

Lemma **3.2.18.** — *The generic* $\{\overline{\mathrm{J}}_\tau\}\in\overline{\mathcal{I}}$ *is regular, provided no points of* $(\overline{\mathbf{a}}\cap\overline{h}(\overline{\mathbf{a}}))-\{z_\infty\}$ *are fixed points of* $\overline{h}$.[6]

Proof. — This is analogous to the proof of regularity for $\overline{\mathrm{J}}_+$, with one caveat: the family $\{\overline{\mathrm{J}}_\tau\}$ is not sufficiently generic to achieve the transversality of negative index horizontal sections, where by a horizontal section we mean the restriction of a cylinder over an orbit that passes through a fixed point y of $\overline{h}$. The additional assumption on the fixed points of $\overline{h}$ eliminates the horizontal sections besides σ^τ_∞, which we do not consider because of the superscript †. □

Next we discuss the regularity of moduli spaces passing through $\overline{\mathfrak{m}}$. Since the point constraints $\overline{\mathfrak{m}}(\tau)$ are nongeneric, we need to introduce a perturbation of the family $\{\overline{\mathrm{J}}_\tau\}$:

[6] Here (and henceforth) we make this additional assumption on $\overline{h}$, which we are free to do.

Definition **3.2.19.** — *Let* $\varepsilon > 0$ *and let* $\{U_\tau\}_{\tau\in\mathbf{R}}$, $U_\tau \subset \overline{W}_\tau$, *be a family of open sets such that* $U_\tau \not\ni \overline{\mathfrak{m}}(\tau)$. *Then a family* $\{\overline{J}^\diamondsuit_\tau\}_{\tau\in\mathbf{R}}$ *of almost complex structures on* $\{\overline{W}_\tau\}$ *is* $(\varepsilon, \{U_\tau\})$-close *to a regular* $\{\overline{J}_\tau\}$ *if:*

- $\overline{J}^\diamondsuit_\tau = \overline{J}_\tau$ *on* $\overline{W}_\tau - U_\tau$;
- $\overline{J}^\diamondsuit_\tau$ *is* ε*-close to* $\overline{J}_\tau$ *on* U_τ*; and*
- $\nabla\overline{J}^\diamondsuit_\tau$ *is* ε*-close to* $\nabla\overline{J}_\tau$ *on* U_τ.

Here the ε-closeness is measured with respect to a family $\{g_\tau\}_{\tau\in\mathbf{R}}$ of Riemannian metrics which is defined as follows: Let h be an s-invariant metric on $\mathbf{R}\times\overline{N}_2$ such that h, viewed as a metric on $\mathbf{R}\times[0,2]\times\overline{S}$, is also t-invariant on $\mathbf{R}\times[1,2]\times\overline{S}$. There is an extension of h to h_τ on $\mathbf{R}\times[0,r(\tau)]\times\overline{S}$ which is s- and t-invariant on $\mathbf{R}\times[2,r(\tau)]\times\overline{S}$. We then view h_τ as a metric on $\mathbf{R}\times\overline{N}_{r(\tau)}$ and define g_τ as the restriction of h_τ to $\overline{W}_\tau\subset\mathbf{R}\times\overline{N}_{r(\tau)}$.

Let $\mathfrak{p}(\tau)\subset int(B_\tau)$, $\tau\in[-\infty,\infty]$, be a family of points, where:

- the cardinality $\#\mathfrak{p}(\tau)$ is finite and independent of $\tau\in[-\infty,\infty]$;
- $\mathfrak{p}(\tau)$ is smooth for $\tau\in(-\infty,\infty)$;
- $\lim_{\tau\to+\infty}\mathfrak{p}(\tau)$ exists and equals $\mathfrak{p}(+\infty)$;
- $\lim_{\tau\to-\infty}\mathfrak{p}(\tau)$ exists and equals $\mathfrak{p}(-\infty)$.

In order for $\lim_{\tau\to+\infty}\mathfrak{p}(\tau)$ to be defined we require the existence of $C>0$ such that, for all $\tau\gg 0$, $\mathfrak{p}(\tau)$ is contained in a C-neighborhood of ∂B_τ. Then $\mathfrak{p}(\tau)$ can be viewed as a subset of $B_+\cup B_-$ and we are asking $\lim_{\tau\to+\infty}\mathfrak{p}(\tau)=\mathfrak{p}(+\infty)$ in $B_+\cup B_-$. $\lim_{\tau\to-\infty}\mathfrak{p}(\tau)$ is defined analogously.

- $\mathfrak{p}(+\infty)$ is a nontrivial union of points of $int(B_+)$ and $int(B_-)$; similarly, $\mathfrak{p}(-\infty)$ is a nontrivial union of points of $int(B_{-\infty,1})$ and $int(B_{-\infty,2})$;
- for each $\tau\in[-\infty,\infty]$, $\overline{\mathfrak{m}}^b(\tau)\notin\mathfrak{p}(\tau)$.

We will use the following specific open sets U_τ: For $\tau\in\mathbf{R}$, let U_τ be an open δ-neighborhood of $K_\tau=\pi_{B_\tau}^{-1}(\mathfrak{p}(\tau))-\{\rho<2\delta\}$, where $\delta>0$ is arbitrarily small. Then let $U_{\pm\infty}$ and $K_{\pm\infty}$ be the limits of U_τ and K_τ as $\tau\to\pm\infty$. When we want to emphasize (ε,U_τ) or $(\varepsilon,\delta,\mathfrak{p}(\tau))$, we write $\overline{J}^\diamondsuit_\tau(\varepsilon,U_\tau)$ or $\overline{J}^\diamondsuit_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))$ for $\overline{J}^\diamondsuit_\tau$, $U_{\varepsilon,\delta,\mathfrak{p}(\tau)}$ for U_τ, and $K_{\mathfrak{p}(\tau),\delta}$ for K_τ.

We define a degree k *almost multisection* $\overline{u}$ *of* $(\overline{W}_\tau,\overline{J}^\diamondsuit_\tau)$ in the same way as a degree k multisection of $(\overline{W}_\tau,\overline{J}_\tau)$, with the following difference: $\overline{u}$ is a degree k multisection when restricted to

$$\pi_{B\tau}:\overline{W}_\tau-\pi_{B_-}^{-1}(V_\tau)\to B_\tau-V_\tau,\quad V_\tau=\pi_{B_-}(U_\tau),$$

but $\overline{u}$ is just a degree k map when restricted to $\pi_{B_-}^{-1}(V_\tau)\to V_\tau$. The moduli spaces of almost multisections are defined in the same way as the moduli spaces of multisections, with $\overline{\mathrm{J}}_\tau^\diamondsuit$ replacing $\overline{\mathrm{J}}_\tau$. If $\{\mathrm{K}_\tau \not\ni \overline{\mathfrak{m}}(\tau)\}$ is a family of compact sets of $\overline{\mathrm{W}}_\tau$, then the modifier $\{\mathrm{K}_\tau\}$ means that the image of $\overline{u}$ in $\overline{\mathrm{W}}_\tau$ intersects K_τ. We stress the fact that, unlike the basepoint $\overline{\mathfrak{m}}$, the modifier $\{\mathrm{K}_\tau\}$ is not meant to define a moduli space with a point constraint at K_τ, but only to select a subset of the moduli space.

Almost complex structures $\overline{\mathrm{J}}_{-\infty,i}^\diamondsuit$, almost multisections on $(\overline{\mathrm{W}}_{-\infty,i}, \overline{\mathrm{J}}_{-\infty,i}^\diamondsuit)$, and moduli spaces of almost multisections are defined similarly.

Definition **3.2.20.** — *The family* $\{\overline{\mathrm{J}}_\tau^\diamondsuit\}$ *is* $\{\mathrm{K}_\tau\}$-regular with respect to $\overline{\mathfrak{m}}$ *if all the* $\{\overline{\mathrm{J}}_\tau^\diamondsuit\}$*-holomorphic maps in* $\mathcal{M}^{\dagger,ext,\{\mathrm{K}_\tau\}}_{\{\overline{\mathrm{J}}_\tau^\diamondsuit\}}(\mathbf{z},\mathbf{z}';\overline{\mathfrak{m}})$ *are regular, i.e., their linearized Cauchy-Riemann operator, taking into account the variation of domain complex structure and the parameter* τ*, is surjective.*

If $\{\overline{\mathrm{J}}_\tau^\diamondsuit\}$ is $\{\mathrm{K}_\tau\}$-regular with respect to $\overline{\mathfrak{m}}$, then $\mathcal{M}^{\dagger,ext,\{\mathrm{K}_\tau\}}_{\{\overline{\mathrm{J}}_\tau^\diamondsuit\}}(\mathbf{z},\mathbf{z}';\overline{\mathfrak{m}})$ is an open subset of $\mathcal{M}^{\dagger,ext}_{\{\overline{\mathrm{J}}_\tau^\diamondsuit\}}(\mathbf{z},\mathbf{z}';\overline{\mathfrak{m}})$ and is a transversely cut out manifold.

Lemma **3.2.21.** — *A generic family* $\{\overline{\mathrm{J}}_\tau^\diamondsuit\}$ *is* $\{\mathrm{K}_\tau\}$*-regular with respect to* $\overline{\mathfrak{m}}$.

Proof. — The proof is similar to the combination of Theorems 3.1.7 and 3.4.1 of [MS], with modifications as in Proposition I.5.8.8. □

The following can also be proved using a standard regularity and compactness argument:

Lemma **3.2.22.** — *If* $\{\overline{\mathrm{J}}_\tau\}$ *is a generic family, then for* $\varepsilon,\delta>0$ *sufficiently small, there exist a generic family* $\{\overline{\mathrm{J}}_\tau^\diamondsuit(\varepsilon,\delta,\mathfrak{p}(\tau))\}$ *which is* $\{\mathrm{K}_{\mathfrak{p}(\tau),\delta}\}$*-regular with respect to* $\overline{\mathfrak{m}}$ *and disjoint finite subsets* $\mathcal{T}_1,\mathcal{T}_2\subset\mathbf{R}$ *with the following properties:*

(1) $\tau\in\mathcal{T}_1$ *if and only if there exists* $\overline{v}_\tau\in\mathcal{M}^{\dagger,s,irr,\mathrm{ind}=-1}_{\overline{\mathrm{J}}_\tau}(\mathbf{z},\mathbf{z}')$ *for some* $\mathbf{z}$ *and* $\mathbf{z}'$.
(2) $\tau\in\mathcal{T}_2$ *if and only if there exists* $\overline{v}_\tau\in\mathcal{M}^{\dagger,s,irr,\{\mathrm{K}_\tau\},\mathrm{ind}=1}_{\overline{\mathrm{J}}_\tau^\diamondsuit(\varepsilon,\delta,\mathfrak{p}(\tau))}(\mathbf{z},\mathbf{z}';\overline{\mathfrak{m}})$ *for some* $\mathbf{z}$ *and* $\mathbf{z}'$.

Moreover, for each $\tau\in\mathcal{T}_i$ *there is a unique such irreducible curve* $\overline{v}_\tau$.

Sketch of proof. — The existence of *discrete* sets $\mathcal{T}_1$ and $\mathcal{T}_2$ as in the lemma follows from Lemmas 3.2.18 and 3.2.21. Note that the only section which is not transversely cut out is the section at infinity σ_∞^*, which is excluded from the moduli space by the modifier †.

The finiteness of $\mathcal{T}_1$ and $\mathcal{T}_2$ relies on an SFT compactness result which will be discussed in the next section; see Equations (3.4.1) and (3.5.1). If either $\mathcal{T}_1$ or $\mathcal{T}_2$ is infinite, then there exists a limit $\overline{\mathrm{J}}_{\pm\infty}$-holomorphic building of the same total index by SFT compactness. However, since $\overline{\mathrm{J}}_{\pm\infty}$ are regular, at least one level in the building will have index which is too low to exist. □

By shrinking $\overline{\mathcal{I}}$, we assume that all $\{\overline{J}_\tau\} \in \overline{\mathcal{I}}$ satisfy Lemma 3.2.22.

3.3. *Proof of half of Theorem 1.0.1.* — In this subsection we prove that $\Psi \circ \Phi$ is an isomorphism on the level of homology.

In the next several paragraphs we briefly recall the chain complexes

$$(\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})), \partial'), \quad (\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), \partial), \quad (\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})), \widetilde{\partial}),$$

from Sections I.4.9.3 and I.6.6, where the first and third have isomorphic homology groups; the quotient map

$$q : \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}));$$

and the maps $\widetilde{\Phi}$, Ψ, and Ψ' from Sections I.6.6 and I.7.1.

The chain complex $\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$ is generated by the set $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ of $2g$-tuples of intersection points of $\mathbf{a}$ and $h(\mathbf{a})$, where each $\mathbf{y} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ intersects a_i and $h(a_i)$ exactly once, and the differential ∂' counts $I = 1$, degree $2g$ multisections of $W = \mathbf{R} \times [0, 1] \times S$. The chain complex $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ is the quotient of $\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$ under the equivalence relation $\sim$ which identifies $\mathbf{y} \sim \mathbf{y}'$ if $\mathbf{y}'$ can be obtained from $\mathbf{y}$ by successively replacing x_i by x_i' or x_i' by x_i for any $i = 1, \dots, 2g$; the map q is the corresponding quotient map, which is a chain map. Here x_i and x_i' are intersection points of a_i and $h(a_i)$ on ∂S given in Section I.4.9.1.

The chain complex $\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$ is a variant of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ (with isomorphic homology groups) generated by $2g$-tuples of intersection points $\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}'$ of $\overline{\mathbf{a}}$ and $\overline{h}(\overline{\mathbf{a}})$ with $\mathcal{I} \subset \{1, \dots, 2g\}$, where:

- z_∞ can be used more than once, but is always viewed as an intersection point of $\overline{a}_i$ and $\overline{h}(\overline{a}_i)$ and hence is written as $z_{\infty,i}$;
- writing $\mathbf{y}' = \{y_i'\}_{i \in \mathcal{I}^c}$ where $\mathcal{I}^c = \{1, \dots, 2g\} - \mathcal{I}$, there exists a permutation σ of $\mathcal{I}^c$ such that $y_i' \in a_i \cap h(a_{\sigma(i)})$ for all $i \in \mathcal{I}^c$.

The differential $\widetilde{\partial}$ counts $I = 1$, degree $2g$ multisections $\overline{u} = \overline{u}' \cup \overline{u}''$ of $\overline{W}$ with $n^*(\overline{u}) \leq 1$ such that $\overline{u}'$ has empty branch locus; see Definition I.6.6.1.

The map

$$\widetilde{\Phi} : \widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N),$$

defined in Section I.6.6, is a variant of the map $\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N)$ and

$$\langle \widetilde{\Phi}(\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}'), \boldsymbol{\gamma} \rangle,$$

counts $I = 0$, degree $2g$ multisections of $\overline{W}_+$ with $n^* \leq |\mathcal{I}|$ from $\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}'$ to $\boldsymbol{\gamma}$.

The map

$$\Psi' : PFC_{2g}(N) \to \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})),$$

defined in Section I.7.1, counts $I = 2$, degree $2g$ almost multisections of $(\overline{W}_-, \overline{J}_-^\diamondsuit)$ (cf. Definition I.5.8.3) with $n^* = m$ that pass through $\overline{\mathfrak{m}}(+\infty)$. The map

$$\Psi : PFC_{2g}(N) \to \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$$

is then given as the composition $q \circ \Psi'$.

The following is proved in this subsection:

Theorem **3.3.1.** — *Suppose $m \gg 0$. Then there exist maps*

$$H', \Theta_0' : \widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}'(S, \mathbf{b}, h(\mathbf{b})),$$
$$\widetilde{V} : \widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})) \to \widetilde{CF}(S, \mathbf{b}, h(\mathbf{b})),$$

which satisfy the following:

$$\Psi' \circ \widetilde{\Phi} - \Theta_0' = (\partial' H' + H' \widetilde{\partial}) + \widetilde{\partial}_1 \circ \widetilde{V}, \quad (\mathbf{3.3.1})$$

where $\widetilde{\partial}_1$, defined in Equation (I.7.1.1), satisfies:

$$\widetilde{\partial}_1(\{z_{\infty,i}\} \cup \mathbf{y}') = \{x_i\} \cup \mathbf{y}' + \{x_i'\} \cup \mathbf{y}'$$

for all $i = 1, \dots, 2g$ and is zero for any other generator. (Here $z_{\infty,i}$, x_i, and x_i' are for the basis **b***.) Postcomposing with q (for the basis* **b***) and writing $H = q \circ H'$ and $\Theta_0 = q \circ \Theta_0'$, we obtain the chain homotopy*

$$\Psi \circ \widetilde{\Phi} - \Theta_0 = \partial H + H \widetilde{\partial}, \quad (\mathbf{3.3.2})$$

where Θ_0 induces an isomorphism on homology.

To go from Equation (3.3.1) to Equation (3.3.2), we observe that $q \circ \widetilde{\partial}_1 = 0$ and $q \circ \partial' = \partial \circ q$.

Since Θ_0 is an isomorphism on the level of homology, so is $\Psi \circ \widetilde{\Phi}$. In view of Corollary I.6.6.7, $\Psi \circ \Phi$ is also an isomorphism on the level of homology.

Proof. — We prove Theorem 3.3.1, assuming the results of Sections 3.4–3.9. In Steps 1–3 we consider the situation where the holomorphic curves $\overline{u}$ in $\overline{W}_\tau$ are asymptotic to some $\mathbf{y} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ at the positive end. In Step 4 we describe the modifications needed for the situation where $\overline{u}$ is asymptotic to some $\mathbf{z} = \{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}$ at the positive end and $\mathcal{I}$ is a subset of $\{1, \dots, 2g\}$ with $|\mathcal{I}| > 0$. Steps 1–4 prove Equation (3.3.1) and hence Equation (3.3.2). In Step 5 we prove that Θ_0 induces an isomorphism on homology by further degenerating $\overline{W}_{-\infty,1}$.

Suppose $m \gg 0$. Choose $\mathfrak{p}(\tau)$ and $\{\overline{J}_\tau\} \in \overline{\mathcal{I}}^{reg}$. For sufficiently small $\varepsilon, \delta > 0$ (which depend on the choices of m and $\{\overline{J}_\tau\}$), there exists $\{\overline{J}_\tau^\diamondsuit(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ so that Lemma 3.2.22 holds.

Fix $\mathbf{y} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$, $\mathbf{y}' \in \mathcal{S}_{\mathbf{b},h(\mathbf{b})}$ and abbreviate

$$\mathcal{M} = \mathcal{M}^{I=2,n^*=m}_{\{\bar{J}^{\diamondsuit}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y},\mathbf{y}';\overline{\mathfrak{m}}),\ \ \mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}} = \mathcal{M}^{I=2,n^*=m,\{K_{\mathfrak{p}(\tau),\delta}\}}_{\{\bar{J}^{\diamondsuit}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y},\mathbf{y}';\overline{\mathfrak{m}}).$$

Let $\overline{\mathcal{M}}$ be the SFT compactification of $\mathcal{M}$. The limit SFT buildings can be described in a manner analogous to Definition I.7.3.1 and the proof of existence follows the same steps as that of Proposition I.7.3.2. This is because:

- the ends of $\overline{W}_{\tau}$ and $\overline{W}_{-\infty,i}$ are basically the same as the strip-like ends of $\overline{W}_{+}$ and $\overline{W}_{-}$ and
- the Lagrangian boundaries of $\overline{W}_{\tau}$ and $\overline{W}_{-\infty,i}$ are locally identical to those of $\overline{W}_{+}$ and $\overline{W}_{-}$.

The main point is that for each component of the limit SFT building the boundary punctures either map to points on the singular Lagrangian or to Reeb chords, including chords over z_{∞}.

Let $\partial\mathcal{M} = \overline{\mathcal{M}} - \mathcal{M}$ be the boundary of $\mathcal{M}$. If $U \subset [-\infty,+\infty]$, then we write $\partial_{U}\mathcal{M}$ for the set of $\bar{u}_{\infty} \in \partial\mathcal{M}$ where $\bar{u}_{\infty}$ is a building which corresponds to some $\tau \in U$. By Lemma 3.2.21, we may take $\mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}}$ to be regular.

Step 1 (Breaking at $+\infty$). Recall the definition of a *bad radial ray* $\mathcal{R}_{\phi}$ from Definition I.7.7.10. We now enlarge the class of bad radial rays as follows: Let $(\bar{J}_{+})_{\infty}$ be the limit of $(\bar{J}_{+})_{m}$ as $m \to \infty$. Let

$$\coprod_{\mathbf{y},\delta_0^{l_i}\boldsymbol{\gamma}'} \mathcal{M}^{I=0,(l_i)}_{(\bar{J}_{+})_{\infty}}(\mathbf{y},\delta_0^{l_i}\boldsymbol{\gamma}') = \{\mathcal{C}_1,\dots,\mathcal{C}_r\},$$

where the disjoint union is over all $\mathbf{y}$ and $\delta_0^{l_i}\boldsymbol{\gamma}'$ with $l_i > 0$ and (l_i) is the "trivial" partition of l_i.[7] Let $f_i : \mathbf{R}/2l_i\mathbf{Z} \to \mathbf{C}$ be the asymptotic eigenfunction corresponding to the end $\delta_0^{l_i}$ of $\mathcal{C}_i$. (We remark that $\mathcal{C}_1,\dots,\mathcal{C}_r$ and f_i also appear in Section I.7.7.2, but denote similar but different things.) We then add the radial rays which pass through

$$\{f_i(t) \mid i = 1,\dots,r;\ 0 < t < 2l_i;\ t \equiv 3/2 \bmod 2\}$$

to the class of bad radial rays. We can still assume that $\mathcal{R}_{\pi}$ is a good radial ray.

Recall the set $\widehat{\mathcal{O}}_k$ of orbit sets constructed from $\widehat{\mathcal{P}}$ (the set of simple Reeb orbits in *int*(N), together with h and e) which intersect $\overline{S} \times \{0\}$ exactly k times. We then have the following:

---

[7] Note that the orbit δ_0 is degenerate when $m = \infty$ and the moduli spaces with $\delta_0^{l_i}$ at the negative end are treated in the same way as in Section I.7.7.4.

Lemma **3.3.2.** — $\partial_{\{+\infty\}}\mathcal{M} \subset \mathrm{A}_1 \sqcup \mathrm{A}_2$,[8] *where*

$$\mathrm{A}_1 = \coprod_{\boldsymbol{\gamma}\in\widehat{\mathcal{O}}_{2g}} \left(\mathcal{M}^{\mathrm{I}=0}_{\mathrm{J}^{\Diamond}_{+}(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\mathbf{y},\boldsymbol{\gamma}) \times \mathcal{M}^{\mathrm{I}=2,n^*=m}_{\bar{\mathrm{J}}^{\Diamond}_{-}(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\boldsymbol{\gamma},\mathbf{y}';\overline{\mathfrak{m}}(+\infty))\right);$$

$$\mathrm{A}_2 = \coprod_{\delta_0\boldsymbol{\gamma},\{z_\infty\}\cup\mathbf{y}''} \left(\mathcal{M}^{\mathrm{I}=1,n^*=m-1,f_{\delta_0}}_{\bar{\mathrm{J}}^{\Diamond}_{+}(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\mathbf{y},\delta_0\boldsymbol{\gamma}) \times \mathcal{M}^{\mathrm{I}=0,n^*=0}_{\bar{\mathrm{J}}^{\Diamond}_{-}(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\delta_0\boldsymbol{\gamma},\{z_\infty\}\cup\mathbf{y}'')\right.$$
$$\left. \times \quad \mathcal{M}^{\mathrm{I}=1,n^*=1}_{\bar{\mathrm{J}}}(\{z_\infty\}\cup\mathbf{y}'',\mathbf{y}')\right),$$

if $\mathbf{y}' = \{x_i^j\}\times\mathbf{y}''$ *for some* x_i^j *and* $\mathrm{A}_2 = \varnothing$ *otherwise. The disjoint union for* A_2 *ranges over all* $\delta_0\boldsymbol{\gamma}$ *such that* $\boldsymbol{\gamma}\in\widehat{\mathcal{O}}_{2g-1}$ *and all* $\{z_\infty\}\cup\mathbf{y}''$ *such that* $\mathbf{y}'$ *can be written as* $\{x_i^j\}\cup\mathbf{y}''$ *for some* x_i^j. *Here we have omitted the potential contributions of connector components and we are writing* $x_i^0 := x_i$ *and* $x_i^1 := x_i'$.

We will explain the moduli spaces that are involved in A_2: f_{δ_0} is a nonzero normalized asymptotic eigenfunction of δ_0 at the negative end such that $f_{\delta_0}(\frac{3}{2})$ lies on the good radial ray $\mathcal{R}_\pi$. Used as a modifier, f_{δ_0} stands for "the normalized asymptotic eigenfunction at the negative end δ_0 is f_{δ_0}". If

$$\overline{u} = \overline{u}'\cup\overline{u}'' \in \mathcal{M}^{\mathrm{I}=0,n^*=0}_{\bar{\mathrm{J}}^{\Diamond}_{-}(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\delta_0\boldsymbol{\gamma},\{z_\infty\}\cup\mathbf{y}''),$$

then $\overline{u}$ consists of $\overline{u}' = \sigma_\infty^-$ and a curve $\overline{u}''$ from $\boldsymbol{\gamma}$ to $\mathbf{y}''$ which is arbitrarily close to a curve with image in W_-. If

$$\overline{u}\in\mathcal{M}^{\mathrm{I}=1,n^*=1}_{\bar{\mathrm{J}}}(\{z_\infty\}\cup\mathbf{y}'',\mathbf{y}'),$$

then $\mathbf{y}' = \{x_i^j\}\cup\mathbf{y}''$ for some i,j and $\overline{u}$ consists of one thin strip from z_∞ to x_i^j and $2g-1$ trivial strips.

Remark **3.3.3.** — As in the proof of the chain map property for Ψ from Section I.7.2, the point constraint of passing through $\overline{\mathfrak{m}}(+\infty)$ is converted to an asymptotic constraint of the normalized asymptotic eigenfunction being f_{δ_0} when a section at infinity σ_∞^- is present.

Lemma 3.3.2 will be proved in Section 3.4. Gluing the pairs in A_1 using the Hutchings-Taubes gluing theorem [HT1, HT2] (see Section I.6.5) accounts for the term $\Psi'\circ\widetilde{\Phi}$ in Equation (3.3.1).

Gluing the triples in A_2 accounts for the term $\partial_1\circ\widetilde{\mathrm{V}}$. This is similar to Section I.7.12, and the details will be omitted. For each triple $(\overline{v}_+,\overline{v}_-,\overline{v}_{-1,1})\in\mathrm{A}_2$, where

[8] Lemma 3.3.2 and its later analogs are compactness statements, which is why we write "⊂" even when equality holds. The equality will follow from the discusson of gluing, which appears later in this step.

the nontrivial component of $\overline{v}_{-1,1}$ is a thin strip from $z_{\infty,i,0}$ to x_i, there is another triple $(\overline{v}_+, \overline{v}_-, \overline{v}^*_{-1,1}) \in A_2$, where the nontrivial component of $\overline{v}^*_{-1,1}$ is a thin strip from $z_{\infty,i,1}$ to x'_i.

Since $\mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}}$ is regular but $\mathcal{M} - \mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}}$ is not a priori regular, it remains to verify the following:

Claim **3.3.4.** — *For $\varepsilon, \delta > 0$ sufficiently small, then there exists a truncation $\mathcal{M}'$ of $\mathcal{M}$ containing $\mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}}$ and whose boundary, when restricted to a neighborhood of $\tau = +\infty$, is $A_1 \cup \widetilde{A}_2$, where $\#A_2 \equiv \#\widetilde{A}_2$ mod 2.*

Note that there are two types of boundary points: (i) points of the SFT compactification and (ii) boundary points of a manifold with boundary. (i) gives A_1 and (ii) gives $\widetilde{A}_2$.

Proof. — There exist $\varepsilon, \delta > 0$ sufficiently small such that $\overline{v}_-$ passes through $K_{\mathfrak{p}(\tau),\delta}$ whenever $(\overline{v}_+, \overline{v}_-) \in A_1$. Hence if $\overline{u} \in \mathcal{M}$ is close to a curve in A_1, then $\overline{u} \in \mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}}$. This accounts for the term A_1 in the claim.

Next suppose that $\overline{u} \in \mathcal{M} - \mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}}$ for τ near $+\infty$. By the argument of Lemma I.7.2.3, $\overline{u}$ is arbitrarily close to a building of type A_2 and $\mathcal{M}'$ is obtained from $\mathcal{M}$ by truncating ends that are close to A_2. Then, by the adaptation of Theorem I.7.2.2 to our case, $\widetilde{A}_2 := \partial\mathcal{M}' - A_1$ satisfies $\#A_2 \equiv \#\widetilde{A}_2$ mod 2. ☐

Step 2 (Breaking at $-\infty$).

Lemma **3.3.5.** — $\partial_{\{-\infty\}}\mathcal{M} \subset A_3$, *where*

$$A_3 = \coprod_{\mathbf{y}_2,\mathbf{y}_4} \Big(\mathcal{M}^{I=0,n^*=0}_{\overline{J}^{\diamondsuit}_{-\infty,1}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\mathbf{y}, \mathbf{y}_2, \mathbf{y}', \overline{h}(\mathbf{y}_4)) \times \mathcal{M}^{I=2,n^*=m}_{\overline{J}^{\diamondsuit}_{-\infty,2}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\mathbf{y}_4, \mathbf{y}_2; \overline{\mathfrak{m}}(-\infty)) \Big).$$

Here the union is over all $\mathbf{y}_2, \mathbf{y}_4$ such that $\mathbf{y}_4 = \{x^{\#}_{ij(i)}\}_{i=1}^{2g}$ and $\mathbf{y}_2 = \{x^{\#}_{ik(i)}\}_{i=1}^{2g}$, where $j(i)$ is odd and $k(i)$ is even.

Lemma 3.3.5 will be proved in Section 3.5. Gluing the pairs $(\overline{v}_1, \overline{v}_2)$ in A_3 accounts for the term Θ'_0 in Equation (3.3.1). The map Θ'_0 is given by:

$$\langle \Theta'_0(\mathbf{y}), \mathbf{y}' \rangle = \sum_{\mathbf{y}_2,\mathbf{y}_4} \#\mathcal{M}^{I=0,n^*=0}_{\overline{J}^{\diamondsuit}_{-\infty,1}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\mathbf{y}, \mathbf{y}_2, \mathbf{y}', \overline{h}(\mathbf{y}_4)),$$

where $\mathbf{y}_2, \mathbf{y}_4$ are as in A_3. Write $\Theta_{\overline{\mathbf{b}},\overline{\mathbf{a}}}$ for the sum of all $\mathbf{y}_2$ from Lemma 3.3.5 and write $\Theta_{\overline{h}(\overline{\mathbf{a}}),\overline{h}(\overline{\mathbf{b}})}$ for the sum of all $\overline{h}(\mathbf{y}_4)$ from Lemma 3.3.5. Theorem 2.4.2 can be rephrased as (verification left to the reader):

Theorem **3.3.6.** — *Suppose* $\mathbf{y}_4 = \{x^{\#}_{ij(i)}\}_{i=1}^{2g}$ *and* $\mathbf{y}_2 = \{x^{\#}_{ik(i)}\}_{i=1}^{2g}$, *where* $j(i)$ *is odd and* $k(i)$ *is even. Then*

$$\#\mathcal{M}^{\mathrm{I}=2,n^*=m}_{\bar{\mathrm{J}}^{\Diamond}_{-\infty,2}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\mathbf{y}_4,\mathbf{y}_2;\overline{\mathfrak{m}}(-\infty)) \equiv 1 \ mod\ 2.$$

The argument of Claim 3.3.4 gives:

Claim **3.3.7.** — *For* $\varepsilon,\delta>0$ *sufficiently small, the restriction of* $\partial\mathcal{M}'$ *to a neighborhood of* $-\infty$ *is* A_3.

Step 3 (Breaking in the middle).

Lemma **3.3.8.** — $\partial_{(-\infty,+\infty)}\mathcal{M}\subset \mathrm{A}_4\sqcup\mathrm{A}_5$, *where:*

$$\mathrm{A}_4=\coprod_{\mathbf{y}''\in\mathcal{S}_{\mathbf{a},\hbar(\mathbf{a})}}\left(\mathcal{M}^{\mathrm{I}=1}_{\mathrm{J}}(\mathbf{y},\mathbf{y}'')\times\mathcal{M}^{\mathrm{I}=1,n^*=m}_{\{\bar{\mathrm{J}}^{\Diamond}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y}'',\mathbf{y}';\overline{\mathfrak{m}})\right);$$

$$\mathrm{A}_5=\coprod_{\mathbf{y}'''\in\mathcal{S}_{\mathbf{b},\hbar(\mathbf{b})}}\left(\mathcal{M}^{\mathrm{I}=1,n^*=m}_{\{\bar{\mathrm{J}}^{\Diamond}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y},\mathbf{y}''';\overline{\mathfrak{m}})\times\mathcal{M}^{\mathrm{I}=1}_{\mathrm{J}}(\mathbf{y}''',\mathbf{y}')\right).$$

Lemma 3.3.8 will be proved in Section 3.6. Using the technique of [Li, Prop. A.1 and A.2], we can glue each of the pairs in A_4 and A_5. This gluing accounts for the term $\partial'\mathrm{H}'+\mathrm{H}'\widetilde{\partial}$ in Equation (3.3.1), where the map H' is given by:

$$\langle\mathrm{H}'(\mathbf{y}),\mathbf{y}'\rangle=\sum_{\tau\in\mathcal{T}_1\cup\mathcal{T}_2}\#\mathcal{M}^{\mathrm{I}=1,n^*=m}_{\{\bar{\mathrm{J}}^{\Diamond}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y},\mathbf{y}';\overline{\mathfrak{m}}).$$

Claim **3.3.9.** — *For* $\varepsilon,\delta>0$ *sufficiently small,*

$$\partial\mathcal{M}'=\mathrm{A}_1\cup\widetilde{\mathrm{A}}_2\cup\mathrm{A}_3\cup\mathrm{A}_4\cup\mathrm{A}_5.$$

Step 4 (Additional degenerations). In this step we give the necessary modifications for

$$\mathcal{M}=\mathcal{M}^{\mathrm{I}=2,n^*=m+|\mathcal{I}|}_{\{\bar{\mathrm{J}}^{\Diamond}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{z},\mathbf{y}';\overline{\mathfrak{m}}),$$

where $\mathbf{z}=\{z_{\infty,i}\}_{i\in\mathcal{I}}\cup\mathbf{y}$, $\mathbf{y}$ and $\mathbf{y}'$ are tuples in S, and $\mathcal{I}\subset\{1,\dots,2g\}$ with $|\mathcal{I}|>0$.

The following is proved in Section 3.7:

Lemma **3.3.10.** — $\partial_{\{+\infty\}}\mathcal{M}\subset\mathrm{A}_2'$, *where*

$$\mathrm{A}_2'=\coprod_{\delta_0\boldsymbol{\gamma},\{z_\infty\}\cup\mathbf{y}''}\Big(\mathcal{M}^{\mathrm{I}=1,n^*=m+|\mathcal{I}|-1,f_{\delta_0},\dagger}_{\bar{\mathrm{J}}^{\Diamond}_{+}(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\mathbf{z},\delta_0\boldsymbol{\gamma})$$
$$\times\mathcal{M}^{\mathrm{I}=0,n^*=0}_{\bar{\mathrm{J}}^{\Diamond}_{-}(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\delta_0\boldsymbol{\gamma},\{z_\infty\}\cup\mathbf{y}'')$$

$$\times \mathcal{M}_{\overline{J}}^{I=1,n^*=1}(\{z_\infty\} \cup \mathbf{y}'', \mathbf{y}')\Big),$$

if $\mathbf{y}' = \{x_i^j\} \times \mathbf{y}''$ *for some* x_i^j *and* $A_2' = \varnothing$ *otherwise. The disjoint union for* A_2' *ranges over all* $\delta_0\boldsymbol{\gamma}$ *such that* $\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_{2g-1}$ *and all* $\{z_\infty\} \cup \mathbf{y}''$ *such that* $\mathbf{y}'$ *can be written as* $\{x_i^j\} \cup \mathbf{y}''$ *for some* x_i^j*. Here we have omitted the potential contributions of connector components and we are writing* $x_i^0 := x_i$ *and* $x_i^1 := x_i'$*.*

If $\mathcal{I} \neq \varnothing$ (i.e., $|\mathcal{I}| \geq 1$), then $\widetilde{\Phi}(\mathbf{z}) = 0$ by Lemma I.6.6.5 and we have $\Psi \circ \widetilde{\Phi}(\mathbf{z}) = 0$; this is consistent with the analog of A_1 being empty. On the other hand, gluing the triples in A_2' accounts for the term $\widetilde{\partial}_1 \circ \widetilde{V}$.

Next, the following is proved in Section 3.8:

Lemma **3.3.11.** — $\partial_{\{-\infty\}}\mathcal{M} \subset A_3'$*, where:*

$$A_3' = \coprod_{\mathbf{y}_2, \mathbf{y}_4} \Big(\mathcal{M}^{I=0,n^*=|\mathcal{I}|}_{\overline{J}^{\diamondsuit}_{-\infty,1}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\mathbf{z}, \mathbf{y}_2, \mathbf{y}', \overline{h}(\mathbf{y}_4))$$
$$\times \mathcal{M}^{I=2,n^*=m}_{\overline{J}^{\diamondsuit}_{-\infty,2}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\mathbf{y}_4, \mathbf{y}_2; \overline{\mathfrak{m}}(-\infty))\Big)$$

and the summation is over $\mathbf{y}_2$ *and* $\mathbf{y}_4$ *as in Lemma 3.3.5.*

We define:

$$\langle \Theta_0'(\mathbf{z}), \mathbf{y}'\rangle = \sum_{\mathbf{y}_2, \mathbf{y}_4} \#\mathcal{M}^{I=0,n^*=|\mathcal{I}|}_{\overline{J}^{\diamondsuit}_{-\infty,1}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\mathbf{z}, \mathbf{y}_2, \mathbf{y}', \overline{h}(\mathbf{y}_4)),$$

where $\mathbf{y}_2$, $\mathbf{y}_4$ are summands of $\Theta_{\overline{\mathbf{b}},\overline{\mathbf{a}}}$ and $\Theta_{\overline{h}(\overline{\mathbf{a}}),\overline{h}(\overline{\mathbf{b}})}$.

The following is proved in Section 3.9. The corresponding gluing accounts for the term $\partial' H' + H'\widetilde{\partial}$ in Equation (3.3.1) when $|\mathcal{I}| \geq 1$. Here the map H' is given by:

$$\langle H'(\mathbf{z}), \mathbf{y}'\rangle = \sum_{\tau \in \mathcal{T}_1 \cup \mathcal{T}_2} \#\mathcal{M}^{I=1,n^*=m+|\mathcal{I}|}_{\{\overline{J}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{z}, \mathbf{y}'; \overline{\mathfrak{m}}).$$

Lemma **3.3.12.** — $\partial_{(-\infty,+\infty)}\mathcal{M} \subset A_4' \sqcup A_5'$*, where:*

$$A_4' = \coprod_{\mathbf{z}'} \Big(\mathcal{M}^{I=1,n^*\leq|\mathcal{I}|}_{\overline{J}}(\mathbf{z}, \mathbf{z}') \times \mathcal{M}^{I=1,n^*\leq m+|\mathcal{I}|}_{\{\overline{J}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{z}', \mathbf{y}'; \overline{\mathfrak{m}})\Big);$$

$$A_5' = \coprod_{\mathbf{y}''' \in \mathcal{S}_{\mathbf{b},h(\mathbf{b})}} \Big(\mathcal{M}^{I=1,n^*=m+|\mathcal{I}|}_{\{\overline{J}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{z}, \mathbf{y}'''; \overline{\mathfrak{m}}) \times \mathcal{M}^{I=1}_{J}(\mathbf{y}''', \mathbf{y}')\Big).$$

Moreover, if $\overline{v}_{1,1} \in \mathcal{M}^{I=1,n^*\leq|\mathcal{I}|}_{\overline{J}}(\mathbf{z}, \mathbf{z}')$*, then either (i)* $\overline{v}^{\sharp}_{1,1}$ *is a thin strip and* $\overline{v}^{\flat}_{1,1}$ *is a union of trivial strips, or (ii)* $\overline{v}^{\sharp}_{1,1} = \varnothing$ *and* $\overline{v}^{\flat}_{1,1}$ *has image in* W*.*

As before, the analog of Claim 3.3.9 holds.

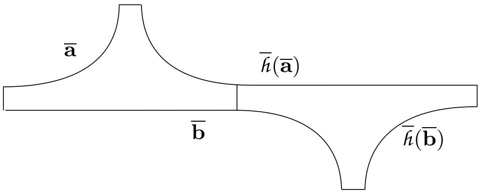


FIG. 7. — Degeneration of the base $B_{-\infty,1}$

Step 5 (The map Θ_0). The map Θ_0 is defined as follows:

$$\Theta_0 : \widetilde{CF}(\mathbf{a}, h(\mathbf{a})) \to \widehat{CF}(\mathbf{b}, h(\mathbf{b})),$$
$$\langle \Theta_0(\mathbf{z}), [\mathbf{y}'] \rangle = \sum_{\mathbf{y}_2,\mathbf{y}_4,\mathbf{y}'\in[\mathbf{y}']} \#\mathcal{M}^{I=0,n^*=|\mathcal{I}|}_{\overline{J}_{-\infty,1}}(\mathbf{z}, \mathbf{y}_2, \mathbf{y}', \overline{h}(\mathbf{y}_4)).$$

where $\mathbf{y}_2$, $\mathbf{y}_4$ are summands of $\Theta_{\overline{\mathbf{b}},\overline{\mathbf{a}}}$ and $\Theta_{\overline{h}(\overline{\mathbf{a}}),\overline{h}(\overline{\mathbf{b}})}$. It is clear that $\Theta_0 = q \circ \Theta_0'$.

Lemma **3.3.13.** — *The map Θ_0 induces an isomorphism on the level of homology.*

Proof of Lemma 3.3.13. We degenerate the base $B_{-\infty,1}$ as given in Figure 7. Slightly more precisely, we take $B_{-\infty,1,\tau'}$, $\tau' \in [0,+\infty)$, which is obtained from

$$\{-2 \leq s \leq 2\} \cup \{0 \leq t \leq 1, s \leq 2\} \cup \{\tau' \leq t \leq \tau' + 1, s \geq -2\} \subset \mathbf{R}^2 = \mathbf{C}$$

by smoothing the corners and use the complex structure $j_{-\infty,1,\tau'}$ induced from the standard complex structure on $\mathbf{C}$; then $B_{-\infty,1,\tau'=0} = B_{-\infty,1}$ and $j_{-\infty,1,\tau'=0} = j_{-\infty,1}$. Let $\overline{W}_{-\infty,1,\tau'} = B_{-\infty,1,\tau'} \times \overline{S}$ and define the almost complex structures $\overline{J}_{-\infty,1,\tau'}$ on $\overline{W}_{-\infty,1,\tau'}$ in the same way as on $\overline{W}_{-\infty,1}$ with $j_{-\infty,1}$ replaced by $j_{-\infty,1,\tau'}$.

The 1-parameter family $(\overline{W}_{-\infty,1,\tau'}, \overline{J}_{-\infty,1,\tau'})$ induces a chain homotopy between Θ_0 and the composition $\Theta_B \circ \Theta_T$, where:

$$\Theta_T : \widetilde{CF}(\mathbf{a}, h(\mathbf{a})) \to \widehat{CF}(\mathbf{b}, h(\mathbf{a})),$$
$$\Theta_B : \widehat{CF}(\mathbf{b}, h(\mathbf{a})) \to \widehat{CF}(\mathbf{b}, h(\mathbf{b}))$$

are defined by by counting holomorphic multisections of $\Delta \times \overline{S}$, where Δ is a triangle (i.e., a disk with three boundary punctures), which are asymptotic to the top generators $\Theta_{\overline{\mathbf{b}},\overline{\mathbf{a}}} \in \widehat{CF}(\overline{\mathbf{b}}, \overline{\mathbf{a}})$ and $\Theta_{\overline{h}(\overline{\mathbf{a}}),\overline{h}(\overline{\mathbf{b}})} \in \widehat{CF}(\overline{h}(\overline{\mathbf{a}}), \overline{h}(\overline{\mathbf{b}}))$, respectively, at one of the vertices. Usually in the definitions of chain maps such as Θ_T and Θ_B we require $\Theta_{\overline{\mathbf{b}},\overline{\mathbf{a}}}$ and $\Theta_{\overline{h}(\overline{\mathbf{a}}),\overline{h}(\overline{\mathbf{b}})}$ to be cycles. In our case they are not, but we have workarounds. In the Θ_T case we must consider one undesirable type of breaking which can a priori occur as we vary τ': a two-level building $\overline{u}_1 \cup \overline{u}_2$, where

- $\overline{u}_1$ is an index $I = -1$, $n^* = |\mathcal{I}|$ curve in $\overline{W}_{-\infty,1,\tau'}$ with ends $\mathbf{z}, \mathbf{y}_2', \mathbf{y}', \overline{h}(\mathbf{y_4})$;

– $\overline{u}_2 \in \mathcal{M}^{I=1,n^*=0}_{\overline{J}_{-\infty,2}}(\mathbf{y}_2, \mathbf{y}'_2)$; and
– $\mathbf{y}_2$ and $\mathbf{y}'_2$ differ by replacing one $x^{\#}_{ik(i)}$ by $x^{\#}_{ij(i)}$ where $j(i)$ is odd and $k(i)$ is even.

One can verify that the only possible component of $\overline{u}_1$ which has left end $x^{\#}_{i1}$ and which satisfies $n^* \leq |\mathcal{I}|$, projects to the quadrilateral Q with edges $\overline{a}_i$, $\overline{b}_i$, $\overline{h}(\overline{b}_i)$, $\overline{h}(\overline{a}_i)$ in Figure 6. However, the component corresponding to Q has ECH index $I = 0$, which is a contradiction. The index calculation basically follows from the fact that Q has three angles smaller than π and one larger. This implies that undesirable breakings do not exist and that all the breakings contribute toward the chain homotopy.

Both Θ_T and Θ_B — and hence Θ_0 — induce isomorphisms on the level of homology. □

This completes the proof of Theorem 3.3.1, assuming the results from Sections 3.4–3.9. □

3.4. *Degeneration at* $+\infty$. — In this subsection we study the limits of holomorphic maps to $\overline{W}_\tau$ as $\tau \to \infty$, i.e., when $\overline{W}_\tau$ degenerates into the concatenation of $\overline{W}_+$ with $\overline{W}_-$ along the ECH-type end, in order to prove Lemma 3.3.2.

We assume that $m \gg 0$; $\varepsilon, \delta > 0$ are sufficiently small; and $\{\overline{J}_\tau\} \in \overline{\mathcal{I}}^{reg}$ and $\{\overline{J}^{\Diamond}_\tau(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ satisfies Lemma 3.2.22. Fix $\mathbf{y} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$, $\mathbf{y}' \in \mathcal{S}_{\mathbf{b},h(\mathbf{b})}$ and let

$$\mathcal{M} = \mathcal{M}^{I=2,n^*=m}_{\{\overline{J}^{\Diamond}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y}, \mathbf{y}'; \overline{\mathfrak{m}}), \quad \mathcal{M}_\tau = \mathcal{M}^{I=2,n^*=m}_{\overline{J}^{\Diamond}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))}(\mathbf{y}, \mathbf{y}'; \overline{\mathfrak{m}}(\tau)).$$

We will analyze $\partial_{\{+\infty\}}\mathcal{M}$.

Let $\overline{u}_i$, $i \in \mathbf{N}$, be a sequence of curves in $\mathcal{M}$ such that $\overline{u}_i \in \mathcal{M}_{\tau_i}$ and $\lim_{i\to\infty} \tau_i = +\infty$, and let

$$\overline{u}_\infty = (\overline{v}_{-1,1} \cup \cdots \cup \overline{v}_{-1,c}) \cup \overline{v}_- \cup (\overline{v}_{0,1} \cup \cdots \cup \overline{v}_{0,b}) \cup \overline{v}_+ \cup (\overline{v}_{1,1} \cup \cdots \cup \overline{v}_{1,a}) \quad (\mathbf{3.4.1})$$

be the limit holomorphic building, where each $\overline{v}_*$ is an SFT level (recall Notation 1.0.3 regarding the use of subscripts $*$), the levels are ordered from bottom to top as we go from left to right, $\overline{v}_{-1,j}, j = 1, \dots, c$, maps to $\overline{W}$; $\overline{v}_-$ maps to $\overline{W}_-$; $\overline{v}_{0,j}, j = 1, \dots, b$, maps to $\overline{W}'$; $\overline{v}_+$ maps to $\overline{W}_+$; and $\overline{v}_{1,j}, j = 1, \dots, a$, maps to $\overline{W}$. Here we are allowing the possibility that a, b, or $c = 0$. For notational convenience, sometimes we will refer to $\overline{v}_+$ as $\overline{v}_{0,b+1}$ or $\overline{v}_{1,0}$ and $\overline{v}_-$ as $\overline{v}_{-1,c+1}$ or $\overline{v}_{0,0}$.

Notation **3.4.1.** — We will be using the conventions established in Section 3.2.3 (and in particular Notation 3.2.10).

– We write $\dot{F}_*$, $\dot{F}'_*$, $\dot{F}''_*$ for the domains of $\overline{v}_*$, $\overline{v}'_*$, $\overline{v}''_*$.
– We write p_* for the covering degree of $\overline{v}'_*$.

– If $\widetilde{v}_{*_1}$ is a union of components of a level $\overline{v}_{*_1}$ and $\widetilde{v}_{*_2}$ is a union of components of a possibly different level $\overline{v}_{*_2}$, then we write $\widetilde{v}_{*_1} \succ \widetilde{v}_{*_2}$ (resp. $\widetilde{v}_{*_1} \succeq \widetilde{v}_{*_2}$) to indicate that the level $\overline{v}_{*_1}$ is above (resp. equal to or above) the level $\overline{v}_{*_2}$.

Since ghost components can be eliminated by the discussion in Lemma I.6.1.8 they will not be explicitly mentioned in the rest of the paper.

We have the following two constraints:

$$n^*(\overline{u}_i) = \sum_{\overline{v}_*} n^*(\overline{v}_*) = m; \tag{3.4.2}$$

$$\mathrm{I}(\overline{u}_i) = \sum_{\overline{v}_*} \mathrm{I}(\overline{v}_*) = 2, \tag{3.4.3}$$

where the summations are over all the levels $\overline{v}_*$ of $\overline{u}_\infty$.

Outline of proof of Lemma 3.3.2. The proof of Lemma 3.3.2 follows the same general outline of Sections I.7.4–I.7.11: First we calculate the contributions to n^* of the ends that limit to multiples of z_∞ or δ_0 in Section 3.4.1 and obtain lower bounds on the ECH indices of the levels $\overline{v}_*$ in Section 3.4.2, under the assumption that there are no boundary points at z_∞. Boundary points at z_∞ are treated in Sections 3.4.3 and 3.4.4. The main new difficulty is to show that $\mathrm{I}(\overline{v}_*) \geq \mathrm{I}(\overline{v}'_*) + \mathrm{I}(\overline{v}''_*)$ for $\overline{v}_* \succeq \overline{v}_+$; this uses the more complicated version of the ECH index inequality given in Lemma I.5.7.21. We then use Equations (3.4.2) and (3.4.3) to obtain Lemma 3.4.21, which describes the case when $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_*$, and Lemma 3.4.25, which gives a preliminary list when $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some $\overline{v}_*$. The renormalization argument from Sections I.7.8–7.10, given in Lemma 3.4.28, eliminates all the possibilities with the exception of Case (2_1) of Lemma 3.4.25 when $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some $\overline{v}_*$.

At this point the reader is strongly encouraged to review Section I.5.7 on holomorphic curves with ends at z_∞.

3.4.1. *Intersection numbers.* — In this subsection we give the analogs of Lemmas I.7.4.1–I.7.4.5 for $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$:

Lemma **3.4.2.**

(1) *If $\overline{v}''_*$ has a negative end $\mathcal{E}_-$ that converges to δ_0^p, then $n^*(\mathcal{E}_-) \geq m - p$.*
(2) *If $\overline{v}''_*$ has a positive end $\mathcal{E}_+$ that converges to δ_0^p, then $n^*(\mathcal{E}_+) \geq p$.*

Proof. — This is analogous to Lemma I.7.4.1 and is proved in the same way. □

Definition **3.4.3.** — *An end $\mathcal{E}$ is* nontrivial *if it is not an end of a trivial cylinder or trivial strip.*

Convention. In this paper we assume that an end of a holomorphic curve is connected, unless stated otherwise.

Recall the sequence $\overline{u}_i \in \mathcal{M}_{\tau_i}$ with $\tau_i \to +\infty$ and its limit SFT building $\overline{u}_\infty$ given by Equation (3.4.1). Let $\dot{G}_i$ be the domain of $\overline{u}_i$. Fix $k \in \{1, \dots, 2g\}$. The following Lemma 3.4.4 describes the breaking/"SFT limit" of the component of $\overline{u}_i|_{\partial \dot{G}_i}$ which maps to $L^{\tau_i,+}_{\overline{a}_k}$, into a sequence of paths $g^1_a, \dots, g^1_1, g_0, g^0_1, \dots, g^0_a, g^*_j \subset \mathrm{Im}(\overline{v}_{1,j}), j \geq 0, * = 0, 1, \varnothing$, as $\tau_i \to \infty$.

Let $\partial_+ B_\tau$ (resp. $\partial_+ \overline{W}_\tau$) be the $s > 0$ boundary of B_τ (resp. $\overline{W}_\tau$) and let $\mathcal{C}_{1,j}, j \geq 0$, be the data for $\overline{v}'_{1,j}$; see Section I.5.7.2 for the definition of the data $\mathcal{C}_{1,j}$. Then we have the following, whose proof is immediate.

Lemma **3.4.4.** — *Given a sequence $\overline{u}_i$ and a choice of $k \in \{1, \dots, 2g\}$, the components of $\overline{u}_i|_{\partial \dot{G}_i}$ which map to $L^{\tau_i,+}_{\overline{a}_k}$ converge uniquely (in the "SFT sense") to a sequence of paths*

$$g^1_a, \dots, g^1_1, g_0, g^0_1, \dots, g^0_a \tag{3.4.4}$$

which satisfies the following:

(1) $g^*_j = \overline{v}_{1,j}(f^*_j)$ *where* $j \geq 0$, $* = 0, 1, \varnothing$, *and* f^*_j *is a (connected) component of* $\partial \dot{F}_{1,j}$, *where* $\dot{F}_*$ *is as given in Notation 3.4.1.*

(2) *If* f^*_j *is a component of* $\partial \dot{F}''_{1,j}$, *then* $g^1_j \subset \mathbf{R} \times \{1\} \times \overline{a}_k$ *and* $g^0_j \subset \mathbf{R} \times \{0\} \times \overline{h}(\overline{a}_k)$ *if* $j > 0$ *and* $g_0 \subset L^+_{\overline{a}_k}$ *if* $j = 0$.

(3) *If* f^*_j *is a component of* $\partial \dot{F}'_{1,j}$, *then* f^*_j and also g^*_j *come with extra data* $\mathcal{C}_{1,j}$ *which assigns:*

$$f^1_j, g^1_j \mapsto L_{\overline{a}_{k,l}} = \mathbf{R} \times \{1\} \times \overline{a}_{k,l},$$

$$f^0_j, g^0_j \mapsto L_{\overline{h}(\overline{a}_{k,l'})} = \mathbf{R} \times \{0\} \times \overline{h}(\overline{a}_{k,l'}),$$

$$f_0, g_0 \mapsto L^+_{\overline{a}_{k,l''}},$$

for some $l, l', l'' = 0$ *or* 1.

For convenience we write $g_0 = g^1_0 = g^0_0$. We say an element g^*_j is *trivial* if the corresponding f^*_j satisfies (3).

Definition **3.4.5** *(Continuations).* — *If* g^*_j *is any element of Sequence* (3.4.4), *then Sequence* (3.4.4) *is the* continuation of g^*_j along $\partial_+ B_\tau$ *and the terms to the right of* g^*_j *in Sequence* (3.4.4) *form the* continuation of g^*_j in the direction of $\partial_+ B_\tau$.

Lemma 3.4.4 and Definition 3.4.5 play an important role in the proof of Lemma 3.4.7, which is a refined version of Lemma I.7.4.2, where the ends are considered collectively as well as individually. The proof strategy will usually be referred to as the *continuation argument.*

Let $\rho_0 > 0$ be small and let $\pi_{D^2_{\rho_0}}$ be the projection of

$$(\mathbf{3.4.5})\quad \mathfrak{D}_1 := \{\rho \le \rho_0\} \subset \overline{W} - int(W) \text{ or } \mathfrak{D}_2 := \{\rho \le \rho_0, |s| \ge l(\tau)+1\} \subset \overline{W}_\tau - int(W_\tau)$$

to $D^2_{\rho_0} = \{(\rho,\phi) \mid \rho \le \rho_0\} \subset \overline{S}$ along the stable Hamiltonian vector field $\overline{R}_\tau$ which was defined in Section 3.1.5.

Remark **3.4.6.** — $\pi_{D^2_{\rho_0}}$ maps the intersection of $\mathbf{R} \times \delta_{\rho_0,\phi_0^+}$ (which appears in the definition of n^* in Equation (3.2.3)) and one of $\mathfrak{D}_1$ or $\mathfrak{D}_2$ to m equally spaced points on $\partial D^2_{\rho_0}$.

Lemma **3.4.7.** — *Suppose $\overline{v}'_{1,j} \cup \overline{v}^\sharp_{1,j} \neq \varnothing$ for some $j > 0$. Let $\mathcal{E}_{-,i}$, $i = 1, \dots, q$, be the negative ends of $\cup_{j=1}^a \overline{v}^\sharp_{1,j}$ that converge to z_∞ and let $\mathcal{E}_{+,i}$, $i = 1, \dots, r$, be the positive ends of $\cup_{j=0}^{a-1} \overline{v}^\sharp_{1,j}$ that converge to z_∞.*

(1) *For each i,*

$$(\mathbf{3.4.6})\qquad n^*(\mathcal{E}_{-,i}) \ge k_0 - 1 \gg 2g,$$

where the constant k_0 is as given in Section I.5.2.2.

(2) *If there are no boundary points at z_∞, then*

$$(\mathbf{3.4.7})\qquad n^*((\cup_{i=1}^q \mathcal{E}_{-,i}) \cup (\cup_{i=1}^r \mathcal{E}_{+,i})) \ge m - p_+,$$

where $p_+ = \deg(\overline{v}'_+)$, and

$$D^2_{\rho_0} - (\cup_{i=1}^q \pi_{D^2_{\rho_0}}(\mathcal{E}_{-,i})) \cup (\cup_{i=1}^r \pi_{D^2_{\rho_0}}(\mathcal{E}_{+,i}))$$

consists of at most p_+ thin sectors.

Recall that a sector $\mathfrak{S}$ of $D^2_{\rho_0}$ from ϕ_0 to ϕ_1 is the map

$$[0,\rho_0] \times [\phi_0,\phi_1] \to D^2_{\rho_0}, \quad (\rho,\phi) \mapsto \rho e^{i\phi},$$

or its image (by abuse of notation). A *thin sector* (also referred to as a *thin wedge* in [I]) is the smallest counterclockwise sector from $\overline{a}_{i,j} \cap D^2_{\rho_0}$ to $\overline{h}(\overline{a}_{i,j}) \cap D^2_{\rho_0}$ for some i,j and has angle $\frac{2\pi}{m}$.

Following Definition I.7.4.4, a point $p \in \partial \dot{F}''_*$ is a *boundary point at z_∞* if $\overline{v}''_*(p) \in (L_{\overline{h}(\overline{\mathbf{a}})} - L_{\overline{h}(\widehat{\mathbf{a}})}) \cup (L_{\overline{\mathbf{a}}} - L_{\widehat{\mathbf{a}}}) = \mathbf{R} \times \{0,1\} \times \{z_\infty\}$ for $* = (1,j)$, $j > 0$, and $\overline{v}''_*(p) \in L^+_{\overline{\mathbf{a}}} - L^+_{\widehat{\mathbf{a}}}$ for $* = +$.

Proof. — We consider the $\pi_{D^2_{\rho_0}}$-projections of the positive and negative ends of $\overline{v}_{1,j}$, $j > 0$, and the positive ends of $\overline{v}_{1,0}$ that limit to z_∞.

(1) Given a nontrivial negative end $\mathcal{E}_{-,i}$ limiting to z_∞, its projection can be written as:

$$\pi_{D^2_{\rho_0}}(\mathcal{E}_{-,i}) = \mathfrak{S}(\overline{h}(\overline{a}_{k,l}), \overline{a}_{k',l'}),$$

where $\mathfrak{S}(A, B)$ denotes the smallest counterclockwise sector from the radial ray A to the radial ray B. Then Equation (3.4.6) follows from Remark 3.4.6 and the fact that the angle between $\overline{h}(\overline{a}_{k,l})$ and $\overline{a}_{k',l'}$ is at least $\frac{2\pi(k_0-1)}{m}$ as defined in Section I.5.2.2. (This is more or less the same as Lemma I.7.4.2(2); in fact "$> 2g$" in the statement can be replaced by "$\geq k_0 - 1 \gg 2g$".)

(2) We start at a nontrivial negative end of some $\overline{v}_{1,j_1}$ limiting to z_∞, which we call $\mathcal{E}_{-,1}$ without loss of generality. Then

$$\pi_{D^2_{\rho_0}}(\mathcal{E}_{-,1}) = \mathfrak{S}(\overline{h}(\overline{a}_{k_1,l_1}), \overline{a}_{k_2,l_2}).$$

We analyze the continuation

$$g^1_{j_1-1,1}, \dots, g^1_{1,1}, g_{0,1}, g^0_{1,1}, \dots, g^0_{a,1} \tag{3.4.8}$$

of $g^1_{j_1,1} \supset \partial_1\mathcal{E}_{-,1}$ in the direction of $\partial_+ B_\tau$. Here $\partial_k\mathcal{E}_{-,i}$, $k = 0, 1$, is the $t = k$ boundary of $\mathcal{E}_{-,i}$, and $f^{*_1}_{*_2}$ corresponds to $g^{*_1}_{*_2}$ as in Definition 3.4.5.

(i) Suppose there is some $0 \leq j < j_1$ such that $g^1_{j,1}$ is nontrivial. Let $j_2 \geq 0$ be the first such occurrence in the continuation. Then $\overline{v}^\sharp_{1,j_2}$ has some nontrivial end which we call $\mathcal{E}_{+,1}$, such that

$$\pi_{D^2_{\rho_0}}(\mathcal{E}_{+,1}) = \mathfrak{S}(\overline{a}_{k_2,l_2}, \overline{h}(\overline{a}_{k_3,l_3})).$$

This is because all the terms of Equation (3.4.8) are assigned $L_{\overline{a}_{k_2}}$ or $L^+_{\overline{a}_{k_2}}$ and all the terms between $g^1_{j_1,1}$ and $g^1_{j_2,1}$ correspond to the same $L_{\overline{a}_{k_2,l_2}}$.

(ii) On the other hand, if $g^1_{j,1}$ is trivial for all $0 \leq j \leq j_1$, then we set $j_2 = 0$ and $\overline{h}(\overline{a}_{k_2,l_2}) = \overline{h}(\overline{a}_{k_3,l_3})$ and skip $\mathfrak{S}(\overline{a}_{k_2,l_2}, \overline{h}(\overline{a}_{k_3,l_3}))$, which is a thin sector.

Next we consider the continuation

$$g^0_{j_2+1,2}, \dots, g^0_{a,2}$$

of $g^0_{j_2,2}$ in the direction of $\partial_+ B_\tau$. Here $g^0_{j_2,2} \supset \partial_0\mathcal{E}_{+,1}$ if (i) holds; and $j_2 = 0$, $g_{0,2} = g_{0,1}$, and $f_{0,2} = f_{0,1}$ if (ii) holds. There must exist some nontrivial $g^0_{j,2}$, $j > j_2$, and we write j_3 for the first such occurrence in the continuation. Then $\overline{v}^\sharp_{1,j_3}$ has some nontrivial end which we call $\mathcal{E}_{-,2}$, such that

$$\pi_{D^2_{\rho_0}}(\mathcal{E}_{-,2}) = \mathfrak{S}(\overline{h}(\overline{a}_{k_3,l_3}), \overline{a}_{k_4,l_4}).$$

Continuing in the same manner, we eventually return to $\mathcal{E}_{-,1}$, and the sectors

$$\mathfrak{S}(\overline{h}(\overline{a}_{k_1,l_1}), \overline{a}_{k_2,l_2}), \mathfrak{S}(\overline{a}_{k_2,l_2}, \overline{h}(\overline{a}_{k_3,l_3})), \mathfrak{S}(\overline{h}(\overline{a}_{k_3,l_3}), \overline{a}_{k_4,l_4}), \dots, \tag{3.4.9}$$

with some thin sectors of type $\mathfrak{S}(\overline{a}_{k_{2i},l_{2i}}, \overline{h}(\overline{a}_{k_{2i+1},l_{2i+1}}))$ omitted if they correspond to (ii), sweep out a neighborhood of z_∞ in $D^2_{\rho_0}$, possibly more than once and with the possible exception of p_+ thin sectors. This proves (2). □

We remark that, a posteriori, the sectors from Equation (3.4.9) sweep out a neighborhood of z_∞ only once in view of Equation (3.4.7).

Lemma **3.4.8.** — *If* $\mathfrak{p} \in \partial F''_*$ *is a boundary point of* $\overline{v}''_*$ *at* z_∞*, then* $n^*(\overline{v}''_*(N(\mathfrak{p}))) \geq k_0 - 1 \gg 2g$*, where* $N(\mathfrak{p}) \subset F''_*$ *is a sufficiently small neighborhood of* $\mathfrak{p}$.

Proof. — The proof is the same as that of Lemma I.7.4.5. □

3.4.2. *Bounds on ECH indices.* — The goal of this subsection is to show the nonnegativity of $I(\overline{v}_*)$ except when $\overline{v}_* = \overline{v}_+$, *under the assumption that there are no boundary points at* z_∞; see Lemma 3.4.14. The main new difficulty is to show that $I(\overline{v}_*) \geq 0$ for $\overline{v}_* \succ \overline{v}_+$. If $\overline{v}_* \succ \overline{v}_+$ and $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$, then we need to apply the version of the ECH index inequality in the presence of ends that limit to z_∞ (Lemma I.5.7.21). To apply Lemma I.5.7.21, we need to verify a certain "alternating property" for the ends of $\overline{v}_*$ that limit to z_∞; this is the content of Lemma 3.4.13.

Remark **3.4.9.** — The ECH index in the presence of ends that limit to z_∞ was given in Definition I.5.7.3. Although the definition of the ECH index involves a groomed multivalued trivialization τ (Definition I.5.7.5), by Lemma I.5.7.20 it is independent of the choice of groomed multivalued trivialization and we may write $I_\tau(\overline{v}_*)$ or $I(\overline{v}_*)$.

Let $A_\varepsilon = \partial D^2_\varepsilon \times [0,1]$ for $0 < \varepsilon < \rho_0$ small and let $\pi_{[0,1]\times\overline{S}}$ be the projection of $\overline{W}$ or the positive end of $\overline{W}_\tau$ to $[0,1]\times\overline{S}$. The following is a corollary of the proof of Lemma 3.4.7:

Corollary **3.4.10.** — *The intersection* $\mathfrak{c} := \pi_{[0,1]\times\overline{S}}(\cup_i \mathcal{E}_{-,i}) \cap A_\varepsilon$ *is groomed and the sets* P_0 *and* P_1 *of initial and terminal points of* $\mathfrak{c}$ *alternate along* $(0, 2\pi)$.

Note that we will often view P_0 *and* P_1 *(and analogous defined points on* A_ε*) as subsets of* ∂D^2_ε.

Definition **3.4.11.** — *A* cycle $\mathcal{Z} = (\mathfrak{z}_1 \to \mathfrak{z}_2 \to \cdots \to \mathfrak{z}_k \to \mathfrak{z}_1)$ on $\partial D^2_\varepsilon = \mathbf{R}/2\pi\mathbf{Z}$ *is a sequence of points* $\mathfrak{z}_i \in \mathbf{R}/2\pi\mathbf{Z}$*, together with chords in* $\mathbf{R}/2\pi\mathbf{Z}$ *from* $\mathfrak{z}_i$ *to* $\mathfrak{z}_{i+1}$*,*[9] *where the indices are taken modulo* k.

[9] By "chord" we mean a path with positive derivative in the **R**-direction.

The continuation method from Lemma 3.4.7 gives a cycle

$$\mathcal{Z} = (\mathfrak{z}_{10} \to \mathfrak{z}_{11} \to \cdots \to \mathfrak{z}_{k0} \to \mathfrak{z}_{k1} \to \mathfrak{z}_{10}),$$

satisfying the following:

(1) the cycle winds around $\mathbf{R}/2\pi\mathbf{Z}$ once;
(2) $P_i = \{\mathfrak{z}_{1i}, \dots, \mathfrak{z}_{ki}\}$ for $i = 0, 1$ and each point of P_i appears only once in $\mathcal{Z}$; and
(3) the chords correspond to the sectors listed in (3.4.9) as well as thin sectors of type $\mathfrak{S}(\overline{a}_{k',l'}, \overline{h}(\overline{a}_{k',l'}))$ that are skipped in Step (ii) of Lemma 3.4.7.

Next let $\overrightarrow{\mathcal{D}}_{\pm,j}$ be the data at z_∞ for the $\pm$ end of $\overline{v}_{1,j}$ and let $P_{\pm,j,0}$ and $P_{\pm,j,1}$ be the initial and terminal points on A_ε determined by $\overrightarrow{\mathcal{D}}_{\pm,j}$. Then we write

$$P_{\pm,j,i} = P'_{\pm,j,i} \sqcup P''_{\pm,j,i}, \qquad (\mathbf{3.4.10})$$

where $P'_{\pm,j,i}$ corresponds to $\overline{v}'_{1,j}$ and $P''_{\pm,j,i}$ corresponds to $\overline{v}''_{1,j}$. Note that

$$P_{+,j-1,i} = P_{-,j,i} \text{ and } P'_{+,j-1,i} = P'_{-,j-1,i}. \qquad (\mathbf{3.4.11})$$

Definition **3.4.12.** — *Let* P *be a finite subset of* $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. *Denoting an element of* S^1 *by an equivalence class* $[c]$, *where* $c \in \mathbf{R}$, *if* $[a] \neq [b] \in P$, *then we write* $[a] \lessdot_P [b]$ *if there exist* $a' \in [a]$, $b' \in [b]$ *such that* $a' < b' < a' + 2\pi$ *and there are no representatives of* P *in the open interval* (a', b').

Observe that the relation $\lessdot_P$ is not symmetric, i.e., $[a] \lessdot_P [b]$ does not necessarily imply that $[b] \lessdot_P [a]$.

Lemma **3.4.13.** — *For each* $* = (\pm, j)$, $P_{*,0} \subset P_0$ *and* $P_{*,1} \subset P_1$ *and the points of* $P_{*,0}$ *and* $P_{*,1}$ *alternate along* $(0, 2\pi)$.

Proof. — The proof is by induction on the level; see Figure 8 for an example. We will inductively define $P_i^{(0)} \supset P_i^{(1)} \supset P_i^{(2)} \supset \dots$, $i = 0, 1$, and the corresponding cycles $\mathcal{Z}^{(0)}, \mathcal{Z}^{(1)}, \dots$ and show that the following hold for each $j = 0, 1, \dots$:

(j0) the points of $P_0^{(j)}$ and $P_1^{(j)}$ alternate along $(0, 2\pi)$;
(j1) $P_{-,j_0-j,i} = P_{+,j_0-j-1,i} \subset P_i^{(j)}$, $i = 0, 1$;
(j2) there is a partition of $P_{-,j_0-j,0} \cup P_{-,j_0-j,1}$ into pairs of type $\{\mathfrak{p}_0, \mathfrak{p}_1\}$, $\mathfrak{p}_i \in P_{-,j_0-j,i}$, such that $\mathfrak{p}_0 \lessdot_{P_0^{(j)} \cup P_1^{(j)}} \mathfrak{p}_1$; in particular, the points of $P_{-,j_0-j,0}$ and $P_{-,j_0-j,1}$ alternate along $(0, 2\pi)$;
(j3) $P'_{+,j_0-j-1,i} = P'_{-,j_0-j-1,i} \subset P_i^{(j+1)}$, $i = 0, 1$;
(j4) there is a partition of $P'_{+,j_0-j-1,0} \cup P'_{+,j_0-j-1,1}$ into pairs of type $\{\mathfrak{p}_0, \mathfrak{p}_1\}$, $\mathfrak{p}_i \in P'_{+,j_0-j-1,i}$, such that $\mathfrak{p}_0 \lessdot_{P_0^{(j+1)} \cup P_1^{(j+1)}} \mathfrak{p}_1$; in particular, the points of $P'_{+,j_0-j-1,0}$ and $P'_{+,j_0-j-1,1}$ alternate along $(0, 2\pi)$.

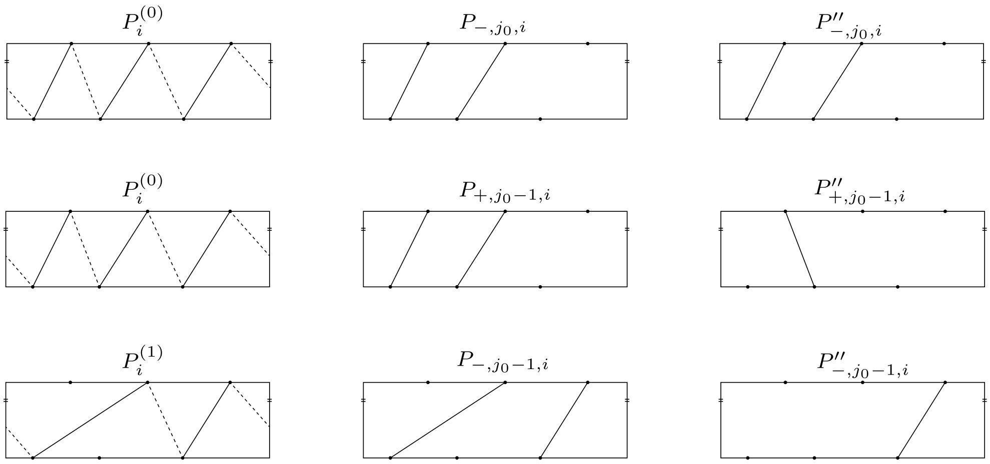


FIG. 8. — Each rectangular box, with the sides identified, is A_ε; the top is $\partial D^2_\varepsilon \times \{1\}$ and the bottom is $\partial D^2_\varepsilon \times \{0\}$. For each figure, $P^\star_{*,0}$ and $P^\star_{*,1}$ are the sets of initial and terminal points of the arcs drawn, where the arcs are always oriented from bottom to top. The extra dots are drawn only for reference. The kth row corresponds to Step k. The solid and dashed arcs in the left column together indicate $\mathcal{Z}^{(0)}$ and $\mathcal{Z}^{(1)}$. The dashed arcs are potential arcs whose endpoints give pairs of points in $P''_{+,j,0}$ and $P''_{+,j,1}$

Note that $P_{-,j_0-j,i} = P_{+,j_0-j-1,i}$ in $(j1)$ and $P'_{+,j_0-j-1,i} = P'_{-,j_0-j-1,i}$ in $(j3)$ by Equation (3.4.11).

Step 1. Let $\overline{v}_{1,j_0}$ be the highest level which has a negative end at z_∞, let $P_i^{(0)} = P_i$, $i = 0, 1$, and let $\mathcal{Z}^{(0)} = \mathcal{Z}$. The construction of $\mathcal{Z}^{(0)}$ (following the proof of Lemma 3.4.7) immediately implies (00)–(02).

Step 2. We consider the positive ends of $\overline{v}_{1,j_0-1}$ that limit to z_∞. There is a partition of $P''_{+,j_0-1,0} \cup P''_{+,j_0-1,1}$ into pairs of type $\{\mathfrak{p}_0, \mathfrak{p}_1\}$, $\mathfrak{p}_i \in P''_{+,j_0-1,i}$, such that $\mathfrak{p}_1 \lessdot_{P_0^{(0)} \cup P_1^{(0)}} \mathfrak{p}_0$, i.e., there is a chord $\mathfrak{p}_1 \to \mathfrak{p}_0$ in $\mathcal{Z}^{(0)}$. Let $P_i^{(1)} = P_i^{(0)} - P''_{+,j_0-1,i}$ and let $\mathcal{Z}^{(1)}$ be obtained from $\mathcal{Z}^{(0)}$ by inductively replacing $\mathfrak{q} \to \mathfrak{p}_1 \to \mathfrak{p}_0 \to \mathfrak{q}'$ by $\mathfrak{q} \to \mathfrak{q}'$, given by concatenation. This makes sense since each point of $P_i^{(0)}$ appears only once in $\mathcal{Z}^{(0)}$. Then $P_0^{(1)} \cup P_1^{(1)}$ is the set of (alternating) points of $\mathcal{Z}^{(1)}$, each point of $P_i^{(1)}$ appears only once in $\mathcal{Z}^{(1)}$, and $\mathcal{Z}^{(1)}$ winds around $\mathbf{R}/2\pi\mathbf{Z}$ once. Hence (10) follows from the description of $\mathcal{Z}^{(1)}$. Similarly, since $P'_{+,j_0-1,i} = P_{+,j_0-1,i} - P''_{+,j_0-1,i}$ and (02) holds, (03) and (04) follow immediately.

Step 3. We consider the negative ends of $\overline{v}_{1,j_0-1}$ that limit to z_∞. There is a partition of $P''_{-,j_0-1,0} \cup P''_{-,j_0-1,1}$ into pairs of type $\{\mathfrak{p}_0, \mathfrak{p}_1\}$, $\mathfrak{p}_i \in P''_{-,j_0-1,i}$, such that $\mathfrak{p}_0 \lessdot_{P_0^{(1)} \cup P_1^{(1)}} \mathfrak{p}_1$, i.e., there is a chord $\mathfrak{p}_0 \to \mathfrak{p}_1$ in $\mathcal{Z}^{(1)}$. Since $P_{-,j_0-1,i} = P'_{-,j_0-1,i} \cup P''_{-,j_0-1,i}$ and (04) holds, (11) and (12) follow immediately.

Repeated application of the above then implies the lemma. □

Lemma **3.4.14.** — *If fiber components are removed from* $\overline{u}_\infty$ *and there are no boundary points at* z_∞*, then:*

(1) *the only components of* $\overline{v}'_*$ *with negative ECH index are the branched covers of* σ_∞^+*;*
(2) *the ECH index of each level* $\overline{v}_* \neq \overline{v}_+$ *is nonnegative; and*
(3) $I(\overline{v}_{1,j}) \geq I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j})$ *for* $0 \leq j \leq a$*, with equality for* $j > 0$.

Proof. — The proof is analogous to that of Lemma I.7.5.5.

(1) If a component $\widetilde{v}$ of $\overline{u}_\infty$ is a branched cover of some σ_∞^* with possibly empty branch locus, then, by Lemmas I.5.7.15 and I.5.7.16, $I(\widetilde{v}) = 0$ with the exception of $I(\widetilde{v}) = -\deg(\widetilde{v})$ if $\widetilde{v}$ covers σ_∞^+. For all other components $\widetilde{v}$ (i.e., those that do not branch cover any σ_∞^*), the regularity of $\{\overline{J}_\tau\}$ and the ECH-type index inequalities imply that $I(\widetilde{v}) \geq 0$.

(2) for $\overline{v}_{0,j}$, $1 \leq j \leq b$. By [HT1, Proposition 7.15(a)] and the regularity of $\{\overline{J}_\tau\}$, we have $I(\overline{v}_{0,j}) \geq 0$ for $1 \leq j \leq b$, where equality holds if and only if $\overline{v}_{0,j}$ is a connector.

(2) and (3) for $\overline{v}_{1,j}$. We claim that $I(\overline{v}_{1,j}) \geq 0$ for $0 < j \leq a$ and $I(\overline{v}_{1,j}) \geq I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j})$ for $0 \leq j \leq a$, with equality for $j > 0$.

Suppose z_∞ does not appear at an end of $\overline{v}_{1,j}$. With the possible exception of fiber components, $\overline{v}_{1,j}$ is simply-covered and regular since there is at least one HF end. Hence the claim follows from the regularity of $\{\overline{J}_\tau\}$ and the usual index inequality (Lemmas I.4.5.13 and I.5.6.9).

Suppose z_∞ appears at an end of $\overline{v}_{1,j}$. Let $\mathfrak{c}_{\pm,j}$ (resp. $\mathfrak{c}'_{\pm,j}$, $\mathfrak{c}''_{\pm,j}$) be the groomings corresponding to the $\pm$ ends of $\overline{v}_{1,j}$ (resp. $\overline{v}'_{1,j}$, $\overline{v}''_{1,j}$) at z_∞, such that:

(α_1) $\mathfrak{c}_{\pm,j} = \mathfrak{c}'_{\pm,j} \cup \mathfrak{c}''_{\pm,j}$;
(α_2) $\mathfrak{c}_{-,j}$ has winding $w(\mathfrak{c}_{-,j}) \geq 0$ and $\mathfrak{c}_{+,j}$ has winding $w(\mathfrak{c}_{+,j}) \leq 0$;
(α_3) $\mathfrak{c}''_{-,j} = \pi_{[0,1]\times \overline{S}}(\mathcal{E}^j_-) \cap A_\varepsilon$ and $\mathfrak{c}''_{+,j} = \pi_{[0,1]\times \overline{S}}(\mathcal{E}^j_+) \cap A_\varepsilon$, where $\mathcal{E}^j_+$ (resp. $\mathcal{E}^j_-$) is the set of positive (resp. negative) ends of $\overline{v}^\sharp_{1,j}$ converging to z_∞.

Recall that the matchings defined by $\mathfrak{c}_{\pm,j}$ do not need to coincide with the matchings in the data $\overrightarrow{\mathcal{D}}_{\pm,j}$ at z_∞ for the $\pm$ ends of $\overline{v}_{1,j}$; only their endpoints do. What (α_3) says is that we are requiring $\mathfrak{c}''_{\pm,j}$ to coincide with the data at z_∞ for $\overline{v}''_{1,j}$. On the other hand, in view of (α_1) the data at z_∞ for $\overline{v}'_{1,j}$ does not necessarily coincide with $\mathfrak{c}'_{\pm,j}$.

By Lemma 3.4.13, the sets $P_{\pm,j,0}$ and $P_{\pm,j,1}$ of initial and terminal points of $\mathfrak{c}_{\pm,j}$ alternate along $(0, 2\pi)$. Hence, by the version of the ECH index inequality given by Lemma I.5.7.21,

$$I(\overline{v}''_{1,j}) \geq \operatorname{ind}(\overline{v}''_{1,j}) \geq 0. \tag{3.4.12}$$

Also, by (α_1) and an easy unwinding of the definition of $I(\overline{v}_{1,j})$,

$$I(\overline{v}_{1,j}) \geq I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j}) \quad \text{for} \quad j \geq 0; \tag{3.4.13}$$

moreover, equality holds for $j > 0$ since $\overline{v}'_{1,j}$ and $\overline{v}''_{1,j}$ do not intersect (if they did, then $\sum_{\overline{v}_*} n^*(\overline{v}_*) > m$). Finally, $I(\overline{v}'_{1,j}) = 0$ for $j > 0$ by Lemma I.5.7.15; together with Equations (3.4.12) and (3.4.13), this gives $I(\overline{v}_{1,j}) \geq 0$ for $j > 0$. This proves the claim.

(2) and (3) for $\overline{v}_{-1,j}$ and $\overline{v}_-$. The case of $\overline{v}_{-1,j}$, $1 \leq j \leq c+1$, (this includes $\overline{v}_-$) is similar, and the lemma follows. □

3.4.3. *Boundary points at z_∞.* — In this subsection we describe the necessary modifications when $\overline{u}_\infty$ has boundary points at z_∞.

Let us suppose the following:

(S) There is only one boundary point $\mathfrak{r} \in \dot{F}''_{*_0}$ at z_∞ and $\overline{v}''_{*_0}(\mathfrak{r}) \in \{(0,1)\} \times \overline{\mathbf{a}} \subset \overline{W} = \mathbf{R} \times [0,1] \times \overline{S}$.

We are assuming (S) for notational simplicity; the general case of multiple boundary points at z_∞ is treated in exactly the same way (except for more complicated notation).

For the purposes of computing the Fredholm and ECH indices, we make the following modifications which allow us to use the considerations from Sections 3.4.1, 3.4.2, and I.5.7. We view the base $B = \mathbf{R} \times [0,1]$ with the preferred point $(0,1)$ as a two-level building consisting of a disk B_1° with three boundary punctures and a disk B_2° with one boundary puncture. (What we are doing here is bubbling off a neighborhood of the point $(0,1) \in B$ when taking the limit $\overline{u}_i \to \overline{u}_\infty$.) All the punctures are viewed as strip-like ends: the punctures of B_1° are called the positive, left negative, and right negative ends, corresponding to the positive end, $(0,1)$, and the negative end of B, and the puncture of B_2° is called the positive end and is identified with the left negative end of B_1°. See Figure 9. We write $\partial B_1^\circ = \sqcup_{i=1}^3 \partial_i B_1^\circ$, arranged in counterclockwise order, such that $\partial_1 B_1^\circ$, $\partial_2 B_1^\circ$, and ∂B_2° correspond to $\mathbf{R} \times \{1\}$ and $\partial_3 B_1^\circ$ corresponds to $\mathbf{R} \times \{0\}$. The cobordism $\overline{W} = B \times \overline{S}$ decomposes into $\overline{W}_1^\circ = B_1^\circ \times \overline{S}$ and $\overline{W}_2^\circ = B_2^\circ \times \overline{S}$, the Lagrangian submanifold $\mathbf{R} \times \{1\} \times \overline{\mathbf{a}}$ decomposes into $\partial_1 B_1^\circ \times \overline{\mathbf{a}}$, $\partial B_2^\circ \times \overline{\mathbf{a}}$, and $\partial_2 B_1^\circ \times \overline{\mathbf{a}}$, and the Lagrangian submanifold $\mathbf{R} \times \{0\} \times \overline{h}(\overline{\mathbf{a}})$ corresponds to $\partial_3 B_1^\circ \times \overline{h}(\overline{\mathbf{a}})$. We identify the positive end of $\overline{W}_1^\circ$ with $[0,\infty) \times [0,1] \times \overline{S}$, the left negative end with $(-\infty, 0] \times [\frac{1}{2}, 1] \times \overline{S}$, and the right negative end with $(-\infty, 0] \times [0, \frac{1}{2}] \times \overline{S}$.

Denote the sections at infinity of $\overline{W}_i^\circ$, $i = 1, 2$, by $\sigma_\infty^{\circ,i} = B_i^\circ \times \{z_\infty\}$. A curve $\overline{v}_{*_0}$ in $\overline{W}$ decomposes into $\overline{v}_{*_0}^{\circ,1} \cup \overline{v}_{*_0}^{\circ,2}$, where $\overline{v}_{*_0}^{\circ,i} = \overline{v}_{*_0}^{\circ,i,\prime} \cup \overline{v}_{*_0}^{\circ,i,\prime\prime}$, $i = 1, 2$, is a curve in $\overline{W}_i^\circ$, $\overline{v}_{*_0}^{\circ,i,\prime}$ is a possibly branched cover of $\sigma_\infty^{\circ,i}$, and $\overline{v}_{*_0}^{\circ,2,\prime\prime}$ is a union of constant sections $B_1^\circ \times \{pt\}$, $pt \in \widehat{\mathbf{a}}$.

(R) We view the chords at the positive ends of $\overline{v}_{*_0}^{\circ,2}$ and the left negative ends of $\overline{v}_{*_0}^{\circ,1}$, including z_∞, as Reeb chords in a Morse-Bott family and, for the purposes of calculating Maslov indices, we pretend that the Morse-Bott family has been perturbed so that z_∞ is the bottom generator of each $\overline{a}_i$. In particular, at z_∞ we assume that, when we go around the boundary of $\overline{W}_2^\circ$ in the counterclockwise direction, each $\overline{a}_i$ is rotated slightly in the negative ϕ-direction by a Hamiltonian isotopy.

We emphasize that $\overline{v}_{*_0}^{\circ,1} \cup \overline{v}_{*_0}^{\circ,2}$ is essentially the same thing as $\overline{v}_{*_0}$, described in a slightly different way.

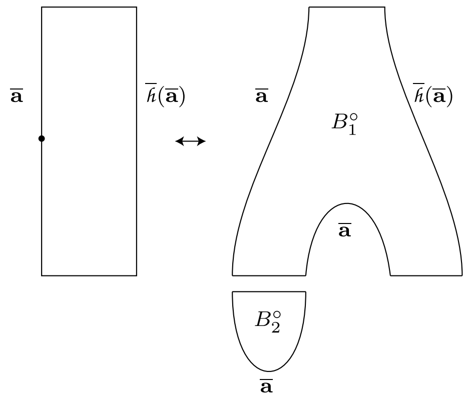


FIG. 9. — The dot in the left-hand picture is the preferred point (0, 1)

The reader can verify that, with our convention (R),

$$\textbf{(3.4.14)}\qquad \begin{aligned} &\mathrm{ind}(\sigma_\infty^{\circ,i})=0,\ i=1,2, \quad \mathrm{I}(\overline{v}_{*_0}^{\circ,2,'})=0,\\ &\mathrm{I}(\overline{v}_{*_0}^{\circ,2,''})=\mathrm{ind}(\overline{v}_{*_0}^{\circ,2,''})=0, \quad \mathrm{ind}(\overline{v}''_{*_0})=\mathrm{ind}(\overline{v}_{*_0}^{\circ,1,''}). \end{aligned}$$

We apply the continuation method to $\overline{v}_{1,j}$, $0\leq j\leq a$, $(1,j)\neq *_0=(1,j_0)$, and $\overline{v}_{*_0}^{\circ,i}$, $i=1,2$: We adapt the definition of a continuation along $\partial_+ \mathrm{B}_\tau$ in a natural way by replacing $g^1_{j_0}$ by $g^1_{j_0,1}, g_{j_0,2}, g^{1/2}_{j_0,1}$, which correspond to $\partial_1 \mathrm{B}_1^\circ$, $\partial \mathrm{B}_2^\circ$, $\partial_2 \mathrm{B}_1^\circ$, respectively, in Equation (3.4.4). As in Lemma 3.4.7, start with a nontrivial negative end $\mathcal{E}_1$ of some $\overline{v}_{1,j_1}$ or $\overline{v}_{*_0}^{\circ,1}$ limiting to z_∞ and continue the corresponding $g^1_{j_1}$, $g^1_{j_0,1}$, or $g^{1/2}_{j_0,1}$ in the direction of $\partial_+\mathrm{B}_\tau$, until some $g^{\star_1}_{\star_2}\not\subset\sigma^*_\infty$ is reached. Suppose $\mathcal{E}_2$ is the end of some component of $\overline{u}_\infty$ which "lies between" $g^{\star_1}_{\star_2}$ and some $g^{\star_1'}_{\star_2'}$. We then switch from $g^{\star_1}_{\star_2}$ to $g^{\star_1'}_{\star_2'}$ and continue. This gives a sequence of sectors of type $\mathfrak{S}(\overline{a}_{k,l},\overline{a}_{k',l'})$ (corresponding to the left negative end of $\overline{\mathrm{W}}_1^\circ$; here if $(k,l)=(k',l')$, then we view the sector as the full 2π sector), $\mathfrak{S}(\overline{h}(\overline{a}_{k,l}),\overline{a}_{k',l'})$, or $\mathfrak{S}(\overline{a}_{k',l'},\overline{h}(\overline{a}_{k,l}))$, and hence a *unique* cycle

$$\mathcal{Z}=(\mathfrak{z}_1\to\mathfrak{z}_2\to\cdots\to\mathfrak{z}_k\to\mathfrak{z}_1)$$

that winds around $\mathbf{R}/2\pi\mathbf{Z}$ once. Note that if there is a boundary point at z_∞, then there is at least one sector of type $\mathfrak{S}(\overline{a}_{k,l},\overline{a}_{k',l'})$.

We leave it to the reader to make the proper generalizations of $\mathcal{Z}$ when we do not assume (S); $*_0=+$ or $(-1,j_0)$, $1\leq j_0\leq c+1$; and/or the boundary point at z_∞ lies on $\mathrm{L}_{\overline{h}(\overline{\mathbf{a}})}$, $\mathrm{L}^+_{\overline{\mathbf{a}}}$, $\mathrm{L}_{\overline{\mathbf{b}}}$, $\mathrm{L}_{\overline{h}(\overline{\mathbf{b}})}$, or $\mathrm{L}^-_{\overline{\mathbf{b}}}$.

Let $\mathrm{A}^{[a,b]}_\varepsilon$ be the annulus $\{\rho=\varepsilon\}\subset[a,b]\times\mathrm{D}^2_{\rho_0}$. Writing $*_0=(1,j_0)$, let $\overrightarrow{\mathcal{D}}_{+,j_0}$, $\overrightarrow{\mathcal{D}}_{\mathrm{L}-,j_0}$, $\overrightarrow{\mathcal{D}}_{\mathrm{R}-,j_0}$ be the data at z_∞ of the positive, left negative, and right negative ends, and let $\mathrm{P}_{\mathrm{L}-,j_0,1/2}$ and $\mathrm{P}_{\mathrm{L}-,j_0,1}$ (resp. $\mathrm{P}_{\mathrm{R}-,j_0,0}$ and $\mathrm{P}_{\mathrm{R}-,j_0,1/2}$) be the initial and terminal points of $\mathrm{A}^{[1/2,1]}_\varepsilon$ (resp. $\mathrm{A}^{[0,1/2]}_\varepsilon$) determined by $\overrightarrow{\mathcal{D}}_{\mathrm{L}-,j_0}$ (resp. $\overrightarrow{\mathcal{D}}_{\mathrm{R}-,j_0}$). We also decompose $\mathrm{P}_{*,i}=$

$P'_{*,i} \cup P''_{*,i}$, $* = (L-, j_0)$, $(R-, j_0)$, or $(+, j_0)$, as before so that $P'_{*,i}$ corresponds to $\overline{v}_*^{\circ,1,'}$ and $P''_{*,i}$ corresponds to $\overline{v}_*^{\circ,1,''}$.

The following definition does not assume (S).

Definition **3.4.15.** — *A boundary point* $\mathfrak{r} \in \partial F_{*_0}$, $*_0 = (1, j_0)$, $0 \leq j_0 \leq a$, *at* z_∞ *falls into one of three (mutually exclusive) types:*

- (P_1) $\overline{v}'_{*_0} = \varnothing$;
- (P_2) $\overline{v}'_{*_0} \neq \varnothing$ *and all the vertices of* $\mathcal{Z}$ *lie on arcs of type* $\overline{a}_{k,l}$ *or all the vertices of* $\mathcal{Z}$ *lie on arcs of type* $\overline{h}(\overline{a}_{k,l})$;
- (P_3) $\overline{v}'_{*_0} \neq \varnothing$ *and* $\mathcal{Z}$ *has vertices that lie on arcs of both types* $\overline{a}_{k,l}$ *and* $\overline{h}(\overline{a}_{k,l})$.

The boundary points $\mathfrak{r} \in \partial F_{*_0}$, $*_0 = (-1, j_0)$, $1 \leq j_0 \leq c+1$, *at* z_∞ *are classified similarly.*

The following is a strengthening of Lemma 3.4.7 when $\overline{v}'_{1,j} \cup \overline{v}^\sharp_{1,j} \neq \varnothing$ for some $j > 0$ and there are boundary points of type (P_3) at z_∞:

Lemma **3.4.16.** — *Suppose* $\overline{v}'_{1,j} \cup \overline{v}^\sharp_{1,j} \neq \varnothing$ *for some* $j > 0$. *Let* $\mathcal{E}_{-,i}$, $i = 1, \dots, q$, *be the negative ends of* $\cup_{j=1}^a \overline{v}^\sharp_{1,j}$ *that converge to* z_∞, *let* $\mathcal{E}_{+,i}$, $i = 1, \dots, r$, *be the positive ends of* $\cup_{j=0}^{a-1} \overline{v}^\sharp_{1,j}$ *that converge to* z_∞, *and let* $\mathcal{E}'_i$, $i = 1, \dots, s$, *be the neighborhoods of the boundary points of type* (P_3). *Then*

$$n^*((\cup_{i=1}^q \mathcal{E}_{-,i}) \cup (\cup_{i=1}^r \mathcal{E}_{+,i}) \cup (\cup_{i=1}^s \mathcal{E}'_i)) \geq m - p_+. \tag{3.4.15}$$

Proof. — Similar to that of Lemma 3.4.7. □

Lemma **3.4.17.** — *If* $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ *are the boundary points of type* (P_1) *and* (P_2) *and* $N(\mathfrak{p}_i) \subset F_*$ *is a small neighborhood of* $\mathfrak{p}_i$, *then* $\sum_{i=1}^s n^*(\overline{v}_*(N(\mathfrak{p}_i))) \geq m$.

Proof. — (P_2) follows immediately from the description of $\mathcal{Z}$; (P_1) is similar. □

3.4.4. *Bounds on ECH indices, part II.* — In this subsection we give bounds on ECH indices in the presence of boundary points at z_∞.

For simplicity we are still assuming (S). The ECH indices of $\overline{v}^{\circ,i}_{*_0}$ are defined in a manner similar to that of Definition I.5.7.3.

Lemma **3.4.18.** — *If* $\overline{v}'_{*_0}, \overline{v}''_{*_0} \neq \varnothing$, *(S) holds, and the boundary point at* z_∞ *is of type* (P_3), *then there exist contributions* $I_+ \geq 0$, $I_{L-} = 2$, *and* $I_{R-} \geq 0$ *from the ends that limit to* z_∞ *at the positive, left negative, and right negative ends, such that*

$$I(\overline{v}_{*_0}) \geq I(\overline{v}^{\circ,1,'}_{*_0}) + I(\overline{v}^{\circ,1,''}_{*_0}) + I_+ + I_{L-} + I_{R-}. \tag{3.4.16}$$

Proof. — The calculation of I_{L-} is similar to the ECH index calculations of Section I.5.7 and in particular that of Lemma I.5.7.22.

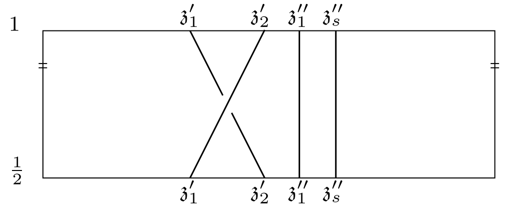


FIG. 10. — $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ for $\mathfrak{r}$ of type (P_3). Here $\mathfrak{c}''_{L-}$ is the strand in the front

First observe that $P_{L-,j_0,1/2} = P_{L-,j_0,1}$. For simplicity suppose that the only chord of $\mathcal{Z}$ corresponding to a sector of type $\mathfrak{S}(\overline{a}_{k,l}, \overline{a}_{k',l'})$ is $\mathfrak{z}'_1 \to \mathfrak{z}'_2$. Observe that there are no points of $\mathcal{Z}$ between $\mathfrak{z}'_1$ and $\mathfrak{z}'_2$, since otherwise $\mathcal{Z}$ winds more than once around $\mathbf{R}/2\pi\mathbf{Z}$. Hence $P''_{L-,j_0,1/2} = \{\mathfrak{z}'_1\}$, $P'_{L-,j_0,1/2} = \{\mathfrak{z}'_2, \mathfrak{z}''_1, \dots, \mathfrak{z}''_s\}$, $P''_{L-,j_0,1} = \{\mathfrak{z}'_2\}$, and $P'_{L-,j_0,1} = \{\mathfrak{z}'_1, \mathfrak{z}''_1, \dots, \mathfrak{z}''_s\}$, where $P_{L-,j_0,1/2} = P_{L-,j_0,1}$ is written as $\{\mathfrak{z}'_1, \mathfrak{z}'_2, \mathfrak{z}''_1, \dots, \mathfrak{z}''_s\}$ in cyclic order around $\mathbf{R}/2\pi\mathbf{Z}$, and the projection of the left negative end of $\overline{v}^{\circ,1,''}_{*0}$ that limits to z_∞ intersects $A^{[1/2,1]}_{\varepsilon}$ along an arc $\mathfrak{c}''_{L-}$ with winding number $w(\mathfrak{c}''_{L-}) = 0$ or 1, depending on whether $\mathfrak{S}(\overline{a}_{k,l}, \overline{a}_{k',l'})$ is a large sector; see Figure 10. The left negative ends of $\overline{v}^{\circ,1,'}_{*0}$ that limit to z_∞ give a grooming $\mathfrak{c}'_{L-}$ on $A^{[1/2,1]}_{\varepsilon/2}$ such that the winding number $w(\mathfrak{c}'_{L-}) = 0$ or -1 and $\mathfrak{c}'_{L-}$ connects $\mathfrak{z}''_i$ to $\mathfrak{z}''_i$, $i = 1, \dots, s$, by vertical arcs. Then the writhe of $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ is $+1$ and resolving the positive crossing of $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ yields a grooming $\widetilde{\mathfrak{c}}_{L,-}$ by vertical arcs from $P_{L-,j_0,1/2}$ to $P_{L-,j_0,1}$.

We now consider the disk $\check{D}$ corresponding to resolving the positive crossing that we "append" to the left negative end of $\overline{v}^{\circ,1}_{*0}$ as in the proof of Lemma I.5.7.22. The disk $\check{D}$ contributes $1, 0, 1$, and 0 to Q, c_1, μ, and the discrepancy: Q is equal to the writhe $+1$. The calculations for μ assume convention (R). Suppose $w(\mathfrak{c}'_{L-}) = w(\mathfrak{c}''_{L-}) = 0$ (the case $w(\mathfrak{c}'_{L-}) = -1$ and $w(\mathfrak{c}''_{L-}) = 1$ is similar and is left to the reader). Then μ of the positive ends of $\check{D}$ are $1, 0, \dots, 0$ and μ of the negative ends are all 0. The discrepancy contribution at the negative end of $\check{D}$ is zero since $\widetilde{\mathfrak{c}}_{L,-}$ is a grooming by (almost) vertical arcs. Hence $I_{L-} = 2$.

Finally, since $I(\overline{v}_{*0}) = I(\overline{v}^{\circ,1}_{*0}) + I(\overline{v}^{\circ,2}_{*0})$ and $I(\overline{v}^{\circ,2}_{*0}) = 0$, we have $I(\overline{v}_{*0}) = I(\overline{v}^{\circ,1}_{*0}) = I(\overline{v}^{\circ,1,'}_{*0}) + I(\overline{v}^{\circ,1,''}_{*0}) + I_+ + I_{L-} + I_{R-}$. Lemma 3.4.14 implies that $I_+ \geq 0$ and $I_{R-} \geq 0$. ☐

Remark **3.4.19.** — In general, each collection of boundary points of type (P_3) that map to the same point on the base contributes at least $+2$ towards I. The proof is similar to but slightly more complicated than that of Lemma 3.4.18, and is left to the reader.

The following is a strengthening of Lemma 3.4.14 in the presence of boundary points at z_∞:

Lemma **3.4.20.** — *If fiber components are removed from $\overline{u}_\infty$ and the only boundary points are of type (P_3), then*

(1) *the only components of* $\overline{v}'_*$ *with negative ECH index are the branched covers of* σ_∞^+*;*
(2) *the ECH index of each level* $\overline{v}_* \neq \overline{v}_+$ *is nonnegative; and*
(3) *For* $0 \leq j \leq a$*,*

$$\mathrm{I}(\overline{v}_{1,j}) \geq \begin{cases} \mathrm{I}(\overline{v}'_{1,j}) + \mathrm{I}(\overline{v}''_{1,j}) & \text{if } \mathfrak{bp}_{1,j} = 0; \\ \mathrm{I}(\overline{v}'_{1,j}) + \mathrm{I}(\overline{v}''_{1,j}) + 2 & \text{if } \mathfrak{bp}_{1,j} > 0. \end{cases}$$

Here $\mathfrak{bp}_*$ *is the number of boundary points of type* (P_3) *on* $\overline{v}_*$.

Proof. — We explain the modifications that need to be made when there are boundary points at z_∞ in view of Lemma 3.4.18 and Remark 3.4.19. We assume (S) for simplicity. The cycles $\mathcal{Z}^{(a-j)}$ are defined as in the proof of Lemma 3.4.13, for $j \geq j_0$. We define $\mathcal{Z}^{(a-j_0,+)} = \mathcal{Z}^{(a-j_0)}$ and $\mathcal{Z}^{(a-j_0,-)}$ as $\mathcal{Z}^{(a-j_0,+)}$ with $\mathfrak{z}'_1 \to \mathfrak{z}'_2$ replaced by $\mathfrak{z}'_2$. Then $\mathcal{Z}^{(a-(j_0-1))}$ is obtained using $\mathcal{Z}^{(a-j_0,-)}$ instead of $\mathcal{Z}^{(a-j_0)}$. Also, $\mathrm{P}^\star_{\mathrm{R}-,j_0,i}$, $\star = \varnothing, ', ''$, is obtained from $\mathrm{P}^\star_{+,j_0,i}$ by replacing $\mathfrak{z}'_1$ by $\mathfrak{z}'_2$. The rest of the argument of Lemma 3.4.14 carries over. □

3.4.5. *Case of* $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ *for all* $\overline{v}_*$.

Lemma **3.4.21.** — *If* $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$ *and* $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ *for all levels* $\overline{v}_*$ *of* $\overline{u}_\infty$*, then* $a = c = 0$*;* $\mathrm{I}(\overline{v}_+) = 0$*;* $\mathrm{I}(\overline{v}_-) = 2$*;* $\overline{v}_+$ *is a* W_+*-curve;* $\overline{v}_-$ *is a* $\overline{\mathrm{W}}_-$*-curve; and there may be connectors* $\overline{v}_{0,j}$ *in between.*

Proof. — Suppose that $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$ and $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$. Then there is a point $\mathfrak{q} \in int(\mathrm{F}_-)$ and a sufficiently small neighborhood $\mathrm{N}(\mathfrak{q}) \subset \dot{\mathrm{F}}_-$ of $\mathfrak{q}$ such that $\overline{v}_-(\mathfrak{q}) = \overline{\mathfrak{m}}(\infty)$ and $n^*(\overline{v}_-(\mathrm{N}(\mathfrak{q}))) \geq m$. By Equation (3.4.2) and Lemma 3.4.8, there are no boundary points at z_∞ and the only possible fiber component passes through $\overline{\mathfrak{m}}(\infty)$. Hence every level $\overline{v}_*$, $* \neq -$, has image inside W', W, or W_+ and $\overline{v}_-$ is a $\overline{\mathrm{W}}_-$-curve or a *degenerate* $\overline{\mathrm{W}}_-$-curve by the analog of Lemma I.7.5.2.

The ECH index of each level $\neq \overline{v}_+$ and which has no fiber component is nonnegative by Lemma 3.4.14. Since $\overline{v}_+$ is a W_+-curve, $\mathrm{I}(\overline{v}_+) \geq 0$. On other hand, by the previous paragraph, if there is a fiber component, then it is a component of $\overline{v}_-$. We claim that $\mathrm{I}(\overline{v}_-) \geq 2$, with equality if and only if $\overline{v}_-$ is not a degenerate $\overline{\mathrm{W}}_-$-curve. Indeed, if $\overline{v}_-$ is not degenerate, then $\mathrm{I}(\overline{v}_-) \geq 2$ by the point constraint (this is the only place where we use the fact that $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$) and the ECH index inequality, and if $\overline{v}_-$ is degenerate, then $\mathrm{I}(\overline{v}_-) \geq 4$, as computed in the proof of Lemma I.7.5.5.

The lemma then follows from Equation (3.4.3). In particular, degenerate $\overline{\mathrm{W}}_-$-curves are not allowed. □

3.4.6. *Preliminary restrictions when* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some* $\overline{v}_*$. — We now consider the case where $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$.

Lemma **3.4.22.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$ of $\overline{u}_\infty$, then:*

(1) $p_- = \deg(\overline{v}'_-) > 0$;
(2) *$\overline{u}_\infty$ has no boundary point of type (P_1) or (P_2);*
(3) *$\overline{u}_\infty$ has no fiber components and no components of $\overline{v}''_*$ that intersect the interior of a section at infinity;*
(4) *each component of $\overline{v}^\sharp_{-1,j}$, $j = 1, \dots, c$, is a thin strip from z_∞ to some x_i or x'_i with $\mathrm{I} = 1$;*
(5) *each component of $\overline{v}^\sharp_-$ is a section of $\overline{\mathrm{W}_-} - \mathrm{W}_-$ from δ_0 to some x_i or x'_i with $\mathrm{I} = 1$;*
(6) *each component of $\overline{v}^\sharp_{0,j}$, $j = 1, \dots, b$, which has a positive end at a multiple of δ_0 is a cylinder from δ_0 to h or e with $\mathrm{I} = 1$ or 2;*
(7) *the only boundary points at z_∞ are type (P_3) points of $\overline{v}''_{1,j}$, $j = 0, \dots, a$.*

Proof. — The lemma is a consequence of Equation (3.4.2). Suppose $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$. Then $\overline{v}^\sharp_* \neq \varnothing$ for some $\overline{v}_*$ and each end of $\overline{v}^\sharp_*$ that limits to some multiple of z_∞ or δ_0 contributes positively to n^* by Lemma 3.4.2 and the proof of Lemma I.7.4.2 (also see Lemma 3.4.7(1)).

(1) If $\overline{v}'_- = \varnothing$, then the restriction of $\overline{v}''_-$ to a neighborhood of $\overline{\mathfrak{m}}(+\infty)$ contributes m towards $n^*(\overline{v}_-)$. This contradicts the discussion from the previous paragraph, proving (1).

Since $\overline{v}'_- \neq \varnothing$, some $\overline{v}^\sharp_{0,j}$, $j = 1, \dots, b+1$, or $\overline{v}^\sharp_{1,j}$, $j = 1, \dots, a$, has a negative end at z_∞ or a multiple of δ_0. By Lemmas 3.4.2(1) and 3.4.16, it follows that:

$$\textbf{(3.4.17)} \qquad \sum_{\overline{v}_* \succ \overline{v}_-} n^*(\overline{v}_*) \geq m - 2g,$$

where we are counting contributions from the ends and the boundary points of type (P_3).

(2)–(7) are consequences of Equation (3.4.17). We explain (2) and (3), leaving the rest to the reader. By Lemma 3.4.17, the neighborhoods of the boundary points of type (P_1) or (P_2) contribute at least m towards $n^*(\overline{v}''_*)$ in total. Also, a non-ghost fiber component or a component of $\overline{v}''_*$ that intersects the interior of a section at infinity contributes m towards n^*. They both contradict Equation (3.4.17). □

Lemma **3.4.23.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$, then the following alternative holds:*

(a) *either some $\overline{v}^\sharp_{0,j_0}$, $j_0 = 1, \dots, b+1$, has a negative end at a multiple of δ_0, in which case $\overline{v}'_{0,j_0} = \varnothing$, $\overline{v}'_* = \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_* \succ \overline{v}_{0,j_0}$, and $\overline{v}'_{0,j} \neq \varnothing$ for all levels $\overline{v}'_- \preceq \overline{v}_{0,j} \prec \overline{v}'_{0,j_0}$; or*
(b) *no $\overline{v}^\sharp_{0,j}$, $j = 1, \dots, b+1$, has a negative end at a multiple of δ_0, in which case $\overline{v}'_* \neq \varnothing$ for all levels $\overline{v}'_- \preceq \overline{v}'_* \preceq \overline{v}'_+$ and $\overline{v}'_{1,j} \cup \overline{v}^\sharp_{1,j} \neq \varnothing$ for some $j > 0$.*

Proof. — This follows from Lemmas 3.4.2(1) and 3.4.16 by observing that either case contributes at least $m - 2g$ towards n^* and that it is not possible to have both since

$m \gg 2g$. It is also not possible to have multiple occurrences of negative ends of $\overline{v}^\sharp_{0,j}, j > 0$, that converge to some multiple of δ_0. □

3.4.7. *List of possibilities when $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some $\overline{v}_*$.* — We first start with a useful definition.

Definition **3.4.24.**

(1) *Let $X_1 \cup \dots \cup X_r$ be an r-level split almost complex manifold with cylindrical ends, ordered from bottom to top, and let*

$$v = v_1 \cup \dots \cup v_r, \quad \mathrm{Im}(v_i) \subset X_i,$$

be a corresponding r-level holomorphic building, where each level has finite energy. If $\mathcal{E}$ is some collection of positive ends of v_r, then the holomorphic building hanging from $\mathcal{E}$ *is the union of irreducible components $\widetilde{v}_{i_0}$ for which there exist irreducible components $\widetilde{v}_i$, $i = i_0 + 1, \dots, r$, where the limit of a positive end of $\widetilde{v}_i$ agrees with the limit of a negative end of $\widetilde{v}_{i+1}$ for all $i = i_0, \dots, r-1$ and a positive end of $\widetilde{v}_r$ is contained in $\mathcal{E}$.*

(2) *If $X_1 \cup \dots \cup X_{r+1}$ is an $(r+1)$-level split almost complex manifold, $v = v_1 \cup \dots \cup v_{r+1}$ is a corresponding $(r+1)$-level holomorphic building and $\mathcal{E}$ is some collection of negative ends of v_{r+1}, then the* holomorphic building hanging from $\mathcal{E}$ *is defined similarly.*

(3) *If $X_1 \cup \dots \cup X_r$ is an r-level split almost complex manifold, $v = v_1 \cup \dots \cup v_r$ is a corresponding r-level holomorphic building and $\mathcal{E}$ is some collection of negative ends of v_1, then the* holomorphic building sitting above $\mathcal{E}$ *is the union of irreducible components $\widetilde{v}_{i_0}$ for which there exist irreducible components $\widetilde{v}_i$, $i = 1, \dots, i_0 - 1$, where the limit of a negative end of $\widetilde{v}_i$ agrees with the limit of a positive end of $\widetilde{v}_{i-1}$ for all $i = 2, \dots, i_0$ and a negative end of $\widetilde{v}_1$ is contained in $\mathcal{E}$.*

(4) *If $X_0 \cup \dots \cup X_r$ is an $(r+1)$-level split almost complex manifold, $v = v_0 \cup \dots \cup v_r$ is a corresponding $(r+1)$-level holomorphic building and $\mathcal{E}$ is some collection of positive ends of v_0, then the* holomorphic building sitting above $\mathcal{E}$ *is defined similarly.*

Lemma **3.4.25.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$, then $\overline{v}'_- \neq \varnothing$, there are no boundary points at z_∞, and $\overline{u}_\infty$ contains one of the following subbuildings:*

(1) *A 3-level building consisting of a component of $\overline{v}^\sharp_{0,1}$ with $I = 1$ from $\boldsymbol{y}$ to $\delta_0\boldsymbol{\gamma}'$; $\overline{v}'_- = \sigma^-_\infty$; and a thin strip.*

(2_i) *A 3-level building consisting of a component of $\overline{v}^\sharp_+$ with $I = i$, $i = 0, 1$, from $\mathbf{y}$ or $\mathbf{y}''$ to $\delta_0\boldsymbol{\gamma}'$; $\overline{v}'_- = \sigma^-_\infty$; and a thin strip.*

(3) *A 4-level building consisting of a component of $\overline{v}^\sharp_+$ with $I = 0$ from $\mathbf{y}$ to $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}'_{0,1} = \sigma'_\infty$; a component of $\overline{v}^\sharp_{0,1}$ which is an $I = 1$ cylinder from δ_0 to h; $\overline{v}'_- = \sigma^-_\infty$; and a thin strip.*

(4) *A 4-level building consisting of a component of $\overline{v}^\sharp_+$ with $I = 0$ from $\mathbf{y}$ to $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}'_{0,1}$ with $I = 0$ and $\deg = 2$; $\overline{v}'_- = \sigma^-_\infty$; a component of $\overline{v}^\sharp_-$ which is an $I = 1$ curve from δ_0 to some x_i or x'_i; and a thin strip.*

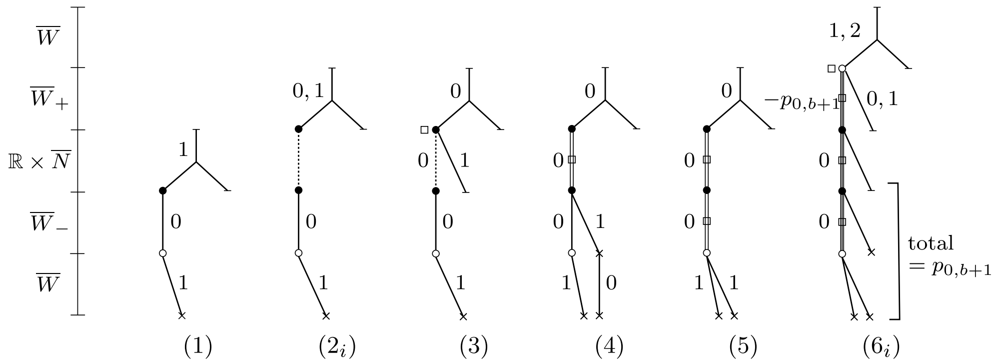


FIG. 11. — Schematic diagrams for the possible types of degenerations. Here • represents δ_0, ∘ represents z_∞, □ represents a possible location of a branch point and × represents some x_i or x_i'. A vertical line indicates the restriction of a trivial cylinder, a dotted vertical line indicates a trivial cylinder, a double vertical line indicates a degree 2 branched cover of a trivial cylinder or a restriction of a trivial cylinder, and a triple vertical line indicates a degree $p \geq 2$ branched cover of a trivial cylinder or a restriction of a trivial cylinder. The labels on the graphs are ECH indices of each component. All the thin strips of $\overline{v}^\sharp_{-1,j}$ are drawn on the same level for convenience

(5) *A* ≥ 3*-level building consisting of a component of* $\overline{v}^\sharp_+$ *with* $\mathrm{I}=0$ *from* $\mathbf{y}$ *to* $\delta_0^2\boldsymbol{\gamma}'$*;* $\overline{v}'_{0,1}$ *and* $\overline{v}'_-$ *with* $\mathrm{I}=0$ *and* $\deg = 2$*, where* $\overline{v}'_{0,1}=\varnothing$ *is possible; and two thin strips.*

(6_i) *A* ≥ 4*-level building consisting of a component of* $\overline{v}^\sharp_{1,1}$ *with* $\mathrm{I}=i$*,* $i=1,2$*, from* $\mathbf{y}$ *to* $\{z_\infty^r\}\cup\mathbf{y}'$*;* $0<p_{0,0}\leq p_{0,1}\leq\cdots\leq p_{0,b+1}\leq r$*;* $\mathrm{I}(\overline{v}'_+)=-p_{0,b+1}$*;* $\mathrm{I}(\overline{v}'_{0,j})=0$ *for* $j=1,\dots,b$*;* $\overline{v}^\sharp_+$ *with* $\mathrm{I}=2-i$ *which has no negative ends at a multiple of* δ_0*;* $p_{0,b+1}-p_{0,1}$ *cylinders of* $\overline{v}^\sharp_{0,j}$*,* $j\in\{1,\dots,b\}$*, with* $\mathrm{I}=1$ *each from* δ_0 *to* h*;* $p_{0,1}-p_{0,0}$ *components of* $\overline{v}^\sharp_-$ *with* $\mathrm{I}=1$ *each from* δ_0 *to some* x_i *or* x_i'*; and* $p_{0,0}$ *thin strips. The total ECH index of the building hanging from the negative end of* $\overline{v}'_+$ *is* $p_{0,b+1}$.

Here we are omitting levels which are connectors. If there is more than one thin strip, then the thin strips could be on the same level or on different levels.

See Figure 11.

Proof. — The lemma is a consequence of Equations (3.4.2) and (3.4.3). By Lemmas 3.4.20 and 3.4.22(2), (3), all the levels $\overline{v}_*$ besides $\overline{v}_+$ have nonzero ECH index, the only components of $\overline{u}_\infty$ which have negative ECH index are the branched covers of σ^+_∞, and $\mathrm{I}(\overline{v}_{1,j})\geq\mathrm{I}(\overline{v}'_{1,j})+\mathrm{I}(\overline{v}''_{1,j})$ or $\mathrm{I}(\overline{v}'_{1,j})+\mathrm{I}(\overline{v}''_{1,j})+2$ for $0\leq j\leq a$, depending on whether $\mathfrak{bp}_{1,j}=0$ or >0. In particular, all the components of $\overline{u}_\infty$ which are left out of the subbuildings in (1)–(6_i) and are not drawn in Figure 11 have $\mathrm{I}\geq 0$.

By Lemma 3.4.22(1), $\overline{v}'_-\neq\varnothing$. By Lemma 3.4.22(2), (7), there are no boundary points of types (P_1) and (P_2) and no boundary points of type (P_3) except for $\overline{v}''_{1,j}$, $j=0,\dots,a$. Cases (1)–(5) have no boundary points at z_∞ (since the ECH index would add up to more than 2) and Case (6_i) may have boundary points of type (P_3).

First suppose that some $\overline{v}^\sharp_{0,j_0}$, $j_0\in\{1,\dots,b\}$, has a negative end at δ_0^p for some $p>0$. This is the situation of Lemma 3.4.23(a) with $j_0\in\{1,\dots,b\}$. In this case, all the components and levels have nonnegative ECH index. Each component of $\overline{v}^\sharp_* \prec \overline{v}_{0,j_0}$ described in Lemma 3.4.22(4)–(6) satisfies $\mathrm{I}\geq 1$ and there must be $p>0$ such components since $\overline{v}_{0,j_0}$ has multiplicity p at δ_0. We then have the following contributions towards I:

(a) $\sum_{\overline{v}^\sharp_*\prec\overline{v}_{0,j_0}} \mathrm{I}(\overline{v}^\sharp_*)\geq p$ and there is at least one thin strip.
(b) $\mathrm{I}(\overline{v}^\sharp_{0,j_0})\geq 1$ since $\overline{v}_{0,j_0}$ is nontrivial.

Then $p=1$ and all the components and levels besides those of (a) and (b) satisfy $\mathrm{I}=0$. We are in Case (1).

Next suppose that $\overline{v}^\sharp_{0,b+1}=\overline{v}^\sharp_+$ has a negative end at δ_0^p. This is the situation of Lemma 3.4.23(a) with $j=b+1$. Since $\overline{v}'_+=\varnothing$, all the components and levels satisfy $\mathrm{I}\geq 0$. Here $\overline{v}^\sharp_+$ cannot have any multiple of z_∞ at the positive end, since otherwise we contradict Equation (3.4.2). Since $\mathrm{I}(\overline{v}^\sharp_+)\geq 0$ and $\sum_{\overline{v}^\sharp_*\prec\overline{v}_+}\mathrm{I}(\overline{v}^\sharp_*)\geq p$, either $p=1$ and we are in Case (2_i) or $p=2$ and we are in Cases (3)–(5).

Finally suppose that no $\overline{v}^\sharp_{0,j}$, $j=1,\dots,b+1$, has a negative end at a multiple of δ_0. This is the situation of Lemma 3.4.23(b). We have the following contributions:

(α) $\deg\overline{v}'_+=p_{0,b+1}$ and $\mathrm{I}(\overline{v}'_+)=-p_{0,b+1}$.
(β) By Lemma 3.4.22(4)–(6), the holomorphic building hanging from the negative end of $\overline{v}'_+$ satisfies $\sum_{\overline{v}_*\prec\overline{v}_+}\mathrm{I}(\overline{v}'_*\cup\overline{v}^\sharp_*)\geq p_{0,b+1}$, and equality holds if and only if there is no cylinder from δ_0 to e.
(γ) $\mathrm{I}=2$ components of $\overline{v}^\sharp_{0,j}$ from δ_0 to e can be eliminated by observing that it is followed by an $\mathrm{I}=-1$ component of $\overline{v}^\flat_-$ from e to some x_i or x'_i, which is a contradiction.[10]
(δ) Some $\overline{v}^\sharp_{1,j_0}$, $j_0\geq 0$, must have an end at z_∞ (or some boundary point of type (P_3) must have a neighborhood) which projects to a large sector of $D^2_{\rho_0}$. Since $\overline{v}^\sharp_{1,j_0}$ generically lies on a codimension 1 stratum of some $\mathrm{ind}(\overline{v}^\sharp_{1,j_0})$-dimensional moduli space, the large sector contributes an additional $+1$ to the Fredholm and ECH indices, cf. Lemma I.5.7.21.
(ε) If $\mathfrak{bp}$ is the number of boundary points of type (P_3) and $\mathfrak{bp}>0$, then the contribution to I is at least 2.

(α), (β), and (γ) together give:

$$\mathrm{I}(\overline{v}'_+)+\sum_{\overline{v}_*\prec\overline{v}_+}\mathrm{I}(\overline{v}'_*\cup\overline{v}^\sharp_*)=0.$$

Now, $\mathrm{I}(\overline{v}''_{1,j})\geq 1$ for each $j>0$,[11] (δ) contributes at least 1 to I, and (ε) contributes at least 2 to I if $\mathfrak{bp}>0$. Hence $\mathfrak{bp}=0$, $a=1$, and we are in Case (6_i), $i=1,2$. □

[10] Strictly speaking, $\overline{J}_-$ is the restriction of a Morse-Bott $\overline{J}'$. To give a proper treatment of regularity, we must perturb $\overline{J}'$ using an arbitrarily small Morse function to obtain $(\overline{J}')^\diamond$ and then restrict $(\overline{J}')^\diamond$ to $\overline{W}_-$.
[11] Recall that we are ignoring connectors.

3.4.8. *Elimination of some cases when $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some $\overline{v}_*$.* — The goal of this subsection is to eliminate all but one of the possibilities (namely Case (2_1)) given in Lemma 3.4.25 and to prove Lemma 3.3.2.

Definition **3.4.26.** — *If $\pi : \Sigma \to \Sigma'$ is a branched cover and $\mathcal{B}(\pi)$ is the branch locus of π, then we define*

$$\mathfrak{b}(\pi) = \sum_{x \in \mathcal{B}(\pi)} (\deg(x) - 1),$$

where $\deg(x)$ is the degree of the branch point x. We also write $\mathfrak{b}(\overline{u}) = \mathfrak{b}(\overline{\pi}_* \circ \overline{u})$, where $\overline{\pi}_*$ is the projection to the base B, B', etc., and refer to it informally as "the number of branch points of $\overline{u}$."

Let $\mathcal{E}_{-,i}$, $i = 1, \dots, q$, be the negative ends of $\cup_{j=1}^a \overline{v}^\sharp_{1,j}$ that converge to z_∞ and let $\mathcal{E}_{+,i}$, $i = 1, \dots, r$, be the positive ends of $\cup_{j=0}^{a-1} \overline{v}^\sharp_{1,j}$ that converge to z_∞ as in Lemma 3.4.7.

Lemma **3.4.27.** — *Suppose we are in Case (6_i) of Lemma 3.4.25. Then $\sum_{\overline{v}'_*} \mathfrak{b}(\overline{v}'_*)$, including branched points of connector components, is equal to the number of negative ends $\mathcal{E}_{-,i}$. Moreover, if F is the surface obtained by gluing all the domains of $\overline{v}'_*$, then F is a connected planar surface whose compactification (after filling in the interior and boundary punctures) has a single positive boundary component, i.e., coming from levels $\overline{v}_{1,j}$, $j = 0, \dots, a$, together with some number of negative boundary components, i.e., coming from levels $\overline{v}_{-1,j}$, $j = 1, \dots, c+1$.*

Proof. — Suppose we are in Case (6_i) of Lemma 3.4.25. We claim that we may make the following simplifying assumptions after rearranging some levels:

(1) $\overline{v}_{1,1}$ is a connector;
(2) all the ends $\mathcal{E}_{-,i}$ are negative ends of $\overline{v}^\sharp_{1,2}$;
(3) all the ends $\mathcal{E}_{+,i}$ are positive ends of $\overline{v}^\sharp_+$;
(4) all the branch points of $\overline{v}'_*$ lie on $\overline{v}'_{1,1}$.

Note that in Lemma 3.4.25 we omitted connectors, but in the present analysis we need to keep track of connectors which are branched covers. Since the lemma is of topological nature, we may assume that all the connectors between $\overline{v}_+$ and $\overline{v}_{1,1}$ have been merged and, after renaming, the connector level is $\overline{v}_{1,1}$ and the old $\overline{v}_{1,1}$ becomes $\overline{v}_{1,2}$. This gives (1). (2) is immediate and (3) can be arranged by topologically moving up/down the ends $\mathcal{E}_{+,i}$ if necessary. (4) can be obtained by pushing all the branched points of $\overline{v}'_*$ from the lower levels to $\overline{v}'_{1,1}$; this operation is possible because $\overline{v}^\sharp_* \preceq \overline{v}_+$ does not have a negative end that limits to z_∞ in Case (6_i). Note that this operation does not affect:

– the total Fredholm index $\sum_{\overline{v}'_*} \mathrm{ind}(\overline{v}'_*)$; and
– the topological type of F.

As a consequence of (1)–(4), each component of $\overline{v}'_* \preceq \overline{v}'_+$ is an unbranched cover of the appropriate σ^*_∞.

We now compute the Fredholm index of $\overline{v}'_{1,1}:\dot{F}_{1,1}\to\overline{W}$ using the Fredholm index formula

$$\operatorname{ind}(\overline{v}'_{1,1})=-\chi(\dot{F}_{1,1})+p_{1,1}+\mu_\tau(\overline{v}'_{1,1})+2c_1((\overline{v}'_{1,1})^*T\overline{S},\tau);$$

see Equation (I.5.7.2). Recall we are writing $p_*=\deg(\overline{v}'_*)$. The groomed multivalued trivialization τ is defined as follows: Let $\mathfrak{S}_{\pm,i}$ be the sector of $D^2_{\rho_0}$ given by $\pi_{D^2_{\rho_0}}(\mathcal{E}_{\pm,i})$. If (ii) and its analogs occur in the proof of Lemma 3.4.7, then we add the thin counterclockwise sectors $\mathfrak{S}(\overline{a}_{k_2,l_2},\overline{h}(\overline{a}_{k_2,l_2}))$, etc. to the set $\{\mathfrak{S}_{+,i}\}$. The sets $\{\mathfrak{S}_{+,i}\}$ and $\{\mathfrak{S}_{-,i}\}$ correspond to the data $\overrightarrow{\mathcal{D}}_-$ and $\overrightarrow{\mathcal{D}}_+$ at the negative and positive ends of $\overline{v}'_{1,1}$[12] and we let τ be the induced groomed multivalued trivialization.

By a calculation similar to that of Lemma I.5.7.15, $c_1((\overline{v}'_{1,1})^*T\overline{S},\tau)=1$ and the ends of $\overline{v}'_{1,1}$ contribute the following to $\mu_\tau(\overline{v}'_{1,1})$:

- 0 if $\mathfrak{S}_{-,i}$ is a small sector;
- -1 if $\mathfrak{S}_{-,i}$ is a large sector;
- -1 if $\mathfrak{S}_{+,i}$ is a small sector; and
- -2 if $\mathfrak{S}_{+,i}$ is a large sector.

Hence $\mu_\tau(\overline{v}'_{1,1})=-p_{1,1}-1$. Since $\chi(\dot{F}_{1,1})=p_{1,1}-\mathfrak{b}(\overline{v}'_{1,1})$, we obtain:

$$\begin{aligned}\operatorname{ind}(\overline{v}'_{1,1})&=(\mathfrak{b}(\overline{v}'_{1,1})-p_{1,1})+p_{1,1}+(-p_{1,1}-1)+2\\&=\mathfrak{b}(\overline{v}'_{1,1})-(p_{1,1}-1).\end{aligned}\tag{3.4.18}$$

Next we claim that $\partial F_{1,1}$ is connected. Indeed, if $\partial F_{1,1}$ is disconnected, then the method of Lemma 3.4.7 implies that the union of all the $\mathfrak{S}_{\pm,i}$ covers $D^2_{\rho_0}$ more than once; this contradicts Equation (3.4.2). The claim in turn implies that $\mathfrak{b}(\overline{v}'_{1,1})\geq p_{1,1}-1$, since otherwise $F_{1,1}$ is disconnected and $\partial F_{1,1}$ will have more than one component. Hence $\operatorname{ind}(\overline{v}'_{1,1})\geq 0$; moreover, if $\operatorname{ind}(\overline{v}'_{1,1})>0$, then $\operatorname{ind}(\overline{v}'_{1,1})\geq 2$.

We claim that $\operatorname{ind}(\overline{v}'_{1,1})\geq 2$ is not possible. Indeed, we add up the Fredholm indices of all the remaining components as in the proof of Lemma 3.4.25:

(a) $\operatorname{ind}(\overline{v}^\sharp_{1,2})\geq 1$ and $\operatorname{ind}(\overline{v}'_+)=-p_+$;
(b) $\sum_{\widetilde{v}}\operatorname{ind}(\widetilde{v})\geq p_+$, where the summation is over all components $\widetilde{v}$ of $\overline{u}_\infty$ that are hanging from the negative end of $\overline{v}'_+$;
(c) a large sector of D^2 contributes an additional $+1$ to the Fredholm index; and
(d) all the other components of $\overline{u}_\infty$ have nonnegative Fredholm index.

(a), (b), (d) are clear, and (c) was explained in Lemma 3.4.25. The total of (a)–(d) is ≥ 2, which is an index excess of $+2$, and the claim follows. The claim then implies that $\operatorname{ind}(\overline{v}'_{1,1})=0$, $\mathfrak{b}(\overline{v}'_{1,1})=p_{1,1}-1$, and $F_{1,1}$ is a disk.

[12] Note that the plus and minus signs are switched. This is due to the fact that, for example, $\{\mathfrak{S}_{-,i}\}$ gives the data for the negative ends of $\overline{v}_{1,2}$ that limit to z_∞, which agrees with the data for the positive ends of $\overline{v}_{1,1}$ that limit to z_∞.

Finally, since F is diffeomorphic to a surface obtained by gluing $\dot{F}_{1,1}$ and $p_{1,1}$ surfaces which are diffeomorphic to B_τ or B_+, it must be connected and planar, and its compactification is as described in the lemma. □

Lemma **3.4.28.** — *Suppose* $m \gg 0$. *If* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$, *then Cases (1), (2_0), (3)–(6_i) of Lemma 3.4.25 are not possible. This leaves us with Case (2_1).*

Proof. — Arguing by contradiction, suppose there exist sequences $m_i \to \infty$, $\varepsilon_i, \delta_i \to 0$, and $\overline{u}_{ij} \to \overline{u}_{i\infty}$, where $\overline{u}_{ij} \in \mathcal{M}^{(m_i)}$, $\overline{u}_{i\infty} \in \partial_{\{+\infty\}}\mathcal{M}^{(m_i)}$, $\mathcal{M}^{(m_i)}$ is $\mathcal{M}$ with respect to the family $\{\overline{J}^\diamondsuit_\tau(\varepsilon_i, \delta_i, \mathfrak{p}(\tau); m_i)\}$, and $\overline{u}_{i\infty}$ falls into one of Cases (1), (2_0), (3)–(6_i). We consider a diagonal subsequence $\overline{u}_i := \overline{u}_{ij(i)}$.

Cases (1) and (2_0) are eliminated by an argument similar to that of Case (2) of Theorem I.7.10.1 and Cases (3)–(5) are eliminated by an argument similar to that of Cases (3)–(6) of Theorem I.7.10.1.

Case (6_i). Suppose for simplicity that $b = 1$, including connectors.

As in the proof of Theorem I.7.10.1, for $i \gg 0$, we consider a truncation $\widetilde{u}_i : \Sigma_i \to \overline{W}_{\tau_i}$ of $\overline{u}_i$ (i.e., a restriction of $\overline{u}_i$ to a neighborhood of $\sigma_\infty^{\tau_i}$) such that there exist real numbers

$$R_{0,i} < R_{1,i} < R_{2,i} < R_{3,i}, \quad R_{0,i} \ll -l(\tau_i), \quad R_{3,i} \gg l(\tau_i)$$

and a map

$$\pi_i : \Sigma_i \to B_{\tau_i} \cap \{R_{0,i} \leq s \leq R_{3,i}\},$$

such that the restriction of π_i to $\pi_i^{-1}(\{R_{j,i} \leq s \leq R_{j+1,i}\})$, $j = 0, 1, 2$, is a degree $p_{0,j}$ branched cover. Here $p_{0,0} \leq p_{0,1} \leq p_{0,2}$. We project $\widetilde{u}_i$ to $D^2_{\rho_0} \subset \overline{S}$ for $\rho_0 > 0$ small using balanced coordinates and then apply the ansatz from Equation (I.7.8.1) to obtain $w_i : \Sigma_i \to \mathbf{C}$.

Applying the method of Section I.7.8, we rescale w_i by a positive real constant and take the limit $m_i \to \infty$ to obtain a 3-level holomorphic building

$$w_\infty = w_- \cup w_{0,1} \cup w_+.$$

We write $w_* : \Sigma_* \to \mathbf{CP}^1$ for the components of the building w_∞ and $\pi_* : \Sigma_* \to cl(B_*)$ for the corresponding branched covers, where $* = +, -$, or $(0, 1)$, and $B_{0,1} = \mathbf{R} \times S^1$. We may also use subscripts $(0, 0)$ and $(0, 2)$ to mean $-$ and $+$. Note that $\deg \pi_{0,j} = p_{0,j}$. By Lemma 3.4.27, the surface Σ_∞, obtained by gluing the Σ_*, is connected and planar. The next few paragraphs are devoted to the description of w_* and π_*.

Suppose for simplicity that Σ_-, $\Sigma_{0,1}$, and Σ_+ are connected. Starting from the bottom, w_- and π_- satisfy the following:

(i_-) $w_-(\partial\Sigma_-) \subset \{\phi = 0, \rho > 0\}$;

(ii$_-$) $\pi_-^{-1}(+\infty)$ is a single point and $w_-(\pi_-^{-1}(+\infty)) = \infty$;
(iii$_-$) $w_-(z_0) = 0$ for one of the points $z_0 \in \pi_-^{-1}(\overline{\mathfrak{m}}^b(\infty))$; and
(iv$_-$) $w_-|_{int(\Sigma_-)}$ is a biholomorphism onto its image.

(i$_-$) and (iii$_-$) are clear, (ii$_-$) follows from the fact that Σ_∞ (and hence Σ_-) is connected and planar, and (iv$_-$) is a consequence of Equation (3.4.2). Let us write $f_* = \pi_* \circ w_*^{-1}$, where defined. Then f_- maps the asymptotic marker $\dot{\mathcal{R}}_\pi(\infty)$ for $\infty \in \mathbf{CP}^1$, corresponding to the radial ray $\mathcal{R}_\pi$, to the asymptotic marker $\dot{\mathcal{L}}_{3/2}(+\infty)$ for $+\infty \in cl(\mathrm{B}_-)$, corresponding to the half-line $\mathcal{L}_{3/2}$, by the Involution Lemma I.7.9.3.

We then move up to $w_{0,1}$, which satisfies the following:

(i$_{0,1}$) $w_{0,1}(\pi_{0,1}^{-1}(-\infty)) = \{d_1 = 0, d_2, \dots, d_k\}$, where $d_i \in \mathbf{R}^{\geq 0}$ and $d_i < d_{i+1}$;
(ii$_{0,1}$) $\pi_{0,1}^{-1}(+\infty)$ is a single point and $w_{0,1}(\pi_{0,1}^{-1}(+\infty)) = \infty$;
(iii$_{0,1}$) $w_{0,1}$ is a biholomorphism; and
(iv$_{0,1}$) $f_{0,1} : \mathbf{CP}^1 \to cl(\mathrm{B}_{0,1})$ maps $\dot{\mathcal{R}}_\pi(0)$ to $\dot{\mathcal{L}}_{3/2}(-\infty)$.

The placement of the points $d_2, \dots, d_k$ in (i$_{0,1}$) follows from observing that $\overline{v}_-^\sharp$ consists of $\mathrm{I} = 1$ components from δ_0 to x_i or x_i'. In particular, the asymptotic eigenfunctions of $\overline{v}_-^\sharp$ at the positive end are close to constant functions with values on $\mathbf{R}^+ \subset \mathbf{C}$. (ii$_{0,1}$) follows from the fact that Σ_∞ is connected and planar and (iii$_{0,1}$) follows from Equation (3.4.2). (iv$_{0,1}$) is a consequence of the fact that f_- maps $\dot{\mathcal{R}}_\pi(\infty)$ to $\dot{\mathcal{L}}_{3/2}(+\infty)$. Then $f_{0,1}$ maps $\dot{\mathcal{R}}_\pi(\infty)$ to $\dot{\mathcal{L}}_{3/2}(+\infty)$ by the Involution Lemma I.7.9.4.

We now describe the construction of w_+ in some detail. By translating

$$\pi_i^{-1}(\mathrm{B}_{\tau_i} \cap \{\mathrm{R}_{2,i} \leq s \leq \mathrm{R}_{3,i}\})$$

down by T_i, we obtain the branched cover

$$\pi_i^+ : \Sigma_i^+ \to \mathrm{B}_+ \cap \{\mathrm{R}_{2,i} - \mathrm{T}_i \leq s \leq \mathrm{R}_{3,i} - \mathrm{T}_i\}$$

and the holomorphic map $w_i^+ : \Sigma_i^+ \to \mathbf{C}$. We may assume that $\mathrm{R}_{2,i} - \mathrm{T}_i \to -\infty$ and $\mathrm{R}_{3,i} - \mathrm{T}_i \to +\infty$ have been chosen so that there is no sequence of branch points of π_i^+ that limits to $s = \pm\infty$ as $i \to \infty$. Indeed, the branch points that "escape to $s = \pm\infty$" properly belong to a different level. Then w_+ is the limit of w_i^+, after suitably rescaling by positive real constants.

Let $\overrightarrow{\mathcal{D}} = \{(i_k', j_k') \to (i_k, j_k)\}_{k=1}^p$ be the data at the positive end of w_+. For $i \gg 0$, the component of $w_i^+|_{(\pi_i^+)^{-1}(\{s=\mathrm{R}_{3,i}-\mathrm{T}_i\})}$ corresponding to $(i_k', j_k') \to (i_k, j_k)$ is arbitrarily close to a normalized asymptotic eigenfunction ϕ_k from $\overline{h}(\overline{a}_{i_k',j_k'})$ to $\overline{a}_{i_k,j_k}$, after multiplying by some positive real constant. (Here ϕ_k is an eigenfunction of an asymptotic operator and we are not making any *a priori* assumptions on the corresponding eigenvalues.) Now we claim that some ϕ_{k_0} must sweep out a large sector of D^2, since otherwise $\mathrm{Im}(w_i^+)$ cannot pass through the origin, contradicting the "continuity" to $w_{0,1}$. In the limit $i \to \infty$, ϕ_{k_0} will sweep out an angle of 2π.

The top level w_+ satisfies the following:

(i$_+$) $w_+(\pi_+^{-1}(-\infty))=\{d_1^+=0,d_2^+,\dots,d_{k_+}^+\}$, where $d_i^+\in\mathbf{R}^{\geq 0}$ and $d_i^+<d_{i+1}^+$;
(ii$_+$) $w_+(\partial\Sigma_+)\subset\{\phi=0,0<\rho\leq\infty\}$;
(iii$_+$) w_+ maps some point of $\pi_+^{-1}(+\infty)$ to ∞;
(iv$_+$) f_+ maps $\dot{\mathcal{R}}_\pi(0)$ to $\dot{\mathcal{L}}_{3/2}(-\infty)$; and
(v$_+$) w_+ is a biholomorphism onto its image.

The placement of the points $d_2^+,\dots,d_{k_+}^+$ in (i$_+$) follows from observing that $\overline{v}_{0,1}^\sharp$ consists of $\mathrm{I}=1$ components from δ_0 to h and that $\mathcal{R}_{\phi_h}\to\mathcal{R}_0$ as $m\to\infty$ by our choice of h from Section 3.2.3. (ii$_+$) and (iii$_+$) are immediate consequences of the construction of w_+ from the previous paragraphs. (iv$_+$) is a consequence of the fact that $f_{0,1}$ maps $\dot{\mathcal{R}}_\pi(\infty)$ to $\dot{\mathcal{L}}_{3/2}(+\infty)$. We now apply the Involution Lemma I.7.9.5 to conclude that $f_+(\infty)=\mathcal{L}_{3/2}\cap\partial\mathrm{B}_+$. Note that the Lemma I.7.9.5 applies because the compactification of Σ_+ is a closed disk by Lemma 3.4.27. This contradicts (ii$_+$). We have eliminated Case (6$_i$). ☐

We are now in a position to prove Lemma 3.3.2.

Proof of Lemma 3.3.2. — Suppose $\overline{u}_\infty\in\partial_{\{+\infty\}}\mathcal{M}$. By Lemma 3.4.21, if $\overline{v}'_*\cup\overline{v}^\sharp_*=\varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$, then $\overline{u}_\infty\in\mathrm{A}_1$. If $\overline{v}'_*\cup\overline{v}^\sharp_*\neq\varnothing$ for some level $\overline{v}_*$, then $\overline{u}_\infty$ is as given in Lemma 3.4.25. Now, by Lemma 3.4.28, the only possibility left is Case (2$_1$), which implies that $\overline{u}_\infty\in\mathrm{A}_2$. ☐

3.5. *Degeneration at $-\infty$.* — In this subsection we study the limit of holomorphic maps to $\overline{\mathrm{W}}_\tau$ as $\tau\to-\infty$, i.e., when $\overline{\mathrm{W}}_\tau$ degenerates into $\overline{\mathrm{W}}_{-\infty,1}\cup\overline{\mathrm{W}}_{-\infty,2}$. This will prove Lemma 3.3.5.

We assume that $m\gg 0$; $\varepsilon,\delta>0$ are sufficiently small; and $\{\overline{J}_\tau\}\in\overline{\mathcal{I}}^{reg}$ and $\{\overline{J}_\tau^\Diamond(\varepsilon,\delta,\mathfrak{p}(\tau))\}$ satisfy Lemma 3.2.22. Fix $\mathbf{y}\in\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$, $\mathbf{y}'\in\mathcal{S}_{\mathbf{b},h(\mathbf{b})}$ and let

$$\mathcal{M}=\mathcal{M}^{\mathrm{I}=2,n^*=m}_{\{\overline{J}_\tau^\Diamond(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y},\mathbf{y}';\overline{\mathfrak{m}}),\quad \mathcal{M}_\tau=\mathcal{M}^{\mathrm{I}=2,n^*=m}_{\overline{J}_\tau^\Diamond(\varepsilon,\delta,\mathfrak{p}(\tau))}(\mathbf{y},\mathbf{y}';\overline{\mathfrak{m}}(\tau)).$$

We will analyze $\partial_{\{-\infty\}}\mathcal{M}$.

Let $\overline{u}_i$, $i\in\mathbf{N}$, be a sequence of curves in $\mathcal{M}$ such that $\overline{u}_i\in\mathcal{M}_{\tau_i}$ and $\lim_{i\to\infty}\tau_i=-\infty$, and let

$$\begin{aligned}\textbf{(3.5.1)}\qquad \overline{u}_\infty=\overline{v}_2\cup(\overline{v}_{\mathrm{L},1}\cup\dots\cup\overline{v}_{\mathrm{L},a})\cup\overline{v}_1\cup(\overline{v}_{\mathrm{R},1}\cup\dots\cup\overline{v}_{\mathrm{R},b})\\ \cup(\overline{v}_{\mathrm{B},1}\cup\dots\cup\overline{v}_{\mathrm{B},c})\cup(\overline{v}_{\mathrm{T},1}\cup\dots\cup\overline{v}_{\mathrm{T},d}),\end{aligned}$$

be the limit holomorphic building, where each $\overline{v}_*$ is an SFT-type level, $\overline{v}_j$ maps to $\overline{\mathrm{W}}_{-\infty,j}$, $j=1,2$; $\overline{v}_{\mathrm{L},j}$ and $\overline{v}_{\mathrm{R},j}$ map to $[-2,2]\times\mathbf{R}\times\overline{S}$; $\overline{v}_{\mathrm{B},j}$ and $\overline{v}_{\mathrm{T},j}$ map to $\mathbf{R}\times[0,1]\times\overline{S}$. The levels in the first row, called *the horizontal levels*, are arranged in cyclic order from left to right, the levels in the second row, called *the vertical levels*, are arranged in order from bottom to top, and $\overline{v}_1$ is between $\overline{v}_{\mathrm{B},c}$ and $\overline{v}_{\mathrm{T},1}$. The terms “horizontal” and “vertical” refer to the positions of the levels in Figure 5, and do not imply a difference in the geometry. Both refer to fairly standard SFT degenerations and the existence of the limit $\overline{u}_\infty$ is a consequence of the SFT compactness discussion from Section I.7.3 with minimal change. For notational convenience we refer to $\overline{v}_1$ as $\overline{v}_{L,a+1}$, $\overline{v}_{R,0}$, $\overline{v}_{B,c+1}$, or $\overline{v}_{T,0}$, and $\overline{v}_2$ as $\overline{v}_{L,0}$ or $\overline{v}_{R,b+1}$. As usual, we write $p_* = \deg \overline{v}'_*$.

Terminology. Thin counterclockwise sectors in $\mathbf{C}$ from $\overline{b}_{i,j}$ to $\overline{a}_{i,j}$ and from $\overline{h}(\overline{b}_{i,j})$ to $\overline{h}(\overline{a}_{i,j})$ will also be referred to as *thin sectors*.

Outline of proof of Lemma 3.3.5. The initial steps of the proof are similar to those of Section I.7.4–I.7.11 and Section 3.4. However, the authors were unable to prove that $I(\overline{v}_{L,j})$ and $I(\overline{v}_{R,j})$ were nonnegative when $\overline{v}'_1 \neq \varnothing$, since we could not sufficiently control the groomings in order to apply the ECH index inequality (Lemma I.5.7.21). Note that Lemma 3.5.2(4) specifically excludes the case $\overline{v}'_1 \neq \varnothing$. Sections 3.5.3–3.5.6 are intended as a substitute, and involve ideas from tropical geometry (see for example Parker [Pa]).

3.5.1. *Continuation argument.* — We discuss the continuation argument in the current case; this is similar to but more complicated than those of Sections 3.4.1 and 3.4.3. Suppose that $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$ of $\overline{u}_\infty$. For simplicity we assume that there are no boundary points at z_∞; we leave it to reader to make the appropriate modifications when there are boundary points at z_∞.

Case 1. Suppose that $\overline{v}'_{T,j} \cup \overline{v}^\sharp_{T,j} \neq \varnothing$ for some $j > 0$. We start at a nontrivial negative end $\mathcal{E}_1$ of some $\overline{v}_{T,j_1}$, $j_1 > 0$, limiting to z_∞. We consider the continuation

$$g^1_{j_1-1,1}, \dots, g^1_{1,1}, g_{-a-1,1}, \dots, g_{b+1,1}, g^0_{1,1}, \dots, g^0_{d,1}$$

of the $t = 1$ boundary of $\mathcal{E}_1$ in the direction of $\partial_+ B_\tau$. Here Definition 3.4.5 needs to be adapted in the obvious way to the breaking of a component of $\overline{u}_i|_{\partial \dot{G}_i}$ as $\tau_i \to -\infty$, where $\dot{G}_i$ is the domain of $\overline{u}_i$. The components $g_{-a-1,1}$ and $g_{b+1,1}$ correspond to $\overline{v}_1$ and the components $g_{j,1}$, $j = -a, \dots, b$, correspond to $\overline{v}_{L,j}$, $j = a, a-1, \dots, 0$, and $\overline{v}_{R,j}$, $j = b, b-1, \dots, 1$, in that order. There are three possibilities:

(i) there is some $g^1_{j,1}$, $0 \leq j \leq j_1 - 1$, which is nontrivial;
(ii) all the $g^1_{j,1}$ are trivial but some $g_{j,1}$ is nontrivial;
(iii) all the $g^1_{j,1}$ and $g_{j,1}$ are trivial.

Cases (i) and (iii) have already been treated in the proof of Lemma 3.4.7. If we are in Case (ii), then the nontrivial component $g_{j,1}$ contains the $s = 2$ boundary of a right end $\mathcal{E}_2$ that limits to z_∞. We then consider the continuation of the $s = -2$ boundary of $\mathcal{E}_2$ in the direction of $\partial_- B_\tau$. The details of the continuation are left to the reader, but in the end the sectors $\pi_{D^2_{\rho_0}}(\mathcal{E}_i)$ will sweep out a neighborhood of z_∞ with the exception of thin sectors.

Case 2. Suppose that $\overline{v}'_{\mathrm{T},j} \cup \overline{v}^\sharp_{\mathrm{T},j} = \varnothing$ for all $j > 0$. Then $\overline{v}'_1 = \varnothing$. Consider the horizontal levels:

$$\overline{v}_1 = \overline{v}_{\mathrm{R},0}, \dots, \overline{v}_{\mathrm{R},b}, \overline{v}_2, \overline{v}_{\mathrm{L},1} \dots, \overline{v}_{\mathrm{L},a+1} = \overline{v}_1. \tag{3.5.2}$$

Suppose that $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some horizontal level $\overline{v}_*$. Let $\overline{v}_*$ be the leftmost level in Equation (3.5.2) such that $\overline{v}^\sharp_*$ has a right end $\mathcal{E}_1$ at z_∞. The sector $\pi_{\mathrm{D}^2_{\rho_0}}(\mathcal{E}_1)$ is not a thin sector and we apply the usual continuation argument to the levels of Equations (3.5.2).

Case 3. The case where $\overline{v}'_{\mathrm{T},j} \cup \overline{v}^\sharp_{\mathrm{T},j} = \varnothing$ for all $j > 0$, $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all horizontal levels $\overline{v}_*$, and $\overline{v}'_{\mathrm{B},j} \cup \overline{v}^\sharp_{\mathrm{B},j} \neq \varnothing$ for some $j \leq c$ is treated similarly.

Let $\mathcal{Z} = (\mathfrak{z}_1 \to \cdots \to \mathfrak{z}_k \to \mathfrak{z}_1)$ be the cycle constructed using the above continuation argument. Boundary points of type (P_1), (P_2), and (P_3) are defined in the same way: for (P_2) all the vertices of $\mathcal{Z}$ lie on arcs of the same type (Type $\overline{\mathbf{a}}$, Type $\overline{\mathbf{b}}$, Type $\overline{h}(\overline{\mathbf{a}})$, or Type $\overline{h}(\overline{\mathbf{b}})$), and for ($P_3$) the vertices of $\mathcal{Z}$ must lie on arcs of more than one type.

3.5.2. *Some restrictions on $\overline{u}_\infty$.* — The following are analogs of Lemmas 3.4.16, 3.4.20, 3.4.21 and 3.4.22.

Lemma **3.5.1.** — *Suppose that $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$ of $\overline{u}_\infty$. If $\mathcal{E}_i$, $i = 1, \dots, q$, are the ends of all the $\overline{v}^\sharp_*$ that converge to z_∞ and $\mathcal{E}'_i$, $i = 1, \dots, r$ are the neighborhoods of the boundary points of type (P_3), then*

$$\sum_{i=1}^{q} n^*(\mathcal{E}_i) + \sum_{i=1}^{r} n^*(\mathcal{E}'_i) \geq m - 2g.$$

Lemma **3.5.2.** — *If fiber components are removed from $\overline{u}_\infty$ and the only boundary points at z_∞ are of type (P_3), then the following hold:*

1. *the ECH index of each $\overline{v}''_*$ is nonnegative;*
2. *the only components of $\overline{u}_\infty$ which have negative ECH index are those of $\overline{v}'_1$, i.e., the branched covers of $\sigma^{-\infty,1}_\infty$;*
3. *the ECH index of each level $\overline{v}_{\mathrm{T},j}$, $\overline{v}_{\mathrm{B},j} \neq \overline{v}_1$ is nonnegative;*
4. *if $\overline{v}'_1 = \varnothing$, then the ECH index of each $\overline{v}_{\mathrm{L},j}$, $\overline{v}_{\mathrm{R},j} \neq \overline{v}_1, \overline{v}_2$ is nonnegative;*
5. *there is an additional contribution of $\mathfrak{bp}_* + 1$ towards* I*, where $\mathfrak{bp}_*$ is the number of boundary points of type (P_3) on $\overline{v}_*$.*

Proof. — (1) and (2) are consequences of the index inequality. (3) and (4) follow from the proof of Lemma 3.4.14. In the proof of (4), the vertical levels $\overline{v}_{1,a}, \dots, \overline{v}_{1,0}$ and the arcs $\overline{\mathbf{a}}$, $\overline{h}(\overline{\mathbf{a}})$ are replaced by the horizontal levels given by Equation (3.5.2) and the arcs $\overline{\mathbf{b}}$, $\overline{\mathbf{a}}$. (5) is argued in the same way as Lemma 3.4.20. □

Lemma **3.5.3.** — *If $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$ and $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$, then $\overline{u}_\infty$ satisfies the following: $a=b=c=d=0$; $\mathrm{I}(\overline{v}_1)=\mathrm{ind}(\overline{v}_1)=0$ and $\mathrm{I}(\overline{v}_2)=\mathrm{ind}(\overline{v}_2)=2$; and $\overline{v}_1$ is a $\mathrm{W}_{-\infty,1}$-curve and $\overline{v}_2$ is a $\overline{\mathrm{W}}_{-\infty,2}$-curve.*

Proof. — Similar to that of Lemma 3.4.21 and is omitted. □

Lemma **3.5.4.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$ of $\overline{u}_\infty$, then:*

(1) *$p_2 = \deg(\overline{v}'_2) > 0$;*
(2) *there is no boundary point of type (P_1) or (P_2), i.e., the only boundary points at z_∞ are of type (P_3);*
(3) *$\overline{u}_\infty$ has no fiber components and no components $\overline{v}''_*$ that intersect the interior of a section at infinity;*
(4) *each of $\overline{v}_{\mathrm{L},j}$, $j=1,\dots,a$, and $\overline{v}_{\mathrm{R},j}$, $j=1,\dots,b+1$, consists of thin strips and trivial strips; in particular, $\overline{v}_{\mathrm{L},j}$, $j=1,\dots,a$, and $\overline{v}_{\mathrm{R},j}$, $j=1,\dots,b+1$ have no boundary points at z_∞.*

Proof. — (1), (2), (3) We have the following contributions towards n^*: (1) the restriction of $\overline{v}''_2$ to a neighborhood of $\overline{\mathfrak{m}}(-\infty)$ contributes m if $\overline{v}'_2 = \varnothing$; (2) boundary points of type (P_1) or (P_2) contribute at least m in total; and (3) a fiber component or a component of $\overline{v}''_*$ that intersects a section at infinity contributes at least m. All cases contradict Lemma 3.5.1.

(4) Arguing by contradiction, suppose that some level $\overline{v}_{\mathrm{R},j_0}$, $1 \leq j_0 \leq b$, has a component which is not a thin strip or a trivial strip. (The case of $\overline{v}_{\mathrm{L},j_0}$, $0 \leq j_0 \leq a$, only differs in notation.) Here $\overline{v}_{\mathrm{R},j_0}$ may have a boundary point of type (P_3).

We claim that the following hold:

(a) $n^*(\overline{v}_{\mathrm{R},j_0}) = m$ and $n^{*,alt}(\overline{v}_{\mathrm{R},j_0}) \geq m-2g$;
(b) $\overline{v}^\sharp_{\mathrm{R},j_0} \neq \varnothing$ and some end of $\overline{v}^\sharp_{\mathrm{R},j_0}$ corresponds to a large sector;
(c) $\overline{v}'_1 = \varnothing$ and $\overline{v}^\sharp_{\mathrm{T},j} = \varnothing$ for all $j>0$;
(d) no left end of $\overline{v}_{\mathrm{R},j_0}$ limits to a multiple of z_∞; in particular $\overline{v}'_{\mathrm{R},j_0} = \varnothing$;
(e) there are no boundary points of type (P_3);
(f) the ECH index of each $\overline{v}_{\mathrm{L},j}$, $1 \leq j \leq a$, and $\overline{v}_{\mathrm{R},j}$, $1 \leq j \leq b+1$ is nonnegative and $\mathrm{I}(\overline{v}_{\mathrm{R},j_0}) \geq 2$.

Recall the definition of $n^{*,alt}$ from Section 3.2.3.

We first prove (a). Consider the projection

$$\pi_{\overline{\mathrm{S}}} : [-2,2] \times \mathbf{R} \times \overline{\mathrm{S}} \to \overline{\mathrm{S}}.$$

Since we are only dealing with compactness issues, we may assume without loss of generality that $\overline{\mathrm{J}}^{\diamondsuit}_{-\infty,2}(\varepsilon,\delta,\mathfrak{p}(-\infty))$ is a product complex structure and the projection $\pi_{\overline{\mathrm{S}}}$ is holomorphic. Since $\overline{v}_{\mathrm{R},j_0}$ has a component which is not a thin strip or a trivial strip,

$\mathrm{Im}(\pi_{\overline{S}} \circ \overline{v}_{R,j_0})$ must contain the complement of all the thin strips between the $\overline{b}_i$ and the $\overline{a}_i$. This proves (a).

We now prove (b)–(f). If (c) does not hold, then there is some negative end $\mathcal{E}$ of $\cup_{j=1}^{a} \overline{v}^\sharp_{T,j}$ that limits to z_∞ and $n^*(\mathcal{E}) \geq k_0 - 1 \gg 2g$. This is a contradiction of (a). If (d) does not hold, i.e., a left end of $\overline{v}_{R,j_0}$ limits to a multiple of z_∞, then in view of (c) there is some right end $\mathcal{E}$ of $\overline{v}^\sharp_*$ to the left of $\overline{v}_{R,j_0}$ that limits to z_∞ and satisfies $n^*(\mathcal{E}) \geq k_0 - 1 \gg 2g$. This is a contradiction of (a). (e) is a consequence of (d). (b) follows from (a) and (e) by excluding some possibilities using (2), (3). Finally we prove (f). By Lemma 3.5.2(4) and (c), the ECH indices of $\overline{v}_{L,j}$, $j = 1, \dots, a$, and $\overline{v}_{R,j}$, $j = 1, \dots, b+1$, are nonnegative. Since $\overline{v}^\sharp_{R,j_0}$ has a large sector by (b), its ECH index is increased by one. Hence $\mathrm{I}(\overline{v}_{R,j_0}) \geq 2$.

We now return to the proof of (4). By (d), the right end of $\overline{v}_{R,j_0}$ limits to a multiple of z_∞. By (a), each component of $\overline{v}^\sharp_* \neq \overline{v}^\sharp_1$ to the right of $\overline{v}_{R,j_0}$ must be a thin strip and the projection to $\overline{S}$ of the union of all the ends of $\overline{v}^\sharp_1$ limiting to z_∞ is a union of thin wedges of type $\mathfrak{S}(\overline{b}_{i,j}, \overline{a}_{i,j})$. Using Figure 6 one can verify that such a curve $\overline{v}^\sharp_1$ does not exist. This implies that $\overline{v}^\sharp_1 = \varnothing$ and that the ECH index of $\overline{v}_1$ is ≥ 0. Finally, since the ECH index of each thin strip is 1, the analog of Equation (3.4.3) for our case, Lemma 3.5.2, and (f) together imply that no component of $\overline{v}_* \neq \overline{v}_1$ to the right of $\overline{v}_{R,j_0}$ can be a thin strip. This is a contradiction and (4) follows. □

3.5.3. *Truncations.* — In the rest of Section 3.5 until the proof of Lemma 3.5.13, we consider the case where $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some $\overline{v}_*$. We have $\overline{v}'_2 \neq \varnothing$ in view of Lemma 3.5.4(1).

In Sections 3.5.3–3.5.5 we consider the case $\overline{v}'_1 = \varnothing$. By Lemma 3.5.4(4), $\overline{u}_\infty$ has no boundary points at z_∞.

Definition **3.5.5.** — *Let* X *be set and* $\varepsilon > 0$ *be a positive real number. Then two functions* $f, g : \mathrm{X} \to \mathbf{C}^\times$ *are* ε-approximate *if*

$$|f(x) - g(x)| < \varepsilon \cdot |g(x)| \quad \text{and} \quad |f(x) - g(x)| < \varepsilon \cdot |f(x)|$$

for all $x \in \mathrm{X}$.

We now define the following sequence of truncations, analogous to those which appear in the proof of Theorem I.7.10.1.

Definition **3.5.6.** — *With respect to the above assumptions on* $\overline{u}_i : \dot{\mathrm{G}}_i \to \overline{\mathrm{W}}_{\tau_i}$, *an* ε_i-truncation *of* $\overline{u}_i$ *with* $0 < \varepsilon_i < \frac{\rho_0}{2}$ *is the restriction of* $\overline{u}_i$ *to a subsurface* $\Sigma_i \subset \dot{\mathrm{G}}_i$ *which satisfies the following: First write*

$$\partial_v \Sigma_i = \partial \Sigma_i - (\pi_{\mathrm{B}_{\tau_i}} \circ \overline{u}_i)^{-1}(\partial \mathrm{B}_{\tau_i})$$

$$\partial_h \Sigma_i = \partial \Sigma_i \cap (\pi_{B_{\tau_i}} \circ \overline{u}_i)^{-1}(\partial B_{\tau_i}).$$

Then

(T1) $\overline{u}_i(\Sigma_i)$ *is contained in a* $2\varepsilon_i$*-neighborhood of* $\sigma_\infty^{\tau_i}$*;*

(T2) *if* $\overline{u}_i(x)$ *is contained in an* $\frac{\varepsilon_i}{2}$*-neighborhood of* $\sigma_\infty^{\tau_i}$*, then* $x \in \Sigma_i$*;*

(T3) $\pi_{B_{\tau_i}} \circ \overline{u}_i$ *maps each component* c *of* $\partial_v \Sigma_i$ *to some* $t = const$ *and* $\partial_h \Sigma_i$ *to* $\{s = \pm 2\}$*;*

(T4) *there exist constants* $r(\tau_i) - 2 > R_i^{(1)} > \cdots > R_i^{(\iota+1)} > 2$ *and a decomposition* $\Sigma_i = \Sigma_i^{(1)} \cup \cdots \cup \Sigma_i^{(\iota)}$ *such that*

$$\Sigma_i^{(j)} = (\pi_{B_{\tau_i}} \circ \overline{u}_i)^{-1}([-2,2] \times [R_i^{(j+1)}, R_i^{(j)}]) \cap \Sigma_i$$

and each

$$\pi_{B_{\tau_i}} \circ \overline{u}_i : \Sigma_i^{(j)} \to [-2,2] \times [R_i^{(j+1)}, R_i^{(j)}]$$

is a branched cover with possibly empty branch locus;

(T5) *for each component* c *of* $\partial_v \Sigma_i$, $\pi_{D^2_{\rho_0}} \circ \overline{u}_i|_c$ *is* $\frac{\varepsilon_i}{10}$*-approximate to a positive multiple* ε_i *of a* normalized *eigenfunction of* $\overline{v}^\sharp_{L,j}$, $j = 0, \dots, a$, *or* $\overline{v}^\sharp_{R,j}$, $j = 0, \dots, b$*; moreover, for all* c, $\max_c |\pi_{D^2_{\rho_0}} \circ \overline{u}_i| = \varepsilon_i$.

Here

$$\pi_{D^2_{\rho_0}} : \pi^{-1}_{B_{\tau_i}}([-2,2] \times [R_i^{(\iota+1)}, R_i^{(1)}]) \cap \{\rho \le \rho_0\} \to D^2_{\rho_0}$$

is obtained by projecting out the ∂_s*- and* $\overline{R}_{\tau_i}$*-directions.*

From now on we assume that $\overline{u}_i|_{\Sigma_i}$ *is a sequence of* ε_i*-truncations, where* $\varepsilon_i \to 0$ *and* $R_i^{(1)} - R_i^{(\iota+1)} \to \infty$ *as* $i \to \infty$.

Let $\widetilde{\pi}_i$ be the map obtained by postcomposing

$$\pi_{B_{\tau_i}} \circ \overline{u}_i : \Sigma_i \to [-2,2] \times [R_i^{(\iota+1)}, R_i^{(1)}]$$

with a $-\frac{r(\tau_i)+1}{2}$-translation in the t-direction, and let $\widetilde{w}_i = \pi_{D^2_{\rho_0}} \circ \overline{u}_i|_{\Sigma_i}$. Also let

$$R_i^+ = R_i^{(1)} - \frac{r(\tau_i)+1}{2}, \quad R_i^- = R_i^{(\iota+1)} - \frac{r(\tau_i)+1}{2}.$$

Notation. When we want to distinguish the t-coordinates for $B_{-\infty,1}$ and $B_{-\infty,2}$, we write t_i for the t-coordinate for $B_{-\infty,i}$. Note that $\widetilde{\pi}_i$ can be viewed as a map to $B_{-\infty,2}$ with (s, t_2)-coordinates.

Lemma **3.5.7.** — *If* $i \gg 0$*, then* Σ_i *is a disk with* $\ge 2p_2$ *boundary punctures.*

Proof. — This is a consequence of (T5) and the following:

(i) $n^*(\overline{u}_i|_{\Sigma_i}) = m$;
(ii) $\widetilde{w}_i(z_0) = 0$ for some $z_0 \in (\pi_{B_{\tau_i}} \circ \overline{u}_i)^{-1}(\overline{\mathfrak{m}}^b(\tau_i))$;
(iii) $\widetilde{w}_i$ maps each component of $\widetilde{\pi}_i^{-1}(\{s=2\})$ to a different $\mathcal{R}_{\phi(\overline{a}_*)}$ and each component of $\widetilde{\pi}_i^{-1}(\{s=-2\})$ to a different component of $\mathcal{R}_{\phi(\overline{b}_*)}$;
(iv) $\widetilde{w}_i|_{int(\Sigma_i)}$ is a biholomorphism onto its image.

Here $\phi(\overline{a}_*)$ is the ϕ-coordinate for $\overline{a}_*$, etc. □

3.5.4. *Large-scale behavior of* Ξ_i. — When taking the limits of $\widetilde{\pi}_i$ and $\widetilde{w}_i$, we are simultaneously stretching in two directions t_2 and $\log\rho$. In this subsection we study the large-scale behavior of the map

$$\Xi_i = (t_2 \circ \widetilde{\pi}_i, \log\rho \circ \widetilde{w}_i) : \Sigma_i \to [R_i^-, R_i^+] \times [-\infty, \infty)$$

as $i \to \infty$. We use coordinates $(x' = t_2, y')$ on $[R_i^-, R_i^+] \times [-\infty, \infty)$. The goal is to construct a "tropical curve"

$$\overline{\Xi}_i : \Gamma_i \to [-1, 1] \times [0, d_i],$$

with some $d_i \geq 1$ which approximates Ξ_i when viewed from "far away"; see Figure 12. Here Γ_i is a finite graph whose topological type is independent of $i \gg 0$. The analysis is of the same type as that of Parker [Pa].

Step 1. We start with the following lemma, which is a consequence of Gromov compactness and which describes the behavior of $\widetilde{w}_i$ and $\widetilde{\pi}_i$ for large i.

Lemma **3.5.8.** — *Given* $\varepsilon > 0$ *small, after passing to a subsequence and possibly shrinking* $\varepsilon_i > 0$ *subject to the condition* $R_i^{(1)} - R_i^{(\iota+1)} \to \infty$*, there exist constants* $L > 0$*,* $\kappa, \kappa' \in \mathbf{Z}^+$ *such that for each i there exist:*

– *disjoint compact subsurfaces* $K_{i1}, \dots, K_{i\kappa} \subset \Sigma_i$ *and*
– *components* $C_{i1}, \dots, C_{i\kappa'}$ *of* $\Sigma_i - \cup_j K_{ij}$ *which are strips*

such that:

(1) $\widetilde{\pi}_i|_{K_{ij}}$ *is a branched cover over* $[-2, 2] \times [\tau_{ij}, \tau_{ij} + L]$ *for some* τ_{ij} *and* $\widetilde{\pi}_i$ *has no branch points outside* $K_{i1}, \dots, K_{i\kappa}$*;*
(2) K_{ij} *is disjoint from* $\partial_v \Sigma_i$*;*
(3) *there is a component* K_{ij_0} *such that* $(0, -\infty) \in \Xi_i(K_{ij_0})$*;*
(4) *for each j, the sequence* $\{\widetilde{w}_i|_{K_{ij}}\}_{i=1}^{\infty}$*, after rescaling by positive constants, limits to some* $\widetilde{w}_{\infty j} : K_{\infty j} \to \mathbf{C}$*;*
(5) $\widetilde{w}_i|_{C_{ij}}$ *is* ε*-approximate to a multiple of* $e^{\lambda_{ij}(t-is)}$*, where* $\lambda_{ij} \in \mathbf{R} - \{0\}$ *is* $\pm\frac{1}{4}$ *times the angle of a sector of type* $\mathfrak{S}(\overline{b}_{k,l}, \overline{a}_{k',l'})$ *(here* $\pm\frac{1}{4}$ *comes from the fact that* $s \in [-2, 2]$*);*
(6) $\Xi_i(C_{ij})$ *is* ε*-close to a line segment* $\{y' = \lambda_{ij} x' + \beta_{ij} \mid x' \in t_2 \circ \widetilde{\pi}_i(C_{ij})\}$*, where* β_{ij} *is a constant;*

(7) *there exists* $d'_i \in \mathbf{R}$ *such that, for each component* c *of* $\partial_v \Sigma_i$, $y' \circ \Xi_i(c)$ *is* ε*-close to* d'_i *and* $\max_c y' \circ \Xi_i(c) = d'_i$.

Proof. — (1)–(5) follow from Gromov compactness, once we observe using the ECH compactness theorem (cf. Section I.3.4) that there is an upper bound on the number of branch points of $\widetilde{\pi}_i$ which is independent of i and $m \gg 0$. (6) follows from (5). (7) is a consequence of (6) and (T5) in Definition 3.5.6. □

Step 2. We now construct the "tropical curve" $\overline{\Xi}_i : \Gamma_i \to [-1, 1] \times [0, d_i]$, where (x, y) are coordinates on $[-1, 1] \times [0, d_i]$. We first define the map

$$\Xi'_i : \Gamma_i \to [R_i^-, R_i^+] \times [-\infty, \infty),$$

where $\Gamma_i = (V_{\Gamma_i}, E_{\Gamma_i})$, $V_{\Gamma_i} = V_{\Gamma_i,i} \sqcup V_{\Gamma_i,e}$ is the set of vertices, E_{Γ_i} is the set of edges, and the following hold:

(1) $V_{\Gamma_i,i}$ is in one-to-one correspondence with the set $\{K_{i1}, \dots, K_{i\kappa}\}$ and $V_{\Gamma_i,e}$ is in one-to-one correspondence with the components of $\partial_v \Sigma_i$;
(2) the set E_{Γ_i} is in one-to-one correspondence with the set $\{C_{i1}, \dots, C_{i\kappa'}\}$;
(3) Ξ'_i maps the vertices corresponding to K_{ij} and the component c of $\partial_v \Sigma_i$ to

$$(x', y') = (\tau_{ij}, \max(\log \rho \circ \widetilde{w}_i|_{K_{ij}})) \text{ and } (t_2 \circ \widetilde{\pi}_i(c), \max(\log \rho \circ \widetilde{w}_i|_c));$$

and each $e \in E_{\Gamma_i}$ to a straight line segment.

Note that Ξ'_i has image in $\{y'_{ij_0} \leq y' \leq d'_i\}$, where $y'_{ij_0} = \max(\log \rho \circ \widetilde{w}_i|_{K_{ij_0}})$. The map $\overline{\Xi}_i$ is obtained by postcomposing $\Xi'_i : \Gamma_i \to [R_i^-, R_i^+] \times [y'_{ij_0}, d'_i]$ by an affine transformation

$$[R_i^-, R_i^+] \times [y'_{ij_0}, d'_i] \xrightarrow{\sim} [-1, 1] \times [0, d_i],$$

where $d_i > 0$ is chosen so that $\max_{e \in E_{\Gamma_i}} |\lambda'_i(e)| = 1$, where $\lambda'_i : E_{\Gamma_i} \to \mathbf{R}$ maps e to the slope of $\overline{\Xi}_i(e)$.

Lemma **3.5.9.** — *Fix* $\varepsilon, \delta > 0$ *small. Then, after passing to a subsequence, the map* $\overline{\Xi}_i$ *satisfies the following:*

(1) $\max_{e \in E_{\Gamma_i}} |\lambda'_i(e)| = 1$;
(2) *if* $\lambda_i : E_{\Gamma_i} \to \mathbf{R}$ *maps* $e \mapsto \lambda_{ij}$, *where* e *corresponds to* C_{ij} *and* λ_{ij} *is as in Lemma 3.5.8(5), then* λ'_i *and a constant multiple of* λ_i *are* ε*-approximate;*
(3) *if* $E'_{\Gamma_i} \subset E_{\Gamma_i}$ *is the set of edges* e *such that* $|\lambda'_i(e)| < 1 - \delta$, *then* $|\lambda'_i(e)| \leq K/m$ *for some constant* $K > 0$ *which is independent of* $m \gg 0$ *and* i;
(4) *each vertex of* $V_{\Gamma_i,i}$, *has the same number* ≥ 1 *of adjacent edges whose interiors have larger* x*-coordinate and whose interiors have smaller* x*-coordinate;*

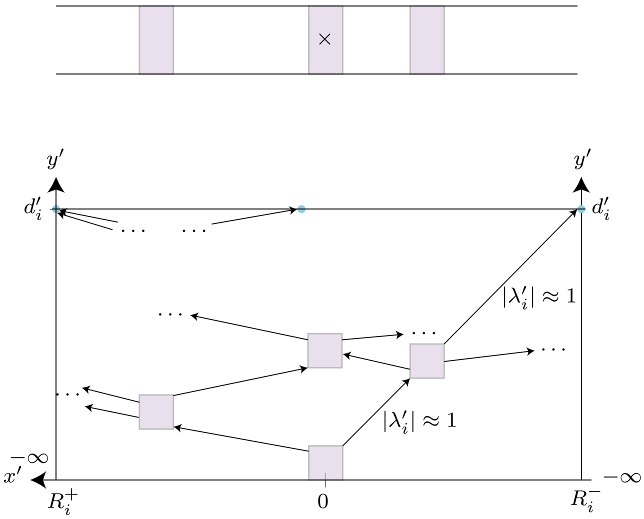


FIG. 12. — The top figure represents $[-2,2] \times [R_i^-, R_i^+]$, where $\times$ indicates the location of $\overline{\mathfrak{m}}^b(\tau_i)$ after translation and the shaded regions are K_{ij}. The bottom is a schematic diagram for the image of Ξ_i which becomes the image of $\overline{\Xi}_i$ "when seen from far away". The dots along the line $y' = d_i'$ correspond to the endpoints of $V_{\Gamma_i,e}$ and the shaded regions are $t_2(K_{ij}) \times \log \rho(\widetilde{w}_i \circ \widetilde{\pi}_i^{-1}(K_{ij}))$. *When viewed in* $[-1,1] \times [0, d_i]$, the edges with the largest slope (in absolute value) satisfy $|\lambda_i'| \approx 1$ and the remaining edges have much smaller slope when $m \gg 0$. (Color figure online)

(5) *$y \circ \overline{\Xi}_i(p) = d_i$ for all $p \in V_{\Gamma_i,e}$;*
(6) *there is a vertex $q_0 \in V_{\Gamma_i,i}$ corresponding to K_{ij_0} (cf. Lemma 3.5.8(3)) such that $y \circ \overline{\Xi}_i(q_0) = 0$;*
(7) *there is a path $e_0 e_1 \dots e_{l-1}$ consisting of edges such that $\lambda_i'(e_j) \geq 1-\delta$ for $j = 0, \dots, l-1$, e_0 starts at q_0, and e_{l-1} ends near $(1, d_i)$ or $(-1, d_i)$, and $d_i > 1-\delta$.*

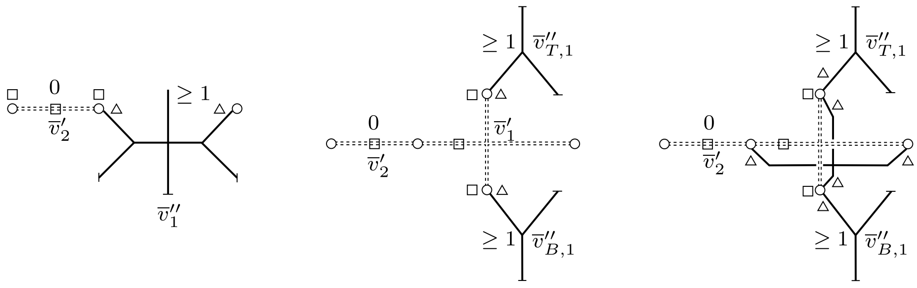
Proof. — (1), (4), (5), and (6) follow from the construction. (2) is a consequence of Lemma 3.5.8(5) and (3) follows from (2). (7) follows from the construction, Lemma 3.5.8(5), and the fact that $\overline{v}_1^\sharp$ must have a large sector by Lemma 3.5.4. □

See Figure 12 for an example. The following lemma is immediate from the construction.

Lemma **3.5.10.** — *After passing to a subsequence, we may assume that the following do not depend on the choice of i:*

- *the graph Γ_i, the function λ_i, and the set E'_{Γ_i};*
- *given an edge E_{Γ_i} with endpoints p, q, whether $x \circ \overline{\Xi}_i(p) \geq x \circ \overline{\Xi}_i(q)$ and whether $y \circ \overline{\Xi}_i(p) \geq y \circ \overline{\Xi}_i(q)$.*

In view of Lemma 3.5.10, we may write $\Gamma = (V_\Gamma, E_\Gamma)$ for $\Gamma_i = (V_{\Gamma_i}, E_{\Gamma_i})$.



(1) (2) (3)

FIG. 13. — Schematic diagrams for the possible types of degenerations corresponding to Lemmas 3.5.12 and 3.5.13. Here ○ represents z_∞ or $z^\#_\infty$, □ represents a branch point, and △ represents an end with a large sector. Double dotted lines indicate multiple covers of σ^*_∞. The labels on the graphs indicate the components and their ECH indices. If there is more than one □ or △ in a diagram, then we interpret it as one of the possible locations for □ or △

Finally, we orient the edges $p \xrightarrow{e} q \in E_\Gamma$ so that $y \circ \overline{\Xi}_i(q) - y \circ \overline{\Xi}_i(p) > 0$; we denote the corresponding orientation by $\mathfrak{o}$. Given $p, q \in V_\Gamma$, we write $p \succeq q$ (resp. $p \simeq q$) to mean $y \circ \overline{\Xi}_i(p) - y \circ \overline{\Xi}_i(q) \geq 0$ (resp. $= 0$) for all i.

3.5.5. *The case $\overline{v}'_1 = \varnothing$, $\overline{v}'_2 \neq \varnothing$.* — We continue to assume that $\overline{v}'_1 = \varnothing$, $\overline{v}'_2 \neq \varnothing$. See Figure 13(1). We write $\approx$ to mean "is close to".

We start with the following useful lemma:

Lemma **3.5.11** *(Comparison Lemma). — Suppose $m \gg 0$, $i = i(m) \gg 0$, and $\varepsilon = \varepsilon(m) > 0$ is small. Let $q \xrightarrow{e} q'$ be an edge of Γ such that $|\lambda'_i(e)| \approx 1$ and let*

$$\delta' = (p_1 \xrightarrow{f_1} \cdots \xrightarrow{f_k} p_{k+1})$$

be an oriented path of Γ from p_1 to p_{k+1} such that $p_1 \preceq q \preceq q' \preceq p_{k+1}$ and $0 < |\lambda'_i(f_j)| \leq K/m$ for $j = 1, \dots, k$. Then

$$|x \circ \overline{\Xi}_i(q') - x \circ \overline{\Xi}_i(q)| \leq K'/m,$$

where K' is independent of m and i.

Note that, by Lemma 3.5.9(3), $0 < |\lambda'_i(f_j)| \leq K/m$ if and only if $|\lambda'_i(f_j)|$ is not close to 1.

Proof. — The horizontal variation $\sum_{j=1}^k |x \circ \overline{\Xi}_i(p_{j+1}) - x \circ \overline{\Xi}_i(p_j)|$ is bounded above by 2 times the maximal covering degree of $\overline{v}'_{L,j}$, $j = 1, \dots, a$, and $\overline{v}'_{R,j}$, $j = 1, \dots, b+1$, which we denote by $\mathcal{K}$. (Here the 2 comes from the width of the interval $[-1, 1]$.) On the other hand, since $p_1 \preceq q \preceq q' \preceq p_{k+1}$, we have:

$$|x \circ \overline{\Xi}_i(q') - x \circ \overline{\Xi}_i(q)| \approx |y \circ \overline{\Xi}_i(q') - y \circ \overline{\Xi}_i(q)|$$

$$\leq \sum_{j=1}^{k} |\lambda_i'(f_j)| \cdot |x \circ \overline{\Xi}_i(p_{j+1}) - x \circ \overline{\Xi}_i(p_j)|$$

$$\leq \left(\max_{j=1,\dots,k} |\lambda_i'(f_j)| \right) (2\mathcal{K}) \leq \mathrm{K}'/m,$$

where $\mathrm{K}' = 2\mathcal{K}\mathrm{K}$. □

Given $(x_0, y_0) \in [-1, 1] \times [0, d_i]$, let

$$l_{(x_0,y_0)} = \{x = x_0, 0 \leq y \leq y_0\}$$

be a line segment oriented in the positive y-direction. Next choose an orientation $\widetilde{\mathfrak{o}}$ of Γ so that the oriented edges of $\overline{\Xi}_i(\Gamma)$ are pointing in the positive x-direction and write $\widetilde{\Gamma} = (\Gamma, \widetilde{\mathfrak{o}})$. (This is different from the orientation $\mathfrak{o}$ used previously.)

Consider the *weight function*

$$\mathcal{W}_i : ([-1, 1] \times [0, d_i]) - \overline{\Xi}_i(\Gamma) \to \mathbf{Z}^{\geq 0},$$

which is defined as follows:

(1) $\mathcal{W}_i$ is locally constant;
(2) $\mathcal{W}_i(x, y)$ is the signed intersection number $\langle \overline{\Xi}_i(\widetilde{\Gamma}), l_{(x,y)} \rangle$ for generic (x, y), with respect to the orientations (∂_x, ∂_y) for $[-1, 1] \times [0, d_i]$ and $\widetilde{\mathfrak{o}}$.

The function $\mathcal{W}_i$ is well-defined by Lemma 3.5.9(4).

Lemma **3.5.12.** — *If $m \gg 0$, then there is no $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$ such that $\overline{v}_1' = \varnothing$ and $\overline{v}_2' \neq \varnothing$.*

Proof. — The proof we give is not the most efficient, but carries over more easily to other situations. We choose $m \gg 0$ so that $0 < \mathrm{K}'/m \ll 1$.

Let $\delta = (q_0 \stackrel{e_0}{\to} \cdots \stackrel{e_{l-1}}{\to} q_l)$ be the oriented path from q_0 to $q_l \in \mathrm{V}_{\Gamma,e}$ given by Lemma 3.5.9(7), i.e., such that $|\lambda_i'(e_j)| \approx 1$ for each i and each edge e_j, $j = 0, \dots, l-1$. We may assume that $x \circ \overline{\Xi}_i(q_l) = \pm 1$ in view of Lemma 3.5.4(4), since the only end $\mathcal{E}$ of $\overline{v}^\sharp_{\mathrm{L},j}$, $j = 1, \dots, a+1$, or $\overline{v}^\sharp_{\mathrm{R},j}$, $j = 1, \dots, b+1$, whose $\pi_{\mathrm{D}^2_{\rho_0}}$-projection is a sector with angle $> \pi$ is an end of $\overline{v}^\sharp_1 = \overline{v}^\sharp_{\mathrm{L},a+1}$.

We claim that, for each $l_0 = 0, \dots, l-1$, there is an oriented path

$$\delta' = (p_1 \stackrel{f_1}{\to} \cdots \stackrel{f_k}{\to} p_{k+1})$$

such that $p_1 \preceq q_{l_0} \preceq q_{l_0+1} \preceq p_{k+1}$ and $0 < |\lambda_i'(f_j)| < \mathrm{K}'/m$ for $j = 1, \dots, k$. Arguing by contradiction, if the claim does not hold, then $\overline{\Xi}_i(\Gamma) \cap \{y = y_0\}$ consists of only one point

$(x_0,y_0)\in\overline{\Xi}_i(e_{l_0})\cap\{y=y_0\}$ for any constant y_0 in the interval $(y\circ\overline{\Xi}_i(q_{l_0}),y\circ\overline{\Xi}_i(q_{l_0+1}))$. This means that there is an integer κ such that $\mathcal{W}_i(x,y_0)=\kappa$ for $-1\leq x<x_0$ and $\mathcal{W}_i(x,y_0)=\kappa\pm 1$ for $1\geq x>x_0$. This contradicts $\mathcal{W}_i(-1,y_0)=\mathcal{W}_i(1,y_0)=0$, which is due to the fact that all the exterior vertices $p\in V_{\Gamma_i,e}$ satisfy $y\circ\overline{\Xi}_i(p)=d_i$ by Lemma 3.5.9(5).

The claim, together with the Comparison Lemma, implies that

$$|x\circ\overline{\Xi}_i(q_{k+1})-x\circ\overline{\Xi}_i(q_k)|\leq K'/m$$

for each $k=0,\dots,l-1$. Hence,

$$\sum_{k=0}^{l-1}|x\circ\overline{\Xi}_i(q_{k+1})-x\circ\overline{\Xi}_i(q_k)|\leq lK'/m\ll 1$$

for $m\gg 0$. This contradicts Lemma 3.5.9(5) with $d_i\geq 1$. □

3.5.6. *The case* $\overline{v}'_1\neq\varnothing$, $\overline{v}'_2\neq\varnothing$. — In this subsection we consider the case $\overline{v}'_1\neq\varnothing$, $\overline{v}'_2\neq\varnothing$. For simplicity assume that there are no boundary points of type (P_3). Some of the possibilities are given by Figure 13(2) and (3).

We outline the necessary modifications in the current case:

(1) We consider the truncation $\overline{u}_i:\Sigma_i\to\overline{W}_{\tau_i}$ of $\overline{u}_i$ so that (T1) and (T2) in Definition 3.5.6 hold. (T4) becomes:

(T4′) there exists a decomposition $\Sigma_i=\Sigma_i^{(1)}\cup\dots\cup\Sigma_i^{(\iota)}$ such that each $\pi_{B_{\tau_i}}\circ\overline{u}_i|_{\Sigma_i^{(j)}}$ is a branched cover with possible empty branch locus over a component of B_{τ_i} which is cut up by (possibly multiple) arcs of type $t=R$ with $2<R<r(\tau_i)-2$ and $s=R'$ with $R'<-3$ or $R'>3$.

(T5) is slightly modified so that the normalized eigenfunction is that of $\overline{v}^\sharp_*$ for any $*$.

(2) Given $\varepsilon>0$ small, there exists $L>0$ so that the analog of Lemma 3.5.8 holds; here we restrict Σ_i to $|s|\leq L$, while keeping the same notation. To the list of compact subsets K_{ij} of Lemma 3.5.8 (viewed as subsets of B_{τ_i} instead of $[-2,2]\times[R_i^-,R_i^+]$), we add the compact subset K_{ij} of the following type, which we call "type $\overline{v}_1$":

$$K_{ij}=B_{\tau_i}\cap\{|s|\leq L,|t-1/2|\leq L\}.$$

(3) After suitable translations, contractions of components of $\pi_{B_{\tau_i}}^{-1}(K_{ij})$ to points, and rescalings, we obtain the "tropical curves"

$$\overline{\Xi}_i:\Gamma\to[-1,1]/(-1\sim 1)\times[0,d_i],$$

where the equivalence relation $\sim$ is consistent with contracting each component of $\pi_{B_{\tau_i}}^{-1}(K_{ij})$ of type $\overline{v}_1$ to a point and the graph $\Gamma=(V_\Gamma,E_\Gamma)$ is independent of i.

(4) The set V_Γ of vertices admits the decomposition $V_{\Gamma,i}\sqcup V_{\Gamma,e}$, where $V_{\Gamma,i}$ is in one-to-one correspondence with the set of components of $\pi_{B_{\tau_i}}^{-1}(K_{ij})$ and $V_{\Gamma,e}$ is in one-to-one correspondence with the set of left and right ends of $\overline{v}^\sharp_{L,j}$, $j=1,\dots,a+1$, and $\overline{v}^\sharp_{R,j}$,

$j=1,\dots,b+1$, that limit to z_∞. There is a subset $V'_{\Gamma,i}\subset V_{\Gamma,i}$ consisting of vertices which are not initial points of any oriented edge; $V'_{\Gamma,i}$ is obtained by contracting components of $\pi_{B_{\tau_i}}^{-1}(K_{ij})$ of type $\overline{v}_1$ to points.

(5) In Lemma 3.5.9, (1)–(3) and (5) still hold, in (4) we replace $V_{\Gamma,i}$ by $V_{\Gamma_i}-V'_{\Gamma,i}$, and (6) becomes: there is a vertex $q_0\in V_{\Gamma,i}$ corresponding to K_{ij_0} which satisfies $\overline{\Xi}_i(q_0)=(0,0)$.

(6) The weight function $\mathcal{W}_i$ is now a function

$$\mathcal{W}_i:(([-1,1]/\sim)\times[0,d_i])-\overline{\Xi}_i(\Gamma)\to\mathbf{Z}^{\geq 0},$$

i.e., it is periodic in the x-direction.

Lemma **3.5.13.** — *If $m\gg 0$, then there is no $\overline{u}_\infty\in\partial_{\{-\infty\}}\mathcal{M}$ such that $\overline{v}'_1\neq\varnothing$ and $\overline{v}'_2\neq\varnothing$.*

Proof. — The proof is similar to that of Lemma 3.5.12 and uses the Comparison Lemma.

Suppose that there are no boundary points of type (P_3). Let $\delta=(q_0\stackrel{e_0}{\to}\cdots\stackrel{e_{l-1}}{\to}q_l)$ be a maximal oriented path which starts from q_0, has $|\lambda_i(e)|\approx 1$ for each edge, and ends at some q_l with $x\circ\overline{\Xi}_i(q_l)=\pm 1$. We claim that, for each $l_0=0,\dots,l-1$, there is an oriented path

$$\delta'=(p_1\stackrel{f_1}{\to}\cdots\stackrel{f_k}{\to}p_{k+1})$$

such that $p_1\preceq q_{l_0}\preceq q_{l_0+1}\preceq p_{k+1}$ and $0<|\lambda'_i(f_j)|\leq K'/m$ for $j=1,\dots,k$. Indeed, using the same notation as that of Lemma 3.5.12, there is a point (x_0,y_0) and an integer κ such that $\mathcal{W}_i(x,y_0)=\kappa$ for $-1\leq x<x_0$ and $\mathcal{W}_i(x,y_0)=\kappa\pm 1$ for $x_0\leq x\leq 1$, which is impossible by the x-periodicity. The claim gives a contradiction as in the proof of Lemma 3.5.12.

Suppose there are boundary points of type (P_3). Then, by Lemma 3.5.4(4), there are no boundary points of type (P_3) on $\overline{v}_{L,j}$, $j=1,\dots,a$, and $\overline{v}_{R,j}$, $j=1,\dots,b+1$. The maximal oriented path δ from the previous paragraph has $y\circ\overline{\Xi}_i(q_l)$ which is much larger than $y\circ\overline{\Xi}_i$ of the endpoint of another path. This is a contradiction. ☐

Proof of Lemma 3.3.5. — Suppose $\overline{u}_\infty\in\partial_{\{-\infty\}}\mathcal{M}$. If $\overline{v}'_*\cup\overline{v}^\sharp_*=\varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$, then, by Lemma 3.5.3, $\overline{u}_\infty$ is a 2-level building $\overline{v}_1\cup\overline{v}_2$, where $\overline{v}_1$ is a $W_{-\infty,1}$-curve with $I=0$ and $\overline{v}_2$ is a $\overline{W}_{-\infty,2}$-curve with $I=2$ which passes through $\overline{\mathfrak{m}}(-\infty)$. By Lemma 3.2.15, $\mathbf{y}_2$ and $\mathbf{y}_4$ satisfy the conditions of A_3.

On the other hand, it is not possible that $\overline{v}'_*\cup\overline{v}^\sharp_*\neq\varnothing$ for some level $\overline{v}_*$ by Lemmas 3.5.12 and 3.5.13. ☐

3.6. *Breaking in the middle.* — In this subsection we study the limit of holomorphic maps to $\overline{W}_\tau$ as $\tau\to T'$ for some $T'\in(-\infty,\infty)$. This will prove Lemma 3.3.8.

We assume that $m \gg 0$; $\varepsilon, \delta > 0$ are sufficiently small; and $\{\bar{J}_\tau\} \in \overline{\mathcal{I}}^{reg}$ and $\{\bar{J}_\tau^\diamondsuit(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ satisfy Lemma 3.2.22. Fix $\mathbf{y} \in \mathcal{S}_{\mathbf{a},\hbar(\mathbf{a})}$, $\mathbf{y}' \in \mathcal{S}_{\mathbf{b},\hbar(\mathbf{b})}$ and let

$$\mathcal{M} = \mathcal{M}^{I=2,n^*=m}_{\{\bar{J}_\tau^\diamondsuit(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{y}, \mathbf{y}'; \overline{\mathfrak{m}}), \quad \mathcal{M}_\tau = \mathcal{M}^{I=2,n^*=m}_{\bar{J}_\tau^\diamondsuit(\varepsilon,\delta,\mathfrak{p}(\tau))}(\mathbf{y}, \mathbf{y}'; \overline{\mathfrak{m}}(\tau)).$$

We will analyze $\partial_{(-\infty,\infty)}\mathcal{M}$.

Let $\overline{u}_i$, $i \in \mathbf{N}$, be a sequence of curves in $\mathcal{M}$ such that $\overline{u}_i \in \mathcal{M}_{\tau_i}$ and $\lim_{i\to\infty} \tau_i = T'$, and such that its limit

$$\overline{u}_\infty = (\overline{v}_{-1,1} \cup \cdots \cup \overline{v}_{-1,c}) \cup \overline{v}_0 \cup (\overline{v}_{1,1} \cup \cdots \cup \overline{v}_{1,a})$$

is a holomorphic building in $\partial_{\{T'\}}\mathcal{M}$, ordered from bottom to top, where each $\overline{v}_*$ is an SFT level, $\overline{v}_{-1,j}$, $j = 1, \dots, c$, and $\overline{v}_{1,j}$, $j = 1, \dots, a$, map to $\overline{W}$ and $\overline{v}_0$ maps to $\overline{W}_{T'}$. Sometimes we will refer to $\overline{v}_0$ as $\overline{v}_{-1,c+1}$ or $\overline{v}_{1,0}$.

The following are analogs of Lemmas 3.4.20–3.4.22, stated without proof.

Lemma **3.6.1.** — *If fiber components are removed from $\overline{u}_\infty$ and the only boundary points at z_∞ are of type (P_3), then the ECH index of each level $\overline{v}_* \neq \overline{v}_0$ is nonnegative,*

$$I(\overline{v}_{1,j}) \geq \begin{cases} I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j}) & \text{if } \mathfrak{bp}_{1,j} = 0; \\ I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j}) + 2 & \text{if } \mathfrak{bp}_{1,j} > 0, \end{cases}$$

for $0 \leq j \leq a$, and the only components of $\overline{u}_\infty$ which have negative ECH index are the following:

(1) *branched covers of $\sigma_\infty^{T'}$; and*
(2) *at most one component $\widetilde{v}$ of $\overline{v}''_0$ with $I(\widetilde{v}) = -1$.*

Here (2) occurs when $T' \in \mathcal{T}_1$ and $\widetilde{v} \in \mathcal{M}^{\dagger,s,irr,\mathrm{ind}=-1}_{\bar{J}_{T'}^\diamondsuit(\varepsilon,\delta,\mathfrak{p}(T'))}(\mathbf{z}, \mathbf{z}')$, as described in Lemma 3.2.22(1).

Lemma **3.6.2.** — *If $\overline{u}_\infty \in \partial_{(-\infty,\infty)}\mathcal{M}$ and $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$, then $\overline{u}_\infty$ is one of the following:*

(1) *$a = 0$, $c = 1$; $\overline{v}_0$ is a $\overline{W}_{T'}$-curve with $I = 1$ which passes through $\overline{\mathfrak{m}}(T')$; and $\overline{v}_{-1,1}$ is a $\overline{W}$-curve with $I = 1$; or*
(2) *$a = 1$, $c = 0$; $\overline{v}_{1,1}$ is a $\overline{W}$-curve with $I = 1$; and $\overline{v}_0$ is a $\overline{W}_{T'}$-curve with $I = 1$ which passes through $\overline{\mathfrak{m}}(T')$.*

Here either $T' \in \mathcal{T}_2$ and there is a component of $\overline{v}_0$ which is in

$$\mathcal{M}^{\dagger,s,irr,\mathrm{ind}=1,n^*=m}_{\bar{J}_{T'}^\diamondsuit(\varepsilon,\delta,\mathfrak{p}(T'))}(\mathbf{z}, \mathbf{z}', \overline{\mathfrak{m}}(T'))$$

from Lemma 3.2.22(2), for some $\mathbf{z}, \mathbf{z}'$; or $T' \in \mathcal{T}_1$ and there is a component of $\overline{v}_0$ which does not pass through $\overline{\mathfrak{m}}(T')$ but is in

$$\mathcal{M}^{\dagger,s,irr,\mathrm{ind}=-1,n^*=0}_{\bar{J}_{T'}^\diamondsuit(\varepsilon,\delta,\mathfrak{p}(T'))}(\mathbf{z}, \mathbf{z}')$$

from Lemma 3.2.22(1).

Lemma **3.6.3.** — *If* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$ *of* $\overline{u}_\infty$, *then:*

(1) $p_0 = \deg(\overline{v}'_0) > 0$;
(2) $\overline{u}_\infty$ *has no boundary point of type* (P_1) *or* (P_2);
(3) $\overline{u}_\infty$ *has no fiber components and no components of* $\overline{v}''_*$ *that intersect the interior of a section at infinity;*
(4) *each component of* $\overline{v}^\sharp_{-1,j}$, $j = 1, \dots, c$, *is a thin strip from* z_∞ *to some* x_i *or* x'_i *with* $I = 1$;
(5) *the only boundary points at* z_∞ *are type* (P_3) *points of* $\overline{v}''_{1,j}$, $j = 0, \dots, a$.

The following is the analog of Lemma 3.4.25:

Lemma **3.6.4.** — *If* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$, *then there are no boundary points at* z_∞ *and* $\overline{u}_\infty$ *contains a subbuilding consisting of* $\overline{v}^\sharp_{1,a}$ *with* $I \geq 1$; $\overline{v}'_{1,j}$, $1 \leq j < a$, *which branch cover* σ_∞; $\overline{v}'_0$ *with* $I = -p$ *which is a degree* p *branched cover of* $\sigma^{T'}_\infty$; $\cup_{j=1}^c \overline{v}^\sharp_{-1,j}$ *which is a union of* p *thin strips; and possibly the following:*

- $\overline{v}^\sharp_0$ *with positive ends at multiples of* z_∞ *and no negative ends at multiples of* z_∞; *and*
- $\overline{v}^\sharp_{1,j}$, $1 \leq j < a$, *with* $I(\overline{v}^\sharp_{1,j}) \geq 1$ *and (positive or negative) ends at multiples of* z_∞.

Here at most one component of $\overline{v}''_0$ *satisfies* $I = -1$ *and the remaining components of* $\overline{v}''_*$ *satisfy* $I \geq 0$.

Proof. — We explain why there are no boundary points of type (P_3); the rest of the proof is similar to that of Lemma 3.4.25. A boundary point of type (P_3) contributes $+2$ towards I by Lemma 3.4.20, there is a large sector which contributes $+1$, and some $\overline{v}_{1,j}$, $j > 0$, contributes $+1$, for a total of $I = 4$. Since there is at most one component which contributes negatively to I, namely a component of $\overline{v}''_0$ with $I = -1$, we have a total of $I \geq 3$, a contradiction. ☐

See Figure 14 for some possibilities. Observe that in the current case there is at most one component of $\overline{v}''_0$ with $I = -1$ whereas there are none in Lemma 3.4.25.

Lemma **3.6.5.** — *For each interval* $[-T, T]$, *there exists* $m \gg 0$ *such that there is no sequence of curves* $\overline{u}_i \in \mathcal{M}_{\tau_i}$, $\tau_i \to T' \in [-T, T]$, *that limits to* $\overline{u}_\infty$ *for which* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$.

Proof. — This is similar to Case (6_i) of Lemma 3.4.28. We apply the usual rescaling argument with $m \to \infty$ and obtain $w_0 : \Sigma_0 \to \mathbf{CP}^1$ and a branched cover $\pi_0 : \Sigma_0 \to cl(B_{T'})$ such that:

(i) $w_0(\partial \Sigma_0) \subset \{\phi = 0, \rho > 0\} \cup \{\infty\}$;
(ii) $w_0(z_0) = \infty$ for some point $z_0 \in \pi_0^{-1}(+\infty)$;
(iii) $w_0(z_1) = 0$ for some point $z_1 \in \pi_0^{-1}(\overline{\mathfrak{m}}^b(T'))$;
(iv) $w_0|_{int(\Sigma_0)}$ is a biholomorphism onto its image.

Let us write $f_0 = \pi_0 \circ w_0^{-1}$ where defined. We now apply the Involution Lemma I.7.9.6. Using the notation of Lemma I.7.9.6, let $\overline{\Sigma}_1$ be the compact Riemann surface with

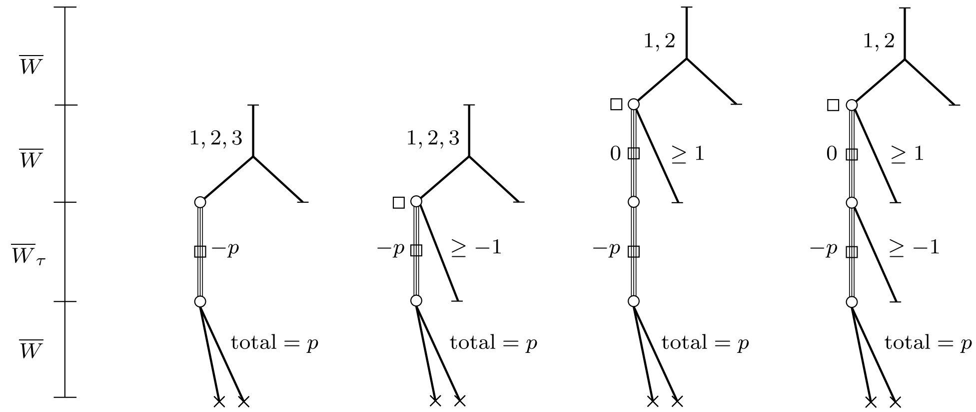


FIG. 14. — Schematic diagrams for some possible types of degenerations. Here $\circ$ represents z_∞, $\square$ represents one or more branch points and $\times$ represents some x_i or x_i'. A vertical line indicates a trivial cylinder or a restriction of a trivial cylinder, and a triple vertical line indicates a degree p branched cover of a trivial cylinder or a restriction of a trivial cylinder. The labels on the graphs are ECH indices of each component

boundary whose interior is biholomorphic to $w_0(int(\Sigma_0))$ and let $\overline{\Sigma}_2 = cl(B_{T'})$. We then extend f_0 to a holomorphic map $\overline{\Sigma}_1 \to \overline{\Sigma}_2$. By the Involution Lemma I.7.9.6 and (i), (iii) and (iv), f_0 maps the point on $\overline{\Sigma}_1$ which corresponds to $\infty \in \mathbf{CP}^1$ to $\mathcal{L}_{(r(T')+1)/2} \cap \partial cl(B_{T'})$. This contradicts (ii). $\square$

We are now in a position to prove Lemma 3.3.8.

Proof of Lemma 3.3.8. — Suppose $\overline{u}_\infty \in \partial_{(-\infty,+\infty)}\mathcal{M}$. By Lemma 3.6.2, if $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$, then $\overline{u}_\infty \in A_4$ or A_5. By Lemma 3.6.5, for any $T > 0$, there exists $m \gg 0$ such that if $\overline{u}_\infty \in \partial_{[-T,T]}\mathcal{M}$, then $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all $\overline{v}_*$.

It remains to consider the case where there exist sequences $m_i \to \infty$, $\varepsilon_i, \delta_i \to 0$, $T_i \to \infty$, and $\overline{u}_{ij} \to \overline{u}_{i\infty}$, where $\overline{u}_{ij} \in \mathcal{M}^{(m_i)}$, $\overline{u}_{i\infty} \in \partial_{\{\pm T_i\}}\mathcal{M}^{(m_i)}$, $\mathcal{M}^{(m_i)}$ is $\mathcal{M}$ with respect to the family $\{\overline{J}^\diamondsuit_\tau(\varepsilon_i, \delta_i, \mathfrak{p}(\tau); m_i)\}$, and $\overline{u}_{i\infty}$ satisfies $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some $*$. We then take a diagonal subsequence $\overline{u}_{ij(i)}$. The proofs of Lemmas 3.4.28 and 3.5.13 carry over to give a contradiction. $\square$

3.7. *Degeneration at $+\infty$, part II.* — In Sections 3.7–3.9 we study the limit of holomorphic maps to $\overline{W}_\tau$ whose positive end is of the form $\mathbf{z} = \{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}$, i.e., we are in Step 4 of the proof of Theorem 3.3.1.

Let us write

$$\mathcal{M} := \mathcal{M}^{I=2, n^* \leq m+|\mathcal{I}|}_{\{\overline{J}^\diamondsuit_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\mathbf{z}, \mathbf{y}'; \overline{\mathfrak{m}}), \text{ and}$$

$$\mathcal{M}_\tau := \mathcal{M}^{I=2, n^* \leq m+|\mathcal{I}|}_{\overline{J}^\diamondsuit_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))}(\mathbf{z}, \mathbf{y}'; \overline{\mathfrak{m}}),$$

where $|\mathcal{I}| \geq 1$.

Let $\overline{u}_\infty \in \partial_{+\infty}\mathcal{M}$ be the limit of $\overline{u}_i \in \mathcal{M}_{\tau_i}$, where $\tau_i \to +\infty$. We use the notation from Section 3.4 for the levels and components of $\overline{u}_\infty$ and write $p_* = \deg(\overline{v}'_*)$ as before.

We now have two constraints

$$n^*(\overline{u}_i) = \sum_{\overline{v}_*} n^*(\overline{v}_*) = m + |\mathcal{I}|; \tag{3.7.1}$$

$$\mathrm{I}(\overline{u}_i) = \sum_{\overline{v}_*} \mathrm{I}(\overline{v}_*) = 2, \tag{3.7.2}$$

where the summations are over all the levels $\overline{v}_*$ of $\overline{u}_\infty$.

Outline of proof of Lemma 3.3.10. The proof is similar to that of Lemma 3.3.2, with the following differences: The proof of Lemma 3.7.10, which is the analog of Lemma 3.4.22, is more involved and the proof of Lemma 3.7.4, which is the analog of Lemma 3.4.14, uses a slightly more complicated notion of an "almost alternating" pair $(\mathrm{P}_{*,0}, \mathrm{P}_{*,1})$.

3.7.1. *Continuation argument.* — First observe that Lemma 3.4.2 carries over verbatim. The analog of Lemmas 3.4.7 and 3.4.16 hinge on the continuation argument: Let $\mathcal{E}_{-,i}$, $i = 1, \dots, q$, be the negative ends of $\cup_{j=1}^{a} \overline{v}^\sharp_{1,j}$ that converge to z_∞ and let $\mathcal{E}_{+,i}$, $i = 1, \dots, r$, be the positive ends of $\cup_{j=0}^{a-1} \overline{v}^\sharp_{1,j}$ that converge to z_∞. Suppose that $q \neq 0$ (i.e., some $\mathcal{E}_{-,i}$ exists) or not all $\mathcal{E}_{+,i}$ project to thin sectors. Also for simplicity we assume that there are no boundary points of type (P_3). By assumption, we may start with an end $\mathcal{E}_{-,i}$ or $\mathcal{E}_{+,i}$ that projects to a non-thin sector. The continuation argument (i.e., the proof of Lemma 3.4.7) carries over with one modification: When we are considering the continuation

$$g^0_{j_2+1,2}, \dots, g^0_{a,2}$$

of $g^0_{j_2,2}$, it is possible that $g^0_{j,2}$ is trivial for all $j_2 + 1 \leq j \leq a$. In other words, there is no nontrivial negative end $\mathcal{E}_{-,2}$ such that

$$\pi_{\mathrm{D}^2_{\rho_0}}(\mathcal{E}_{-,2}) = \mathfrak{S}(\overline{h}(\overline{a}_{k_3,l_3}), \overline{a}_{k_4,l_4})$$

for some (k_4, l_4). This happens when $(k_3, l_3) \to (k_3, l_3)$ belongs to the data $\overrightarrow{\mathcal{D}}$ at the positive end of $\overline{v}'_{1,a}$. In this case we set $j_3 = a$ and $\overline{a}_{k_3,l_3} = \overline{a}_{k_4,l_4}$ and skip $\mathfrak{S}(\overline{h}(\overline{a}_{k_3,l_3}), \overline{a}_{k_3,l_3})$. The rest of the argument is the same.

3.7.2. *Bounds on ECH indices.* — The goal of this subsection is to show the nonnegativity of $\mathrm{I}(\overline{v}_*)$ except when $\overline{v}_* = \overline{v}_+$, *under the assumption that there are no boundary points at z_∞.*

Let $\mathrm{A}_\varepsilon = \partial \mathrm{D}^2_\varepsilon \times [0,1]$ for $0 < \varepsilon < \rho_0$ small and let $\pi_{[0,1]\times\overline{\mathrm{S}}}$ be the projection of $\overline{\mathrm{W}}$ or the positive end of $\overline{\mathrm{W}}_\tau$ to $[0,1] \times \overline{\mathrm{S}}$. Let $\mathfrak{c}'$ be the grooming on A_ε corresponding

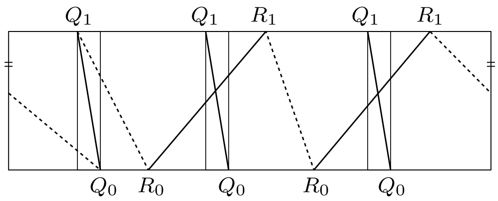


FIG. 15. — The rectangular box, with the sides identified, is A_ε. The grooming $\mathfrak{c}'$ connects Q_0 to Q_1 and the grooming $\mathfrak{c}''$ connects R_0 to R_1. The vertical lines corresponding to Q_1 (resp. Q_0) are the intersections $(\overline{a}_{i,j} \cap \partial D^2_\varepsilon) \times [0,1]$ (resp. $(\overline{h}(\overline{a}_{i,j}) \cap \partial D^2_\varepsilon) \times [0,1]$). The dotted lines, together with some solid lines, give the main cycle $\mathcal{Z}_{\text{main}}$. There are also two auxiliary cycles of $\mathcal{Z}_{\text{aux}}$ corresponding to the middle and right arcs of $\mathfrak{c}'$

to the data $\overrightarrow{\mathcal{D}}'_{+,a}$ at z_∞ for the positive end of $\overline{v}'_{1,a}$, such that the winding number $w(\mathfrak{c}')$ is zero. Here the data $\overrightarrow{\mathcal{D}}'_{+,a}$ satisfies $(\mathcal{D}'_{+,a})^{to} = (\mathcal{D}'_{+,a})^{from}$. Let $\mathfrak{c}'' = \pi_{[0,1]\times\overline{S}}(\cup_i \mathcal{E}_{-,i}) \cap A_\varepsilon$. By Section 3.7.1, $\mathfrak{c}''$ is groomed and the sets of initial and terminal points of $\mathfrak{c}''$ alternate along $(0, 2\pi)$. We then define P_0 and P_1 as the initial and terminal points of $\mathfrak{c}' \cup \mathfrak{c}''$.

Remark **3.7.1.** — It is possible for initial/terminal points of $\mathfrak{c}'$ to also appear as initial/terminal points of $\mathfrak{c}''$.

Definition **3.7.2.** — *A pair of points* $(\mathfrak{q}_0, \mathfrak{q}_1) \subset \partial D^2_\varepsilon$ *is a* thin pair *if* $\mathfrak{q}_0 = \overline{h}(\overline{a}_{i,j}) \cap \partial D^2_\varepsilon$ *and* $\mathfrak{q}_1 = \overline{a}_{i,j} \cap \partial D^2_\varepsilon$ *for the same* i, j. *A pair* $(P_{*,0}, P_{*,1})$ *consisting of disjoint finite subsets of* $(0, 2\pi)$ *with the same cardinality is* almost alternating along $(0, 2\pi)$ *if each* $P_{*,i}$, $i = 0, 1$, *admits a splitting* $P_{*,i} = Q_{*,i} \sqcup R_{*,i}$ *such that the pair* $(R_{*,0}, R_{*,1})$ *is alternating along* $(0, 2\pi)$ *and there is a partition of* $Q_{*,0} \cup Q_{*,1}$ *into thin pairs* $(\mathfrak{q}_0, \mathfrak{q}_1)$, $\mathfrak{q}_i \in Q_{*,i}$.

By definition, (P_0, P_1) is almost alternating along $(0, 2\pi)$. See Figure 15 for an example.

Section 3.7.1 gives the main cycle

$$\mathcal{Z}_{\text{main}} = (\mathfrak{z}_0 \to \mathfrak{z}_1 \to \cdots \to \mathfrak{z}_{k-1} \to \mathfrak{z}_0),$$

where $\{\mathfrak{z}_1, \dots, \mathfrak{z}_k\}$ can be decomposed into an almost alternating pair and the cycle winds around $\mathbf{R}/2\pi\mathbf{Z}$ once. If we apply the continuation method to the positive ends of $\overline{v}'_{1,a}$, then we also obtain a union $\mathcal{Z}_{\text{aux}}$ of auxiliary cycles of the form $(\mathfrak{z}_0 \to \mathfrak{z}_1 \to \mathfrak{z}_0)$, where $(\mathfrak{z}_0, \mathfrak{z}_1)$ is a thin pair and the chords are short chords from $\mathfrak{z}_i$ to $\mathfrak{z}_{1-i}$, $i = 0, 1$. The sets of initial and terminal points of $\mathcal{Z}_{\text{main}} \cup \mathcal{Z}_{\text{aux}}$ are P_0 and P_1.

Let $\overrightarrow{\mathcal{D}}_{\pm,j}$ be the data at z_∞ for the $\pm$ end of $\overline{v}_{1,j}$ and let $P_{\pm,j,0}$ and $P_{\pm,j,1}$ be the initial and terminal points on A_ε determined by $\overrightarrow{\mathcal{D}}_{\pm,j}$. Then we write

$$P_{\pm,j,i} = P'_{\pm,j,i} \cup P''_{\pm,j,i},$$

where $P'_{\pm,j,i}$ corresponds to $\overline{v}'_{1,j}$ and $P''_{\pm,j,i}$ corresponds to $\overline{v}''_{1,j}$. For convenience we write $P_{-,a+1,i} = P'_{-,a+1,i}$, $i = 0, 1$, for the endpoints of $\mathfrak{c}'$. Also let $Q^\star_*$ and $R^\star_*$ be the thin and alternating parts of $P^\star_*$ as given in Definition 3.7.2.

The following lemmas are analogs of Lemma 3.4.13 and 3.4.14:

Lemma **3.7.3.** — *If $\overline{u}_\infty$ has no boundary point at z_∞, then, for each $* = (\pm, j)$, $P_{*,0} \subset P_0$ and $P_{*,1} \subset P_1$ and the pair $(P_{*,0}, P_{*,1})$ is almost alternating along $(0, 2\pi)$.*

Proof. — Similar to but slightly more complicated than that of Lemma 3.4.13. We set $P_i^{(0)} = P_i$ and $\mathcal{Z}_*^{(0)} = \mathcal{Z}_*$, where $* =$ main or aux. Then $P_{-,a+1,i} = P'_{-,a+1,i} \subset P_i^{(0)}$ and the pair $(P_{-,a+1,0}, P_{-,a+1,1})$ is almost alternating. In $(j0)$–$(j4)$ we replace "alternating" by "almost alternating", j_0 by $a+1$, and $(j2)$ and $(j4)$ by:

- $(j2)$ there is a partition of $P_{-,a-j+1,0} \cup P_{-,a-j+1,1}$ into pairs of type $\{\mathfrak{p}_0, \mathfrak{p}_1\}$, $\mathfrak{p}_i \in P_{-,a-j+1,i}$, such that $\mathfrak{p}_0 \lessdot_{R_0^{(j)} \cup R_1^{(j)}} \mathfrak{p}_1$ or $\{\mathfrak{p}_0, \mathfrak{p}_1\}$ is a thin pair of $Q_0^{(j)} \cup Q_1^{(j)}$; in particular, the points of $P_{-,a-j+1,0}$ and $P_{-,a-j+1,1}$ almost alternate along $(0, 2\pi)$;
- $(j4)$ there is a partition of $P'_{+,a-j,0} \cup P'_{+,a-j,1}$ into pairs of type $\{\mathfrak{p}_0, \mathfrak{p}_1\}$, $\mathfrak{p}_i \in P'_{+,a-j,i}$, such that $\mathfrak{p}_0 \lessdot_{R_0^{(j+1)} \cup R_1^{(j+1)}} \mathfrak{p}_1$ or $\{\mathfrak{p}_0, \mathfrak{p}_1\}$ is a thin pair of $Q_0^{(j+1)} \cup Q_1^{(j+1)}$; in particular, the points of $P'_{+,a-j,0}$ and $P'_{+,a-j,1}$ almost alternate along $(0, 2\pi)$.

We inductively define $P_i^{(j)}$ and $\mathcal{Z}_*^{(j)}$ as follows: For each pair $\{\mathfrak{p}_0, \mathfrak{p}_1\}$ of $P''_{+,a-j,0} \cup P''_{+,a-j,1}$, if $(\mathfrak{p}_1 \to \mathfrak{p}_0)$ is a chord of $\mathcal{Z}^{(j-1)}_{\rm aux}$, then we remove $(\mathfrak{p}_1 \to \mathfrak{p}_0 \to \mathfrak{p}_1)$ from $\mathcal{Z}^{(j-1)}_{\rm aux}$; otherwise, in $\mathcal{Z}^{(j-1)}_{\rm main}$, we replace $\mathfrak{q} \to \mathfrak{p}_1 \to \mathfrak{p}_0 \to \mathfrak{q}'$ by $\mathfrak{q} \to \mathfrak{q}'$, given by concatenation. Then $P_0^{(j)}$ and $P_1^{(j)}$ are the sets of endpoints of $\mathcal{Z}^{(j)}_{\rm main} \cup \mathcal{Z}^{(j)}_{\rm aux}$.

The verification of $(j0)$–$(j4)$ is left to the reader. □

Lemma **3.7.4.** — *If $\overline{u}_\infty$ has no fiber components and there are no boundary points at z_∞, then the ECH index of each level $\overline{v}_* \neq \overline{v}_+$ is nonnegative, the only components of $\overline{u}_\infty$ which have negative ECH index are branched covers of σ^+_∞, and $I(\overline{v}_{1,j}) \geq I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j})$ for $0 \leq j \leq a$.*

Before embarking on the proof of Lemma 3.7.4 we encourage the reader to review Section I.5.7.10 and in particular Lemmas I.5.7.22 and I.5.7.23.

Proof. — Similar to the proof of Lemma 3.4.14. The levels $\overline{v}_{0,j}$, $1 \leq j \leq b$, satisfy $I(\overline{v}_{0,j}) \geq 0$ by [HT1, Proposition 7.15(a)].

In order to treat the case of $\overline{v}_{1,j}$, $0 \leq j \leq a$, we use Lemma 3.7.3 and Lemmas I.5.7.22 and I.5.7.23 to compute $I(\overline{v}_{1,j})$. Let

$$\pi_{[0,1]\times\overline{S}} : \check{\overline{W}} = [-1, 1] \times [0, 1] \times \overline{S} \to [0, 1] \times \overline{S}$$

be the projection along $[-1, 1]$.

We claim that there exist representatives $\check{C}_1, \check{C}_2, \check{C}_3 \subset \check{\overline{W}}$ such that the following hold:

(0) $\check{C}_1 \cup \check{C}_2$ is a representative of $\overline{v}'_{1,j}$, $\check{C}_3$ is a representative of $\overline{v}''_{1,j}$, $\check{C}_1 \subset [-1,1] \times [0,1] \times D^2_{\varepsilon/2}$, and $\check{C}_2 \subset [-1,1] \times [0,1] \times D^2_\varepsilon$;

(1) $\mathfrak{c}_1^\pm := \pi_{[0,1]\times\overline{S}}(\check{C}_1|_{s=\pm 1}) \subset A_{\varepsilon/2}$, $\mathfrak{c}_1^+ = \mathfrak{c}_1^-$ is groomed, and the endpoints of each component of $\mathfrak{c}_1^\pm$ forms a thin pair;

(2) the components of $\pi_{[0,1]\times\overline{S}}(\check{C}_i|_{s=\pm 1})$, $i = 2, 3$, corresponding to z_∞ are contained in A_ε; let us write $\mathfrak{c}_i^\pm$ for their unions;

(3) $\mathfrak{c}_2^+ \cup \mathfrak{c}_3^+$ and $\mathfrak{c}_2^- \cup \mathfrak{c}_3^-$ are groomed;

(4) $w(\mathfrak{c}_2^+ \cup \mathfrak{c}_3^+) = 0$ or -1, and $w(\mathfrak{c}_2^- \cup \mathfrak{c}_3^-) = 0$ or 1;

(5) $\mathfrak{c}_2^+ \cup \mathfrak{c}_3^+$ satisfies the first condition of (G'_3) from Section I.5.7.10 and $\mathfrak{c}_2^- \cup \mathfrak{c}_3^-$ satisfies (G_3) from Section I.5.7.10.

We first choose a representative $\check{C}_3$ of $\overline{v}''_{1,j}$ such that (2) holds for $i = 3$ and

$$\mathfrak{c}_3^\pm = \pi_{[0,1]\times\overline{S}}(\cup_k \mathcal{E}_{\pm,k}) \cap A_\varepsilon,$$

where $\mathcal{E}_{\pm,k}$ are the $\pm$ ends of $\overline{v}''_{1,j}$ that limit to z_∞. We then choose the representative $\check{C}_1 \cup \check{C}_2$ of $\overline{v}'_{1,j}$ as follows: Let $\check{C}_1 = [-1,1] \times \mathfrak{c}_1^+$ so that (1) holds and the endpoints of $\mathfrak{c}_1^+ = \mathfrak{c}_1^-$ are partitioned into thin pairs of $(P'_{+,j,0}, P'_{+,j,1})$. Let $\check{C}_2 \subset [-1,1] \times [0,1] \times D^2_\varepsilon$ be a disk of the type used in the proof of Lemma I.5.7.15, such that (2) holds for $i = 2$, $\check{C}_2|_{s=\pm 1} = \mathfrak{c}_2^\pm$, $\mathfrak{c}_2^\pm$ are groomed, $w(\mathfrak{c}_2^+) = 0$ or -1, $w(\mathfrak{c}_2^-) = 0$ or 1, and the endpoints of $\mathfrak{c}_2^\pm$ are alternating along $(0, 2\pi)$. This is possible by (j4) in the proof of Lemma 3.7.3. (3)–(5), in particular the fact that adding $\mathfrak{c}_3^+$ to $\mathfrak{c}_2^+$ — and similarly adding $\mathfrak{c}_3^-$ to $\mathfrak{c}_2^-$ — leaves the grooming property invariant, follow from the proof of Lemma 3.7.3.

We now prove that $I(\overline{v}_{1,j}) \geq 0$ for $j > 0$; the verification of $I(\overline{v}_{1,j}) \geq I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j})$ for $0 \leq j \leq a$ is similar and is left to the reader. First observe that $I(\check{C}_1) = 0$ and $I(\check{C}_2) = 0$, where the latter follows from the proof of Lemma I.5.7.15. Also $I(\check{C}_3) \geq 0$ by the ECH index inequality. Next observe that $\langle \check{C}_1, \check{C}_2 \rangle = -l$, where l is the degree of $\check{C}_1$. Hence the contribution of $\check{C}_1 \cap \check{C}_2$ to $I(\overline{v}_{1,j})$ is $-2l$. We now apply Lemmas I.5.7.22 and I.5.7.23, where we split into $\check{C}_1$ and $\check{C}_2 \cup \check{C}_3$ instead of $\overline{v}'_{1,j}$ and $\overline{v}''_{1,j}$. Observe that $q_2 = 0$ (resp. $q_2 = 1$) in Lemma I.5.7.22 corresponds to $q_2 = -1$ (resp. $q_2 = 0$) in Lemma I.5.7.23. One can verify that the extra contributions to I from the rightmost terms of Equations (I.5.7.10) and (I.5.7.11) add up to at least $2l$. Hence $I(\overline{v}_{1,j}) \geq 0$ for $j > 0$.

The case of $\overline{v}_{-1,j}$, $1 \leq j \leq c+1$, is easier and will be omitted. □

3.7.3. *Boundary points at z_∞.* — In this subsection we describe the necessary modifications when $\overline{u}_\infty$ has boundary points at z_∞. We use the notation from Section 3.4.3, with some modifications: The continuation method gives rise to a main cycle $\mathcal{Z}_{\text{main}}$ which winds around $\mathbf{R}/2\pi\mathbf{Z}$ once and a union $\mathcal{Z}_{\text{aux}}$ of auxiliary cycles. Also, in Definition 3.4.15 we replace $\mathcal{Z}$ by $\mathcal{Z}_{\text{main}}$.

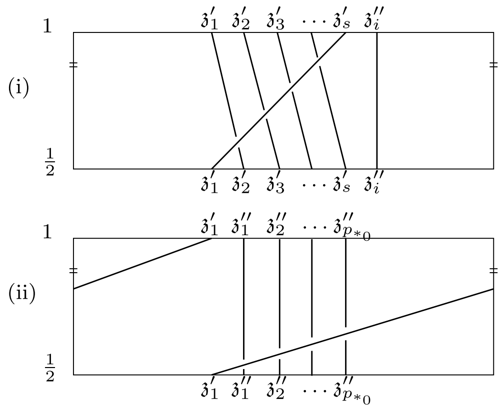


FIG. 16. — $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ for $\mathfrak{r}$ of (i) type (P_3) and (ii) type (P_2). Here $\mathfrak{c}''_{L-}$ is the strand in the front

Lemma **3.7.5.** — *If $\overline{u}_\infty$ has no fiber components, then the ECH index of each level $\overline{v}_* \neq \overline{v}_+$ is nonnegative, the only components of $\overline{u}_\infty$ which have negative ECH index are branched covers of σ^+_∞, and there exist constants $\delta_{1,j}$ such that*

$$(\mathbf{3.7.3}) \qquad I(\overline{v}_{1,j}) = I(\overline{v}'_{1,j}) + I(\overline{v}''_{1,j}) + \delta_{1,j}$$

for $0 \leq j \leq a$, where:

(i) *if there is a boundary point of type (P_3) on $\overline{v}_{1,j}$, then $\delta_{1,j} \geq 2$;*
(ii) *if there is a boundary point of type (P_2) on $\overline{v}_{1,j}$, then $\delta_{1,j} \geq 2p_{1,j}$, where $p_* = \deg(\overline{v}'_*)$;*
(iii) *otherwise, $\delta_{1,j} = 0$.*

We remark that we have not tried to obtain the best lower bound, just one that suffices for our purposes of eliminating boundary points at z_∞.

Proof. — Let $\mathfrak{r}$ be a boundary point at z_∞ on $\overline{v}_{*_0}$ with $*_0 = (1, j_0)$. We compute the extra contribution I_{L-} to $I(\overline{v}^{\circ,1}_{*_0})$ that comes from the left negative end at z_∞ as in Lemma 3.4.18.

(i) Suppose $\mathfrak{r}$ is of type (P_3). We assume (S) from Section 3.4.3 for simplicity. First observe that $P_{L-,j_0,1/2} = P_{L-,j_0,1}$. By (S), $\mathcal{Z}_{\rm main}$ has only one chord $\mathfrak{z}'_1 \to \mathfrak{z}'_s$ corresponding to a sector of type $\mathfrak{S}(\overline{a}_{k,l}, \overline{a}_{k',l'})$. Hence we have:

$$P''_{L-,j_0,1/2} = \{\mathfrak{z}'_1\}, \quad P'_{L-,j_0,1/2} = \{\mathfrak{z}'_2, \dots, \mathfrak{z}'_s, \mathfrak{z}''_1, \dots, \mathfrak{z}''_t\},$$
$$P''_{L-,j_0,1} = \{\mathfrak{z}'_s\}, \quad P'_{L-,j_0,1} = \{\mathfrak{z}'_1, \dots, \mathfrak{z}'_{s-1}, \mathfrak{z}''_1, \dots, \mathfrak{z}''_t\},$$

where $P_{L-,j_0,1/2} = P_{L-,j_0,1}$ is written as $\{\mathfrak{z}'_1, \dots, \mathfrak{z}'_s, \mathfrak{z}''_1, \dots, \mathfrak{z}''_t\}$ in cyclic order around $\mathbf{R}/2\pi\mathbf{Z}$; see Figure 16(i). Note that $\mathfrak{z}'_2, \dots, \mathfrak{z}'_{s-1}$ are points of $\mathcal{Z}_{\rm aux}$ but not $\mathcal{Z}_{\rm main}$; otherwise $\mathcal{Z}_{\rm main}$ winds more than once around $\mathbf{R}/2\pi\mathbf{Z}$. The projection of the left negative end of $\overline{v}^{\circ,1,''}_{*_0}$ that limits to z_∞ intersects $A^{[1/2,1]}_\varepsilon$ along an arc $\mathfrak{c}''_{L-}$ with winding number

$w(\mathfrak{c}''_{L-}) = 0$ or 1, depending on whether $\mathfrak{S}(\overline{a}_{k,l}, \overline{a}_{k',l'})$ is a large sector. The left negative ends of $\overline{v}^{\circ,1,'}_{*_0}$ that limit to z_∞ can modified to give a grooming $\mathfrak{c}'_{L-}$ on $A^{[1/2,1]}_{\varepsilon/2}$ such that the winding number $w(\mathfrak{c}'_{L-}) = 0$ or -1 and $\mathfrak{c}'_{L-}$ connects $\mathfrak{z}''_i$ to $\mathfrak{z}''_i$, $i = 1, \dots, t$, by vertical arcs.

We now consider the disk $\check{D}$ that we "append" to the left negative end of $\overline{v}^{\circ,1}_{*_0}$ as in the proof of Lemma I.5.7.22. The writhe of $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ is equal to $s-1$. Resolving the (positive) crossings of $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ yields a grooming with vertical arcs from $\mathfrak{z}'_i$ to $\mathfrak{z}'_i$; this corresponds to a disk $\check{D}$ whose contributions to Q, c_1, μ are $s-1, 0, s-1$. [The calculations for μ assume (R) from Section 3.4.3. For example, if $w(\mathfrak{c}'_{L-}) = w(\mathfrak{c}''_{L-}) = 0$, then μ of the positive ends are 1 for $s-1$ arcs and 0 for the rest and μ of the negative ends are all 0.] The discrepancy contribution to I is ≥ 0. Hence $I_{L-} = 2(s-1) \geq 2$.

In general, each collection of boundary points of type (P_3) that map to the same point on the base contributes at least $+2$ towards I; this is analogous to Remark 3.4.19.

The cycles $\mathcal{Z}^{(a-j)}_{\text{main}}$ and $\mathcal{Z}^{(a-j)}_{\text{aux}}$ are defined as before, for $j \geq j_0$. We define $\mathcal{Z}^{(a-j_0,+)}_{*} = \mathcal{Z}^{(a-j_0)}_{*}$, $\mathcal{Z}^{(a-j_0,-)}_{\text{aux}} = \mathcal{Z}^{(a-j_0,+)}_{\text{aux}}$, and $\mathcal{Z}^{(a-j_0,-)}_{\text{main}}$ as $\mathcal{Z}^{(a-j_0,+)}_{\text{main}}$ with $\mathfrak{z}'_1 \to \mathfrak{z}'_s$ replaced by $\mathfrak{z}'_s$. Also, $P^{\star}_{R-,j_0,i}$, $\star = \varnothing, ', ''$, is obtained from $P^{\star}_{+,j_0,i}$ by replacing $\mathfrak{z}'_1$ by $\mathfrak{z}'_s$. The rest of the arguments of Lemmas 3.7.3 and 3.7.4 carry over.

(ii) Suppose $\mathfrak{r}$ is the only boundary point of type (P_2). In this case $\mathfrak{z}'_1 = \mathfrak{z}'_s$ and $\mathcal{Z}_{\text{main}} = (\mathfrak{z}'_1 \to \mathfrak{z}'_1)$. Then:

$$P''_{L-,j_0,1/2} = \{\mathfrak{z}'_1\}, \quad P'_{L-,j_0,1/2} = \{\mathfrak{z}''_1, \dots, \mathfrak{z}''_{p_{*_0}}\},$$
$$P''_{L-,j_0,1} = \{\mathfrak{z}'_1\}, \quad P'_{L-,j_0,1} = \{\mathfrak{z}''_1, \dots, \mathfrak{z}''_{p_{*_0}}\}.$$

Also $\mathfrak{c}''_{L-}$ is an arc from $\mathfrak{z}'_1$ to itself with $w(\mathfrak{c}''_{L-}) = 1$ and $\mathfrak{c}'_{L-}$ consists of p_{*_0} vertical arcs from $\mathfrak{z}''_i$ to itself; see Figure 16(ii).

In order to groom $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ so that the result $\mathfrak{c}$ satisfies $w(\mathfrak{c}) = 0$, we switch the (positive) crossings of $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ while keeping the same endpoints. This gives rise to $\check{D}$ which is a union of $p_{*_0} + 1$ disks. The total contributions to Q, c_1, μ, and the discrepancy, are $2p_{*_0} + 1, 1, -2, 0$. [Again recall (R) from Section 3.4.3. The writhe of $\mathfrak{c}'_{L-} \cup \mathfrak{c}''_{L-}$ is $2p_{*_0}$ which contributes $2p_{*_0}$ towards Q. The writhe of $\mathfrak{c}''_{L-}$ and its pushoff is 1, which contributes an additional 1 towards Q. μ of the positive (resp. negative) ends are $0, \dots, 0, -2$ (resp. all 0).] Hence $I_{L-} = 2p_{*_0}$.

We also obtain a lower bound of $2p_{*_0}$ in the general case of more boundary points of type (P_2); the details are left to the reader. □

The following analog of Lemma 3.4.16 is a consequence of Section 3.7.1:

Lemma **3.7.6.** — *Suppose $\overline{v}'_{1,j} \cup \overline{v}^{\sharp}_{1,j} \neq \varnothing$ for some $j > 0$. Let $\mathcal{E}_{-,i}$, $i = 1, \dots, q$, be the negative ends of $\cup_{j=1}^{a} \overline{v}^{\sharp}_{1,j}$ that converge to z_∞, let $\mathcal{E}_{+,i}$, $i = 1, \dots, r$, be the positive ends of $\cup_{j=0}^{a-1} \overline{v}^{\sharp}_{1,j}$*

that converge to z_∞, and let $\mathcal{E}'_i$, $i=1,\dots,s$, be the neighborhoods of the points of type (P_3). Then

$$n^+(\mathcal{E}_{-,i}) \geq k_0 - 1 \gg 2g \tag{3.7.4}$$

for each i, where the constant k_0 is as given in Section I.5.2.2. If $q\neq 0$ (i.e., some $\mathcal{E}_{-,i}$ exists) or not all $\mathcal{E}_{+,i}$ project to thin sectors, then

$$n^+((\cup_{i=1}^q \mathcal{E}_{-,i}) \cup (\cup_{i=1}^r \mathcal{E}_{+,i}) \cup (\cup_{i=1}^s \mathcal{E}'_i)) \geq m - p_+, \tag{3.7.5}$$

where $p_+ = \deg(\overline{v}'_+)$. Moreover,

$$\mathrm{D}^2_{\rho_0} - (\cup_{i=1}^q \pi_{\mathrm{D}^2_{\rho_0}}(\mathcal{E}_{-,i})) \cup (\cup_{i=1}^r \pi_{\mathrm{D}^2_{\rho_0}}(\mathcal{E}_{+,i})) \cup (\cup_{i=1}^s \pi_{\mathrm{D}^2_{\rho_0}}(\mathcal{E}'_i))$$

consists of at most p_+ thin sectors.

The following is the analog of Lemma 3.4.17:

Lemma **3.7.7.** — *If $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ are the boundary points of type (P_1) and (P_2) and $\mathrm{N}(\mathfrak{p}_i)\subset \mathrm{F}_*$ is a small neighborhood of $\mathfrak{p}_i$, then $\sum_{i=1}^s n^*(\overline{v}_*(\mathrm{N}(\mathfrak{p}_i)))\geq m$.*

3.7.4. *Asymptotic eigenfunctions.* — Fix $m\gg 0$ and let $\overline{u}_i \to \overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$.

Lemma **3.7.8.** — *There is no $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$ such that:*

(a) *for all $\overline{v}_* \succeq \overline{v}_+$, $\deg(\overline{v}'_*) = p_+ \geq 1$, $\overline{v}^\sharp_* = \varnothing$, and $\overline{v}'_* \cap \overline{v}''_* = \varnothing$; and*
(b) *$\overline{v}^\sharp_{0,b}$ is a union of p_+ cylinders from δ_0 to h.*

Proof. — Arguing by contradiction, we restrict $\overline{u}_i$ to a neighborhood of $\sigma^{\tau_i}_\infty$ and further restrict it to the components that are close to $\overline{v}'_+$. After projecting to $\mathrm{D}^2_{\rho_0}$ *using balanced coordinates,* applying the ansatz given by Equation (I.7.8.1), rescaling, and taking the SFT limit with $m\gg 0$ fixed, we obtain a holomorphic map $w_+ : \Sigma_+ \to \mathbf{CP}^1$ together with the branched cover $\pi_+ : \Sigma_+ \to cl(\mathrm{B}_+)$. By (a), w_+ maps each component of $\partial\Sigma_+$ to a distinct thin sector $\mathfrak{S}(\{\phi = \phi(\overline{a}_{i,j}) + \frac{\pi}{m}\}, \overline{h}(\overline{a}_{i,j}))$.[13] By (a) and (b), the image of w_+ cannot contain $\infty \in \mathbf{CP}^1$. Hence, by the open mapping property, w_+ maps each component of Σ_+ to a distinct $\mathfrak{S}(\{\phi = \phi(\overline{a}_{i,j}) + \frac{\pi}{m}\}, \overline{h}(\overline{a}_{i,j}))$.

From now on assume without loss of generality that $p_+ = 1$ and $\Sigma_+ = \mathrm{B}_+$. We identify $cl(\mathrm{B}_+)$ with the closed unit disk $\overline{\mathbf{D}} = \{|z|\leq 1\}$ by sending $+\infty$ to 1 and $-\infty$ to 0.

The Lagrangian boundary condition for $\overline{u}_i$, when projected to $\mathrm{D}^2_{\rho_0}$ using balanced coordinates, descends to the boundary condition $w_+(e^{i\theta}) \in \mathcal{R}_{\varphi(e^{i\theta})}$, where the map $\varphi : \partial\mathbf{D} - \{1\} \to \mathrm{S}^1 = \mathbf{R}/2\pi\mathbf{Z}$ satisfies the following:

[13] This complicated expression is the result of using balanced coordinates instead of $\pi_{\mathrm{D}^2_{\rho_0}}$.

– $\lim_{\theta\to 0^+}\varphi(e^{i\theta})=\phi(\overline{a}_{i_0,j_0})+\frac{\pi}{m}$ and $\lim_{\theta\to 0^-}\varphi(e^{i\theta})=\phi(\overline{h}(\overline{a}_{i_0,j_0}))=\phi(\overline{a}_{i_0,j_0})+\frac{2\pi}{m}$ for some (i_0,j_0);
– $\varphi(e^{i\theta})\in(0,2\pi)$ is a nondecreasing function of $\theta\in(0,2\pi)$; and
– $\varphi\circ i_0=i_1\circ\varphi$, where $i_0:\partial\mathbf{D}-\{1\}\xrightarrow{\sim}\partial\mathbf{D}-\{1\}$ is the reflection across the x-axis and $i_1:S^1\xrightarrow{\sim}S^1$ is the reflection across $\mathcal{R}_{\phi_0}$ where $\phi_0=\phi(\overline{a}_{i_0,j_0})+\frac{3\pi}{2m}$.

Lemma **3.7.9.** — *The holomorphic map* $w_+:\overline{\mathbf{D}}\to\mathbf{CP}^1$ *satisfies the following:*

(i) $w_+(1)=0$ *and* $w_+(e^{i\theta})\in\mathcal{R}_{\varphi(e^{i\theta})}$ *for* $e^{i\theta}\neq 1$*;*
(ii) $w_+(0)\in\mathcal{R}_{\phi_0}$ *and* w_+ *maps* $\mathcal{R}_0\cup\mathcal{R}_\pi$ *to* $\mathcal{R}_{\phi_0}$*;*
(iii) $\mathrm{Im}(w_+)\subset\mathfrak{S}(\{\phi=\phi(\overline{a}_{i_0,j_0})+\frac{\pi}{m}\},\overline{h}(\overline{a}_{i_0,j_0}))$*;*
(iv) w_+ *is a biholomorphism onto its image.*

Moreover, w_+ *is the unique holomorphic map* $\overline{\mathbf{D}}\to\mathbf{CP}^1$ *which satisfies (i), (iii) and (iv), up to multiplication by a positive real constant.*

Proof. — (i), (iii), and (iv) are immediate from the construction. To prove the uniqueness statement, suppose there exists $\widetilde{w}_+$ satisfying (i), (iii) and (iv). Then we consider $\frac{\widetilde{w}_+}{w_+}$ as in Section I.7.9. It is easy to see that $\frac{\widetilde{w}_+}{w_+}$ is a positive real constant on $\partial\mathbf{D}$ and has no zeros or poles on $\mathbf{D}$, hence is a constant map. (ii) then follows from the uniqueness, Observation I.7.9.1, and the fact that i_0 and i_1 extend to involutions of $\mathbf{D}$ and $\mathbf{CP}^1$. Here we are viewing S^1 as the unit circle in $\mathbf{CP}^1$. □

The proof of Lemma 3.7.8 now follows from the first part of Lemma 3.7.9(ii): Suppose $\overline{v}^\sharp_{0,b}$ is a cylinder from δ_0 to h. Then by the choice of h from Section 3.2.3 we have a contradiction, since the asymptotic eigenfunction of $\overline{v}^\sharp_{0,b}$ at the positive end δ_0 is a constant $c\in\mathcal{R}_{\phi=-2\pi/m}$. □

3.7.5. *Preliminary restrictions.*

Lemma **3.7.10.** — *If* $m\gg 0$ *and* $\overline{u}_\infty\in\partial_{\{+\infty\}}\mathcal{M}$*, then the following hold:*

(1) $\overline{u}_\infty$ *has no fiber components;*
(2) *there is no level* $\overline{v}_*$ *such that* $\overline{v}'_*\cap\overline{v}''_*\neq\varnothing$ *and* $\overline{v}'_*\cap\overline{v}''_*\subset int(\overline{v}'_*)$*; and*
(3) $\overline{u}_\infty$ *has no boundary point at* z_∞*.*

Proof. — First observe that "not (1)", "not (2)", and "not (3)" are mutually exclusive by considerations of n^*.

(1) Arguing by contradiction, suppose that $\overline{u}_\infty$ has a fiber component $\widetilde{v}:\dot{F}\to\overline{W}_*$. The fiber component $\widetilde{v}$ satisfies $n^*(\widetilde{v})\geq m$.

We first claim the following:

(a) The only ends of $\overline{v}^\sharp_*$ that limit to multiples of z_∞ or δ_0 are positive ends.

(b) All the positive ends $\mathcal{E}_+$ of $\overline{v}^\sharp_{0,j}$, $j = 0, \dots, b$, that limit to δ_0^r satisfy $n^*(\mathcal{E}_+) = r$.
(c) All the positive ends of $\overline{v}^\sharp_{1,j}$, $j = 0, \dots, a$, and $\overline{v}^\sharp_{-1,j}$, $j = 1, \dots, c$, that limit to z_∞ project to thin sectors under $\pi_{D^2_{\rho_0}}$.
(d) $p_{*_1} \le p_{*_2}$ if $\overline{v}_{*_1} \preceq \overline{v}_{*_2}$.

(a) Arguing by contradiction, let $\mathcal{E}_-$ be a negative end of some $\overline{v}^\sharp_*$ that limits to a multiple of z_∞ or δ_0. Then $n^*(\mathcal{E}_-) \gg 2g$ by Lemmas 3.4.2(1) and 3.7.6. Since $n^*(\widetilde{v}) \ge m$, such a negative end $\mathcal{E}_-$ cannot exist by Equation (3.7.1). The arguments for (b) and (c) are similar and (d) is a consequence of (a)–(c).

Next we have the following contributions towards I:

(β_1) $\mathrm{I}(\overline{v}'_+) = -p_{0,b+1}$.
(β_2) $\cup_{j=1}^b \overline{v}^\sharp_{0,j}$ is a union of $p_{0,b+1} - p_{0,1}$ cylinders from δ_0 to h or e and

$$\sum_{j=1}^{b} \mathrm{I}(\overline{v}^\sharp_{0,j}) \ge p_{0,b+1} - p_{0,1}.$$

(β_3) $\overline{v}^\sharp_-$ is a union of $p_{0,1} - p_{0,0}$ sections from δ_0 to x_i or x'_i and

$$\mathrm{I}(\overline{v}^\sharp_-) = p_{0,1} - p_{0,0}.$$

(β_4) $\cup_{j=1}^c \overline{v}^\sharp_{-1,j}$ is a union of $p_{0,0}$ trivial strips and

$$\sum_{j=1}^{c} \mathrm{I}(\overline{v}^\sharp_{-1,j}) = p_{0,0}.$$

(β_5) If the fiber $\widetilde{v}$ is a component of $\overline{v}_*$, then $\widetilde{v}$ and the intersection $\widetilde{v} \cap (\overline{v}_* - \widetilde{v})$ contribute $2g + 2 \ge 4$ towards I.
(β_6) After removing any fiber components, all the levels $\neq \overline{v}_+$ have nonnegative ECH index, $\mathrm{I}(\overline{v}''_+) \ge 0$, and $\mathrm{I}(\overline{v}_+) \ge \mathrm{I}(\overline{v}'_+) + \mathrm{I}(\overline{v}''_+)$.

(β_1)–(β_4) are clear.

We now prove (β_5). Suppose that $\widetilde{v}$ is a component of $\overline{v}_{*_0}$. If $\overline{v}''_{*_0}$ has a boundary point at z_∞, then a neighborhood of the boundary point contributes $n^* \gg 2g$. Since $n^*(\widetilde{v}) \ge m$, we have a contradiction of Equation (3.7.1). Hence $\overline{v}''_{*_0}$ does not have a boundary point at z_∞.

When $\widetilde{v}$ is an interior fiber, (β_5) follows from Equation (I.7.5.6).

Next suppose that $\widetilde{v}$ is a boundary fiber, i.e., $\widetilde{v}(\widetilde{F}) \subset \{p\} \times \overline{S}$, where $\widetilde{F}$ is the domain of $\widetilde{v}$ and $p \in \partial B_{*_0}$. If $(p, z_\infty) \notin \widetilde{v}(\partial \widetilde{F})$, i.e., $\widetilde{v}(\partial \widetilde{F})$ consists of $2g$ slits along $\widehat{\mathbf{a}}$, for example, then the Fredholm index of $\widetilde{v}$ is:

$$\mathrm{ind}(\widetilde{v}) = -\chi(\widetilde{F}) + \mu_\tau(\partial \widetilde{F}) + 2c_1(\widetilde{v}^* T\overline{S}, \tau)$$

$$= -(2-4g) + 0 + 2(2-2g) = 2,$$

where τ is a partially defined trivialization of $T\overline{S}$ along $\widehat{\mathbf{a}}$ which directs $T\widehat{\mathbf{a}}$. Since there are $2g$ arcs of $\overline{u}_i$ that are pinched to points when taking the limit, $\widetilde{v}$ and the "pinched points" contribute a total of $2g+2$ towards I.

Now suppose that $(p, z_\infty) \in \widetilde{v}(\partial\widetilde{F})$. Suppose that $\overline{v}''_{*_0} = \varnothing$ and that $*_0 = (1, j)$ with $j \geq 1$. By considerations of n^*,

(*) all the components of the data $\overrightarrow{\mathcal{D}}_\pm$ of the positive and negative ends of $\overline{v}_{*_0}$ are of the form $(i,j) \to (i,j)$.

Let τ_0 be a groomed multivalued trivialization which is compatible with $\overrightarrow{\mathcal{D}}_\pm$ and let $\mathfrak{c}_+ = \mathfrak{c}_- \subset A_\varepsilon = [0,1] \times \partial D^2_\varepsilon$ be the corresponding groomings. Then a τ_0-trivial representative $\check{C}$ of $\widetilde{v} \cup \overline{v}'_{*_0}$ satisfies

$$[\check{C}] = [[-1,1] \times \mathfrak{c}] + [\overline{S}] \in \pi_2(\mathbf{z}_+, \mathbf{z}_-, \tau_0),$$

where $\mathbf{z}_\pm = \{z_\infty^{2g}(\overrightarrow{\mathcal{D}}_\pm)\}$. (β_5) then follows from the calculation for a closed interior fiber $\widetilde{v}$ when $\overline{v}''_{*_0} = \varnothing$ and $*_0 = (1, j)$ with $j \geq 1$. The general case follows by incorporating considerations of the previous paragraph and is left to the reader.

(β_6) follows from Lemma 3.7.4 by observing that the proof carries over when fibered components are removed from $\overline{u}_\infty$: Removing interior fibers and boundary fibers with $(p, z_\infty) \notin \widetilde{v}(\partial\widetilde{F})$ do not affect groomings. If $\overline{u}_\infty$ has a boundary fiber with $(p, z_\infty) \in \widetilde{v}(\partial\widetilde{F})$, then (*) holds, which also is sufficient.

Summing (β_1)–(β_6), the total ECH index is ≥ 4, which contradicts Equation (3.7.2). Hence (1) follows.

(2) Arguing by contradiction, suppose there exist sequences $m_l \to \infty$ and $\overline{u}_{li} \to \overline{u}_{l\infty}$, where $\overline{u}_{li} \in \mathcal{M}^{(m_l)}$, $\overline{u}_{l\infty} \in \partial_{\{+\infty\}}\mathcal{M}^{(m_l)}$, and $\mathcal{M}^{(m_l)}$ is $\mathcal{M}$ with respect to m_l, such that each $\overline{u}_{l\infty}$ has some $\overline{v}_{l,*_0}$ such that $\overline{v}'_{l,*_0} \cap \overline{v}''_{l,*_0} \neq \varnothing$. Unless indicated otherwise, we fix $l \gg 0$ and suppress l from the notation.

The same argument as in (1) implies that (a)–(d) hold. Since $n^*(\overline{u}_i) = m + |\mathcal{I}| \leq m + 2g$, there is only one intersection point of $\overline{v}'_{*_0}$ and $\overline{v}''_{*_0}$, which we denote by $\mathfrak{r} = (\mathfrak{r}^b, z_\infty)$.

(2A) Suppose that $*_0 = +$. As in the proof of (1), the holomorphic building hanging from the positive end of $\overline{v}'_+$ has total ECH index ≥ 0 — this is obtained by adding all the ECH index contributions from (β_1)–(β_4). We also have (β'_5) instead of (β_5):

(β'_5) $\overline{v}'_{*_0} \cap \overline{v}''_{*_0}$ contributes $2 \cdot m(\mathfrak{r})$ towards I, where $m(\mathfrak{r}) \geq 1$ is the multiplicity of $\mathfrak{r}$.

Hence $\sum_{\overline{v}_*} I(\overline{v}_*) \geq 2 \cdot m(\mathfrak{r})$. This implies that $m(\mathfrak{r}) = 1$ and $p_+ = \deg \overline{v}'_+ = 1$. The sum of the ECH indices from (β_1)–(β_4), (β'_5), (β_6), and $I(\overline{v}''_+) \geq 0$ is at least $+2$.

We claim that $p_- \geq 1$ by considerations of n^*. Indeed, if $\overline{v}'_- = \varnothing$, then there exist a point $\mathfrak{q} \in int(F_-)$ and a sufficiently small neighborhood $N(\mathfrak{q}) \subset \dot{F}_-$ of $\mathfrak{q}$ such that $\overline{v}_-(\mathfrak{q}) = \overline{\mathfrak{m}}(+\infty)$ and $n^*(\overline{v}_-(N(\mathfrak{q}))) \geq m$. This is a contradiction.

Since $p_+ = 1$ and $p_- \geq 1$, it follows that $p_{0,j} = 1$ for all $j = 0, \dots, b+1$ by (d). For the purposes of applying the rescaling argument, we may assume that $b = 0$. By restricting $\overline{u}_{li}$ to a neighborhood of $\sigma_\infty^{\tau_{li}}$, taking the limit of a diagonal subsequence $\overline{u}_{li(l)}$, and applying the rescaling argument, we obtain a 2-level holomorphic building $w_+ \cup w_-$, where $w_\pm : cl(B_\pm) \to \mathbf{CP}^1$ satisfy the following:

(i) $w_-(\partial B_-) \subset \{\phi = 0, \rho > 0\}$;
(ii) $w_-(\overline{\mathfrak{m}}^b(+\infty)) = 0$ and $w_-(+\infty) = \infty$;
(iii) $w_+(\partial B_+) \subset \{\phi = 0, \rho > 0\}$;
(iv) $w_+(-\infty) = 0$ and $w_+(\mathfrak{r}^b) = \infty$;
(v) $w_\pm|_{int(B_\pm)}$ is a biholomorphism onto its image.

Then, by the Involution Lemma I.7.9.3, w_- maps the marker $\dot{\mathcal{L}}_{3/2}(+\infty)$ to $\dot{\mathcal{R}}_\pi(\infty)$ and w_+ maps the ray $\mathcal{L}_{3/2}$ to the ray $\mathcal{R}_\pi$. This gives rise to two constraints for $\overline{v}''_+$: $\mathfrak{r}^b$ is restricted on $\mathcal{L}_{3/2}$ and the derivative of $\overline{v}''_+$ at $\mathfrak{r}$ in the $D^2_{\rho_0}$-direction is also restricted. Hence $I(\overline{v}''_+) \geq 2$ and the total ECH index is at least 4, a contradiction.

(2B) Suppose that $*_0 = (1, j_0)$ for some $1 \leq j_0 \leq a$. Summing the ECH indices using (β_1)–(β_4), (β_5'), (β_6), and $I(\overline{v}''_{*_0}) \geq 1$, the total ECH index is at least 3, a contradiction. However, we want to do slightly better so that the argument carries over in the case of Lemma 3.9.2: The rescaling argument with $m_l \to \infty$ gives $w_+ : cl(B_+) \to \mathbf{CP}^1$ satisfying the following:

(i) $w_+(\partial B_+) \subset \{\phi = 0, \rho > 0\}$;
(ii) $w_+(-\infty) = 0$ and $w_+(+\infty) = \infty$;
(iii) w_+ maps the marker $\dot{\mathcal{L}}_{3/2}(-\infty)$ to $\dot{\mathcal{R}}_\pi(0)$;
(iv) $w_+|_{int(B_+)}$ is a biholomorphism onto its image.

By the Involution Lemmas, w_+^{-1} maps $\dot{\mathcal{R}}_0(0)$ to $\dot{\mathcal{L}}_{3/2}(-\infty)$. This contradicts (iii).

(2C) Suppose that $*_0 = (0, j_0)$ for some $1 \leq j_0 \leq b$. We replace (β_2) by:

(β_2') If $j \geq j_0$, then each positive end of $\overline{v}^\sharp_{0,j}$ that limits to δ_0 contributes an additional $+1$ towards I.
(β_2'') All the other components of $\overline{v}^\sharp_{0,j}$, $1 \leq j < j_0$, are cylinders from δ_0 to h or e with $I = 1$ or 2.

(β_2') is a consequence of the argument of Lemma 3.7.8: there is an independent condition imposed on the asymptotic expansion of each positive end of $\overline{v}^\sharp_{0,j}$ with $j \geq j_0$ that limits to δ_0. (β_2'') is clear. Summing the ECH indices using (β_1), (β_2'), (β_2'') (β_3), (β_4), (β_5'), (β_6), and $I(\overline{v}''_{*_0}) \geq 1$, the total ECH index is at least 3, a contradiction.

(2D) Suppose that $*_0 = -$. In a manner similar to (2C) we replace (β_3) by:

(β_3') Each positive end of $\overline{v}^\sharp_-$ that limits to δ_0 and is not the end of a section from δ_0 to x_i or x_i' contributes an additional $+1$ towards I.

(β_3'') All the other components of $\overline{v}^\sharp_-$ are sections from δ_0 to x_i or x_i' with $I = 1$.

Summing the ECH indices using (β_1), (β_2), (β_3'), (β_3''), (β_4), (β_5'), (β_6), and $I(\overline{v}''_-) \geq 0$, the total ECH index is at least 2.

Suppose that $\mathfrak{r}^b \neq \overline{\mathfrak{m}}^b(+\infty)$. Then the rescaling argument gives $w_- : cl(B_-) \to \mathbf{CP}^1$ which satisfies the following:

(i) $w_-(\partial B_-) \subset \{\phi = 0, \rho > 0\}$;
(ii) $w_-(\overline{\mathfrak{m}}^b(+\infty)) = 0$ and $w_-(\mathfrak{r}^b) = \infty$;
(iii) $w_-(+\infty) \subset \{\phi = 0, \rho > 0\}$; and
(iv) $w_-|_{int(B_-)}$ is a biholomorphism onto its image.

Here (iii) follows from considering the continuation of w_{-,m_l} to upper levels w_{*,m_l} and π_{*,m_l} (here w_{*,m_l} is the limit of $\overline{u}_{li}$ for $m_l \gg 0$, without taking $m_l \to \infty$) and the fact that $w_{+,m_l} : \Sigma_+ \to \mathbf{CP}^1$ must map $\partial \Sigma_+$ to a thin sector near $\{\phi = 0, \rho > 0\}$.

By the Involution Lemma I.7.9.3 and (i)–(iv), $\mathfrak{r}^b$ must lie on $\mathcal{L}_{1/2} \cup \mathcal{L}_{3/2}$ and the map w_-^{-1} maps the marker $\dot{\mathcal{R}}_\pi(\infty)$ to the marker $\dot{\mathcal{L}}_{1/2}$ or $\dot{\mathcal{L}}_{3/2}$ at $\mathfrak{r}^b$, as appropriate. This gives rise to two constraints for $\overline{v}''_-$. Hence $I(\overline{v}''_-) \geq 2$ and the total ECH index is ≥ 4, a contradiction.

Now suppose that $\mathfrak{r}^b = \overline{\mathfrak{m}}^b(+\infty)$. Since passing through $\overline{\mathfrak{m}}^b(+\infty)$ is a codimension 2 condition, we have $I(\overline{v}''_-) \geq 2$. The total ECH index is ≥ 4, a contradiction.

(2E) Suppose that $*_0 = (-1, j_0)$ for some $1 \leq j_0 \leq c$. Then (β_1)–(β_3) hold and the rescaling argument with $m_l \to \infty$ gives $w_- : \Sigma_- \to \mathbf{CP}^1$ and $\pi_- : \Sigma_- \to cl(B_-)$ such that:

(i) $w_-(\partial \Sigma_-) \subset \{\phi = 0, \rho > 0\}$;
(ii) $w_-(z_0) = 0$ for some $z_0 \in \pi_-^{-1}(\overline{\mathfrak{m}}^b(+\infty))$;
(iii) $w_-(z_1) = \infty$ for some $z_1 \in \pi_-^{-1}(-\infty)$;
(iv) $w_-(\pi_-^{-1}(+\infty)) \subset \{\phi = 0, \rho > 0\}$;
(v) $w_-|_{int(\Sigma_-)}$ is a biholomorphism onto its image.

Here (iv) follows from the considerations similar to those of Lemma 3.7.8, together with (a)–(d). By the Involution Lemmas, $\pi_- \circ w_-^{-1}$, where defined, maps the component of $\{\mathrm{Im}(z) = 0\}$ passing through 0 to $\mathcal{L}_{3/2}$. This contradicts (iii).

(3) Suppose there is a boundary point $\mathfrak{r} \in \partial F_{*_0}$ at z_∞. Then the neighborhood of $\overline{v}''_{*_0}(\mathfrak{r})$ contributes at least $k_0 - 1 \gg 2g$ towards n^* by Lemma 3.4.8. We remark that a priori there could be more than one boundary point at z_∞.

(3A) Suppose that $\overline{v}'_- = \varnothing$. Then there is a point $\mathfrak{q} \in int(F_-)$ and a sufficiently small neighborhood $N(\mathfrak{q}) \subset \dot{F}_-$ of $\mathfrak{q}$ such that $\overline{v}_-(\mathfrak{q}) = \overline{\mathfrak{m}}(+\infty)$ and $n^*(\overline{v}_-(N(\mathfrak{q}))) \geq m$. This is a contradiction.

(3B) Suppose that $\overline{v}'_- \neq \varnothing$ and $\overline{v}'_{*_0} = \varnothing$, i.e., $\mathfrak{r}$ is of type (P_1). Then there exist restrictions $\widetilde{u}_{i,-}$ and $\widetilde{u}_{i,*_0}$ of $\overline{u}_i$ such that $\widetilde{u}_{i,-}$ is close to $\overline{v}'_-$ after translation and passes through $\overline{\mathfrak{m}}(\tau_i)$, $\widetilde{u}_{i,*_0}$ is close to $\overline{v}''_{*_0}$ after translation and nontrivially intersects $\sigma_\infty^{\tau_i}$, and the images of $\widetilde{u}_{i,-}$ and $\widetilde{u}_{i,*_0}$ are disjoint. This is a contradiction since we have an excess of n^*.

(3C) Suppose that $\overline{v}'_- \neq \varnothing$, $\overline{v}'_{*_0} \neq \varnothing$, and $*_0 = (1,j), j = 0, \dots, a$.

We claim that (a)–(d) from the proof of (1) hold, where we are only considering levels $\overline{v}_* \prec \overline{v}_+$. Indeed, if (a)–(c) do not hold, then the sum of n^* of the ends is $\geq m - 2g$ by Lemmas 3.4.2 and 3.7.6, a contradiction. (The situation of $\overline{v}^\sharp_{-1,j}$ is not explicitly covered by Lemma 3.7.6, but the same proof works.) (d) is a consequence of (a)–(c). The claim implies that (β_1)–(β_4) hold.

Next we claim that there must be a boundary point at z_∞ on $\overline{v}_+$. Arguing by contradiction, if there is no boundary point at z_∞ on $\overline{v}_+$, then a rescaling argument similar to (2B) gives $w_+ : \Sigma_+ \to \mathbf{CP}^1$ and $\pi_+ : \Sigma_+ \to cl(B_+)$ such that:

(i) $w_+(\partial \Sigma_+) \subset \{\phi = 0, \rho > 0\}$;
(ii) $w_+(z_0) = 0$ for some $z_0 \in \pi_+^{-1}(-\infty)$;
(iii) $w_+(\pi_+^{-1}(-\infty)) \subset \{\phi = 0, \rho \geq 0\}$;
(iv) $w_+(z_1) = \infty$ for some $z_1 \in \pi_+^{-1}(+\infty)$;
(v) $w_+|_{int(\Sigma_+)}$ is a biholomorphism onto its image.

Here (iii) follows from observing that:

(β_2''') There is no component of $\overline{v}^\sharp_{0,j}$ which is a cylinder from δ_0 to e.

Indeed, $\mathfrak{r}$ contributes at least 2 towards I by Lemma 3.7.5, $I(\overline{v}''_{*_0}) \geq 1$ if $*_0 \neq +$, and a component of $\overline{v}^\sharp_{0,j}$ which is a cylinder from δ_0 to e contributes $I = 2$. Together with (β_1)–(β_4), the total ECH index is at least 4, if there is a cylinder from δ_0 to e, a contradiction.

By the Involution Lemmas, $\pi_+ \circ w_+^{-1}$, where defined, maps $\dot{\mathcal{R}}_0(0)$ to $\dot{\mathcal{L}}_{3/2}(-\infty)$, a contradiction. Hence there must be a boundary point at z_∞ on $\overline{v}_+$. A similar argument implies that a boundary point at z_∞ must lie on $\mathcal{L}_{3/2}$ and project to a large sector.

By Lemma 3.7.5, the boundary points at z_∞ contribute at least 2 towards I. In addition, the constraint of lying in $\mathcal{L}_{3/2}$ contributes 1 towards I and the large sector condition contributes 1 towards I. These add up to at least $I = 4$, a contradiction.

(3D) Suppose that $\overline{v}'_- \neq \varnothing$, $\overline{v}'_{*_0} \neq \varnothing$, and $*_0 = (-1,j), j = 1, \dots, c+1$.

We would like apply the argument from (3C) and (2E). The slight complication is that we may have ends of $\overline{v}^\sharp_{-1,j}$ that limit to z_∞ but map to non-thin sectors. In particular, there may be components of $\overline{v}^\sharp_{-1,j}$ have at least two ends that limit to z_∞ but has ind $= 1$ and it is not a priori clear that (β_4) holds.

We claim that:

(β_4') Each positive end of $\overline{v}^\sharp_{-1,j}$, $j = 1, \dots, c$, that limits to z_∞ contributes $+1$ towards I.

For ease of notation assume that $c = 1$ and $p_{0,0} = 2$ during the proof of the claim. Applying the rescaling argument without taking $m \to \infty$, we obtain $w_- : \Sigma_- \to \mathbf{CP}^1$ and $\pi_- : \Sigma_- \to cl(B_-)$ such that:

(i) $w_-(z_0) = 0$ for some $z_0 \in \pi_-^{-1}(\overline{\mathfrak{m}}^b(+\infty))$;
(ii) w_- maps the negative ends of Σ_- to sectors $\mathfrak{S}(\overline{b}_{k,l}, \overline{h}(\overline{b}_{k',l'}))$.

We now make crucial use of the fact that the sectors $\mathfrak{S}(\overline{b}_{k,l}, \overline{h}(\overline{b}_{k',l'}))$ corresponding to the negative ends of Σ_- have different angles, provided the sectors are not both thin sectors; this follows from the definition of $\overline{\mathbf{a}}$ from Section I.5.2.2. Since each thin sector corresponds to a thin strip of $\overline{v}^\sharp_{-1,1}$ which contributes $I = 1$, let us assume that there are no thin sectors in (ii). We use coordinates $(s, t) \in (-\infty, c] \times [0, 1]$ for the two negative ends $\mathcal{E}_{-,1}$, $\mathcal{E}_{-,2}$ of Σ_- which agree with the coordinates (s, t) on B_- and write $w^i_- := w_-|_{\mathcal{E}_{-,i}}$. Then $w^i_-(s, t) \approx c_i e^{\lambda_i(s+it)}$, where $\lambda_1 > \lambda_2 > 0$ without loss of generality; $c_1, c_2 \neq 0$; and the approximation gets better as $s \to -\infty$. On the other hand, suppose that $\operatorname{ind}(\overline{v}^\sharp_{-1,1}) = 1$ and $\overline{v}^\sharp_{-1,1}$ is connected and has two positive ends $\mathcal{E}_{+,i}$, $i = 1, 2$, that limit to z_∞. Here $\mathcal{E}_{+,i}$ is to be paired with $\mathcal{E}_{-,i}$. We use coordinates $(s', t') \in [c', \infty) \times [0, 1]$ for the positive ends $\mathcal{E}_{+,i}$ which agree with the coordinates on $B = \mathbf{R} \times [0, 1]$. Then $\mathcal{E}_{+,1}$ approaches z_∞ much faster than $\mathcal{E}_{+,2}$ does, as $s' \to +\infty$. This is inconsistent with our description of w_-, when we change coordinates $s' = s + \tilde{c}$ for some $\tilde{c}$. Hence each end $\mathcal{E}_{+,i}$ should be allowed to move independently and each positive end of $\overline{v}^\sharp_{-1,1}$ that limits to z_∞ contributes $+1$ towards I.

(β_4') replaces (β_4). We now take the limit $m \to \infty$ and argue as in (3C), which implies that the boundary point at z_∞ must be on $\overline{v}_-$ and gives a total ECH contribution of at least 4. □

3.7.6. *The case $\overline{v}'_- \neq \varnothing$.*

Lemma **3.7.11.** — *If $m \gg 0$, $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$, and $\overline{v}'_- \neq \varnothing$, then no negative end of $\overline{v}^\sharp_{0,j}$, $1 \leq j \leq b$, is asymptotic to a multiple of δ_0.*

Proof. — Arguing by contradiction, suppose there exists some $\overline{v}^\sharp_{0,j_0}$, $1 \leq j_0 \leq b$, which has a negative end asymptotic to a multiple of δ_0, say $\delta_0^{q_{0,j_0}}$. Since the negative end of $\overline{v}^\sharp_{0,j_0}$ contributes at least $m - q_{0,j_0}$ towards n^*, the following hold:

(i) at most one $\overline{v}^\sharp_{0,j}, j > 0$, has a negative end asymptotic to a multiple of δ_0;
(ii) only one negative end of $\overline{v}^\sharp_{0,j_0}$ limits to a multiple of δ_0;
(iii) $p_{0,0} \leq p_{0,1} \leq \cdots \leq p_{0,j_0-1} \leq p_{0,j_0} + q_{0,j_0}$ and $p_{0,j_0} \leq p_{0,j_0+1} \leq \cdots \leq p_{0,b+1}$;
(iv) if $\mathcal{E}$ is a positive end of $\overline{v}^\sharp_{0,j}, j \geq 0$, that limits to δ_0^r, then $n^*(\mathcal{E}) = r$; and
(v) $\cup_{j=1}^c \overline{v}^\sharp_{-1,j}$ is a union of $p_{0,0} > 0$ thin strips.

By Lemma 3.7.5, all the components besides $\overline{v}'_+$ have nonnegative ECH index. We have contributions (β_1), (β_3), (β_4), (β_6) as in Lemma 3.7.10, as well as the following variant of (β_2):

$(\widetilde{\beta}'_2)$ For $1 \leq j < j_0$, $\overline{v}^\sharp_{0,j}$ is a union of cylinders from δ_0 to h or e and

$$\sum_{j=1}^{j_0-1} \mathrm{I}(\overline{v}^\sharp_{0,j}) \geq p_{0,j_0} + q_{0,j_0} - p_{0,1}.$$

$(\widetilde{\beta}''_2)$ For $j_0 + 1 \leq j \leq b$, $\overline{v}^\sharp_{0,j}$ is a union of cylinders from δ_0 to h or e and

$$\sum_{j=j_0+1}^{b} \mathrm{I}(\overline{v}^\sharp_{0,j}) \geq p_{0,b+1} - p_{0,j_0+1}.$$

$(\widetilde{\beta}'''_2)$ $\mathrm{I}(\overline{v}^\sharp_{0,j_0}) \geq p_{0,j_0+1} - p_{0,j_0} + 1.$

$(\widetilde{\beta}'_2)$ and $(\widetilde{\beta}''_2)$ are immediate from $n^*(\overline{v}^\sharp_{0,j_0}) \geq m - q_{0,j_0}$. $(\widetilde{\beta}'''_2)$ follows from the proof of Lemma 3.7.8, which implies that each positive end of $\overline{v}^\sharp_{0,j_0}$ that limits to δ_0 additionally contributes one constraint.

Since $\overline{v}_{0,j_0}$ does not satisfy the partition condition at $\delta_0^{p_{0,j_0}+q_{0,j_0}}$, a straightforward calculation which takes into account the braiding near $\delta_0^{p_{0,j_0}+q_{0,j_0}}$ gives:

$$\mathrm{I}(\overline{v}_{0,j_0}) \geq \mathrm{I}(\overline{v}'_{0,j_0}) + \mathrm{I}(\overline{v}''_{0,j_0}) + 2p_{0,j_0}. \tag{3.7.6}$$

Summing the contributions of (β_1), $(\widetilde{\beta}'_2)$–$(\widetilde{\beta}'''_2)$, (β_3), (β_4), and Equation (3.7.6), we obtain:

$$\sum_{\overline{v}_*} \mathrm{I}(\overline{v}_*) \geq q_{0,j_0} + 2p_{0,j_0} + 1. \tag{3.7.7}$$

Here $q_{0,j_0} \geq 1$ by assumption. If $p_{0,j_0} > 0$ or $p_{0,j_0+1} > 0$, then $\sum_{\overline{v}_*} \mathrm{I}(\overline{v}_*) \geq 4$, a contradiction. On the other hand, if $p_{0,j_0} = \cdots = p_{0,b+1} = 0$, then $\mathrm{I}(\overline{v}^\sharp_{0,j_0}) \geq 2$ by the usual rescaling argument (cf. Cases (3)–(6) of Theorem I.7.10.1). In this case the right-hand side of Equation (3.7.7) can be increased by one and $\sum_{\overline{v}_*} \mathrm{I}(\overline{v}_*) \geq 3$, a contradiction. □

Lemma **3.7.12.** — *If $m \gg 0$, $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$, $\overline{v}'_- \neq \varnothing$, and $\overline{v}^\sharp_+$ has a negative end asymptotic to a multiple of δ_0, then $\overline{u}_\infty \in A'_2$.*

Here A'_2 is as given in Step 4 of the proof of Theorem 3.3.1.

Proof. — This is similar to Lemma 3.7.11. The contributions of (β_1), $(\widetilde{\beta}'_2)$, (β_3), (β_4), and Equation (3.7.6) with $j_0 = b + 1$ add up to $q_+ + 2p_+$.

If $p_+ > 0$, then $\sum_{\overline{v}_*} \mathrm{I}(\overline{v}_*) \geq q_+ + 2p_+ \geq 3$, a contradiction.

If $p_+ = 0$, then $q_+ + 2p_+ \geq 1$. By the usual rescaling argument, the negative end of $\overline{v}^\sharp_+$ that limits to δ_0 additionally contributes $+1$ to I. Hence if $\sum_{\overline{v}_*} \mathrm{I}(\overline{v}_*) = 2$, then $q_+ = 1$. This implies that $\overline{u}_\infty \in \mathrm{A}'_2$. ☐

Lemma **3.7.13.** — *If $m \gg 0$, $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$, and $\overline{v}'_- \neq \varnothing$, then $\overline{u}_\infty \in \mathrm{A}'_2$.*

Proof. — Consider $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$ such that $\overline{v}'_- \neq \varnothing$. By Lemma 3.7.10, there is no boundary point at z_∞, no fiber component and no $\overline{v}''_*$ such that $\overline{v}'_* \cap \overline{v}''_* \neq \varnothing$. By Lemmas 3.7.11 and 3.7.12, if $\overline{v}^\sharp_{0,j_0}$, $1 \leq j_0 \leq b+1$, has a negative end asymptotic to a multiple of δ_0, then $\overline{u}_\infty \in \mathrm{A}'_2$.

It remains to consider three possibilities, all of which satisfy $p_{0,0} \leq \cdots \leq p_{0,b+1}$:

(1) Some $\overline{v}^\sharp_{-1,j_0}$, $j_0 = 1, \dots, c+1$, has a positive or negative end $\mathcal{E}$ at z_∞ such that $n^*(\mathcal{E}) > m/2$.
(2) Some $\overline{v}^\sharp_{1,j_0}$, $j_0 = 0, \dots, a$, has a positive or negative end $\mathcal{E}$ at z_∞ such that $n^*(\mathcal{E}) > m/2$.
(3) Some $\overline{v}^\sharp_{0,j_0}$, $j_0 = 0, \dots, b$, has a positive end $\mathcal{E}$ at a multiple of δ_0 such that $n^*(\mathcal{E}) > m/2$.

It is clear that (1), (2), and (3) are mutually exclusive.

(1) is eliminated in a manner similar to (2E) of Lemma 3.7.10 and (2) in a manner similar to Case (6_i) of Lemma 3.4.28.

(3) In this case, (a)–(c) in the proof of Lemma 3.7.10 hold, with the exception of the level $\overline{v}_{0,j_0}$. Observe that $\overline{v}'_{0,j_0} \neq \varnothing$ and $\overline{v}^\sharp_{0,j_0}$ consists of:

– cylinders from δ_0 to e or h which contribute $\mathrm{I} = 1$ or 2 each; and
– a curve $\overline{v}^{\sharp 1}_{0,j_0}$ with one positive end $\mathcal{E}^\sharp_1$ which is asymptotic to $\delta_0^{c_1}$ and satisfies $n^*(\mathcal{E}^\sharp_1) = m + c_1$ and other positive ends which in total are asymptotic to $\delta_0^{c_2}$ and satisfy $n^* = c_2$.

We compute the contributions to the ECH index from the ends of $\overline{v}'_{0,j_0} \cup \overline{v}^\sharp_{0,j_0}$ that limit to multiples of δ_0 at the positive end. Here is the list of such ends:

– the union $\mathcal{E}'$ of positive ends of $\overline{v}'_{0,j_0}$;
– $\mathcal{E}^\sharp_1$ satisfying $n^*(\mathcal{E}^\sharp_1) = m + c_1$;
– the union $\mathcal{E}^\sharp_2$ of positive ends of $\overline{v}^\sharp_{0,j_0}$ that correspond to punctures of w_{0,j_0+1};
– the union $\mathcal{E}^\sharp_3$ of all other positive ends.

We use the formula:

$$\mathrm{ind}(\overline{u}) + (\widetilde{\mu}_\tau(\overline{u}) - \mu_\tau(\overline{u}) - w_\tau(\overline{u})) \leq \mathrm{I}(\overline{u}), \tag{3.7.8}$$

where the notation is as in Section I.3.4 and the equation follows from the adjunction inequality and [Hu2, Lemma 4.20]. The end $\mathcal{E}_1^\sharp$ has contributions $\widetilde{\mu}_\tau = c_1$, $\mu_\tau = 1$, and $w_\tau = 1 - c_1$. Moreover, since the vanishing of the leading asymptotic eigenfunction corresponding to the end $\mathcal{E}_1^\sharp$ is a codimension two condition which contributes 2 to the Fredholm index, the extra contribution to I from $\mathcal{E}_1^\sharp$ is $c_1 - 1 - (1 - c_1) + 2 = 2c_1$. The contributions to I from $\mathcal{E}'$ and $\mathcal{E}_2^\sharp$, arising from a writhe computation, is $2(\deg \mathcal{E}' + \deg \mathcal{E}_2^\sharp) = 2(p_{0,j_0} + \deg \mathcal{E}_2^\sharp)$. Finally, the contributions to I from $\mathcal{E}_3^\sharp$ is $\deg \mathcal{E}_3^\sharp$ by the argument of Lemma 3.7.8.

The total ECH index of all the levels is $\geq 2c_1 + 2p_{0,j_0} \geq 4$, a contradiction. □

3.7.7. *The case $\overline{v}'_- = \varnothing$.*

Lemma **3.7.14.** — *If $m \gg 0$, then there is no $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$ such that $\overline{v}'_- = \varnothing$.*

Proof. — Arguing by contradiction, suppose there exists $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$ such that $\overline{v}'_- = \varnothing$. By Lemma 3.7.10, there is no boundary point at z_∞, no fiber component, and no $\overline{v}''_*$ such that $\overline{v}'_* \cap \overline{v}''_* \neq \varnothing$.

The point constraint gives $n^*(\overline{v}''_-) \geq m$ and $I(\overline{v}''_-) \geq 2$. This immediately implies that (a)–(d) in the proof of Lemma 3.7.10 hold. In particular, (β_2) from Lemma 3.7.10 holds.

If $p_{0,b+1} > 0$, then the proof of Lemma 3.7.8 gives us a contradiction. Hence $p_{0,b+1} = 0$. If a positive end of $\overline{v}''_+$ limits to z_∞, then a component $\widetilde{v}$ of $\overline{v}''_+$ must be a section of $\overline{W}_+$ from z_∞ to h or e. However, by the choice of h from Section 3.2.3 and the proof of Lemma I.6.6.5, sections of $\overline{W}_+$ from z_∞ to h cannot exist, leaving us with the possibility that $\widetilde{v}$ is a section from z_∞ to e. Hence $I(\widetilde{v}) \geq 1$ and the total ECH index is ≥ 3, a contradiction. On the other hand, if $\overline{v}''_+ = \overline{v}^\flat_+$, then $I(\overline{v}_{1,j}) \geq 1$ for some $j > 0$, which is again a contradiction. □

Proof of Lemma 3.3.10. — Lemma 3.3.10 follows from Lemmas 3.7.13 and 3.7.14. □

3.8. *Degeneration at $-\infty$, part II.* — Let $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$ be the limit of $\overline{u}_i \in \mathcal{M}_{\tau_i}$, where $\tau_i \to -\infty$.

Lemma **3.8.1.** — *If $\overline{u}_\infty$ has no fiber components and $\overline{v}'_1 = \varnothing$, then the ECH index of each level is nonnegative.*

Proof. — The proof is similar to that of Lemma 3.5.2 and uses the considerations of Lemma 3.7.5. □

3.8.1. *The case $\overline{v}'_2 = \varnothing$.*

Lemma **3.8.2.** — *If* $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$ *and* $\overline{v}'_2 = \varnothing$*, then:*

(1) $\overline{u}_\infty$ *is a* 2*-level building consisting of* $\overline{v}_1$ *with* $\mathrm{I}(\overline{v}_1) = 0$*,* $\overline{v}_2$ *with* $\mathrm{I}(\overline{v}_2) = 2$*, and* $\overline{v}'_j = \varnothing$ *for* $j = 1, 2$*;*
(2) *the left and right ends of* $\overline{v}_1$ *(= left and right ends of* $\overline{v}_2$*) do not limit to* z_∞*;*
(3) $\overline{u}_\infty$ *has no fiber component;*
(4) $\overline{u}_\infty$ *has no boundary point at* z_∞*.*

Proof. — Suppose that $\overline{v}'_2 = \varnothing$. First observe that

$$(\mathbf{3.8.1}) \qquad n^*(\overline{v}_2(\mathrm{N}(\mathfrak{q}))) \geq m,$$

where $\mathfrak{q} \in \dot{\mathrm{F}}_2$ is the point such that $\overline{v}_2(\mathfrak{q}) = \overline{\mathfrak{m}}(-\infty)$.

We also have $\mathrm{I}(\overline{v}_2) \geq 2$: If $\overline{u}_\infty$ has a fiber component $\widetilde{v}$, then $\widetilde{v}$ must pass through $\overline{\mathfrak{m}}(-\infty)$ and $\mathrm{I}(\overline{v}_2) \geq 2g+2 \geq 4$ by an argument similar to that of Lemma I.7.5.5. On the other hand, if $\overline{u}_\infty$ does not have a fiber component, then $\mathrm{I}(\overline{v}_2) \geq 2$, since passing through $\overline{\mathfrak{m}}(-\infty)$ is a generic codimension two condition.

(1) We claim that $\overline{v}'_1 = \varnothing$. Arguing by contradiction, if $\overline{v}'_1 \neq \varnothing$, then at least one of $\overline{v}^\sharp_{\mathrm{L},j}$, $j = 0, \dots, a$, has a right end $\mathcal{E}_\mathrm{R}$ that limits to z_∞. On the other hand, the end $\mathcal{E}_\mathrm{R}$ satisfies

$$(\mathbf{3.8.2}) \qquad n^*(\mathcal{E}_\mathrm{R}) \geq k_0 - 1 \gg 2g,$$

which contradicts Equation (3.8.1). This proves the claim.

The claim and Lemma 3.8.1 imply that each level of $\overline{u}_\infty$ has nonnegative ECH index. Hence $\mathrm{I}(\overline{v}_2) = 2$, $\mathrm{I}(\overline{v}_1) = 0$, and $a = b = c = d = 0$ since a level $\overline{v}_{\mathrm{L},j}$, $j = 1, \dots, a$, is not a connector if and only if $\mathrm{I}(\overline{v}_{\mathrm{L},j}) > 0$ (and the same holds for $\overline{v}_{\mathrm{R},j}$, $j = 1, \dots, b$, $\overline{v}_{\mathrm{B},j}$, $j = 1, \dots, c$, and $\overline{v}_{\mathrm{T},j}$, $j = 1, \dots, d$).

(2) If $\overline{v}_2$ has a right end $\mathcal{E}_\mathrm{R}$ that limits to z_∞, then $\mathcal{E}_\mathrm{R}$ satisfies Equation (3.8.2), and we have a contradiction of Equation (3.8.1). If $\overline{v}_2$ has a left end that limits to z_∞, then $\overline{v}_1$ has a right end $\mathcal{E}_\mathrm{R}$ that limits to z_∞ and satisfies Equation (3.8.2) (since $\overline{v}'_1 = \varnothing$), which is again a contradiction.

(3) Since $\mathrm{I}(\overline{v}_2) = 2$, we cannot have a fiber component.

(4) A boundary point at z_∞ contradicts Equation (3.8.1). □

3.8.2. *The case* $\overline{v}'_1 = \varnothing$, $\overline{v}'_2 \neq \varnothing$.

Lemma **3.8.3.** — *If* $m \gg 0$*, then there is no* $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$ *such that* $\overline{v}'_1 = \varnothing$ *and* $\overline{v}'_2 \neq \varnothing$*.*

Proof. — Arguing by contradiction, suppose there exists $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$ such that $\overline{v}'_1 = \varnothing$ and $\overline{v}'_2 \neq \varnothing$. The analog of Lemma 3.5.1 holds, i.e.,

$$\sum_{i=1}^{q} n^*(\mathcal{E}_i) + \sum_{i=1}^{r} n^*(\mathcal{E}'_i) = m, \quad \sum_{i=1}^{q} n^{*,alt}(\mathcal{E}_i) + \sum_{i=1}^{r} n^{*,alt}(\mathcal{E}'_i) \geq m - 2g,$$

where $\mathcal{E}_i$, $i = 1, \dots, q$, are the ends that limit to z_∞ and $\mathcal{E}'_i$, $i = 1, \dots, r$, are the neighborhoods of boundary points of type (P_3) of all the $\overline{v}^\sharp_{\mathrm{L},j}$, $j = 0, \dots, a$, and $\overline{v}^\sharp_{\mathrm{R},j}$, $j = 0, \dots, b$. The conclusion of Lemma 3.5.4 then holds by an almost identical argument. The proof of Lemma 3.5.12 then carries over without modification. □

3.8.3. *The case* $\overline{v}'_1 \neq \varnothing$, $\overline{v}'_2 \neq \varnothing$.

Lemma **3.8.4.** — *Suppose* $\overline{v}'_1 \neq \varnothing$ *and* $\overline{v}'_2 \neq \varnothing$. *Let* $\mathcal{E}_i$, $i = 1, \dots, q$, *be the ends of all the* $\overline{v}^\sharp_*$ *that limit to* z_∞ *and let* $\mathcal{E}'_i$, $i = 1, \dots, r$, *be the neighborhoods of the boundary points of type* (P_3). *Then one of the following holds:*

(a) $\sum_{i=1}^q n^*(\mathcal{E}_i) + \sum_{i=1}^r n^*(\mathcal{E}'_i) \geq m - 2g$.
(b) *Each end* $\mathcal{E}_i$ *is an end of* $\overline{v}^\sharp_{\mathrm{B},j}$, $j = 1, \dots, c$, *or* $\overline{v}^\sharp_{\mathrm{T},j}$, $j = 0, \dots, d$, *that projects to a thin sector of type* $\mathfrak{S}(\overline{a}_{i',j'}, \overline{h}(\overline{a}_{i',j'}))$ *or* $\mathfrak{S}(\overline{b}_{i',j'}, \overline{h}(\overline{b}_{i',j'}))$. *In particular,* $\mathcal{E}_i$ *is not a left or right end of* $\overline{v}^\sharp_*$, *where* $* = (\mathrm{L}, j)$, $j = 0, \dots, a$, *or* (R, j), $j = 0, \dots, b$.

Proof. — We apply the continuation argument. If some $\mathcal{E}_i$, $i = 1, \dots, q$, does not project to a thin sector (of type $\mathfrak{S}(\overline{a}_{i',j'}, \overline{h}(\overline{a}_{i',j'}))$, $\mathfrak{S}(\overline{b}_{i',j'}, \overline{h}(\overline{b}_{i',j'}))$, or $\mathfrak{S}(\overline{b}_{i',j'}, \overline{a}_{i',j'})$), then the sectors $\pi_{\mathrm{D}^2_{\rho_0}}(\mathcal{E}_i)$, $i = 1, \dots, q$, and $\pi_{\mathrm{D}^2_{\rho_0}}(\mathcal{E}'_i)$, $i = 1, \dots, r$, will sweep out a neighborhood of z_∞ with the exception of some thin sectors, implying (a).

On the other hand, suppose that all the ends $\mathcal{E}_i$, $i = 1, \dots, r$, project to thin sectors. We claim that there are no thin sectors of type $\mathfrak{S}(\overline{b}_{i',j'}, \overline{a}_{i',j'})$. Indeed, the number of left and right ends of all the $\overline{v}^\sharp_*$ that limit to z_∞ must be equal and the right ends cannot map to thin sectors. This gives (b). □

Lemma **3.8.5.** — *If* $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$, $\overline{v}'_1 \neq \varnothing$, *and* $\overline{v}'_2 \neq \varnothing$, *then:*

(1) $\overline{u}_\infty$ *has no fiber components;*
(2) *if* $\overline{v}''_*$ *satisfies* $\overline{v}'_* \cap \overline{v}''_* \neq \varnothing$ *and* $\overline{v}'_* \cap \overline{v}''_* \subset int(\overline{v}'_*)$, *then* $* = 1$ *and* $p_* = 1$;
(3) $\overline{u}_\infty$ *has no boundary point of type* (P_1) *or* (P_2).

Proof. — This is similar to the proof of Lemma 3.7.10 and uses Equations (3.7.1) and (3.7.2).

First observe that if $\overline{u}_\infty$ has a fiber component, a boundary point of type (P_1) or (P_2), or some $\overline{v}'_* \cap \overline{v}''_* \neq \varnothing$ with $\overline{v}'_* \cap \overline{v}''_* \subset int(\overline{v}'_*)$, then we are in the situation of Lemma 3.8.4(b). It is not hard to verify that $\mathrm{I}(\overline{v}_*) \geq 0$ for all $*$ with the exception of $\overline{v}_1$.

(1) Suppose $\overline{u}_\infty$ has a fiber component $\widetilde{v}$. Then we have the following contributions towards I:

(β_1) $\mathrm{I}(\overline{v}'_1) = -p_1 = -\deg(\overline{v}'_1)$.
(β_2) $\sum_{j=1}^c \mathrm{I}(\overline{v}^\sharp_{\mathrm{B},j}) = p_1$.
(β_3) If the fiber $\widetilde{v}$ is a component of $\overline{v}_*$, then $\widetilde{v}$ and the intersection $\widetilde{v} \cap (\overline{v}_* - \widetilde{v})$ contribute $2g + 2 \geq 4$ towards I.

The argument is similar to that of Lemma 3.7.10(1). This gives a total of $I > 2$, which is a contradiction.

(2) Suppose $\overline{v}'_{*_0} \cap \overline{v}''_{*_0} \neq \varnothing$ and $\overline{v}'_{*_0} \cap \overline{v}''_{*_0} \subset int(\overline{v}'_{*_0})$. If $*_0 \neq 1$, then $I(\overline{v}''_{*_0}) \geq 1$ and the intersection points contribute at least $2p_{*_0} \geq 2$. If $*_0 = 1$ and $p_{*_0} > 1$, then $I(\overline{v}''_{*_0}) \geq 0$ and the intersection points contribute at least $2p_{*_0} \geq 4$. Combined with the ECH contributions from (β_1) and (β_2), we have a contradiction.

(3) If $\overline{u}_\infty$ has a boundary point of type (P_1), then the argument from (3B) of Lemma 3.7.10 implies an excess of n^*. If $\overline{u}_\infty$ has a boundary point of type (P_2), then the analog of Lemma 3.7.5 implies that the boundary points at z_∞ contribute $2p_{*_0} \geq 2$; there is also a large sector which contributes an additional $+1$. Combined with (β_1) and (β_2) we obtain a total of $I > 2$, a contradiction. □

Lemma **3.8.6.** — *If $m \gg 0$, $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$, $\overline{v}'_1 \neq \varnothing$, and $\overline{v}'_2 \neq \varnothing$, then there is no level $\overline{v}_*$ such that $\overline{v}'_* \cap \overline{v}''_* \neq \varnothing$ and $\overline{v}'_* \cap \overline{v}''_* \subset int(\overline{v}'_*)$.*

Proof. — Arguing by contradiction, suppose that $\overline{v}'_{*_0} \cap \overline{v}''_{*_0} \neq \varnothing$ and $\overline{v}'_{*_0} \cap \overline{v}''_{*_0} \subset int(\overline{v}'_{*_0})$ for some $\overline{v}_{*_0}$. Then $*_0 = 1$ and $p_1 = 1$ by Lemma 3.8.5(2). By Lemma 3.8.4(b), we have $p_{L,j} = 1$ and $p_{R,j} = 1$ for all j. Hence we may assume that $a = b = 0$. Equation (3.7.2) then implies that $c = 1$ and $d = 0$.

By the rescaling argument with $m \to \infty$, we obtain a 2-level building $w_1 \cup w_2$, where

$$w_2 : cl(B_{-\infty,2}) \to \mathbf{CP}^1, \quad w_1 : cl(B_{-\infty,1}) \to \mathbf{CP}^1,$$

and (i)–(viii), given below, hold. Here $cl(B_{-\infty,2})$ is obtained from $B_{-\infty,2}$ by adding the left and right points at infinity, denoted $t = \pm\infty$; similarly $cl(B_{-\infty,1})$ is obtained from $B_{-\infty,1}$ by adding $s = \pm\infty$ and $t = \pm\infty$. The map w_2 satisfies the following:

(i) $w_2(\overline{\mathfrak{m}}^b(-\infty)) = 0$;
(ii) $w_2(\partial cl(B_{-\infty,2})) \subset \{\phi = 0, \rho > 0\} \cup \{\infty\}$;
(iii) $w_2(t = +\infty) = \infty$ and $w_2(t = -\infty) \subset \{\phi = 0, \rho > 0\}$ (or vice versa);
(iv) $w_2|_{int(B_{-\infty,2})}$ is a biholomorphism onto its image.

The map w_1 satisfies the following:

(v) $w_1(t = -\infty) = 0$ (or $w_1(t = +\infty) = 0$; for simplicity assume the former);
(vi) $w_1(\partial cl(B_{-\infty,1})) \subset \{\phi = 0, \rho > 0\} \cup \{\infty\}$;
(vii) $w_1(\mathfrak{r}) = \infty$, where $\mathfrak{r} \in \overline{v}'_1 \cap \overline{v}''_1$;
(viii) $w_1|_{int(B_{-\infty,1})}$ is a biholomorphism onto its image.

In view of the above, we consider the "tropical curves"

$$\overline{\Xi}_i : \Gamma \to [-1, 1]/(-1 \sim 1) \times [0, d_i]$$

as in Section 3.5.6. The proof strategy of Lemmas 3.5.12 and 3.5.13 carry over to give us a contradiction; this is due to the disparity in the growth rates of the left and right ends of $\overline{u}_i$ when restricted to $[-2, 2] \times [1 + \mathrm{L}, r(\tau_i) - \mathrm{L}]$. □

The combination of Lemmas 3.8.5 and 3.8.6 give the following, which is the analog of Lemma 3.5.4(2),(3):

Corollary **3.8.7.** — *If $m \gg 0$, $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$, $\overline{v}'_1 \neq \varnothing$, and $\overline{v}'_2 \neq \varnothing$, then there is no fiber component, no boundary point of type (P_1) or (P_2), and no intersection point $\mathfrak{r} \in \overline{v}'_* \cap \overline{v}''_*$ such that $\mathfrak{r} \in \overline{v}'_* \cap \overline{v}''_* \subset int(\overline{v}'_*)$.*

Before embarking on the proof of Lemma 3.8.11, we prove Lemma 3.8.8, which is the analog of Lemma 3.5.4(4) and is a bit involved.

First we recall and modify some notation from Section I.5.7. We use the convention that $\star = (\mathrm{L}, j), j = 1, \dots, a$, or $(\mathrm{R}, j), j = 1, \dots, b+1$. The data $\overrightarrow{\mathcal{D}}_{\star,\pm}$ at z_∞ for $\overline{v}_\star$ is given by a $p = p_{\star,\pm}$-tuple of matchings

$$\{(i'_1, j'_1) \to (i_1, j_1), \dots, (i'_p, j'_p) \to (i_p, j_p)\}, \tag{3.8.3}$$

where $i_k, i'_k \in \{1, \dots, 2g\}$, $j_k, j'_k \in \{0, 1\}$ for $k = 1, \dots, p$ and $i_k \neq i_l$, $i'_k \neq i'_l$ for $k \neq l$. Here the subscript $+$ (resp. $-$) in $\overrightarrow{\mathcal{D}}_{\star,\pm}$ refers to the left (resp. right) end, (i'_k, j'_k) corresponds to $\overline{b}_{i'_k j'_k}$, and (i_k, j_k) corresponds to $\overline{a}_{i_k j_k}$. We write

$$\overrightarrow{\mathcal{D}}_{\star,\pm} = \overrightarrow{\mathcal{D}}'_{\star,\pm} \cup \overrightarrow{\mathcal{D}}''_{\star,\pm},$$

where $\overrightarrow{\mathcal{D}}'_{\star,\pm}$ and $\overrightarrow{\mathcal{D}}''_{\star,\pm}$ correspond to the ends of $\overline{v}'_\star$, $\overline{v}''_\star$, respectively.

Lemma **3.8.8.** — *If $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$, $\overline{v}'_1 \neq \varnothing$ and $\overline{v}'_2 \neq \varnothing$, then every component of $\overline{v}^\sharp_\star$, $\star = (\mathrm{L}, j), j = 1, \dots, a$, or $(\mathrm{R}, j), j = 1, \dots, b+1$, is a thin strip.*

Proof. — Arguing by contradiction, suppose there exists $\overline{v}^\sharp_{\star_0}$ which is not a union of thin strips; we take $\star_0 = (\mathrm{R}, 1)$ without loss of generality. Then the following hold:

(a) $n^*(\overline{v}^\sharp_{\star_0}) = m$ and $n^{*,alt}(\overline{v}^\sharp_{\star_0}) \geq m - 2g$;

(b) each component $\widetilde{v}$ of $\overline{v}^\sharp_{\mathrm{T},j}, j = 1, \dots, d$, or $\overline{v}^\sharp_{\mathrm{B},j}, j = 1, \dots, c$, is a thin strip with z_∞ at the positive end and $\mathrm{I}(\widetilde{v}) = 1$;

(c) each component $\widetilde{v}$ of $\overline{v}^\sharp_\star$, $\star \neq \star_0$, is a thin strip with z_∞ at the left end and $\mathrm{I}(\widetilde{v}) = 1$;

(d) each component $\widetilde{v}$ of $\overline{v}^\sharp_1$ limits to z_∞ only at the left end and the top end, the left and top ends project to thin sectors, $\deg(\widetilde{v}) = 1$, and $\mathrm{I}(\widetilde{v}) = 1$.

(a) follows from the argument of Lemma 3.5.4(4)(a) and (b) and (c) follow immediately from (a). (d) The assertions about the ends of $\widetilde{v}$ follow from (a). This then implies that $\widetilde{v}$

projects to the domain bounded by $\overline{b}_i$, $\overline{h}(\overline{b}_i)$, and $\overline{h}(\overline{a}_i)$ in $\overline{S}$, in view of the positions of the arcs $\overline{b}_{i,j}$, $\overline{a}_{i,j}$, $\overline{h}(\overline{b}_{i,j})$, and $\overline{h}(\overline{a}_{i,j})$ from Figure 6.

We now analyze the ends of $\overline{v}^{\sharp}_{\star_0}$ at z_∞ in more detail. Let l_+ and l_- be the number of left and right ends of $\overline{v}^{\sharp}_{\star_0}$ at z_∞ and let $\mathfrak{bp}$ be the number of boundary points of type (P_3) on $\overline{v}_{\star_0}$. In view of (c) and (d), the total number of left ends of $\overline{v}_*$ that limit to z_∞ besides those of $\overline{v}^{\sharp}_{\star_0}$ is $l_- - l_+$. By (a slight modification of) the continuation method, we obtain a cycle $\mathcal{Z}$ consisting of $2l_- + \mathfrak{bp}$ chords, where l_- of the chords are of type $\mathfrak{S}(\overline{a}_{k,l}, \overline{b}_{k',l'})$, l_+ of the chords are of type $\mathfrak{S}(\overline{b}_{k,l}, \overline{a}_{k',l'})$, $\mathfrak{bp}$ of the chords are of type $\mathfrak{S}(\overline{a}_{k,l}, \overline{a}_{k',l'})$ or $\mathfrak{S}(\overline{b}_{k,l}, \overline{b}_{k',l'})$, and we are eliding the chords from $\overline{h}(\overline{a}_{k,l})$ to $\overline{a}_{k,l}$ and chords from $\overline{b}_{k,l}$ to $\overline{h}(\overline{b}_{k,l})$.

Let us write $p_* = \deg \overline{v}'_*$, $q_* = \deg \overline{v}^{\sharp}_*$, and $r_* = \deg \overline{v}^{\flat}_*$. We have $\operatorname{ind}(\overline{v}^{\sharp}_{\star_0}) \geq 2$ since there is an end that limits to z_∞ or a boundary point of type (P_3) that projects to a large sector. Then $I(\overline{v}^{\sharp}_{\star_0}) \geq 2$ by the ECH index inequality (cf. Lemma I.5.7.21) and considerations of $\mathcal{Z}$.

(1) Suppose that $\overline{v}'_{\star_0} \neq \varnothing$. We claim the following:

Claim **3.8.9.** — $I(\overline{v}_{\star_0}) \geq I(\overline{v}'_{\star_0}) + I(\overline{v}''_{\star_0}) + \delta_{\star_0}$, *where* $\delta_{\star_0} \geq 2$ *if* $\mathfrak{bp} > 0$ *and* $\delta_{\star_0} = 2p_{\star_0}$ *if* $\mathfrak{bp} = 0$.

Proof of Claim 3.8.9. — Each collection of boundary points of type (P_3) that map to the same point on the base contributes at least $+2$ towards I by the argument from Lemma 3.7.5. The inequality for $\mathfrak{bp} > 0$ then follows.

Next assume that $\mathfrak{bp} = 0$. Also assume that $l_+ = l_-$, since otherwise there exist thin strips of $\overline{v}_*$ from (c) or components from (d), all of which contribute > 0 to I, yielding a total of $I \geq 3$, a contradiction. Let (s, t_2) be coordinates on $[-2, 2] \times [-1, 1]$. Let

$$\check{C}'_{[-1,1]}, \check{C}''_{[-1,1]} \subset [-2, 2] \times [-1, 1] \times \overline{S}$$

be representatives of $\overline{v}'_{\star_0}$ and $\overline{v}''_{\star_0}$ and let $\mathfrak{c}'_{\pm 1}$, $\mathfrak{c}''_{\pm 1}$ be groomings on $A_{\varepsilon/2} = \partial D^2_{\varepsilon/2} \times [-2, 2]$ and $A_\varepsilon = \partial D^2_\varepsilon \times [-2, 2]$ corresponding to $\check{C}'_{[-1,1]}|_{t_2=\pm 1}$ and $\check{C}''_{[-1,1]}|_{t_2=\pm 1}$, such that the following hold:

– $\mathfrak{c}'_{+1} = \mathfrak{c}'_{-1}$ and $w(\mathfrak{c}'_{\pm 1}) = 0$;
– $\mathfrak{c}''_{\pm 1}$ is obtained by intersecting $\pi_{[-2,2]\times\overline{S}}$-projections of the ends of $\overline{v}''_{\star_0}$ with A_ε.

Here $\pi_{[-2,2]\times\overline{S}}$ is the projection $[-2, 2] \times \mathbf{R} \times \overline{S} \to [-2, 2] \times \overline{S}$. We also remark that, for sign consistency with the computations in Section I.5.7, we use the standard orientations on $A_{\varepsilon/2}$ and A_ε but view $\mathfrak{c}'_{+1}$ and $\mathfrak{c}''_{+1}$ to be at the *negative end* and $\mathfrak{c}'_{-1}$ and $\mathfrak{c}''_{-1}$ to be at the *positive end* during the proof of this claim. By the above description of $\mathcal{Z}$:

– $w(\mathfrak{c}''_{+1}) = 0$ or -1 and $w(\mathfrak{c}''_{-1}) = 0$ or 1;
– the endpoints of $\mathfrak{c}''_{+1}$ and $\mathfrak{c}''_{-1}$ agree and are alternating.

We now extend $\check{C}'_{[-1,1]}, \check{C}''_{[-1,1]}$ by concatenating with

$$\check{C}'_{[1,2]}, \check{C}''_{[1,2]} \subset [-2,2] \times [1,2] \times \overline{S},$$
$$\check{C}'_{[-2,-1]}, \check{C}''_{[-2,-1]} \subset [-2,2] \times [-2,-1] \times \overline{S}$$

such that the ends of $\check{C}'_{[-2,2]} \cup \check{C}''_{[-2,2]}$ are groomed and have zero winding number. Writing $\mathfrak{w}$ for the writhe, Lemmas I.5.7.22 and I.5.7.23 imply that we have an additional contribution of at least

$$2p_{\star_0} = 2(\mathfrak{w}(\mathfrak{c}'_{+1} \cup \mathfrak{c}''_{+1}) - \mathfrak{w}(\mathfrak{c}'_{-1} \cup \mathfrak{c}''_{-1}))$$

towards I. □

By Claim 3.8.9, $I(\overline{v}_{\star_0}) \geq 4$. Since the ECH indices of the other levels add up to at least 0 in view of (b), (c), and (d), we have a contradiction.

(2) Suppose that $\overline{v}'_{\star_0} = \varnothing$. Then there are no boundary points of type (P_3) by definition. Since $I(\overline{v}_{\star_0}) = I(\overline{v}''_{\star_0}) \geq 2$, the only nontrivial levels besides $\overline{v}_{\star_0}$ and $\overline{v}_1$ are $\overline{v}_{B,j}$, $j = 1, \dots, c$, which consist of thin strips and trivial strips. Again we have $l_+ = l_-$, since otherwise there exist thin strips of $\overline{v}_*$ from (c) or components from (d), all of which contribute > 0 to I, yielding a total of $I \geq 3$, a contradiction. Let us write $l_{\star_0} := l_+ = l_-$. We claim the following:

Claim **3.8.10.** — $I(\overline{v}''_{\star_0}) \geq 2 + l_{\star_0}$.

Proof of Claim 3.8.10. — Let us first consider the case where:

(1) $\pi_{\overline{S}} \circ \overline{v}^\sharp_{\star_0}|_{\dot{F}^\sharp_{\star_0}}$ is a diffeomorphism onto its image, which contains $\overline{S} - \overline{\mathbf{a}} - \overline{\mathbf{b}}$;

(2) $\overline{v}^\sharp_{\star_0}$ limits to $z_\infty^{l_{\star_0}} \cup \mathbf{y}'$ to the left and to $z_\infty^{l_{\star_0}} \cup \mathbf{y}''$ to the right, where $\mathbf{y}' = \mathbf{y}''$.

If we choose a multivalued trivialization τ to be compatible with $\overrightarrow{\mathcal{D}}''_{\star_0,\pm}$ (cf. Section I.5.7.3), then

$$\begin{aligned}\operatorname{ind}(\overline{v}^\sharp_{\star_0}) &= -\chi(F^\sharp_{\star_0}) + \deg \overline{v}^\sharp_{\star_0} + \mu_\tau(\overline{v}^\sharp_{\star_0}) + 2c_1((\overline{v}^\sharp_{\star_0})^* T\overline{S}, \tau)\\ &= -(1 - 2g - (\deg \overline{v}^\sharp_{\star_0} - l_{\star_0})) + \deg \overline{v}^\sharp_{\star_0} + (l_{\star_0} + 1) + 2(1 - 2g)\\ &= 2 - 2g + 2\deg \overline{v}^\sharp_{\star_0}.\end{aligned}$$

Hence $I(\overline{v}^\sharp_{\star_0}) \geq 2 - 2g + 2\deg \overline{v}^\sharp_{\star_0}$. We also have $I(\overline{v}^\flat_{\star_0}) \geq 0$ and $\langle \overline{v}^\sharp_{\star_0}, \overline{v}^\flat_{\star_0} \rangle = \deg \overline{v}^\flat_{\star_0}$, where $\langle , \rangle$ is the algebraic intersection number. Summing the contributions, we obtain:

$$\begin{aligned}I(\overline{v}_{\star_0}) &= I(\overline{v}^\sharp_{\star_0}) + I(\overline{v}^\flat_{\star_0}) + 2\langle \overline{v}^\sharp_{\star_0}, \overline{v}^\flat_{\star_0} \rangle\\ &\geq 2 - 2g + 2\deg \overline{v}^\sharp_{\star_0} + 2\deg \overline{v}^\flat_{\star_0} \geq 2 + \deg \overline{v}^\sharp_{\star_0}.\end{aligned} \tag{3.8.4}$$

In general, the cases with smallest ECH indices occur when elements of $\mathbf{y}'$ are of type $x_{i1}^{\#}$ or $x_{i3}^{\#}$ and elements of $\mathbf{y}''$ are of type $x_{i2}^{\#}$. This has the effect of decreasing the lower bound in Equation (3.8.4) by the cardinality of $\mathbf{y}'$. Hence $\mathrm{I}(\overline{v}_{\star_0}) \geq 2 + l_{\star_0}$. □

Since $l_{\star_0} \geq 1$, we have $\mathrm{I}(\overline{v}_{\star_0}) \geq 3$, which is a contradiction. □

Lemma **3.8.11.** — *If $m \gg 0$, then there is no $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$ such that $\overline{v}_1' \neq \varnothing$ and $\overline{v}_2' \neq \varnothing$.*

Proof. — Corollary 3.8.7 and Lemma 3.8.8 imply the analog of Lemma 3.5.4. The proof of Lemma 3.5.13 then carries over with no change. □

Proof of Lemma 3.3.11. — Suppose $\overline{u}_\infty \in \partial_{\{-\infty\}}\mathcal{M}$. If $\overline{v}_2' = \varnothing$, then we are in the situation of Lemma 3.3.11 by Lemma 3.8.2. On the other hand, Lemmas 3.8.3 and 3.8.11 imply that $\overline{v}_2' = \varnothing$. □

3.9. *Breaking in the middle, part II.* — Let $\overline{u}_\infty \in \partial_{(-\infty,+\infty)}\mathcal{M}$ be the limit of $\overline{u}_i \in \mathcal{M}_{\tau_i}$, where $\tau_i \to \mathrm{T}'$.

Lemma **3.9.1.** — *If $\overline{u}_\infty$ has no fiber components, then the ECH index of each level $\overline{v}_* \neq \overline{v}_0$ is nonnegative and the only components of $\overline{u}_\infty$ which have negative ECH index are the following:*

- *branched covers of $\sigma_\infty^{\mathrm{T}'}$; and*
- *at most one component $\widetilde{v}$ of $\overline{v}_0''$ with $\mathrm{I}(\widetilde{v}) = -1$.*

Proof. — Similar to the proofs of Lemmas 3.7.4 and 3.7.5. □

The following lemma is analogous to Lemma 3.7.10.

Lemma **3.9.2.** — *If $m \gg 0$ and $\overline{u}_\infty \in \partial_{(-\infty,+\infty)}\mathcal{M}$, then the following hold:*

(1) *$\overline{u}_\infty$ has no fiber components;*
(2) *there is no level $\overline{v}_*$ such that $\overline{v}_*' \cap \overline{v}_*'' \neq \varnothing$ and $\overline{v}_*' \cap \overline{v}_*'' \subset int(\overline{v}_*')$; and*
(3) *$\overline{u}_\infty$ has no boundary point at z_∞.*

Proof. — (1) Arguing by contradiction, suppose that $\overline{u}_\infty$ has a fiber component $\widetilde{v}$. Then $n^*(\widetilde{v}) \geq m$ and $\widetilde{v}$ can be eliminated by arguing as in Lemma 3.7.10(1), in view of the following contributions towards I:

(γ_1) $\mathrm{I}(\overline{v}_0') = -p_0$.
(γ_2) $\cup_{j=1}^{c}\overline{v}_{-1,j}^{\sharp}$ is a union of p_0 trivial strips and $\sum_{j=1}^{c}\mathrm{I}(\overline{v}_{-1,j}^{\sharp}) = p_0$.
(γ_3) If $\widetilde{v}$ is a component of $\overline{v}_*$, then $\widetilde{v}$ and the intersection $\widetilde{v} \cap (\overline{v}_* - \widetilde{v})$ contribute $2g + 2 \geq 4$ towards I.

(γ_4) After removing any fiber components $\widetilde{v}$, all the levels $\neq \overline{v}_0$ have nonnegative ECH index and $I(\overline{v}_0'') \geq -1$ by Lemma 3.9.1.

The sum of the above ECH indices is at least 3, a contradiction.

(2), (3) Similar to (2) and (3) of Lemma 3.7.10. Observe that the sum of ECH indices of the levels was at least 4 in all the cases of Lemma 3.7.10 that were not ruled out by other means; in the present case the sum will be at least 3. (Note that the ECH indices added up to only 3 in (2C) of Lemma 3.7.10, but we do not have levels $\overline{v}_{0,j}$, $j = 1, \dots, b$, in the current situation.) ☐

Lemma **3.9.3.** — *If $\overline{u}_\infty \in \partial_{(-\infty,+\infty)}\mathcal{M}$ and $\overline{v}_0' = \varnothing$, then $\overline{u}_\infty$ is one of the following 2-level buildings:*

(1) $\overline{v}_{1,1}$ *with* $I = 1$ *from* $\mathbf{z}$ *to some* $\mathbf{z}'$ *consisting of*
   (i) *one thin strip and trivial strips or*
   (ii) *one nontrivial component of $\overline{v}_{1,1}''$ with image in* W *and trivial strips;*
   and $\overline{v}_0 = \overline{v}_0''$ with $I = 1$ and $n^* \leq m + |\mathcal{I}|$ from $\mathbf{z}'$ to $\mathbf{y}'$.

(2) $\overline{v}_0 = \overline{v}_0''$ *with $I = 1$ and $n^* \leq m + |\mathcal{I}|$ from $\mathbf{z}$ to some $\mathbf{y}''$; and $\overline{v}_{-1,1}$ with $I = 1$ from $\mathbf{y}''$ to $\mathbf{y}'$.*

Proof. — Suppose that $\overline{v}_0' = \varnothing$. Since passing through $\overline{\mathfrak{m}}(T')$ is a codimension two condition, we have $I(\overline{v}_0) \geq 1$. Hence there can be only one other nontrivial level — either $\overline{v}_{1,1}$ or $\overline{v}_{-1,1}$ — and $I(\overline{v}_{\pm 1,1}) \geq 1$. By Lemma 3.9.1, $I(\overline{v}_0) = 1$ and $I(\overline{v}_{\pm 1,1}) = 1$. Moreover, $n^*(\overline{v}_{\pm 1,1}) \leq |\mathcal{I}|$ since $n^*(\overline{v}_0) \geq m$. This means that there are no components $\overline{v}_*^\sharp$ that limit to z_∞ at the negative end and all the positive ends $\mathcal{E}_+$ that limit to z_∞ satisfy $n^*(\mathcal{E}_+) = 1$. The lemma follows. ☐

Lemma **3.9.4.** — *For each interval $[-T, T]$, there exists $m \gg 0$ such that there is no sequence of curves $\overline{u}_i \in \mathcal{M}_{\tau_i}$, $\tau_i \to T' \in [-T, T]$, that limits to $\overline{u}_\infty$ for which $\overline{v}_0' \neq \varnothing$.*

Proof. — This is similar to Lemma 3.6.5 and will be omitted. ☐

Proof of Lemma 3.3.12. — This is argued in the same way as Lemma 3.3.8. Suppose $\overline{u}_\infty \in \partial_{(-\infty,+\infty)}\mathcal{M}$. If $\overline{v}_0' = \varnothing$, then Lemma 3.9.3 implies that $\overline{u}_\infty$ is as described in Lemma 3.3.12. On the other hand, by Lemma 3.9.4, for any $T > 0$ there exists $m \gg 0$ such that $\overline{v}_0' = \varnothing$ in $\overline{u}_\infty \in \partial_{[-T,+T]}\mathcal{M}$. The possibilities where $m_i \to \infty$, $T_i \to \infty$, $\overline{u}_{ij} \to \overline{u}_{i\infty} \in \partial_{\{\pm T_i\}}\mathcal{M}$ are treated in the same way as in Lemmas 3.7.14, 3.8.3, and 3.8.11. ☐

4. Homotopy of cobordisms II

In this section we consider the homotopy of cobordisms corresponding to $\Phi \circ \Psi$. Many of the constructions for $\overline{W}_\tau^\mp$ are similar to those of $\overline{W}_\tau^\pm$ with minor modifications.

We first give a brief description of $\overline{W}_\tau = \overline{W}_\tau^\mp$. (If $\mp$ is understood, as it will be in the rest of this section, then it will be omitted.) The base B_τ of $\overline{W}_\tau$ is biholomorphic to an infinite cylinder with a disk removed. As $\tau \to +\infty$, $\overline{W}_\tau$ degenerates to the stacking of $\overline{W}_-$ "on top of" $\overline{W}_+$. On the other hand, as $\tau \to -\infty$, $\overline{W}_\tau$ degenerates to $\overline{W}_{-\infty}$, whose base $B_{-\infty}$ is given by:

$$B_{-\infty} = ((\mathbf{R} \times (\mathbf{R}/2\mathbf{Z})) \sqcup E) / \sim, \tag{4.0.1}$$

where $E = \{|z| \le 1\} \subset \mathbf{C}$ and $\sim$ identifies $(0, \frac{3}{2}) \in \mathbf{R} \times (\mathbf{R}/2\mathbf{Z})$ with $0 \in E$.

The degeneration for $\tau \to -\infty$ can be described in an equivalent way as a neck-stretching along a stable Hamiltonian hypersurface which is the preimage of a boundary-parallel loop in the base.

4.1. *Construction of the homotopy of cobordisms for $\Phi \circ \Psi$.*

4.1.1. *Definition of the family $\overline{W}_\tau$.* — Let $l \in (0, \infty)$ and $r \in (0, 1]$. Consider the fibration

$$\pi : \mathbf{R} \times \overline{N} \to \mathbf{R} \times (\mathbf{R}/2\mathbf{Z}),$$

where $\overline{N}$ is viewed as $(\overline{S} \times [0, 2])/(x, 2) \sim (\overline{h}(x), 0)$ and (s, t) are coordinates on $\mathbf{R} \times (\mathbf{R}/2\mathbf{Z})$. We define $\overline{W}_{l,r} = \pi^{-1}(B_{l,r})$, where the base $B_{l,r}$ is obtained by smoothing the corners of

$$(\mathbf{R} \times (\mathbf{R}/2\mathbf{Z})) - ((-l/2, l/2) \times ((3 - r)/2, (3 + r)/2)).$$

Next choose a function

$$\eta = (l, r) : \mathbf{R} \to (0, \infty) \times (0, 1], \tag{4.1.1}$$

which is obtained by smoothing

$$\eta_0(\tau) = \begin{cases} (\tau + 1, 1), & \text{for } \tau \ge 0; \\ (e^\tau, e^\tau), & \text{for } \tau \le 0; \end{cases}$$

near $\tau = 0$.

We then define $\overline{W}_\tau = \overline{W}_{\eta(\tau)}$ and $B_\tau = B_{\eta(\tau)}$. Let $\pi_{B_\tau} : \overline{W}_\tau \to B_\tau$ be the projection along $\{(s, t)\} \times \overline{S}$.

As $\tau \to +\infty$, the cobordism $\overline{W}_\tau$ approaches the concatenation of $\overline{W}_-$ and $\overline{W}_+$. See Figure 17. On the other hand, as $\tau \to -\infty$, the cobordism $\overline{W}_\tau$ can be viewed as degenerating to $\overline{W}_{-\infty} = (\overline{W}_{-\infty,1} \sqcup \overline{W}_{-\infty,2})/\sim$, which we describe now: Consider the base $B_{-\infty} = (B_{-\infty,1} \sqcup B_{-\infty,2})/\sim$, where $B_{-\infty,1} = \mathbf{R} \times (\mathbf{R}/2\mathbf{Z})$, $B_{-\infty,2} = E = \{|z| \le 1\}$, and $\sim$ identifies $(0, \frac{3}{2}) \in B_{-\infty,1}$ with $0 \in B_{-\infty,2}$. We also identify the asymptotic markers ∂_s at $(0, \frac{3}{2})$ and $\{x = 0, y > 0\}$ at 0. We then set $\overline{W}_{-\infty,1} = \pi^{-1}(B_{-\infty,1})$ and $\overline{W}_{-\infty,2} = B_{-\infty,2} \times \overline{S}$,

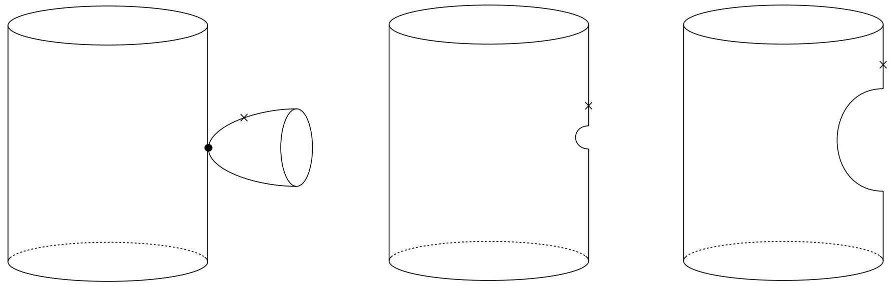


FIG. 17. — The bases of the family $\overline{W}^{\mp}_{\tau}$. The leftmost diagram represents $B_{-\infty}$, i.e., $\tau = -\infty$, and the parameter τ increases as we go to the right. The location of $\overline{\mathfrak{m}}^b(\tau)$ is indicated by $\times$

and identify $(0, \frac{3}{2}, x) \sim (0, x)$, where $(0, \frac{3}{2}, x) \in \overline{W}_{-\infty,1}$ and $(0, x) \in \overline{W}_{-\infty,2}$. We write $\pi_{B_{-\infty,i}} : \overline{W}_{-\infty,i} \to B_{-\infty,i}$ for the projection along $\overline{S}$.

The alternate description of the degeneration as $\tau \to -\infty$ mentioned at the beginning of the section relies on the conformal equivalence between annuli $\{e^{\tau} \leq |z| \leq 1\} \subset \mathbf{C}$ and finite cylinders $[\tau, 0] \times S^1$ for $\tau < 0$. The details are left to the reader.

4.1.2. *Marked points.* — We choose a 1-parameter family of marked points

$$\overline{\mathfrak{m}}(\tau) = (\overline{\mathfrak{m}}^b(\tau), \overline{\mathfrak{m}}^f(\tau)) = (\overline{\mathfrak{m}}^b(\tau), z_\infty) \in \overline{W}_\tau$$

for $\tau \in \mathbf{R}$, such that the following hold:

(i) $\overline{\mathfrak{m}}^b(\tau)$ is on the segment $\{s > \frac{l(\tau)}{2}, t = \frac{3}{2}\}$;
(ii) as $\tau \to +\infty$, $\overline{\mathfrak{m}}(\tau)$ limits to $\overline{\mathfrak{m}}(+\infty) = (\overline{\mathfrak{m}}^b(+\infty), \overline{\mathfrak{m}}^f(+\infty))$, where $\overline{\mathfrak{m}}^b(+\infty) = (0, \frac{3}{2}) \in B_-$ and $\overline{\mathfrak{m}}^f(+\infty) = z_\infty$;
(iii) as $\tau \to -\infty$, $\overline{\mathfrak{m}}(\tau)$ limits to $\overline{\mathfrak{m}}(-\infty) = (\overline{\mathfrak{m}}^b(-\infty), \overline{\mathfrak{m}}^f(-\infty))$, where $\overline{\mathfrak{m}}^b(-\infty) = \frac{i}{2} \in B_{-\infty,2}$ and $\overline{\mathfrak{m}}^f(-\infty) = z_\infty$.

Remark **4.1.1.** — Here is a slightly more detailed description of the convergence of the marked points: For $\tau \gg 0$ we take $\overline{\mathfrak{m}}^b(\tau) = \left(\frac{\tau+5}{2}, 0\right)$. For $\tau \ll 0$ we define sets $Q_\tau = B_\tau - B_{-\sqrt{|\tau|}}$ and maps

$$\mathfrak{r}_\tau : Q_\tau \to \mathbf{R}^2, \quad \mathfrak{r}_\tau(s,t) = 2e^{-\tau}\left(s, t - \tfrac{3}{2}\right).$$

The images $\mathfrak{r}_\tau(Q_\tau)$ form an exhaustion of a set $Q_{-\infty}$ obtained by smoothing the boundary of $\mathbf{R}^2 - ((-1,1) \times (-1,1))$. Let $\mathfrak{u} : D^2 - \{0\} \to Q_{-\infty}$ be a diffeomorphism which is holomorphic in the interior. By smoothing the boundary of $\mathbf{R}^2 - ((-1,1) \times (-1,1))$ symmetrically, we choose $\mathfrak{u}$ so that the imaginary axis is mapped to the axis $\{t = 0\}$. Let $\mathfrak{u}\left(\frac{i}{2}\right) = (s_{-\infty}, 0)$. Then for $\tau \ll 0$ we take $\overline{\mathfrak{m}}^b(\tau) = \mathfrak{r}_\tau^{-1}(s_{-\infty}, 0) = \left(\frac{e^\tau}{2} s_{-\infty}, \frac{3}{2}\right)$.

4.1.3. *Stable Hamiltonian structures and symplectic forms.* — We first consider $\overline{W}_\tau$. The stable Hamiltonian structure on $\overline{N}=(\overline{S}\times[0,2])/\sim$ is obtained from $(dt,\overline{\omega})$ on $\overline{S}\times[0,2]$ by passing to the quotient. The 2-plane field is $\xi_\tau=T\overline{S}$ and the Hamiltonian vector field is $R_\tau=\partial_t$. The symplectic form $\overline{\Omega}_\tau$ is the restriction to $\overline{W}_\tau$ of $ds\wedge dt+\overline{\omega}$, defined on $\mathbf{R}\times\overline{S}\times[0,2]$ and sent to the quotient $\mathbf{R}\times\overline{N}$.

Next we consider $\overline{W}_{-\infty}$. For $\overline{W}_{-\infty,1}=\overline{N}$ the Hamiltonian structure is $(dt,\overline{\omega})$ and the symplectic form is $\overline{\Omega}_{-\infty,1}=ds\wedge dt+\overline{\omega}$. The symplectic form on $\overline{W}_{-\infty,2}=B_{-\infty,2}\times\overline{S}$ is a product symplectic form $\overline{\Omega}_{-\infty,2}=\omega_{-\infty,2}\oplus\overline{\omega}$, where $\omega_{-\infty,2}$ is the standard area form on the unit disk.

4.2. *Holomorphic curves and moduli spaces.*

4.2.1. *Lagrangian boundary conditions.* — Consider the cobordism $(\overline{W}_\tau,\overline{\Omega}_\tau)$. We place a copy of $\overline{\mathbf{a}}$ on $\pi_{B_\tau}^{-1}(\frac{l(\tau)}{2},\frac{3}{2})$ and parallel transport along the vertical boundary $\partial_v\overline{W}_\tau:=\pi^{-1}(\partial B_\tau)$ using the symplectic connection $\overline{\Omega}_\tau$ to obtain $L^\tau_{\overline{\mathbf{a}}}$. Similarly, we place a copy of $\overline{\mathbf{a}}$ over $1\in B_{-\infty,2}$ and parallel transport along $\partial B_{-\infty,2}$ to obtain $L^{-\infty,2}_{\overline{\mathbf{a}}}=\partial B_{-\infty,2}\times\overline{\mathbf{a}}$. The Lagrangian submanifolds $L^\tau_{\widehat{\mathbf{a}}}$, $L^\tau_{\overline{a}_i}$, $L^\tau_{\widehat{a}_i}$ are defined similarly. Note that the Lagrangians close up because the monodromy of the fibration $\pi_{B_\tau}:\overline{W}_\tau\to B_\tau$ is trivial along ∂B_τ.

4.2.2. *Almost complex structures.*

Definition **4.2.1.**

(1) *An almost complex structure* $\overline{J}_\tau$ *on* $\overline{W}_\tau$ *is* admissible *if* $\overline{J}_\tau$ *is the restriction of some* $\overline{J}'\in\mathcal{J}_{\overline{W}'}$. *In this case* $\overline{J}_\tau$ *is said to be* compatible *with* $\overline{J}'$.

(2) *An almost complex structure* $\overline{J}_{-\infty,1}$ *on* $\mathbf{R}\times\overline{N}$ *is* admissible *if* $\overline{J}_{-\infty,1}\in\mathcal{J}_{\overline{W}'}$.

(3) *An almost complex structure* $\overline{J}_{-\infty,2}$ *on* $B_{-\infty,2}\times\overline{S}$ *is* admissible *if it is a product complex structure.*

The space of admissible $\overline{J}_\tau$, $\overline{J}_{-\infty,1}$ *and* $\overline{J}_{-\infty,2}$ *will be denoted by* $\mathcal{J}_{\overline{W}_\tau}$, $\mathcal{J}_{\overline{W}_{-\infty,1}}$ *and* $\mathcal{J}_{\overline{W}_{-\infty,2}}$, *respectively.*

Definition **4.2.2.** — *A family* $\{\overline{J}_\tau\in\overline{\mathcal{J}}_\tau\}_{\tau\in\mathbf{R}}$ *of almost complex structures is* admissible *if there exist* $\overline{J}'\in\mathcal{J}_{\overline{W}'}$, $\overline{J}\in\mathcal{J}_{\overline{W}}$, $\overline{J}_+\in\mathcal{J}_{\overline{W}_+}$, $\overline{J}_-\in\mathcal{J}_{\overline{W}_-}$, $\overline{J}_{-\infty,1}\in\mathcal{J}_{\overline{W}_{-\infty,1}}$ *and* $\overline{J}_{-\infty,2}\in\mathcal{J}_{\overline{W}_{-\infty,2}}$ *such that the following hold:*

(1) $\overline{J}_\tau$ *converges to* $(\overline{J}_{-\infty,1},\overline{J}_{-\infty,2})$ *as* $\tau\to-\infty$;

(2) $\overline{J}_\tau$ *converges to* $(\overline{J}_+,\overline{J}_-)$ *as* $\tau\to+\infty$;

(3) $\overline{J}_+$ *and* $\overline{J}_-$ *are compatible with* $\overline{J}'$ *and* $\overline{J}$; *and*

(4) $\overline{J}_\tau$ *is compatible with* $\overline{J}'=\overline{J}_{-\infty,1}$.

For $\tau\to+\infty$ *the convergence of almost complex structures is to be understood in the sense of neck-stretching as in [BEHWZ, Section 3.4]. For* $\tau\to-\infty$ *we gave two equivalent descriptions of the* *limit: one as nodal degeneration and one as a neck-stretching. If the limit is viewed as a neck-stretching, then the convergence of the almost complex structures is again as in [BEHWZ, Section 3.4]. If the limit is viewed as a nodal degeneration, then* $\overline{J}_{-\infty,1}$ *is the* C^∞_{loc} *limit of* $\overline{J}_\tau$ *and* $\overline{J}_{-\infty,1}$ *is the pullback by the uniformization map* $\mathfrak{u}$ *of the* C^∞_{loc} *limit of the restriction of* $\overline{J}_\tau$ *to* Q_τ *after applying the rescaling map* $\mathfrak{r}_\tau$*; see Remark 4.1.1 for the definitions of* $\mathfrak{u}$ *and* $\mathfrak{r}_\tau$*. Strictly speaking, the limit produces almost complex structures on* $\overline{W}_{-\infty,1} - \pi^{-1}((0,\frac{3}{2}))$ *and* $\overline{W}_{-\infty,2} - (\{0\}\times \overline{S})$*, but they can be extended over the missing points. The space of all admissible* $\{\overline{J}_\tau \in \overline{\mathcal{J}}_\tau\}_{\tau\in\mathbf{R}}$ *will be denoted by* $\overline{\mathcal{I}}$*.*

4.2.3. *Notation and conventions.* — We will be using the notation and conventions from Section 3.2.3, with the exception of intersection numbers $n^*(\overline{u})$.

Intersection numbers. Let δ_{ρ_0,ϕ_0} be a closed orbit of the Hamiltonian vector field ∂_t which lies on the torus $\{\rho=\rho_0\}\subset \overline{S}\times[0,2]/\sim$ for $\rho_0>0$ sufficiently small and which passes through the point $(t,\rho,\phi)=(0,\rho_0,\phi_0)$. We assume additionally that δ_{ρ_0,ϕ_0} does not intersect the projections of the Lagrangians of $\overline{W}_\tau$, $\overline{W}_+$ and $\overline{W}_-$ to $\overline{N}$.

We then write $(\sigma^*_\infty)^\dagger$ for the restriction of $\mathbf{R}\times\delta_{\rho_0,\phi_0}$ to $\overline{W}_*$, where $*=\varnothing,',\tau,+,-$ or $(-\infty,1)$. For $\overline{W}_{-\infty,2}$ we write

$$(\sigma^{-\infty,2}_\infty)^\dagger = B_{-\infty,2}\times\{\rho=\rho_0,\phi=\phi_0+3\pi/2m+2\pi k/m, k\in\mathbf{Z}\}.$$

Finally we define $n^*(\overline{u})=\langle\overline{u},(\sigma^*_\infty)^\dagger\rangle$, where $*=\varnothing,',\tau,+,-$ or $(-\infty,1)$.

4.2.4. *Holomorphic maps to* $\overline{W}_\tau$*.* — Let (F,j) be a compact Riemann surface, possibly disconnected, with $2g$ boundary components and two sets of interior punctures $\mathbf{p}^+=\{p_1^+,\dots,p_{k_+}^+\}$ and $\mathbf{p}^-=\{p_1^-,\dots,p_{k_-}^-\}$ such that each component of F has nonempty boundary, at least one puncture from $\mathbf{p}^+$, and at least one puncture from $\mathbf{p}^-$. We write $\dot{F}=F-\mathbf{p}^+-\mathbf{p}^-$.[14]

Definition **4.2.3.** — *Let* $\overline{J}_\tau\in\mathcal{J}_{\overline{W}_\tau}$ *and let* $\boldsymbol{\gamma}=\prod\gamma_i^{m_i}$, $\boldsymbol{\gamma}'=\prod(\gamma_i')^{m_i'}\in\widehat{\mathcal{O}}_k$.
A degree k multisection of $(\overline{W}_\tau,\overline{J}_\tau)$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$ *is a holomorphic map*

$$\overline{u}:(\dot{F},j)\to(\overline{W}_\tau,\overline{J}_\tau)$$

which is a degree k *multisection of* $\pi_{B_\tau}:\overline{W}_\tau\to B_\tau$ *and which additionally satisfies the following:*

(1) $\overline{u}(\partial F)\subset L^\tau_{\hat{\mathbf{a}}}$;
(2) $\overline{u}$ *maps each connected component of* ∂F *to a different* $L^\tau_{\hat{a}_i}$;
(3) $\lim_{w\to p_i^+}\pi_{\mathbf{R}}\circ\overline{u}(w)=+\infty$ *and* $\lim_{w\to p_i^-}\pi_{\mathbf{R}}\circ\overline{u}(w)=-\infty$;
(4) $\overline{u}$ *converges to a cylinder over a multiple of some* γ_j *near each puncture* p_i^+ *so that the total multiplicity of* γ_j *over all the* p_i^+*'s is* m_j *(and similarly for* p_i^-*).*

Here $\pi_{\mathbf{R}}:\overline{W}_\tau\to\mathbf{R}$ *is the projection to the s-coordinate.*

[14] Observe that $\dot{F}$ has no boundary punctures.

A $(\overline{W}_\tau,\overline{J}_\tau)$-curve from $\boldsymbol\gamma$ to $\boldsymbol\gamma'$ *is a degree* $2g$ *multisection of* $(\overline{W}_\tau,\overline{J}_\tau)$ *satisfying* $n^*(\overline{u})=m$.

Let $\mathcal{M}_{\overline{J}_\tau}(\boldsymbol\gamma,\boldsymbol\gamma')$ be the moduli space of degree k multisections of $(\overline{W}_\tau,\overline{J}_\tau)$ from $\boldsymbol\gamma$ to $\boldsymbol\gamma'$ and let $\mathcal{M}_{\overline{J}_\tau}(\boldsymbol\gamma,\boldsymbol\gamma';\overline{\mathfrak{m}}(\tau))$ be the moduli space of curves with a point constraint $\overline{\mathfrak{m}}(\tau)$; as usual, they are called moduli spaces of curves *passing through* $\overline{\mathfrak{m}}(\tau)$. Topological considerations based on the intersection numbers n^* imply that the forgetful map

$$\mathcal{M}^{n^*=m}_{\overline{J}_\tau}(\boldsymbol\gamma,\boldsymbol\gamma';\overline{\mathfrak{m}}(\tau))\to\mathcal{M}^{n^*=m}_{\overline{J}_\tau}(\boldsymbol\gamma,\boldsymbol\gamma')$$

is injective.

We write

$$\mathcal{M}_{\{\overline{J}_\tau\}}(\boldsymbol\gamma,\boldsymbol\gamma')=\{(\tau,u)\mid\tau\in\mathbf{R},u\in\mathcal{M}_{\overline{J}_\tau}(\boldsymbol\gamma,\boldsymbol\gamma')\},$$
$$\mathcal{M}_{\{\overline{J}_\tau\}}(\boldsymbol\gamma,\boldsymbol\gamma';\overline{\mathfrak{m}})=\{(\tau,u)\mid\tau\in\mathbf{R},u\in\mathcal{M}_{\overline{J}_\tau}(\boldsymbol\gamma,\boldsymbol\gamma';\overline{\mathfrak{m}}(\tau))\}.$$

4.2.5. *Holomorphic maps to* $\overline{W}_{-\infty}$. — We now describe the curves in $\overline{W}_{-\infty,i}$.

Definition **4.2.4.** — *Let* $\overline{J}_{-\infty,1}\in\mathcal{J}_{\overline{W}_{-\infty,1}}$ *and let* $\boldsymbol\gamma=\prod\gamma_i^{m_i}$, $\boldsymbol\gamma'=\prod(\gamma_i')^{m_i'}\in\widehat{\mathcal{O}}_k$. *A* degree k multisection of $(\overline{W}_{-\infty,1},\overline{J}_{-\infty,1})$ from $\boldsymbol\gamma$ to $\boldsymbol\gamma'$ *is a holomorphic map*

$$\overline{u}:(\dot{F},j)\to(\overline{W}_{-\infty,1},\overline{J}_{-\infty,1})$$

which is a degree k multisection of $\pi_{B_{-\infty,1}}:\overline{W}_{-\infty,1}\to B_{-\infty,1}$ *and which is asymptotic to* $\boldsymbol\gamma$ *and* $\boldsymbol\gamma'$ *at the positive and negative ends. Here* $(\dot{F},j)$ *is a punctured Riemann surface. A* $(\overline{W}_{-\infty,1},\overline{J}_{-\infty,1})$-curve from $\boldsymbol\gamma$ to $\boldsymbol\gamma'$ *is a degree* $2g$ *multisection of* $(\overline{W}_{-\infty,1},\overline{J}_{-\infty,1})$ *satisfying* $n^*(\overline{u})=0$.

Definition **4.2.5.** — *Let* $\overline{J}_{-\infty,2}\in\mathcal{J}_{\overline{W}_{-\infty,2}}$. *A* degree k multisection of $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$ *is holomorphic map*

$$\overline{u}:(F,j)\to(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$$

which is a degree k multisection of $\pi_{B_{-\infty,2}}:\overline{W}_{-\infty,2}\to B_{-\infty,2}$ *and which additionally satisfies the following:*

(1) (F,j) *is a compact Riemann surface with k boundary components;*
(2) $\overline{u}$ *maps each component of* ∂F *to a different* $L^{-\infty,2}_{\widehat{a}_i}$.

A $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$-curve *is a degree* $2g$ *multisection of* $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$ *satisfying* $n^*(\overline{u})=m$. *A* degenerate $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$-curve *consists of* $2g$ *copies of* $B_{-\infty,2}\times\{pt\}$ *and a singular fiber* $\{0\}\times\overline{S}$.

Let $\mathcal{M}_{\overline{J}_{-\infty,1}}(\boldsymbol\gamma,\boldsymbol\gamma')$ be the moduli space of degree $2g$ multisections of $(\overline{W}_{-\infty,1},\overline{J}_{-\infty,1})$ from $\boldsymbol\gamma$ to $\boldsymbol\gamma'$, let $\mathcal{M}_{\overline{J}_{-\infty,2}}$ be the moduli space of degree $2g$ multisections of $(\overline{W}_{-\infty,2},\overline{J}_{-\infty,2})$, and let $\mathcal{M}_{\bar{J}_{-\infty,2}}(\overline{\mathfrak{m}}(-\infty)) \subset \mathcal{M}_{\bar{J}_{-\infty,2}}$ be the subset consisting of multisections satisfying a point constraint at $\overline{\mathfrak{m}}(-\infty)$.

We define the evaluation maps

$$ev_{\overline{S},1} : \mathcal{M}_{\bar{J}_{-\infty,1}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}') \to \mathrm{Sym}^{2g}(\overline{S}), \quad ev_{\overline{S},2} : \mathcal{M}_{\bar{J}_{-\infty,2}} \to \mathrm{Sym}^{2g}(\overline{S})$$

as in Definition 2.1.1, by intersecting the holomorphic curves in $\mathcal{M}_{\bar{J}_{-\infty,1}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ (resp. in $\mathcal{M}_{\bar{J}_{-\infty,2}}$) with the fiber of the fibration $\overline{W}_{-\infty,1} \to \mathbf{R} \times \mathbf{R}/2\mathbf{Z}$ over $(0, \frac{3}{2})$ which we identify with $\overline{S}$ (resp. with the fiber $\{0\} \times \overline{S}$). Given $\mathfrak{z} \in \mathrm{Sym}^{2g}(\overline{S})$, we define

$$\begin{aligned}
&\mathcal{M}_{\bar{J}_{-\infty,1}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathfrak{z}) = ev_{\overline{S},1}^{-1}(\mathfrak{z}) \subset \mathcal{M}_{\bar{J}_{-\infty,1}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}'),\\
&\mathcal{M}_{\bar{J}_{-\infty,2}}(\mathfrak{z}) = ev_{\overline{S},2}^{-1}(\mathfrak{z}) \subset \mathcal{M}_{\bar{J}_{-\infty,2}},\\
&\mathcal{M}_{\bar{J}_{-\infty,2}}(\mathfrak{z}, \overline{\mathfrak{m}}(-\infty)) = \mathcal{M}_{\bar{J}_{-\infty,2}}(\overline{\mathfrak{m}}(-\infty)) \cap \mathcal{M}_{\bar{J}_{-\infty,2}}(\mathfrak{z}).
\end{aligned}$$

In order to highlight the special role of the point z_∞ we will write

$$\mathfrak{z} = \zeta_0^{r_0} \zeta_1^{r_1} \dots \zeta_l^{r_l},$$

where $r_0 \in \mathbf{Z}^{\geq 0}$, $r_1 \dots, r_l \in \mathbf{Z}^{>0}$, $r_0 + \dots + r_l = 2g$, $\zeta_0 = z_\infty$, and $\zeta_1, \dots, \zeta_l \in \overline{S} - \{z_\infty\}$. Also, for sake of brevity, we will denote $\mathfrak{Z} = \mathrm{Sym}^{2g}(\overline{S})$.

4.2.6. *Indices.* — We now briefly discuss the Fredholm index $\mathrm{ind}(\overline{u})$ and the ECH index $I(\overline{u})$ of a $\overline{W}_\tau$-curve $\overline{u} : \dot{F} \to \overline{W}_\tau$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$. This is similar to Section 3.2.6.

Let $\check{\overline{W}}_\tau = \overline{W}_\tau - \{s > \frac{l(\tau)}{2} + 1\} - \{s < -\frac{l(\tau)}{2} - 1\}$, where $l(\tau)$ is given in Equation (4.1.1). Let $\check{\overline{u}} : \check{F} \to \check{\overline{W}}_\tau$ be the compactification of $\overline{u}$, where $\check{F}$ is obtained by performing a real blow-up of F at its boundary punctures.

The trivialization $\tau^\star$ of $T\overline{S}$ along $L^\tau_{\hat{\mathbf{a}}}$ is defined as in Section I.4.4.2: We pick a point $p \in \partial B_\tau$, define $\tau^\star$ of $T\overline{S}|_{\pi_{B_\tau}^{-1}(p)}$ along $\widehat{\mathbf{a}}$ by choosing a nonsingular tangent vector field along $\widehat{\mathbf{a}}$, and then parallel transport $\tau^\star$ along $\partial \overline{W}_\tau$. We also choose $\tau^\star$ along $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$.

Let $Q_{\tau^\star}(\check{\overline{u}})$ be the relative intersection form given by normalizing $\check{\overline{u}}$ near $s = \pm(\frac{l(\tau)}{2} + 1)$ as in [Hu1, Definition 2.4] and intersecting $\check{\overline{u}}$ and a pushoff of $\check{\overline{u}}$. Here the pushoff along $\partial \check{\overline{W}}_\tau$ is in the direction of $J\tau^\star$. Then

$$(\mathbf{4.2.1}) \qquad I(\overline{u}) = c_1(\check{\overline{u}}^* T\overline{S}, \tau^*) + Q_{\tau^\star}(\check{\overline{u}}) + \widetilde{\mu}_{\tau^\star}(\boldsymbol{\gamma}) - \widetilde{\mu}_{\tau^\star}(\boldsymbol{\gamma}') - 2g,$$

$$(\mathbf{4.2.2}) \qquad \mathrm{ind}(\overline{u}) = -\chi(\dot{F}) + 2c_1(\check{\overline{u}}^* T\overline{S}, \tau^\star) + \mu_{\tau^\star}(\boldsymbol{\gamma}) - \mu_{\tau^\star}(\boldsymbol{\gamma}') - 4g,$$

These index formulas are obtained by adding the index formulas for holomorphic curves in $\overline{W}_\pm$ of [I]. In deriving Equation (4.2.2) from Propositions I.5.5.2 and I.5.5.5 we should observe that every term in the Fredholm index formula is additive except for the Euler characteristic: if a surface $\dot{F}$ is obtained by gluing surfaces $\dot{F}_+$ and $\dot{F}_-$ along $2g$ strip-like ends, then $\chi(\dot{F}_+) + \chi(\dot{F}_-) = \chi(\dot{F}) + 2g$. The index inequality holds as usual:

$$\text{(4.2.3)} \qquad \operatorname{ind}(\overline{u}) + 2\delta(\overline{u}) \le I(\overline{u}),$$

where $\delta(\overline{u}) \ge 0$ and equals zero if and only if $\overline{u}$ is an embedding. We also have

$$\text{(4.2.4)} \qquad I(\sigma_\infty^\tau) = \operatorname{ind}(\sigma_\infty^\tau) = -1.$$

The Fredholm and ECH indices for $\overline{W}_{-\infty,i}$-curves can be defined and computed similarly.

Remark **4.2.6.** — The Fredholm and ECH indices for $\overline{W}_\tau$ and $\overline{W}_{-\infty,i}$ do not take into account the point constraint $\overline{\mathfrak{m}}(\tau)$ and the condition "passing through $\overline{\mathfrak{m}}(\tau)$" is a codimension 2 condition. Moreover, the indices for $\overline{W}_{-\infty,i}$, $i = 1, 2$, do not take into account the constraints $\mathfrak{z}$.

We now have the following:

Lemma **4.2.7.** — *Let $\overline{u}$ be a $\overline{W}_{-\infty,2}$-curve. Then*

$$[\overline{u}] = 2g[B_{-\infty,2} \times \{pt\}] + [\{pt\} \times \overline{S}] \in H_2(\overline{W}_{-\infty,2}, \partial B_{-\infty,2} \times \overline{S}),$$

$\overline{u}$ consists of $0 \le k < 2g$ copies of $B_{-\infty,2} \times \{pt\}$ together with an irreducible component in the class $(2g-k)[B_{-\infty,2} \times \{pt\}] + [\{pt\} \times \overline{S}]$, and $I(\overline{u}) = 4g + 2$.

Proof. — Let $\pi_{\overline{S}} : \overline{W}_{-\infty,2} \to \overline{S}$ be the projection along $B_{-\infty,2}$. Since $\overline{J}_{-\infty,2}$ is a split complex structure, $\pi_{\overline{S}}$ is a holomorphic map. Hence if $\overline{u}$ is a $\overline{W}_{-\infty,2}$-curve, then $\pi_{\overline{S}} \circ \overline{u}$ either maps to a point on some $\widehat{a}_i$ or to all of $\overline{S}$. The lemma follows by listing all the possibilities. □

We also have the following, which is stated without proof.

Lemma **4.2.8.** — *Let $\overline{u}$ be a degree $2g$ multisection of $(\overline{W}_{-\infty,2}, \overline{J}_{-\infty,2})$ satisfying $n^*(\overline{u}) = 0$. Then $\overline{u} = \overline{u}' \cup \overline{u}''$ consists of $\overline{u}'$ which is a degree k cover of $\sigma_\infty^{-\infty,2}$ and $\overline{u}''$ which is the union of $2g-k$ copies of $B_{-\infty,2} \times \{pt\}$, and $I(\overline{u}) = 2g$.*

4.2.7. *Regularity.*

Definition **4.2.9.** — *The family $\{\overline{J}_\tau\}_{\tau\in\mathbf{R}} \in \overline{\mathcal{I}}$ is* regular *if:*

(1) *$\mathcal{M}^\dagger_{\{\overline{J}_\tau\}}(\delta_0^p\boldsymbol{\gamma}, \delta_0^q\boldsymbol{\gamma}')$ is transversely cut out for all $\delta_0^p\boldsymbol{\gamma}, \delta_0^q\boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_k$, $k \le 2g$;*
(2) *the restriction $\overline{J}'$ of $\overline{J}_\tau$ to the positive and negative ends (the restriction is independent of τ) is regular; and*

(3) $\bar{J}_-$ *and* $\bar{J}_+$ *in the limit* $\tau \to +\infty$ *are regular.*

Let $\overline{\mathcal{I}}^{reg}$ be the space of regular $\{\bar{J}_\tau\} \in \overline{\mathcal{I}}$. As usual we have:

Lemma **4.2.10.** — *The generic* $\{\bar{J}_\tau\} \in \overline{\mathcal{I}}$ *is regular.*

We also introduce the perturbations of $\{\bar{J}_\tau\}$ to ensure that passing through $\overline{\mathfrak{m}}(\tau)$ is a generic condition. Let $\mathfrak{p}(\tau) \in B_\tau$ be a family of points such that:

$$\lim_{\tau\to+\infty} \mathfrak{p}(\tau) = \mathfrak{p}(+\infty) \in B_-, \qquad \lim_{\tau\to-\infty} \mathfrak{p}(\tau) = \mathfrak{p}(-\infty) \in B_{-\infty,1}$$

and $\mathfrak{p}(\tau) \neq \overline{\mathfrak{m}}^b(\tau)$ for all $\tau \in [-\infty, \infty]$. We then define the families $\{U_\tau = U_{\varepsilon,\delta,\mathfrak{p}(\tau)}\}$, $\{K_\tau = K_{\mathfrak{p}(\tau),\delta}\}$, and $\{\bar{J}_\tau^\diamond = \bar{J}_\tau^\diamond(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ as in Section 3.2.7. Also, the modifier $\{K_\tau\}$ means "passing through K_τ for an appropriate τ."

Definition **4.2.11.** — *The family* $\{\bar{J}_\tau^\diamond\}$ *is* $\{K_\tau\}$-regular with respect to $\overline{\mathfrak{m}}$ *if the moduli spaces* $\mathcal{M}^{\dagger,\{K_\tau\}}_{\{\bar{J}_\tau\}}(\delta_0^p\boldsymbol{\gamma}, \delta_0^q\boldsymbol{\gamma}'; \overline{\mathfrak{m}})$ *are transversely cut out.*

The following lemmas are analogous to Lemmas 3.2.21 and 3.2.22:

Lemma **4.2.12.** — *A generic* $\{\bar{J}_\tau^\diamond\}_{\tau\in\mathbf{R}}$ *is* $\{K_\tau\}$*-regular with respect to* $\overline{\mathfrak{m}}$.

Lemma **4.2.13.** — *If* $\{\bar{J}_\tau\}$ *is a generic family, then for* $\varepsilon, \delta > 0$ *sufficiently small, there exist a generic family* $\{\bar{J}_\tau^\diamond(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ *which is* $\{K_{\mathfrak{p}(\tau),\delta}\}$*-regular with respect to* $\overline{\mathfrak{m}}$ *and disjoint finite subsets* $\mathcal{T}_1, \mathcal{T}_2 \subset \mathbf{R}$ *with the following properties:*

(1) $\tau \in \mathcal{T}_1$ *if and only if there exists* $\overline{v}_\tau \in \mathcal{M}^{\dagger,s,irr,\mathrm{ind}=-1}_{\bar{J}_\tau}(\delta_0^p\boldsymbol{\gamma}, \delta_0^q\boldsymbol{\gamma}')$ *for some* $\delta_0^p\boldsymbol{\gamma}$ *and* $\delta^q\boldsymbol{\gamma}'$.

(2) $\tau \in \mathcal{T}_2$ *if and only if there exists* $\overline{v}_\tau \in \mathcal{M}^{\dagger,s,irr,\{K_\tau\},\mathrm{ind}=1}_{\bar{J}_\tau^\diamond(\varepsilon,\delta,\mathfrak{p}(\tau))}(\delta_0^p\boldsymbol{\gamma}, \delta_0^q\boldsymbol{\gamma}'; \overline{\mathfrak{m}})$ *for some* $\delta_0^p\boldsymbol{\gamma}$ *and* $\delta^q\boldsymbol{\gamma}'$.

Moreover, for each $\tau \in \mathcal{T}_i$ *there is a unique such irreducible curve* $\overline{v}_\tau$.

4.3. *Proof of the other half of Theorem 1.0.1.* — In this subsection we prove the "other half" of Theorem 1.0.1, i.e., $\Phi \circ \Psi = id$ on the level of homology, assuming the results of Sections 4.4–4.5.

We make the following simplifying assumption, which is possible by Theorem I.2.5.2.

$(\dagger\dagger)_{2g}$ All the elliptic orbits of the stable Hamiltonian vector field corresponding to $h : S \xrightarrow{\sim} S$ that intersect $S \times \{0\}$ at most $2g$ times have linearized first return maps which are ε-rotations with $0 < |\varepsilon| < \frac{\pi}{g}$ with respect to some trivialization.

The partition conditions for k-fold iterates of such elliptic orbits with $k \leq 2g$ are particularly simple and have the form $(1, \dots, 1)$ or (k).

Theorem **4.3.1.** — *Suppose* $(\dagger\dagger)_{2g}$ *holds. If* $m \gg 0$*, then there exists a chain homotopy*

$$K : PFC_{2g}(N) \to PFC_{2g}(N),$$

such that the following holds:

$$\partial_{ECH} \circ K + K \circ \partial_{ECH} = \Phi \circ \Psi + id. \quad (\mathbf{4.3.1})$$

Proof. — Suppose $m \gg 0$. Fix $\mathfrak{p}(\tau)$ and choose $\{\bar{J}_\tau\} \in \overline{\mathcal{I}}^{reg}$. For sufficiently small $\varepsilon, \delta > 0$ (which depend on the choices of m and $\{\bar{J}_\tau\}$), there exists $\{\bar{J}^{\Diamond}_\tau(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ such that Lemma 4.2.13 holds.

Fix $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_{2g}$ and abbreviate

$$\mathcal{M} = \mathcal{M}^{I=2, n^*=m}_{\{\bar{J}^{\Diamond}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}'; \overline{\mathfrak{m}}), \;\; \mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}} = \mathcal{M}^{I=2, n^*=m, \{K_{\mathfrak{p}(\tau),\delta}\}}_{\{\bar{J}^{\Diamond}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}'; \overline{\mathfrak{m}}).$$

Let $\overline{\mathcal{M}}$ be the SFT compactification of $\mathcal{M}$ and let $\partial\mathcal{M} = \overline{\mathcal{M}} - \mathcal{M}$ be the boundary of $\mathcal{M}$. As in the $\overline{W}^{\pm}_\tau$ case, the limit SFT buildings can be described in a manner analogous to Definition I.7.3.1 and the proof of existence follows the same steps as that of Proposition I.7.3.2. The main point is that for each component of the limit SFT building the boundary punctures either map to points on the singular Lagrangian or to Reeb chords, including chords over z_∞.

If $U \subset [-\infty, +\infty]$, then we write $\partial_U\mathcal{M}$ for the set of $\overline{u}_\infty \in \partial M$ where $\overline{u}_\infty$ is a building which corresponds to some $\tau \in U$. By Lemma 4.2.12, we may take $\mathcal{M}^{\{K_{\mathfrak{p}(\tau),\delta}\}}$ to be regular.

In view of the considerations from Claim 3.3.4, we assume that all of $\mathcal{M}$ *is regular.*

Step 1 (Breaking at $+\infty$*).* The following is proved in Section 4.4.

Lemma **4.3.2.** — $\partial_{\{+\infty\}}\mathcal{M} \subset A_1$, *where*

$$A_1 = \coprod_{\mathbf{y} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}} \left(\mathcal{M}^{I=2, n^*=m}_{\bar{J}^{\Diamond}_-(\varepsilon,\delta,\mathfrak{p}(+\infty))}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}}(+\infty)) \times \mathcal{M}^{I=0}_{J_+}(\mathbf{y}, \boldsymbol{\gamma}') \right).$$

Gluing the pairs in A_1 accounts for the term $\Phi \circ \Psi$ in Equation (4.3.1).

Step 2 (Breaking at $-\infty$*).* The following lemmas are proved in Section 4.5.

Lemma **4.3.3.** — $\partial_{\{-\infty\}}\mathcal{M} \subset A_2 \sqcup A_3$, *where*

$$A_2 = \coprod_{\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_{2g}, \mathfrak{z} \in \mathfrak{Z}, n_0 = 0} \left(\mathcal{M}^{I=0, n^*=0}_{\bar{J}^{\Diamond}_{-\infty,1}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\boldsymbol{\gamma}, \boldsymbol{\gamma}, \mathfrak{z}) \right.$$

$$\times \mathcal{M}^{\mathrm{I}=4g+2,n^*=m}_{\bar{\mathrm{J}}_{-\infty,2}}(\mathfrak{z};\overline{\mathfrak{m}}(-\infty))\Big);$$

$$\mathrm{A}_3 = \coprod_{\boldsymbol{\gamma},\boldsymbol{\gamma}'\in\widehat{\mathcal{O}}_{2g},\mathfrak{z}\in\mathfrak{Z},n_0=1}\Big(\mathcal{M}^{\mathrm{I}=2g+2,n^*=m}_{\bar{\mathrm{J}}^{\diamondsuit}_{-\infty,1}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\boldsymbol{\gamma},\boldsymbol{\gamma}',\mathfrak{z})\mathcal{M}^{\mathrm{I}=2g,n^*=0}_{\bar{\mathrm{J}}_{-\infty,2}}(\mathfrak{z})\Big).$$

Note that if $\overline{v}_2 \in \mathcal{M}^{\mathrm{I}=2g,n^*=0}_{\bar{\mathrm{J}}_{-\infty,2}}(\mathfrak{z})$, then $\overline{v}_2 = \mathrm{B}_{-\infty,2}\times\mathfrak{z}$.

Lemma **4.3.4.** — *Gluing the pairs in* A_2 *accounts for the term id in Equation* (4.3.1).

Lemma **4.3.5.** — *Gluing the pairs in* A_3 *gives a total of* 0 *mod* 2.

Step 3 (Breaking in the middle). The following is proved in Section 4.6.

Lemma **4.3.6.** — $\partial_{(-\infty,+\infty)}\mathcal{M}\subset \mathrm{A}_4\sqcup\mathrm{A}_5\sqcup\mathrm{A}_6\sqcup\mathrm{A}_7$, *where*

$$\mathrm{A}_4 = \coprod_{\boldsymbol{\gamma}''\in\widehat{\mathcal{O}}_{2g}}\Big(\mathcal{M}^{\mathrm{I}=1,n^*=m}_{\{\bar{\mathrm{J}}^{\diamondsuit}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma},\boldsymbol{\gamma}'';\overline{\mathfrak{m}})\times\mathcal{M}^{\mathrm{I}=1}_{\mathrm{J}'}(\boldsymbol{\gamma}'',\boldsymbol{\gamma}')\Big);$$

$$\mathrm{A}_5 = \coprod_{\boldsymbol{\gamma}''\in\widehat{\mathcal{O}}_{2g}}\Big(\mathcal{M}^{\mathrm{I}=1}_{\mathrm{J}'}(\boldsymbol{\gamma},\boldsymbol{\gamma}'')\times\mathcal{M}^{\mathrm{I}=1,n^*=m}_{\{\bar{\mathrm{J}}^{\diamondsuit}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma}'',\boldsymbol{\gamma}';\overline{\mathfrak{m}})\Big);$$

$$\mathrm{A}_6 = \coprod_{\boldsymbol{\gamma}'',\boldsymbol{\gamma}'''\in\widehat{\mathcal{O}}_{2g-1}}\Big(\mathcal{M}^{\mathrm{I}=2,n^*=m-1,f\delta_0}_{\mathrm{J}'}(\boldsymbol{\gamma},\delta_0\boldsymbol{\gamma}'')\times\mathcal{M}^{\mathrm{I}=-2,n^*=0}_{\{\bar{\mathrm{J}}^{\diamondsuit}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\delta_0\boldsymbol{\gamma}'',\delta_0\boldsymbol{\gamma}''')$$
$$\times\mathcal{M}^{\mathrm{I}=2,n^*=1}_{\mathrm{J}'}(\delta_0\boldsymbol{\gamma}''',e\boldsymbol{\gamma}''')\Big);$$

$$\mathrm{A}_7 = \coprod_{\boldsymbol{\gamma}'',\boldsymbol{\gamma}'''\in\widehat{\mathcal{O}}_{2g-2}}\Big(\mathcal{M}^{\mathrm{I}=2,n^*=m-2,f\delta_0}_{\mathrm{J}'}(\boldsymbol{\gamma},\delta_0^2\boldsymbol{\gamma}'')\times\mathcal{M}^{\mathrm{I}=-3,n^*=0}_{\{\bar{\mathrm{J}}^{\diamondsuit}_{\tau}(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\delta_0^2\boldsymbol{\gamma}'',\delta_0^2\boldsymbol{\gamma}''')$$
$$\times\mathcal{M}^{\mathrm{I}=3,n^*=2}_{\mathrm{J}'}(\delta_0^2\boldsymbol{\gamma}''',eh\boldsymbol{\gamma}''')\Big).$$

Gluing the pairs in A_4 and A_5 accounts for the terms $\partial_{\mathrm{ECH}}\circ\mathrm{K}$ and $\mathrm{K}\circ\partial_{\mathrm{ECH}}$ in Equation (4.3.1).

Lemma **4.3.7.** — *Gluing the triples in* A_6 *and* A_7 *gives a total of* 0 *mod* 2.

This completes the proof of Theorem 4.3.1, modulo the results that will be proved in Sections 4.4–4.6. □

4.4. *Degeneration at* $+\infty$. — In this subsection we study the limit of holomorphic maps to $\overline{\mathrm{W}}_\tau$ as $\tau\to+\infty$, i.e., when $\overline{\mathrm{W}}_\tau$ degenerates into a concatenation of $\overline{\mathrm{W}}_-$ with $\overline{\mathrm{W}}_+$ along the HF-type end. This will prove Lemma 4.3.2.

We assume that $m \gg 0$; $\varepsilon, \delta > 0$ are sufficiently small; and $\{\bar{J}_\tau\} \in \overline{\mathcal{I}}^{reg}$ and $\{\bar{J}_\tau^\Diamond(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ satisfy Lemma 4.2.13. Fix $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_{2g}$ and let

$$\mathcal{M} = \mathcal{M}^{I=2, n^*=m}_{\{\bar{J}_\tau^\Diamond(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}'; \overline{\mathfrak{m}}), \quad \mathcal{M}_\tau = \mathcal{M}^{I=2, n^*=m}_{\bar{J}_\tau^\Diamond(\varepsilon,\delta,\mathfrak{p}(\tau))}(\boldsymbol{\gamma}, \boldsymbol{\gamma}'; \overline{\mathfrak{m}}).$$

We will analyze $\partial_{\{+\infty\}}\mathcal{M}$.

Let $\overline{u}_i$, $i \in \mathbf{N}$, be a sequence of curves in $\mathcal{M}$ such that $\overline{u}_i \in \mathcal{M}_{\tau_i}$ and $\lim_{i\to\infty} \tau_i = +\infty$, and let

$$\overline{u}_\infty = (\overline{v}_{-1,1} \cup \dots \cup \overline{v}_{-1,c}) \cup \overline{v}_+ \cup (\overline{v}_{0,1} \cup \dots \cup \overline{v}_{0,b}) \cup \overline{v}_- \cup (\overline{v}_{1,1} \cup \dots \cup \overline{v}_{1,a})$$

be the limit holomorphic building in order from bottom to top, where each $\overline{v}_*$ is an SFT level, $\overline{v}_{-1,j}$, $j = 1, \dots, c$, maps to $\overline{W}'$; $\overline{v}_+$ maps to $\overline{W}_+$; $\overline{v}_{0,j}$, $j = 1, \dots, b$, maps to $\overline{W}$; $\overline{v}_-$ maps to $\overline{W}_-$; and $\overline{v}_{1,j}$, $j = 1, \dots, a$ maps to $\overline{W}'$. Here we are allowing the possibility that a, b, or $c = 0$. For notational convenience, sometimes we refer to $\overline{v}_+$ as $\overline{v}_{-1,c+1}$ or $\overline{v}_{0,0}$ and $\overline{v}_-$ as $\overline{v}_{0,b+1}$ or $\overline{v}_{1,0}$.

As before, we have the following constraints:

$$n^*(\overline{u}_i) = \sum_{\overline{v}_*} n^*(\overline{v}_*) = m; \tag{4.4.1}$$

$$I(\overline{u}_i) = \sum_{\overline{v}_*} I(\overline{v}_*) = 2, \tag{4.4.2}$$

where the summations are over all the levels $\overline{v}_*$ of $\overline{u}_\infty$.

The following is the analog of Lemma 3.4.21, with a similar proof (omitted):

Lemma **4.4.1.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$, then $a = b = c = 0$; $I(\overline{v}_+) = 0$; $I(\overline{v}_-) = 2$; $\overline{v}_+$ is a W_+-curve; and $\overline{v}_-$ is a $\overline{W}_-$-curve.*

Lemma **4.4.2.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$ of $\overline{u}_\infty$, then:*

(1) *$p_- = \deg(\overline{v}'_-) > 0$;*
(2) *some $\overline{v}_{1,j_0}$, $j_0 > 0$, has a negative end $\mathcal{E}_-$ that limits to δ_0^p for some $p > 0$ and satisfies $n^*(\mathcal{E}_-) \geq m - p$;*
(3) *$\overline{u}_\infty$ has no boundary point at z_∞;*
(4) *$\overline{u}_\infty$ has no fiber components and no components of $\overline{v}''_*$ that intersect the interior of a section at infinity;*
(5) *each component of $\overline{v}^\sharp_{0,j}$, $1 \leq j \leq b$, is a thin strip;*
(6) *each component of $\overline{v}^\sharp_+$ is an $n^* = 1$, $I = 0$ or 1 section from z_∞ to h or e which is contained in $\overline{W}_+ - W_+$;*
(7) *each component of $\overline{v}^\sharp_{-1,j}$, $0 \leq j \leq c$, is an $n^* = 1$, $I = 1$ or 2 cylinder from δ_0 to h or e which is contained in $\mathbf{R} \times (\overline{N} - N)$;*

(8) *h appears at most once at the negative end of $\overline{v}^\sharp_{-1,1}$.*

Proof. — The proof is based on Equation (4.4.1). First observe that:

(*) either there is a negative end $\mathcal{E}_-$ that limits to δ_0^p for some $p > 0$ and satisfies $n^*(\mathcal{E}_-) \geq m - p$ by Lemma 3.4.2; or

(**) there is a negative end $\mathcal{E}_-$ that limits to z_∞ and the sum of $n^*(\mathcal{E}_i)$ over all the ends $\mathcal{E}_i$ that limit to z_∞ and $n^*(\mathcal{E}'_i)$ over all the neighborhoods $\mathcal{E}'_i$ of the boundary points at z_∞ is $\geq m - 2g$.

(1) If (*) or (**) holds, then $\overline{v}'_- \neq \varnothing$, since otherwise the neighborhood of $\overline{\mathfrak{m}}(+\infty)$ contributes m towards $n^*(\overline{v}_-)$.

(2) is a consequence of (1) and is a subcase of (*). In particular (**) does not hold. (3)–(7) follow from (2). (8) follows from the definition of the ECH differential. □

Lemma **4.4.3.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$, then $\overline{u}_\infty$ cannot have the following subbuildings:*

(1) *a degree one component of $\overline{v}^\sharp_+$ from z_∞ to h; and*
(2) *$\overline{v}'_+$ which has degree p_+ and $\overline{v}^\sharp_{-1,1}$ which is a union of p_+ cylinders from δ_0 to h.*

Proof. — This is due to the positioning of h, given in Section 3.2.3. (1) was proved in Lemma I.6.6.5. (2) is due to Lemma 3.7.8: the usual rescaling procedure with fixed $m \gg 0$, together with Lemma 4.4.2, gives rise to an SFT limit $w_+ : \Sigma_+ \to \mathbf{CP}^1$, $\pi_+ : \Sigma_+ \to cl(B_+)$, where Σ_+ consists of p_+ copies of $cl(B_+)$ (and hence π_+ is a trivial branched cover) and the restriction of w_+ to each component of Σ_+ satisfies the conditions of Lemma 3.7.9. □

Lemma **4.4.4.** — *If $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$, then $\overline{v}'_- \neq \varnothing$ and $\overline{u}_\infty$ contains one of the following subbuildings:*

(1) *A 3-level building consisting of $\overline{v}^\sharp_{1,1}$ with $I = 1$ and a negative end $\delta_0\boldsymbol{\gamma}'$; $\overline{v}'_- = \sigma^-_\infty$; and a thin strip of $\overline{v}^\sharp_{0,1}$.*
(2) *A 3-level building consisting of $\overline{v}^\sharp_{1,1}$ with $I = 1$ and a negative end $\delta_0\boldsymbol{\gamma}'$; $\overline{v}'_- = \sigma^-_\infty$; and a component of $\overline{v}^\sharp_+$ with $I = 1$ from z_∞ to e.*
(3) *A 4-level building consisting of $\overline{v}^\sharp_{1,1}$ with $I = 1$ and a negative end $\delta_0\boldsymbol{\gamma}'$; $\overline{v}'_- = \sigma^-_\infty$; $\overline{v}'_+ = \sigma^+_\infty$; and a cylinder component of $\overline{v}^\sharp_{-1,1}$ from δ_0 to e.*

Here we are omitting levels which are connectors.

Proof. — The lemma is a consequence of Lemmas 4.4.2 and 4.4.3, the positivity of the ECH indices of all the components with the exception of $\overline{v}'_+$, and the conditions given by Equations (4.4.1) and (4.4.2). □

Lemma **4.4.5.** — *If $m \gg 0$, then there is no $\overline{u}_\infty \in \partial_{\{+\infty\}}\mathcal{M}$ such that $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ for some level $\overline{v}_*$.*

Proof. — The proofs to eliminate Cases (1)–(3) of Lemma 4.4.4 are similar to those of Theorem I.7.10.1 and will be omitted. □

Proof of Lemma 4.3.2. — This is a combination of Lemmas 4.4.1 and 4.4.5. □

4.5. *Degeneration at $-\infty$.* — In this subsection we study the limit of holomorphic maps to $\overline{W}_\tau$ as $\tau \to -\infty$. This will prove Lemma 4.3.3.

We assume that $m \gg 0$; $\varepsilon, \delta > 0$ are sufficiently small; and $\{\overline{J}_\tau\} \in \overline{\mathcal{I}}^{reg}$ and $\{\overline{J}^\Diamond_\tau(\varepsilon, \delta, \mathfrak{p}(\tau))\}$ satisfy Lemma 4.2.13. Fix $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_{2g}$ and let

$$\mathcal{M} = \mathcal{M}^{\mathrm{I}=2, n^*=m}_{\{\overline{J}^\Diamond_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}'; \overline{\mathfrak{m}}), \quad \mathcal{M}_\tau = \mathcal{M}^{\mathrm{I}=2, n^*=m}_{\overline{J}^\Diamond_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))}(\boldsymbol{\gamma}, \boldsymbol{\gamma}'; \overline{\mathfrak{m}}).$$

We will analyze $\partial_{\{-\infty\}}\mathcal{M}$.

Let $\overline{u}_i$, $i \in \mathbf{N}$, be a sequence of curves in $\mathcal{M}$ such that $\overline{u}_i \in \mathcal{M}_{\tau_i}$ and $\lim_{i\to\infty} \tau_i = -\infty$, and let

$$\overline{u}_\infty = (\overline{v}_{-1,1} \cup \cdots \cup \overline{v}_{-1,c}) \cup \overline{v}_1 \cup \overline{v}_2 \cup (\overline{v}_{1,1} \cup \cdots \cup \overline{v}_{1,a})$$

be the limit holomorphic building in order from bottom to top, where each $\overline{v}_*$ is an SFT level, $\overline{v}_{-1,j}, j = 1, \dots, c$, maps to $\overline{W'}$; $\overline{v}_j$ maps to $\overline{W}_{-\infty,j}$; and $\overline{v}_{1,j}, j = 1, \dots, a$ maps to $\overline{W'}$. Sometimes we refer to $\overline{v}_1$ as $\overline{v}_{-1,c+1}$ or $\overline{v}_{1,0}$.

We write $\overline{v}_* = \overline{v}'_* \cup \overline{v}''_*$, $\overline{v}''_* = \overline{v}^\sharp_* \cup \overline{v}^\flat_*$, where:

- $\overline{v}'_*$ is the union of branched covers of a section at infinity;
- $\overline{v}^\sharp_*$ is the union of components that are not in $\overline{v}'_*$ and are asymptotic to some multiple of δ_0 or pass through $\mathfrak{z} = z_\infty^{r_0}\zeta_1^{r_1}\cdots\zeta_l^{r_l}$ with $r_0 > 0$; and
- $\overline{v}^\flat_*$ is the union of the remaining components of $\overline{v}_*$.

Remark **4.5.1.** — The best way to prove the existence of the limit for $\tau \to -\infty$ is probably by regarding the degeneration of $\overline{W}_\tau$ as a neck stretching as mentioned at the beginning of Section 4. Strictly speaking, in the limit $\tau \to -\infty$, there exist levels between $\overline{v}_1$ and $\overline{v}_2$, i.e., levels that map to $\mathbf{R} \times S^1 \times \overline{S}$. By considerations of n^*, these levels are connectors (i.e., map to $\mathbf{R} \times S^1 \times \{pt\}$) and will be ignored until we consider gluing. From the point of view of nodal degenerations, these intermediate levels correspond to curves mapped to the fiber over the node of $B_{-\infty}$, equipped with meromorphic functions *à la* Ionel and Parker [IP2, Section 5].

The following are the analogs of Lemmas 4.4.1 and 4.4.2:

Lemma **4.5.2.** — *If* $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ *for all levels* $\overline{v}_*$ *of* $\overline{u}_\infty$, *then* $a = c = 0$; $I(\overline{v}_1) = 0$; $I(\overline{v}_2) = 4g + 2$; *and* $\overline{v}_2$ *is a* $\overline{W}_{-\infty,2}$*-curve.*

Proof. — Suppose $\overline{v}'_* \cup \overline{v}^\sharp_* = \varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$. The following are immediate from considerations of n^*:

(1) $\overline{v}_2$ is a $\overline{W}_{-\infty,2}$-curve or a degenerate $\overline{W}_{-\infty,2}$-curve; and
(2) $\overline{v}_* = \overline{v}''_*$ and $n^*(\overline{v}_*) = 0$ for $* = (-1, j)$, 1, and $(1, j)$.

(1) implies that $I(\overline{v}_2) = 4g + 2$ by Lemma 4.2.7. Note that:

(3) a degenerate $\overline{W}_{-\infty,2}$-curve consists of a fiber component with ind $= 2 - 2g$ and $2g$ components of the type $B_{-\infty,2} \times \{q_l\}$ for some $q_l \in \widehat{a}_l$, each with ind $= 1$, for a total of $I(\overline{v}_2) = 4g + 2$.

Since there are $2g$ codimension two gluing conditions between $\overline{v}_1$ and $\overline{v}_2$, it follows that

$$\sum_* I(\overline{v}_*) = 4g + 2, \qquad (\mathbf{4.5.1})$$

where the summation is over all the levels $*$. Hence $a = c = 0$, $I(\overline{v}_1) = 0$, and each component $\widetilde{v}$ of $\overline{v}_1$ is a branched cover of a trivial cylinder with possibly empty branch locus. (Here we are assuming without loss of generality that the almost complex structure on $\overline{W}_{-\infty,1}$ is $\overline{J}_{-\infty,1}$.)

We eliminate degenerate $\overline{W}_{-\infty,2}$-curves as follows: Since we may assume that $\widehat{\mathcal{O}}_{2g}$ is disjoint from $\overline{\mathbf{a}} \times \{1\} \subset \overline{S} \times \{1\}$ by genericity, the gluing condition for $\overline{v}_1$ and $\overline{v}_2$ is not satisfied in view of (3). □

Lemma **4.5.3.** — *If* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$ *of* $\overline{u}_\infty$, *then:*

(1) $p_2 = \deg(\overline{v}'_2) > 0$;
(2) $\overline{u}_\infty$ *has no boundary point at* z_∞;
(3) $\overline{u}_\infty$ *has no fiber components and no components of* $\overline{v}''_*$ *that intersect the interior of a section at infinity; and*
(4) $\overline{v}''_2$ *is the union of* $2g - p_2$ *copies of* $B_{-\infty,2} \times \{pt\}$, *and* $I(\overline{v}_2) = 2g$.

Proof. — (1)–(3) are analogs of Lemma 4.4.2 and (4) is a consequence of Lemma 4.2.8. □

The following is the analog of Lemma 4.4.4.

Lemma **4.5.4.** — *If* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$, *then* $\overline{v}'_2 \neq \varnothing$ *and* $\overline{v}''_2$ *is a union of components* $B_{-\infty,2} \times \{pt\}$, *and* $\overline{u}_\infty$ *contains one of the following subbuildings:*

(1_i) *A* 2*-level building consisting of* $\overline{v}^\sharp_1$ *with* $I = i + (\deg(\overline{v}^\sharp_1) - 1)$, $i = 2, 3$, *which passes through* $\mathfrak{z}$ *with multiplicity* $r_0 = 1$; *and* $\overline{v}'_2 = \sigma_\infty^{-\infty,2}$ *with* $I = 1$.

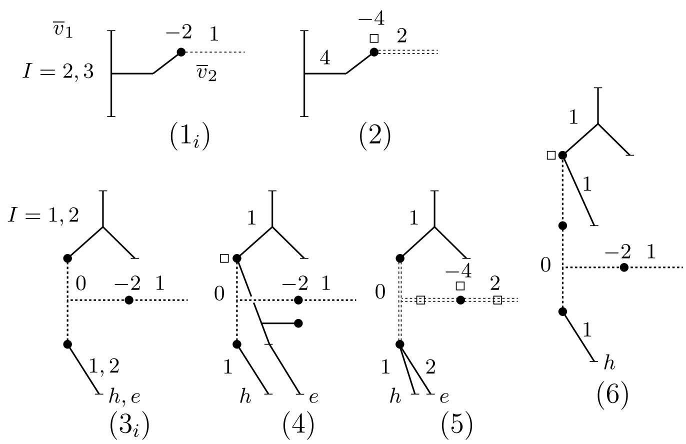


Fig. 18. — Schematic diagrams for the possible types of degenerations. The dotted lines indicate the section at infinity. The quantities -2 and -4 along the intersections of $\overline{v}_1$ and $\overline{v}_2$ indicate the reduction of I due to the gluing conditions. For simplicity we are assuming that $\deg \overline{v}_1^\sharp = 1$ in (1$_i$) and $\deg \overline{v}_1^\sharp = 2$ in (2)

(2) *A 2-level building consisting of $\overline{v}_1^\sharp$ with $\mathrm{I} = 4 + (\deg(\overline{v}_1^\sharp) - 2)$ which passes through $\mathfrak{z}$ with multiplicity $r_0 = 2$; and $\overline{v}_2'$ with $\mathrm{I} = 2$ which is a branched double cover of $\sigma_\infty^{-\infty,2}$.*

(3$_i$) *A 4-level building consisting of $\overline{v}_{1,1}^\sharp$ with $\mathrm{I} = 1$ or 2 and a negative end at δ_0; $\overline{v}_1' = \sigma_\infty^{-\infty,1}$; $\overline{v}_2' = \sigma_\infty^{-\infty,2}$; and a cylinder of $\overline{v}_{-1,1}^\sharp$ from δ_0 to h or e with $\mathrm{I} = 1$ or 2.*

(4) *A 4-level building consisting of $\overline{v}_{1,1}^\sharp$ with $\mathrm{I} = 1$ and a negative end at δ_0^2; $\overline{v}_1' = \sigma_\infty^{-\infty,1}$; a cylinder of $\overline{v}_1^\sharp$ from δ_0 to h or e with $\mathrm{I} = 1, 2$; $\overline{v}_2' = \sigma_\infty^{-\infty,2}$; a cylinder of $\overline{v}_{-1,1}^\sharp$ from δ_0 to h with $\mathrm{I} = 1$; and a cylinder of $\overline{v}_{-1,1}''$ from h or e to e with $\mathrm{I} = 1, 0$.*

(5) *A 4-level building consisting of $\overline{v}_{1,1}^\sharp$ with $\mathrm{I} = 1$ and a negative end at δ_0^2; $\overline{v}_1'$ with $\mathrm{I} = 0$ which is a branched double cover of $\sigma_\infty^{-\infty,1}$; $\overline{v}_2'$ with $\mathrm{I} = 2$ which is a branched double cover of $\sigma_\infty^{-\infty,2}$; and two cylinder components of $\cup_{j=1}^c \overline{v}_{-1,j}^\sharp$ from δ_0 to h or e, each with $\mathrm{I} = 1$ or 2.*

(6) *A 5-level building consisting of $\overline{v}_{1,2}^\sharp$ with $\mathrm{I} = 1$ and a negative end at δ_0^2; $\overline{v}_{1,1}' = \mathbf{R} \times \delta_0$; a component of $\overline{v}_{1,1}^\sharp$ with $\mathrm{I} = 1$ which is a cylinder from δ_0 to h; $\overline{v}_1' = \sigma_\infty^{-\infty,1}$; $\overline{v}_2' = \sigma_\infty^{-\infty,2}$; and a cylinder of $\overline{v}_{-1,1}^\sharp$ from δ_0 to h with $\mathrm{I} = 1$.*

We are omitting levels which are branched covers of trivial cylinders. Moreover, each gluing condition reduces the sum of ECH indices by 2.

See Figure 18.

Proof. — The proof is similar to that of Lemma 4.4.4 and we only provide a sketch.

We have $\overline{v}_2' \neq \varnothing$ by Lemma 4.5.3(1). By Lemma 4.5.3(4), $\overline{v}_2''$ is a union of components $B_{-\infty,2} \times \{q_l\}$, $q_l \in \widehat{a}_l$, and $\mathrm{I}(\overline{v}_2) = 2g$.

First suppose that $\overline{v}_1' = \varnothing$. Then $\overline{v}_1^\sharp$ passes through $((0, \frac{3}{2}), z_\infty)$ exactly once and $\pi_{B_{-\infty,1}} \circ \overline{v}_1^\sharp$ has a k-fold branch point (here a "1-fold branch point" is a regular point) at

$(0,\frac{3}{2})$ for some $k>0$, where $\pi_{B_{-\infty,1}}:\overline{W}_{-\infty,1}\to B_{-\infty,1}$ is the projection along $\overline{S}$. The k-fold branch point at $((0,\frac{3}{2}),z_\infty)$ contributes $2k$ towards $\operatorname{ind}(\overline{v}_1^\sharp)$ and each of the remaining $2g-k$ intersection points with $\{(0,\frac{3}{2})\}\times\overline{S}$ passes through some $\widehat{a}_l$ and contributes $+1$ towards $\operatorname{ind}(\overline{v}_1^\sharp)$.

We restate the ind calculation from the previous paragraph in terms of ECH indices $I'(\overline{v}_1)$ and $I'(\overline{v}_2)$ when:

(I_1) $\overline{v}_1,\overline{v}_2$ are viewed as curves in the fibrations

$$\overline{W}_{-\infty,1}-(\{(0,3/2)\}\times\overline{S})\to B_{-\infty,1}-\{(0,3/2)\},$$
$$\overline{W}_{-\infty,2}-(\{0\}\times\overline{S})\to B_{-\infty,2}-\{0\};$$

(I_2) $\{(0,\frac{3}{2})\}\times\overline{S}$ is in one-to-one correspondence with a Morse-Bott family of orbits, viewed as orbits at the *positive end of* $\overline{v}_1$*;*

(I_3) the orbit corresponding to z_∞ is viewed as a negative elliptic orbit, i.e., an elliptic orbit with Conley-Zehnder index -1 with respect to the framing coming from the Morse-Bott fibration, and the orbits corresponding to q_l are viewed as hyperbolic orbits.

Then $I'(\overline{v}_1)\geq 0$, $I'(\overline{v}_2')=k$, and $I'(\overline{v}_2'')=0$, and the ECH index of the curve $\tilde{v}$ obtained by (pre-)gluing $\overline{v}_1$ and $\overline{v}_2$ is given by:

$$I(\tilde{v})\geq I'(\overline{v}_1)+I'(\overline{v}_2)\geq 0+(k+0)=k.$$

This implies that $k\leq 2$, giving us (1_i), $i=2,3$, or (2).

Next suppose that $\overline{v}_1'\neq\varnothing$. Then the ECH index $I(\tilde{v})$ of the curve $\tilde{v}$ obtained by (pre-)gluing $\overline{v}_1'$ and $\overline{v}_2'$ is $-k$, where $k=\deg(\overline{v}_1')=\deg(\overline{v}_2')$. Hence (3_i), $i=1,2$, and (4)–(6) follow from enumerating all the possibilities, subject to the condition that h appear only once at the negative end of $\overline{v}_{-1,1}$. □

Lemma **4.5.5.** — *If* $m\gg 0$, $\overline{u}_\infty\in\partial_{\{-\infty\}}\mathcal{M}$, *and* $\overline{v}_*'\cup\overline{v}_*^\sharp\neq\varnothing$ *for some level* $\overline{v}_*$, *then the only possibility is Case* (1_3).

Proof. — *Cases* (1_2) *and (2).* We will eliminate Case (1_2); Case (2) is similar. We apply the usual rescaling argument with $m\gg 0$ fixed to obtain a function $w_2:B_{-\infty,2}\to\mathbf{CP}^1$ satisfying the following:

(i_2) $w_2(0)=\infty$;

(ii_2) $w_2(\frac{i}{2})=0$;

(iii_2) $w_2(\partial B_{-\infty,2})\subset\mathbf{R}^+\cdot e^{i\phi(\overline{a}_{k,l})}$ for some (k,l);

(iv_2) w_2 is a biholomorphism away from $\partial B_{-\infty,2}$.

We now observe that w_2 is uniquely determined by (i_2)–(iv_2), up to multiplication by a positive real constant. This implies that:

(v_2) w_2 maps the asymptotic marker ∂_y at $0 \in B_{-\infty,2}$ to the asymptotic marker $\dot{\mathcal{R}}_{\pi+\phi(\overline{a}_{k,l})}(\infty)$.

Here $z = x + iy$ is the complex coordinate on $B_{-\infty,2}$. (v_2) translates into an asymptotic condition for $\overline{v}_1^\sharp$ at $((0, \frac{3}{2}), z_\infty)$. Hence $I(\overline{v}_1^\sharp)$ is at least 3.

Cases (3_i), (4)–(6). We will eliminate Case (3_i); the remaining cases are similar. The rescaling argument gives $w_2 \cup w_1$, where w_2 is as in the previous paragraph and $w_1 : cl(B_{-\infty,1}) \to \mathbf{CP}^1$ satisfies the following:

(i_1) $w_1(0, \frac{3}{2}) = 0$ and $w_1(+\infty) = +\infty$;
(ii_1) w_1 is a biholomorphism.

(v_2) implies the following asymptotic condition for w_1:

(iii_1) w_1 maps the marker ∂_s at $(0, \frac{3}{2})$ to the marker $\dot{\mathcal{R}}_{\pi+\phi(\overline{a}_{k,l})}(0)$.

Hence w_1 is uniquely determined by (i_1)–(iii_1) up to multiplication by a positive real constant.

As a consequence of the uniqueness of w_1 up to multiplication by a positive real constant, the following are uniquely determined:

(a) the asymptotic eigenfunction of $\overline{v}_{1,1}^\sharp$ at the negative end δ_0;
(b) the asymptotic eigenfunction of $\overline{v}_{-1,1}^\sharp$ at the positive end δ_0;

(a) is determined by the image of the asymptotic marker $\dot{\mathcal{L}}_{3/2}(+\infty)$ at $+\infty \in cl(B_{-\infty,1})$ and (b) by the radial ray that contains $w_1(-\infty)$. (a) and (b) give rise to one constraint each on $\overline{v}_{1,1}^\sharp$ and $\overline{v}_{-1,1}^\sharp$. Hence $I(\overline{v}_{1,1}^\sharp) \geq 2$ and $I(\overline{v}_{-1,1}^\sharp) \geq 2$, which is a contradiction. □

Proof of Lemma 4.3.3. — This is a combination of Lemmas 4.5.2 and 4.5.5. □

Proof of Lemma 4.3.4. — We first claim that the mod 2 cardinality of

$$\mathcal{M}_{\overline{J}_{-\infty,2}}^{I=4g+2,n^*=m}(\mathfrak{z};\overline{\mathfrak{m}}(-\infty))$$

is 1 when $r_0 = 0$ (i.e., $\mathfrak{z}$ does not contain z_∞) and the arcs $\overline{\mathbf{a}}$ are chosen generically. This is proved by reducing to the calculation of Theorem 2.3.3 as follows: Degenerate $B_{-\infty,2}$ into a sphere $B_{-\infty,21}$ and a disk $B_{-\infty,22}$ which are identified at one point and degenerate $\overline{W}_{-\infty,2} = B_{-\infty,2} \times \overline{S}$ into $(B_{-\infty,21} \times \overline{S}) \cup (B_{-\infty,22} \times \overline{S})$. We assume that the marked point is in $B_{-\infty,21} \times \overline{S}$. Then a curve

$$\overline{v}_2 \in \mathcal{M}_{\overline{J}_{-\infty,2}}^{I=4g+2,n^*=m}(\mathfrak{z};\overline{\mathfrak{m}}(-\infty))$$

degenerates into a pair $(\overline{v}_{21}, \overline{v}_{22})$, where $\overline{v}_{2i}$, $i = 1, 2$, maps to $B_{-\infty,2i} \times \overline{S}$ and $\overline{v}_{22}$ is a union of constant sections $B_{-\infty,22} \times \{q_l\}$, where $q_l \in \widehat{a}_l$. Hence $\overline{v}_{21}$ is a curve in $B_{-\infty,21} \times \overline{S}$ with exactly the same type of constraints as in Theorem 2.3.3. This implies the claim.

In the rest of the proof we discuss how to glue pairs $(\overline{v}_1, \overline{v}_2) \in A_2$. For simplicity we work with $\overline{J}_{-\infty,1}$ instead of $\overline{J}^{\diamondsuit}_{-\infty,1}(\varepsilon, \delta, \mathfrak{p}(-\infty)))$. Since $I(\overline{v}_1) = 0$, each component $\widetilde{v}$ of $\overline{v}_1$ is a branched cover of a trivial cylinder with possibly empty branch locus. If $\widetilde{v}$ is simple, then it glues to $\overline{v}_2$ in the usual manner. On the other hand, if $\widetilde{v}$ is multiply covered, then by $(\dagger\dagger)_{2g}$ it is a branched cover of degree $k \leq 2g$ of the cylinder over an elliptic orbit γ with $\mu_{\tau^\star}(\gamma^l) = \pm 1$ for all $l = 1, \dots, k$ (here γ^l denotes the lth cover of the orbit γ) and partitions (k) and $(1, \dots, 1)$; this is the same partition condition as that of $\overline{u}_i$ for $i \gg 0$ where $\overline{u}_i \to \overline{u}_\infty$. If $\mu_{\tau^\star}(\gamma^i) = 1$ the partition $(1, \dots, 1)$ is at the positive end, and if $\mu_{\tau^\star}(\gamma^i) = -1$ it is at the negative end. The partition (k) is at the other end. The component $\widetilde{v}$ intersects ζ_1 at $(0, \frac{3}{2})$ with total multiplicity $r_1 = k$.

Suppose the elliptic orbit γ has partition (k) at the positive end and $(1, \dots, 1)$ at the negative end; the other case is analogous. Then $\widetilde{v}$ has 1 positive end, k negative ends, and $\mathfrak{b} \geq k - 1$ branch points (in the sense of Definition 3.4.26). Since the image of $\overline{v}_2$ in $\overline{W}_{-\infty,2} = B_{-\infty,2} \times \overline{S}$ is graphical over $\overline{S}$, there are local holomorphic coordinates (w, z) about $(0, \zeta_1) \in \overline{W}_{-\infty,2}$ with respect to which $\mathrm{Im}(\overline{v}_2)$ has the form $w = z^k$. Hence $\widetilde{v}$ must have a branch point at $(0, \frac{3}{2})$ which contributes $k - 1$ towards $\mathfrak{b}$.

Next we define $\mathrm{ind}'(\overline{v}_1)$ and $\mathrm{ind}'(\overline{v}_2)$ as the Fredholm indices when (I_1) and (I_2) from the proof of Lemma 4.5.4 and (I_3') hold, where:

- (I_3') the ends of $\overline{v}_1$ corresponding to $\zeta \in \{(0, \frac{3}{2})\} \times \overline{S}$ are viewed as limiting to positive elliptic orbits, i.e., elliptic orbits with Conley-Zehnder index 1 with respect to the framing coming from the Morse-Bott fibration.

By the usual Fredholm index calculation, we obtain $\mathrm{ind}'(\widetilde{v}) \geq 0$, with equality if and only if $\mathfrak{b} = k - 1$. We also compute $\mathrm{ind}'(\overline{v}_2) = 2$. When we use ind' instead of ind, the gluing conditions between $\overline{v}_1$ and $\overline{v}_2$ are 0-dimensional instead of $2g$-dimensional. This implies that $\mathrm{ind}'(\widetilde{v}) = 0$. Since the index calculation is topological, this also implies that $\overline{u}_\infty$ has no ghost bubble at $(\zeta_1, (0, \frac{3}{2}))$.

Finally we discuss the automatic transversality of $\widetilde{v}$. Recall from [We2, Theorem 1] that automatic transversality holds if

$$\mathrm{ind}'(\widetilde{v}) \geq 2g + \#\Gamma_0 - 1, \tag{4.5.2}$$

where g is the genus of $\widetilde{v}$ and $\#\Gamma_0$ is the number of punctures (of $\widetilde{v}$ with $(0, \frac{3}{2})$ removed) with even Conley-Zehnder index. Observe that the Conley-Zehnder index of the Morse-Bott ends is odd by [We2, Section 3.2] because they are constrained. Since the right-hand side of Equation (4.5.2) is equal to -1, automatic transversality holds and $\widetilde{v}$ glues in the usual manner (without any concerns of inserting branch-covered cylinders) to $\overline{v}_2$. This implies the lemma. □

Sketch of proof of Lemma 4.3.5. — We use some considerations of Section I.7.12. For simplicity we assume we are gluing degree one curves and the curve in $\overline{W}_{-\infty,2}$ is $\sigma_\infty^{-\infty,2}$, the section at ∞; this is justified by noting that the gluing of $\sigma_\infty^{-\infty,2}$ to $\overline{v}_1$ can be done essentially independently of the gluings of the remaining components of $\overline{W}_{-\infty,2}$ to $\overline{v}_1$.

Fix $\boldsymbol{\gamma},\boldsymbol{\gamma}'\in\widehat{\mathcal{O}}_1$, $k\in\{1,\dots,2g\}$, and $l\in\{0,1\}$. We consider the gluing parameter space

$$\mathfrak{P}_{k,l}:=\coprod_{\mathfrak{z}_0\in\mathfrak{u}_{k,l}}\left((-\infty,-r]\times\mathcal{M}_1(\mathfrak{z}_0)\times\mathcal{M}_2(\mathfrak{z}_0)\right),$$

where $\mathfrak{u}_{k,l}\subset\vec{a}_{k,l}$ is a small open interval containing z_∞, $\mathfrak{z}_0$ is a formal product consisting of a single point on $\mathfrak{u}_{k,l}$,

$$\mathcal{M}_1(\mathfrak{z}_0):=\mathcal{M}^{\mathrm{I}=3,n^*=m}_{\overline{\mathrm{J}}^{\diamondsuit}_{-\infty,1}(\varepsilon,\delta,\mathfrak{p}(-\infty))}(\boldsymbol{\gamma},\boldsymbol{\gamma}',\mathfrak{z}_0),$$

$$\mathcal{M}_2(\mathfrak{z}_0):=\mathcal{M}^{\mathrm{I}=1,n^*=0,ext}_{\overline{\mathrm{J}}_{-\infty,2}}(\mathfrak{z}_0),$$

ext means the boundary of the holomorphic curve is mapped to $\mathrm{L}^{-\infty,2}_{\overline{a}_k\cup\vec{a}_{k,l}}$, and $(-\infty,-r]$ is viewed as a subset of $(-\infty,\infty)$ with parameter τ. Note that $\mathcal{M}_2(\mathfrak{z}_0)$ consists of a single point $\mathrm{B}_{-\infty,2}\times\{\mathfrak{z}_0\}$. For each $\mathfrak{z}_0\in\mathfrak{u}_{k,l}$ there is a covering map

$$\pi_{\mathfrak{z}_0}:\mathcal{M}_1(\mathfrak{z}_0)\to\mathrm{S}^1=\mathbf{R}/2\pi\mathbf{Z}\simeq\mathrm{T}_{\mathfrak{z}_0}\overline{\mathrm{S}}/\mathbf{R}^+,$$

such that at $((0,\frac{3}{2}),\mathfrak{z}_0)$, $\overline{v}_1\in\mathcal{M}_1(\mathfrak{z}_0)$ is tangent to $\frac{\partial}{\partial s}+\pi_{\mathfrak{z}_0}(\overline{v}_1)$ (where the second term is defined up to a positive real constant). The maps $\pi_{\mathfrak{z}_0}$ are continuous in $\mathfrak{z}_0$.

Let

$$\mathrm{G}_{k,l}:\mathfrak{P}_{k,l}\to\coprod_{\tau\in(-\infty,r]}\mathcal{M}^{\mathrm{I}=2,n^*=m,ext}_{\{\overline{\mathrm{J}}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma},\boldsymbol{\gamma}'),\qquad\mathfrak{d}=(\tau,\overline{v}_1,\overline{v}_2)\mapsto\overline{u}(\mathfrak{d})$$

be the gluing map, defined as usual. Also let $\mathfrak{P}'_{k,l}\subset\mathfrak{P}_{k,l}$ be the subset of gluing parameters $\mathfrak{d}$ such that $\mathrm{G}_{k,l}(\mathfrak{d})\in\coprod_{\tau\in(-\infty,r]}\mathcal{M}^{\mathrm{I}=2,n^*=m}_{\{\overline{\mathrm{J}}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma},\boldsymbol{\gamma}')$.

We now apply the rescaling argument from Section I.7.2.2: if a pair in A_3 is the limit of a sequence $\overline{u}_1,\overline{u}_2,\dots$ of curves in $\mathcal{M}$, $\overline{u}_i\in\mathcal{M}_{\tau_i}$ (in particular $\overline{u}_i$ passes through the marked point $\overline{\mathfrak{m}}(\tau_i)$), and $\tau_i\to-\infty$, then there is a unique *transverse limit profile*, which is encoded by an element of $\widetilde{\mathcal{N}}/\mathbf{R}^+$ that we describe now.

Let $\widetilde{\mathcal{N}}$ be the space of holomorphic maps $w_2:\mathrm{B}_{-\infty,2}=\overline{\mathbf{D}}\to\mathbf{CP}^1$ satisfying (i_2) and (iv_2) of Lemma 4.5.5 and

(iii'_2) $w_2(\partial\mathrm{B}_{-\infty,2})\subset\mathbf{R}\cdot e^{i\phi(\overline{a}_{k,l})}$ for some (k,l).

For simplicity we assume that $\phi(\overline{a}_{k,l})=0$. We leave it to the reader to verify the following:

Claim **4.5.6.** — $\widetilde{\mathcal{N}}=\{w_2(z)=az+b+\frac{\overline{a}}{z}\mid a\in\mathbf{C}^\times,b\in\mathbf{R}\}$. *Hence* $\dim\widetilde{\mathcal{N}}=3$ *and* $\widetilde{\mathcal{N}}$ *admits an* $\mathbf{R}^\times$*-action which is multiplication by* $c\in\mathbf{R}^\times$.

Let $D(\overline{\mathfrak{m}}^b(-\infty)) \subset B_{-\infty,2}$ be a small disk about $\frac{i}{2}$ and let $\widetilde{\mathcal{N}}_x := \{w_2 \in \widetilde{\mathcal{N}} \mid w_2(x) = 0\}$, where $x \in D(\overline{\mathfrak{m}}^b(-\infty))$. The total space of the bundle

$$\widetilde{\mathcal{N}}_{D(\overline{\mathfrak{m}}^b(-\infty))} := \sqcup_{x \in D(\overline{\mathfrak{m}}^b(-\infty))} \widetilde{\mathcal{N}}_x \to D(\overline{\mathfrak{m}}^b(-\infty))$$

smoothly embeds into $\widetilde{\mathcal{N}}$. We will call this the "transversality of the constraint $\overline{\mathfrak{m}}^b(-\infty)$". Also, the transverse limit profile is an element of $\widetilde{\mathcal{N}}_{\overline{\mathfrak{m}}^b(-\infty)}/\mathbf{R}^+$.

For $\widetilde{\tau} \ll 0$, the restrictions of $\overline{u}(\mathfrak{d})$ to a neighborhood of the section at infinity are approximated by elements of $\widetilde{\mathcal{N}}$. More precisely, for $\widetilde{\tau} \ll 0$ we define:

$$g_{\widetilde{\tau},k,l} : \mathfrak{P}_{k,l} \cap \{\tau = \widetilde{\tau}\} \to \widetilde{\mathcal{N}}/\mathbf{R}^+,$$

as follows: Given $\mathfrak{d} \in \mathfrak{P}_{k,l} \cap \{\tau = \widetilde{\tau}\}$, restrict $G_{k,l}(\mathfrak{d})$ to a neighborhood of the section at infinity so that the domain of $G_{k,l}(\mathfrak{d})$ is

$$\overline{\mathbf{D}}^{\varepsilon(\widetilde{\tau})} := \{\varepsilon(\widetilde{\tau}) \leq |z| \leq 1\} \subset \overline{\mathbf{D}}, \quad \varepsilon(\widetilde{\tau}) > 0 \quad \text{small}.$$

Let us write $\pi_{\overline{S}} \circ G_{k,l}(\mathfrak{d})|_{\overline{\mathbf{D}}^{\varepsilon(\widetilde{\tau})}}$ as a Laurent series $\sum_{i=-\infty}^{\infty} c_i(\mathfrak{d}) z^i$. Then we set

$$g_{\widetilde{\tau},k,l}(\mathfrak{d}) = \left[\overline{c_{-1}(\mathfrak{d})} \cdot z + \mathrm{Re}(c_0(\mathfrak{d})) + \frac{c_{-1}(\mathfrak{d})}{z}\right].$$

The definition makes sense in view of the following consequence of Gromov-Hofer compactness:

Claim **4.5.7.**

(1) $\lim_{\widetilde{\tau} \to -\infty} \frac{c_i(\mathfrak{d})}{c_{-1}(\mathfrak{d})} = 0$ *if* $i > 1$ *or* $i < -1$ *and* $\lim_{\widetilde{\tau} \to -\infty} \frac{\overline{c_1(\mathfrak{d})}}{c_{-1}(\mathfrak{d})} = 1$, *where the convergence is uniform in* $\coprod_{\mathfrak{z}_0 \in \overline{\mathfrak{U}}_{k,l}} (\mathcal{M}_1(\mathfrak{z}_0) \times \mathcal{M}_2(\mathfrak{z}_0))$.

(2) *If* $\overline{v}_2 = B_{-\infty,2} \times \{\mathfrak{z}_0\}$, *where* $\mathfrak{z}_0$ *is a fixed point* $\neq z_\infty$, *then*

$$\lim_{\widetilde{\tau} \to -\infty} \frac{c_{-1}(\mathfrak{d})}{c_0(\mathfrak{d})} = 0 \quad \textit{and} \quad \lim_{\widetilde{\tau} \to -\infty} \frac{\mathrm{Im}(c_0(\mathfrak{d}))}{\mathrm{Re}(c_0(\mathfrak{d}))} = 0,$$

where the convergence is uniform in $\mathcal{M}_1(\mathfrak{z}_0)$.

What Claim 4.5.7(2) is saying is that when $\mathfrak{z}_0 \neq z_\infty$ and $\widetilde{\tau} = \widetilde{\tau}(\mathfrak{z}_0) \ll 0$, then $\pi_{\overline{S}} \circ G_{k,l}(\mathfrak{d})$ maps $\partial B_{-\infty,2}$ to a small slit on $\mathbf{R}$ which is disjoint from 0 but close to $\mathfrak{z}_0$.

Now define the evaluation map:

$$ev_{\widetilde{\tau},k,l} : \mathfrak{P}_{k,l} \cap \{\tau = \widetilde{\tau}\} \to \mathbf{C},$$

which sends $\mathfrak{d}$ to $g_{\widetilde{\tau},k,l}(\mathfrak{d})(\frac{i}{2})$. By Gromov-Hofer compactness $\frac{c_{-1}(\mathfrak{d})}{|c_{-1}(\mathfrak{d})|}$ and $\pi_{\mathfrak{z}_0}(\overline{v}_2)$ are close, up to an overall rotation and possibly a reflection. Together with Claim 4.5.7(2) and the transversality of the constraint $\overline{\mathfrak{m}}^b(-\infty)$, the local degree of $(ev_{\widetilde{\tau},k,l})|_{\mathfrak{P}'_{k,l}\cap\{\tau=\widetilde{\tau}\}}$ near $0\in\mathbf{C}$ is $\deg\pi_{\mathfrak{z}_0}$. We will informally say that "$\mathcal{M}$ has $\deg\pi_{\mathfrak{z}_0}$ ends near $\mathfrak{P}'_{k,l}$".

Finally observe that an element of $\mathcal{M}_2(\mathfrak{z}_0)$ can be viewed as a map to $\overline{a}_{i,0}$ or to $\overline{a}_{i,1}$. Hence $\mathcal{M}$ has $\deg\pi_{\mathfrak{z}_0}$ ends near $\mathfrak{P}'_{k,l}$ and $\deg\pi_{\mathfrak{z}_0}$ ends near $\mathfrak{P}'_{k,1-l}$ for a total of $2\cdot\deg\pi_{\mathfrak{z}_0}$ ends mod 2. This proves Lemma 4.3.5. □

4.6. *Breaking in the middle.* — In this subsection we study the limit of holomorphic maps to $\overline{W}_\tau$ as $\tau\to T'$ for some $T'\in(-\infty,\infty)$. This will prove Lemma 4.3.6.

We assume that $m\gg 0$; $\varepsilon,\delta>0$ are sufficiently small; and $\{\overline{J}_\tau\}\in\overline{\mathcal{I}}^{reg}$ and $\{\overline{J}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}$ satisfy Lemma 4.2.13. Fix $\boldsymbol{\gamma},\boldsymbol{\gamma}'\in\widehat{\mathcal{O}}_{2g}$ and let

$$\mathcal{M}=\mathcal{M}^{I=2,n^*=m}_{\{\overline{J}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))\}}(\boldsymbol{\gamma},\boldsymbol{\gamma}';\overline{\mathfrak{m}}),\qquad \mathcal{M}_\tau=\mathcal{M}^{I=2,n^*=m}_{\overline{J}^{\diamondsuit}_\tau(\varepsilon,\delta,\mathfrak{p}(\tau))}(\boldsymbol{\gamma},\boldsymbol{\gamma}';\overline{\mathfrak{m}}).$$

We will analyze $\partial_{(-\infty,\infty)}\mathcal{M}$.

Let $\overline{u}_i$, $i\in\mathbf{N}$, be a sequence of curves in $\mathcal{M}$ such that $\overline{u}_i\in\mathcal{M}_{\tau_i}$ and $\lim_{i\to\infty}\tau_i=T'$, and let

$$\overline{u}_\infty=(\overline{v}_{-1,1}\cup\cdots\cup\overline{v}_{-1,c})\cup\overline{v}_0\cup(\overline{v}_{1,1}\cup\cdots\cup\overline{v}_{1,a})$$

be the limit holomorphic building in order from bottom to top, where each $\overline{v}_*$ is an SFT-type level, $\overline{v}_{-1,j}$ and $\overline{v}_{1,j}$ map to $\overline{W}'$ and $\overline{v}_0$ maps to $\overline{W}_{T'}$. Sometimes we refer to $\overline{v}_0$ as $\overline{v}_{-1,c+1}$ or $\overline{v}_{1,0}$.

The following is the analog of Lemma 4.4.1, and is stated without proof.

Lemma **4.6.1.** — *If $\overline{v}'_*\cup\overline{v}^\sharp_*=\varnothing$ for all levels $\overline{v}_*$ of $\overline{u}_\infty$, then $\overline{u}_\infty$ is one of the following:*

(1) *$a=0$, $c=1$; $\overline{v}_0$ is a $\overline{W}_{T'}$-curve with $I=1$ which passes through $\overline{\mathfrak{m}}(T')$; and $\overline{v}_{-1,1}$ is a W'-curve with $I=1$; or*
(2) *$a=1$, $c=0$; $\overline{v}_{1,1}$ is a W'-curve with $I=1$; and $\overline{v}_0$ is a $\overline{W}_{T'}$-curve with $I=1$ which passes through $\overline{\mathfrak{m}}(T')$.*

Here either $T'\in\mathcal{T}_2$ and a component of $\overline{v}_0$ is in

$$\mathcal{M}^{\dagger,s,irr,\mathrm{ind}=1,n^*=m}_{\overline{J}^{\diamondsuit}_{T'}(\varepsilon,\delta,\mathfrak{p}(T'))}(\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2;\overline{\mathfrak{m}}(T'))$$

from Lemma 4.2.13(2), for some $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$; or $T'\in\mathcal{T}_1$ and a component of $\overline{v}_0$ does not pass through $\overline{\mathfrak{m}}(T')$ but is in

$$\mathcal{M}^{\dagger,s,irr,\mathrm{ind}=-1,n^*=0}_{\overline{J}^{\diamondsuit}_{T'}(\varepsilon,\delta,\mathfrak{p}(T'))}(\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2)$$

from Lemma 4.2.13(1).

The following is the analog of Lemma 4.4.2 and is stated without proof.

Lemma **4.6.2.** — *If* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$ *of* $\overline{u}_\infty$*, then:*

(1) $p_0 = \deg(\overline{v}'_0) > 0$;
(2) *some* $\overline{v}_{1,j_0}$, $j_0 > 0$, *has a negative end* $\mathcal{E}_-$ *that limits to* δ_0^p *for some* $p > 0$ *and satisfies* $n^*(\mathcal{E}_-) \geq m - p$;
(3) $\overline{u}_\infty$ *has no boundary point at* z_∞;
(4) $\overline{u}_\infty$ *has no fiber components and no components of* $\overline{v}''_*$ *that intersect the interior of a section at infinity;*
(5) *each component of* $\overline{v}^\sharp_{-1,j}$, $0 \leq j \leq c$, *is an* $n^* = 1$, $\mathrm{I} = 1$ *or* 2 *cylinder from* δ_0 *to* h *or* e *which is contained in* $\mathbf{R} \times (\overline{\mathrm{N}} - \mathrm{N})$;
(6) h *appears at most once at the negative end of* $\overline{v}^\sharp_{-1,1}$.

The following is the analog of Lemma 4.4.4 and is stated without proof.

Lemma **4.6.3.** — *If* $\overline{v}'_* \cup \overline{v}^\sharp_* \neq \varnothing$ *for some level* $\overline{v}_*$*, then* $\overline{v}'_0 \neq \varnothing$ *and* $\overline{u}_\infty$ *contains one of the following subbuildings, subject to two conditions:*

– *the sum of the ECH indices of the components of the subbuildings is at most* 3;
– *the total multiplicity of* h *at the negative end of* $\overline{v}^\sharp_{-1,1}$ *is at most* 1.

(1_i) *A* 3*-level building consisting of* $\overline{v}^\sharp_{1,1}$ *with* $\mathrm{I} = i$, $i = 1, 2$, *and a negative end* $\delta_0\boldsymbol{\gamma}'$; $\overline{v}'_0 = \sigma_\infty^{\mathrm{T}'}$; *and a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *with* $\mathrm{I} = 1$ *or* 2 *from* δ_0 *to* h *or* e.
(2_i) *A* 4*-level building consisting of* $\overline{v}^\sharp_{1,2}$ *with* $\mathrm{I} = i$, $i = 1, 2$, *and a negative end* $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}'_{1,1} = \mathbf{R} \times \delta_0$; *a cylinder component of* $\overline{v}^\sharp_{1,1}$ *with* $\mathrm{I} = 1$ *or* 2 *from* δ_0 *to* h *or* e; $\overline{v}'_0 = \sigma_\infty^{\mathrm{T}'}$; *and a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *with* $\mathrm{I} = 1$ *or* 2 *from* δ_0 *to* h *or* e.
(3_i) *A* 3*-level building consisting of* $\overline{v}^\sharp_{1,1}$ *with* $\mathrm{I} = i$, $i = 1, 2$, *and a negative end* $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}'_0 = \sigma_\infty^{\mathrm{T}'}$; *a component of* $\overline{v}^\sharp_0$ *with* $\mathrm{I} = 0$ *or* 1 *from* δ_0 *to* h *or* e; *and a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *with* $\mathrm{I} = 1$ *or* 2 *from* δ_0 *to* h *or* e.
(4_i) *A* 3*-level building consisting of* $\overline{v}^\sharp_{1,1}$ *with* $\mathrm{I} = i$, $i = 1, 2$, *and a negative end* $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}'_0$ *with* $\mathrm{I} = -2$ *which is a degree* 2 *branched cover of* $\sigma_\infty^{\mathrm{T}'}$; *and two cylinder components of* $\cup_{j=1}^c \overline{v}^\sharp_{-1,j}$ *from* δ_0 *to* h *or* e, *each with* $\mathrm{I} = 1$ *or* 2.
(5) *A* 5*-level building consisting of* $\overline{v}^\sharp_{1,3}$ *with* $\mathrm{I} = 1$ *and a negative end* $\delta_0^3\boldsymbol{\gamma}'$; $\overline{v}'_{1,2}$ *which is a degree* 2 *branched cover of* $\mathbf{R} \times \delta_0$; *a cylinder component of* $\overline{v}^\sharp_{1,2}$ *with* $\mathrm{I} = 1$ *from* δ_0 *to* h; $\overline{v}'_{1,1} = \mathbf{R} \times \delta_0$; *a cylinder component of* $\overline{v}^\sharp_{1,1}$ *with* $\mathrm{I} = 1$ *from* δ_0 *to* h; $\overline{v}'_0 = \sigma_\infty^{\mathrm{T}'}$; *and a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *with* $\mathrm{I} = 1$ *from* δ_0 *to* h.
(6) *A* 4*-level building consisting of* $\overline{v}^\sharp_{1,2}$ *with* $\mathrm{I} = 1$ *and a negative end* $\delta_0^3\boldsymbol{\gamma}'$; $\overline{v}'_{1,1} = \mathbf{R} \times \delta_0$; *two cylinder components of* $\overline{v}^\sharp_{1,1}$ *from* δ_0 *to* h, *each with* $\mathrm{I} = 1$; $\overline{v}'_0 = \sigma_\infty^{\mathrm{T}'}$; *and a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *with* $\mathrm{I} = 1$ *from* δ_0 *to* h.
(7) *A* 4*-level building consisting of* $\overline{v}^\sharp_{1,2}$ *with* $\mathrm{I} = 1$ *and a negative end* $\delta_0^3\boldsymbol{\gamma}'$; $\overline{v}'_{1,1}$ *which is a degree* 2 *branched cover of* $\mathbf{R} \times \delta_0$; *a cylinder component of* $\overline{v}^\sharp_{1,1}$ *with* $\mathrm{I} = 1$ *or* 2 *from* δ_0

to h or e; $\overline{v}'_0=\sigma_\infty^{T'}$*; a component of* $\overline{v}^\sharp_0$ *with* $I=0$ *or* 1 *from* δ_0 *to h or e; and a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *with* $I=1$ *or* 2 *from* δ_0 *to h or e.*

(8) *A* 4*-level building consisting of* $\overline{v}^\sharp_{1,2}$ *with* $I=1$ *and a negative end* $\delta_0^3\boldsymbol{\gamma}'$*;* $\overline{v}'_{1,1}$ *which is a degree* 2 *branched cover of* $\mathbf{R}\times\delta_0$*; a cylinder component of* $\overline{v}^\sharp_{1,1}$ *with* $I=1$ *from* δ_0 *to h;* $\overline{v}'_0$ *with* $I=-2$ *which is a degree* 2 *branched cover of* $\sigma_\infty^{T'}$*; and two cylinder components of* $\cup_{j=1}^c\overline{v}^\sharp_{-1,j}$ *from* δ_0 *to h or e, each with* $I=1$ *or* 2*.*

(9) *A* 3*-level building consisting of* $\overline{v}^\sharp_{1,1}$ *with* $I=1$ *and a negative end* $\delta_0^3\boldsymbol{\gamma}'$*;* $\overline{v}'_0=\sigma_\infty^{T'}$*; two components of* $\overline{v}^\sharp_0$ *from* δ_0 *to h or e, each with* $I=0$ *or* 1*; and a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *with* $I=1$ *or* 2 *from* δ_0 *to h or e.*

(10) *A* 3*-level building consisting of* $\overline{v}^\sharp_{1,1}$ *with* $I=1$ *and a negative end* $\delta_0^3\boldsymbol{\gamma}'$*;* $\overline{v}'_0$ *with* $I=-2$ *which is a degree* 2 *branched cover of* $\sigma_\infty^{T'}$*; a component of* $\overline{v}^\sharp_0$ *with* $I=0$ *or* 1 *from* δ_0 *to h or e; and two cylinder components of* $\cup_{j=1}^c\overline{v}^\sharp_{-1,j}$ *from* δ_0 *to h or e, each with* $I=1$ *or* 2*.*

(11) *A* 3*-level building consisting of* $\overline{v}^\sharp_{1,1}$ *with* $I=1$ *and a negative end* $\delta_0^3\boldsymbol{\gamma}'$*;* $\overline{v}'_0$ *with* $I=-3$ *which is a degree* 3 *branched cover of* $\sigma_\infty^{T'}$*; and three cylinder components of* $\cup_{j=1}^c\overline{v}^\sharp_{-1,j}$ *from* δ_0 *to h or e, each with* $I=1$ *or* 2*.*

We are omitting levels which are connectors.

See Figure 19. We will write $(1_{i,e})$, $(1_{i,h})$, etc. to indicate that we are in Case (1_i) and the negative ends of the lowest level are e, h, etc.

Lemma **4.6.4.** — *If* $m\gg 0$, $\overline{u}_\infty\in\partial_{(-\infty,+\infty)}\mathcal{M}$*, and* $\overline{v}'_*\cup\overline{v}^\sharp_*\neq\varnothing$ *for some level* $\overline{v}_*$*, then the only possibilities are* (1_2) *with a cylinder component of* $\overline{v}^\sharp_{-1,1}$ *from* δ_0 *to e and* (4_2) *with two cylinder components of* $\overline{v}^\sharp_{-1,1}$*, one from* δ_0 *to h and another from* δ_0 *to e.*

Proof. — *Cases* (1_i)*,* (2_i)*,* (3_i)*, (5), (6), (7) and (9).* We will treat Case (1_i); the rest of the cases are similar and can be eliminated. The key observation here is that $\deg(\overline{v}'_0)=1$ and $\overline{v}^\sharp_{-1,1}$ is a cylinder from δ_0 to h or e. Applying the usual rescaling argument with $m\gg 0$ fixed, we obtain a holomorphic map $w_0: cl(B_{T'})\to\mathbf{CP}^1$ which satisfies the following:

(i) $w_0(+\infty)=\infty$ and $w_0(\overline{\mathfrak{m}}^b(T'))=0$;
(ii) $w_0(s,t)\in int(\mathcal{R}_{\eta(t)})$ for all $(s,t)\in\partial B_{T'}$;
(iii) $\deg(w_0)=1$ away from $\mathfrak{S}(\overline{a}_{i_1,j_1},\overline{h}(\overline{a}_{i_1,j_1}))$ for some (i_1,j_1).

Here $\eta(t)=\phi_0+\frac{\pi}{m}(t-1)$, where ϕ_0 is the ϕ-coordinate of $\overline{a}_{i_1,j_1}$. (Recall that we are projecting to $\pi_{D^2_{\rho_0}}$ using balanced coordinates.)

We now observe that $w_0: cl(B_{T'})\to\mathbf{CP}^1$ is uniquely determined by (i)–(iii), up to multiplication by a positive real constant; this is argued in the same way as in Lemma 3.7.9. Using the same method as in Case (3_i) of Lemma 4.5.5, we obtain that $I(\overline{v}^\sharp_{1,1})\geq 2$ and $I(\overline{v}^\sharp_{-1,1})\geq 2$. Hence the only possibility is Case (1_2) with a cylinder component of $\overline{v}^\sharp_{-1,1}$ from δ_0 to e.

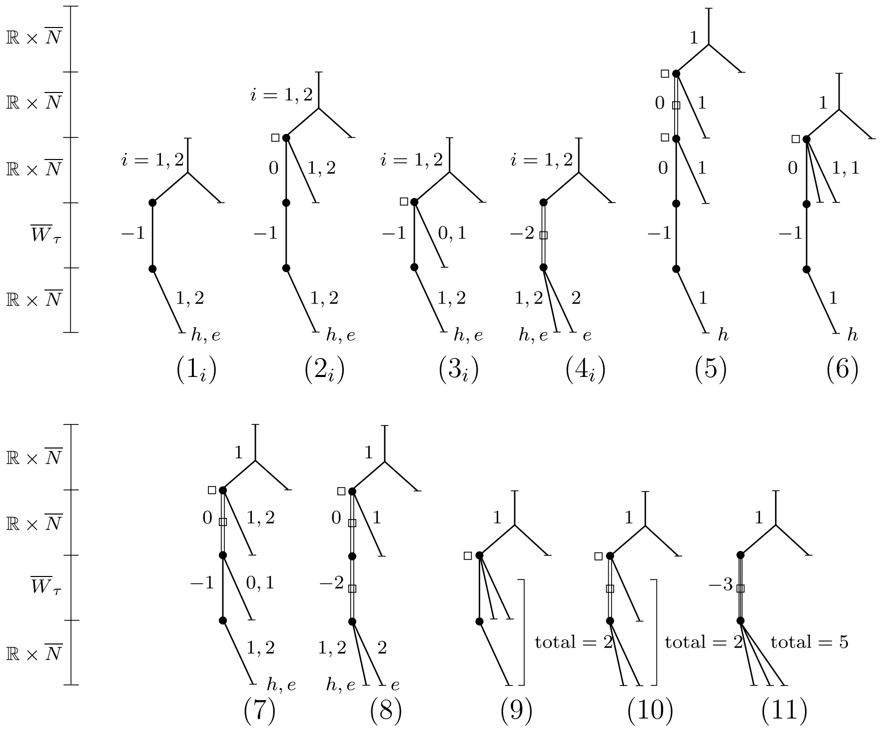


FIG. 19. — Schematic diagrams for the possible types of degenerations. For simplicity we have drawn only one level $\overline{v}_{-1,1}$ to indicate the cylindrical components of $\cup_{j=1}^{c}\overline{v}^{\sharp}_{-1,j}$ from δ_0 to h or e

Cases (4$_i$), (8), (10), and (11). We will treat Case (4$_i$). In this case we first apply the rescaling argument with $m \to \infty$ to obtain a holomorphic map $w_0 : \Sigma_0 \to \mathbf{CP}^1$ and a branched double cover $\pi_0 : \Sigma_0 \to cl(B_{T'})$ such that:

- (i) $w_0(z_0) = \infty$, where $\pi_0^{-1}(+\infty) = \{z_0\}$;
- (ii) $w_0(z_1) = 0$ for some $z_1 \in \pi_0^{-1}(\overline{\mathfrak{m}}^b(T'))$;
- (iii) $w_0(\pi_0^{-1}(\partial B_{T'})) \subset \{\phi = 0, \rho > 0\}$;
- (iv) $w_0|_{int(\Sigma_0)}$ is a biholomorphism onto its image $\mathbf{CP}^1 - ([a_1, a_2] \cup [a_3, a_4])$ with $0 < a_1 < a_2 \leq a_3 < a_4$.

Here an interval $[a, b]$ stands for $\{\phi = 0, a \leq \rho \leq b\}$.

By the Involution Lemma I.7.9.3,

- (v) $\pi_0 \circ w_0^{-1}$ maps both $(-\infty, a_1]$ and $[a_4, \infty)$ to $\mathcal{L}_{3/2} \cap \{s \geq \frac{l(T')}{2}\}$ and $[a_2, a_3]$ to $\mathcal{L}_{1/2} \cup (\mathcal{L}_{3/2} \cap \{s \leq \frac{-l(T')}{2}\})$.

In particular this constrains the asymptotic behavior of $\overline{v}^{\sharp}_{1,1}$ and hence $I(\overline{v}^{\sharp}_{1,1}) \geq 2$. Hence the only possibility is Case (4$_2$) with two cylinder components of $\overline{v}^{\sharp}_{-1,1}$, one from δ_0 to h and another from δ_0 to e. □

Proof of Lemma 4.3.6. — This is a combination of Lemmas 4.6.1 and 4.6.4. $\square$

Proof of Lemma 4.3.7.

Case (1_2). We glue triples in A_6, i.e., we are in Case (1_2). Note that the middle level has $I=-2$: there is a component $\sigma_\infty^{T'}$ (the section at infinity) which has $I = \text{ind} = -1$ and another $I=-1$ component corresponding to $T' \in \mathcal{T}_1$.

We make the following simplification:

- the top level is $\mathcal{M}_+ = \mathcal{M}_{J'}^{I=2, n^*=m-1, f_{\delta_0}}(\boldsymbol{\gamma}, \delta_0)$ consisting of $\overline{v}_{1,1} = \overline{v}_{1,1}^\sharp$ and such that $\partial \mathcal{M}_+ = \varnothing$;
- the middle level is $\sigma_\infty^{T'}$; and
- the bottom level is $\mathcal{M}_- = \mathcal{M}_{J'}^{I=2, n^*=1}(\delta_0, e)$ consisting of cylinders $\overline{v}_{-1,1} = \overline{v}_{-1,1}^\sharp$,

and the gluing is for any fixed $\tau = T'$. This simplification is justified by noting that the gluing of the $I=-1$ component $\neq \sigma_\infty^{T'}$ can be done essentially independently of the above gluing.

The gluing argument is similar to that of Theorem I.7.2.2 but has an additional twist since $I(\sigma_\infty^{T'}) = \text{ind}(\sigma_\infty^{T'}) = -1$ and we need to do an obstruction bundle gluing. The key observation is that the gluing of triples in A_6 come in pairs $(i_1, 0)$ and $(i_1, 1)$, where $\sigma_\infty^{T'}$ is viewed as having boundary mapping to $L_{\overline{a}_{i_1},0}^{T'}$ or $L_{\overline{a}_{i_1},1}^{T'}$. Since we can identify the obstruction bundles of each pair in such a way that the obstructions sections are close, the count corresponding to A_6 will be 0 mod 2.

As in Section I.7.7.1, the normal linearized $\overline{\partial}$-operator for $\sigma_\infty^{T'}$ acts on the appropriate Banach space of maps $B_{T'} \to \mathbf{C}$ and has the form $D = \partial_s - A$, where $A = -i\partial_t - \varepsilon$ and the coordinates are s, t. Without loss of generality we may assume that the boundary condition for $\partial B_{T'}$ is $\mathbf{R} \subset \mathbf{C}$, since the actual boundary condition is very close to $\mathbf{R}$. Its adjoint is $D^* = \partial_s + A$ with boundary condition $\mathbf{R}\langle dx - idy \rangle$ if a collar neighborhood of $\partial B_{T'}$ is given by $\mathbf{R} \times [0,1]$ with holomorphic coordinates $x+iy$ and $\partial B_{T'} = \{y=0\}$.

Claim **4.6.5.** — $\ker D^*$ *is* 1*-dimensional and if we write a nonzero element* $\xi \in \ker D^*$ *as* $f(ds - idt)$ *where* f *is a map* $B_{T'} \to \mathbf{C}$*, then, for* $R \gg 0$*,* $f|_{s=R}$ *has winding* $\text{wind}(f|_{s=R}, 0) = 1$ *about* 0 *and likewise* $\text{wind}(f|_{s=-R}, 0) = 0$.

Proof of Claim 4.6.5. — $\ker D^*$ is 1-dimensional since $\ker D = 0$, where D is the normal linearized $\overline{\partial}$-operator. This is because the normal bundle to $\sigma_\infty^{T'}$ is trivial.

The asymptotic decay conditions for f imply that $\text{wind}(f|_{s=R}, 0) \geq 1$ and $\text{wind}(f|_{s=-R}, 0) \leq 0$. On the other hand, the considerations of Wendl [We2] (the methods of Lemma 1.6 and Theorem 4.1 of [BH] using the negativity of zeros of ξ are more directly applicable here) imply that $\text{wind}(f|_{s=R}, 0) \leq 1$ and $\text{wind}(f|_{s=-R}, 0) \geq 0$. The claim then follows. $\square$

We choose smooth slices $\widetilde{\mathcal{M}}_+$ and $\widetilde{\mathcal{M}}_-$ of $\mathcal{M}_+$ and $\mathcal{M}_-$ transverse to the $\mathbf{R}$-translation as in Section I.7.12.3 so that if $u_+ \in \widetilde{\mathcal{M}}_+$ (resp. $u_- \in \widetilde{\mathcal{M}}_-$), then for $s \leq 0$

(resp. $s \geq 0$) the negative end of u_+ (resp. the positive end of u_-) has image in a small neighborhood of the section at ∞ and is dominated by the term of the form $c(u_+)e^{\lambda_+ s} f_+(t)$ (resp. $c(u_-)e^{\lambda_- s} f_-(t)$) when projected to the $\mathbf{C}$-direction. Here $f_+(t)$ (resp. $f_-(t)$) is a fixed L^2-normalized leading asymptotic eigenfunction of $A = -i\partial_t - \varepsilon$ for the negative end of u_+ (resp. positive end of u_-) with eigenvalue $\lambda_+ > 0$ (resp. $\lambda_- < 0$) and $c(u_\pm) \in \mathbf{C}$ satisfy $|c(u_\pm)| = \varepsilon_0$ for $\varepsilon_0 > 0$ small.

For $R \gg 0$, ξ is dominated by $d_+ e^{-\lambda_+ R} f_+(t)(ds - idt)$ for $s = R$ and is dominated by $d_- e^{\lambda_- R} f_-(t)(ds - idt)$ for $s = -R$, where d_+ and d_- are both nonzero.

As in [HT2], there exist an obstruction section $\mathfrak{s}$ and a linearized obstruction section $\mathfrak{s}_0$ which are both maps

$$(5r, \infty)^2 \times \widetilde{\mathcal{M}}_+ \times \widetilde{\mathcal{M}}_- \to \mathcal{O} := \mathrm{Hom}(\ker D^*, \mathbf{R}),$$

such that the set of gluable (T_+, T_-, u_+, u_-) — by this we mean we are gluing $u_+, \sigma_\infty^{T'}$, and u_- with gluing parameters $T_\pm$ that indicate how much $u_\pm$ is translated up/down when pregluing — is given by $\mathfrak{s}^{-1}(0)$. Following [HT2, Definition 8.1], $\mathfrak{s}_0$ has the form:

$$\begin{aligned}
(\mathbf{4.6.1}) \qquad \mathfrak{s}_0(T_+, T_-, u_+, u_-)(\xi) &= \langle c(u_+) e^{\lambda_+(R - T_+)} f_+(t), d_+ e^{-\lambda_+ R} f_+(t) \rangle \\
&\quad + \langle c(u_-) e^{\lambda_-(-R + T_-)} f_-(t), d_- e^{\lambda_- R} f_-(t) \rangle \\
&= e^{-\lambda_+ T_+} \langle c(u_+) f_+(t), d_+ f_+(t) \rangle \\
&\quad + e^{-|\lambda_-| T_-} \langle c(u_-) f_-(t), d_- f_-(t) \rangle,
\end{aligned}$$

where the brackets are $\mathbf{R}$-linear inner products in $L^2(\mathbf{R}/2\mathbf{Z}, \mathbf{C})$. The specific form of $\mathfrak{s}_0$ is not that important; what matters here is that $\mathfrak{s}_0$ is transverse to the zero section since $d_\pm \neq 0$ and $c(u_\pm) \neq 0$.

On the other hand, the rescaling argument from Section I.7.2.2 implies that, if a triple in A_6 is the limit of a sequence $\overline{u}_1, \overline{u}_2, \dots$ of curves in $\mathcal{M}$ and $\overline{u}_i \in \mathcal{M}_{\tau_i}$ (in particular $\overline{u}_i$ passes through the marked point $\overline{\mathfrak{m}}(\tau_i)$), then there is a unique *transverse limit profile* $w_0 : B_{T'} \to \mathbf{C}$ modulo multiplication by $\mathbf{R}^+$ such that:

(i) $w_0(+\infty) = \infty$;
(ii) $w_0(\overline{\mathfrak{m}}^b(T')) = 0$; and
(iii) $w_0(\partial B_{T'}) \subset \mathbf{R}^+$.

This in turn implies that there exist:

(1) a finite subset $Z_{\overline{\mathfrak{m}}^b(T')} \subset \widetilde{\mathcal{M}}_+ \times \widetilde{\mathcal{M}}_-$; and
(2) a 1-dimensional subset

$$\begin{aligned}
(\mathbf{4.6.2}) \qquad \mathcal{Z}_{\overline{\mathfrak{m}}^b(T')} = \{ &(T_+, T_-(T_+, u_+, u_-), u_+, u_-) \mid T_+ \geq T'_+, \\
&(u_+, u_-) \in Z_{\overline{\mathfrak{m}}^b(T')} \} \\
&\subset (5r, \infty)^2 \times \widetilde{\mathcal{M}}_+ \times \widetilde{\mathcal{M}}_- \quad \text{for some } T'_+ \gg 0,
\end{aligned}$$

such that if $(T_+, T_-, u_+, u_-) \in \mathcal{Z}_{\overline{\mathfrak{m}}^b(T')}$, then there exists a constant $\tilde{c} > 0$ such that:

- the negative end of the T_+-translate of u_+ projected to the $\mathbf{C}$-direction and the positive end of $\tilde{c}e^{-\varepsilon s}w_0$ — recall the factor $e^{-\varepsilon s}$ in the ansatz given by Equation (I.7.8.1) — agree up to first order (i.e., on the level of leading asymptotic eigenfunctions) on their overlap; and
- the same holds for the positive end of the T_--translate of u_- projected to the $\mathbf{C}$-direction and the negative end of $\tilde{c}e^{-\varepsilon s}w_0$.

In particular, all the $\overline{u}_i$, $i \gg 0$, are close to the pregluing of elements in $\mathcal{Z}_{\overline{\mathfrak{m}}^b(T')}$.

We can analogously define $\mathcal{Z}_x$ for x in a small disk $D(\overline{\mathfrak{m}}^b(T'))$ about $\overline{\mathfrak{m}}^b(T') \in \sigma_\infty^{T'}$, form the bundle

$$\mathcal{Z}_{D(\overline{\mathfrak{m}}^b(T'))} := \sqcup_{x \in D(\overline{\mathfrak{m}}^b(T'))} \mathcal{Z}_x \to D(\overline{\mathfrak{m}}^b(T')) \tag{4.6.3}$$

with 3-dimensional total space, and show that the natural map

$$\mathcal{Z}_{D(\overline{\mathfrak{m}}^b(T'))} \to (5r, \infty)^2 \times \widetilde{\mathcal{M}}_+ \times \widetilde{\mathcal{M}}_- \tag{4.6.4}$$

is an embedding. The details are left to the reader.

Note that, for $c \geq 0$ and sufficiently large T'_+, the simultaneous translation

$$\begin{aligned} \mathrm{Tr}_c : (5r, \infty)^2 \times \widetilde{\mathcal{M}}_+ \times \widetilde{\mathcal{M}}_- &\to (5r, \infty)^2 \times \widetilde{\mathcal{M}}_+ \times \widetilde{\mathcal{M}}_-, \\ (T_+, T_-, u_+, u_-) &\mapsto (T_+ + c|\lambda_-|, T_- + c\lambda_+, u_+, u_-) \end{aligned}$$

takes $\mathcal{Z}_x$ to $\mathcal{Z}_x \cap \{T_+ \geq T'_+ + c|\lambda_-|\}$, i.e., is Tr_c*-invariant.*

Let $\mathcal{N}$ be a small Tr_c-invariant neighborhood of a connected component of $\mathcal{Z}_{\overline{\mathfrak{m}}^b(T')}$ in $(5r, \infty)^2 \times \widetilde{\mathcal{M}}_+ \times \widetilde{\mathcal{M}}_-$ which nontrivially intersects $\mathfrak{s}_0^{-1}(0)$. By the form of Equation (4.6.1), $\mathfrak{s}_0$ is transverse to the zero section and, by the estimate [HT2, Lemma 8.7], there exists a Tr_c-invariant tubular neighborhood $(\mathfrak{s}_0^{-1}(0) \cap \mathcal{N}) \times (-\delta, \delta)$ of $\mathfrak{s}_0^{-1}(0) \cap \mathcal{N}$ (after slightly enlarging $\mathcal{N}$ if necessary) such that on $(\mathfrak{s}_0^{-1}(0) \cap \mathcal{N}) \times \{\pm\delta\}$:

(1) $|\mathfrak{s} - \mathfrak{s}_0| \ll |\mathfrak{s}_0|$ and
(2) as $T'_+ \to \infty$, the ratio $|\mathfrak{s} - \mathfrak{s}_0|/|\mathfrak{s}_0|$ goes to zero.

Hence $\mathfrak{s}^{-1}(0) \cap \mathcal{N}$ is arbitrarily close to and limits to $\mathfrak{s}_0^{-1}(0) \cap \mathcal{N}$ as $T'_+ \to +\infty$, and there is a degree 1 map

$$\mathfrak{s}^{-1}(0) \cap \mathcal{N} \cap \{T_+ = T'_+\} \to \mathfrak{s}_0^{-1}(0) \cap \mathcal{N} \cap \{T_+ = T'_+\}.$$

By the construction of the transverse limit profile, $\mathcal{Z}_{D(\overline{\mathfrak{m}}^b(T'))} \cap \mathcal{N} \cap \{T_+ = T'_+\}$ approaches $\mathfrak{s}^{-1}(0) \cap \mathcal{N} \cap \{T_+ = T'_+\}$ as $T'_+ \to \infty$ and hence must agree with $\mathfrak{s}_0^{-1}(0) \cap \mathcal{N} \cap \{T_+ = T'_+\}$ by the Tr_c-invariance. Hence there is a degree 1 map

$$\mathfrak{s}^{-1}(0) \cap \mathcal{N} \cap \{T_+ = T'_+\} \to D(\overline{\mathfrak{m}}^b(T'))$$

and the condition of passing through $\overline{\mathfrak{m}}^b(T')$ is a transverse condition. The gluings in A_6 therefore occur in pairs.

Case (4₂). Next we treat A_7, i.e., Case (4_2) with two cylinder components of $\overline{v}^\sharp_{-1,1}$, one from δ_0 to h and another from δ_0 to e.

As in Case (4_2), there is an obstruction bundle

$$\mathcal{O} := \mathrm{Hom}(\ker D^*, \mathbf{R}) \to (5r, \infty)^3 \times \widetilde{\mathcal{M}}_+ \times \mathcal{M}_0 \times \widetilde{\mathcal{M}}_{-,h} \times \widetilde{\mathcal{M}}_{-,e},$$

where $\mathcal{M}_0$ is the moduli space of branched covers of $\sigma_\infty^{T'}$ with one branch point, $\widetilde{\mathcal{M}}_{-,h}$ is a one-point set consisting of a bottom level cylinder from δ_0 to h, and $\widetilde{\mathcal{M}}_{-,e}$ is a 1-dimensional set consisting of bottom level cylinders from δ_0 to e, $\mathcal{O}$ has rank 4, and the base has dimension 7. There are also obstruction sections $\mathfrak{s}$ and $\mathfrak{s}_0$, where $\mathfrak{s}_0$ is defined as in Equation (4.6.1) and is the sum of three terms, each given by pairing the cokernel elements with the first term (possibly nonzero) of the Fourier expansions of the ends. A winding condition for any nonzero cokernel element ξ analogous to that of Claim 4.6.5 implies that $\mathfrak{s}_0$ is transverse to the zero section. The verification of the claims in this paragraph are left to the reader.

Next we consider the moduli space $\mathcal{M}_\infty$ of pairs (w_0, π_0) consisting of a holomorphic map $w_0 : \Sigma_0 \to \mathbf{CP}^1$ and a branched double cover $\pi_0 : \Sigma_0 \to cl(B_{T'})$ satisfying (i)–(iv) in the proof of Case (4_2) of Lemma 4.6.4 for some $a_1, \dots, a_4$. There is an action of $\mathbf{R}^+$ on $\mathcal{M}_\infty$ given by $(c, (w_0, \pi_0)) \mapsto (cw_0, \pi_0)$.

Claim **4.6.6.** — *The mod 2 count of transverse limit profiles $[(w_0, \pi_0)] \in \mathcal{M}_\infty/\mathbf{R}^+$, i.e., those that arise when taking the limit of a sequence $\overline{u}_1, \overline{u}_2, \dots$ of curves in $\mathcal{M}$ such that $\overline{u}_i$ passes through the marked point $\overline{\mathfrak{m}}(\tau_i)$, is* 1.

Proof of Claim 4.6.6. — One can verify that:

(a) $\dim \mathcal{M}_\infty/\mathbf{R}^+ = 1$ and is parametrized by moving a branched point of w_0 along $\mathcal{L}_{1/2} \cup (\mathcal{L}_{3/2} \cap \{s \le \frac{-l(T')}{2}\})$;
(b) either $w_0(\pi_0^{-1}(-\infty)) \subset [a_2, a_3]$ or is on a line orthogonal to $[a_2, a_3]$ at $a^* \in [a_2, a_3]$, which maps to a branch point of π_0.

Let $\widetilde{\mathcal{M}}_\infty$ be the set of $(w_0, \pi_0) \in \mathcal{M}_\infty$ together with an ordering of the points of $\pi^{-1}(-\infty)$; if $\pi^{-1}(-\infty)$ is a single point, we view it as a duplicate pair of points. The forgetful map $\widetilde{\mathcal{M}}_\infty \to \mathcal{M}_\infty$ is generically a double cover. Then consider the following evaluation map:

$$ev_\infty : \widetilde{\mathcal{M}}_\infty/\mathbf{R}^+ \to S^1 \times S^1,$$

$$(w_0, \pi_0) \mapsto \arg(w_0(\pi^{-1}(-\infty))).$$

Here arg refers to the projection to the ϕ-coordinate, and we are using coordinates (ϕ_1, ϕ_2) on $S^1 \times S^1$.

Let $Z = \{\phi_1 = c\} \subset S^1 \times S^1$, where c is a small negative constant. (b) implies that ev_∞ has intersection number 1 (mod 2) with Z. The claim then follows from noting that specifying c corresponds to choosing a radial direction corresponding to the hyperbolic orbit h. □

The same conclusion also holds if we take $m \gg 0$ fixed and if replace (ii) in the proof of Case (4_2) of Lemma 4.6.4 by "$w_0(z_1) = 0$ for some $z_1 \in \pi_0^{-1}(x)$", where $x \in D(\overline{\mathfrak{m}}^b(T'))$. We can define the 3-dimensional manifold $\mathcal{Z}_{D(\overline{\mathfrak{m}}^b(T'))}$ as in Equation (4.6.3) such that the analog of Equation (4.6.4) is an embedding; the embedding condition implies the transversality of the point condition for $\overline{\mathfrak{m}}^b(T')$.

Once we have the obstruction bundle, the obstruction sections $\mathfrak{s}_0$ and $\mathfrak{s}$, and the embedding property of $\mathcal{Z}_{D(\overline{\mathfrak{m}}^b(T'))}$, the rest of the argument proceeds as in Case (4_2). Again we note that the branched double cover of $\sigma_\infty^{T'}$ can be viewed as having boundary mapping to $L^{T'}_{\overline{a}_{i_k},0}$ or $L^{T'}_{\overline{a}_{i_k},1}$ for $k = 1, 2$. Hence the count corresponding to A_7 is 0 mod 2. □

5. Stabilization

The goal of this section is to prove Theorem 1.0.2. Let $N = N_{(S,h)}$ be the mapping torus of (S, h), where S is a bordered surface of genus g with connected boundary and $h : (S, \omega) \xrightarrow{\sim} (S, \omega)$ is a symplectomorphism which has zero flux and restricts to the identity on ∂S.

The strategy of the proof is to apply two positive stabilizations to (S, h) — corresponding to the connected sum with a trefoil knot — to obtain (S', h'), where S' has connected boundary and genus $g + 1$. We then compare $\Phi_{(S,h)}$ and $\Phi_{(S',h')}$, which both induce isomorphisms on the level of homology. Here $\Phi_{(S,h)}$ is the Φ map for (S, h).

5.1. *The setup.* — Let T be a genus one surface with connected boundary and let η_0, η_1 be two essential simple closed curves on T which intersect transversely in one point. A positive Dehn twist along a closed curve η will be denoted by τ_η. Let $h_T : T \xrightarrow{\sim} T$ be the first return map of a Reeb vector field on the mapping torus

$$N_T = N_{(T,h_T)} = (T \times [0, 1])/((x, 1) \sim (h_T(x), 0))$$

so that the following hold:

(1) h_T is isotopic to $\tau_{\eta_0} \circ \tau_{\eta_1}$ relative to the boundary;
(2) all the Reeb orbits in the interior of N_T which intersect $T \times \{0\}$ at most $2g + 2$ times are nondegenerate;
(3) $h_T|_{\partial T} = id$ and ∂N_T is foliated by a negative Morse-Bott family of slope ∞.

In view of the discussion in Section I.3, h_T will be viewed interchangeably as (i) the first return map of a Reeb vector field or (ii) the time-1 map of a stable Hamiltonian vector field with zero flux.

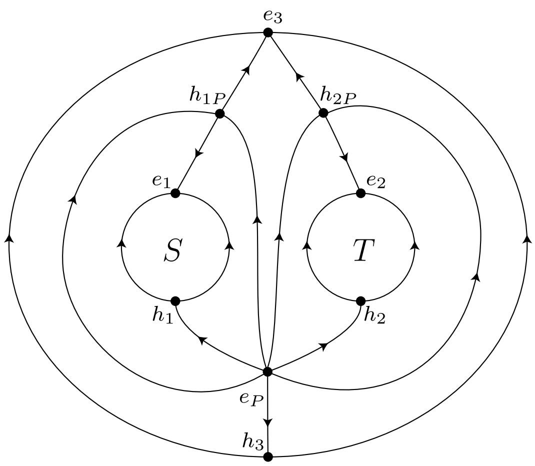


FIG. 20. — The page S′. The gradient trajectories of the Morse function f are given

Now let S′ be the boundary connected sum of S and T. More precisely, if P is a pair-of-pants with boundary $\partial P = \partial_1 P \sqcup \partial_2 P \sqcup \partial_3 P$, then S′ is obtained from $S \sqcup T \sqcup P$ by identifying $\partial S \simeq -\partial_1 P$ and $\partial T \simeq -\partial_2 P$. Observe that $\partial S' = \partial_3 P$ is connected and $g(S') = g + 1$. See Figure 20.

We now define a symplectomorphism $h' : (S', \omega') \stackrel{\sim}{\to} (S', \omega')$ with zero flux (with respect to some ω') as follows: First set $h'|_S = h$ and $h'|_T = h_T$. Then define $h'|_P$ as the first return map of a Reeb flow R_α on

$$N_P = N_{(P,id)} = (P \times [0, 1])/((x, 1) \sim (x, 0)).$$

The contact form α is given by $f_0 dt + \beta$, where f_0 is a function and β is a 1-form on P, and both f_0 and β do not depend on t. We choose a Morse-Bott function $f_0 : P \to \mathbf{R}$ which satisfies the following:

(1) f_0 is C^k-close to 1 for $k \gg 0$;
(2) f_0 attains its minimum along the Morse-Bott family ∂P;
(3) the critical points of f_0 in $int(P)$ are isolated and consist of the maximum e_P and two saddles h_{1P}, h_{2P}.

The Reeb orbits corresponding to e_P, h_{1P}, h_{2P} will also be denoted by e_P, h_{1P}, h_{2P}.

Let $\mathcal{N}_i$, $i = 1, 2, 3$, be the negative Morse-Bott family of Reeb orbits corresponding to $\partial_i P$. If f_0 is C^k-close to 1 for $k \gg 0$, then the only orbits that intersect $P \times \{0\}$ at most $2g + 2$ times are: e_P, h_{iP} and the orbits of $\mathcal{N}_i$. We pick two orbits in each $\mathcal{N}_i$ and label them e_i, h_i; they will become elliptic and hyperbolic when the Morse-Bott function f_0 is perturbed into a Morse function, which we call f. For convenience we write N_S for the mapping torus of (S, h), $N_{S'}$ for the mapping torus of (S', h'), etc.

Since (S', h') is obtained from (S, h) by applying two positive stabilizations, the corresponding contact structures $\xi_{(S,h)}$ and $\xi_{(S',h')}$ are isomorphic. As a special case, observe that if $(S, h) = (D^2, id)$, then $(S', h') = (T, h_T)$ and (T, h_T) is an open book decomposition for the standard tight contact structure on S^3.

5.2. *Morse-Bott theory.* — Let J be an adapted almost complex structure on $\mathbf{R} \times N_{S'}$ and let $\pi : \mathbf{R} \times N_{S'} \to N_{S'}$ be the projection onto the second factor. Also let u be a finite energy Morse-Bott building in $\mathbf{R} \times N_{S'}$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$, where both orbit sets intersect $S' \times \{0\}$ in $2g + 2$ points.

The following lemma partitions the irreducible components of u into three regions. When u is a Morse-Bott building, then we say that u is "irreducible" if the holomorphic curve, obtained by perturbing the Morse-Bott contact form to a nondegenerate one and correspondingly gluing up the levels of the Morse-Bott building, is irreducible.

Lemma **5.2.1.** — *Every irreducible component of the Morse-Bott building u in $\mathbf{R} \times N_{S'}$ has image in one of $\mathbf{R} \times N_S$, $\mathbf{R} \times N_T$, or $\mathbf{R} \times N_P$.*

Proof. — The lemma is an application of the winding number wind_π of [HWZ1] and the positivity of intersections. (For example, see [0, Lemma 5.5.1].)

Suppose without loss of generality that $u : \dot{F} \to \mathbf{R} \times N_{S'}$ is a single-level Morse-Bott building and that $u(\dot{F})$ nontrivially intersects $\mathbf{R} \times N_P$; the case of a multiple-level Morse-Bott building only differs in notation. If u corresponds to a gradient trajectory from h_i to e_i, then the lemma holds. Otherwise let $P_\varepsilon \subset P$ be a slight retraction of P so that $\partial P_\varepsilon = \partial_1^\varepsilon P \sqcup \partial_2^\varepsilon P \sqcup \partial_3 P$. Let $\Pi : N_P = P \times S^1 \to P$ be the projection onto the first factor, let $T_i^\varepsilon = \Pi^{-1}(\partial_i^\varepsilon P)$, $i = 1, 2$, and let $N_{P_\varepsilon} = \Pi^{-1}(P_\varepsilon)$. We may assume that T_i^ε is foliated by (not necessarily closed) Reeb orbits. We consider the intersection $\delta_i = \pi(u(\dot{F})) \cap T_i^\varepsilon$, where δ_i is given the boundary orientation of $\pi(u(\dot{F})) \cap N_{P_\varepsilon}$. The curve δ_i is transverse to the Reeb vector field away from a finite number of points by [0, Lemma 5.5.1]. If δ_i is not homologous to a multiple of $\{pt\} \times S^1$ in $H_1(T_i^\varepsilon)$, then $\Pi(\delta_i)$ is in the homology class $k[\partial_i P_\varepsilon] \in H_1(P_\varepsilon)$, $k > 0$, by the positivity of intersections in dimension four. This implies that $\Pi(\pi(u(\dot{F})) \cap \partial N_{S'})$ is in the class $k[\partial_3 P] \in H_1(P)$ for $k > 0$. Since this is a contradiction, we must have $[\delta_i] = l[\{pt\} \times S^1] \in H_1(T_i^\varepsilon)$, for some $l \geq 0$. This implies that u has negative ends along $\mathcal{N}_i$, $i = 1, 2$. Hence u has image in $\mathbf{R} \times N_P$. □

We now use Morse-Bott theory (cf. Bourgeois [Bo1, Bo2]) to analyze holomorphic curves on $\mathbf{R} \times N_P$. In particular, we consider a perturbation of the Morse-Bott family of orbits on N_P, perturbed by the Morse function $f : P \to \mathbf{R}$ as described above.

Lemma **5.2.2.** — *Let $f : P \to \mathbf{R}$ be C^k-close to 1 for $k \gg 0$. Then there is an adapted almost complex structure J on $\mathbf{R} \times N_P$ such that every $I_{ECH} = 1$ finite energy J-holomorphic curve u whose image is in $\mathbf{R} \times N_P$, is a simply-covered cylinder which corresponds to a gradient flow line between critical points of f of adjacent index. The complete list is as follows:*

(1) *one cylinder from e_P to h_i, $i = 1, 2, 3$, and two cylinders from e_P to h_{iP}, $i = 1, 2$;*

(2) *two cylinders from* h_i *to* e_i, $i = 1, 2, 3$;
(3) *one cylinder from* h_{1P} *to* e_1 *and one cylinder from* h_{1P} *to* e_3;
(4) *one cylinder from* h_{2P} *to* e_2 *and one cylinder from* h_{2P} *to* e_3.

Moreover, all the $I_{ECH} = 1$ *curves above are regular.*

See Figure 20 for the gradient trajectories of f which correspond to the above holomorphic cylinders.

Proof. — The proof is very similar to that of [0, Lemma 9.2.2]. We use the following fact, which can be proved using Morse-Bott theory or by a direct computation:

Fact. There is an adapted almost complex structure J on $\mathbf{R} \times N_P$ such that there is a one-to-one correspondence between (parametrized) gradient trajectories $\delta : \mathbf{R} \to P$ of f and finite energy J-holomorphic cylinders Z_δ in $\mathbf{R} \times N_P$ which intersect each $\{(s, t)\} \times P$ exactly once and project to $\mathrm{Im}(\delta)$ under the projection $\Pi \circ \pi : \mathbf{R} \times N_P \to P$, $(s, x, t) \mapsto x$. Moreover, the cylinders Z_δ, together with the trivial cylinders over the orbits corresponding to the Morse critical points, give a finite energy foliation of $\mathbf{R} \times N_P$.

Fix f and J as above. We will use the notation $s\delta : \mathbf{R} \to P$ for the translation $(s\delta)(\tau) = \delta(\tau + s)$.

Let $u : \dot{F} \to \mathbf{R} \times N_P$ be a finite energy J-holomorphic curve. Let $D_\varepsilon \subset P$ be an arbitrarily small disk centered at the point e_P and let $N(e_P) = \Pi^{-1}(D_\varepsilon)$ be a solid torus neighborhood of the orbit e_P. We assume that $\partial N(e_P)$ is foliated by (not necessarily closed) orbits of the Reeb vector field. We identify $\partial N(e_P) \simeq \mathbf{R}^2/\mathbf{Z}^2$ so that the meridian has slope zero and a fiber $\{pt\} \times S^1$ has slope ∞. Consider $\eta = \pi(u(\dot{F})) \cap \partial N(e_P)$, where η is given the boundary orientation of $\pi(u(\dot{F})) \cap \Pi^{-1}(P - int(D_\varepsilon))$. If the projection $[\Pi(\eta)] \in H_1(P - int(D_\varepsilon))$ is nonzero, then $[\Pi(\eta)] = -k[\partial D_\varepsilon]$, $k > 0$, by the positivity of intersections. This is a contradiction as in the proof of Lemma 5.2.1. Hence $[\eta] = l[\{pt\} \times S^1] \in H_1(\partial N(e_P))$ for some $l \geq 0$, i.e., has slope ∞. In other words, u cannot intersect $\mathbf{R} \times e_P$ and can only have e_P at the positive end. Similar considerations hold for $N(e_i)$, $i = 1, 2, 3$, where $D_{i,\varepsilon} \subset P$ is a half-disk centered at e_i and $N(e_i) = \Pi^{-1}(D_{i,\varepsilon})$.

We now claim that u is some multiple cover of some Z_δ with multiplicity ≥ 1. Arguing by contradiction, suppose u does not multiply cover any Z_δ. Let us first consider the case where $\Pi \circ \pi(u(\dot{F}))$ does not equal $\mathrm{Im}(\delta)$ for any δ. Then there is some Z_δ from e_P to some e_i such that the intersection $u(\dot{F}) \cap Z_\delta$ is nonempty; moreover, in view of the asymptotics on $N(e_P)$ and $N(e_i)$, we may assume that $K = \pi(u(\dot{F}) \cap Z_\delta)$ is compact. This implies that $u(\dot{F})$ and $Z_{s\delta}$ do not intersect for sufficiently large s. On the other hand, since the intersection pairing $\langle u(\dot{F}), Z_{s\delta}\rangle$ is a homological quantity and does not depend on s due to the asymptotics, it follows that $K = \varnothing$, which is a contradiction. This implies that $\Pi \circ \pi(u(\dot{F})) = \mathrm{Im}(\delta)$ for some δ. Now $\mathbf{R} \times \Pi^{-1}(\delta)$ is a 3-manifold which is foliated by $Z_{s\delta}$, $s \in \mathbf{R}$, and if u does not multiply cover any $Z_{s\delta}$, then u intersects some $Z_{s\delta}$ along a 1-manifold, a contradiction. We conclude that u is a multiple cover of some Z_δ.

Now Fredholm index one cylinders Z_δ — i.e., those that correspond to δ connecting two Morse critical points of adjacent index — are regular by the automatic transversality results of Wendl [We1, We2]. Moreover, u cannot multiply cover Z_δ with multiplicity > 1 by [HT1, Proposition 7.15], since otherwise $I(u) > 1$. This implies that u is equal to some Z_δ, thereby completing the proof of the lemma. □

5.3. *Computation of* PFH($N_{S'}$). — In this section we compute PFH($N_{S'}$) in terms of PFH(N_S) and PFH(N_T). From now on we will write PFC(S, h) for PFC(N_S), with the understanding that the periodic Floer homology group is defined using a stable Hamiltonian structure induced by the fibration, and whose stable Hamiltonian vector field has first return map h. Similar notation will be used for PFC(N_T) and PFC($N_{S'}$). Sometimes we will even drop the monodromy from the notation for the periodic Floer homology groups.

5.3.1. *Description of the differential of* PFH(S′, h'). — Given two orbit sets $\boldsymbol{\gamma}' = \prod \gamma_i^{m'_i}$ and $\boldsymbol{\gamma} = \prod \gamma_i^{m_i}$, we set $\boldsymbol{\gamma}/\boldsymbol{\gamma}' = \prod \gamma_i^{m_i - m'_i}$ if $m'_i \leq m_i$ for all i; otherwise we set $\boldsymbol{\gamma}/\boldsymbol{\gamma}' = 0$.

The chain group $PFC_k(S')$ can be written as:

$$PFC_k(S') = \bigoplus_{m+i+j=k} \mathbf{F}[h_{1P}, h_{2P}, h_3, e_P, e_3]_m \otimes PFC_i(S) \otimes PFC_j(T).$$

$\mathbf{F}[h_{1P}, h_{2P}, h_3, e_P, e_3]$ is a polynomial ring where h_{1P}, h_{2P}, h_3 (resp. e_P, e_3) are considered as Grassmann variables of odd degree (resp. even degree) and the subscript m indicates the subspace spanned by monomials with total exponent m.

Let us write a generator of $PFC_k(S')$ as $\boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2$, where $\boldsymbol{\Gamma}_1 \in PFC_i(S)$, $\boldsymbol{\Gamma}_2 \in PFC_j(T)$, and $\boldsymbol{\gamma}$ is constructed from orbits passing through $S' - S - T$. Using Lemma 5.2.1 and the description of ECH index one curves in $\mathbf{R} \times N_P$ from Lemma 5.2.2, we write the differential ∂ of $PFC_k(S')$ as follows:

$$\begin{aligned}
&\partial(\boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2)\\
&\quad = \boldsymbol{\gamma} \otimes (\partial_S \boldsymbol{\Gamma}_1) \otimes \boldsymbol{\Gamma}_2 + \boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes (\partial_T \boldsymbol{\Gamma}_2)\\
&\qquad + (\boldsymbol{\gamma}/e_P)(h_3 \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2 + 1 \otimes h_1 \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2 + 1 \otimes \boldsymbol{\Gamma}_1 \otimes h_2 \boldsymbol{\Gamma}_2)\\
&\qquad + (\boldsymbol{\gamma}/h_{1P})(e_3 \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2 + 1 \otimes e_1 \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2)\\
&\qquad + (\boldsymbol{\gamma}/h_{2P})(e_3 \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2 + 1 \otimes \boldsymbol{\Gamma}_1 \otimes e_2 \boldsymbol{\Gamma}_2).
\end{aligned}$$

Here ∂_S and ∂_T are the differentials on PFC(S) and PFC(T).

5.3.2. *Spectral sequence calculation.* — In this subsection we use spectral sequences to prove the following:

Lemma **5.3.1.** — *The inclusion*

$$\bigoplus_{i+j=k} \mathrm{PFC}_i(\mathrm{S}) \otimes \mathrm{PFH}_j(\mathrm{T}) \to \mathrm{PFC}_k(\mathrm{S}')$$

induces an isomorphism between the quotient $\left(\bigoplus_{i+j=k} \mathrm{PFH}_i(\mathrm{S}) \otimes \mathrm{PFH}_j(\mathrm{T})\right) / \sim$ *and* $\mathrm{PFH}_k(\mathrm{S}')$, *where the equivalence relation* $\sim$ *is given by* $e_1\boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2 \sim \boldsymbol{\Gamma}_1 \otimes e_2\boldsymbol{\Gamma}_2$.

Proof. — Let $\mathcal{F}$ be a filtration on $(\mathrm{PFC}_k(\mathrm{S}'), \partial)$ which, on the generators, counts the multiplicity of $h_{1\mathrm{P}}$. This means that $\mathcal{F}$ takes values in $\{0, 1\}$. We write $(\mathrm{E}^r(\mathcal{F}), \partial_r)$ for the E^r-term and the E^r-differential of the spectral sequence for $\mathcal{F}$. Each page $\mathrm{E}^r(\mathcal{F})$ has a grading coming from $\mathcal{F}$ and $\mathrm{E}^r_\ell(\mathcal{F})$ is the degree ℓ component of $\mathrm{E}^r(\mathcal{F})$ with respect to this grading. We remark that the spectral sequence associated to $\mathcal{F}$ is nothing but a long exact sequence in homology, and its use is motivated by our wish to give a parallel treatment of the cases where we filter by the multiplicity of a hyperbolic orbit or by the multiplicity of an elliptic orbit.

Next let $\mathcal{G}$ be a filtration on $(\mathrm{E}^0(\mathcal{F}), \partial_0)$ which counts the multiplicity of $h_{2\mathrm{P}}$. Again, $\mathcal{G}$ takes values in $\{0, 1\}$. We write $(\mathrm{E}^r(\mathcal{G}), \partial_{0r})$ for the E^r-term and the E^r-differential of the spectral sequence for $\mathcal{G}$. Finally, let $\mathcal{H}$ be the filtration on $(\mathrm{E}^0(\mathcal{G}), \partial_{00})$ which counts the multiplicity of e_P, and let $(\mathrm{E}^r(\mathcal{H}), \partial_{00r})$ be the E^r-term and the E^r-differential of the spectral sequence for $\mathcal{H}$.

We first consider $(\mathrm{E}^0(\mathcal{H}), \partial_{000})$, where:

$$\partial_{000}(\boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2) = \boldsymbol{\gamma} \otimes (\partial_\mathrm{S}\boldsymbol{\Gamma}_1) \otimes \boldsymbol{\Gamma}_2 + \boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes (\partial_\mathrm{T}\boldsymbol{\Gamma}_2).$$

By the Künneth formula, we have:

$$\mathrm{E}^1(\mathcal{H}) = \bigoplus_{m+i+j=k} \mathbf{F}[h_{1\mathrm{P}}, h_{2\mathrm{P}}, h_3, e_\mathrm{P}, e_3]_m \otimes \mathrm{PFH}_i(\mathrm{S}) \otimes \mathrm{PFH}(\mathrm{T})_j.$$

Next consider $(\mathrm{E}^1(\mathcal{H}), \partial_{001})$, where:

$$\begin{aligned}\partial_{001}(e_\mathrm{P}^n\boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2) = {} & e_\mathrm{P}^{n-1}h_3\boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2 + e_\mathrm{P}^{n-1}\boldsymbol{\gamma} \otimes (h_1\boldsymbol{\Gamma}_1) \otimes \boldsymbol{\Gamma}_2 \\ & + e_\mathrm{P}^{n-1}\boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes (h_2\boldsymbol{\Gamma}_2),\end{aligned}$$

Here $\boldsymbol{\Gamma}_1 \in \mathrm{PFH}_i(\mathrm{S})$, $\boldsymbol{\Gamma}_2 \in \mathrm{PFH}_j(\mathrm{T})$, and $\boldsymbol{\gamma}$ has no e_P term. Note that any $e_\mathrm{P}^{n-1}h_3\boldsymbol{\gamma} \otimes \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_2$ is homologous (with respect to the differential ∂_{001}) to a linear combination of $e_\mathrm{P}^{n-1}\boldsymbol{\gamma}' \otimes \boldsymbol{\Gamma}'_1 \otimes \boldsymbol{\Gamma}'_2$, where $\boldsymbol{\gamma}'$ does not have any e_P and h_3 terms, $\boldsymbol{\Gamma}'_1$ is in some $\mathrm{PFH}_i(\mathrm{S})$, and $\boldsymbol{\Gamma}'_2$ is in some $\mathrm{PFH}(\mathrm{T})_j$. Hence every element of $\mathrm{E}^1(\mathcal{H})/\mathrm{Im}(\partial_{001})$ can be represented by $w = \sum_{i\geq 0} e_\mathrm{P}^i w_i$, where w_i has no e_P and h_3 terms, and we can write

$$\partial_{001}(w) = \sum_{i>0} e_\mathrm{P}^{i-1}h_3w_i + w',$$

where w' has no terms which contain h_3. If $\partial_{001}(w)=0$, then all the w_i, $i>0$, must be zero. We therefore obtain:

$$\mathrm{E}^2(\mathcal{H})=\bigoplus_{m+i+j=k}\mathbf{F}[h_{1\mathrm{P}},h_{2\mathrm{P}},e_3]_m\otimes \mathrm{PFH}_i(\mathrm{S})\otimes \mathrm{PFH}_j(\mathrm{T}).$$

Since $\mathrm{E}^2(\mathcal{H})$ is supported in degree 0, the spectral sequence for $\mathcal{H}$ converges at the E^2-term and we have $\mathrm{E}^1(\mathcal{G})\simeq \mathrm{E}^2(\mathcal{H})$. Moreover, $\mathrm{E}^2(\mathcal{H})=\mathrm{E}^2_0(\mathcal{H})$ is naturally isomorphic to $\mathrm{E}^1(\mathcal{G})$ since 0 is the lowest filtration level.

Now consider $(\mathrm{E}^1(\mathcal{G}),\partial_{01})$, where

$$\partial_{01}(\boldsymbol{\gamma}\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2)=e_3(\boldsymbol{\gamma}/h_{2\mathrm{P}})\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2+(\boldsymbol{\gamma}/h_{2\mathrm{P}})\otimes\boldsymbol{\Gamma}_1\otimes e_2\boldsymbol{\Gamma}_2.$$

The calculation of $\mathrm{E}^2(\mathcal{G})$ is similar to the calculation of $\mathrm{E}^2(\mathcal{H})$ in the previous paragraph. Any $e_3^{n+1}\boldsymbol{\gamma}\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2$ is homologous to a linear combination of $e_3^n\boldsymbol{\gamma}'\otimes\boldsymbol{\Gamma}_1'\otimes\boldsymbol{\Gamma}_2'$, where $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}'$ do not have any $h_{2\mathrm{P}}$ and e_3 terms. Hence every element of $\mathrm{E}^1(\mathcal{G})/\operatorname{Im}(\partial_{01})$ can be represented by $w=w'+\sum_{i=0}^k h_{2\mathrm{P}}e_3^i w_i$, where w' and w_i have no $h_{2\mathrm{P}}$ or e_3 terms, and we can write

$$\partial_{01}w=\sum_{i=0}^k e_3^{i+1}w_i+\sum_{i=1}^k e_3^i e_2 w_i.$$

If $\partial_{01}w=0$, then all the w_i must be zero. Only the w' term remains, and we have:

$$\mathrm{E}^2(\mathcal{G})=\bigoplus_{m+i+j=k}\mathbf{F}[h_{1\mathrm{P}}]_m\otimes\mathrm{PFH}_i(\mathrm{S})\otimes\mathrm{PFH}_j(\mathrm{T}),$$

which we can write as a direct sum $\mathfrak{L}_0\oplus\mathfrak{L}_1$, where:

$$\mathfrak{L}_0=\bigoplus_{i+j=k}\mathrm{PFH}_i(\mathrm{S})\otimes\mathrm{PFH}_j(\mathrm{T}),$$

$$\mathfrak{L}_1=\bigoplus_{i+j=k-1}\mathbf{F}\{h_{1\mathrm{P}}\}\otimes\mathrm{PFH}_i(\mathrm{S})\otimes\mathrm{PFH}_j(\mathrm{T}).$$

Since $\mathrm{E}^2(\mathcal{G})$ is supported in the lowest degree 0, the spectral sequence for $\mathcal{G}$ converges at the E^2-term and $\mathrm{E}^2(\mathcal{G})=\mathrm{E}^2_0(\mathcal{G})$ is naturally isomorphic to $\mathrm{E}^1(\mathcal{F})$.

Finally, $\mathrm{E}^1(\mathcal{F})\simeq\mathfrak{L}_0\oplus\mathfrak{L}_1$ has differential ∂_1 given by:

$$\partial_1(\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2)=0, \tag{5.3.1}$$

$$\partial_1(h_{1\mathrm{P}}\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2)=e_1\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_1\otimes e_2\boldsymbol{\Gamma}_2, \tag{5.3.2}$$

since $\partial(h_{1\mathrm{P}}\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2)=e_1\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2+e_3\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2$ and $\partial(h_{2\mathrm{P}}\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2)=\partial_0(h_{2\mathrm{P}}\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2)=e_3\otimes\boldsymbol{\Gamma}_1\otimes\boldsymbol{\Gamma}_2+\boldsymbol{\Gamma}_1\otimes e_2\boldsymbol{\Gamma}_2$. Viewing the spectral sequence as a long exact sequence with connecting homomorphism $\mathfrak{L}_1 \xrightarrow{\delta} \mathfrak{L}_0$, where δ is given by Equation (5.3.2), we see that $E^2(\mathcal{F})$ is isomorphic to $\mathfrak{L}_0/\operatorname{Im}(\delta) = \mathfrak{L}_0/\sim$, where the equivalence relation is given by $e_1\mathbf{\Gamma}_1 \otimes \mathbf{\Gamma}_2 \sim \mathbf{\Gamma}_1 \otimes e_2\mathbf{\Gamma}_2$. As argued previously, the spectral sequence converges at the E^2-term and $E^2(\mathcal{F})$ is naturally isomorphic to $PFH_k(S')$. □

5.4. *Stabilization maps.* — For every integer $k \geq g$ we define

$$\mathfrak{i}_k : PFC_{2k}(S, \hbar) \to PFC_{2k}(S', \hbar')$$

as the map induced by the inclusion, and

$$\mathfrak{j}_k : PFC_{2k}(S, \hbar) \to PFC_{2k+2}(S, \hbar)$$

as $\mathfrak{j}_k(\boldsymbol{\gamma}) = e_1^2\boldsymbol{\gamma}$ for every orbit set $\boldsymbol{\gamma} \in PFC_{2k}(S, \hbar)$. We also define the stabilization map $\mathfrak{S}_k = \mathfrak{i}_{k+1} \circ \mathfrak{j}_k$.

The maps $\mathfrak{j}_k$ are chain maps by [0, Lemma 5.3.2] and the maps $\mathfrak{i}_k$ are chain maps by Lemma 5.2.1; therefore the maps $\mathfrak{S}_k$ are also chain maps.

Lemma **5.4.1.** — *If $(\mathfrak{S}_k)_*$ is an isomorphism, then $(\mathfrak{j}_k)_*$ is an isomorphism.*

Proof. — If $(\mathfrak{S}_k)_*$ is an isomorphism, then $(\mathfrak{j}_k)_* : PFH_{2g}(S) \to PFH_{2g+2}(S)$ is injective. It remains to show that it is surjective.

First note that $PFH_0(T) = \mathbf{F}\{1\}$. We also have $PFH_2(T) = \mathbf{F}\{e_2^2\}$, since $\widehat{HF}(S^3) \simeq \mathbf{F}$, generated by the contact class $c(\xi_{std})$ of the standard tight contact structure ξ_{std} on S^3, and e_2^2 is the image of $c(\xi_{std})$ under the isomorphism

$$(\Phi_{(T,\hbar_T)})_* : \widehat{HF}(S^3) = \widehat{HF}(T, \hbar_T) \xrightarrow{\sim} PFH_2(T).$$

Now, since the isomorphism $PFH_0(T) \xrightarrow{\sim} PFH_2(T)$, $1 \mapsto e_2^2$, factors as

$$PFH_0(T) \to PFH_1(T) \to PFH_2(T), \quad 1 \mapsto e_2 \mapsto e_2^2,$$

it follows that $PFH_1(T) = \mathbf{F}\{e_2\} \oplus W$ for some $\mathbf{F}$-vector space W.

By Lemma 5.3.1,

$$PFH_{2k+2}(S') \simeq \left(\bigoplus_{i+j=2k+2} PFH_i(S) \otimes PFH_j(T) \right) / \sim .$$

Moreover, $PFH_{2k}(S) \otimes PFH_2(T) = PFH_{2k}(S) \otimes \mathbf{F}\{e_2^2\}$ is the image of $PFH_{2k}(S)$ under the map $(\mathfrak{S}_k)_*$. Since this map is an isomorphism, every element of $PFH_{2k+2}(S) \otimes PFH_0(T) = PFH_{2k+2}(S) \otimes \mathbf{F}\{1\}$ is equivalent to some element of $PFH_{2k}(S) \otimes PFH_2(T)$, i.e., if $v_{2k+2} \in \mathrm{PFH}_{2k+2}(S)$, then $v_{2k+2} \otimes 1 \sim v_{2k} \otimes e_2^2$ for some $v_{2k} \in \mathrm{PFH}_{2k}(S)$. More explicitly,

$$v_{2k+2} \otimes 1 + v_{2k} \otimes e_2^2 = (e_1 v'_{2k+1} \otimes 1 + v'_{2k+1} \otimes e_2) + (e_1 v'_{2k} \otimes e_2 + v'_{2k} \otimes e_2^2) + \sum_i (e_1 v''_{2k,i} \otimes w_i + v''_{2k,i} \otimes e_2 w_i) + \cdots ,$$

where $v'_{2k+1} \in \mathrm{PFH}_{2g+1}(S)$; $v'_{2k}, v''_{2k,i} \in \mathrm{PFH}_{2g}(S)$; $\{w_i\}$ is a basis for W; and $\dots$ is a linear combination of terms which are not in $\mathrm{PFH}_{2k+2}(S) \otimes \mathrm{PFH}_0(T)$ and $\mathrm{PFH}_{2k+1}(S) \otimes \mathrm{PFH}_1(T)$. A term-by-term comparison gives $v_{2k+2} = e_1 v'_{2k+1}$ and $v'_{2k+1} = e_1 v'_{2k}$. Hence $v_{2+2} = e_1^2 v'_{2k}$ and v_{2k+2} is in the image of $(\mathfrak{j}_k)_*$. □

Corollary **5.4.2.** — *If* $(\mathfrak{S}_k)_*$ *is an isomorphism, then* $(\mathfrak{i}_{k+1})_*$ *is an isomorphism.*

5.5. *Isomorphisms from stabilisation.* — In this section we prove the following:

Proposition **5.5.1.** — *If* $k \geq g$ *and* h *satisfies Condition* $(\dagger\dagger)_{2k+2}$, *then the map*

$$(\mathfrak{S}_k)_* : \mathrm{PFH}_{2k}(S, h) \to \mathrm{PFH}_{2k+2}(S', h')$$

is an isomorphism.

The proof will be by induction on $k-g$. First we consider the base step $g = k$, which proved by comparing the stabilization maps in Heegaard Floer homology and periodic Floer homology.

Let $\mathbf{a} = \{a_1, \dots, a_{2g}\}$ be a basis of S. We extend $\mathbf{a}$ to $\overline{\mathbf{a}} = \{\overline{a}_1, \dots, \overline{a}_{2g}\}$ so that $\overline{a}_i$ is a properly embedded arc with boundary on $\partial S'$ and then complete $\overline{\mathbf{a}}$ to a basis $\mathbf{a}'$ of S' by adding the extensions $\overline{a}_{2g+1}$ and $\overline{a}_{2g+2}$ of the basis arcs a_{2g+1} and a_{2g+2} of T, subject to the following conditions:

- $\overline{a}_i - a_i$ and $h'(\overline{a}_j - a_j)$ are disjoint for $1 \leq i \neq j \leq 2g+2$;
- $\overline{a}_i$ and $h'(\overline{a}_i)$ have two extra pairs of canceling intersections $\overline{x}_i, \theta_i$ and $\overline{x}'_i, \theta'_i$ in $S' - (S \cup T)$ for $i = 1, \dots, 2g+2$;
- $\overline{x}_1, \overline{x}'_1, \dots \overline{x}_{2g+2}, \overline{x}'_{2g+2}$ lie on $\partial S'$.

We then define

$$\mathfrak{S}'_{\mathrm{HF}} : \widehat{\mathrm{CF}}'(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{\mathrm{CF}}'(S', \mathbf{a}', h'(\mathbf{a}'))$$

as $\mathfrak{S}_{\mathrm{HF}}(\mathbf{y}) = \mathbf{y} \cup \{\overline{x}_{2g+1}, \overline{x}_{2g+2}\}$. It is easy to see that $\mathfrak{S}_{\mathrm{HF}}$ is a chain map. (Compare with the gluing map from [HKM].)

Lemma **5.5.2.** — *The map* $\mathfrak{S}'_{\mathrm{HF}}$ *induces a map*

$$(\mathfrak{S}_{\mathrm{HF}})_* : \widehat{\mathrm{HF}}(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{\mathrm{HF}}(S', \mathbf{a}', h'(\mathbf{a}'))$$

which is an isomorphism.

Proof. — Let S^e and T^e be extensions of S and T such that:

- $\mathbf{a}'_S = \{\overline{a}_1, \dots, \overline{a}_{2g}\}$ and $h'(\mathbf{a}'_S) = h'(\{h'(\overline{a}_1), \dots, h'(\overline{a}_{2g})\}$ are bases of arcs of S^e,
- $\mathbf{a}'_T = \{\overline{a}_{2g+1}, \overline{a}_{2g+2}\}$ and $h'(\mathbf{a}'_T) = \{h'(\overline{a}_1), h'(\overline{a}_{2g+2})\}$ are bases of arcs of T^e,
- S' can be seen as a boundary connected sum of S^e and T^e.

It is easy to see that

$$\widehat{CF}(S'\mathbf{a}', h'(\mathbf{a}')) \cong \widehat{CF}(S^e\mathbf{a}'_S, h'(\mathbf{a}'_S)) \otimes \widehat{CF}(T^e, \mathbf{a}'_T, h'(\mathbf{a}'_T))$$

and $\widehat{HF}(T^e, \mathbf{a}'_T, h'(\mathbf{a}'_T)) \cong \mathbf{F}$ and is generated by the class of $\{\overline{x}_{2g+1}, \overline{x}_{2g+2}\}$ because $(T^e, \mathbf{a}'_T, h'(\mathbf{a}'_T))$ represents S^3. Hence the map

$$\widehat{CF}(S^e, \mathbf{a}'_S, h'(\mathbf{a}'_S)) \to \widehat{CF}(S'\mathbf{a}', h'(\mathbf{a}')), \quad \mathbf{y} \mapsto \mathbf{y} \cup \{\overline{x}_{2g+1}, \overline{x}_{2g+2}\}$$

induces an isomorphism in homology. In view of the discussion above, it remains to prove that the inclusion $\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}(S^e, \mathbf{a}'_S, h'(\mathbf{a}'_S))$ induces a map $\widehat{HF}(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{HF}(S^e, \mathbf{a}'_S, h'(\mathbf{a}'_S))$ which is an isomorphism. Note that the inclusion does not factor through a map $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}(S^e, \mathbf{a}'_S, h'(\mathbf{a}'_S))$; we will need a slightly more complicated argument.

Let $Q \subset \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$ be the subspace generated by elements of the form $\mathbf{y} \cup \{x_i\} + \mathbf{y} \cup \{x'_i\}$ for some $i = 1, \dots 2g$, where $\mathbf{y}$ is a $(2g-1)$-uple of intersection points none of which is in a_i or $h(a_i)$. Then $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) = \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))/Q$. Similarly, we define $\overline{Q} \subset \widehat{CF}'(S^e, \mathbf{a}'_S, h(\mathbf{a}'_S))$ as the subspace generated by elements of the form $\mathbf{y} \cup \{\overline{x}_i\} + \mathbf{y} \cup \{\overline{x}'_i\}$. Then, as before, $\widehat{CF}(S^e, \mathbf{a}'_S, h(\mathbf{a}'_S)) = \widehat{CF}'(S^e, \mathbf{a}'_S, h(\mathbf{a}'_S))/\overline{Q}$.

We also define $R \subset \widehat{CF}'(S^e, \mathbf{a}'_S, h(\mathbf{a}'_S))$ as the subspace generated by elements of the form $\mathbf{y} \cup \{\overline{x}_i\} + \mathbf{y} \cup \{\overline{x}'_i\}$, $\mathbf{y} \cup \{x_i\} + \mathbf{y} \cup \{x'_i\}$ and $\mathbf{y} \cup \{\theta_i\} + \mathbf{y} \cup \{\theta'_i\}$ for some $i = 1, \dots, 2g$. Since both Q and $\overline{Q}$ are subcomplexes of R, we have chain maps

(**5.5.1**) $$\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}'(S^e, \mathbf{a}'_S, h(\mathbf{a}'_S))/R,$$

(**5.5.2**) $$\widehat{CF}(S^e, \mathbf{a}'_S, h'(\mathbf{a}'_S)) \to \widehat{CF}'(S^e, \mathbf{a}'_S, h(\mathbf{a}'_S))/R.$$

The map (5.5.2) induces an isomorphism in homology because $H(R/\overline{Q}) \cong 0$. This can be proved in a similar manner to Theorem I.4.9.4. The map $(\mathfrak{S}_{HF})_*$ is then the composition of map induced by (5.5.1) with the inverse of the map induced by (5.5.2).

It remains to show that the map induced by (5.5.1) is an isomorphism. We define $\widetilde{R} \subset \widehat{CF}'(S^e, \mathbf{a}'_S, h(\mathbf{a}'_S))$ as the subspace generated by intersection points of the form $\mathbf{y} \cup \{\overline{x}_i\}$ $\mathbf{y} \cup \{\overline{x}'_i\}$, $\mathbf{y} \cup \{\theta_i\}$ and $\mathbf{y} \cup \{\theta'_i\}$ for $i = 1, \dots, 2g$. If we apply the identity of vector spaces

$$X/(A+B) \cong \frac{X/A}{B/(B \cap A)}$$

to $X = \widehat{CF}'(S^e, \mathbf{a}'_S, h'(\mathbf{a}'_S))$, $A = \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$, $B = R$ and observe that $X/A \cong \widetilde{R}$ and $A \cap B \cong Q$, we obtain the short exact sequence

$$0 \to \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to \widehat{CF}'(S^e, \mathbf{a}'_S, h'(\mathbf{a}'_S))/R \to \widetilde{R}/(R/Q) \to 0.$$

Since both $\widetilde{R}$ and R/Q are acyclic (by an argument similar to the proof of Theorem I.4.9.4), it follows that the map (5.5.1) induces an isomorphism in homology. □

Lemma **5.5.3.** — *The diagram of chain complexes*

$$\begin{array}{ccc} \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a})) & \xrightarrow{\mathfrak{S}'_{HF}} & \widehat{CF}'(S', \mathbf{a}', h'(\mathbf{a}')) \\ \Big\downarrow \Phi'_{(S,h)} & & \Big\downarrow \Phi'_{(S',h')} \\ PFC_{2g}(S, h) & \xrightarrow{\mathfrak{S}_{2g}} & PFC_{2g+2}(S', h') \end{array} \tag{5.5.3}$$

commutes up to homotopy.

Proof. — Since x_{2g+1} and x_{2g+2} are components of the contact class, there are no holomorphic curves in W_+, besides restrictions of trivial cylinders, which have a positive limit to x_{2g+1} or x_{2g}. The argument is the same as that of Lemma I.6.2.3.

The restriction of the trivial cylinder over x_i, $i = 2g+1, 2g+2$ has its negative end to a generic point p_i of the Morse-Bott family $\mathcal{N}_3$ and is concatenated with a cylinder corresponding to a gradient trajectory from p_i to e_3 to give a Morse-Bott building. Moreover, by automatic transversality [We1, Theorem 4.5.36] and the discussion from Lemma I.5.8.9, the above Morse-Bott building is Morse-Bott regular. Hence the pair $\{x_{2g+1}, x_{2g+2}\}$ is "mapped" to e_3^2.

By arguments similar to the proof of Lemma 5.2.1, every W_+-curve for (S', h') which is positively asymptotic to $\mathbf{y} \cup \{x_{2g+1}, x_{2g+2}\}$ is the disjoint union of a W_+-curve for (S, h) which is positively asymptotic to $\mathbf{y}$ with the Morse-Bott buildings from x_{2g+1} and x_{2g+2} described above. This implies that

$$\Phi'_{(S',h')}(\mathbf{y} \cup \{x_{2g+1}, x_{2g+2}\}) = e_3^2 \cdot \Phi'_{(S,h)}(\mathbf{y}).$$

Then the map $\mathbf{y} \mapsto h_{1P}(e_1 + e_3)\Phi'_{(S,h)}(\mathbf{y})$ is a chain homotopy between $\Phi'_{(S',h')} \circ \mathfrak{S}_{HF}$ and $\mathfrak{S}_g \circ \Phi'_{(S,h)}$. □

Corollary **5.5.4.** — *The diagram*

$$\begin{array}{ccc} \widehat{\mathrm{HF}}(\mathrm{S}, \mathbf{a}, h(\mathbf{a})) & \xrightarrow{(\mathfrak{S}_{\mathrm{HF}})_*} & \widehat{\mathrm{HF}}(\mathrm{S}', \mathbf{a}', h'(\mathbf{a}')) \\ {\scriptstyle (\Phi_{(\mathrm{S},h)})_*}\downarrow & & \downarrow{\scriptstyle (\Phi_{(\mathrm{S}',h')})_*} \\ \mathrm{PFH}_{2g}(\mathrm{S}, h) & \xrightarrow{(\mathfrak{S}_g)_*} & \mathrm{PFH}_{2g+2}(\mathrm{S}', h') \end{array} \tag{5.5.4}$$

commutes.

Proof. — Since we are working over a field, the map (5.5.2) has an inverse up to homotopy. □

The following lemma provides the base step of the induction.

Lemma **5.5.5.** — *If* h *satisfies Condition* $(\dagger\dagger)_{2g+2}$*, then the map*

$$(\mathfrak{S}_g)_* : \mathrm{PFH}_{2g}(\mathrm{S}, h) \to \mathrm{PFH}_{2g+2}(\mathrm{S}', h')$$

is an isomorphism.

Proof. — Since h satisfies Condition $(\dagger\dagger)_{2g+2}$, we can arrange the construction of h' so that it also satisfies Condition $(\dagger\dagger)_{2g+2}$. Then by Theorem 1.0.1 the vertical arrows of Diagram (5.5.4) are isomorphisms. The map $(\mathfrak{S}_{\mathrm{HF}})_*$ is also an isomorphism since $\widehat{\mathrm{HF}}(\mathrm{S}', \mathbf{a}', h'(\mathbf{a}'))$ can be computed as the tensor product of the S and T sides and the T side gives $\widehat{\mathrm{HF}}(\mathrm{S}^3)$, which is generated by $\{x_{2g+1}, x_{2g+2}\}$. Then $(\mathfrak{S}_g)_*$ is an isomorphism. □

The next lemma provides the inductive step of the induction, which in turn completes the proof of Proposition 5.5.1.

Lemma **5.5.6.** — *If there exists* $k_0 \geq 1$ *such that, for all* $g > 0$*, surfaces* S *of genus* g*, and monodromies* $h : \mathrm{S} \to \mathrm{S}$ *satisfying Condition* $(\dagger\dagger)_{2g+2k_0}$*, the map*

$$(\mathfrak{S}_{g+k_0-1})_* : \mathrm{PFH}_{2g+2k_0-2}(\mathrm{S}, h) \to \mathrm{PFH}_{2g+2k_0}(\mathrm{S}', h')$$

is an isomorphism, then for all $g > 0$ *and monodromies* $h : \mathrm{S} \to \mathrm{S}$ *satisfying Condition* $(\dagger\dagger)_{2g+2k_0+2}$ *the map*

$$(\mathfrak{S}_{g+k_0})_* : \mathrm{PFH}_{2g+2k_0}(\mathrm{S}, h) \to \mathrm{PFH}_{2g+2k_0+2}(\mathrm{S}', h')$$

is an isomorphism.

Proof. — Let (S'', h'') be a stabilization of (S', h'), i.e., a double stabilization of (S, h). If h satisfies Condition $(\dagger\dagger)_{2g+2k_0+2}$, then we can arrange h' and h'' so that they also satisfy Condition $(\dagger\dagger)_{2g+2k_0+2}$. It is easy to see that the diagram

(**5.5.5**)
$$\begin{array}{ccc} PFH_{2g+2k_0}(S,h) & \xrightarrow{(\mathfrak{S}_{g+k_0})_*} & PFH_{2g+2k_0+2}(S',h') \\ \Big\downarrow {\scriptstyle (\mathfrak{i}_{g+k_0})_*} & & \Big\downarrow {\scriptstyle (\mathfrak{i}'_{g+k_0})_*} \\ PFH_{2g+2k_0}(S',h') & \xrightarrow{(\mathfrak{S}'_{g+k_0})_*} & PFH_{2g+2k_0+2}(S'',h'') \end{array}$$

commutes. Here $\mathfrak{S}'_{g+k_0}$ and $\mathfrak{i}'_{g+k_0}$ are the obvious analogues of $\mathfrak{S}_{g+k_0}$ and $\mathfrak{i}_{g+k_0}$. The proof that the corresponding chain level diagram commutes up to homotopy is similar to the proof that Diagram (5.5.3) commutes up to homotopy.

Since $g + k_0 = g(S') + k_0 - 1$, the map $(\mathfrak{S}'_{g+k_0})_*$ is an isomorphism by hypothesis. The map $(\mathfrak{S}_{g+k_0-1})_*$ is also an isomorphism because h satisfies Condition $(\dagger\dagger)_{2g+2k_0}$ *a fortiori*. Then, by Corollary 5.4.2, the maps $(\mathfrak{i}_{g+k_0})_*$ and $(\mathfrak{i}'_{g+k_0})_*$ are isomorphisms. Therefore, the commutativity of Diagram (5.5.5) implies that $(\mathfrak{S}_{g+k_0})_*$ is an isomorphism. □

5.6. *Proof of Theorem 1.0.2.* — The following is similar to Theorem I.2.5.2 with a slightly simpler proof which is left to the reader:

Lemma **5.6.1.** — *Let* $(\alpha_0 = dt, \omega = d\beta)$ *be a stable Hamiltonian structure on* N *with stable Hamiltonian vector field* R *such that all closed orbits of* R *that intersect* int(S) *at most* k *times are hyperbolic. Then, for every* $\delta > 0$ *sufficiently small, there is a stable Hamiltonian structure* $(\alpha_0, \omega_\delta = d\beta_\delta)$ *with stable Hamiltonian vector field* R_δ *such that:*

(1) β *and* β_δ *(and hence also* R *and* R_δ*) coincide outside of a* δ*-neighborhood* V_δ *of the orbits of* R *which intersect* int(S) *exactly* $k+1$ *times,*
(2) $\beta_\delta \to \beta$ *in the* C^1*-topology, and hence* $R_\delta \to R$ *in the* C^0 *topology, as* $\delta \to 0$*, and*
(3) *all orbits of* R_δ *that intersect* int(S) *at most* $k+1$ *times are hyperbolic.*

We perturb the stable Hamiltonian structures (α_0, ω) and $(\alpha_0, \omega_\delta)$ to contact structures

$$\alpha_\varsigma = dt + \varsigma\beta, \quad \alpha_{\delta,\varsigma} = dt + \varsigma\beta_\delta$$

for $\varsigma > 0$ sufficiently small as in Equation (I.3.1.1). The C^1-convergence $\beta_\delta \to \beta$ ensures that ς can be chosen independently of δ. Note that, in the notation of Section I.3.1, $\beta = \beta_t + f_t dt$. The Reeb vector fields R_ς of α_ς and $R_{\delta,\varsigma}$ of $\alpha_{\delta,\varsigma}$ are parallel to R and R_δ respectively. By Theorem I.3.6.1, for every $\delta > 0$ sufficiently small there exists $\varsigma'_\delta > 0$ such that for all $0 < \varsigma_\delta < \varsigma'_\delta$, $\alpha_{\delta,\varsigma_\delta}$ is a contact form and there is an isomorphism of chain complexes

(**5.6.1**)
$$PFC_j(N, \alpha_0, \omega_\delta) \cong ECC_j(N, \alpha_{\delta,\varsigma_\delta})$$

for all $j \leq k+1$. Moreover, there is $\varsigma_0' > 0$ such that for all $0 < \varsigma_0 < \varsigma_0'$, α_{ς_0} is a contact form and there is an isomorphism of chain complexes

$$\text{PFC}_j(\text{N}, \alpha_0, \omega) \cong \text{ECC}_j(\text{N}, \alpha_{\varsigma_0}) \qquad (\mathbf{5.6.2})$$

for all $j \leq k$.

In the next two lemmas we study the continuation maps between the various contact forms appearing in Lemma 5.6.1.

Lemma **5.6.2.** — *There exists $\epsilon > 0$ such that for every $0 < \varsigma, \varsigma' < \epsilon$ the forms α_ς and $\alpha_{\varsigma'}$ are contact forms, the identifications from Equation (5.6.2) hold, and the continuation map*

$$(\mathfrak{K}_j')_* : \text{ECH}_j(\text{N}, \alpha_\varsigma) \to \text{ECH}_j(\text{N}, \alpha_{\varsigma'})$$

given by Equation (I.2.5.6) with α_i and α_j replaced by α_ς and $\alpha_{\varsigma'}$, coincides with the map induced by the identifications from Equation (5.6.2) for $j \leq k$.

Proof. — By the Holomorphic Curves Property in [HT3, Theorem 2.4], the continuation map $(\mathfrak{K}_j)_*'$ is induced by a noncanonical chain map $\mathfrak{K}_j'$ which is supported on $\text{J}_{\varsigma,\varsigma'}$-holomorphic buildings of ECH index $\text{I} = 0$ in an exact symplectic cobordism $(\mathbf{R} \times \text{N}, \Omega_{\varsigma,\varsigma'})$ between α_ς and $\alpha_{\varsigma'}$ for a generic almost complex structure $\text{J}_{\varsigma,\varsigma'}$ which is compatible with $\Omega_{\varsigma,\varsigma'}$. This means that, if $\langle \mathfrak{K}_j'(\boldsymbol{\gamma}), \boldsymbol{\gamma}' \rangle \neq 0$, then there is an $\text{I} = 0$ $\text{J}_{\varsigma,\varsigma'}$-holomorphic building in $\mathbf{R} \times \text{N}$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$. Strictly speaking, the continuation maps are defined by passing to a closure M of N, but the positivity of intersections implies that the holomorphic buildings in $\mathbf{R} \times \text{M}$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$ are contained in $\mathbf{R} \times \text{N}$; see [0, Lemma 5.2.3].

Now we choose sequences $\varsigma_\ell, \varsigma_\ell' \to 0$ as $\ell \to +\infty$ and generic almost complex structures $\text{J}_\ell = \text{J}_{\varsigma_\ell, \varsigma_\ell'}$ such that $\text{J}_\ell \to \text{J}$ as $\ell \to +\infty$, where J is a generic almost complex structure on $\mathbf{R} \times \text{N}$ which is adapted to the stable Hamiltonian structure (α_0, ω). Then $\text{I} = 0$ J_ℓ-holomorphic buildings converge to $\text{I} = 0$ J-holomorphic buildings. Since J is generic and cylindrical, the only $\text{I} = 0$ J-holomorphic buildings are trivial cylinders. This implies that, for ς and ς' small enough, there is no $\text{I} = 0$ $\text{J}_{\varsigma,\varsigma'}$-holomorphic building between $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ unless $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$. Then the Holomorphic Curves Property implies that $\mathfrak{K}_j'(\boldsymbol{\gamma}) = \epsilon(\boldsymbol{\gamma})\boldsymbol{\gamma}$.

Since we are working over $\mathbf{Z}/2\mathbf{Z}$, this implies that (after identifying $\text{ECC}_j(\text{N}, \alpha_\varsigma)$ with $\text{ECC}_j(\text{N}, \alpha_{\varsigma'})$ via Equation (5.6.2)), $(\mathfrak{K}_j')^2 = \mathfrak{K}_j'$. Since $(\mathfrak{K}_j')_*$ is an isomorphism, we obtain that $(\mathfrak{K}_j')_*$ coincides with the map induced by the identification from Equation (5.6.2). □

Remark **5.6.3.** — A similar results holds for the continuation maps

$$(\mathfrak{K}_j')_* : \text{ECH}_j(\text{N}, \alpha_{\delta,\varsigma}) \to \text{ECH}_j(\text{N}, \alpha_{\delta,\varsigma'})$$

for $j \leq k+1$, where ϵ is allowed to depend on δ.

Lemma **5.6.4.** — *For $j \leq k$ and $\delta, \varsigma > 0$ sufficiently small, the continuation map*

$$(\mathfrak{K}_j'')_* : \mathrm{ECH}_j(\mathrm{N}, \alpha_\varsigma) \to \mathrm{ECH}_j(\mathrm{N}, \alpha_{\delta,\varsigma})$$

given by Equation (I.2.5.6) with $\mathfrak{K}_j'$, α_i, and α_j replaced by $\mathfrak{K}_j''$, α_ς, and $\alpha_{\delta,\varsigma}$, is induced by the chain map $\mathfrak{K}_j''$ satisfying $\mathfrak{K}_j''(\boldsymbol{\gamma}) = \boldsymbol{\gamma}$, and is therefore a quasi-isomorphism.

Proof. — Let δ_ℓ and ς_ℓ be sequences converging to 0 as $\ell \to +\infty$. We choose the following generic almost complex structures on $\mathbf{R} \times \mathrm{N}$:

- an almost complex structure J adapted to the stable Hamiltonian structure (α_0, ω);
- almost complex structures J_ℓ adapted to the contact forms α_{ς_ℓ} such that $\mathrm{J}_\ell \to \mathrm{J}$ in the C^∞-topology on $\mathbf{R} \times \mathrm{N}$ as $\ell \to +\infty$;
- almost complex structures J_ℓ' adapted to the contact forms $\alpha_{\delta_\ell,\varsigma_\ell}$ which coincide with J_ℓ on $\mathbf{R} \times (\mathrm{N} - \mathrm{V}_{\delta_\ell})$ and converge to J_0 in the C^0-topology on $\mathbf{R} \times \mathrm{N}$ as $\ell \to +\infty$;
- almost complex structures $\tilde{\mathrm{J}}_\ell$ which are compatible with an exact symplectic cobordism $(\mathbf{R} \times \mathrm{N}, \Omega_\ell)$ between α_{ς_ℓ} and $\alpha_{\delta_\ell,\varsigma_\ell}$, interpolate between J_ℓ at the positive end and J_ℓ' at the negative end, coincides with J_ℓ and J_ℓ' on $\mathbf{R} \times (\mathrm{N} \setminus \mathrm{V}_{\delta_\ell})$, and converge to J in the C^0-topology on $\mathbf{R} \times \mathrm{N}$ as $\ell \to +\infty$.

If u_ℓ is a sequence of $\tilde{\mathrm{J}}_\ell$-holomorphic curves in $\mathbf{R} \times \mathrm{N}$ from an orbit set $\boldsymbol{\gamma}'$ to $\boldsymbol{\gamma}$ where both $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ consist of orbits which intersect a fiber of N at most k times, then, up to passing to a subsequence, u_ℓ converges to a J-holomorphic building. This follows from Gromov compactness for C^0-convergence of almost complex structures, due to Ivashkovich-Shevchishin [IS].

Alternatively, we can argue as follows, using the usual Gromov compactness for C^∞-convergence of almost complex structures. By the compactness argument from Section I.3.4, we may assume that $[u_\ell] \in \mathrm{H}_2(\mathrm{N}, \boldsymbol{\gamma}', \boldsymbol{\gamma})$ and the topological type of the domains F_ℓ of u_ℓ are fixed. We then restrict u_ℓ to the preimage G_ℓ of $\mathbf{R} \times (\mathrm{N} - \mathrm{V}_{\delta_\ell})$. We may assume that G_ℓ is obtained from a compact Riemann surface with boundary by removing interior punctures. (This is possible by taking δ_ℓ to be generic.)

We claim that $|\chi(\mathrm{G}_\ell)|$ is bounded above. This is equivalent to an upper bound on the number of disk components of $\mathrm{F}_\ell - \mathrm{G}_\ell$. Let $\boldsymbol{\gamma}''$ be the union of core orbits of V_{δ_ℓ}, which is independent of ℓ. The number of disk components is bounded above by the intersection number with $\mathbf{R} \times \boldsymbol{\gamma}''$, which in turn is controlled by the homology class $[u_\ell] \in \mathrm{H}_2(\mathrm{N}, \boldsymbol{\gamma}', \boldsymbol{\gamma})$.

Since $\widetilde{\mathrm{J}}_\ell \to \mathrm{J}$ in C^∞_{loc} on $\mathbf{R} \times (\mathrm{N} - \mathrm{V}_{\delta_\ell})$, and the ends $\boldsymbol{\gamma}, \boldsymbol{\gamma}'$ are contained in $\mathrm{N} - \mathrm{V}_{\delta_\ell}$ for all ℓ, we can take the limit of $u_\ell|_{\mathrm{G}_\ell}$ to obtain $u|_\mathrm{G}$. For simplicity assume that u has only one level. Then G is a punctured surface and the punctures which are not mapped to $\boldsymbol{\gamma}$ or $\boldsymbol{\gamma}'$ are removable.

Next we use the Holomorphic Curves Property in [HT3, Theorem 2.4], which states that $\langle \mathfrak{K}''_j(\boldsymbol{\gamma}'), \boldsymbol{\gamma} \rangle$ is supported on $\tilde{J}_\ell$-holomorphic buildings of ECH index $I = 0$. The convergence discussed above implies that, for ℓ large enough, the only $\tilde{J}_\ell$-holomorphic buildings of index $I = 0$ between orbit sets $\boldsymbol{\gamma}'$ and $\boldsymbol{\gamma}$, when both orbit sets consist of Reeb orbits intersecting a fiber of N at most k times, are covers of trivial cylinders contained in a product region. Then [HT3, Theorem 2.4] implies that $\mathfrak{K}''_j(\boldsymbol{\gamma}) = \boldsymbol{\gamma}$ when $j \leq k$. □

Lemma **5.6.5.** — *There are sequences of stable Hamiltonian structures* (α_0, ω^k) *and real numbers* ς_k *(both indexed by k) satisfying the following:*

(1) *The stable Hamiltonian vector field* R_k *of* (α_0, ω^k) *has no elliptic Reeb orbit in int*(N) *that intersects a fiber at most k times.*
(2) R_k *and* R_{k+1} *coincide outside of a neighborhood of the orbits of* R_k *which intersect a fiber of* N *exactly* $k+1$ *times.*
(3) α^k_ς *is a contact form for every* $0 < \varsigma \leq \varsigma_k$.
(4) *For every* $j \leq k$ *and* $0 < \varsigma \leq \varsigma_k$ *there are canonical identifications*

$$PFC_j(N, \alpha_0, \omega^k) \cong ECC_j(N, \alpha^k_\varsigma).$$

(5) *For every* $j \leq k$ *and* $0 < \varsigma, \varsigma' \leq \varsigma_k$ *the continuation maps*

$$(\mathfrak{K}'_j)_* : ECC_j(N, \alpha^k_\varsigma) \to ECC_j(N, \alpha^k_{\varsigma'})$$

are induced by the identifications of (4).
(6) *For every* $j \leq k$ *the continuation maps*

$$(\mathfrak{K}_j)_* : ECH_j(N, \alpha^k_{\varsigma_k}) \to ECH_j(N, \alpha^{k+1}_{\varsigma_{k+1}})$$

are induced by chain maps $\mathfrak{K}_j$ *such that* $\mathfrak{K}_j(\boldsymbol{\gamma}) = \boldsymbol{\gamma}$ *for every orbit set* $\boldsymbol{\gamma}$ *of* R_k *that intersects a fiber of* N j *times.*

Proof. — The proof is by induction on k. For $k = 0$ we choose a stable Hamiltonian structure (α_0, ω) and define $(\alpha_0, \omega^0) = (\alpha_0, \omega)$ and $\alpha^0 = \alpha_\varsigma$ for $\varsigma > 0$ sufficiently small such that the identification of Equation (5.6.2) holds. Suppose we have constructed the sequences up to some k_0. By Lemma 5.6.1, Equation (5.6.2), and Lemma 5.6.2 there exist a stable Hamiltonian structure $(\alpha_0, \omega^{k_0+1})$ and a real number ς_{k_0+1} such that (1)–(5) hold for $k \leq k_0 + 1$, and moreover

$$\mathfrak{K}''_j : ECC(N, \alpha^{k_0+1}_{\varsigma_{k_0+2}}) \to ECC(N, \alpha^{k_0+2}_{\varsigma_{k_0+2}})$$

satisfies $\mathfrak{K}''_j(\boldsymbol{\gamma}) = \boldsymbol{\gamma}$ as in Lemma 5.6.4. Without loss of generality we can assume that $\varsigma_{k_0+2} < \varsigma_{k_0+1}$. Since $(\mathfrak{K}_j)_* = (\mathfrak{K}''_j)_* \circ (\mathfrak{K}'_j)_*$ by [HT3, Theorem 2.4], (6) is satisfied up to $k_0 + 1$. □

Let us write $\alpha^k = \alpha^k_{\varsigma_k}$. In view of Theorem I.2.5.6, we can write

$$\widehat{\mathrm{ECH}}(M) \simeq \lim_{k\to\infty} \mathrm{ECH}_{2k}(N, \alpha^{2k}),$$

where the limit is taken with respect to the maps

$$\mathrm{ECH}_{2k}(N, \alpha^{2k}) \to \mathrm{ECH}_{2k+2}(N, \alpha^{2k+2})$$

induced by the chain map $\boldsymbol{\gamma} \mapsto e^2 \mathfrak{K}_{2k}(\boldsymbol{\gamma})$. The diagram

$$(\mathbf{5.6.3}) \qquad \begin{array}{ccccc} \mathrm{ECH}_{2k}(N, \alpha^{2k}) & \xrightarrow{(\mathfrak{K}_{2k})_*} & \mathrm{ECH}_{2k}(N, \alpha^{2k+2}) & \xrightarrow{(j_k)_*} & \mathrm{ECH}_{2k+2}(N, \alpha^{2k+2}) \\ & & \simeq \downarrow & & \simeq \uparrow \\ & & \mathrm{PFH}_{2k}(N, \alpha_0, \omega^{2k+2}) & \xrightarrow{(\mathrm{j}_k)_*} & \mathrm{PFH}_{2k+2}(N, \alpha_0, \omega^{2k+2}) \end{array}$$

commutes. Therefore, Proposition 5.5.1 and Lemma 5.4.1 imply that $(j_k)_*$ is an isomorphism. This proves Theorem 1.0.2.

Acknowledgements

We are indebted to Michael Hutchings for many helpful conversations and for our previous collaboration which was a catalyst for the present work. We also thank Denis Auroux, Thomas Brown, Tobias Ekholm, Dusa McDuff, Roberto Pignatelli, Ivan Smith and Jean-Yves Welschinger for illuminating exchanges. Part of this work was done while KH and PG visited MSRI during the academic year 2009–2010. We are extremely grateful to MSRI and the organizers of the "Symplectic and Contact Geometry and Topology" and the "Homology Theories of Knots and Links" programs for their hospitality; this work probably would never have seen the light of day without the large amount of free time which was made possible by the visit to MSRI. KH also thanks the Simons Center for Geometry and Physics for their hospitality during his visit in May 2011.

Declarations:

Competing Interests

The authors declare no competing interests.

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Publisher's Note

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Manuscrit reçu le 7 août 2012
Version révisée le 10 avril 2017
Manuscrit accepté le 28 février 2024