THE EQUIVALENCE OF HEEGAARD FLOER HOMOLOGY AND EMBEDDED CONTACT HOMOLOGY VIA OPEN BOOK DECOMPOSITIONS I

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ABSTRACT

Given an open book decomposition (S, h) adapted to a closed, oriented 3-manifold M, we define a chain map Φ from a certain Heegaard Floer chain complex associated to (S, h) to a certain embedded contact homology chain complex associated to (S, h), as defined in (Colin et al. in Geom. Topol., 2024), and prove that it induces an isomorphism on the level of homology. This implies the isomorphism between the hat version of Heegaard Floer homology of $-M$ and the hat version of embedded contact homology of M.

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Notation common to [I]-[III]

$(\alpha_\varsigma, \omega)$ Stable Hamiltonian structures on $N_{(S,h)}$

$\boldsymbol{\gamma}$ An orbit set; from Section 5.3 onwards, $\boldsymbol{\gamma} \subset N$ and does not contain multiples of δ_0

δ_0 Closed orbit in $\overline{N}$ over z_∞

(ε, U) or (ε, δ, p) Perturbation data for $\overline{J}_-^\diamondsuit$; cf. Convention 5.8.12

$\widetilde{\mu}_\tau(\boldsymbol{\gamma})$ Symmetric Conley-Zehnder index

$\pi_B : X \to B$ or $W \to B$ Symplectic fibration with fiber Σ or S used in the definition the Heegaard Floer complex

$\pi_{B'} : W' \to B'$ Symplectic fibration with fiber S used in the definition of $\widehat{ECC}$

$\pi_{B_+} : W_+ \to B_+$ Symplectic fibration with fiber S used in the definition of Φ

$\overline{\pi}_*$ with $* = B, B', B_+, B_-$ Symplectic fibrations with fiber $\overline{S}$ extending π_*

$\overline{\pi}_{D^2} : \overline{W}_* \cap (\mathbf{R} \times V) \to D^2$ Projection with respect to balanced coordinates, where $* = +$ or $-$

$\overline{\pi}_{D^2_{\rho_0}} : \mathbf{R} \times (\mathbf{R}/2\mathbf{Z}) \times D^2_{\rho_0} \to D^2_{\rho_0}$, also $\overline{\pi}$ Projection with respect to balanced coordinates

(ρ, ϕ) Polar coordinates on D^2

$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ Pointed Heegaard diagram for M

$\sigma_\infty, \sigma'_\infty, \sigma^+_\infty, \sigma^-_\infty$ Sections at infinity of $\overline{W}, \overline{W'}, \overline{W}_+, \overline{W}_-$

$\sigma^\dagger_\infty, (\sigma'_\infty)^\dagger, (\sigma^+_\infty)^\dagger, (\sigma^-_\infty)^\dagger$ "Pushoff" of m-fold cover of section at ∞ for $\overline{W}, \overline{W'}, \overline{W}_+, \overline{W}_-$

Φ Chain map from $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ to $PFC_{2g}(N)$

Φ' Map from $\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$ to $PFC_{2g}(N)$
$\widetilde{\Phi}$ Variant of Φ from $\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$ to $PFC_{2g}(N)$
$\Psi' = \Psi'_{\overline{J}^{\diamondsuit}_{-}}(m, \overline{\mathfrak{m}})$ Map from $PFC_{2g}(N)$ to $\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$
$\Psi = \Psi_{\overline{J}^{\diamondsuit}_{-}}(m, \overline{\mathfrak{m}})$ Chain map from $PFC_{2g}(N)$ to $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$
ω Area form on S invariant under h; also viewed as a 2-form on W_+, W_-
$\overline{\omega}$ Area form on $\overline{S}$ which restricts to ω on S and is invariant under $\overline{h}$; also viewed as a 2-form on $\overline{W}_+, \overline{W}_-$
Ω_X Symplectic form on X
Ω and Ω' Symplectic forms on W and W' given by $ds \wedge dt + \omega$
Ω_+ and Ω_- Symplectic forms on W_+ and W_-
$\overline{\Omega}, \overline{\Omega'}, \overline{\Omega}_+, \overline{\Omega}_-$ Extensions of $\Omega, \Omega', \Omega_+, \Omega_-$
$* = \dagger$ Modifier "$\overline{u}' = \varnothing$"
$* = ext$ Modifier "extended moduli space"; cf. Definition 5.7.24
$* = f_{\delta_0}$ Modifier "the normalized asymptotic eigenfunction at the negative end δ_0 is f_{δ_0}"
$* = K$ Modifier "passes through the compact set K"
$* = (l_1, \dots, l_\lambda)$ Modifier "curve with λ ends at δ_0 with covering multiplicities $l_1, \dots, l_\lambda$"
$* = nc$ Modifier "without connector components"
$* = reg$ Modifier "regular"
$* = s$ Modifier "somewhere injective"
$\mathbf{a} = \mathbf{a}(m)$ Basis $\{a_1, \dots, a_{2g}\}$ of arcs for S; depends on m starting from Section 5
$a_i = a_i(m)$ Basis arc in S; depends on m starting from Section 5
$\overline{\mathbf{a}}$ Collection $\{\overline{a}_1, \dots, \overline{a}_{2g}\}$
$\overline{a}_i$ Extension of a_i to $\overline{S}$
$\overline{a}_{i,j}$ (for $j = 0, 1$) Rays from x_i and x_i' to z_∞ in $\overline{a}_i - a_i$
$\vec{a}_{i,j}$ (for $j = 0, 1$) Extension of $\overline{a}_{i,j}$ past z_∞
$\widehat{\mathbf{a}}$ Collection $\{\widehat{a}_1, \dots, \widehat{a}_{2g}\}$
$\widehat{a}_i$ $\overline{a}_i - \{z_\infty\}$
$\partial_h W_+, \partial_v W_+$ Horizontal and vertical boundaries of W_+
B Alternate notation for $\mathbf{R} \times [0, 1]$
B' Alternate notation for $\mathbf{R} \times S^1$
B_+, B_- Bases of the symplectic fibrations $\pi_{B_+} : W_+ \to B_+$ and $\overline{\pi}_{B_-} : \overline{W}_- \to B_-$
$\mathcal{C}$ Data $\mathcal{C}$ as in Definition 5.7.1
$\mathfrak{c}^+, \mathfrak{c}^-$ Groomings at the positive and negative ends as in Definition 5.7.5
$\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ Heegaard Floer hat complex for pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$
$\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$ Subcomplex of Heegaard Floer chain complex $\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ obtained from the open book decomposition (S, h)
$\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ Heegaard Floer chain complex isomorphic to $\widehat{HF}(-M)$, obtained by quotienting $\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$
$\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$ Variant of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ given by Definition 6.6.1
$cl(B_+)$ and $cl(B_-)$ Compactifications of B_+ and B_-

$\overrightarrow{\mathcal{D}}, \mathcal{D}^{to}, \mathcal{D}^{from}, \mathcal{D}$ Data at z_∞^p with initial points $\mathcal{D}^{from}$ and terminal points $\mathcal{D}^{to}$. $\mathcal{D} = (\mathcal{D}^{to}, \mathcal{D}^{from})$

D^2, D^2_δ Unit disk $\overline{S} - int(S)$ with polar coordinates (ρ, ϕ), from Section 5 onwards; $D^2_\delta = \{\rho \leq \delta\}$ for $\delta > 0$ small

e Elliptic orbit of Conley-Zehnder index -1 on $\partial N_{(S,h)}$

$E(u)$ Hofer energy of the holomorphic curve u

$E_\omega(u)$ E_ω-area of the holomorphic curve u

$ECC(N, \alpha)$ ECH chain complex for (N, α)

$ECC_j(N, \alpha)$ Subcomplex of $ECC(N, \alpha)$ generated by orbit sets which intersect a page j times

g Genus of page S

h Hyperbolic orbit of Conley-Zehnder index 0 on $\partial N_{(S,h)}$; assumed to satisfy Convention 6.6.4 from Section 6.6 onwards

$H_2(M, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ Homology classes of chains in N with boundary in $\boldsymbol{\gamma}^-$ and $\boldsymbol{\gamma}^-$

$H_2(\mathbf{y}, \mathbf{y}')$ Homology classes of maps in HF (denoted $\pi_2(\mathbf{y}, \mathbf{y}')$ elsewhere in the literature)

$H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$ Homology classes of continuous multisections for Φ

$H_2(\overline{W}_-, \boldsymbol{\gamma}, \mathbf{y})$ Homology classes of continuous multisections for Ψ

h Monodromy $h : S \xrightarrow{\sim} S$

$\overline{h} = \overline{h}_m$ Extension of h to $\overline{S}$ which depends on m

$\mathfrak{I}_j$ ECH chain maps given by Equation (2.2.1)

I_{ECH} ECH index

I_{HF} ECH-type index for Heegaard Floer homology

$I_{W_+}, I_{\overline{W}_-}$ ECH indices for W_+, $\overline{W}_-$

ind Fredholm index

J_ς Almost complex structure on $\mathbf{R} \times N$ adapted to $(\alpha_\varsigma, \omega)$

$J' = J_0$ Another name for J_ς when $\varsigma = 0$

$\overline{J}_0$ Almost complex structure on $\overline{W}$ which restricts to the standard one on D^2 on each fiber

J Almost complex structure on W that is the restriction of $\overline{J}$

J_+, J_- Almost complex structures on W_+, W_- that are restrictions of $\overline{J}_+, \overline{J}_-$

$\overline{J}, \overline{J}', \overline{J}_+, \overline{J}_-$ Almost complex structures on $\overline{W}$, $\overline{W}'$, $\overline{W}_+$, $\overline{W}_-$ that restrict to J, J', J_+, J_- on W, W', W_+, W_-

$\overline{J}_-^\Diamond = \overline{J}_-^\Diamond(\varepsilon, U)$ Almost complex structure on $\overline{W}_-$ that is (ε, U)-close to $\overline{J}_-$ as in Definition 5.8.11 and Convention 5.8.12

$\mathcal{J}_X$ Space of C^∞-smooth admissible almost complex structures on X

$\mathcal{J}_\star^*$ Subspace of $\mathcal{J}_\star$ satisfying $*$

$\mathcal{J}_W$ Space of C^∞-smooth Ω-admissible almost complex structures on W

$\mathcal{J}_{\overline{W}}$ Space of C^∞-smooth $\overline{\Omega}$-admissible almost complex structures $\overline{J}$ on $\overline{W}$ satisfying Definition 5.3.2

$\mathcal{J}_{W'}$ Space of C^∞-smooth (α_0, ω)-adapted almost complex structures on W'

$\mathcal{J}_{\overline{W'}}$ Space of C^∞-smooth $(\overline{\alpha}_0, \overline{\omega})$-adapted almost complex structures on $\overline{W'}$ satisfying Definition 5.3.14

$\mathcal{J}_{W_+}, \mathcal{J}_{W_-}, \mathcal{J}_{\overline{W}_+}, \mathcal{J}_{\overline{W}_-}$ Spaces of C^∞-smooth admissible almost complex structures on $W_+, W_-, \overline{W}_+, \overline{W}_-$

$K_{p,2\delta}$ $\overline{\pi}_{B_-}^{-1}(p) - \{\rho < 2\delta\}$ with $\delta > 0$ small

$\mathfrak{K}_j$ ECH chain maps given by Equation (2.5.5)

$L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}}$ Lagrangian submanifolds used in the definition of the Heegaard Floer complex $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$

$L_{\mathbf{a}}, L_{h(\mathbf{a})}$ Lagrangian submanifolds of W used in the definition of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$

$L^+_{\mathbf{a}}, L^+_{a_i}$ Lagrangian submanifold of W_+ constructed from $\mathbf{a}$ and a connected component of $L^+_{\mathbf{a}}$ corresponding to the arc a_i

$L^-_{\overline{\mathbf{a}}}, L^-_{\overline{a}_i}$ Singular Lagrangian in $\overline{W}_-$ constructed from $\overline{\mathbf{a}}$ and a connected component of $L^-_{\overline{\mathbf{a}}}$ corresponding to the arc $\overline{a}_i$

$L^\pm_{\widehat{a}_i}, L^\pm_{\overline{a}_i}, L^\pm_{\vec{a}_{i,j}}, L^\pm_{\overline{a}_i \cup \vec{a}_{i,j}}$ Lagrangian submanifolds of $\overline{W}_\pm$ constructed from the arcs $\widehat{a}_i$, $\overline{a}_i$, $\vec{a}_{i,j}$ and $\overline{a}_i \cup \vec{a}_{i,j}$

m Positive integer $\gg 2g$; $\overline{h}$ and $\overline{\mathbf{a}}$ depend on m starting from Section 5

$\overline{\mathfrak{m}} = (\overline{\mathfrak{m}}^b, \overline{\mathfrak{m}}^f) = ((0, \frac{3}{2}), z_\infty)$ Marked point in $\overline{W}_-$

$\mathcal{M}^X_J(\mathbf{y}, \mathbf{y}')$ Moduli space of Heegaard Floer curves in (X,J) from $\mathbf{y}$ to $\mathbf{y}'$

$\widehat{\mathcal{M}}^X_J(\mathbf{y}, \mathbf{y}')$ Moduli space of Heegaard Floer curves in (X,J) from $\mathbf{y}$ to $\mathbf{y}'$ that do not cross the basepoint

$\mathcal{M}_{\overline{J}}(\mathbf{y}, \mathbf{y}')$ Moduli space of Heegaard Floer curves in $(\overline{W}, \overline{J})$ from $\mathbf{y}$ to $\mathbf{y}'$

$\mathcal{M}_{J'}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ Moduli space of J'-holomorphic curves in $W' = \mathbf{R} \times N$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$

$\mathcal{M}_{\overline{J'}}(\delta_0^p \boldsymbol{\gamma}, \delta_0^q \boldsymbol{\gamma}')$ Moduli space of $(\overline{W'}, \overline{J'})$-holomorphic maps from $\delta_0^p \boldsymbol{\gamma}$ to $\delta_0^q \boldsymbol{\gamma}'$ without fiber components

$\mathcal{M}_{\overline{J}_+}(\mathbf{y}, \boldsymbol{\gamma})$ Moduli space of multisections of $(\overline{W}_+, \overline{J}_+)$ from $\mathbf{y}$ to $\boldsymbol{\gamma}$

$\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y})$ Moduli space of multisections in $(\overline{W}_-, \overline{J}_-)$ from $\boldsymbol{\gamma}$ to $\mathbf{y}$

$\mathcal{M}^*_{\overline{J}^\diamond_-}(\star_1, \star_2; \overline{\mathfrak{m}})$ Moduli space of $\overline{J}^\diamond_-$-holomorphic almost multisections that pass through $\overline{\mathfrak{m}}$

$\mathcal{N}$ Negative Morse-Bott family of Reeb orbits on ∂N; from Section 5 onwards, is a negative Morse-Bott family of orbits of the Hamiltonian vector field $\overline{R}_0$ on ∂N

$n_{z_j}(A)$ Intersection number given by Equation (4.5.1)

$n(\overline{u}), n'(\overline{u}), n^+(\overline{u}), n^-(\overline{u})$ Intersection of a curve $\overline{u}$ in $\overline{W}, \overline{W'}, \overline{W}_+, \overline{W}_-$ with the "pushoff" of the section at ∞

$N = N_{(S,h)}$ Mapping torus of $h : S \xrightarrow{\sim} S$; from Section 5 onwards, is given by $N = S \times [0, 2]/(x, 2) \sim (h(x), 0)$

$\overline{N} = \overline{N}_m$ Mapping torus of $\overline{h} : \overline{S} \to \overline{S}$

$\widehat{\mathcal{O}}_k, \overline{\mathcal{O}}_k$ Orbit sets constructed from $\widehat{\mathcal{P}}$ and $\overline{\mathcal{P}}$ which intersect $\overline{S} \times \{0\}$ exactly k times

$\mathcal{P}$ Set of simple Reeb orbits of α in $int(N)$; from Section 5 onwards, is the set of simple orbits of the Hamiltonian vector field $\overline{R}_0$ in $int(N)$

$\widehat{\mathcal{P}}$ $\mathcal{P} \cup \{e, h\}$

$\overline{\mathcal{P}}$ $\widehat{\mathcal{P}} \cup \{\delta_0\}$
$PFC_j(N, \alpha_0, \omega)$ Periodic Floer homology chain complex on N analogous to $ECC_j(N, \alpha)$
$\mathcal{R}_{\phi_0}$ Radial ray $\{\phi = \phi_0, \rho \geq 0\} \subset \mathbf{C}$
$\overline{R}_0$, R_0 Hamiltonian vector field on $\overline{N}$ and its restriction to N
$\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ Set of generators of $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$
$\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ $2g$-tuples of intersection points of $\mathbf{a}$ and $h(\mathbf{a})$
$\mathcal{S}_{\overline{\mathbf{a}},\overline{h}(\overline{\mathbf{a}})}$ $2g$-tuples of intersection points $\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}'$ of $\overline{\mathbf{a}}$ and $\overline{h}(\overline{\mathbf{a}})$
$\mathfrak{S}(R, R')$ Counterclockwise sector in $D^2_{\rho_0}$ from radial ray R to R′
S Compact oriented connected surface of genus g with connected boundary
(S, h) Open book decomposition of M with connected binding, page S, and monodromy h
$\overline{S}$ Capped-off surface $S \cup D^2$
$\overline{u} = \overline{u}' \cup \overline{u}''$ Decomposition of a holomorphic multisection $\overline{u}$ in $\overline{W}$, $\overline{W}'$, $\overline{W}_+$, or $\overline{W}_-$ into a possibly disconnected branched cover $\overline{u}'$ of the section at ∞ and the rest $\overline{u}''$
$\overline{u}_\infty = \overline{v}_{-b} \cup \cdots \cup \overline{v}_a$ Limit of $\overline{W}_-$-curves in Section 7
V $\overline{N} - int(N)$ (starting from Section 5.1.2)
$\overline{v}_j^\flat$ Union of non-fiber components of $\overline{v}_j''$ not asymptotic to any multiple of δ_0 or z_∞
$\overline{v}_j^\sharp$ Union of irreducible components of $\overline{v}_j''$ asymptotic to a multiple of δ_0 or z_∞ at either end
W and W′ $\mathbf{R} \times [0, 1] \times S$ and $\mathbf{R} \times N$
W_+ Symplectic fibration with fiber S used in the definition of Φ
W_- Symplectic fibration with fiber S; $\overline{W}_-$ is used in the definition of Ψ
$\overline{W}$, $\overline{W}'$, $\overline{W}_+$, $\overline{W}_-$ Extensions of W, W′, W_+, W_- with fibers $\overline{S}$
X $\mathbf{R} \times [0, 1] \times \Sigma$
x_i, x_i' Intersection points on ∂S forming the contact class
(y, θ, t) Coordinates of a collared neighborhood of $\partial N_{(S,h)}$
$\mathbf{y}$ A tuple of $\mathbf{a} \cap h(\mathbf{a})$; from Section 5.3 onwards, $\mathbf{y} \subset S$ and does not contain multiples of z_∞
z_∞ Center of the disk $D^2 = \overline{S} - int(S)$
$\mathbf{z} = \{z_\infty^p(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$ A tuple of $\overline{\mathbf{a}} \cap \overline{h}(\overline{\mathbf{a}})$, where z_∞ has multiplicity p and $\mathbf{y} \subset S$

1. Introduction and main results

There is a plethora of Floer-type homology theories that can be associated to a three-manifold. This paper and its sequels are concerned with three specific Floer-type homology theories: *monopole Floer homology*, *embedded contact homology*, and *Heegaard Floer homology*. Monopole Floer homology, constructed by Kronheimer and Mrowka [KrM], counts solutions of the Seiberg-Witten equations. Since its definition requires an auxiliary Riemannian metric, it has strong connections with geometry (e.g., positive curvature). Embedded contact homology, abbreviated ECH, is due to Hutchings [Hu1, Hu2] and Hutchings-Taubes [HT1, HT2] and is dynamical in nature. It counts periodic orbits of a Reeb vector field associated with a contact form. Finally, Heegaard Floer homology, due to Ozsváth and Szabó [OSz1, OSz2], is defined in the most topological way: from a Heegaard diagram. Of the three homologies, it is the easiest to compute and admits a combinatorial description [SW]. These theories have had spectacular applications over the last decade, including a proof of the Gordon conjecture by Kronheimer, Mrowka, Ozsváth and Szabó [KMOS] and progress on the exceptional surgery problem due to Ghiggini [Gh] and Ni [Ni].

In 2006, Taubes obtained a breakthrough result which extended his celebrated correspondence between solutions of the Seiberg-Witten equations and J-holomorphic curves to the relative case. This enabled him to prove the Weinstein conjecture in dimension three [T1] and to establish the equivalence between monopole Floer cohomology and ECH shortly thereafter [T2]. A byproduct of this equivalence was the proof of Arnold's chord conjecture in dimension three [HT3, HT4].

The goal of our series of papers [0, I, II, III] is to prove the equivalence of Heegaard Floer homology and ECH. The first paper [0] can be viewed as a stand-alone paper. In this paper [I] and the sequel [II], we establish an isomorphism between the hat versions of the Heegaard Floer homology and ECH groups associated to a closed, oriented 3-manifold M. This isomorphism is compatible with the splitting of Heegaard Floer homology according to Spin^c-structures and of ECH according to first homology classes. *For simplicity we will use* $\mathbf{F}=\mathbf{Z}/2\mathbf{Z}$ *coefficients (or coefficients in a module* Λ *over* $\mathbf{F}[H_2(M;\mathbf{Z})]$*) for both Heegaard Floer homology and ECH.*

Remark **1.0.1.** — We expect the isomorphism to hold over the integers. To this end, we would need to define coherent orientations for all the moduli spaces used in this paper. This is a little subtle for the following reasons:

(1) The orientations from [OSz1, OSz2] are not the standard ones in Lagrangian Floer homology, defined using spin structures: if M is a closed 3-manifold with a genus g Heegaard splitting, then there are 2^g spin structures on each Heegaard torus, whilst there are $2^{b_1(M)}$ orientation systems in [OSz1, OSz2]. On the other hand, on ECH there is only one orientation system. A preferred orientation system for Heegaard Floer homology is chosen using HF^∞. It would be an interesting future project to relate the preferred orientation systems in Heegaard Floer homology and ECH using the fact that the ECH analog of HF^∞ is algebraically given as the inverse limit of the U-map.

(2) The chain map Ψ from ECH to Heegaard Floer homology works because there are certain cancellations that occur in pairs; see Theorem 7.1.3. One needs to verify that each pair comes with opposite signs. An analogous statement also needs to be proven for the chain homotopies.

The isomorphism between the plus version of Heegaard Floer homology and the usual version of ECH will be proved in [III].

The results of this paper were announced in [CGH1]. An isomorphism between monopole Floer homology and Heegaard Floer homology was proved, more or less simultaneously, by Kutluhan, Lee and Taubes [KLT1, KLT2, KLT3, KLT4, KLT5].

1.1. *Some background.* — Before stating the main result of this paper and describing the ideas involved in its proofs, we will give a brief summary of the definitions of Heegaard Floer homology and ECH.

We first describe Heegaard Floer homology in its "cylindrical reformulation" due to Lipshitz [Li]. The starting point of its construction is a *pointed Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ which describes a three-manifold M. Here Σ is a genus $g > 0$ Heegaard surface associated to some self-indexing Morse function with a unique maximum and a unique minimum, $\boldsymbol{\alpha}$ is the collection of the attaching circles for the index one critical points, $\boldsymbol{\beta}$ is the collection of the attaching circles for the index two critical points, and z is a basepoint in the complement of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The Heegaard Floer complex $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is generated by g-tuples of intersection points between the $\boldsymbol{\alpha}$-curves and the $\boldsymbol{\beta}$-curves and the differential counts certain J-holomorphic curves in $\mathbf{R} \times [0, 1] \times \Sigma$ with boundary on $\mathbf{R} \times \{0\} \times \boldsymbol{\beta}$ and $\mathbf{R} \times \{1\} \times \boldsymbol{\alpha}$; see Section 4 or [Li] for a more detailed exposition. While Heegaard Floer homology *a priori* depends on the choice of a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and an almost complex structure on $\mathbf{R} \times [0, 1] \times \Sigma$, it was shown to be independent of those choices, i.e., is a topological invariant of M.

The starting point for embedded contact homology is a contact form α on M. The contact form determines the *Reeb vector field* R by

$$\iota_R d\alpha = 0 \quad \text{and} \quad \alpha(R) = 1.$$

The complex $ECC(M, \alpha)$ is generated by finite sets of simple Reeb orbits with finite multiplicities and its differential counts certain J-holomorphic curves in the symplectization $(\mathbf{R} \times M, d(e^s\alpha))$; see [Hu1, Hu2, Hu3] for more details. The hat version of ECH is defined as the homology of the mapping cone of a chain map $U : ECC(M, \alpha) \to ECC(M, \alpha)$. ECH *a priori* depends on the choice of a contact form α on M and an adapted almost complex structure J on the symplectization $\mathbf{R} \times M$. There is currently no direct proof of the fact that the ECH groups are topological invariants of M (or even invariants of the contact structure $\ker \alpha$, for that matter); the only known proof is due to Taubes [T1, T2], and is a consequence of the equivalence between Seiberg-Witten Floer cohomology and ECH. A direct proof of the invariance would provide, combined with the present work and computations in Heegaard Floer homology, an alternative proof of the Weinstein conjecture.

A natural setting for relating Heegaard Floer homology and ECH is that of *open book decompositions* (see Definition 2.1.1) because an open book decomposition determines both a Heegaard splitting and a contact structure. The Heegaard splitting is obtained by taking as Heegaard surface the union of two opposite pages. The contact structure is provided by the Thurston-Winkelnkemper construction [TW]. In the foundational work

[Gi], Giroux proved the equivalence of contact structures up to isomorphism and (abstract) open book decompositions modulo positive stabilization. While we are not using its full strength, Giroux's correspondence was an important source of inspiration for this work.

We use open book decompositions and their relationship with both Heegaard splittings and contact structures to build symplectic cobordisms relating the geometric setting of Heegaard Floer homology to that of embedded contact homology. Then the isomorphisms are defined by counting certain J-holomorphic maps in those cobordisms. Their definition is similar in spirit to the definition of the open-closed and closed-open maps in [Abo]. The similarity is more direct for the open-closed map, while our closed-open map require some extra twist, as we will see.

1.2. *Main result.* — Let M be a closed, connected, oriented three-manifold. We fix an open book decomposition of M with connected binding, page S, and monodromy h, and denote by $\xi_{(S,h)}$ the contact structure supported by it. The contact structure $\xi_{(S,h)}$ determines a Spin^c-structure which we denote by $\mathfrak{s}_{(S,h)}$.

The main result of [I] and [II] is the following:

Theorem **1.2.1.** — *There is an isomorphism*

$$\Phi_0 : \widehat{HF}(-M, \mathfrak{s}_{(S,h)} + PD(A)) \xrightarrow{\sim} \widehat{ECH}(M, \xi_{(S,h)}, A),$$

where $A \in H_1(M; \mathbf{Z})$, *defined via the open book decomposition* (S, h) *of* M. *Moreover,* Φ_0 *sends the Heegaard Floer contact invariant for* $\xi_{(S,h)}$ *to the ECH contact invariant for* $\xi_{(S,h)}$.

The definition of Φ_0 depends on many choices, and first of all on the choice of the open book decomposition. In this series of papers we will not address the question of its naturality, but we conjecture that it only depends on the contact structure $\xi_{(S,h)}$. Even the dependence on the contact structure should be very mild, and due only to the fact that ECH groups defined from different contact structures are isomorphic after a shift in their decompositions according to homology classes of orbit sets.

There is a similar map for the so-called twisted coefficients. Let Λ be any module over the group ring $\mathbf{F}[H_2(M; \mathbf{Z})]$. We denote the versions of Heegaard Floer homology and ECH with twisted coefficients in Λ by $\underline{\widehat{HF}}(-M, \mathfrak{s}; \Lambda)$ and $\underline{\widehat{ECH}}(M, \xi, A; \Lambda)$. Twisted coefficients are defined in [OSz2, Section 8] for Heegaard Floer homology and in [HS2, Section 11] for ECH.

Theorem **1.2.2.** — *There is an* $\mathbf{F}[H_2(M; \mathbf{Z})]$*-module isomorphism*

$$\underline{\Phi}_0 : \underline{\widehat{HF}}(-M, \mathfrak{s}_{(S,h)} + PD(A); \Lambda) \xrightarrow{\sim} \underline{\widehat{ECH}}(M, \xi_{(S,h)}, A; \Lambda),$$

where $A \in H_1(M; \mathbf{Z})$, *defined via an open book decomposition* (S, h) *of* M. *Moreover,* $\underline{\Phi}_0$ *sends the Heegaard Floer contact invariant for* $\xi_{(S,h)}$ *to the ECH contact invariant for* $\xi_{(S,h)}$.

On the other hand, Taubes [T2] has proven that Seiberg-Witten Floer cohomology and ECH are isomorphic. Let $\widetilde{\mathrm{HM}}(M)$ be the homology of the mapping cone of $U_\dagger : \check{C}(M) \to \check{C}(M)$, where $\check{C}(M)$ is the chain complex for $\widecheck{HM}(M)$. Combining Taubes' theorem with Theorem 1.2.1, we obtain the following "corollary":

Corollary **1.2.3.** — $\widehat{\mathrm{HF}}(M, \mathfrak{s}) \simeq \widetilde{\mathrm{HM}}(M, \mathfrak{s})$ *for any* $\mathfrak{s} \in$ *Spin*$^c(M)$.

An alternate proof of Corollary 1.2.3 is due to Kutluhan, Lee and Taubes [KLT1, KLT2, KLT3, KLT4, KLT5].

1.3. *Outline of proof.* — Fix an open book decomposition (S, h) for M. The first step in the construction of the map Φ_0 is to adapt the definitions of $\widehat{\mathrm{HF}}(-M)$ and $\widehat{\mathrm{ECH}}(M)$ to the open book decomposition (S, h). For Heegaard Floer homology this is achieved in Section 4, essentially by pushing all interesting intersection points between the $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-curves to one side of the Heegaard surface obtained from (S, h).

On the ECH side this was achieved in the first paper [0] of our series, where we introduced the group $\widehat{\mathrm{ECH}}(N, \partial N)$ for the mapping torus N of (S, h). This group was defined as a direct limit

$$\widehat{\mathrm{ECH}}(N, \partial N) = \lim_{j \to \infty} \mathrm{ECH}_j(N),$$

where j is the number of intersections of an orbit set in N with a page and the direct limit is taken with respect to maps

$$(\mathfrak{I}_j)_* : \mathrm{ECH}_j(N) \to \mathrm{ECH}_{j+1}(N),$$

defined by increasing the multiplicity of an elliptic orbit on ∂N which can be regarded intuitively as a receptacle for the J-holomorphic curves in $\mathbf{R} \times M$ which intersect the cylinder over the binding. The following was proved in [0]:

Theorem **1.3.1.** — $\widehat{\mathrm{ECH}}(M) \simeq \widehat{\mathrm{ECH}}(N, \partial N)$.

The map Φ_0 is, roughly speaking, the composition of a map

$$\Phi_* : \widehat{\mathrm{HF}}(-M) \to \mathrm{ECH}_{2g}(N),$$

induced by a symplectic cobordism W_+, followed by the natural map

$$\mathrm{ECH}_{2g}(N) \to \lim_{j \to \infty} \mathrm{ECH}_j(N),$$

induced by the maps $(\mathfrak{I}_j)_*$. Here g is the genus of S. (Strictly speaking, we need to use a "perturbed" version of $\mathrm{ECH}_j(N)$, as explained in Section 2.5, and replace the ECH groups by certain periodic Floer homology groups, as explained in Section 3.)

The cobordism W_+ is a symplectic fibration with fiber S and monodromy h over a Riemann surface with a strip-like end and a cylindrical end, which is biholomorphic to a disk with a puncture at the center and a puncture on the boundary. On the boundary of W_+ there is a (disconnected) Lagrangian submanifold Λ which, over the strip-like end, roughly speaking coincides with the Lagrangians used in the definition of Heegaard Floer homology (restricted to the nontrivial half of the Heegaard splitting). Then Φ_* is defined by counting degree $2g$ J-holomorphic multisections of W_+ with boundary on Λ which converge to generators of the Heegaard Floer complex over the strip-like end and to generators of the ECH complex over the cylindrical end.

We also define a map $\Psi_* : ECH_{2g}(N) \to \widehat{HF}(-M)$ by counting degree $2g$ J-holomorphic multisections of a symplectic cobordism $\overline{W}_-$ obtained by closing all fibers of W_+ with a disk and turning it upside down. The Lagrangian boundary condition on $\overline{W}_-$ becomes singular, and this leads to many more potential degenerations of J-holomorphic sections. For this reason, the proof that Ψ_* is defined is the longest and most difficult part of this paper.

In the next paper [II] we prove that Φ_* and Ψ_* are inverses of each other by composing the two cobordisms and degenerating them in a different way. The proof that $\Phi_* \circ \Psi_*$ and $\Psi_* \circ \Phi_*$ are the identity is thus reduced to a computation of some relative Gromov-Witten (or relative Gromov-Taubes) invariants. Finally we prove that the maps $(\mathfrak{I}_j)_*$ are isomorphisms for $j \geq 2g$ by an argument based on stabilizing the open book decomposition.

1.4. *Organization of the paper.* — Papers [I] and [II] should be read as a single paper which has been split for practical reasons. References from [II] will be written as "II.x"; for example "Section II.x" will mean "Section x" of [II].

In Section 2 we recall some results of [0], including the definition of $\widehat{ECH}(N, \partial N)$. In Section 3 we replace the ECH chain complexes $ECC_j(N)$ by the periodic Floer homology chain complexes $PFC_j(N)$, which are technically a little easier to use when defining chain maps to and from Heegaard Floer homology. Then in Section 4 we (i) review Lipshitz' reformulation of Heegaard Floer homology, (ii) restrict the Heegaard Floer chain complex to a page S as in [HKM] and obtain the chain group $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ whose homology is isomorphic to $\widehat{HF}(-M)$, and (iii) introduce an ECH-type index I_{HF} for Heegaard Floer homology. Section 5 is devoted to describing the moduli spaces of multisections which are used in the definitions of the chain maps Φ and Ψ between the Heegaard Floer chain complex $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ and the periodic Floer homology chain complex $PFC_{2g}(N)$. Then in Sections 6 and 7 we show that Φ and Ψ are indeed chain maps. The proof that Ψ is a chain map is substantially more involved than the proof that Φ is a chain map.

The proofs of the chain homotopies between the chain maps $\Psi \circ \Phi$ and id, and between the chain maps $\Phi \circ \Psi$ and id, are rather involved and occupy almost all (Sections II.2–II.4) of [II]. The necessary Gromov-Witten (or Gromov-Taubes) type calculations which are used in the proof of the chain homotopy are carried out in Section II.2. Finally, in Section II.5 we prove that (variants of) the maps $(\mathfrak{I}_j)_*$ are isomorphisms for $j \geq 2g$.

2. Adapting ECH to an open book decomposition

In this section we briefly recall the results of [0]. The reader is referred to [0] for a more complete discussion; the notation here is the same as that of [0], unless indicated otherwise.

2.1. *The first return map.* — We start by recalling the definition of an open book decomposition in order to set the notation. Let (S, h) be a pair consisting of a compact oriented surface S with nonempty boundary (sometimes called a *bordered surface*) and a diffeomorphism $h: S \xrightarrow{\sim} S$ which restricts to the identity on ∂S. *In this paper we will always assume that S is connected and ∂S is connected, unless stated otherwise.*

We define the mapping torus

$$N_{(S,h)} = S \times [0,1]/\sim,$$

where $(x, 1) \sim (h(x), 0)$.

Definition **2.1.1.** — *Let $K \subset M$ be a link. Then M admits an* open book decomposition (S, h) with binding K *if there is a diffeomorphism*

$$M \cong N_{(S,h)} \cup V,$$

where $V \cong D^2 \times S^1$ is a tubular neighborhood of K and, if we parametrize ∂D^2 by $t \in [0,1]$, then $\partial S \times \{t\} \subset \partial N_{(S,h)}$ is glued to $\{t\} \times S^1$.

The decomposition we give here is the standard one and is slightly different from the one in [0], where we introduced a "no man's land" diffeomorphic to $T^2 \times [0,1]$ between $N_{(S,h)}$ and V. Of course the two decompositions are equivalent.

Let S be a bordered surface. Let $\nu(\partial S) \simeq [-\varepsilon, 0] \times \mathbf{R}/\mathbf{Z}$ be a neighborhood of ∂S with coordinates (y, θ) so that $\partial S = \{y = 0\}$. (The slight difference with [0] is that, in [0, Section 9.3], $\partial S = \{y = 1\}$ instead of $\{y = 0\}$.)

The following was essentially proved in [0]:

Lemma **2.1.2.** — *Given $h \in \mathrm{Diff}(S, \partial S)$, there exists $h_0 \in \mathrm{Diff}(S, \partial S)$ in the same connected component as h, and which satisfies the following:*

1. *there exists a contact form α on $N_{(S,h)}$ such that the Reeb vector field R_α of α is transverse to $S \times \{t\}$, $t \in [0,1]$, and h_0 is the first return map of R_α on $S \times \{0\}$; and*
2. *the diffeomorphism h_0 restricts to $(y, \theta) \mapsto (y, \theta - y)$ on $\nu(\partial S)$.*

Moreover, $\alpha = f_t dt + \beta_t$, where f_t is a positive function on S and β_t is a Liouville 1-form on S.

Proof. — (1) and the last sentence of the lemma follow from combining Lemmas 9.3.2 and 9.3.3 from [0]. To verify (2), we refer to Section 9.3.1 and the discussion after the proof of Lemma 9.3.3 in [0] and take a slight modification of α of the form

$$\alpha = g(y)d\theta + f(y)dt$$

on a neighborhood $\nu(\partial N_{(S,h)})$ of $\partial N_{(S,h)}$. Here $\nu(\partial N_{(S,h)})$ is the quotient of

$$\nu(\partial S) \times [0,1] \simeq [-\varepsilon, 0] \times (\mathbf{R}/\mathbf{Z}) \times [0,1]$$

with coordinates (y, θ, t), by the equivalence relation $(y,\theta,1) \sim (y,\theta,0)$. If we take

$$(f(y), g(y)) = \left(f(0) + y^2/2, g(0) + y\right),$$

with $(f(0), g(0))$ in the interior of the first quadrant, then the Reeb vector field R_α is parallel to

$$-f'(y)\partial_\theta + g'(y)\partial_t = -y\partial_\theta + \partial_t. \qquad (\mathbf{2.1.1})$$

Its first return map then satisfies (2). □

From now on, we assume that $h = h_0$ as given by Lemma 2.1.2.

2.2. $ECH(N, \partial N, \alpha)$ *and* $\widehat{ECH}(N, \partial N, \alpha)$. — Let $N = N_{(S,h)}$ and α be as in the previous subsection. We recall the definitions of the variants $ECH(N, \partial N, \alpha)$ and $\widehat{ECH}(N, \partial N, \alpha)$ and the main result concerning them from [0]. In particular, we carry over the Morse-Bott terminology from [0, Section 4]. We will assume that the almost complex structure J on $\mathbf{R} \times N$ is Morse-Bott regular.

The boundary ∂N is foliated by a Morse-Bott family $\mathcal{N}$ of simple orbits of R_α of the form $\theta = const$. We may assume without loss of generality that α is nondegenerate away from ∂N, after a C^k-small perturbation for $k \gg 0$. We pick two orbits from $\mathcal{N}$ and label them h and e. The orbits h and e are meant to become hyperbolic and elliptic after a small, controlled perturbation of α. The Morse-Bott family $\mathcal{N}$ is *negative*. Since $\mathcal{N}$ is a Morse-Bott family on ∂N, this means that $\mathcal{N}$ plays the role of a sink and that no holomorphic curve (besides a trivial cylinder) is asymptotic to an orbit of $\mathcal{N}$ at the positive end.

2.2.1. $ECH(N, \partial N, \alpha)$. — Let $\mathcal{P}$ be the set of simple Reeb orbits of α in $int(N)$. We write $ECC_j^\flat(N, \alpha)$ for the chain complex generated by orbit sets $\boldsymbol{\gamma}$ constructed from $\mathcal{P} \cup \{e\}$, whose homology class $[\boldsymbol{\gamma}]$ intersects the page $S \times \{t\}$ exactly j times. The differential for $ECC_j^\flat(N, \alpha)$ counts ECH index 1 Morse-Bott buildings in $(\mathbf{R} \times N, J)$ between orbit sets which are constructed from $\mathcal{P} \cup \{e\}$; for more details see [0, Section 7.3]. In particular, if $\tilde{u}$ is an ECH index 1 Morse-Bott building which is counted in the differential, then e can appear only at a negative end of $\tilde{u}$ and no single end of $\tilde{u}$ can multiply cover e with multiplicity > 1. (It is still possible that there are many ends of $\tilde{u}$ which simply cover e.)

There are inclusions of chain complexes:

$$\mathfrak{I}_j^\flat : \mathrm{ECC}_j^\flat(\mathrm{N}, \alpha) \to \mathrm{ECC}_{j+1}^\flat(\mathrm{N}, \alpha),$$

$$\boldsymbol{\gamma} \mapsto e\boldsymbol{\gamma},$$

where we are using multiplicative notation for orbit sets. Let us write $\mathrm{ECH}_j^\flat(\mathrm{N}, \alpha)$ for the homology of the chain complex $\mathrm{ECC}_j^\flat(\mathrm{N}, \alpha)$. We then define

$$\mathrm{ECH}(\mathrm{N}, \partial\mathrm{N}, \alpha) = \lim_{j\to\infty} \mathrm{ECH}_j^\flat(\mathrm{N}, \alpha).$$

2.2.2. $\widehat{\mathrm{ECH}}(\mathrm{N}, \partial\mathrm{N}, \alpha)$. — Let $\mathrm{ECC}_j(\mathrm{N}, \alpha)$ be the chain complex generated by orbit sets $\boldsymbol{\gamma}$ constructed from $\widehat{\mathcal{P}} = \mathcal{P} \cup \{e, h\}$, whose homology class $[\boldsymbol{\gamma}]$ intersects $\mathrm{S} \times \{t\}$ exactly j times. The differential for $\mathrm{ECC}_j(\mathrm{N}, \alpha)$ counts ECH index 1 Morse-Bott buildings in $(\mathbf{R} \times \mathrm{N}, \mathrm{J})$. There are inclusions:

$$\mathfrak{I}_j : \mathrm{ECC}_j(\mathrm{N}, \alpha) \to \mathrm{ECC}_{j+1}(\mathrm{N}, \alpha), \tag{2.2.1}$$

$$\boldsymbol{\gamma} \mapsto e\boldsymbol{\gamma},$$

as before. Writing $\mathrm{ECH}_j(\mathrm{N}, \alpha)$ for the homology of $\mathrm{ECC}_j(\mathrm{N}, \alpha)$, we define

$$\widehat{\mathrm{ECH}}(\mathrm{N}, \partial\mathrm{N}, \alpha) = \lim_{j\to\infty} \mathrm{ECH}_j(\mathrm{N}, \alpha).$$

The following was the main result of [0]:

Theorem **2.2.1.** — *We have the isomorphisms:*

$$\mathrm{ECH}(\mathrm{M}) \simeq \mathrm{ECH}(\mathrm{N}, \partial\mathrm{N}, \alpha),$$

$$\widehat{\mathrm{ECH}}(\mathrm{M}) \simeq \widehat{\mathrm{ECH}}(\mathrm{N}, \partial\mathrm{N}, \alpha).$$

2.3. *Splitting of* ECH *according to homology classes.* — Given an orbit set $\boldsymbol{\gamma} = \prod_{j=1}^{l} \gamma_j^{m_j}$ in M or N, its *total homology class* $[\boldsymbol{\gamma}]$ is defined as

$$[\boldsymbol{\gamma}] = \sum_{j=1}^{l} m_j[\gamma_j],$$

where $[\gamma_j] \in H_1(M; \mathbf{Z})$ if M is a closed manifold and $[\gamma_j] \in H_1(N; \mathbf{Z})$ or $H_1(N, \partial N; \mathbf{Z})$ (as appropriate) if N has torus boundary. We then have the direct sum decomposition:

$$ECC(M) = \bigoplus_{A \in H_1(M)} ECC(M, A),$$

where ECC(M, A) is the subcomplex of ECC(M) generated by orbit sets with total homology class A. The direct sum of chain groups descends to a direct sum of homology groups ECH(M, A).

There is an analogous splitting for $ECC(N, \partial N, \alpha)$. In fact,

$$ECC_j^\flat(N, \alpha) = \bigoplus_{A \in H_1(N, \partial N)} ECC_j^\flat(N, \alpha, A),$$

and the inclusion $\boldsymbol{\gamma} \mapsto e\boldsymbol{\gamma}$ respects this splitting since $[e] = 0$ in $H_1(N, \partial N)$.

Lemma **2.3.1.** — *If* M *has an open book decomposition with binding* K *and* N *is the mapping torus of a page, then there is an isomorphism*

$$\varpi : H_1(M) \xrightarrow{\sim} H_1(N, \partial N).$$

Proof. — We use the long exact sequence for the pair (M, K):

$$H_1(K) \to H_1(M) \xrightarrow{i} H_1(M, K) \to H_0(K).$$

Since $[K] = 0$ in $H_1(M)$ and K is connected, the map i is an isomorphism. By excision and homotopy invariance, we obtain the isomorphism $H_1(M, K) \xrightarrow{\sim} H_1(N, \partial N)$. Combining the two isomorphisms gives us ϖ. □

The isomorphism $ECH(M) \cong ECH(N, \partial N, \alpha)$ respects the splitting into total homology classes. In fact

$$ECH(M, A) \cong ECH(N, \partial N, \alpha, \varpi(A)).$$

The same also holds for the hat versions.

2.4. *Twisted coefficients in ECH.* — In this subsection we describe the construction of ECH with twisted coefficients. We will adapt the analogous construction in Heegaard Floer homology from [OSz2, Section 8] instead of following the original construction in [HS2, Section 11].

Fix a homology class A and a closed curve $\Gamma \subset M$ such that $[\Gamma] = A$. Let

$$\boldsymbol{\gamma}^+ = \prod_{j=1}^{l_+} \gamma_{j,+}^{m_{j,+}}, \quad \boldsymbol{\gamma}^- = \prod_{j=1}^{l_-} \gamma_{j,-}^{m_{j,-}}$$

be orbit sets with $[\boldsymbol{\gamma}^+]=[\boldsymbol{\gamma}^-]=A$. Here we are assuming that all the multiplicities $m_{j,\pm}$ are positive. We denote by $H_2(M,\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-)$ the set of relative homology classes $[C]$ such that $\partial C=\boldsymbol{\gamma}^+-\boldsymbol{\gamma}^-$, namely $[C]\in H_2(M,\mathcal{O}(\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-))$, such that $\mathcal{O}(\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-)=(\cup_{j=1}^{l_+}\gamma_{j,+})\cup(\cup_{j=1}^{l_-}\gamma_{j,-})$ and $[\partial C]=\sum_{j=1}^{l_+}m_{j,+}[\gamma_{j,+}]-\sum_{j=1}^{l_-}m_{j,-}[\gamma_{j,-}]$. We also denote by $\mathcal{M}^{I=1}(\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-,C)$ the moduli spaces of $I=1$ holomorphic curves in $\mathbf{R}\times M$ from $\boldsymbol{\gamma}^+$ to $\boldsymbol{\gamma}^-$ representing the homology class of C.

A *complete set of paths for* A *based at* Γ is the choice, for every orbit set $\boldsymbol{\gamma}$ such that $[\boldsymbol{\gamma}]=A$, of a surface $C_{\boldsymbol{\gamma}}\subset M$ such that $\partial C_{\boldsymbol{\gamma}}=\boldsymbol{\gamma}-\Gamma$, defined in the same way as the previous paragraph. A complete set of paths for A induces maps

$$\mathfrak{A}':H_2(M,\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-)\to H_2(M)$$

for all $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$ in A by $\mathfrak{A}'(C)=[C_{\boldsymbol{\gamma}^-}\cup C\cup -C_{\boldsymbol{\gamma}^+}]$. This map is compatible with the action of $H_2(M)$ on $H_2(M,\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-)$ and with the concatenation of chains with matching ends.

We denote the group ring of $H_2(M;\mathbf{Z})$ by $\mathbf{F}[H_2(M;\mathbf{Z})]$ and the generator corresponding to $c\in H_2(M;\mathbf{Z})$ by e^c. We define

$$\underline{ECC}(M,\alpha,A)=ECC(M,\alpha,A)\otimes_{\mathbf{F}}\mathbf{F}[H_2(M;\mathbf{Z})]$$

as an $\mathbf{F}[H_2(M;\mathbf{Z})]$-module, with differential

$$\partial\boldsymbol{\gamma}^+=\sum_{\boldsymbol{\gamma}^-}\sum_{C\in H_2(M,\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-)}\#\left(\mathcal{M}^{I=1}(\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-,C)/\mathbf{R}\right)e^{\mathfrak{A}'(C)}\boldsymbol{\gamma}^-.$$

The homology of this complex is ECH with twisted coefficients $\underline{ECH}(M,A)$. The U-map can be defined in a similar manner and $\underline{\widehat{ECH}}(M,A)$ is the homology of its mapping cone. The construction of $\underline{ECH}(N,\partial N,\varpi(A))$ with coefficient ring $\mathbf{F}[H_2(N,e;\mathbf{Z})]\cong\mathbf{F}[H_2(M;\mathbf{Z})]$ is similar, and there are isomorphisms

$$\underline{ECH}(M,A)\simeq\underline{ECH}(N,\partial N,\varpi(A)). \tag{2.4.1}$$

$$\underline{\widehat{ECH}}(M,A)\simeq\underline{\widehat{ECH}}(N,\partial N,\varpi(A)). \tag{2.4.2}$$

Moreover, by considering only orbit sets that intersect a page j times we can define $\underline{ECH}_j(N)$ and we have

$$\underline{\widehat{ECH}}(N,\partial N,\alpha,\varpi(A))=\lim_{j\to\infty}\underline{ECH}_j(N,\varpi(A)).$$

2.5. *Elimination of elliptic orbits.* — The goal of this subsection is to show how to locally replace elliptic orbits by hyperbolic orbits with the same parity (i.e., with negative eigenvalues). The main result, which is used in [II] but is also of independent interest, is Theorem 2.5.2 below. Let us first give the following definition:

Definition **2.5.1** *(Filtration $\mathcal{F}$). — If $N = N_{(S,h)}$ is the mapping torus of (S, h) and $\gamma \subset N$ is a link which is everywhere transverse to $S \times \{t\}$, $t \in [0, 1]$, then we define $\mathcal{F}(\gamma) = \langle \gamma, S \times \{0\}\rangle$, where $\langle , \rangle$ is the algebraic intersection number.*

Theorem **2.5.2** *(Elimination of elliptic orbits). — Let α be a contact form on the mapping torus $N = N_{(S,h)}$, $h \in \mathrm{Diff}(S, \partial S)$, such that the Reeb vector field R_α is transverse to $S \times \{t\}$, $t \in [0, 1]$, and h is the first return map of R_α on $S \times \{0\}$. Then, given $m \in \mathbf{Z}^+$ and $\varepsilon > 0$, there exists a smooth function $f : N \to (0, +\infty)$ which is ε-close to 1 with respect to a fixed C^1-norm and whose Reeb vector field $R_{f\alpha}$ has no elliptic orbits γ in int(N) satisfying $\mathcal{F}(\gamma) \leq m$.*

2.5.1. *Model situation on the solid torus.* — Fix a constant $\delta > 0$. Consider the solid torus

$$V = D^2 \times S^1 = D^2 \times (\mathbf{R}/\mathbf{Z}) = \{(r, \theta, z) \mid r \leq \delta\}$$

with the contact structure $\xi_0 = \ker \alpha_0$, where $\alpha_0 = dz + r^2 d\theta$. We write $D_{z_0} = \{z = z_0\} \subset V$ and $T_{r_0} = \{r = r_0\} \subset V$.

Given a function $f : [0, \delta] \to (0, +\infty)$, the Reeb vector field $R_{f(r)\alpha_0}$ is given by:

$$R_{f\alpha_0} = \frac{1}{2rf^2}((r^2 f' + 2rf)\partial_z - f'\partial_\theta). \qquad (\mathbf{2.5.1})$$

In particular, $R_{f\alpha_0}$ is transverse to each D_z, provided $r^2 f' + 2rf > 0$, and $R_{f\alpha_0}$ is tangent to each T_r. The first return map $\Phi_{f\alpha_0} : D_0 \xrightarrow{\sim} D_0$ is a rotation on each circle $\{r = r_0\}$ and can be written as $(r, \theta) \mapsto (r, \theta + \phi_{f\alpha_0}(r))$.

Claim **2.5.3.** — *Let $C : [0, \delta] \to (0, +\infty)$ be a smooth function. If*

$$f(r) = A \exp\left(-\int_0^r \frac{2sC(s)}{1 + C(s)s^2} ds\right), \qquad (\mathbf{2.5.2})$$

where A is a positive constant, then the first return map of $R_{f\alpha_0}$ satisfies $\phi_{f\alpha_0}(r) = C(r)$ for each $r \in (0, \delta]$.

Here we are ignoring the differentiability at $r = 0$.

Proof. — In view of Equation (2.5.1), $R_{f\alpha_0}$ is parallel to $\partial_z + C(r)\partial_\theta$ if and only if

$$C(r)(r^2 f' + 2rf) = -f' \qquad (\mathbf{2.5.3})$$

is satisfied. □

Let $f_0 : [0, \delta] \to (0, +\infty)$ be a function such that $\phi_{f_0\alpha_0}(r) = \phi_0$, where $\phi_0 \in (0, 2\pi)$. By Claim 2.5.3, we may take

$$f_0(r) = \exp\left(-\int_\delta^r \frac{2s\phi_0}{1+\phi_0 s^2} ds\right) = \frac{1+\phi_0\delta^2}{1+\phi_0 r^2}.$$

In particular, $\gamma_0 = \{0\} \times S^1$ is the only orbit γ of $R_{f_0\alpha_0}$ satisfying $\mathcal{F}(\gamma) = 1$, where $\mathcal{F}(\gamma) = \langle \gamma, D_0 \rangle$, and the orbit γ_0 is elliptic and nondegenerate.

Lemma **2.5.4** *(Modification lemma). — There exists a function $f_2 : V \to (0, +\infty)$ such that $f_2\alpha_0$ is arbitrarily C^1-close to $f_0\alpha_0$ and the Reeb vector field $R_{f_2\alpha_0}$ is equal to $R_{f_0\alpha_0}$ near ∂V, is transverse to D_z for all $z \in S^1$, and has only one orbit γ satisfying $\mathcal{F}(\gamma) = 1$. Moreover the orbit γ is hyperbolic.*

Proof. — Without loss of generality we take $\phi_0 \in (0, \pi]$; the case $\phi_0 \in [\pi, 2\pi)$ is similar.

Let $0 < \delta' \ll \delta$ and let $C_{\delta'} : [0, \delta] \to [\phi_0, \pi]$ be a smooth function such that:

- $C_{\delta'}(r) = \pi$ on $[0, \frac{\delta'}{2}]$; and
- $C_{\delta'}(r) = \phi_0$ on $[\delta', \delta]$.

Let $f_1 : [0, \delta] \to (0, +\infty)$ be the smooth function

$$f_1(r) = \exp\left(-\int_\delta^r \frac{2sC_{\delta'}(s)}{1+C_{\delta'}(s)s^2} ds\right).$$

The function f_1 satisfies the following:

(1) $f_1 = f_0$ and $R_{f_1\alpha_0} = R_{f_0\alpha_0}$ for $r \in [\delta', \delta]$;
(2) $R_{f_1\alpha_0} \pitchfork D_z$ for all $z \in S^1$;
(3) $\phi_{f_1\alpha_0}(r) = C_{\delta'}(r)$ for all $r \in [0, \delta]$.

We compute that

$$\textbf{(2.5.4)} \qquad \frac{f_1(r)}{f_0(r)} = \exp\left(\int_\delta^r \left(\frac{2s\phi_0}{1+\phi_0 s^2} - \frac{2sC_{\delta'}(s)}{1+C_{\delta'}(s)s^2}\right) ds\right).$$

By taking $\delta' > 0$ to be arbitrarily small, the absolute value of the integrand of Equation (2.5.4) can be made arbitrarily small. Hence $f_1(r) \approx f_0(r)$ for all $r \in [0, \delta]$, in view of (1). (Here $f \approx g$ means $|f - g|$ is bounded above by a continuous function of δ' which approaches 0 as $\delta' \to 0$.)

Next we consider $\partial_x f_i = \frac{x f_i'(r)}{r}$ and $\partial_y f_i = \frac{y f_i'(r)}{r}$, where $r = \sqrt{x^2+y^2}$. We appeal to Equation (2.5.3) and write

$$f_1'(r) = \frac{2rf_1(r)}{1+C_{\delta'}(r)r^2}, \quad f_0'(r) = \frac{2rf_0(r)}{1+\phi_0 r^2}.$$

For $\delta' > 0$ sufficiently small, $f_1'(r)$ and $f_0'(r)$ are both arbitrarily close to 0. Moreover $f_1'(r) = f_0'(r)$ for $r \in [\delta', \delta]$. Hence $f_1' \approx f_0'$ for all $r \in [0, \delta]$. This implies that $\partial_* f_1 \approx \partial_* f_0$ and $\partial_*(f_1/f_0) \approx 0$ for $* = x, y$.

The Reeb vector field $R_{f_1\alpha_0}$ has exactly one orbit γ_0 satisfying $\mathcal{F}(\gamma_0) = 1$. The linearized first return map $d\Phi_{f_1\alpha_0}(0)$ of γ_0 has eigenvalues -1.

We now claim there exists a C^k-small perturbation f_2 of f_1 for any $k \gg 0$ such that the linearized first return map $d\Phi_{f_2\alpha_0}(0)$ of the corresponding orbit has eigenvalues $-\lambda$ and $-\frac{1}{\lambda}$ with $\lambda \in \mathbf{R}^{>0} - \{1\}$. This follows from a local model on $D^2 \times [0, 1]$ with coordinates (x, y, z): Suppose the Reeb vector field is $R = \partial_z$. Then the contact 1-form can be written as $\alpha = dz + \beta$, where β is a 1-form on D^2. We consider $R_{\mathcal{G}\alpha}$, where $\mathcal{G}(x, y) = 1 + \varepsilon(x^2 - y^2)$ and $\varepsilon > 0$ is sufficiently small. The component of $R_{\mathcal{G}\alpha}$ in the xy-direction is parallel to $y\partial_x - x\partial_y$. Hence the derivative at zero of the holonomy map $D^2 \times \{0\} \dashrightarrow D^2 \times \{1\}$,[1] obtained by flowing along $R_{\mathcal{G}\alpha}$, has eigenvalues λ_0 and $\frac{1}{\lambda_0}$ with $\lambda_0 \in \mathbf{R}^{>0} - \{1\}$. By appropriately damping $\mathcal{G}$ out to 1 outside a small neighborhood of $(0, 0, 0) \in D^2 \times [0, 1]$, the above model can be grafted into V to give f_2. This procedure does not introduce any extra $\mathcal{F} = 1$ orbits, since the graph of $\Phi_{f_1\alpha_0}$ in $D_0 \times D_0$ intersects the diagonal transversely in one point and this property is stable under a C^k-small perturbation of $\Phi_{f_1\alpha_0}$ for any $k \gg 0$. □

Remark **2.5.5.** — The modification in Lemma 2.5.4 introduces many elliptic orbits satisfying $\mathcal{F} > 1$.

2.5.2. *Proof of Theorem 2.5.2.* — Starting with the contact form α on N, we make a C^2-small perturbation of α relative to ∂N such that the resulting Reeb vector field — also called R_α — satisfies the following:

(1) R_α is nondegenerate away from ∂N;
(2) for each $\mathcal{F} = 1$ elliptic orbit $\gamma \subset int(N)$, the first return map is a rotation by an irrational angle and α is of the form $C_0 f_0 \alpha_0$ on a tubular neighborhood V_γ of γ.

Here f_0 and α_0 are as in Section 2.5.1 and C_0 is some constant. We then use Lemma 2.5.4 on the tubular neighborhoods V_γ to replace the $\mathcal{F} = 1$ elliptic orbits by hyperbolic orbits plus $\mathcal{F} > 1$ orbits. Next, we perturb the form so that the $\mathcal{F} = 1$ orbits and the $\mathcal{F} = 2$ hyperbolic orbits are left unchanged and the $\mathcal{F} = 2$ elliptic orbits satisfy (2), with $\mathcal{F} = 1$ replaced by $\mathcal{F} = 2$. Using Lemma 2.5.4 again, we replace the $\mathcal{F} = 2$ elliptic orbits by hyperbolic orbits and $\mathcal{F} > 2$ orbits. Continuing in this manner, we obtain α without any $\mathcal{F} \leq m$ elliptic orbits in $int(N)$.

[1] The dashed arrow indicates that the map is only partially defined.

2.5.3. *Direct limits.* — Starting with (S, h) and α from Section 2.1, by Theorem 2.5.2 there exists a sequence of contact 1-forms $\alpha_j, j \in \mathbf{N}$, on N such that:

(1) α and α_j agree on a small neighborhood of ∂N and $\ker \alpha$ and $\ker \alpha_j$ are isotopic relative to ∂N;
(2) all the orbit sets of R_{α_j} that satisfy $\mathcal{F} = i$, $i \leq j$, have α_j-action between $i - \epsilon$ and $i + \epsilon$, where $0 < \epsilon \ll 1$;
(3) R_{α_j} and $R_{\alpha_{j+1}}$ are parallel outside of a small neighborhood V_{j+1} of the elliptic orbits of R_{α_j} with $\mathcal{F} = j + 1$;
(4) $R_{\alpha_{j+1}}$ has no elliptic orbits with $\mathcal{F} \leq j + 1$.

We take suitable extensions of α_j and α_{j+1} to M as in [0, Section 9.3], which we still call α_j and α_{j+1}. Then

$$\widehat{ECH}(M) \simeq \lim_{j \to \infty} ECH^{\leq L_j}(M, \alpha_j),$$

where $L_j = j + \epsilon$, the superscript $\leq L_j$ indicates orbit sets of action $\leq L_j$, and the direct limit is the limit of the induced maps of ECH chain maps

$$\overline{\mathfrak{L}}_j : ECC^{\leq L_j}(M, \alpha_j) \to ECC^{\leq L_{j+1}}(M, \alpha_{j+1}),$$

given by [HT3, Theorem 2.4]. It is important to remember that the ECH cobordism maps are defined by using cobordism maps in Seiberg-Witten Floer cohomology.

Next we consider the ECH chain maps

$$\mathfrak{K}_j : ECC_j(N, \alpha_j) \to ECC_{j+1}(N, \alpha_{j+1}), \tag{2.5.5}$$

given by composing the restriction

$$\mathfrak{K}'_j : ECC_j(N, \alpha_j) \to ECC_j(N, \alpha_{j+1}) \tag{2.5.6}$$

of $\overline{\mathfrak{L}}_j$ to N and $\mathfrak{I}_j : ECC_j(N, \alpha_{j+1}) \to ECC_{j+1}(N, \alpha_{j+1})$ given by $\boldsymbol{\gamma} \mapsto e\boldsymbol{\gamma}$. The methods of [0, Section 9] that are used to prove Theorem 1.3.1 show that

$$ECH^{\leq L_j}(M, \alpha_j) \simeq ECH_j(N, \alpha_j),$$

and $(\overline{\mathfrak{L}}_j)_* = (\mathfrak{K}_j)_*$ under this identification.

Hence we have the following theorem which will be used in [II]:

Theorem **2.5.6.** — $\widehat{ECH}(M) \simeq \lim_{j \to \infty} ECH_j(N, f_j\alpha)$, *where the direct limit is taken with respect to the maps* $\mathfrak{K}_j$.

3. Periodic Floer homology

In order to simplify some technicalities, we would like to replace the ECH groups by the *periodic Floer homology groups* of Hutchings [Hu1, Hu2, HS1], abbreviated PFH in this paper. The PFH groups are defined in a manner completely analogous to the ECH groups, with stable Hamiltonian vector fields replacing the Reeb vector fields.

If M is a closed manifold which fibers over the circle, then the PFH groups of M are equivalent to the Seiberg-Witten Floer cohomology groups of M by the work of Lee-Taubes [LT].

3.1. *Interpolating between Reeb and stable Hamiltonian vector fields.* — Consider the contact form $\alpha = f_t dt + \beta_t$ on $S \times [0,1]$, as defined in Section 2.1. We may assume that R_α is parallel to ∂_t on $S \times [0,1]$, after transforming $S \times [0,1]$ by a smooth family of diffeomorphisms parametrized by $t \in [0,1]$ and isotopic to the identity. Since

$$d\alpha = d_S f_t \wedge dt + d_S \beta_t + dt \wedge \frac{d\beta_t}{dt},$$

where d_S is the exterior derivative in the S-direction, it follows that $\frac{d\beta_t}{dt} = d_S f_t$. (Hence $d_S \beta_t$ is an area form which does not depend on t.) Also, the form α is a contact form as long as $d_S \beta_t > 0$. Hence, for $C \gg 0$, the form $(C + f_t)dt + \beta_t$ is a contact form with Reeb vector field parallel to ∂_t.

Now consider the 1-parameter family of 1-forms

$$\alpha_\varsigma = C dt + \varsigma(f_t dt + \beta_t), \tag{3.1.1}$$

$\varsigma \in [0,1]$, on N. It interpolates between the contact form $\alpha_1 = C dt + (f_t dt + \beta_t)$ and the stable Hamiltonian form $\alpha_0 = C dt$. The Reeb vector fields $R_\varsigma = R_{\alpha_\varsigma}$ are directed by ∂_t and hence are parallel for all $\varsigma > 0$.

The pair $(\alpha_\varsigma, \omega = d_S \beta_t)$ is a *stable Hamiltonian structure* on N, i.e., ω is closed, $\alpha_\varsigma \wedge \omega > 0$, and $\ker d\alpha_\varsigma \supset \ker \omega$; see for example [CV]. When $\varsigma = 0$, the Hamiltonian vector field R_0 equals $\frac{1}{C}\partial_t$ and hence is parallel to all the R_ς, $\varsigma > 0$. Also let ξ_ς be the 2-plane field on N given by the kernel of α_ς. The closed 2-form ω can either be viewed as an area form on S or as a (maximally nondegenerate) 2-form on N.

3.2. *Definitions.* — Consider the infinite cylinder $\mathbf{R} \times N$ with coordinates (s, x). We will also use the notation $W' = \mathbf{R} \times N$.

Definition **3.2.1.** — *An almost complex structure* J_ς *on* $\mathbf{R} \times N$ *is* adapted to the stable Hamiltonian structure $(\alpha_\varsigma, \omega)$ *if* J_ς *satisfies the following:*

- J_ς *is s-invariant;*
- $J_\varsigma(\partial_s) = R_\varsigma$ *and* $J_\varsigma(\xi_\varsigma) = \xi_\varsigma$*;*

– J_ς *is tamed by the symplectic form* $\Omega' = ds \wedge dt + \omega$.

Our goal is to replace the ECH chain complexes

$$ECC_j(N, \alpha_\varsigma, J_\varsigma), \quad ECC_j^\flat(N, \alpha_\varsigma, J_\varsigma)$$

for $\varsigma > 0$, by the analogously defined PFH chain complexes

$$PFC_j(N, \alpha_0, \omega, J_0), \quad PFC_j^\flat(N, \alpha_0, \omega, J_0).$$

The orbit sets of the ECH chain groups are constructed using the Reeb vector fields R_ς and the orbit sets of the PFH chain groups are constructed using the Hamiltonian vector field R_0.

We introduce some notation. Let J_ς, $\varsigma \in [0,1]$, be a smooth family of $(\alpha_\varsigma, \omega)$-adapted almost complex structures, let $\boldsymbol{\gamma} = \prod_i \gamma_i^{m_i}$, $\boldsymbol{\gamma}' = \prod_i \gamma_i^{m_i'}$ be orbit sets of $PFC_j(N, \alpha_0, \omega)$ or $ECC_j(N, \alpha_\varsigma)$, and let $Z \in H_2(N, \boldsymbol{\gamma}, \boldsymbol{\gamma}')$.

A J_ς*-holomorphic curve (or map) from* $\boldsymbol{\gamma}$ *to* $\boldsymbol{\gamma}'$ is a holomorphic map

$$u : (\dot{F}, j) \to (\mathbf{R} \times N, J_\varsigma)$$

satisfying the following:

(1) $(\dot{F}, j)$ is a closed Riemann surface with a finite number of punctures removed;
(2) the neighborhoods of the punctures are mapped asymptotically to cylinders over Reeb orbits;
(3) at the positive end of $\mathbf{R} \times N$, u is asymptotic to $\mathbf{R} \times \gamma_i$ with total multiplicity m_i (more precisely, if we list the positive ends of u that are asymptotic to some multiple cover of $\mathbf{R} \times \gamma_i$ and the covering degrees are $m_{i1}, \dots, m_{ij_i}$, then $m_i = m_{i1} + \dots + m_{ij_i}$); and
(4) at the negative end of $\mathbf{R} \times N$, u is asymptotic to $\mathbf{R} \times \gamma_i$ with total multiplicity m_i'.

We write:

– $\mathcal{M}_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ for the moduli space of J_ς-holomorphic curves from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$;
– $\mathcal{M}_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', Z) \subset \mathcal{M}_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ for the subset of curves in the class Z;
– $\mathcal{M}^s_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}') \subset \mathcal{M}_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ for the subset of simply-covered curves; and
– $\mathcal{M}^{nc}_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', Z) \subset \mathcal{M}_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', Z)$ for the subset of curves without connector components.

Convention **3.2.2.** — *The curves in all the moduli spaces are defined only up to automorphisms of the domain, the complex structures of their domains are not fixed, and the curves are allowed to be disconnected.*

A *connector over an orbit set* $\delta = \prod_i \delta_i^{m_i}$ is a collection of branched covers of trivial cylinders where the branching is optional and the total covering degree over $\mathbf{R} \times \delta_i$ is m_i.

Given two orbit sets $\delta = \prod_i \delta_i^{m_i}$ and $\delta' = \prod_i \delta_i^{m_i'}$, we set $\delta/\delta' = \prod_i \delta_i^{m_i - m_i'}$ if $m_i' \leq m_i$ for all i; otherwise we set $\delta/\delta' = 0$. Here some m_i and m_i' may be zero. We then write $u \in \mathcal{M}_{J_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', Z)$ as $u^0 \cup u^1$, where u^0 is a connector over the orbit set $\boldsymbol{\gamma}_0$ and $u^1 \in \mathcal{M}^{nc}_{J_\varsigma}(\boldsymbol{\gamma}/\boldsymbol{\gamma}_0, \boldsymbol{\gamma}'/\boldsymbol{\gamma}_0)$. By [HT1, Proposition 7.5], if $I_{ECH}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', Z) = 1$ (resp. 2), then $u^1 \in \mathcal{M}^{s}_{J_\varsigma}(\boldsymbol{\gamma}/\boldsymbol{\gamma}_0, \boldsymbol{\gamma}'/\boldsymbol{\gamma}_0)$ and has Fredholm index 1 (resp. 2).

3.3. *The flux.* — For more details, see for example [CHL, Section 2]. Let

$$h_* : H_1(S; \mathbf{Z}) \to H_1(S; \mathbf{Z})$$

be the map on homology induced by $h : (S, \omega) \xrightarrow{\sim} (S, \omega)$ and let K be the kernel of $h_* - id$. Then the *flux* $F_h : K \to \mathbf{R}$ is defined as follows: Let $[\zeta] \in K$. Since $[\zeta] = [h(\zeta)] \in H_1(S; \mathbf{Z})$, there is a 2-chain $\mathcal{S} \subset S$ such that $\partial \mathcal{S} = \zeta - h(\zeta)$. We then define:

$$F_h(\zeta) = \int_{\mathcal{S}} \omega.$$

Since the map h is the first return map of a Reeb vector field R_ς, it has zero flux (cf. [CHL, Lemma 2.2]). This implies that $\int_{[C]} \omega = 0$ for every $[C] \in H_2(N)$, where ω is now viewed as a closed 2-form on N. Indeed, [C] can be represented by a surface of the form $(\zeta \times [0, 1]) \cup (\mathcal{S} \times \{0\})$, where the relevant boundary components are glued. Hence, if $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ are orbit sets of $PFC_j(N, \alpha_0, \omega_0, J_0)$, then the ω-area of any $Z \in H_2(N, \boldsymbol{\gamma}, \boldsymbol{\gamma}')$ only depends on $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$.

3.4. *Compactness.* — The vanishing of the flux is an important ingredient in establishing that $\mathcal{M}_{J_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ admits a compactification in the sense of [EGH, BEHWZ].

We briefly outline the argument from [Hu1, Section 9]: (1) There is a bound on the ω-area for all elements of $\mathcal{M}_{J_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$. This was done above. (2) Given a sequence of holomorphic curves $u_i \in \mathcal{M}_{J_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$, $i \in \mathbf{N}$, there is a subsequence which converges weakly as currents to a holomorphic building u_∞. This is due to the Gromov-Taubes compactness theorem [T3], which works in dimension four and does not require any a priori bound on the genus of u_i. The extraction of the holomorphic building is treated in some detail in [Hu1, Lemma 9.8]. Hence we may assume that the homology classes $[u_i] \in H_2(N, \boldsymbol{\gamma}, \boldsymbol{\gamma}')$ are fixed. (3) There is a bound on the genus of the curve, provided the homology classes $[u_i]$ are fixed. This follows from the adjunction inequality and will be discussed below. (4) Once there is a genus bound, apply the SFT compactness theorem of [BEHWZ].

We now explain in some detail how to obtain genus bounds from bounds on the homology classes $[u_i]$, especially since similar arguments will appear in later sections. But first let us introduce some notation.

Let (F,j) be a closed Riemann surface and $\mathbf{p}^+$ and $\mathbf{p}^-$ be disjoint finite sets of punctures of F. Then let

$$u : \dot{F} = F - \mathbf{p}^+ - \mathbf{p}^- \to \mathbf{R} \times N,$$

be a (j, J_0)-holomorphic map from $\boldsymbol{\gamma} = \prod_i \gamma_i^{m_i}$ to $\boldsymbol{\gamma}' = \prod_i \gamma_i^{n_i}$. Here the punctures of $\mathbf{p}^\pm$ are asymptotic to the $\pm$ ends of u. The positive ends of u partition m_i into $(m_{i1}, m_{i2}, \dots)$ and the negative ends of u partition n_i into $(n_{i1}, n_{i2}, \dots)$. (We ignore the partition terms that are zero.) Pick a trivialization τ of TS in a neighborhood of all the γ_i, and let $\mu_\tau(\gamma_i^k)$ be the usual Conley-Zehnder index of the k-fold cover of γ_i with respect to τ. Then we define the *total Conley-Zehnder indices* at the positive and negative ends of u as follows:

$$\mu_\tau^+(u) = \sum_i \sum_r \mu_\tau(\gamma_i^{m_{ir}}),$$

$$\mu_\tau^-(u) = \sum_i \sum_r \mu_\tau(\gamma_i^{n_{ir}}),$$

and also write $\mu_\tau(u) = \mu_\tau^+(u) - \mu_\tau^-(u)$. The *symmetric Conley-Zehnder index* of Hutchings [Hu1], so called because of its motivation from studying the "symplectomorphism" $Sym^k(h)$ of $Sym^k(S)$ induced by h,[2] is defined as:

$$\widetilde{\mu}_\tau(\boldsymbol{\gamma}) = \sum_i \sum_{r=1}^{m_i} \mu_\tau(\gamma_i^r),$$

and does not depend on the choice of u from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$. We write $\widetilde{\mu}_\tau(u) = \widetilde{\mu}_\tau(\boldsymbol{\gamma}) - \widetilde{\mu}_\tau(\boldsymbol{\gamma}')$. We also recall the *writhe*

$$w_\tau(u) = w_\tau^+(u) - w_\tau^-(u),$$

where $w_\tau^+(u)$ is the total writhe of braids $u(\dot{F}) \cap (\{s\} \times N)$, $s \gg 0$, viewed in the union of solid torus neighborhoods of γ_i and computed with respect to the framing τ; and $w_\tau^-(u)$ is defined similarly.

The key ingredient in establishing genus bounds is the relative adjunction formula from [Hu1, Equation (18)] for simple curves u with a finite number of singularities and no connector components:

(**3.4.1**) $$c_1(u^*TS, \tau) = \chi(\dot{F}) + w_\tau(u) + Q_\tau(u) - 2\delta(u),$$

where $Q_\tau(u)$ is the relative intersection pairing with respect to τ and $\delta(u)$ is a nonnegative integer which is a count of the singularities. In particular, $\delta(u) = 0$ if and only if u is an

[2] $Sym^k(h)$ is a symplectomorphism away from the (multi)-diagonal.

embedding (see [M, MW]). Together with the writhe bounds

$$(\mathbf{3.4.2})\qquad \begin{aligned} w_\tau^+(u) &\leq \widetilde{\mu}_\tau(\boldsymbol{\gamma}) - \mu_\tau^+(u),\\ w_\tau^-(u) &\geq \widetilde{\mu}_\tau(\boldsymbol{\gamma}') - \mu_\tau^-(u), \end{aligned}$$

from [Hu2, Lemma 4.20], we obtain:

$$(\mathbf{3.4.3})\qquad \chi(\dot{\mathrm{F}}) \geq c_1(u^*\mathrm{TS},\tau) + \mu_\tau(u) - \widetilde{\mu}_\tau(u) - \mathrm{Q}_\tau(u) + 2\delta(u).$$

(See [Hu1, Theorem 10.1].) Since all of the terms on the right-hand side are either homological quantities or depend on the data near $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$, we have a lower bound on $\chi(\dot{\mathrm{F}})$, which implies an upper bound on the genus of $\dot{\mathrm{F}}$.

3.5. *Transversality.* — Let $\mathcal{J}_{\mathrm{W}',\varsigma}$ be the space of almost complex structures J_ς on $\mathrm{W}' = \mathbf{R}\times\mathrm{N}$ in the class C^∞ which are adapted to $(\alpha_\varsigma,\omega)$.

Definition **3.5.1.** — *An almost complex structure* $\mathrm{J}_\varsigma \in \mathcal{J}_{\mathrm{W}',\varsigma}$ *is* j-regular *if, for all orbit sets* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\gamma}'$ *of* R_ς *which intersect* $\mathrm{S}\times\{0\}$ *at most* j *times, the moduli space* $\mathcal{M}^s_{\mathrm{J}_\varsigma}(\boldsymbol{\gamma},\boldsymbol{\gamma}')$ *is transversely cut out.*

Let $\mathcal{J}^{reg,j}_{\mathrm{W}',\varsigma} \subset \mathcal{J}_{\mathrm{W}',\varsigma}$ be the subset of j-regular J_ς. In order to simplify the notation, from now on we will write $\mathcal{J}_{\mathrm{W}'}$ for $\mathcal{J}_{\mathrm{W}',0}$ and $\mathcal{J}^{reg,j}_{\mathrm{W}'}$ for $\mathcal{J}^{reg,j}_{\mathrm{W}',0}$. The following lemma states that $\mathcal{J}^{reg,j}_{\mathrm{W}'} \subset \mathcal{J}_{\mathrm{W}'}$ is dense.

Lemma **3.5.2** *(Transversality).* — *A generic* $\mathrm{J}_0 \in \mathcal{J}_{\mathrm{W}'}$ *is* j*-regular.*

Proof. — This follows from [Hu1, Lemma 9.12(b)], which states that a generic $\mathrm{J}_0 \in \mathcal{J}_{\mathrm{W}'}$ is regular away from holomorphic curves which have a fiber $\{(s,t)\}\times\mathrm{S}$ as an irreducible component. (Observe that the fibers are holomorphic for any $\mathrm{J}_0 \in \mathcal{J}_{\mathrm{W}'}$.) In our case, the fibers are not closed and cannot occur as irreducible components of curves in $\mathcal{M}^s_{\mathrm{J}_0}(\boldsymbol{\gamma},\boldsymbol{\gamma}')$. ☐

3.6. *The equivalence of certain ECH and PFH groups.* — In this section we prove the following theorem:

Theorem **3.6.1.** — *Given* $j>0$*, there exist* $\mathrm{J}_0 \in \mathcal{J}^{reg,j}_{\mathrm{W}'}$ *and* $\varsigma_0 = \varsigma_0(j,\mathrm{J}_0) > 0$ *such that there are isomorphisms of chain complexes*

$$\mathrm{PFC}_j(\mathrm{N},\alpha_0,\omega,\mathrm{J}_0) \simeq \mathrm{ECC}_j(\mathrm{N},\alpha_\varsigma,\mathrm{J}_\varsigma),$$

$$\mathrm{PFC}^\flat_j(\mathrm{N},\alpha_0,\omega,\mathrm{J}_0) \simeq \mathrm{ECC}^\flat_j(\mathrm{N},\alpha_\varsigma,\mathrm{J}_\varsigma),$$

for all $0<\varsigma\leq\varsigma_0$*. Here* $\mathrm{J}_\varsigma \in \mathcal{J}^{reg,j}_{\mathrm{W}',\varsigma}$ *is sufficiently close to* J_0*. Similar isomorphisms hold with twisted coefficients.*

Proof. — We will prove the first equivalence, leaving the second to the reader.

Since there is a one-to-one correspondence between the generators of the chain groups $\mathrm{PFC}_j(\mathrm{N}, \alpha_0, \omega)$ and $\mathrm{ECC}_j(\mathrm{N}, \alpha_\varsigma)$, we have

$$\mathrm{PFC}_j(\mathrm{N}, \alpha_0, \omega) \simeq \mathrm{ECC}_j(\mathrm{N}, \alpha_\varsigma)$$

as $\mathbf{F}$-vector spaces, but not necessarily as chain complexes. In other words, we may view any orbit set $\boldsymbol{\gamma}$ for R_ς as an orbit set of any other $\mathrm{R}_{\varsigma'}$.

Let J_0 be an almost complex structure in $\mathcal{J}^{reg,j}_{\mathrm{W}'}$. By Lemma 3.5.2, $\mathcal{J}^{reg,j}_{\mathrm{W}'}$ is a dense subset of $\mathcal{J}_0$, and in particular is nonempty. The moduli spaces $\mathcal{M}^{nc}_{\mathrm{J}_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$ of ECH index 1 and 2 are transversely cut out since they are simple by the ECH index inequality. Now let J_ς, $\varsigma \in [0, 1]$, be a smooth family of $(\alpha_\varsigma, \omega)$-adapted almost complex structures which extend J_0.

We claim that the ECH index 1 moduli spaces $\mathcal{M}^{nc}_{\mathrm{J}_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$ are transversely cut out and diffeomorphic to $\mathcal{M}^{nc}_{\mathrm{J}_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$, when $\varsigma > 0$ is sufficiently small. Indeed, if $u_\varsigma \in \mathcal{M}^{nc}_{\mathrm{J}_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$ is sufficiently close to $\mathcal{M}^{nc}_{\mathrm{J}_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$, then the moduli space is regular at u_ς. Hence it suffices to prove that, if $\varsigma > 0$ is sufficiently small, then every $u_\varsigma \in \mathcal{M}^{nc}_{\mathrm{J}_\varsigma}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$ is sufficiently close to $\mathcal{M}^{nc}_{\mathrm{J}_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$. Indeed, this follows from the compactness argument from Section 3.4. Let $u_i \in \mathcal{M}^{nc}_{\mathrm{J}_{\varsigma_i}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$ be a sequence of ECH index 1 holomorphic curves with $\varsigma_i \to 0$. By the compactness theorem and incoming/outgoing partition considerations, u_i converges to $u \in \mathcal{M}_{\mathrm{J}_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$ with $\mathrm{I}_{\mathrm{ECH}}(u) = 1$, after possibly taking a subsequence. In particular, the limit u is not a holomorphic building with multiple levels. If u has connector components u^0 over $\boldsymbol{\gamma}_0$, then u_i must also have connector components u_i^0 over $\boldsymbol{\gamma}_0$, a contradiction. Hence $u \in \mathcal{M}^{nc}_{\mathrm{J}_0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \mathrm{Z})$, which proves the claim.

Since the chain groups are isomorphic as vector spaces and the differentials agree for sufficiently small $\varsigma > 0$, the theorem follows. □

3.7. *Remarks about Morse-Bott theory.* — There is an equivalent approach to the definition of the groups $\mathrm{PFC}_j(\mathrm{N}, \alpha_0, \omega)$ which does not use the Morse-Bott language. A similar construction is possible also for the groups $\mathrm{PFC}^\flat_j(\mathrm{N}, \alpha_0, \omega)$, but will be left to the reader.

We add a small collar $[0, \eta] \times \partial\mathrm{S}$ to S and extend the monodromy such that h restricts to $(y, \theta) \mapsto (y, \theta - y)$ on $[0, \eta] \times \partial\mathrm{S}$. (See Lemma 2.1.2). We denote by (S^+, h^+) the extensions. The mapping torus of (S^+, h^+) is a manifold N^+ such that $\mathrm{N} \subset \mathrm{N}^+$. The Hamiltonian structure (α_0, ω) can be extended in an obvious way from N to N^+, and the extension will still be denoted by (α_0, ω). The groups $\mathrm{PFC}_j(\mathrm{N}^+, \alpha_0, \omega)$ are defined because $\partial\mathrm{N}^+$ is foliated by orbits of the Hamiltonian vector field. If η is small enough, all Hamiltonian orbits in $\mathrm{N}^+ - \mathrm{N}$ intersect a fiber more than j times, and therefore $\mathrm{PFC}_j(\mathrm{N}^+, \alpha_0, \omega) \cong \mathrm{PFC}_j(\mathrm{N}, \alpha_0, \omega)$ and $\mathrm{PFC}^\flat_j(\mathrm{N}^+, \alpha_0, \omega) \cong \mathrm{PFC}^\flat_j(\mathrm{N}, \alpha_0, \omega)$.

Remark **3.7.1.** — In the above construction, η depends on j.

Given a function $f : N^+ \to \mathbf{R}$ and a real number $\lambda > 0$, we define a new stable Hamiltonian structure

$$\alpha_0' = \alpha_0, \quad \omega' = \omega + \lambda df \wedge \alpha_0.$$

Note that (α_0', ω') is stable Hamiltonian since $d\alpha_0 = 0$. If R_0 is the Hamiltonian vector field of (α_0, ω) and R_0' of (α_0', ω'), then $R_0' = R_0 + \lambda X$, where X is a vector field satisfying the equations

$$\begin{cases} \alpha_0(X) = 0, \\ \iota_X \omega = df - df(R_0)\alpha_0. \end{cases}$$

We construct a function f on N^+ as follows: We fix a Morse function on ∂S with a unique minimum and a unique maximum, pull it back to ∂N, and extend it to a nonpositive function f supported on a small neighborhood of ∂N using a bump function centered at ∂N. We choose f to be supported away from the (finitely many and nondegenerate) orbits of R_0 in *int*(N) with $\mathcal{F} \leq j$, so that those orbits will also be orbits of R_0'.

On ∂N, the vector field R_0' has a pair of nondegenerate orbits e and h, where e corresponds to the minimum of the Morse function on ∂S, and h to the maximum. We will choose λ small enough so that the perturbation creates no new orbits with $\mathcal{F} \leq j$. Since ∂N^+ is foliated by the flow of R_0', we can define the periodic Floer homology group $PFC_j(N^+, \alpha_0', \omega')$ and, as explained in [0, Section 4], there is an isomorphism $PFC_j(N^+, \alpha_0, \omega) \cong PFC_{2g}(N^+, \alpha_0', \omega')$. ([0, Section 4] is mostly concerned with contact structures, but there is no problem in applying those arguments here.) The isomorphism holds already on the level of complexes because there is no direct limit involved in its definition. This is a consequence of the topological constraint $\mathcal{F} \leq j$ on the Hamiltonian orbits, which implies that only finitely many of them enter the definition of the groups PFH_j.

4. A variation of $\widehat{HF}(M)$ adapted to open book decompositions

In this section we recall the cylindrical reformulation of Heegaard Floer homology. This reformulation was suggested by Eliashberg and worked out in detail by Lipshitz [Li]. The discussion is slightly different from that of [Li] in that we introduce an ECH-type index I_{HF} and define the Heegaard Floer groups in terms of I_{HF}. We then use the work of [HKM] to restrict the Heegaard Floer data to the page S of an open book decomposition.

4.1. *Heegaard data.* — A *pointed Heegaard diagram* is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ which consists of the following:

– a closed oriented surface Σ of genus k;

– two collections $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_k\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_k\}$ of k pairwise disjoint simple closed curves in Σ; and
– a point $z \in \Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$;

where each of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ forms a basis of $H_1(\Sigma; \mathbf{Z})$ and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ intersect transversely in Σ.

If M is a closed oriented 3-manifold, then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is a *pointed Heegaard diagram for* M if Σ decomposes M into two handlebodies, i.e., $M = H_{\boldsymbol{\alpha}} \cup_\Sigma H_{\boldsymbol{\beta}}$ and $\Sigma = \partial H_{\boldsymbol{\alpha}} = -\partial H_{\boldsymbol{\beta}}$, where the $\boldsymbol{\alpha}$-curves (resp. $\boldsymbol{\beta}$-curves) bound compression disks in the handlebody $H_{\boldsymbol{\alpha}}$ (resp. $H_{\boldsymbol{\beta}}$).

Let ω be an area form on Σ. We consider $[0,1] \times \Sigma$ with the stable Hamiltonian structure (dt, ω), where t is the $[0,1]$-coordinate. The Hamiltonian vector field is ∂_t, and the 2-plane field $\ker dt$ will be written as $T\Sigma$, at the risk of some confusion. (If Y is a topological space and $f : Y \to \Sigma$ is a continuous map, then we write $T\Sigma_Y$ or $T\Sigma$ for the pullback bundle $f^* T\Sigma$.)

Let $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be the set of k-tuples of chords $\{[0,1] \times \{y_1\}, \dots, [0,1] \times \{y_k\}\}$ in $[0,1] \times \Sigma$, where there exists a permutation $\sigma \in \mathfrak{S}_k$ for which $y_i \in \alpha_i \cap \beta_{\sigma(i)}$, $i = 1, \dots, k$. Here the chords $[0,1] \times \{y_i\}$ are orbits of the Hamiltonian vector field ∂_t which connect from $\{0\} \times \beta$ to $\{1\} \times \alpha$.

Terminology. We will often write elements of $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as $\mathbf{y} = \{y_1, \dots, y_k\}$ and refer to $\mathbf{y}$ as a *k-tuple of intersection points*. Also, if $l \leq k$, then an *l-tuple of chords/intersection points* $\mathbf{y} = \{y_1, \dots, y_l\}$ is a collection of points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ where each α_i is used at most once and each β_i is used at most once.

4.2. *Almost complex structures.* — Consider the natural projection $\pi_B : X \to B$, where $X = \mathbf{R} \times [0,1] \times \Sigma$ and $B = \mathbf{R} \times [0,1]$. We also write $\pi_{\mathbf{R}}$, $\pi_{[0,1]\times\Sigma}$ and π_Σ for the natural projections of X onto $\mathbf{R}$, $[0,1] \times \Sigma$ and Σ. Let (s,t) be the coordinates on the base $B = \mathbf{R} \times [0,1]$. We then define the symplectic form

$$\Omega_X = ds \wedge dt + \omega$$

on X. The submanifolds $L_{\boldsymbol{\alpha}} = \mathbf{R} \times \{1\} \times \boldsymbol{\alpha}$ and $L_{\boldsymbol{\beta}} = \mathbf{R} \times \{0\} \times \boldsymbol{\beta}$ are Lagrangian submanifolds of the symplectic manifold (X, Ω_X). We denote by L_{α_i} or L_{β_i} their connected components.

Definition **4.2.1** *(Ω_X-admissibility).* — *An almost complex structure* J *on* X *is* Ω_X-admissible *(or simply* admissible*) if it satisfies the following:*

(1) J *is s-invariant;*
(2) $J(\partial_s) = \partial_t$ *and* $J(T\Sigma) = T\Sigma$*;*
(3) J *is tamed by the symplectic form* Ω_X*;*
(4) *there is a point* z_i *in each component of* $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ *such that* J *is a product complex structure* $j_B \times j_\Sigma$ *in a small neighborhood of* $\mathbf{R} \times [0,1] \times \{z_i\}$ *in* X*; some* z_i *coincides with the basepoint* z.

Remark **4.2.2.** — The fibers $\{(s,t)\}\times\Sigma$ of an admissible J on X are holomorphic and the projection π_B is (J, j_B)-holomorphic, where j_B is the standard complex structure on $B=\mathbf{R}\times[0,1]$.

We write $\mathcal{J}_X$ for the space of C^∞-smooth Ω_X-admissible almost complex structures J on X.

4.3. *Holomorphic curves and moduli spaces.* — Let (F, j) be a compact Riemann surface, possibly disconnected, with two sets of punctures $\mathbf{q}^+=\{q_1^+,\dots,q_k^+\}$ and $\mathbf{q}^-=\{q_1^-,\dots,q_k^-\}$ on ∂F, such that (i) each component of F has nonempty boundary, (ii) on each boundary component there is at least one puncture from each of $\mathbf{q}^+$ and $\mathbf{q}^-$, and (iii) the punctures on $\mathbf{q}^+$ and $\mathbf{q}^-$ alternate around each boundary component. We write $\dot F = F-\mathbf{q}^+-\mathbf{q}^-$ and $\partial\dot F=\partial F-\mathbf{q}^+-\mathbf{q}^-$.

Definition **4.3.1.** — *Let* $J\in\mathcal{J}_X$. *A* degree $l\leq k$ multisection of (X, J) *is a holomorphic map*

$$u:(\dot F, j)\to(X,J)$$

which is a degree l *multisection of* $\pi_B: X\to B=\mathbf{R}\times[0,1]$ *and which* additionally satisfies the following:

1. $u(\partial\dot F)\subset L_{\boldsymbol\alpha}\cup L_{\boldsymbol\beta}$*;*
2. *for each* $i\in\{1,\dots,k\}$, $u^{-1}(L_{\alpha_i})$ *(resp.* $u^{-1}(L_{\beta_i})$*) consists of at most one component of* $\partial\dot F$*, which we call* α_i^* *(resp.* β_i^**);*
3. $\lim_{w\to q_i^-}\pi_{\mathbf{R}}\circ u(w)=-\infty$ *and* $\lim_{w\to q_i^+}\pi_{\mathbf{R}}\circ u(w)=+\infty$*;*
4. *the energy of* u *(see Definition 4.3.2 below) is finite.*

A Heegaard Floer curve *(or* HF curve*) is a degree* k *multisection of* (X, J).

By the compactness theorem of [BEHWZ] (adapted to the Lagrangian case), a holomorphic curve u satisfying (1), (2) and (4) converges to cylinders over Reeb chords as $s\to\pm\infty$. By the work of Abbas [Abb1], an HF curve u converges exponentially to cylinders over Reeb chords near the ends. Components of u may map to $\mathbf{R}\times[0,1]\times\{y_i\}$; such components will be called *trivial strips*.

Definition **4.3.2.** — *The* energy *of* u *is the quantity*

$$E(u)=\int_{\dot F}u^*\omega+\sup_{\phi\in\mathcal{C}}\int_{\dot F}u^*d(\phi(s)dt), \tag{4.3.1}$$

where $\mathcal{C}$ *is the set of nondecreasing smooth functions* $\phi:\mathbf{R}\to[0,1]$.

We now define some moduli spaces of HF curves with respect to $J \in \mathcal{J}_X$. Let $\mathbf{y} = \{y_1, \dots, y_k\}$ and $\mathbf{y}' = \{y'_1, \dots, y'_k\}$ be k-tuples of $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$. Let $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')$ be the moduli space of HF curves u which are asymptotic to $\mathbf{R} \times [0,1] \times \{y_i\}$ near q_i^+ and to $\mathbf{R} \times [0,1] \times \{y'_i\}$ near q_i^-.[3] Such a curve u is said to be an *HF curve from* $\mathbf{y}$ *to* $\mathbf{y}'$. Also let $\widehat{\mathcal{M}}_J^X(\mathbf{y}, \mathbf{y}') \subset \mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')$ be the subset consisting of curves u which additionally satisfy $\pi_\Sigma \circ u(\dot{F}) \cap \{z\} = \varnothing$.

Let $\check{X} = [-1,1] \times [0,1] \times \Sigma$ be the compactification of $X = \mathbf{R} \times [0,1] \times \Sigma$, obtained by attaching $[0,1] \times \Sigma$ at the positive and negative ends, and let $\check{L}_{\boldsymbol{\alpha}} = [-1,1] \times \{1\} \times \boldsymbol{\alpha}$ and $\check{L}_{\boldsymbol{\beta}} = [-1,1] \times \{0\} \times \boldsymbol{\beta}$ be the compactifications of $L_{\boldsymbol{\alpha}}$ and $L_{\boldsymbol{\beta}}$. We then define $Z_{\mathbf{y},\mathbf{y}'} \subset \check{X}$ as the subset

$$Z_{\mathbf{y},\mathbf{y}'} = \check{L}_{\boldsymbol{\alpha}} \cup \check{L}_{\boldsymbol{\beta}} \cup (\{1\} \times [0,1] \times \mathbf{y}) \cup (\{-1\} \times [0,1] \times \mathbf{y}'). \tag{4.3.2}$$

Similarly define

$$Z_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \check{L}_{\boldsymbol{\alpha}} \cup \check{L}_{\boldsymbol{\beta}} \cup (\{-1,1\} \times [0,1] \times (\boldsymbol{\alpha} \cap \boldsymbol{\beta})). \tag{4.3.3}$$

The exponential decay of HF curves in $\mathbf{R} \times [0,1] \times \Sigma$ implies that an HF curve $u : \dot{F} \to X$ from $\mathbf{y}$ to $\mathbf{y}'$ can be compactified to a continuous map

$$\check{u} : (\check{F}, \partial\check{F}) \to (\check{X}, Z_{\mathbf{y},\mathbf{y}'}).$$

Here $\check{F}$ is obtained from $\dot{F}$ by performing a real blow-up at its boundary punctures.

We define $H_2(X, \mathbf{y}, \mathbf{y}')$ as the set of *homology* classes of continuous maps $u : \dot{F} \to X$ which satisfy (1), (2) and (3) of Definition 4.3.1 and are positively asymptotic to $[0,1] \times \mathbf{y}$ and negatively asymptotic to $[0,1] \times \mathbf{y}'$; here two maps u_1 and u_2 are equivalent in $H_2(X, \mathbf{y}, \mathbf{y}')$ if their compactifications $\check{u}_1$ and $\check{u}_2$ are homologous in $H_2(\check{X}, Z_{\mathbf{y},\mathbf{y}'})$. To any HF curve from $\mathbf{y}$ to $\mathbf{y}'$ we can then associate a class in $H_2(X, \mathbf{y}, \mathbf{y}')$. If we consider moduli spaces of HF curves u in the homology class $A \in H_2(X, \mathbf{y}, \mathbf{y}')$, we will write $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}', A)$ or $\widehat{\mathcal{M}}_J^X(\mathbf{y}, \mathbf{y}', A)$.

Remark **4.3.3.** — The sets $H_2(X, \mathbf{y}, \mathbf{y}')$ are denoted $\pi_2(\mathbf{y}, \mathbf{y}')$ in the Heegaard Floer literature. We opted for this change of notation in order to be consistent with the notation we introduced for ECH and for the maps to the open-closed cobordisms introduced in Section 5.

4.4. *The Fredholm index.* — In this subsection and the next, we fix $J \in \mathcal{J}_X$. Now we discuss the Fredholm index of an HF curve $u : \dot{F} \to X$, which is the expected dimension of a neighborhood $\mathcal{U}$ of $u \in \mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')$, modulo reparametrizations of the domain. The Fredholm index of u will be denoted by $\operatorname{ind}(u) = \operatorname{ind}_{HF}(u)$.

[3] In [Li], W has the opposite orientation, $\mathbf{y}$ is at $-\infty$, and $\mathbf{y}'$ is at $+\infty$. The moduli spaces, however, are diffeomorphic.

4.4.1. *The Fredholm index, first version.* — We start with Lipshitz's formula [Li, Equation 5] for the Fredholm index of u:

$$\text{ind}(u) = -\chi(\text{F}) + k + \sum_{i=1}^{k} \mu(\alpha_i^*) - \sum_{i=1}^{k} \mu(\beta_i^*), \tag{4.4.1}$$

where k is the genus of Σ, $\chi(\text{F}) = \chi(\dot{\text{F}})$ is the Euler characteristic of F or $\dot{\text{F}}$, and α_i^* and β_i^* are as in Definition 4.3.1.

We now define the Maslov indices $\mu(\alpha_i^*)$ and $\mu(\beta_i^*)$ which appear in Equation (4.4.1): Choose a trivialization τ_0' of $\text{T}\Sigma \simeq \mathbf{C}$ on a neighborhood of the points $w \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ so that $\mathbf{R}$ corresponds to $\text{T}_w\boldsymbol{\beta}$ and $i\mathbf{R}$ corresponds to $\text{T}_w\boldsymbol{\alpha}$. Then let τ_0 be a trivialization of $u^*\text{T}\Sigma_{\text{X}}$ which coincides with the one already given near $\mathbf{p}$ and $\mathbf{q}$ by pulling back τ_0'. Along each component of $\partial\dot{\text{F}}$ — called α_i^* or β_i^*, depending on whether it is mapped to α_i or to β_i, and oriented in the same way as ∂F for α_i and in the opposite way for β_i — we have a loop of unoriented real lines in $\mathbf{C}$, given by the pullback of $\text{T}\alpha_i$ or $\text{T}\beta_i$. The Maslov index $\mu(\alpha_i^*)$ (resp. $\mu(\beta_i^*)$) is the degree of the loop along α_i^* (resp. β_i^*) with respect to τ_0.

4.4.2. *The Fredholm index, second version.* — For the purposes of computing indices, we replace $\text{X} = \mathbf{R} \times [0, 1] \times \Sigma$ by the compactification $\check{\text{X}} = [-1, 1] \times [0, 1] \times \Sigma$ from Section 4.3.

Recall $\text{Z}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \subset \check{\text{X}}$ given by Equation (4.3.3). We define a trivialization τ of $\text{T}\Sigma_{\check{\text{X}}}$ along $\text{Z}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \subset \check{\text{X}}$ as follows: First choose nonzero sections of $\text{T}\boldsymbol{\alpha} \subset \text{T}\Sigma|_{\boldsymbol{\alpha}}$ and $\text{T}\boldsymbol{\beta} \subset \text{T}\Sigma|_{\boldsymbol{\beta}}$. The nonzero sections induce a trivialization τ' of $\text{T}\Sigma_{[0,1]\times\Sigma}$ on $(\{0\} \times \boldsymbol{\beta}) \cup (\{1\} \times \boldsymbol{\alpha})$. We then extend τ' arbitrarily to $[0, 1] \times (\boldsymbol{\alpha} \cap \boldsymbol{\beta})$. Finally we pull τ' back to $\text{Z}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \subset \check{\text{X}}$ using the projection $\pi_{[0,1]\times\Sigma} : \check{\text{X}} \to [0, 1] \times \Sigma$ to obtain τ.

Given an HF curve $u : \dot{\text{F}} \to \text{X}$, we define its *Maslov index* $\mu_\tau(u)$ as follows: Let

$$\check{u} : (\check{\text{F}}, \partial\check{\text{F}}) \to (\check{\text{X}}, \text{Z}_{\boldsymbol{\alpha},\boldsymbol{\beta}})$$

be the compactification of u. We then construct a (not necessarily oriented) real rank one subbundle $\mathcal{L}$ of $\check{u}^*\text{T}\Sigma$ on $\partial\check{\text{F}}$. The bundle $\mathcal{L}$ is given by $\check{u}^*\text{T}\boldsymbol{\alpha}$ and $\check{u}^*\text{T}\boldsymbol{\beta}$ along $\partial\dot{\text{F}}$. We extend $\mathcal{L}$ to $\partial\check{\text{F}} - \partial\dot{\text{F}}$ by rotating in the counterclockwise direction from $\check{u}^*\text{T}\boldsymbol{\beta}$ to $\check{u}^*\text{T}\boldsymbol{\alpha}$ by the minimum amount possible. (Assuming orthogonal intersections, this is a $\frac{\pi}{2}$-rotation.) Then $\mu_\tau(u)$ is the sum of the Maslov indices of $\mathcal{L}$ with respect to the trivialization τ, where the sum is over all the connected components of $\partial\check{\text{F}}$.

Lemma **4.4.1.** — *If* $u : \dot{\text{F}} \to \text{X}$ *is an HF curve, then*

$$\mu_\tau(u) + 2c_1(u^*\text{T}\Sigma, \tau) = \sum_{i=1}^{k} \mu(\alpha_i^*) - \sum_{i=1}^{k} \mu(\beta_1^*). \tag{4.4.2}$$

Proof. — By standard Maslov index theory, we have

$$\mu_\tau(u) + 2c_1(u^*T\Sigma, \tau) = \mu_{\tau_0}(u) + 2c_1(u^*T\Sigma, \tau_0),$$

where τ_0 denotes the trivialization of $u^*T\Sigma$ from Section 4.4.1. We immediately obtain $c_1(u^*T\Sigma, \tau_0) = 0$ since τ_0 is a trivialization on all of $\dot{F}$. Hence it suffices to prove that:

$$\mu_{\tau_0}(u) = \sum_{i=1}^{k} \mu(\alpha_i^*) - \sum_{i=1}^{k} \mu(\beta_1^*). \tag{4.4.3}$$

The difference between the two sides of Equation (4.4.3) is the total amount of rotation of the real lines introduced at $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ in the definition of μ_{τ_0}: if we go from $\boldsymbol{\beta}$ to $\boldsymbol{\alpha}$ we rotate by $\frac{\pi}{2}$, while if we go from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ we rotate by $-\frac{\pi}{2}$; hence the total amount of rotation is 0. □

We can now rephrase the Fredholm index as follows:

$$\mathrm{ind}(u) = -\chi(F) + k + \mu_\tau(u) + 2c_1(u^*T\Sigma, \tau). \tag{4.4.4}$$

Remark **4.4.2.** — Since the Maslov index of $\mathcal{L}$ with respect to τ is an integer along each chord $[0, 1] \times \{y_i\}$, it makes sense to write $\mu_\tau(y_i) \in \mathbf{Z}$. If we let

$$\mu_\tau(\mathbf{y}) = \sum_{i=1}^{k} \mu_\tau(y_i),$$

then $\mu_\tau(u) = \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}')$. In particular, $\mu_\tau(u)$ only depends on $\mathbf{y}$, $\mathbf{y}'$, and the choice of τ.

4.5. *The ECH-type index.* — In this subsection we define an ECH-type index $I_{HF}(u)$ and prove an index inequality which is analogous to the ECH index inequality of [Hu1].

4.5.1. *The relative intersection form.* — Let τ be a trivialization of $T\Sigma_{\check{X}}$ along $Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as defined in Section 4.4.2. We will define the notions of a *representative* and a *τ-trivial representative* of a homology class $A \in H_2(X, \mathbf{y}, \mathbf{y}')$, where $\mathbf{y} = \{y_1, \dots, y_k\}$ and $\mathbf{y}' = \{y_1', \dots, y_k'\}$ are k-tuples in $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Definition **4.5.1.** — *An oriented immersed compact surface*

$$\check{C} \subset \check{X} = [-1, 1] \times [0, 1] \times \Sigma$$

in the homology class $A \in H_2(X, \mathbf{y}, \mathbf{y}')$ *is an* immersed representative of A *if:*

(1) $\check{C}$ *is positively transverse to the fibers* $\{(s, t)\} \times \Sigma$ *along all of* $\partial\check{C}$.

(2) $(\check{\pi}_{[0,1]\times\Sigma})|_{\check{C}}$ *is an embedding near* $\partial\check{C}\cap(\{-1,1\}\times[0,1]\times\Sigma)$*, where* $\check{\pi}_{[0,1]\times\Sigma}$ *is the projection* $\check{X}\to[0,1]\times\Sigma$.

If $\check{C}$ *is embedded in addition, then* $\check{C}$ *is a* representative *of* A.

Definition **4.5.2** *(τ-trivial representative). — A representative* $\check{C}$ *of* A *is* τ-trivial *if, for all sufficiently small* $\varepsilon>0$, $\check{C}\cap\{s=\pm(1-\varepsilon)\}$ *is the union of single-stranded braids* $\zeta_i^{\pm}$, $i=1,\dots,k$, *where* ζ_i^+ *(resp.* ζ_i^-*) lies in a tubular neighborhood of* $[0,1]\times\{y_i\}$ *(resp.* $[0,1]\times\{y_i'\}$*), is disjoint from* $[0,1]\times\{y_i\}$ *(resp.* $[0,1]\times\{y_i'\}$*), and induces a framing which agrees with* τ *along* $[0,1]\times\{y_i\}$ *(resp.* $[0,1]\times\{y_i'\}$*).*

Let A be a homology class in $H_2(X,\mathbf{y},\mathbf{y}')$. Then we define

$$n_{z_j}(A)=\langle A,[-1,1]\times[0,1]\times\{z_j\}\rangle, \tag{4.5.1}$$

where $z_j\in\Sigma-\boldsymbol{\alpha}-\boldsymbol{\beta}$ are given in Definition 4.2.1 and $\langle\cdot,\cdot\rangle$ is the signed intersection number. We say that $A\in H_2(X,\mathbf{y},\mathbf{y}')$ is *positive* if $n_{z_j}(A)\geq 0$ is nonnegative for all z_j.

Lemma **4.5.3.** — *A positive* $A\in H_2(X,\mathbf{y},\mathbf{y}')$ *admits a* τ*-trivial representative* $\check{C}$.

Proof. — Let A be a positive homology class in $H_2(X,\mathbf{y},\mathbf{y}')$. Then we can glue closures of connected components of $\Sigma-\boldsymbol{\alpha}-\boldsymbol{\beta}$ with multiplicity $n_{z_j}(A)$ as in Rasmussen [Ra, Lemma 9.3] (also see [Li, Lemma 4.1] and its correction [Li2, Lemma 4.1']) to construct a smooth map $u=(u_1,u_2)$, where $u_1:\dot{F}\to\mathbf{R}\times[0,1]$ is a branched cover with interior branch points and $u_2:\dot{F}\to\Sigma$ admits an extension $u_2:F\to\Sigma$ such that

- $u_2(q_i^+)=y_i$ and $u_2(q_i^-)=y_i'$;
- each component of $\partial\dot{F}$ is mapped to some α_i or β_i so that each α_i,β_i, $i=1,\dots,k$, is used exactly once; and
- u_2 is holomorphic on a neighborhood of ∂F and is locally given by $w\mapsto w^{\ell(q_i^\pm)/2}$ for some positive odd integer $\ell(q_i^\pm)$ near $q_i^\pm$, provided $q_i^\pm$ is not the end of a trivial strip of u_2.

We extend u to $\check{u}:\check{F}\to X$, where $\check{F}$ is the real blow-up of F given in Section 4.3. Condition (1) of Definition 4.5.1 is immediately satisfied. We can resolve all the (interior) singularities to make $\check{u}$ embedded. It is a local exercise to modify $\check{u}$ in a neighborhood of $\partial\check{F}-\partial\dot{F}$ so that $\check{u}$ becomes τ-trivial. □

Lemma **4.5.4.** — *If* $u:\dot{F}\to X$ *is an HF curve and* $\check{C}$ *is the image of the compactification* $\check{u}:\check{F}\to\check{X}$*, then the following hold:*

(1) $\pi_B\circ u:\dot{F}\to B$ *has no branch points along* ∂B.
(2) $\check{u}$ *is positively transverse to the fibers* $\{(s,t)\}\times\Sigma$ *along all of* $\partial\check{F}$ *and* $\check{\pi}_{[0,1]\times\Sigma}|_{\check{C}}$ *is an embedding near* $\partial\check{C}$.

In other words, $\check{C}$ *satisfies all the conditions of a* τ*-trivial representative for some* τ*, with the exception of the embeddedness of* $\check{C}$.

Proof. — (1) follows from the fact that $\pi_B \circ u$ is a k-fold branched cover of B. Let $\mathbf{H} = \{\mathrm{Im}(z) \geq 0\}$ be the upper half-plane and $U \subset \mathbf{H}$ be an open subset which contains 0. If f is a holomorphic map $U \to \mathbf{R} \times [0,1]$ which maps 0 to $(0,0)$ and $U \cap \partial\mathbf{H}$ to $\mathbf{R} \times \{0\}$, then it can be extended to a holomorphic map $f : U \cup \overline{U} \to \mathbf{R} \times [-1,1]$ by Schwarz reflection, where $\overline{U} = \{\overline{z} \mid z \in U\}$. If $df(0) = 0$, then f is locally a composition of $z \mapsto z^l$ for some integer $l > 1$ and a biholomorphism. This contradicts the requirement that $f(\mathbf{H})$ stay on one side of $\mathbf{R} \times \{0\}$.

(2) follows from (1), together with the asymptotics of u as $s \to \pm\infty$. □

We now define the relative intersection form $Q_\tau(A)$, which is analogous to the relative intersection form which appears in the definition of the ECH index I_{ECH}, but is easier.

Definition **4.5.5** *(Relative intersection form* $Q_\tau(A)$*). — Let* $A \in H_2(X, \mathbf{y}, \mathbf{y}')$ *be a positive homology class and let* $\check{C}$ *be a* τ*-trivial representative of* A. *Let* ψ *be a section of the normal bundle* ν *to* $\check{C}$ *such that* $\psi|_{\partial\check{C}} = J\tau$*, and let* $\check{C}'$ *be a pushoff of* $\check{C}$ *in the direction of* ψ. *Then the* relative intersection form $Q_\tau(A)$ *is given by:*

$$Q_\tau(A) = \langle \check{C}, \check{C}' \rangle$$

where $\langle \cdot, \cdot \rangle$ *denotes the algebraic count of intersection points.*

Note that, since a representative $\check{C}$ is positively transverse to the fibers $\{(s,t)\} \times \Sigma$ along all of $\partial\check{C}$, we may take the normal bundle ν to $\check{C}$ to satisfy $\nu|_{\partial\check{C}} = T\Sigma|_{\partial\check{C}}$. Also, since J is Ω_X-admissible, it takes $T\Sigma$ to itself. Hence $(\tau, J\tau)$ is a trivialization of $\nu|_{\partial\check{C}}$. Although τ and $J\tau$ are homotopic, we will often use $J\tau$ due to its appearance in the definition of $Q_\tau(A)$.

Let τ' and τ be trivializations of $T\Sigma_{\check{X}}$ along $Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ which differ only on $\{\pm 1\} \times [0,1] \times (\boldsymbol{\alpha} \cap \boldsymbol{\beta})$. Let $\deg(\tau, \tau', y_i)$ (resp. $\deg(\tau, \tau', y_i')$) be the degree of τ with respect to the trivialization τ' along $[0,1] \times \{y_i\}$ (resp. $[0,1] \times \{y_i'\}$), oriented in the ∂_t-direction. We then have the following:

Lemma **4.5.6** *(Change of trivialization). — If* $A \in H_2(X, \mathbf{y}, \mathbf{y}')$ *is positive, then*

$$Q_\tau(A) - Q_{\tau'}(A) = \sum_{i=1}^{k} \deg(\tau, \tau', y_i) - \sum_{i=1}^{k} \deg(\tau, \tau', y_i'). \quad (\mathbf{4.5.2})$$

Proof. — Let $\check{C}_{\tau'}$ be a τ'-trivial representative of A. Let $\varepsilon > 0$ be small and let

$$\check{C}_{\tau',0} = \check{C}_{\tau'} \cap ([-1+\varepsilon, 1-\varepsilon] \times [0,1] \times \Sigma).$$

We can extend $\check{C}_{\tau',0}$ to $\check{C}_\tau$ in $\check{X}$ by gluing disks D_i, D_i', $i=1,\dots,k$, corresponding to y_i, y_i', so that $\check{C}_\tau$ becomes τ-trivial.

Let ψ be a section of the normal bundle to $\check{C}_\tau$ such that $\psi|_{\partial\check{C}_\tau} = J\tau$ and $\psi|_{\partial\check{C}_{\tau',0}} = J\tau'$. (Here we are assuming that τ' has been extended to a neighborhood of the $[0,1]\times\{y_i\}$.) Then

$$Q_\tau(A) - Q_{\tau'}(A) = \sum_{i=1}^{k} \#(\psi|_{D_i})^{-1}(0) + \sum_{i=1}^{k} \#(\psi|_{D_i'})^{-1}(0),$$

where # is a signed count. A local calculation gives

$$\#(\psi|_{D_i})^{-1}(0) = \deg(\tau,\tau',y_i), \quad \#(\psi|_{D_i'})^{-1}(0) = -\deg(\tau,\tau',y_i'),$$

which proves the lemma. □

The following is immediate from the definition of $Q_\tau(A)$.

Lemma **4.5.7** *(Additivity). — If* $A_1 \in H_2(X,\mathbf{y},\mathbf{y}')$ *and* $A_2 \in H_2(X,\mathbf{y}',\mathbf{y}'')$ *are positive, then*

$$Q_\tau(A_1\#A_2) = Q_\tau(A_1) + Q_\tau(A_2),$$

where $A_1\#A_2 \in H_2(X,\mathbf{y},\mathbf{y}'')$ *is obtained from stacking two copies of* $\check{X}$.

4.5.2. *The relative adjunction formula.* — In this subsection we prove a relative adjunction formula for HF curves (Lemma 4.5.9).

If $\check{C}\subset\check{X}$ is a properly immersed surface with only transverse double points in its interior, we denote by $\delta(\check{C})$ the signed count of the double points of $\check{C}$.

Lemma **4.5.8.** — *If* $\check{C}$ *is an immersed representative of* $A\in H_2(X,\mathbf{y},\mathbf{y}')$ *with only transverse double points in its interior, then*

$$c_1(\nu, J\tau) = Q_\tau(A) - 2\delta(\check{C}).$$

Proof. — Let us first assume that $\check{C}$ is embedded. If $\check{C}$ is τ-trivial, then there is a section ψ of the normal bundle to $\check{C}$ such that $\psi|_{\partial\check{C}} = J\tau$, and

$$Q_\tau(A) = \#\psi^{-1}(0) = c_1(\nu, J\tau).$$

Next let τ' and τ be trivializations of $T\Sigma_{\check{X}}$ along $Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, which differ only on $\{\pm1\}\times[0,1]\times(\boldsymbol{\alpha}\cap\boldsymbol{\beta})$. By Equation (4.5.2), together with an analogous equation for $c_1(\nu,J\tau) - c_1(\nu,J\tau')$, we have

$$Q_\tau(A) - Q_{\tau'}(A) = c_1(\nu,J\tau) - c_1(\nu,J\tau'),$$

which proves the lemma for embedded $\check{C}$.

Suppose now that $\check{C}$ has a single positive transverse double point d. (The case of a negative double point is similar.) We resolve the intersection in the following way: Let $\mathcal{B} \subset \check{X}$ be a small ball centered at d. Then $\check{C} \cap \partial\mathcal{B}$ is a Hopf link, and $\check{C} \cap \mathcal{B}$ is the union of two slice disks for the components which intersect at d. We can construct a new surface $\check{C}_{sm}$ by replacing the two disks with a Hopf band connecting the two components of the Hopf link. By definition, we have $Q_\tau(A) = \langle \check{C}_{sm}, \check{C}'_{sm} \rangle$. On the other hand, if ν_{sm} is the normal bundle to $\check{C}_{sm}$, then

$$c_1(\nu_{sm}, J\tau) = c_1(\nu, J\tau) + 2.$$

This can be seen easily by embedding $\mathcal{B}$ into $S^2 \times S^2$ and using the properties of the intersection product for closed 4-manifolds.

In general,

$$c_1(\nu, J\tau) + 2\delta(\check{C}) = c_1(\nu_{sm}, J\tau) = \langle \check{C}_{sm}, \check{C}'_{sm} \rangle = Q_\tau(A),$$

and the lemma follows. □

Let $u : \dot{F} \to X$ be an HF curve. Then u has no singular points on $\partial\dot{F}$ because $\pi_B \circ u$ has no branch point on the boundary. By [M, MW] (see also [MS, Appendix E]), there exists a modification $v : \dot{F} \to X$ of $u : \dot{F} \to X$ in a neighborhood of its finitely many singular points so that v is an immersion with only positive transverse double points. We define $\delta(u)$ the number of positive double points of v, which depends only on u and not on the modification v used to define it. Then $\delta(u) \geq 0$ and $\delta(u) = 0$ if and only if u is an embedding. We can now state and prove the relative adjunction formula:

Lemma **4.5.9** *(Relative adjunction formula). — If $u : \dot{F} \to X$ is an HF curve in the homology class $A \in H_2(X, \mathbf{y}, \mathbf{y}')$, then*

$$(\mathbf{4.5.3}) \qquad \begin{aligned} c_1(\check{u}^* T\Sigma, \tau) &= c_1(T\check{F}, \partial_t) + Q_\tau(A) - 2\delta(u) \\ &= \chi(F) - k + Q_\tau(A) - 2\delta(u). \end{aligned}$$

Here ∂_t is the pullback to $\partial\check{F}$ of the trivialization ∂_t on $[-1, 1] \times [0, 1]$.

Proof. — Let $v : \dot{F} \to X$ be the immersion with transverse double points obtained by modifying $u : \dot{F} \to X$. Since the modification is purely local and is away from $\partial\dot{F}$, it follows that $\check{u}$ and $\check{v}$ belong to the same homology class in $H_2(X, \mathbf{y}, \mathbf{y}')$ and $c_1(\check{u}^* T\Sigma, \tau) = c_1(\check{v}^* T\Sigma, \tau)$. Hence Equation (4.5.3) for u is equivalent to Equation (4.5.3) for v, and we may assume without loss of generality that u is immersed with positive transverse double points.

The vector field ∂_t is a global trivialization of the complex line bundle $T([-1, 1] \times [0, 1])$ over $\check{X} = [-1, 1] \times [0, 1] \times \Sigma$. Hence

$$c_1(\check{u}^* T\check{X}, (\tau, \partial_t)) = c_1(\check{u}^* T\Sigma, \tau).$$

On the other hand,

$$c_1(\check{u}^*T\check{X}, (\tau, \partial_t)) = c_1(T\check{F}, \partial_t) + c_1(\nu, J\tau).$$

The first line of the relative adjunction formula now follows from Lemma 4.5.8. The equivalence of the first and second lines is a consequence of Claim 4.5.10, proved below. □

Claim **4.5.10.** — $c_1(T\check{F}, \partial_t) = \chi(F) - k$.

Proof. — Let $\tau_{\partial F}$ be the trivialization of $TF|_{\partial F}$ which is given by an oriented nonsingular vector field tangent to ∂F. We then have

$$c_1(T\check{F}, \partial_t) = \chi(F) + \deg(\partial_t, \tau_{\partial F}),$$

where $\deg(\partial_t, \tau_{\partial F})$ is the degree of ∂_t with respect to $\tau_{\partial F}$. By an easy direct calculation we obtain $\deg(\partial_t, \tau_{\partial F}) = -k$. □

4.5.3. *The index* I_{HF} *and the index inequality.* — We are now ready to define the ECH-type index I_{HF} and prove the ECH-type index inequality (Theorem 4.5.13).

Definition **4.5.11** *(ECH-type index).* — *Let* $A \in H_2(X, \mathbf{y}, \mathbf{y}')$ *be a positive homology class. Then the* ECH-type index I_{HF} *of* A *is given as follows:*

$$I_{HF}(A) = c_1(T\Sigma|_A, \tau) + Q_\tau(A) + \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}'). \tag{4.5.4}$$

We observe that $I_{HF}(A)$ does not depend on the choice of τ: Suppose τ and τ' differ only at y_i and $\deg(\tau, \tau', y_i) = 1$. Then we compute that (i) $Q_\tau(A) = Q_{\tau'}(A) + 1$ by Lemma 4.5.6, (ii) $\mu_\tau(\mathbf{y}) = \mu_{\tau'}(\mathbf{y}) - 2$ and (iii) $c_1(T\Sigma|_A, \tau) = c_1(T\Sigma|_A, \tau') + 1$. Next we reverse the direction of τ (viewed as a vector field) along a single component $[-1, 1] \times \{1\} \times \alpha_i$ of $[-1, 1] \times \{1\} \times \boldsymbol{\alpha}$, keeping in mind that τ is tangent to $\{s\} \times \{1\} \times \boldsymbol{\alpha}$ and $\{s\} \times \{0\} \times \boldsymbol{\beta}$ for all $s \in [-1, 1]$. More precisely, let

- $\tau' = -\tau$ on $[-1, 1] \times \{1\} \times \alpha_i$,
- on each $\{-1, 1\} \times [0, 1] \times y_j \subset Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that $y_j \in \alpha_i$, τ' is obtained by adding an extra π-rotation to τ near $\{-1, 1\} \times \{1\} \times y_j$, and
- $\tau' = \tau$ elsewhere.

Then it is easy to verify that (i) $Q_\tau(A) = Q_{\tau'}(A)$, (ii) $\mu_\tau(\mathbf{y}) = \mu_{\tau'}(\mathbf{y})$, and (iii) $c_1(T\Sigma|_A, \tau) = c_1(T\Sigma|_A, \tau')$. Any two trivializations τ and τ' can be related to each other by a sequence of differences of the above two types.

The index I_{HF} satisfies the following additivity property under the stacking operation $\# : H_2(X, \mathbf{y}, \mathbf{y}') \times H_2(X, \mathbf{y}', \mathbf{y}'') \to H_2(X, \mathbf{y}, \mathbf{y}'')$.

Lemma **4.5.12** *(Additivity of* I_{HF}*). — If* $A_1 \in H_2(X, \mathbf{y}, \mathbf{y}')$ *and* $A_2 \in H_2(X, \mathbf{y}', \mathbf{y}'')$ *are positive, then*

$$I_{HF}(A_1 \# A_2) = I_{HF}(A_1) + I_{HF}(A_2).$$

Proof. — Each of the terms $c_1(T\Sigma|_A, \tau)$, $Q_\tau(A)$, and $\mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}')$ in the definition of $I_{HF}(A)$ is additive under stacking; see Lemma 4.5.7 for the additivity of $Q_\tau(A)$. □

The following index inequality is analogous to (but much easier than) the ECH index inequality, due to Hutchings [Hu1, Theorem 1.7]. We remark that u is required to be simply-covered in the statement of the usual ECH index inequality. This is automatically satisfied for HF-curves.

Theorem **4.5.13** *(ECH-type index inequality). — Let* $u : \dot{F} \to X$ *be an HF curve in the class* $A \in H_2(X, \mathbf{y}, \mathbf{y}')$. *Then*

$$\text{ind}(u) + 2\delta(u) = I_{HF}(A), \tag{4.5.5}$$

where $\delta(u) \geq 0$ *is an integer count of the singularities. Hence*

$$\text{ind}(u) \leq I_{HF}(A), \tag{4.5.6}$$

with equality if and only if u is an embedding.

Note that u has no boundary singularities because $\pi_B \circ u$ has no branch points at the boundary.

Proof. — We calculate:

$$\begin{aligned} \text{ind}(u) &= -\chi(F) + k + \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}') + 2c_1(\check{u}^* T\Sigma, \tau) \\ &= c_1(\check{u}^* T\Sigma, \tau) + Q_\tau(A) + \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}') - 2\delta(u). \end{aligned}$$

The first line is Equation (4.4.4). The equivalence of the first and second lines follows from the relative adjunction formula. Hence

$$\text{ind}(u) + 2\delta(u) = I_{HF}(A).$$

The index inequality (4.5.6) follows immediately. □

4.6. *Compactness.* — We now discuss the requisite compactness issues. The key notion is that of *weak admissibility* from [OSz1, Definition 4.10], which is analogous to the vanishing of the flux in the PFH situation (see Section 3.3). A pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is *weakly admissible* if, for every Spin^c-structure $\mathfrak{s}$ and nontrivial periodic domain $\mathcal{Q}$ which satisfies $\langle c_1(\mathfrak{s}), \mathcal{Q}\rangle = 0$, there exist j_1 and j_2 for which $n_{z_{j_1}}(\mathcal{Q}) > 0$ and $n_{z_{j_2}}(\mathcal{Q}) < 0$. (The points z_{j_1} and z_{j_2} are the points from Definition 4.2.1(4).) Equivalently, by [OSz1, Lemma 4.12], $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is weakly admissible if and only if there is an area form ω on Σ such that each periodic domain has total signed ω-area zero.

Let $N > 0$ be a fixed constant. We consider the subset $H_2^N(X, \mathbf{y}, \mathbf{y}')$ consisting of homology classes of $H_2(X, \mathbf{y}, \mathbf{y}')$ which intersect $[-1, 1] \times [0, 1] \times \{z\}$ at most N times. (This is sufficient for $\widehat{CF}$ and CF^+, defined in Section 4.8.) The difference of two homology classes $A_1, A_2 \in H_2^0(X, \mathbf{y}, \mathbf{y}')$ is a periodic domain $\mathcal{Q}$ and has zero ω-area. This implies that the ω-areas of any two $A_1, A_2 \in H_2^N(X, \mathbf{y}, \mathbf{y}')$ differ by $i \cdot \omega(\Sigma)$ where $0 \le i \le N$. Let $\phi_1, \dots, \phi_r$ be the connected components of $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ with $z_j \in \phi_j$. If A is represented by a holomorphic curve, then the projection of A to Σ can be written as $\sum_j n_{z_j}(A)\phi_j$ with $n_{z_j}(A) \ge 0$. Since each ϕ_i has finite area, there must only be a finite number of homology classes $A \in H_2^N(X, \mathbf{y}, \mathbf{y}')$ for which the moduli space $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}', A)$ is nonempty.

We now prove the existence of a compactification of $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}', A)/\mathbf{R}$. It suffices to show that if $u : \dot{F} \to X$ is an element of $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}', A)$, then the genus of $\dot{F}$ is bounded as long as A is fixed. This will be carried out in Lemma 4.6.1. Once we have a genus bound, the SFT compactness theorem from [BEHWZ] can be applied to give a compactification of $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}', A)/\mathbf{R}$.

Lemma **4.6.1.** — *There is an upper bound on the genus of a holomorphic curve* $u : \dot{F} \to X$ *in a fixed homology class* $A \in H_2(X, \mathbf{y}, \mathbf{y}')$.

Proof. — The proof is analogous to the proof in the PFH case. In view of the relative adjunction formula (Lemma 4.5.9) and the nonnegativity of $\delta(u)$, we have

$$\chi(\dot{F}) \ge c_1(\check{u}^* T\Sigma, \tau) + k - Q_\tau(A). \qquad (\mathbf{4.6.1})$$

The lemma follows by observing that the terms on the right-hand side depend only on the homology class A. □

4.7. *Transversality.*

Definition **4.7.1.** — *An almost complex structure* $J \in \mathcal{J}_X$ *is* regular *if the moduli spaces* $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')$ *are transversely cut out for all* $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\alpha,\beta}$.

Note that if u is an HF curve, then it does not have any closed irreducible components by definition. In particular, u cannot have any fibers $\{(s, t)\} \times \Sigma$ as irreducible components.

We write $\mathcal{J}_X^{reg} \subset \mathcal{J}_X$ for the subset of regular almost complex structures J. For $J \in \mathcal{J}_X^{reg}$, the dimension of $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')$ near u is equal to the *Fredholm index* $\operatorname{ind}(u)$. The moduli space $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')$ carries a natural $\mathbf{R}$-action given by translations in the s-direction, and the quotient $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')/\mathbf{R}$ is a manifold.

Lemma **4.7.2.** — *A generic* $J \in \mathcal{J}_X$ *is regular.*

Proof. — This follows from [Li, Proposition 3.8] (see also [Li2, Section 2]), by noting that an HF curve u does not have any fibers as irreducible components. Lemma 4.7.2 can also be proved in the same way as in [Hu1, Lemma 9.12(b)]. Note that the transversality theory is relatively straightforward because HF curves are never multiply-covered, i.e., all the moduli spaces $\mathcal{M}_J^X(\mathbf{y}, \mathbf{y}')$ consist of simple curves. □

We will use the notation $\mathcal{M}_J^{X,I=r}(\mathbf{y}, \mathbf{y}')$ to denote the moduli space of HF curves from $\mathbf{y}$ to $\mathbf{y}'$ with ECH index $I_{HF} = r$.

Corollary **4.7.3** *(Corollary of Theorem 4.5.13).* — *If* $I_{HF}(A) = 0, 1$ *and* $J \in \mathcal{J}_X^{reg}$, *then every HF curve u in the class* A *satisfies* $\operatorname{ind}(u) = I_{HF}(A)$ *and is therefore embedded.*

Proof. — This follows from Equation (4.5.5) by observing that the term $2\delta(u)$ is even and nonnegative and that $\operatorname{ind}(u) \geq 0$ since J is regular and u is not multiply-covered. □

4.8. *Definition of the Heegaard Floer homology groups.* — Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a weakly admissible Heegaard diagram and let $J \in \mathcal{J}_X^{reg}$. We define the Heegaard Floer chain complexes

$$(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J), \widehat{\partial}) \quad \text{and} \quad (CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J), \partial^+),$$

whose corresponding homology groups are

$$\widehat{HF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J) \quad \text{and} \quad HF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J).$$

The hat group $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J)$ is the $\mathbf{F}$-vector space generated by $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and the plus group $CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J)$ is the $\mathbf{F}$-vector space generated by $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \times \mathbf{Z}^{\geq 0}$. Elements of $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ will be written as $\mathbf{y}$ and elements of $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \times \mathbf{Z}^{\geq 0}$ will be written as $[\mathbf{y}, i]$.

We now define the differentials $\widehat{\partial}$ and ∂^+. The differential $\widehat{\partial}$ is given by

$$\widehat{\partial}\mathbf{y} = \sum_{\mathbf{y}' \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \langle \widehat{\partial}\mathbf{y}, \mathbf{y}' \rangle \cdot \mathbf{y}',$$

where $\langle \widehat{\partial}\mathbf{y}, \mathbf{y}' \rangle$ is the count of $\widehat{\mathcal{M}}_J^{X,I=1}(\mathbf{y}, \mathbf{y}')/\mathbf{R}$. The differential ∂^+ is given by

$$\partial^+([\mathbf{y}, i]) = \sum_{[\mathbf{y}', j] \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \times \mathbf{Z}^{\geq 0}} \langle \partial^+([\mathbf{y}, i]), [\mathbf{y}', j] \rangle \cdot [\mathbf{y}', j],$$

where $\langle \partial^+([\mathbf{y}, i]), [\mathbf{y}', j] \rangle$ is the count of $\mathcal{M}_J^{X,I=1}(\mathbf{y}, \mathbf{y}')/\mathbf{R}$ whose representatives have intersection number $i-j$ with $\mathbf{R} \times [0,1] \times \{z\}$. By Theorem 4.5.13, the count of $I_{HF}(u) = 1$

curves is equivalent to the count of embedded $\mathrm{ind}(u) = 1$ curves. Hence our definition is the same as that of Lipshitz.

The differentials $\widehat{\partial}$ and ∂^+ indeed satisfy $\widehat{\partial}^2 = 0$ and $(\partial^+)^2 = 0$ by [Li]. A tricky issue which arises for ∂^+ (but not for $\widehat{\partial}$) is that an element u of the boundary of $\mathcal{M}_J^{X,I=2}(\mathbf{y}, \mathbf{y}')/\mathbf{R}$ might a priori have a fiber $\{(s,t)\} \times \Sigma$ as an irreducible component. (In that case, u consists of one copy of Σ, together with k trivial strips.) This possibility is eliminated in [Li, Lemma 8.2].

Both $\widehat{HF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J)$ and $HF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J)$ are independent of the choices and can be written as $\widehat{HF}(M)$ and $HF^+(M)$. In this paper, we are interested in $\widehat{HF}(-M)$, where $-M$ is the manifold M with the opposite orientation. The group $\widehat{HF}(-M)$ is the homology of the chain complex $\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, J)$, i.e., the complex for which the roles of the $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-curves are exchanged.

4.9. *Restricting the complex to a page.* — In this subsection we describe a pointed Heegaard diagram for $-M$ which is adapted to an open book and which has the property that $\widehat{HF}(-M)$ can be computed from a single page. The Heegaard diagram constructed here is a slight modification of the Heegaard diagram constructed by Honda, Kazez and Matić in [HKM].

Let S be a bordered surface of genus g and connected boundary, and let (S, h) be an open book decomposition of M with binding K. Then there is a homeomorphism

$$((S \times [0,1])/\sim, (\partial S \times [0,1])/\sim) \simeq (M, K),$$

where $(x, 1) \sim (h(x), 0)$ for $x \in S$ and $(y, t) \sim (y, t')$ for $y \in \partial S$ and $t, t' \in [0,1]$.

4.9.1. *A Heegaard diagram compatible with* (S, h). — We define a pointed Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ for $-M$ which is compatible with (S, h); this means that $L_{\boldsymbol{\beta}} = \mathbf{R} \times \{1\} \times \boldsymbol{\beta}$ and $L_{\boldsymbol{\alpha}} = \mathbf{R} \times \{0\} \times \boldsymbol{\alpha}$. Recall from above that switching the roles of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ has the effect of reversing the orientation of the ambient manifold.

The open book decomposition (S, h) gives a natural Heegaard decomposition of M into two handlebodies $H_1 = (S \times [0, \frac{1}{2}])/\sim$ and $H_2 = (S \times [\frac{1}{2}, 1])/\sim$. The Heegaard surface Σ is $S_{1/2} \cup -S_0$ and has genus $2g$. Here we abbreviate $S \times \{t\}$ by S_t.

A *basis* of S is a collection of properly embedded pairwise disjoint arcs $\mathbf{a} = \{a_1, \dots, a_{2g}\}$ of S such that $S - \mathbf{a}$ is a connected $8g$-gon. Given a basis $\mathbf{a}$ of S, there is a natural collection of compression curves $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ for H_1. We write $\alpha_i = a_i^\dagger \cup a_i$, where the presence of † indicates a copy of an arc in $S_{1/2}$ and the absence indicates a copy of the arc in S_0. Recall the monodromy h maps $(y, \theta) \mapsto (y, \theta - y)$ near ∂S. We then construct a collection of compression curves $\beta_i = b_i^\dagger \cup h(a_i)$ for H_2, where b_i is the simplest arc (i.e., with the fewest number of intersections with the a_j) in $S_{1/2}$ which is parallel to a_i and extends $h(a_i)$ to smooth curve in Σ. See Figure 1.

The arcs a_i and $h(a_i)$ intersect at their endpoints x_i and x_i' by the definition of h near ∂S, and the arcs $a_i^\dagger$ and $b_i^\dagger$ intersect at a unique point x_i'' in $int(S_{1/2})$. This means that all the intersection points of $\alpha_i \cap \beta_j$ lie in S_0, except for one intersection point x_i'' of $\alpha_i \cap \beta_i$ for each i. We then place the basepoint z on the binding, away from all the intersection points x_i, x_i'. The regions of $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ which nontrivially intersect $S_{1/2}$ are the following:

– the "forbidden region" containing the basepoint z;
– for each $i = 1, \dots, 2g$, a bigon D_i from x_i'' to x_i and a bigon D_i' from x_i'' to x_i'.

By the placement of the basepoint z, it is clear that any periodic domain must have terms of the form $k(D_i - D_i')$, where k is an integer. This implies the weak admissibility of the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$.

Remark **4.9.1.** — The point x_i or x_i' (either one) is a component of the contact class $c(\xi_{(S,h)}) \in \widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$, where $\xi_{(S,h)}$ is the contact structure which corresponds to (S, h).

4.9.2. *Holomorphic curves in the region* $\mathbf{R} \times [0,1] \times S_{1/2}$. — Let $J \in \mathcal{J}_X$ with the additional property:

(&) J is a product complex structure on $\mathbf{R} \times [0,1] \times S_{1/2}$.

All the holomorphic curves and moduli spaces in this subsection are for the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$.

Claim **4.9.2.** — *Let $u \in \widehat{\mathcal{M}}_J^X(\mathbf{y}, \mathbf{y}')$ for some $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}}$. Then the following hold:*

(1) *If u is not asymptotic to any x_i'', then its image is contained in* $\mathbf{R} \times [0,1] \times S_0$.
(2) *If u is asymptotic to some x_i'', then u has x_i'' at the positive end and a component of u is either (i) a trivial strip over x_i'' or (ii) a "thin strip" from x_i'' to x_i or x_i', whose projection to Σ is* D_i *or* D_i'.
(3) *If u is asymptotic to x_i or x_i' at the positive end, then a component of u is a trivial strip over x_i or x_i'.*

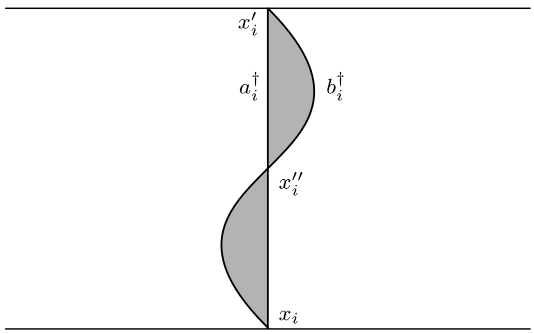

FIG. 1. — A portion of $S_{1/2}$ whose normal orientation points out of the page. The shaded regions are the disks D_i and D_i'

The only nontrivial components of u which intersect $\mathbf{R}\times[0,1]\times S_{1/2}$ are the "thin strips" in (2) and are easily seen to be regular since their projections to $\mathbf{R}\times[0,1]$ and $S_{1/2}$ are regular; see for example [Se2, Section (13a)] for the automatic transversality of holomorphic curves with boundary punctures in dimension 2. Hence a generic J which satisfies (&) is in $\mathcal{J}_X^{reg}$.

4.9.3. *The variant* $\widehat{CF}(S,\mathbf{a},h(\mathbf{a}))$. — Let $J\in\mathcal{J}_X^{reg}$ which satisfies (&). We now define the chain complex $\widehat{CF}(S,\mathbf{a},h(\mathbf{a}),J)$, which can be defined on a page of the open book (S,h) and whose homology is isomorphic to $\widehat{HF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha},z,J)$. The almost complex structure J will usually be suppressed from the notation.

Let $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ be the set of $2g$-tuples of intersection points of $\mathbf{a}$ and $h(\mathbf{a})$; equivalently, $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}=\{\mathbf{y}\in\mathcal{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}}\mid \mathbf{y}\subset S_0\}$. Then we define $(\widehat{CF'}(S,\mathbf{a},h(\mathbf{a})),\partial')$ as the subcomplex of $(\widehat{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha},z),\partial)$ generated by $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$. The differential ∂ restricts to ∂' by Claim 4.9.2(1).

Next define an equivalence relation $\sim$ on $\widehat{CF'}(S,\mathbf{a},h(\mathbf{a}))$ as follows: if we write $\mathbf{y}=\mathbf{y}_0\cup\mathbf{y}_1$, where $\mathbf{y}_0$ consists of chords of type x_i,x_i', $i=1,\dots,2g$, and $\mathbf{y}_1$ does not contain any x_i,x_i', $i=1,\dots,2g$, then $\mathbf{y}\sim\mathbf{y}'$ if and only if $\mathbf{y}=\mathbf{y}_0\cup\mathbf{y}_1$, $\mathbf{y}'=\mathbf{y}_0'\cup\mathbf{y}_1'$ and $\mathbf{y}_1=\mathbf{y}_1'$. We then take the quotient complex

$$\widehat{CF}(S,\mathbf{a},h(\mathbf{a}))=\widehat{CF'}(S,\mathbf{a},h(\mathbf{a}))/\sim,$$

with the differential $\widehat{\partial}$ induced from ∂'. The differential ∂' descends to the quotient $\widehat{\partial}$ by Claim 4.9.2(3).

Remark **4.9.3.** — Since Σ and $S=S_0$ have opposite orientations, the order $(\boldsymbol{\beta},\boldsymbol{\alpha})$ is switched to $(\mathbf{a},h(\mathbf{a}))$.

The following theorem allows us to restrict from the Heegaard surface Σ to the page S:

Theorem **4.9.4.** — $H_*(\widehat{CF}(S,\mathbf{a},h(\mathbf{a})),\widehat{\partial})\simeq\widehat{HF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha},z)$.

Proof. — Let us write $\widehat{CF}$ for $\widehat{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha},z)$. Also let $\widehat{CF}_k$ be the subgroup of $\widehat{CF}$ generated by $2g$-tuples of chords, exactly k of which are of the form x_i''. Using Claim 4.9.2, we can write the differential ∂ on $\widehat{CF}$ as $\partial=\partial_0+\partial_1$, where $\partial_0:\widehat{CF}_k\to\widehat{CF}_k$ counts $I_{HF}=1$ curves whose nontrivial part is contained in S_0 and $\partial_1:\widehat{CF}_k\to\widehat{CF}_{k-1}$ counts $I_{HF}=1$ curves whose nontrivial part is contained in $S_{1/2}$. In particular, ∂_1 counts HF curves which correspond to the domains D_i and D_i'. Since $\partial^2=0$, it follows that

$$\partial_0^2=\partial_1^2=\partial_0\partial_1+\partial_1\partial_0=0,$$

i.e., $\widehat{CF}$ becomes a double complex.

The ∂_1-homology of $\widehat{CF}$ is:

$$H_k(\widehat{CF}, \partial_1) = \begin{cases} \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), & \text{if } k = 0; \\ 0, & \text{if } k > 0. \end{cases}$$

This claim will be proved in Lemma 4.9.5. For the moment we assume it to finish the proof of the theorem. The double complex gives rise to a spectral sequence converging to $\widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ such that:

$$E^1 = H_*(\widehat{CF}, \partial_1) = \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$$
$$E^2 = H_*(H_*(\widehat{CF}, \partial_1), [\partial_0]) = H_*(\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), \widehat{\partial}).$$

Since E^2 is concentrated in degree $k = 0$, the spectral sequence degenerates at the second step and $\widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z) \cong E^2$. This proves the theorem. □

Before proceeding to Lemma 4.9.5, let us introduce some notation. Let $\mathcal{I}$ be a k-element subset of $\{1, \dots, 2g\}$ and let $\mathcal{I}^c$ be its complement. Then let $\mathcal{S}_{\mathcal{I}^c}$ be the set of $(2g-k)$-tuples $\mathbf{y}_1$ of chords from $h(\mathbf{a})$ to $\mathbf{a}$ such that each a_j and $h(a_j)$, $j \in \mathcal{I}^c$, is used exactly once and no x_j, x_j', x_j'' is in $\mathbf{y}_1$. In particular, a_i and $h(a_i)$ remain unoccupied for all $i \in \mathcal{I}$.

Lemma **4.9.5.** — *The homology of* $(\widehat{CF}, \partial_1)$ *is:*

$$H_k(\widehat{CF}, \partial_1) = \begin{cases} \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), & \text{if } k = 0; \\ 0, & \text{if } k > 0. \end{cases}$$

Proof. — Let $(\widehat{CF}(\mathbf{y}_1), \partial_1) \subset (\widehat{CF}, \partial_1)$ be the subcomplex generated by $2g$-tuples of chords of the form $\mathbf{y}_0 \cup \mathbf{y}_1$, where $\mathbf{y}_0$ is a k-tuple of chords consisting of one of x_j, x_j', x_j'' for each $j \in \mathcal{I}$ and $\mathbf{y}_1 \in \mathcal{S}_{\mathcal{I}^c}$. Since $(\widehat{CF}, \partial_1)$ is the direct sum of chain complexes of the form $(\widehat{CF}(\mathbf{y}_1), \partial_1)$, it suffices to treat each $(\widehat{CF}(\mathbf{y}_1), \partial_1)$ separately.

Consider the chain complexes $(C(j), d) = (C_0(j) \oplus C_1(j), d)$, where

$$C_1(j) = \mathbf{F}\{x_j''\}, \quad C_0(j) = \mathbf{F}\{x_j, x_j'\}, \quad d(x_j'') = x_j - x_j'.$$

The homology groups of those complexes are:

$$H_k(C(j), d) = \begin{cases} \langle x_j, x_j' \rangle / \langle x_j - x_j' \rangle, & \text{if } k = 0; \\ 0, & \text{if } k = 1. \end{cases} \tag{4.9.1}$$

By Claim 4.9.2(2), we have

$$(\widehat{CF}(\mathbf{y}_1), \partial_1) \cong \bigotimes_{j \in \mathcal{I}} (C(j), d).$$

By the Künneth formula, $H_*(\widehat{CF}(\mathbf{y}_1), \partial_1)$ is generated by the equivalence class $\{\mathbf{y}'_0 \cup \mathbf{y}_1\}$, where $\mathbf{y}_1$ is fixed and $\mathbf{y}'_0$ ranges over all k-tuples of chords which consist of one of x_j, x'_j for each $j \in \mathcal{I}$. The lemma then follows. □

4.10. Spin^c*-structures.* — Let $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be the set of k-tuples of intersection points of the pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and let $\mathrm{Spin}^c(M)$ be the set of Spin^c-structures on M. In [OSz1, Section 2.6], Ozsváth and Szabó defined a map

$$s_z : \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \to \mathrm{Spin}^c(M).$$

Although the precise definition of s_z will not be given here, we review an important property of s_z which is more or less equivalent to the definition. Given $\mathbf{y} = \{y_i\}_{i=1}^k, \mathbf{y}' = \{y'_i\}_{i=1}^k \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, the difference between the Spin^c-structures corresponding to $\mathbf{y}$ and $\mathbf{y}'$ is given by:

$$\epsilon(\mathbf{y}, \mathbf{y}') = \mathrm{PD}(s_z(\mathbf{y}) - s_z(\mathbf{y}')) \in H_1(M),$$

where a cycle representing $\epsilon(\mathbf{y}, \mathbf{y}')$ can be constructed on the Heegaard diagram as follows: For each $i = 1, \dots, k$, choose an arc $\alpha_i^\star$ on α_i from y_i to y'_i, where $y_i, y'_i \in \alpha_i$. Similarly, we choose arcs $\beta_i^\star$ on β_i, $i = 1, \dots, k$, which connect $\mathbf{y}'$ to $\mathbf{y}$. Then $\epsilon(\mathbf{y}, \mathbf{y}')$ is the homology class of $\cup_{i=1}^k (\alpha_i^\star \cup \beta_i^\star)$, which is a union of closed curves; see [OSz1, Definition 2.11 and Lemma 2.19]. It is easy to verify that $\epsilon(\mathbf{y}, \mathbf{y}')$ does not depend on the choice of arcs $\alpha_i^\star$ and $\beta_i^\star$ and provides a topological obstruction to the existence of HF curves connecting $\mathbf{y}$ and $\mathbf{y}'$.

The Heegaard Floer chain complex $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ therefore splits into a direct sum

$$\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M)} \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, \mathfrak{s}),$$

where the subgroup $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, \mathfrak{s})$ is generated by $\mathbf{y} \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ with $s_z(\mathbf{y}) = \mathfrak{s}$ and is a subcomplex.

We now interpret the above discussion in a way which relates more easily to the splitting of ECH in terms of homology classes of orbit sets. Consider the chain complex $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ which is generated by the set $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ of $2g$-tuples of intersection points of $\mathbf{a}$ and $h(\mathbf{a})$, i.e., we are restricting to a page S. The homology groups $H_1(M; \mathbf{Z}) \simeq H_1(N, \partial N; \mathbf{Z})$ are identified via the isomorphism ϖ, given in Lemma 2.3.1.

We then define the map

$$\mathfrak{h} : \mathcal{S}_{\mathbf{a},h(\mathbf{a})} \to H_1(M)$$

by assigning a cycle $\mathfrak{h}(\mathbf{y})$ to $\mathbf{y} = \{y_i\}_{i=1}^{2g} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ as follows: Suppose $y_i \in a_i \cap h(a_{\sigma(i)})$ for some $\sigma \in \mathfrak{S}_{2g}$. On $[0,1] \times S$, we consider the union of the following oriented arcs:

– $[0,1] \times \{y_i\}$, $i = 1, \dots, 2g$, where the orientation is given by ∂_t;

– $\{0\} \times c_i$, $i = 1, \dots, 2g$, where c_i is a subarc of $h(a_i)$ which goes from $h(y_{\sigma(i)})$ to y_i.

With the identification $(x, 1) \sim (h(x), 0)$, the arcs glue to give a cycle in N which represents $\mathfrak{h}(\mathbf{y})$.

Proposition **4.10.1.** — *Let ξ be the contact structure supported by the open book decomposition* (S, h) *of* M, *and let $\mathfrak{s}_\xi$ be the canonical* Spin^c*-structure determined by ξ. Then for any $\mathbf{y} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ we have*

$$s_z(\mathbf{y}) = \mathfrak{s}_\xi + \mathrm{PD}(\mathfrak{h}(\mathbf{y})).$$

Proof. — The equality holds for any $2g$-tuple $\mathbf{x}_0$ which represents the contact class. In fact, $s_z(\mathbf{x}_0) = \mathfrak{s}_\xi$ by the definition of the contact class and $\mathfrak{h}(\mathbf{x}_0) = 0$ since the cycle representing it is parallel to ∂N. Hence, in order to prove the proposition, it suffices to prove that

$$\mathfrak{h}(\mathbf{y}) - \mathfrak{h}(\mathbf{y}') = \epsilon(\mathbf{y}, \mathbf{y}')$$

for all $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$. One can check that $\mathfrak{h}(\mathbf{y}) - \mathfrak{h}(\mathbf{y}')$ is homologous to the union δ of the following types of arcs:

– $[0, 1] \times \{y_i\}$ with orientation ∂_t;
– $[0, 1] \times \{y'_i\}$ with orientation $-\partial_t$;
– subarcs of $\{1\} \times a_i$ connecting from $\mathbf{y}$ to $\mathbf{y}'$; and
– subarcs of $\{0\} \times h(a_i)$ connecting from $\mathbf{y}'$ to $\mathbf{y}$.

By homotoping δ to a page S, we see that $[\delta] = \epsilon(\mathbf{y}, \mathbf{y}')$ with respect to the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ given in Section 4.9.1. ☐

4.11. *Twisted coefficients in Heegaard Floer homology.* — In this subsection we review the definition of Heegaard Floer homology with twisted coefficients, originally defined in [OSz2, Section 8], and prove a twisted coefficient analog of Theorem 4.9.4. We describe the construction for $\widehat{\mathrm{HF}}$; the construction for HF^+ — which will be used in [III] — can be treated in a similar manner.

Fix a Spin^c-structure $\mathfrak{s}$ and a k-tuple of intersection points $\mathbf{y}_0$ such that $s_z(\mathbf{y}_0) = \mathfrak{s}$. A *complete set of paths for $\mathfrak{s}$ based at* $\mathbf{y}_0$ is the choice, for each k-tuple of intersection points $\mathbf{y}$ such that $s_z(\mathbf{y}) = \mathfrak{s}$, of a surface $C_{\mathbf{y}}$ which is the projection to $[0, 1] \times \Sigma$ of a surface representing an element of $H_2(X, \mathbf{y}, \mathbf{y}_0)$.[4]

A complete set of paths determines maps

$$\mathfrak{A} : H_2(X, \mathbf{y}, \mathbf{y}') \to H_2([0, 1] \times \Sigma, \{0\} \times \boldsymbol{\beta} \cup \{1\} \times \boldsymbol{\alpha}) \simeq H_2(M)$$

[4] We are identifying relative homology classes in $\check{X}$ with relative homology classes in $[0, 1] \times \Sigma$ as necessary.

for all $\mathbf{y}$ and $\mathbf{y}'$ such that $s_z(\mathbf{y}) = s_z(\mathbf{y}') = \mathfrak{s}$ by $\mathfrak{A}(A) = [C_{\mathbf{y}'} \cup A \cup -C_{\mathbf{y}}]$. This map is compatible with the action of $H_2(M)$ on $H_2(X, \mathbf{y}, \mathbf{y}')$ and with the concatenation of chains with matching ends.

We define

$$\underline{\widehat{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, \mathfrak{s}) = \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, \mathfrak{s}) \otimes_{\mathbf{F}} \mathbf{F}[H_2(M; \mathbf{Z})]$$

as an $\mathbf{F}[H_2(M; \mathbf{Z})]$-module, with differential

$$\widehat{\partial}\mathbf{y} = \sum_{s_z(\mathbf{y}')=\mathfrak{s}} \sum_{A \in H_2(X, \mathbf{y}, \mathbf{y}')} \#\left(\widehat{\mathcal{M}}^{X, I=1}(\mathbf{y}, \mathbf{y}', A)/\mathbf{R}\right) e^{\mathfrak{A}(A)} \mathbf{y}'.$$

The homology of this complex is the Heegaard Floer homology with twisted coefficients $\underline{\widehat{HF}}(M, \mathfrak{s})$.

Consider the special Heegaard diagram constructed in Section 4.9.1. For every Spin^c-structure $\mathfrak{s} \in \text{Spin}^c(M)$ we define the complex

$$\underline{\widehat{CF}}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}) = \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}) \otimes_{\mathbf{F}} \mathbf{F}[H_2(M; \mathbf{Z})]$$

with the differential induced by the differential on $\underline{\widehat{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, \mathfrak{s})$.

Theorem **4.11.1.** — $H_*(\underline{\widehat{CF}}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}))$ *is isomorphic to* $\underline{\widehat{HF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, \mathfrak{s})$ *as* $\mathbf{F}[H_2(M; \mathbf{Z})]$*-modules.*

Proof. — Fix a distinguished $2g$-tuple of generators $\mathbf{y}_0$ such that $s_z(\mathbf{y}_0) = \mathfrak{s}$. We choose a complete set of paths $C_{\mathbf{y}}$ with the following property: if $\mathbf{y} = \widetilde{\mathbf{y}} \cup \{x_i\}$, $\mathbf{y}' = \widetilde{\mathbf{y}} \cup \{x_i'\}$ and $\mathbf{y}'' = \widetilde{\mathbf{y}} \cup \{x_i''\}$, then $C_{\mathbf{y}} = C_{\mathbf{y}''} \cup D_i$ and $C_{\mathbf{y}'} = C_{\mathbf{y}''} \cup D_i'$, where D_i and D_i' are the surfaces corresponding to the thin strips connecting x_i'' to x_i and x_i', respectively; see Figure 1. With this choice of complete set of paths, we have $\mathfrak{A}(D_i) = \mathfrak{A}(D_i') = 0$ for all i, so the proof of Theorem 4.9.4 goes through unchanged. □

5. Moduli spaces of multisections

The goal of this section is to introduce the moduli spaces which will be used to define the chain maps

$$\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N, \alpha_0, \omega)$$

and

$$\Psi : PFC_{2g}(N, \alpha_0, \omega) \to \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})).$$

The definition of these chain maps can be viewed as a melding of ideas of Seidel [Se1, Se2] and Donaldson-Smith [DS].

Let $\mathbf{y} = \{y_1, \dots, y_{2g}\}$ be a generator of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$, where $y_i \in a_i \cap h(a_{\sigma(i)})$. Intuitively, $\mathbf{y}$ is mapped to an element of $PFC_{2g}(N)$ through an intermediary called a *broken closed string* $\gamma_{\mathbf{y}}$. It is a union of closed curves in $N = S \times [0, 1]/\sim$, obtained by taking the union of $y_i \times [0, 1]$, $i = 1, \dots, 2g$, and $c_i \times \{0\}$, $i = 1, \dots, 2g$, where c_i is a subarc of $h(a_i)$ which connects $h(y_{\sigma(i)})$ to y_i. Note that there is a unique homotopy class of arcs from $h(y_{\sigma(i)})$ to y_i, since $h(a_i)$ is an arc (and not a closed curve). The arcs $y_i \times [0, 1]$, $c_i \times \{0\}$, $i = 1, \dots, 2g$, glue up to give a union of closed curves since $(h(y_{\sigma(i)}), 0) \sim (y_{\sigma(i)}, 1)$.

5.1. *Symplectic cobordisms.* — We recall the stable Hamiltonian structure (α_0, ω) on N from Section 3. The mapping torus N is given by:

$$N = (S \times [0, 2])/\sim, \quad (x, 2) \sim (h(x), 0)$$

where h is an area-preserving map of S with zero flux. The projection $S \times [0, 2] \to [0, 2]$ defines a projection $\pi_{S^1} : N \to S^1$, where we identify $S^1 \cong [0, 2]/0 \sim 2$. Here we made a slight modification to the definition of N: the interval $[0, 1]$ in Section 2.1 is now replaced by $[0, 2]$.

The area form ω on S, the form dt on $[0, 2]$, and the vector field ∂_t on $S \times [0, 2]$ give well-defined objects on N which we still denote by ω, dt, and ∂_t. By Equation (3.1.1) the form α_0 is a positive multiple of dt. For simplicity we assume that $\alpha_0 = dt$. Then the stable Hamiltonian vector field is $R_0 = \partial_t$ and, by construction, the monodromy h is the first return map of R_0. In view of Lemma 2.1.2 we may assume that:

- all orbits of R_0 in *int*(N) satisfying $\mathcal{F} \leq 2g$ are nondegenerate; and
- R_0 is negative Morse-Bott along ∂N.

Remark **5.1.1.** — Indeed, the stable Hamiltonian vector field R_0 on N has the same first return map as a Reeb vector field R_τ, $\tau > 0$, by construction, and we could have taken R_τ to be Morse-Bott nondegenerate.

Let us write $W = \mathbf{R} \times [0, 1] \times S$ and $W' = \mathbf{R} \times N$. Let

$$\Omega = ds \wedge dt + \omega, \qquad \Omega' = ds \wedge dt + \omega$$

be the symplectic forms on W and W′, where s is the $\mathbf{R}$-coordinate.

In this section we introduce the symplectic cobordisms (W_+, Ω_+) and (W_-, Ω_-), as well as their "compactifications" $(\overline{W}_+, \overline{\Omega}_+)$ and $(\overline{W}_-, \overline{\Omega}_-)$. The cobordism (W_+, Ω_+) interpolates from the stable Hamiltonian structure $([0, 1] \times S, (dt, \omega))$ at the positive end to the stable Hamiltonian structure $(N, (\alpha_0, \omega))$ at the negative end, whereas the cobordism (W_-, Ω_-) goes from $(N, (\alpha_0, \omega))$ at the positive end to $([0, 1] \times S, (dt, \omega))$ at the negative end.

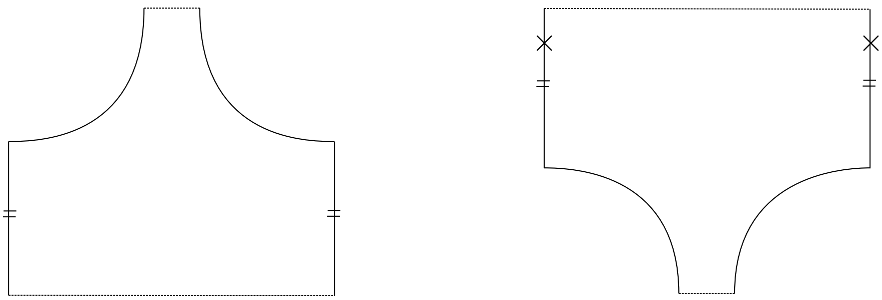

FIG. 2. — The bases B_+ and B_-. The sides are identified. The dotted lines on the top and bottom indicate cylindrical ends. Both B_+ and B_- are biholomorphic to a disk with an interior puncture and a boundary puncture. Here the location of $\overline{\mathfrak{m}}^b$ on B_- is indicated by $\times$

5.1.1. *The symplectic cobordisms* (W_+, Ω_+) *and* (W_-, Ω_-). — Consider the infinite cylinder $\mathbf{R} \times S^1 \simeq \mathbf{R} \times (\mathbf{R}/2\mathbf{Z})$ with coordinates (s, t). Let $\pi_{S^1} : N \to S^1$ be the fibration $(x, t) \mapsto t$ and let

$$\pi_{B'} : \mathbf{R} \times N = W' \to B' = \mathbf{R} \times S^1$$

be the extension $(s, x, t) \mapsto (s, \pi_{S^1}(x, t))$. Let us write $N_s = \pi_{B'}^{-1}(\{s\} \times S^1)$ for $s \in \mathbf{R}$. We define $W_+ = \pi_{B'}^{-1}(B_+)$, where $B_+ = (\mathbf{R} \times (\mathbf{R}/2\mathbf{Z})) - B_+^c$ and B_+^c is the subset $[2, \infty) \times [1, 2] \subset \mathbf{R} \times (\mathbf{R}/2\mathbf{Z})$ with the corners rounded. See the left-hand side of Figure 2. We write

$$\pi_{B_+} : W_+ \to B_+$$

for the restriction of $\pi_{B'}$. Note that the boundary of W_+ can be decomposed into two parts that meet along a codimension two corner: the *vertical boundary* $\partial_v W_+ = \pi_{B_+}^{-1}(\partial B_+)$, and the *horizontal boundary* $\partial_h W_+$, which is equal to the union of the boundaries of the fibers.

Similarly, we define $W_- = \pi_{B'}^{-1}(B_-)$, where $B_- = (\mathbf{R} \times (\mathbf{R}/2\mathbf{Z})) - B_-^c$ and B_-^c is $(-\infty, -2] \times [1, 2]$ with the corners rounded. The projection

$$\pi_{B_-} : W_- \to B_-,$$

the vertical boundary $\partial_v W_-$, and the horizontal boundary $\partial_h W_-$ are defined analogously.

The symplectic form Ω_+ (resp. Ω_-) is the restriction of

$$ds \wedge dt + \omega = ds \wedge dt + d_S \beta_t$$

to W_+ (resp. W_-). By this we mean the following: On $\mathbf{R} \times S \times [0, 2]$, we take the symplectic form $ds \wedge dt + \omega$. Then the symplectic form glues under the identification $(s, x, 2) \sim (s, h(x), 0)$.

We also write $cl(B_+)$, $cl(B_-)$ for the compactifications of B_+, B_-, obtained by adjoining the points at infinity $\mathfrak{p}_+$ corresponding to $s=+\infty$, and $\mathfrak{p}_-$ corresponding to $s=-\infty$. Therefore $cl(B_+)$ and $cl(B_-)$ are isomorphic to the closed unit disk with one marked point on the interior and one marked point on the boundary.

5.1.2. *The extended cobordisms* $(\overline{W}_+, \overline{\Omega}_+)$ *and* $(\overline{W}_-, \overline{\Omega}_-)$. — We now extend (W_+, Ω_+) to $(\overline{W}_+, \overline{\Omega}_+)$, which corresponds to capping off each fiber S by a disk; the definition of $(\overline{W}_-, \overline{\Omega}_-)$ is analogous.

We first define the capped-off surface $\overline{S}$: Let $D^2=\{\rho\leq 1\}$ be a disk with polar coordinates (ρ,ϕ). We write z_∞ for the origin $\rho=0$. Let (y,θ) be the coordinates on a neighborhood $\nu(\partial S)\simeq[-\varepsilon,\varepsilon]\times \mathbf{R}/\mathbf{Z}$ of $\partial S=\{0\}\times\mathbf{R}/\mathbf{Z}$ (inside a slight extension of S), as before. Then $\overline{S}=(S\sqcup D^2)/\sim$, where $(y,\theta)\in\nu(\partial S)$ is identified with $(\frac{1}{y+1},-2\pi\theta)\in D^2$.

For every integer $m>2g$ we define

$$\overline{h}_m:\overline{S}\xrightarrow{\sim}\overline{S}$$

as a smooth extension of $h:S\xrightarrow{\sim}S$, depending on m, such that $\overline{h}_m|_{D^2}$ is the diffeomorphism of D^2 given by

$$(\rho,\phi)\mapsto(\rho,\phi+\nu_m(\rho)),$$

where $\nu_m:[0,1]\to\mathbf{R}$ is a smooth function which satisfies the following:

- $\nu_m(\rho)=\frac{2\pi}{m}$ for $\rho\leq\frac{1}{2}$;
- $\nu_m(\rho)$ is increasing for $\frac{1}{2}<\rho<\frac{3}{4}$;
- $\nu_m(\rho)$ is decreasing and independent of m for $\frac{3}{4}<\rho<1$;
- $\nu_m(\frac{3}{4})\ll\frac{2\pi}{2g}$ and $\nu_m(1)=0$;
- $\nu_\infty:=\lim_{m\to\infty}\nu_m$ exists in the C^k-topology for $k\gg 0$.

In particular, $\nu_\infty(\rho)=0$ for $\rho\leq\frac{1}{2}$. Taking the limit $m\to\infty$ becomes important starting from Section 7.7, but until then we just need $m\gg 0$ and we simply write $\overline{h}=\overline{h}_m$ and $\nu=\nu_m$.

We then define the mapping torus

$$\overline{N}_m=(\overline{S}\times[0,2])/(x,2)\sim(\overline{h}(x),0).$$

When we do not need to specify m, we will simplify the notation by writing $\overline{N}_m=\overline{N}$. Note that, although $\overline{N}_m$ depends on m and ν_m, all the $\overline{N}_m$ are diffeomorphic. The closed manifold $\overline{N}$ is obtained from M by a 0-surgery along the binding of the open book.

Let $\overline{\omega}$ be an area form on $\overline{S}$ which extends ω and equals $\rho d\rho\wedge d\phi$ on D^2. We then extend the stable Hamiltonian structure (α_0,ω) on N to the stable Hamiltonian structure

$(\overline{\alpha}_0, \overline{\omega})$ with Hamiltonian vector field $\overline{R}_0$ on $\overline{N}$, where we still have $\overline{\alpha}_0 = dt$ and $\overline{R}_0 = \partial_t$. Moreover, the orbit

$$\delta_0 = \{z_\infty\} \times [0, 2]/\sim$$

is the only simple closed orbit of $\overline{R}_0$ on $\overline{N} - N$ which intersects $\overline{S}$ at most $2g$ times; it is called δ_0 since it lies on the level set $\rho = 0$. The 2-plane field of the stable Hamiltonian structure is $\ker \overline{\alpha}_0 = T\overline{S}$.

We now define $\overline{W}_+$ and $\overline{W}_-$. First define extensions $\overline{W} = \mathbf{R} \times [0, 1] \times \overline{S}$ of $W = \mathbf{R} \times [0, 1] \times S$ and $\overline{W'} = \mathbf{R} \times \overline{N}$ of $W' = \mathbf{R} \times N$. Let $\overline{\pi}_{S^1} : \overline{N} \to S^1$ be the fibration $(x, t) \mapsto t$ and let

$$\overline{\pi}_{B'} : \overline{W'} = \mathbf{R} \times \overline{N} \to B' = \mathbf{R} \times S^1$$

be the extension $(s, x, t) \mapsto (s, \overline{\pi}_{S^1}(x, t))$. For $* = +$ or $-$, we set $\overline{W}_* = \overline{\pi}_{B'}^{-1}(B_*)$ and write

$$\overline{\pi}_{B_*} : \overline{W}_* \to B_*$$

for the restriction of $\overline{\pi}_{B'}$ to $\overline{W}_*$. We also write

$$\overline{\pi}_B : \overline{W} \to B$$

for the restriction of $\overline{\pi}_{B'}$ to $\overline{W}$.

The symplectic forms $\overline{\Omega}_+$, $\overline{\Omega}_-$, $\overline{\Omega}$, $\overline{\Omega'}$ for $\overline{W}_+$, $\overline{W}_-$, $\overline{W}$, $\overline{W'}$ are obtained from $ds \wedge dt + \overline{\omega}$ by gluing and restricting as necessary.

Let us write

$$V := \overline{N} - int(N) = (D^2 \times [0, 2])/(x, 2) \sim (\overline{h}(x), 0).$$

We then identify $\varphi : V \xrightarrow{\sim} D^2 \times (\mathbf{R}/2\mathbf{Z})$ via $(\rho e^{i\phi}, t) \mapsto (\rho e^{i(\phi + t\nu(\rho)/2)}, t)$. Note that φ relates two coordinate systems:

(i) the "Reeb coordinates" (x, t) on $D^2 \times [0, 2]$ such that $\overline{R}_0 = \partial_t$ and $(x, 2) \sim (\overline{h}(x), 0)$, and
(ii) the "balanced coordinates" on $D^2 \times (\mathbf{R}/2\mathbf{Z})$ such that $\overline{R}_0 = \partial_t + \frac{\nu(\rho)}{2}\partial_\phi$ and $(x, 2) \sim (x, 0)$.

For $* = +$ or $-$, let

$$\overline{\pi}_{D^2} : \overline{W}_* \cap (\mathbf{R} \times V) \to D^2$$

be the projection of $\overline{W}_* \cap (\mathbf{R} \times V)$ to V, followed by the projection of V to D^2 via the identification φ, i.e., with respect to the balanced coordinates.

We also choose the marked point $\overline{\mathfrak{m}} = (\overline{\mathfrak{m}}^b, \overline{\mathfrak{m}}^f) \in \overline{W}_-$, where $\overline{\mathfrak{m}}^b = (0, \frac{3}{2}) \in B_-$ and $\overline{\mathfrak{m}}^f = z_\infty \in \overline{S}$. The marked point will play a crucial role in the definition of the chain map Ψ.

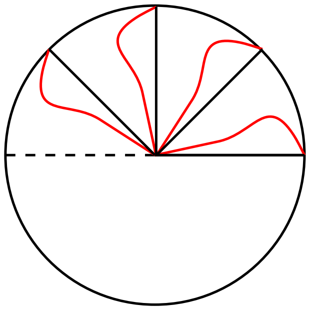

FIG. 3. — Extended arcs in D^2. The black arcs are portions of the $\overline{a}_i$ and the red ones are portions of the $\overline{h}(\overline{a}_i)$. The dashed arc is one of the $\vec{a}_{i,j} - \overline{a}_i$. (Color figure online)

5.2. *Lagrangian boundary conditions.* — Let us first write

$$L_{\mathbf{a}} = \mathbf{R} \times \{1\} \times \mathbf{a} \quad \text{and} \quad L_{h(\mathbf{a})} = \mathbf{R} \times \{0\} \times h(\mathbf{a})$$

for the Lagrangians in W that are used in the definition of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$.

5.2.1. *Lagrangian boundary conditions for* $W_\pm$. — The symplectic fibration

$$\pi_{B_+} : (W_+, \Omega_+) \to (B_+, ds \wedge dt)$$

admits a *symplectic connection*, defined as the Ω_+-orthogonal of the tangent plane to the fibers. The symplectic connection is spanned by ∂_s and ∂_t if we consider $\Omega = ds \wedge dt + \omega$ on $\mathbf{R} \times S \times [0, 2]$ before the identification $(s, x, 2) \sim (s, h(x), 0)$. (We recall that W_+ is defined as a subset of $\mathbf{R} \times N = \mathbf{R} \times (S \times [0, 2])/\sim$.)

We first place a copy of the basis $\mathbf{a}$ on the fiber $\pi_{B_+}^{-1}(3, 1)$ and take its parallel transport along ∂B_+ using the symplectic connection. The parallel transport sweeps out a Lagrangian submanifold $L_{\mathbf{a}}^+$ of (W_+, Ω_+). Let $L_{a_i}^+$ be the connected component of $L_{\mathbf{a}}^+$ given by parallel transport of a_i. Since the symplectic connection is spanned by ∂_s and ∂_t on $\mathbf{R} \times S \times [0, 2]$, over the strip $\{s \geq 3, t \in [0, 1]\}$ we have:

$$L_{\mathbf{a}}^+ \cap \{s \geq 3, t = 0\} = \{s \geq 3\} \times h(\mathbf{a}) \times \{0\},$$
$$L_{\mathbf{a}}^+ \cap \{s \geq 3, t = 1\} = \{s \geq 3\} \times \mathbf{a} \times \{1\}.$$

Similarly, the Lagrangian submanifold $L_{\mathbf{a}}^-$ on the vertical boundary of (W_-, Ω_-) is obtained by taking the parallel transport of a copy of $\mathbf{a}$ — placed on the fiber $\pi_{B_-}^{-1}(-3, 1)$ — by the symplectic connection.

5.2.2. *Extended Lagrangian boundary conditions.* — Fix $k_0 > 2g$. In what follows we assume that the basis arcs a_i, $i = 1, \dots, 2g$, depend on an integer $m \gg 0$ (i.e., sufficiently large that the conditions below make sense) and satisfy some additional conditions. Let $E \subset \partial D^2$ be the set of endpoints of $\cup_{i=1,\dots,2g} a_i$ and let $y_1(m), \dots, y_{4g}(m)$ be the points of E in counterclockwise order. We denote by $\phi(y_i(m))$ the ϕ-coordinate of the endpoint $y_i(m)$. Then we assume the following:

- $a_i = a_i(m)$ is oriented;
- $0 < \phi(y_1(m)) < \frac{2\pi}{m}$;
- for $1 \leq j_1, j_2 \leq 4g$,

$$\phi^{j_1,j_2}(m) := \phi(y_{j_1}(m)) - \phi(y_{j_2}(m))$$

  is an integer multiple of $\frac{2\pi}{m}$;
- $\frac{\phi^{j_1,j_2}(m)\cdot m}{2\pi} \geq k_0$ for all $j_1 > j_2$, i.e., the spacing between any two $\phi(y_{j_1}(m))$ and $\phi(y_{j_2}(m))$ with $j_1 > j_2$ is at least k_0 times $\frac{2\pi}{m}$;
- $\lim_{m\to+\infty} \frac{\phi^{j_1,j_2}(m)\cdot m}{2\pi} = +\infty$ and $\lim_{m\to+\infty} \phi^{j_1,j_2}(m) = 0$;
- for all quadruples $j_1 > j_2, j_3 > j_4$, $\lim_{m\to+\infty} \frac{\phi^{j_1,j_2}(m)}{\phi^{j_3,j_4}(m)} \neq 1$ unless $(j_1, j_2) = (j_3, j_4)$.

Assume without loss of generality that the initial point of a_i is x_i and the terminal point of a_i is x_i'. Let $\overline{a}_i$ be the (oriented) extension of $a_i \subset S$ to $\overline{S}$, obtained by attaching two radial rays $\overline{a}_{i,j} = \{0 \leq \rho \leq 1, \phi = \phi_{i,j}\}$, $j = 0, 1$, where $\phi_{i,j}$ is a constant. Here $\overline{a}_{i,0}$ (resp. $\overline{a}_{i,1}$) is the initial (resp. terminal) segment of $\overline{a}_i$. We also define the extension $\vec{a}_{i,j} = \{-1 < \rho \leq 1, \phi = \phi_{i,j}\}$, $j = 0, 1$, of $\overline{a}_{i,j} = \{0 \leq \rho \leq 1, \phi = \phi_{i,j}\}$. Here (ρ, ϕ) with $\rho < 0$ is a notation for $(-\rho, \phi + \pi)$.

We write $\overline{\mathbf{a}} = \{\overline{a}_1, \dots, \overline{a}_{2g}\}$. Then $L^{\pm}_{\overline{\mathbf{a}}}$ is the extension of $L^{\pm}_{\mathbf{a}}$, obtained by the parallel transport of a copy of $\overline{\mathbf{a}}$, placed at $\overline{\pi}^{-1}_{B_+}(3, 1)$ or $\overline{\pi}^{-1}_{B_-}(-3, 1)$. We similarly define $L^{\pm}_{\widehat{\mathbf{a}}}$, $L^{\pm}_{\widehat{a}_i}$, $L^{\pm}_{\overline{a}_i}$, $L^{\pm}_{\vec{a}_{i,j}}$, and $L^{\pm}_{\overline{a}_i \cup \vec{a}_{i,j}}$, where $\widehat{a}_i = \overline{a}_i - \{z_\infty\}$ and $\widehat{\mathbf{a}} = (\widehat{a}_1, \dots, \widehat{a}_{2g})$.

Definition **5.2.1.** — *A bigon (with acute angles) contained in* D^2 *and bounded by* $\overline{a}_{i,j}$ *and* $\overline{h}(\overline{a}_{i,j})$ *will be called a* thin strip. *The portion of a thin strip contained in* $D_{1/2} = \{\rho \leq \frac{1}{2}\}$ *will be called a* thin wedge. *See Figure 3.*

5.3. *Almost complex structures and moduli spaces for* W, $\overline{W}$, W', *and* $\overline{W'}$. — In this subsection we specify the almost complex structures and moduli spaces for $W = \mathbf{R} \times [0, 1] \times S$, $\overline{W} = \mathbf{R} \times [0, 1] \times \overline{S}$, $W' = \mathbf{R} \times N$, and $\overline{W'} = \mathbf{R} \times \overline{N}$.

Convention **5.3.1.** — *When we write* $\mathbf{y}$ *or* $\boldsymbol{\gamma}$ *(with possible superscripts, subscripts and other decorations), it is assumed that every intersection point in* $\mathbf{y}$ *is contained in* S *and every orbit in* $\boldsymbol{\gamma}$ *is contained in* N. *(By abuse of notation, we will write* $\mathbf{y} \subset S$ *and* $\boldsymbol{\gamma} \subset N$.*) In particular,* $\mathbf{y}$ *and* $\boldsymbol{\gamma}$ *do not contain any multiples of* z_∞ *or* δ_0.

5.3.1. *Holomorphic maps to* W *and* $\overline{W}$. — Let $W = \mathbf{R} \times [0, 1] \times S$ and $\overline{W} = \mathbf{R} \times [0, 1] \times \overline{S}$. Also let $\Omega = ds \wedge dt + \omega$ and $\overline{\Omega} = ds \wedge dt + \overline{\omega}$. Then $\mathcal{J}_W$ is defined as the set of C^∞-smooth Ω-admissible almost complex structures on W.

The analogous space $\mathcal{J}_{\overline{W}}$ for $\overline{W}$ is slightly more complicated:

Definition **5.3.2.** — *Fix* $\varepsilon, \delta > 0$ *sufficiently small and* $k \gg 0$. *Then* $\mathcal{J}_{\overline{W}}$ *is the space of* C^∞*-smooth* $\overline{\Omega}$*-admissible almost complex structures* $\overline{J}$ *on* $\overline{W}$ *which satisfy the following:*

(1) *there exists an* $\overline{\Omega}$*-admissible almost complex structure* $\overline{J}_0$ *which restricts to the standard complex structure on the subsurface* $D^2 \subset \overline{S}$ *of each fiber;*
(2) $\overline{J} = \overline{J}_0$ *on* $\{\rho \leq \varepsilon\} \cup W$ *and* $|\overline{J} - \overline{J}_0|_k < \delta$ *over* $\overline{W}$*;*
(SR) $\overline{J}$ *is a split complex structure on* $\mathbf{R} \times \{0\}$ *times a neighborhood of* $\overline{h}(\overline{\mathbf{a}})$ *and on* $\mathbf{R} \times \{1\}$ *times a neighborhood of* $\overline{\mathbf{a}}$*; moreover* $\overline{h}(\overline{\mathbf{a}})$ *and* $\overline{\mathbf{a}}$ *are real analytic with respect to the fiber complex structures on* $\overline{\pi}_B^{-1}(s,0)$ *and* $\overline{\pi}_B^{-1}(s,1)$*, respectively.*

Here $|\cdot|_k$ *is some fixed* C^k*-norm.*

Remark **5.3.3.** — Condition (SR) and the analogous Condition (SR) for $\overline{J}'$, given later, are technical conditions that allow us to apply Schwarz reflection on a neighborhood of every point of the Lagrangian boundary. It is easily satisfied and the inclusion of this condition does not affect the proofs of regularity.

We denote by J the restriction of $\overline{J}$ to W. (We will be assuming that $J \in \mathcal{J}_W$ and admits an extension to $\overline{J} \in \mathcal{J}_{\overline{W}}$.) Let $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ be the set of $2g$-tuples of intersection points of $\mathbf{a} \cap h(\mathbf{a})$. Given $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$, let $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$ be the moduli space of HF curves from $\mathbf{y}$ to $\mathbf{y}'$ with respect to J which are contained in $\mathbf{R} \times [0,1] \times S$.

We next discuss holomorphic curves in $\overline{W}$.

Definition **5.3.4.** — *Let* $\mathbf{y}, \mathbf{y}'$ *be* k*-tuples of* $\mathbf{a} \cap h(\mathbf{a})$ *and* $\overline{J} \in \mathcal{J}_{\overline{W}}$. *Then a* degree $k \leq 2g$ multisection $\overline{u}$ from $\mathbf{y}$ to $\mathbf{y}'$ in $(\overline{W}, \overline{J})$ *is a holomorphic map* $\overline{u} : \dot{F} \to \overline{W}$ *which is a degree* k *multisection of* $\overline{\pi}_B : \overline{W} \to B$*, satisfies the conditions of Definition 4.3.1 with* $L_{\boldsymbol{\alpha}}$ *and* $L_{\boldsymbol{\beta}}$ *replaced by* $L_{\widehat{\mathbf{a}}} = \mathbf{R} \times \{1\} \times \widehat{\mathbf{a}}$ *and* $L_{\overline{h}(\widehat{\mathbf{a}})} = \mathbf{R} \times \{0\} \times \overline{h}(\widehat{\mathbf{a}})$ *and* (k,l) *replaced by* $(2g,k)$*, and is asymptotic to* $\mathbf{y}$ *and* $\mathbf{y}'$ *at the positive and negative ends.*

Note that we require $\partial\dot{F}$ to be mapped to $L_{\widehat{\mathbf{a}}}$, $L_{\overline{h}(\widehat{\mathbf{a}})}$ and be disjoint from $L_{\overline{\mathbf{a}}} - L_{\widehat{\mathbf{a}}}$, $L_{\overline{h}(\overline{\mathbf{a}})} - L_{\overline{h}(\widehat{\mathbf{a}})}$.

Definition **5.3.5.** — *The* section at infinity σ_∞ *is the map* $\sigma_\infty : \mathbf{R} \times [0,1] \to \overline{W}$ *such that* $\sigma_\infty(s,t) = (s,t,z_\infty)$. *By abuse of notation we will also refer to its image as the section at* ∞.

The section at infinity is $\overline{J}$-holomorphic for every $\overline{J} \in \mathcal{J}_{\overline{W}}$. Let $z_\infty^\dagger \in D^2_{1/2} \subset \overline{S}$ be a point with ρ-coordinate less than $\frac{1}{2}$ and in the complement of all arcs $\vec{a}_{i,j}$ and $\overline{h}(\vec{a}_{i,j})$. The orbit $\mathcal{O}(z_\infty^\dagger)$ of $z_\infty^\dagger$ under the action of $\overline{h}$ consists of m points, and each thin wedge in $D^2_{1/2}$ between $\overline{a}_{i,j}$ and $\overline{h}(\overline{a}_{i,j})$ contains exactly one point of $\mathcal{O}(z_\infty^\dagger)$. Let $\sigma_\infty^\dagger = \mathbf{R} \times [0,1] \times \mathcal{O}(z_\infty^\dagger)$. (Note that this multisection does not have Lagrangian boundary conditions; it will only be used to impose topological constraints on the HF curves.)

Definition **5.3.6.** — *Given a degree k multisection $\overline{u}$ of $\overline{W}$, we define $n(\overline{u}) = \langle \overline{u}, \sigma^\dagger_\infty \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the algebraic intersection between the images.*

The intersection number $n(\overline{u})$ is a homological invariant since $z^\dagger_\infty$ was chosen so that $\mathcal{O}(z^\dagger_\infty)$ is disjoint from the Lagrangian arcs.

Lemma **5.3.7.** — *The intersection number $n(\overline{u})$ satisfies the following properties:*

(1) *$n(\overline{u}) \geq 0$ and $n(\overline{u}) = 0$ if and only if the image of $\overline{u}$ is disjoint from $\sigma^\dagger_\infty$;*
(2) *if $\langle \overline{u}, \sigma_\infty \rangle > 0$, then $n(\overline{u}) \geq m$;*
(3) *if $n(\overline{u}) = 1$, then $\overline{u}$ projects onto a thin strip; and*
(4) *$n(\overline{u})$ is independent of the choice of $z^\dagger_\infty$.*

Proof. — (1) follows from the positivity of intersections of pseudo-holomorphic maps in dimension four, (2) is a consequence of the fact that holomorphic maps are open, and (3) is a consequence of the positivity of intersections and of the fact that every thin wedge contains only one point of $\mathcal{O}(z^\dagger_\infty)$. Finally, (4) follows from the fact that different choices for $\sigma^\dagger_\infty$ are connected by a path of multisections which are disjoint from the Lagrangian boundary conditions. ☐

Definition **5.3.8.** — *Let $\mathbf{y}$ and $\mathbf{y}'$ be k-tuples of $\mathbf{a} \cap h(\mathbf{a})$. Then $\mathcal{M}_{\overline{J}}(\mathbf{y}, \mathbf{y}')$ is the moduli space of degree k multisections $\overline{u}$ of $(\overline{W}, \overline{J})$ from $\mathbf{y}$ to $\mathbf{y}'$.*

Modifiers. For any moduli space $\mathcal{M}_{\star_1}(\star_2)$ we may place modifiers $*$ as in $\mathcal{M}^*_{\star_1}(\star_2)$ to denote the subset of $\mathcal{M}_{\star_1}(\star_2)$ satisfying $*$. Typical self-explanatory modifiers are $I = i$, $n = m$, and $\deg = k$. (Note however that the degree can be inferred from $\star_2$.) $* = s$ means "somewhere injective".

The following lemma is an easy consequence of Lemma 5.3.7, in particular of points (1) and (4).

Lemma **5.3.9.** — *$\mathcal{M}^{n=0}_{\overline{J}}(\mathbf{y}, \mathbf{y}') = \mathcal{M}_J(\mathbf{y}, \mathbf{y}')$ if $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$.*

5.3.2. *Holomorphic maps to* W' *and* $\overline{W'}$. — Recall the stable Hamiltonian structures (α_0, ω) on N and $(\overline{\alpha}_0, \overline{\omega})$ on $\overline{N}$. We will denote by R_0 the Hamiltonian vector field of (α_0, ω) on N and by $\overline{R}_0$ the Hamiltonian vector field of $(\overline{\alpha}_0, \overline{\omega})$ on $\overline{N}$. Clearly R_0 is the restriction of $\overline{R}_0$ to N. Since ∂N is a Morse-Bott torus of $\overline{R}_0$, we briefly review Morse-Bott theory in the context of periodic Floer homology. A more extensive reference is [0, Section 4], where a completely analogous discussion is given for embedded contact homology.

While the closed orbits of $\overline{R}_0$ are nondegenerate on $int(N) \cup int(V) = \overline{N} - \partial N$, the orbits on ∂N form a *negative Morse-Bott torus*, i.e., ∂N is foliated by closed orbits whose linearized first return map at any point $p \in \partial N$ is given by a matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ with $a < 0$ with respect to an oriented basis (v_1, v_2) of the tangent space to the fiber of the fibration $\overline{N} \to S^1$ passing through p. In this basis v_1 is transverse to ∂N and v_2 is tangent to ∂N.

We denote the Morse-Bott orbit family associated to ∂N by $\mathcal{N}$, i.e., $\mathcal{N}$ is the quotient of ∂N by the flow. (Previously we had used $\mathcal{N}$ for the Morse-Bott family for the Reeb vector field R_α on ∂N; for the rest of the paper R_α is replaced by $\overline{R}_0$.) We choose a Morse function $\overline{g} : \mathcal{N} \to \mathbf{R}$ with two critical points: a maximum h and a minimum e. After perturbing the Hamiltonian vector field by $\overline{g}$, the orbits e and h become nondegenerate elliptic and hyperbolic, respectively. For this reason, in constructing orbit sets, e will be treated as an elliptic orbit and h as a hyperbolic orbit.

We define the following set of orbits. Let $\mathcal{P}$ be the set of simple orbits of R_0 in $int(N)$. All orbits in $\mathcal{P}$ are nondegenerate. We also define

$$\widehat{\mathcal{P}} = \mathcal{P} \cup \{e, h\} \quad \text{and} \quad \overline{\mathcal{P}} = \widehat{\mathcal{P}} \cup \{\delta_0\},$$

together with the sets $\widehat{\mathcal{O}}_k$, $\overline{\mathcal{O}}_k$ of orbit sets constructed respectively from $\widehat{\mathcal{P}}$ and $\overline{\mathcal{P}}$ which intersect $\overline{S} \times \{0\}$ exactly k times.

Let $\mathcal{J}_{W'}$ be the space of C^∞-smooth (α_0, ω)-adapted almost complex structures J' on $W' = \mathbf{R} \times N$ satisfying the following local Schwarz reflection condition:

(SR) J' is a split complex structure on $\mathbf{R} \times [1, 2]$ times a neighborhood of $\mathbf{a}$; moreover $\mathbf{a}$ is real analytic with respect to the fiber complex structure on $\overline{\pi}_B^{-1}(s, t)$ for $t \in [1, 2]$.

A $\mathcal{J}_{W'}$-holomorphic Morse-Bott building is one of the following objects:

(i) a union of negative gradient flow trajectories from h to e, or
(ii) a $\mathcal{J}_{W'}$-holomorphic map $u : \dot{F} \to W'$ from an orbit set $\tilde{\boldsymbol{\gamma}}_+ \in \mathcal{P} \cup \mathcal{N}$ to an orbit set $\tilde{\boldsymbol{\gamma}}_- \in \mathcal{P} \cup \mathcal{N}$, together with half-infinite negative gradient flow trajectories connecting h to positive ends of u at $\mathcal{N} - \{e, h\}$ and negative ends of u at $\mathcal{N} - \{e, h\}$ to e, or
(iii) a union of (i) and (ii).

See [0] for more details.

Definition **5.3.10.** — *Given $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_k$ and $J' \in \mathcal{J}_{W'}$, let $\mathcal{M}_{J'}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ be the moduli space of J'-holomorphic Morse-Bott buildings in W' from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$.*

Remark **5.3.11.** — The stable Hamiltonian structure (α_0, ω) can be perturbed to a stable Hamiltonian structure (α_0', ω') as described in Section 3.7; the resulting Hamiltonian vector field has two nondegenerate orbits corresponding to e and h (and denoted by the same names) instead of the Morse-Bott torus $\mathcal{N}$. Any almost complex structure $J' \in \mathcal{J}_{W'}$ is then perturbed to an almost complex structure adapted to (α_0', ω').

Because of the simple nature of the J'-holomorphic Morse-Bott buildings, and since $\widehat{\mathcal{O}}_k$ for $k \leq 2g$ are finite sets, for a sufficiently small perturbation (α_0', ω') of (α_0, ω) and for every $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathcal{O}_k$, there is a bijection between the ECH index one moduli spaces $\mathcal{M}_{J'}^{I=1}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ and the index one moduli spaces of holomorphic maps from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$ for the perturbation of J' adapted to (α_0', ω'). See [0, Lemma 4.4.5] and [0, Lemma 7.1.2].

Since the Morse-Bott situation and its generic perturbation are completely equivalent, we will go from one point of view to the other as more convenient, often without explicit mention. For this reason, *we will still denote by J' a perturbation of J' adapted to (α_0', ω'), and by $\mathcal{M}_{J'}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ the moduli spaces of holomorphic maps with respect to the perturbation of J'.*

The definition of the analogous space $\mathcal{J}_{\overline{W'}}$ for $\overline{W'} = \mathbf{R} \times \overline{N}$ is slightly more complicated, but completely analogous to Definition 5.3.2. We first discuss the Morse-Bott version $\mathcal{J}_{\overline{W'}}^{MB}$. Let $D_\varepsilon^2 = \{|z| \leq \varepsilon\} \subset \mathbf{C}$ and, for $\varepsilon \leq 1$, let V_ε be the mapping torus of $h|_{D_\varepsilon^2}$. For $\varepsilon = 1$ this coincides with the already defined solid torus V.

Definition **5.3.12.** — *Fix $\varepsilon, \delta > 0$ sufficiently small and $k \gg 0$. Then $\mathcal{J}_{\overline{W'}}^{MB}$ is the space of C^∞-smooth $(\overline{\alpha}_0, \overline{\omega})$-adapted almost complex structures $\overline{J'}$ on $\mathbf{R} \times \overline{N}$ which satisfy the following:*

(1) *there exists an $(\overline{\alpha}_0, \overline{\omega})$-adapted almost complex structure $\overline{J_0'}$ which restricts to the standard complex structure on the subsurface $D^2 \subset \overline{S}$ of each fiber;*
(2) *$\overline{J'} = \overline{J_0'}$ on $\mathbf{R} \times (V_\varepsilon \cup N)$ and $|\overline{J'} - \overline{J_0'}|_k < \delta$ over $\mathbf{R} \times \overline{N}$;*
(SR) *$\overline{J'}$ is a split complex structure on $\mathbf{R} \times [1, 2]$ times a neighborhood of $\overline{\mathbf{a}}$; moreover $\overline{\mathbf{a}}$ is real analytic with respect to the fiber complex structure on $\overline{\pi}_B^{-1}(s, t)$ for $t \in [1, 2]$.*

Here $|\cdot|_k$ is some fixed C^k-norm.

The Morse-Bott theory for $\overline{W'}$ is more subtle than that of W', and therefore we will discuss it in more detail. In the following definition, all $\overline{J}'$-holomorphic maps are allowed to be nodal and to have disconnected domains. Let $\boldsymbol{\gamma}_+ = \gamma_{+,1}^{k_1} \dots \gamma_{+,m_+}^{k_{m_+}} h^a$ be an orbit set such that $\gamma_{+,i} \in \mathcal{P} \cup \{e, \delta_0\}$ and $a \in \{0, 1\}$. Let $\boldsymbol{\gamma}_- = \gamma_{-,1}^{l_1} \dots \gamma_{-,m_-}^{l_{m_-}} e^b$ be an orbit set such that $\gamma_{-,i} \in \mathcal{P} \cup \{h, \delta_0\}$ and $b \geq 0$.

Definition **5.3.13** *(Morse-Bott holomorphic buildings). — Let $\overline{J'}$ be an almost complex structure in $\mathcal{J}_{\overline{W'}}^{MB}$ and let $\boldsymbol{\gamma}_+ = \gamma_{+,1}^{k_1} \dots \gamma_{+,m_+}^{k_{m_+}} h^a$ and $\boldsymbol{\gamma}_- = \gamma_{-,1}^{l_1} \dots \gamma_{-,m_-}^{l_{m_-}} e^b$ be orbit sets as above. A $\overline{J}'$-holomorphic Morse-Bott building u from $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$ consists of*

(1) *a set of $\ell = 0$ or 1 negative gradient flow trajectories (with $a \geq \ell$) from h to e in $\mathcal{N}$, and*
(2) *a (possibly empty) set of $\overline{J}'$-holomorphic maps $u_i : \dot{F}_i \to \overline{W'}$, for $i = 1, \dots, n$, such that the following hold:*
    (a) *Positive ends of u_n converge to multiples of $\gamma_{+,i}$ with total multiplicity k_i and to a simple orbit in $\mathcal{N} - \{e\}$ if $a - \ell \neq 0$.*
    (b) *For $i = 1, \dots, n-1$, the negative ends of u_{i+1} are paired with the positive ends of u_i. For each pair, the two ends either converge to the same orbit, or they converge to covers of*

orbits in $\mathcal{N}$ with the same multiplicity and there is a negative gradient flow line of $\overline{g}$ in $\mathcal{N}$ from the limit of u_{i+1} to the limit of u_i.

(c) *Negative ends of u_1 converge to multiples of $\gamma_{-,i}$ with total multiplicity l_i and to multiple covers of orbits in $\mathcal{N} - \{h\}$ with total multiplicity $b - \ell$.*

The stable Hamiltonian structure $(\overline{\alpha}_0, \overline{\omega})$ can be perturbed as described in Section 3.7 — in fact, (N^+, α_0, ω) embeds in $(\overline{N}, \overline{\alpha}_0, \overline{\omega})$. The perturbed stable Hamiltonian structure obtained in this way will be called a *nondegenerate perturbation of* $(\overline{\alpha}_0, \overline{\omega})$. After the perturbation, the Morse-Bott torus $\mathcal{N}$ is replaced by two non-degenerate orbits corresponding to e and h (and still denoted by the same name) and all orbits in $\overline{N} - \partial N$ which intersect a fiber at most $2g$ times remain unchanged.

We fix a nondegenerate perturbation $(\overline{\alpha}'_0, \overline{\omega}')$ of $(\overline{\alpha}_0, \overline{\omega})$ once and for all. We define the following space of almost complex structures on $\overline{W'}$ in analogy to Definition 5.3.12.

Definition **5.3.14.** — *Fix $\varepsilon, \delta > 0$ sufficiently small and $k \gg 0$. Then $\mathcal{J}_{\overline{W'}}$ is the space of C^∞-smooth $(\overline{\alpha}'_0, \overline{\omega}')$-adapted almost complex structures $\overline{J}'$ on $\mathbf{R} \times \overline{N}$ which satisfy the following:*

(1) *there exists an $(\overline{\alpha}'_0, \overline{\omega}')$-adapted almost complex structure $\overline{J}'_0$ which restricts to the standard complex structure on the subsurface $D^2 \subset \overline{S}$ of each fiber;*

(2) *$\overline{J}' = \overline{J}'_0$ on $\mathbf{R} \times (V_\varepsilon \cup N)$ and $|\overline{J}' - \overline{J}'_0|_k < \delta$ over $\mathbf{R} \times \overline{N}$;*

(3) *every $\overline{J}'$-holomorphic map $u : \dot{F} \to \overline{W'}$ connecting two orbit sets in $\overline{\mathcal{O}}_k$, for $k \leq 2g$, is close to breaking to a holomorphic Morse-Bott building for some almost-complex structure in $\mathcal{J}^{MB}_{\overline{W'}}$;*

(SR) *$\overline{J}'$ is a split complex structure on $\mathbf{R} \times [1, 2]$ times a neighborhood of $\overline{\mathbf{a}}$; moreover $\overline{\mathbf{a}}$ is real analytic with respect to the fiber complex structure on $\overline{\pi}_B^{-1}(s, t)$ for $t \in [1, 2]$.*

Here $|\cdot|_k$ is some fixed C^k-norm.

Remark **5.3.15.** — Condition (3) in Definition 5.3.14 will be used to constrain the topological behavior of $\overline{J}'$-holomorphic maps: to show that a map has a certain behavior, we will show that the corresponding Morse-Bott building does, which is often simpler.

Definition **5.3.16.** — *Given $\delta_0^p \boldsymbol{\gamma}, \delta_0^q \boldsymbol{\gamma}' \in \overline{\mathcal{O}}_k$ (with $p, q \geq 0$ and $k \leq 2g$) and $\overline{J}' \in \mathcal{J}_{\overline{W'}}$, let $\mathcal{M}_{\overline{J}'}(\delta_0^p \boldsymbol{\gamma}, \delta_0^q \boldsymbol{\gamma}')$ be the moduli space of $\overline{J}'$-holomorphic maps in $\overline{W'}$ from $\delta_0^p \boldsymbol{\gamma}$ to $\delta_0^q \boldsymbol{\gamma}'$ without fiber components.*

Maps in $\mathcal{M}_{\overline{J}'}(\delta_0^p \boldsymbol{\gamma}, \delta_0^q \boldsymbol{\gamma}')$, with $\delta_0^p \boldsymbol{\gamma}, \delta_0^q \boldsymbol{\gamma}' \in \overline{\mathcal{O}}_k$, will be called *degree k PFH curve* (or *degree k $\overline{W'}$-curve*). A degree k PFH curve can be viewed as a degree k holomorphic multisection of the holomorphic fibration $\mathbf{R} \times \overline{N} \to \mathbf{R} \times S^1$. A degree $2g$ PFH curve will also be called a *PFH curve* (or *$\overline{W'}$-curve*).

Definition **5.3.17.** — *The* section at infinity σ'_∞ *in* $\mathbf{R} \times \overline{N}$ *is* $\mathbf{R} \times \delta_0$.

Choose a point $z_\infty^\dagger \in \overline{S}$ which is sufficiently close to z_∞. Then there is a periodic orbit $\delta_0^\dagger$ of $\overline{R}_0$ with period m which passes through $z_\infty^\dagger$. We then write $(\sigma_\infty')^\dagger = \mathbf{R} \times \delta_0^\dagger$. Both σ_∞' and $(\sigma_\infty')^\dagger$ are $\overline{J}'$-holomorphic sections for every $\overline{J}' \in \mathcal{J}_{\overline{W'}}$.

Definition **5.3.18.** — *Given a degree k PFH curve $\overline{u}$ in $\overline{W'}$, we define $n'(\overline{u}) = \langle \overline{u}, (\sigma_\infty')^\dagger \rangle$.*

The proof of the following is similar to that of Lemma 5.3.7.

Lemma **5.3.19.** — *The intersection number $n'(\overline{u})$ satisfies the following properties:*

(1) *$n'(\overline{u}) \geq 0$ and $n'(\overline{u}) = 0$ if and only if the image of $\overline{u}$ is disjoint from $(\sigma_\infty')^\dagger$;*
(2) *if $\langle \overline{u}, \mathbf{R} \times \delta_0 \rangle > 0$, then $n'(\overline{u}) > 0$; and*
(3) *$n'(\overline{u})$ is independent of the choice of $z_\infty^\dagger$.*

Lemma **5.3.20.** — *If $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ are contained in N, then $\mathcal{M}_{\overline{J}'}^{n'=0}(\boldsymbol{\gamma}, \boldsymbol{\gamma}') = \mathcal{M}_{J'}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$.*

Proof. — Let $\overline{u}$ be a $\overline{J}'$-holomorphic map in $\mathcal{M}_{\overline{J}'}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ satisfying the constraint $n'(\overline{u}) = 0$. By Condition (3) in Definition 5.3.14, we can assume that $\overline{u}$ is a Morse-Bott building to the purpose of finding homological constraints on it.

Let $T_{\rho_0} \subset V$ be the torus $\{\rho = \rho_0\}$, oriented as the boundary of the solid torus $\{\rho \leq \rho_0\}$. The torus T_{ρ_0} is foliated by orbits of the Hamiltonian vector field $\overline{R}_0$ and, for a dense set of ρ_0, those orbits are closed. If δ_{ρ_0} is one such orbit, then the homology class $[\delta_{\rho_0}]$ is a rational multiple of $[\delta_0^\dagger]$ in $H_2(\overline{N} - \boldsymbol{\gamma} \cup \boldsymbol{\gamma}')$. This implies that $\langle \overline{u}, \mathbf{R} \times \delta_{\rho_0} \rangle = 0$ for ρ_0 in a dense set. The positivity of intersections then implies that the image of $\overline{u}$ is contained in N. □

5.4. *Almost complex structures and moduli spaces for* W_+, $\overline{W}_+$, W_-, *and* $\overline{W}_-$. —

5.4.1. *Almost complex structures.* —

Definition **5.4.1.** — *An almost complex structure $\overline{J}_+$ on $\overline{W}_+$ is* admissible *if it is the restriction to $\overline{W}_+$ of an almost complex structure $\overline{J}' \in \mathcal{J}_{\overline{W'}}$ on $\overline{W'}$. If $\overline{J}_+$ agrees with $\overline{J}$ (resp. $\overline{J}'$) at the positive (resp. negative) end, then $\overline{J}_+$ is* compatible *with $\overline{J}$ (resp. $\overline{J}'$).*

An admissible *almost complex structure J_+ on W_+ is the restriction of an admissible almost complex structure on $\overline{W}_+$. The admissibility criteria for J_- and $\overline{J}_-$ on W_- and $\overline{W}_-$ are analogous.*

The space of C^∞-smooth admissible almost complex structures $J_\pm$ on $W_\pm$ will be denoted by $\mathcal{J}_{W_\pm}$. The space of C^∞-smooth admissible almost complex structures $\overline{J}_\pm$ on $\overline{W}_\pm$ will be denoted by $\mathcal{J}_{\overline{W}_\pm}^{MB}$ if $\overline{J}_\pm$ is the restriction of $\overline{J}' \in \mathcal{J}_{\overline{W'}}^{MB}$ and by $\mathcal{J}_{\overline{W}_\pm}$ if $\overline{J}_\pm$ is obtained from an almost complex structure in $\mathcal{J}_{\overline{W}_\pm}^{MB}$ by modifying its positive (resp. negative) end as in Remark 5.3.11.

Remark **5.4.2.** — If $J_+ \in \mathcal{J}_{W_+}$, then the projection $\pi_{B_+} : W_+ \to B_+$ is holomorphic. This is due to the fact that the fibers $\{(s,t)\} \times S$ are holomorphic and J_+ takes ∂_s to ∂_t. The same holds for $\bar{J}_+, J_-$, and $\bar{J}_-$.

5.4.2. *Moduli spaces for* W_+. — Let (F, j) be a compact Riemann surface, possibly disconnected, with an l-tuple of punctures $\mathbf{p} = \{p_1, \dots, p_l\}$ in the interior of F and a k-tuple of punctures $\mathbf{q} = \{q_1, \dots, q_k\}$ on ∂F, such that (i) each component of F has nonempty boundary and at least one interior puncture and (ii) each component of ∂F has at least one boundary puncture. We write $\dot{F} = F - \mathbf{p} - \mathbf{q}$ and $\partial\dot{F} = \partial F - \mathbf{q}$.

Definition **5.4.3.** — *Let* $J_+ \in \mathcal{J}_{W_+}$, $\mathbf{y}$ *be a k-tuple of* $\mathbf{a} \cap h(\mathbf{a})$, *and* $\boldsymbol{\gamma} = \prod_j \gamma_j^{m_j}$ *be an orbit set in* $\widehat{\mathcal{O}}_k$. *Then a* degree $k \leq 2g$ multisection of (W_+, J_+) from $\mathbf{y}$ to $\boldsymbol{\gamma}$ *is either (i) a holomorphic map*

$$u : (\dot{F}, j) \to (W_+, J_+)$$

which is a degree k multisection of $\pi_{B_+} : W_+ \to B_+$ *and which additionally satisfies the following:*

1. $u(\partial\dot{F}) \subset L^+_{\mathbf{a}}$ *and u maps each connected component of* $\partial\dot{F}$ *to a different* $L^+_{a_i}$;
2. $\lim_{w \to q_i} \pi_{\mathbf{R}} \circ u(w) = +\infty$ *and* $\lim_{w \to p_i} \pi_{\mathbf{R}} \circ u(w) = -\infty$;
3. *u converges to a strip over* $[0,1] \times \mathbf{y}$ *near* $\mathbf{q}$;
4. *u converges to a cylinder over a multiple of some* γ_j *near each puncture* p_i *so that the total multiplicity of* γ_j *over all the* p_i*'s is* m_j;
5. *the energy of u given by Equation* (4.3.1) *is finite;*

or is (ii) a Morse-Bott building consisting in a J_+*-holomorphic curve as above, except for some negative ends at Reeb orbits in* $\mathcal{N} - \{e, h\}$, *followed by gradient flow trajectories from those orbits to e. Here* $\pi_{\mathbf{R}}$ *is the projection* $\pi_{B_+} : W_+ \to B_+ \subset \mathbf{R} \times S^1$, *followed by the projection to* $\mathbf{R}$.

A (W_+, J_+)-curve *is a degree* $2g$ *multisection of* (W_+, J_+).

The finiteness of the Hofer energy $E(u)$ implies that u is a cylinder over a Reeb chord or a closed orbit in neighborhoods of punctures p_i and q_i. Hence (5) implies (3) and (4) for some $\mathbf{y}$ and $\boldsymbol{\gamma}$. Moreover, since the orbits are nondegenerate, the convergence is exponential by the work of Abbas [Abb1] for chords and Hofer-Wysocki-Zehnder [HWZ1] for closed orbits.

Remark **5.4.4.** — For all practical purposes, it suffices to assume that the Morse-Bott family on ∂N has been perturbed into a pair h, e of nondegenerate orbits as in 5.3.11 and that J_+ is the restriction of $J' = J_0$ which satisfies Definition 3.2.1 for the perturbed Hamiltonian vector field. Since the two points of view are completely equivalent, we will switch liberally from one to another, often without explicit mention.

Let $\check{W}_+$ be W_+ with the ends $\{s > 3\}$ and $\{s < -1\}$ removed. We can view $\check{W}_+$ as a compactification of W_+, obtained by attaching $[0, 1] \times S$ at $s = +\infty$ and N at $s = -\infty$. We define $Z_{\mathbf{y},\boldsymbol{\gamma}} \subset \check{W}_+$ as the subset

$$Z_{\mathbf{y},\boldsymbol{\gamma}} = (L^+_{\mathbf{a}} \cap \check{W}_+) \cup (\{3\} \times [0, 1] \times \mathbf{y}) \cup (\{-1\} \times \boldsymbol{\gamma}).$$

As in Section 4.3, the W_+-curve $u : \dot{F} \to W_+$ can be compactified to a continuous map

$$\check{u} : (\check{F}, \partial\check{F}) \to (\check{W}_+, Z_{\mathbf{y},\boldsymbol{\gamma}}).$$

We write $\mathcal{M}_{J_+}(\mathbf{y}, \boldsymbol{\gamma})$ for the moduli space of multisections of (W_+,J_+) from $\mathbf{y}$ to $\boldsymbol{\gamma}$. We denote by $H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$ the equivalence classes of continuous degree $2g$ multisections in W_+ satisfying Conditions (1)–(4) of Definition 5.4.3, where two multisections are equivalent if they represent the same element in $H_2(\check{W}_+, Z_{\mathbf{y},\boldsymbol{\gamma}})$, and by $\mathcal{M}_{J_+}(\mathbf{y}, \boldsymbol{\gamma}, A)$ the moduli space of maps $u \in \mathcal{M}_{J_+}(\mathbf{y}, \boldsymbol{\gamma})$ which represent the class $A \in H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$. Then

$$\mathcal{M}_{J_+}(\mathbf{y}, \boldsymbol{\gamma}) = \coprod_{A \in H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})} \mathcal{M}_{J_+}(\mathbf{y}, \boldsymbol{\gamma}, A).$$

5.4.3. *Moduli spaces for* $\overline{W}_+$. — Let $\overline{J}_+ \in \mathcal{J}_{\overline{W}_+}$, $\mathbf{y}$ be a k-tuple of $\mathbf{a} \cap h(\mathbf{a})$ and let $\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_k$. Then a *degree* $k \leq 2g$ *multisection of* $(\overline{W}_+,\overline{J}_+)$ *from* $\mathbf{y}$ *to* $\boldsymbol{\gamma}$ is defined as in Definition 5.4.3, where $W_+, J_+, L^+_{a_i}$ are replaced by $\overline{W}_+, \overline{J}_+, L^+_{\widehat{a}_i}$.[5] We write $\mathcal{M}_{\overline{J}_+}(\mathbf{y}, \boldsymbol{\gamma})$ for the moduli space of multisections of $(\overline{W}_+,\overline{J}_+)$ from $\mathbf{y}$ to $\boldsymbol{\gamma}$.

Remark **5.4.5.** — As in Section 5.3.2, we will assume that $\overline{J}_+$ is sufficiently close to a Morse-Bott almost complex structure for which all degree $k \leq 2g$ multisections of $(\overline{W}_+,\overline{J}_+)$ are close to breaking into a Morse-Bott building. This will allow us to use the Morse-Bott formalism to prove topological constraints on the $\overline{J}_+$-holomorphic multisections. In the context of maps to $\overline{W}_+$, the definition of a Morse-Bott building is the same as in Definition 5.3.13, with the only difference that the topmost holomorphic map u_1 takes values in $\overline{W}_+$ and is positively asymptotic to chords over intersection points in $\mathbf{a} \cap h(\mathbf{a})$.

Definition **5.4.6.** — *The* section at infinity σ^+_∞ *is the restriction of* $\sigma'_\infty = \mathbf{R} \times \delta_0 \subset \overline{W'}$ *to* $\overline{W}_+$.

Let $\delta_0^\dagger$ be the closed orbit of $\overline{R}_0$ used in Definition 5.3.18. We define $(\sigma^+_\infty)^\dagger$ as the restriction of $\mathbf{R} \times \delta_0^\dagger$ to $\overline{W}_+$. Both σ^+_∞ and $(\sigma^+_\infty)^\dagger$ are $\overline{J}_+$-holomorphic for every almost complex structure $\overline{J}_+ \in \mathcal{J}_{\overline{W}_+}$.

Definition **5.4.7.** — *Given a multisection* $\overline{u}$ *of* $\overline{W}_+$, *we define* $n^+(\overline{u}) = \langle \overline{u}, (\sigma^+_\infty)^\dagger \rangle$.

[5] In particular, the image of the multisection is disjoint from the section at infinity σ^+_∞, defined in Definition 5.4.6.

Lemma **5.4.8.** — *The intersection number $n^+(\overline{u})$ satisfies the following properties:*

(1) $n^+(\overline{u}) \geq 0$ *and* $n^+(\overline{u}) = 0$ *if and only if the image of* $\overline{u}$ *is disjoint from* $(\sigma^+_\infty)^\dagger$*;*
(2) *if* $\langle \overline{u}, \sigma^+_\infty \rangle > 0$*, then* $n^+(\overline{u}) \geq m$*; moreover* $n^+(\overline{u}) = m$ *if and only if there is a unique transverse intersection point between the image of* $\overline{u}$ *and* σ^+_∞*; and*
(3) $n^+(\overline{u})$ *is independent of the choice of* $\delta_0^\dagger$.

Proof. — The proof is similar to that of Lemma 5.3.7. The only difference is in (2), which we discuss in more detail. Consider $\overline{u} : \dot{F} \to \overline{W}_+$ and let $p \in \dot{F}$ be a point such that $\overline{u}(p) \in \sigma^+_\infty$. If π_{D^2} is the projection of a neighborhood of $\overline{u}(p) \in \overline{W}_+$ to $D^2 \subset \overline{S}$ along the symplectic connection (not to be confused with the balanced projection $\overline{\pi}_{D^2}$ from Section 5.1.2), then $\pi_{D^2} \circ \overline{u}$ is holomorphic and nonconstant, and therefore maps an open neighborhood of p in $\dot{F}$ to an open neighborhood of z_∞. This implies that $n^+(\overline{u}) \geq d \cdot m$, where d is the multiplicity of the intersection between the image of $\overline{u}$ and σ^+_∞. □

Lemma **5.4.9.** — *If* $\overline{u} \in \mathcal{M}^{n^+=0}_{\overline{J}_+}(\mathbf{y}, \boldsymbol{\gamma})$*, then* $\mathrm{Im}(\overline{u}) \subset W_+$.

The proof is similar to that of Lemma 5.3.20 and will be omitted.

5.4.4. *Moduli spaces for* $\overline{W}_-$. — The moduli space of holomorphic maps which is used to define the map Ψ from $\widehat{PFC}$ to $\widehat{CF}$ is of a slightly different type from the moduli space which is used to define the map Φ from $\widehat{CF}$ to $\widehat{PFC}$. In particular, the target of the holomorphic maps is $\overline{W}_-$ instead of W_-. The reason we need to consider more complicated holomorphic curves instead of curves analogous to W_+-curves is that the naive W_--curves do not have the desired Fredholm index. See the index calculations in Section 5.5.2 and Remark 5.5.6 for an explanation.

Let (F, j) be a compact Riemann surface, possibly disconnected, with an l-tuple of punctures $\mathbf{p} = \{p_1, \dots, p_l\}$ in the interior of F and a k-tuple of punctures $\mathbf{q} = \{q_1, \dots, q_k\}$ on ∂F, such that (i) each component of F has nonempty boundary and has at least one interior puncture and (ii) each component of ∂F has at least one boundary puncture. We write $\dot{F} = F - \mathbf{p} - \mathbf{q}$ and $\partial \dot{F} = \partial F - \mathbf{q}$.

Definition **5.4.10.** — *Let* $\overline{J}_- \in \mathcal{J}_{\overline{W}_-}$*,* $\mathbf{y}$ *be a k-tuple of* $\mathbf{a} \cap h(\mathbf{a})$*, and* $\boldsymbol{\gamma} = \prod_j \gamma_j^{m_j} \in \widehat{\mathcal{O}}_k$*. Then a* degree $k \leq 2g$ multisection of $(\overline{W}_-, \overline{J}_-)$ from $\mathbf{y}$ to $\boldsymbol{\gamma}$ *is a* $\overline{J}_-$*-holomorphic map*

$$\overline{u} : (\dot{F}, j) \to (\overline{W}_-, \overline{J}_-)$$

which is a degree k multisection of $\overline{\pi}_{B_-} : \overline{W}_- \to B_-$ *and which additionally satisfies the following:*

(1) $\overline{u}(\partial \dot{F}) \subset L^-_{\widehat{\mathbf{a}}}$ *and* $\overline{u}$ *maps each connected component of* $\partial \dot{F}$ *to a different* $L^-_{\widehat{a}_i}$*;*
(2) $\lim_{w \to q_i} \overline{\pi}_{\mathbf{R}} \circ \overline{u}(w) = -\infty$ *and* $\lim_{w \to p_i} \overline{\pi}_{\mathbf{R}} \circ \overline{u}(w) = +\infty$*;*
(3) $\overline{u}$ *converges to a strip over* $[0, 1] \times \mathbf{y}$ *near* $\mathbf{q}$*;*

(4) *$\overline{u}$ converges to a cylinder over a multiple of some γ_j near each puncture p_i so that the total multiplicity of γ_j over all the p_i's is m_j;*
(5) *the energy of $\overline{u}$ given by Equation (4.3.1) is finite.*

Here $\overline{\pi}_{\mathbf{R}}$ is the projection $\overline{\pi}_{B_-} : \overline{W}_- \to B_- \subset \mathbf{R} \times S^1$, followed by the projection to $\mathbf{R}$.

Remark **5.4.11.** — As in Section 5.3.2, we will assume that $\overline{J}_-$ is sufficiently close to a Morse-Bott almost complex structure for which all degree $k \leq 2g$ multisections of $(\overline{W}_-, \overline{J}_-)$ are close to breaking into a Morse-Bott building. This will allow us to use the Morse-Bott formalism to prove topological constraints on the $\overline{J}_-$-holomorphic multisections. In the context of maps to $\overline{W}_-$, the definition of a Morse-Bott building is the same as in Definition 5.3.13, with the only difference that the bottommost holomorphic map u_n takes values in $\overline{W}_-$ and is negatively asymptotic to chords over intersection points in $\mathbf{a} \cap h(\mathbf{a})$.

Definition **5.4.12.** — *The* section at infinity σ_∞^- *is the restriction of* $\sigma'_\infty = \mathbf{R} \times \delta_0 \subset \overline{W'}$ *to* $\overline{W}_-$.

Let $\delta_0^\dagger$ be the closed orbit of $\overline{R}_0$ used in Definition 5.3.18. We define $(\sigma_\infty^-)^\dagger$ as the restriction of $\mathbf{R} \times \delta_0^\dagger$ to $\overline{W}_-$. Both σ_∞^- and $(\sigma_\infty^-)^\dagger$ are $\overline{J}_-$-holomorphic for every almost complex structure $\overline{J}_- \in \mathcal{J}_{\overline{W}_-}$.

Definition **5.4.13.** — *Given a multisection $\overline{u}$ of $\overline{W}_-$, we define $n^-(\overline{u}) = \langle \overline{u}, (\sigma_\infty^-)^\dagger \rangle$.*

The proof of the following is similar to that of Lemma 5.4.8.

Lemma **5.4.14.** — *The intersection number $n^-(\overline{u})$ satisfies the following properties:*

(1) *$n^-(\overline{u}) \geq 0$ and $n^-(\overline{u}) = 0$ if and only if the image of $\overline{u}$ is disjoint from $(\sigma_\infty^-)^\dagger$;*
(2) *if $\langle \overline{u}, \sigma_\infty^- \rangle > 0$, then $n^-(\overline{u}) \geq m$; moreover $n^-(\overline{u}) = m$ if and only if there is a unique transverse intersection point between the image of $\overline{u}$ and σ_∞^-; and*
(3) *$n^-(\overline{u})$ is independent of the choice of $\delta_0^\dagger$.*

Definition **5.4.15.** — *A* $(\overline{W}_-, \overline{J}_-)$-curve *is a degree $2g$ multisection of $(\overline{W}_-, \overline{J}_-)$ satisfying $n^-(\overline{u}) = m$.*

Let $\check{\overline{W}}_-$ be $\overline{W}_-$ with the ends $\{s > 1\}$ and $\{s < -3\}$ removed. We can view $\check{\overline{W}}_-$ as a compactification of $\overline{W}_-$, obtained by attaching $[0,1] \times \overline{S}$ at $s = -\infty$ and $\overline{N}$ at $s = +\infty$. Also let

$$Z_{\boldsymbol{\gamma},\mathbf{y}} = (L^-_{\widehat{\mathbf{a}}} \cap \check{\overline{W}}) \cup (\{1\} \times \boldsymbol{\gamma}) \cup (\{-3\} \times [0,1] \times \mathbf{y}).$$

We write $\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y})$ for the moduli space of multisections of $(\overline{W}_-, \overline{J}_-)$ from $\boldsymbol{\gamma}$ to $\mathbf{y}$. We denote by $H_2(\overline{W}_-, \boldsymbol{\gamma}, \mathbf{y})$ the equivalence classes of continuous degree $2g$ multisections in $\overline{W}_-$ satisfying Conditions (1)–(4) of Definition 5.4.10, where two multisections are equivalent if they represent the same element in $H_2(\check{\overline{W}}_-, Z_{\boldsymbol{\gamma},\mathbf{y}})$. Then

$$\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}) = \coprod_{A\in H_2(\overline{W}_-,\boldsymbol{\gamma},\mathbf{y})} \mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}, A).$$

Also let $\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}})$ be the moduli space of multisections of $(\overline{W}_-, \overline{J}_-)$ from $\boldsymbol{\gamma}$ to $\mathbf{y}$ that pass through the marked point $\overline{\mathfrak{m}}$. This moduli space comes equipped with a forgetful map

$$\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}}) \to \mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}).$$

When we restrict our attention to $\overline{J}_-$-holomorphic curves $\overline{u}$ with $n^-(\overline{u}) = m$, the forgetful map $\mathcal{M}^{n^-=m}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}}) \to \mathcal{M}^{n_-=m}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y})$ is injective by Lemma 5.4.14(2) because $\overline{\mathfrak{m}} \in \sigma^-_\infty$. This observation allows us to regard $\mathcal{M}^{n^-=m}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}})$ as a subset of $\mathcal{M}^{n^-=m}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y})$.

The following lemma is similar to Lemma 5.4.9:

Lemma **5.4.16.** — *If* $\overline{u} \in \mathcal{M}^{n^-=0}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y})$, *then* $\mathrm{Im}(\overline{u}) \subset W_-$.

5.5. *The Fredholm index.* — In this subsection we compute the Fredholm index ind_{W_+} of a (W_+,J_+)-curve from $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ to $\boldsymbol{\gamma} = \prod \gamma_j^{m_j} \in \widehat{\mathcal{O}}_{2g}$ and the Fredholm index $\mathrm{ind}_{\overline{W}_-}$ of a $(\overline{W}_-,\overline{J}_-)$-curve from $\boldsymbol{\gamma}$ to $\mathbf{y}$.

The Fredholm indices are computed using the doubling technique of Hofer-Lizan-Sikorav [HLS], which we quickly review, referring to the original paper for the details.

5.5.1. *Doubling.* — Let (F, j) be a compact Riemann surface with boundary, $\mathbf{p}$ and $\mathbf{q}$ be finite sets of interior and boundary punctures of F, and $\dot{F} = F - \mathbf{p} - \mathbf{q}$. We form the *double* $(2\dot{F}, 2j)$ of $(\dot{F}, j)$ by gluing two copies of $\dot{F}$ with opposite complex structures j and $-j$ along their boundary $\partial\dot{F} = \partial F - \mathbf{q}$. By the Schwarz reflection principle, the two complex structures match and the doubled surface becomes a punctured Riemann surface.

Let $E \to \dot{F}$ be a complex vector bundle with fiberwise complex structure i and let $L \to \partial\dot{F}$ be a totally real subbundle of maximal rank. Let $\overline{E} \to \dot{F}$ be a complex vector bundle whose fiber $\overline{E}_p$ at $p \in \dot{F}$ is E_p with complex structure $-i$. We then construct the doubled complex vector bundle $2E \to 2\dot{F}$ by gluing $E \to \dot{F}$ and $\overline{E} \to \dot{F}$ along $\partial\dot{F}$ such that at each $p \in \partial\dot{F}$ the gluing map identifies the fibers $(E_p, i) \simeq (\overline{E}_p, -i)$ via an involution which fixes L_p pointwise. Let $\sigma : 2\dot{F} \xrightarrow{\sim} 2\dot{F}$ be the anti-holomorphic involution σ which switches $(\dot{F}, j)$ and $(\dot{F}, -j)$ and let $\tilde{\sigma} : 2E \xrightarrow{\sim} 2E$ be the anti-$\mathbf{C}$-linear bundle isomorphism which projects to σ and identifies $E_p \simeq \overline{E}_{\sigma(p)}$ by the identity map, where $p \in (\dot{F}, j)$. Finally, given a linear Cauchy-Riemann type operator D on E, we can define the doubled operator 2D on 2E with the property that 2D is $\tilde{\sigma}$-invariant and its restriction to $E \to \Sigma$ is D.

One of the results of [HLS] is the following:

Theorem **5.5.1.** — *Suppose that both* D *and* 2D *are Fredholm operators in some suitable Sobolev spaces. Then:*

- $\mathrm{ind}(D) = \frac{1}{2}\mathrm{ind}(2D)$*; and*
- *if* 2D *is surjective, then* D *is also surjective.*

Our situation is slightly more general than that considered by [HLS], since we are considering boundary punctures and exponential weights. The proof, however, remains largely unmodified.

5.5.2. *The* W_+ *case.* — For the purposes of computing indices, we replace W_+ by $\check{W}_+$, as defined in Section 5.4.2. The tangent space $T\check{W}_+$ splits into the vertical and horizontal subbundles via the symplectic connection. In this section we slightly abuse notation and write TS (resp. TB_+) for the vertical (resp. horizontal) subbundle over $\check{W}_+$.

We define a trivialization τ of TS along $Z_{\mathbf{y},\boldsymbol{\gamma}}$ as follows: First define τ along $L_{\mathbf{a}}^+ \cap \check{W}_+$ by orienting all $\mathbf{a}$-arcs arbitrarily as in Section 4.4.2 and extending the trivialization by parallel transport along the vertical boundary $\partial_v W_+$. We then extend the trivialization of $TS|_{L_{\mathbf{a}}^+\cap\check{W}_+}$ in an arbitrary manner to a trivialization of TS along $\{3\}\times[0,1]\times\mathbf{y}$ and along $\{-1\}\times\boldsymbol{\gamma}$.

Let $u : (\dot{F}, j) \to (W_+, J_+)$ be a W_+-curve. Suppose the negative ends of u partition m_j into $(m_{j1}, m_{j2}, \dots)$. We then write:

$$\mu_\tau(\boldsymbol{\gamma}, u) = \sum_j \sum_i \mu_\tau(\gamma_j^{m_{ji}}),$$

where $\mu_\tau(\gamma_j^{m_{ji}})$ is the Conley-Zehnder index of the m_{ji}-fold cover of γ_j with respect to τ.

We now define a real rank one subbundle $\mathcal{L}_0$ of TS along

$$(L_{\mathbf{a}}^+ \cap \check{W}_+) \cup (\{3\}\times[0,1]\times\mathbf{y}).$$

Let $\mathcal{L}_0 = TL_{\mathbf{a}}^+ \cap TS$ on $L_{\mathbf{a}}^+ \cap \check{W}_+$. In particular, $\mathcal{L}_0 = T\mathbf{a}(y_i)$ at $\{3\}\times\{1\}\times\{y_i\}$ and $\mathcal{L}_0 = Th(\mathbf{a})(y_i)$ at $\{3\}\times\{0\}\times\{y_i\}$. We then extend $\mathcal{L}_0$ across $\{3\}\times[0,1]\times\mathbf{y}$ by rotating in the counterclockwise direction from $Th(\mathbf{a})$ to $T\mathbf{a}$ in TS by the minimum amount possible. Assuming orthogonal intersections between $\mathbf{a}$ and $h(\mathbf{a})$, the angle of rotation is $\frac{\pi}{2}$.

Let $\mu_\tau(y_i)$ be the Maslov index of $\mathcal{L}_0$ along $\{3\}\times[0,1]\times\{y_i\}$ with respect to τ. If u is a W_+-curve and $\mathcal{L} = \check{u}^*\mathcal{L}_0$, then we define $\mu_\tau(\mathbf{y})$ to be the Maslov index of $\mathcal{L}$ with respect to τ. By the definitions of $\mathcal{L}_0$ and τ, it is immediate that

$$\mu_\tau(\mathbf{y}) = \sum_{i=1}^{2g} \mu_\tau(y_i).$$

We then have the following Fredholm index formula for W_+-curves:

Proposition **5.5.2** *(Fredholm index formula for* W_+*-curves). — The Fredholm index of a* (W_+, J_+)*-curve* $u : (\dot{F}, j) \to (W_+, J_+)$ *from* $\mathbf{y}$ *to* $\boldsymbol{\gamma}$ *is given by the formula:*

$$\text{ind}_{W_+}(u) = -\chi(\dot{F}) - 2g + \mu_\tau(\mathbf{y}) - \mu_\tau(\boldsymbol{\gamma}, u) + 2c_1(u^*TS, \tau). \quad (\mathbf{5.5.1})$$

Proof. — We double the surface $\dot{F}$ along $\partial\dot{F} = \partial F - \mathbf{q}$, where $\mathbf{q}$ is the set of boundary punctures, and double the pullback bundle u^*TW_+ along the real subbundle $(u|_{\partial\dot{F}})^*TL^+_{\mathbf{a}}$ on $\partial\dot{F}$, to obtain the doubled surface $2\dot{F}$ with a complex vector bundle $2u^*TW_+ \to 2\dot{F}$. The boundary punctures q_i of $\dot{F}$ are doubled to give positive interior punctures $2q_i$ on $2\dot{F}$.

Since the trivialization τ of $TS|_{Z_{\mathbf{y},\boldsymbol{\gamma}}}$ was chosen to be tangent to $TL^+_{\mathbf{a}} \cap TS$, its pullback to u^*TW_+ is compatible with the doubling operation and gives a trivialization 2τ of $2u^*TS$ over a neighborhood of the punctures of $2\dot{F}$. Also let τ' be a partially defined trivialization of TB_+ which is given by ∂_s at the positive and negative ends of B_+. Then τ' can be doubled to $2\tau'$ on $2u^*TB_+$ which is defined over a neighborhood of the punctures of $2\dot{F}$.

The linearized $\overline{\partial}$-operator at u splits as a sum $D_u = D^0_u \oplus K_u$, where D^0_u is a Cauchy-Riemann type operator on $W^{1,p}$-sections of u^*TW_+ with exponential weights and K_u is a direct sum of finite-dimensional operators, one for each puncture of $\dot{F}$, with index 2 for interior punctures and index 1 for boundary punctures; see [Dr, Section 2.3] for the definition of the operator K_u. We define $2D_u = 2D^0_u \oplus K'_u$, where K'_u is a finite-dimensional operator and contributes by 2 to the index for each puncture of $2\dot{F}$. Technically speaking, K'_u is not the double of K_u; nevertheless $\text{ind}(K_u) = \frac{1}{2}\text{ind}(K'_u)$. By applying Theorem 5.5.1 to D^0_u, we obtain $\text{ind}(D_u) = \frac{1}{2}\text{ind}(2D_u)$.

We now apply the standard Fredholm index formula (see for example Dragnev [Dr]) for holomorphic curves in symplectizations — with slight modifications — to obtain:

$$\begin{aligned}\text{ind}(2D_u) = &-\chi(2\dot{F}) + \mu_{2\tau}(2\mathbf{y}) - \mu_{2\tau}(2\boldsymbol{\gamma}, 2u)\\ &+ 2c_1(2u^*TB_+, 2\tau') + 2c_1(2u^*TS, 2\tau).\end{aligned} \quad (\mathbf{5.5.2})$$

Here $2\mathbf{y}$ and $2\boldsymbol{\gamma}$ stand for the doubles of $\mathbf{y}$ and $\boldsymbol{\gamma}$. Also $\mu_{2\tau}(2\mathbf{y})$ is the sum of the Conley-Zehnder indices, computed with respect to the trivialization 2τ, of the paths of symplectic matrices arising from the asymptotic operators at $2y_i$. The last term is the relative first Chern class of the double of the pullback bundle u^*TS with respect to 2τ. The one term that is not present in the Fredholm index formula for J-holomorphic curves in a symplectization is the penultimate term $2c_1(2u^*TB_+, 2\tau')$.

We then compute the following:

(a) $\chi(2\dot{F}) = 2\chi(\dot{F}) - 2g$;
(b) $\mu_{2\tau}(2y_i) = 2\mu_\tau(y_i) - 1$;
(c) $\mu_{2\tau}(2\boldsymbol{\gamma}, 2u) = 2\mu_\tau(\boldsymbol{\gamma}, u)$;
(d) $c_1(2u^*TB_+, 2\tau') = -2g$;
(e) $c_1(2u^*TS, 2\tau) = 2c_1(u^*TS, \tau)$.

(a), (c) and (e) are straightforward. (b) will be proved in Lemma 5.5.3 below. (d) By arguing in a manner similar to that of Claim 4.5.10 we have $c_1(2u^*TB_+, 2\tau') = 2g\chi(2B_+) = -2g$.

Summarizing, we obtain:

$$\begin{aligned}\mathrm{ind}_{W_+}(u) &= (-\chi(\dot{F}) + g) + (\mu_\tau(\mathbf{y}) - g) - \mu_\tau(\boldsymbol{\gamma}, u) - 2g + 2c_1(u^*TS, \tau)\\ &= -\chi(\dot{F}) - 2g + \mu_\tau(\mathbf{y}) - \mu_\tau(\boldsymbol{\gamma}, u) + 2c_1(u^*TS, \tau).\end{aligned}$$

This completes the proof of the proposition. □

Lemma **5.5.3.** — $\mu_{2\tau}(2y_i) = 2\mu_\tau(y_i) - 1$.

Proof. — Let $q_i \in \partial F$ be a (positive) boundary puncture and let $\mathcal{E} = [0, \infty) \times [0, 1] \subset \dot{F}$ be a strip-like end with coordinates (s, t) which parametrizes a neighborhood of q_i. Suppose u maps the end asymptotically to y_i. Fix a symplectic trivialization

$$\Theta : u|_{\mathcal{E}}^* TS \xrightarrow{\sim} \mathbf{R}^2 \times \mathcal{E}$$

such that $u|_{\partial\mathcal{E}}^*(TL_{\mathbf{a}}^+ \cap TS)$ corresponds to $(\mathbf{R} \oplus 0) \times \partial\mathcal{E}$. Here $\partial\mathcal{E} = \partial\dot{F} \cap \mathcal{E}$.

Let $2\mathcal{E} = ([0, \infty) \times [0, 2])/(s, 0) \sim (s, 2)$ be the end of the doubled surface $2\dot{F}$, which corresponds to the interior puncture $2q_i$ and is obtained by doubling $\mathcal{E}$. The involution σ is given by $\sigma(s, t) = (s, 2 - t)$ with respect to these coordinates. Similarly, we have a symplectic trivialization

$$2\Theta : 2u|_{\mathcal{E}}^* TS \xrightarrow{\sim} \mathbf{R}^2 \times 2\mathcal{E}$$

and $\tilde{\sigma}$ is given by $\tilde{\sigma}((x_1, x_2), s, t) = ((x_1, -x_2), s, 2 - t)$.

Let $J_0\partial_t + S_i(t)$ be the *asymptotic operator* of D_u^0 corresponding to q_i with respect to the trivialization Θ. Here J_0 is the standard complex structure on $\mathbf{R}^2$ and $S_i(t)$ is a symmetric 2×2 matrix with real coefficients. For the definition of the asymptotic operator and its relation with the Fredholm theory of linearized $\overline{\partial}$-operators, see [Dr, Section 3] or [HT2].[6] The solution of the Cauchy problem

$$\dot{\Phi}(t) = J_0 S_i(t)\Phi(t), \quad \Phi(0) = \mathrm{id}$$

[6] What we call S_i here is written as $C^i_{2\infty}$ in [Dr].

is a path of symplectic matrices which represents the linearized Reeb flow along the chord y_i, expressed with respect to the trivialization τ. In our setting, we may assume that $\Phi(t)$ is a path of unitary matrices. By identifying $(\mathbf{R}^2, J_0) = (\mathbf{C}, i)$, we write $\Phi(t) = e^{i\alpha(t)}$ for some function $\alpha : [0, 1] \to \mathbf{R}$. Then

$$\mu_\tau(y_i) = \lfloor\!\lfloor \tfrac{\alpha(1)-\alpha(0)}{\pi} \rfloor\!\rfloor + 1,$$

where $\lfloor\!\lfloor x \rfloor\!\rfloor$ is the greatest integer $\leq x$.

The double of the asymptotic operator, i.e., the asymptotic operator of the doubled operator $2D^0_u$ at the interior puncture $2q_i$, can be written as $J_0\partial_t + \widetilde{S}_i(t)$, where:

$$\widetilde{S}_i(t) = \begin{cases} S_i(t), & t \in [0, 1]; \\ CS_i(2-t)C^{-1}, & t \in [1, 2], \end{cases}$$

and $C = \mathrm{diag}(1, -1)$. The solution of the corresponding Cauchy problem is

$$\widetilde{\Phi}(t) = \begin{cases} \Phi(t), & t \in [0, 1]; \\ C\Phi(2-t)\Phi(1)^{-1}C^{-1}\Phi(1), & t \in [1, 2]. \end{cases}$$

Hence can write $\widetilde{\Phi}(t) = e^{i\tilde{\alpha}(t)}$, where $\tilde{\alpha} : [0, 2] \to \mathbf{R}$ is given by:

$$\tilde{\alpha}(t) = \begin{cases} \alpha(t), & t \in [0, 1]; \\ -\alpha(2-t) + 2\alpha(1), & t \in [1, 2]. \end{cases}$$

The Conley-Zehnder index of the path $\widetilde{\Phi}$ is

$$\mu_{2\tau}(2y_i) = 2 \lfloor\!\lfloor \tfrac{\tilde{\alpha}(2)-\tilde{\alpha}(0)}{2\pi} \rfloor\!\rfloor + 1.$$

Since

$$\begin{aligned} \lfloor\!\lfloor \tfrac{\tilde{\alpha}(2)-\tilde{\alpha}(0)}{2\pi} \rfloor\!\rfloor &= \lfloor\!\lfloor \tfrac{(\tilde{\alpha}(2)-\tilde{\alpha}(1))+(\tilde{\alpha}(1)-\tilde{\alpha}(0))}{2\pi} \rfloor\!\rfloor \\ &= \lfloor\!\lfloor \tfrac{2(\tilde{\alpha}(1)-\tilde{\alpha}(0))}{2\pi} \rfloor\!\rfloor = \lfloor\!\lfloor \tfrac{\alpha(1)-\alpha(0)}{\pi} \rfloor\!\rfloor, \end{aligned}$$

we obtain $\mu_{2\tau}(2y_i) = 2\mu_\tau(y_i) - 1$, as desired. □

5.5.3. *The* $\overline{W}_-$ *case.* — The trivialization τ of $T\overline{S}_{\overline{W}_-}$ is defined on $Z_{\boldsymbol{\gamma},\mathbf{y}}$ in a manner similar to that of W_+, except that the positive and negative ends are reversed. The definition of the real rank one subbundle $\mathcal{L}$ of $T\overline{S}$ along $(L^-_{\hat{\mathbf{a}}} \cap \check{\overline{W}}_-) \cup (\{-3\} \times [0, 1] \times \mathbf{y})$ is also similar and will be omitted.

If $\overline{u}$ is a $\overline{W}_-$-curve from $\boldsymbol{\gamma}$ to $\mathbf{y}$, then the Fredholm index $\mathrm{ind}_{\overline{W}_-}(\overline{u})$ is defined as the expected dimension of the moduli space $\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y})$ near $\overline{u}$.

Remark **5.5.4.** — The expected dimension of $\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}})$ is $\mathrm{ind}_{\overline{W}_-}(\overline{u}) - 2$.

We then have the following Fredholm index formula:

Proposition **5.5.5** *(Fredholm index formula for* $\overline{W}_-$*-curves). — The Fredholm index of a* $(\overline{W}_-,\overline{J}_-)$*-curve* $\overline{u}:(\dot{F},j)\to(\overline{W}_-,\overline{J}_-)$ *from* $\boldsymbol{\gamma}$ *to* $\mathbf{y}$ *is given by the formula:*

$$\operatorname{ind}_{\overline{W}_-}(\overline{u}) = -\chi(\dot{F}) + \mu_\tau(\boldsymbol{\gamma},\overline{u}) - \mu_\tau(\mathbf{y}) + 2c_1(\overline{u}^*T\overline{S},\tau). \quad (\mathbf{5.5.3})$$

Proof. — By a calculation similar to that of Proposition 5.5.2, we obtain

$$\operatorname{ind}_{\overline{W}_-}(\overline{u}) = (-\chi(\dot{F})+g) + \mu_\tau(\boldsymbol{\gamma},\overline{u}) - (\mu_\tau(\mathbf{y})-g) - 2g + 2c_1(\overline{u}^*T\overline{S},\tau),$$

which simplifies to the desired result. □

Remark **5.5.6** *(Reason for considering* $\overline{W}_-$*).* — We will give a rough explanation of the reason for considering holomorphic curves in $\overline{W}_-$ which pass through $\overline{\mathfrak{m}}$. Suppose we have a W_+-curve u from $\mathbf{y}$ to $\boldsymbol{\gamma}$ and a W_--curve v from $\boldsymbol{\gamma}$ to $\mathbf{y}$ (i.e., $\operatorname{Im}(v)\subset W_-$). Then, by taking the sum of Equations (5.5.1) and (5.5.3), we compute the Fredholm index of the glued curve $u\#v$, corresponding to the stacking of W_+ at the top $(s>0)$ and W_- at the bottom $(s<0)$, to be:

$$\operatorname{ind}(u\#v) = -\chi(\dot{F}) + 2c_1((u\#v)^*TS,\tau) - 2g,$$

where $\dot{F}$ is obtained from gluing the domains of u and v. The stacking gives rise to a chain map $\widehat{CF}\to\widehat{CF}$, which we expect to map $\mathbf{y}\mapsto\mathbf{y}$ via restrictions of trivial cylinders (modulo chain homotopy). This would mean $\chi(\dot{F})=0$ and $c_1((u\#v)^*TS,\tau)=0$. This leaves us with a deficiency of $2g$. Introducing the point constraint at $\overline{\mathfrak{m}}$, from the perspective of Fredholm indices, is basically equivalent to applying a multiple connected sum to the holomorphic curve v and the fiber $\overline{S}$ which passes through $\overline{\mathfrak{m}}$. The multiple connected sum is performed at the $2g$ intersection points between v and $\overline{S}$. We effectively increase the Fredholm index by $2g+2$, obtained by adding up the following contributions:

(i) the Fredholm index of the fiber $\overline{S}$, which is $\chi(\overline{S})=2-2g$;
(ii) $2g$ intersection points, each of which contributes $+2$ to minus the Euler characteristic.

The point constraint then cuts the expected dimension of the moduli space by -2, for a net gain of $2g$.

5.6. *The ECH index.* — In this section we present the ECH index I_{W_+}, the relative adjunction formula, and the ECH index inequality for W_+. The situation for $\overline{W}_-$ is analogous, and will not be discussed explicitly, except to point out some differences.

5.6.1. *Definitions.* — Let $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ and $\boldsymbol{\gamma} = \prod_{j=1}^{l} \gamma_j^{m_j} \in \widehat{\mathcal{O}}_{2g}$. Let τ be a trivialization of $TS_{\check{W}_+}$ along $Z_{\mathbf{y},\boldsymbol{\gamma}}$, as given in Section 5.5.2. Using τ, for each simple orbit γ_j of $\boldsymbol{\gamma}$, we choose an identification of a sufficiently small neighborhood $\nu(\gamma_j)$ of γ_j with $\gamma_j \times D^2$, where D^2 has polar coordinates (r, θ).

Definition **5.6.1** *(τ-trivial representative).* — *A* τ-trivial representative $\check{C}$ *of* $A \in H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$ *is an oriented immersed compact surface in the class* A *which satisfies the following:*

(1) $\check{C}$ *is embedded on* $\check{W}_+ - \{s = -1\}$*;*
(2) $\check{C}$ *is positively transverse to the fibers* $\{(s, t)\} \times \overline{S}$ *along all of* $\partial\check{C}$*;*
(3) $\check{C}$ *is* τ*-trivial in the sense of Definition 4.5.2 at the HF end;*
(4) $\check{C}$ *is* τ-trivial *at the ECH end, i.e., for all sufficiently small* $\varepsilon > 0$, $\check{C} \cap \{s = -1 + \varepsilon\}$ *consists of* m_j *disjoint circles* $\{r = \varepsilon, \theta = \theta_{ji}\}$, $i = 1, \dots, m_j$, *in* $\nu(\gamma_j)$ *for all* j*. (See [Hu1, Definition 2.3].)*

Let $\partial_+\check{C} = \partial\check{C} \cap \{s > 0\}$ and $\partial_-\check{C} = \partial\check{C} \cap \{s < 0\}$.

Definition **5.6.2** *(Relative intersection form).* — *Let* $A \in H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$ *be a homology class which is realized by a* τ*-trivial representative* $\check{C}$*. Then the relative intersection form* $Q_\tau(A)$ *is given by* $\langle \check{C}, \check{C}' \rangle$, *where* $\check{C}'$ *is a pushoff of* $\check{C}$ *which satisfies the following:*

(1) $\check{C}$ *is pushed off in the* $J_+\tau$*-direction along* $\partial_+\check{C}$*; and*
(2) *for small* $\varepsilon > 0$, $\check{C}' \cap \{s = -1 + \varepsilon\}$ *consists of* m_j *disjoint circles* $\{r = \varepsilon, \theta = \theta_{ji} + \varepsilon'\}$, $i = 1, \dots, m_j$, *in* $\nu(\gamma_j)$ *for all* j, *where* $\varepsilon' > 0$ *is a sufficiently small constant.*

(See [Hu1, Definition 2.4].)

5.6.2. *Relative adjunction formula.* — Let $u : \dot{F} \to W_+$ be a W_+-curve and let $\check{u} : \check{F} \to \check{W}_+$ be its compactification. Then we write $w_\tau^-(u)$ for the total writhe of the braids $u(\dot{F}) \cap \{s = s_0\}$, $s_0 \ll 0$, with respect to τ. Similarly, if $\overline{u} : \dot{F} \to \overline{W}_-$ is a $\overline{W}_-$-curve, then we write $w_\tau^+(\overline{u})$ for the total writhe of the braids $\overline{u}(\dot{F}) \cap \{s = s_0\}$, $s_0 \gg 0$, with respect to τ.

Lemma **5.6.3** *(Relative adjunction formula).* — *Let* $u : \dot{F} \to W_+$ *be a* W_+*-curve in the homology class* $A \in H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$*. Then*

$$c_1(u^*TW_+, (\tau, \partial_t)) = \chi(\dot{F}) - w_\tau^-(u) + Q_\tau(A) - 2\delta(u), \tag{5.6.1}$$

where ∂_t *trivializes* TB_+.

For a $\overline{W}_-$-curve $\overline{u}$, we replace $-w_\tau^-(u)$ by $w_\tau^+(\overline{u})$.

Proof. — Suppose u is immersed. If ν is the normal bundle of u, then we have the formula:

$$c_1(u^*TW_+, (\tau, \partial_t)) = c_1(T\check{F}, \partial_t) + c_1(\nu, \tau). \tag{5.6.2}$$

In the general case, we combine the calculations of Lemma 4.5.8 and [Hu1, Proposition 3.1] to obtain the equation

$$c_1(u^*TW_+, (\tau, \partial_t)) = c_1(T\check{F}, \partial_t) - w_\tau^-(u) + Q_\tau(A) - 2\delta(u).$$

The analog of Claim 4.5.10 for the present situation is

$$c_1(T\check{F}, \partial_t) = \chi(\dot{F}). \tag{5.6.3}$$

The difference in the shape of the base (i.e., B vs. B_+) accounts for the discrepancy between Equation (5.6.3) and Claim 4.5.10. □

Remark **5.6.4.** — Since $c_1(u^*TB_+, \partial_t) = 0$, we have

$$c_1(u^*TW_+, (\tau, \partial_t)) = c_1(u^*TS, \tau) + c_1(u^*TB_+, \partial_t) = c_1(u^*TS, \tau).$$

5.6.3. *ECH index.* — We now define the ECH indices for W_+ and $\overline{W}_-$.

Definition **5.6.5** *(ECH index for* W_+*). — Given a class* $A \in H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$ *which admits a* τ*-trivial representative* $\check{C}$*, we define*

$$I_{W_+}(A) = c_1(T\check{W}_+|_A, (\tau, \partial_t)) + Q_\tau(A) + \mu_\tau(\mathbf{y}) - \widetilde{\mu}_\tau(\boldsymbol{\gamma}) - 2g, \tag{5.6.4}$$

where $\widetilde{\mu}_\tau(\boldsymbol{\gamma})$ *is the symmetric Conley-Zehnder index at the negative (ECH) end.*

Definition **5.6.6** *(ECH index for* $\overline{W}_-$*). — Given a class* $A \in H_2(\overline{W}_-, \boldsymbol{\gamma}, \mathbf{y})$ *which admits a* τ*-trivial representative* $\check{C}$*, we define*

$$I_{\overline{W}_-}(A) = c_1(T\check{\overline{W}}_-|_A, (\tau, \partial_t)) + Q_\tau(A) + \widetilde{\mu}_\tau(\boldsymbol{\gamma}) - \mu_\tau(\mathbf{y}). \tag{5.6.5}$$

As usual, the ECH indices $I_{W_+}(A)$ and $I_{\overline{W}_-}(A)$ are independent of the choice of trivialization τ.

Remark **5.6.7.** — To obtain a finite count of $\overline{W}_-$-curves which pass through the point $\overline{\mathfrak{m}}$, we count curves $\overline{u}$ with ECH index $I_{\overline{W}_-}(\overline{u}) = 2$.

5.6.4. *Additivity of indices.*

Lemma **5.6.8** *(Additivity of indices). — If* $u \in \mathcal{M}_J(\mathbf{y}, \mathbf{y}')$, $v \in \mathcal{M}_{J_+}(\mathbf{y}', \boldsymbol{\gamma})$*, and* $u\#v$ *is a pre-glued curve, then*

$$\mathrm{ind}_{W_+}(u\#v) = \mathrm{ind}_{HF}(u) + \mathrm{ind}_{W_+}(v), \tag{5.6.6}$$

$$I_{W_+}(u\#v) = I_{HF}(u) + I_{W_+}(v). \tag{5.6.7}$$

Similarly, if $u \in \mathcal{M}_{J_+}(\mathbf{y}, \boldsymbol{\gamma})$, $v \in \mathcal{M}_{J'}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$, *and* $u\#v$ *is a pre-glued curve, then*

$$\text{ind}_{W_+}(u\#v) = \text{ind}_{W_+}(u) + \text{ind}_{ECH}(v), \tag{5.6.8}$$

$$I_{W_+}(u\#v) = I_{W_+}(u) + I_{ECH}(v). \tag{5.6.9}$$

Proof. — The additivity for ind is well-known and the additivity for I is immediate from the definitions. □

5.6.5. *Index inequality.* — Although we will not define it here, given an integer $m_k > 0$ and a simple orbit γ_k, we can define the *incoming partition* $P^{in}_{\gamma_k}(m_k)$ and the *outgoing partition* $P^{out}_{\gamma_k}(m_k)$ as in [Hu2, Definition 4.14].

We have the following index inequality, which is analogous to [Hu1, Theorem 1.7] (also see [Hu2, Theorem 4.15] which is applicable to symplectic cobordisms). Note that a W_+-curve is automatically simply-covered.

Theorem **5.6.9** *(Index inequality). — Let* u *be a* W_+*-curve from* $\mathbf{y}$ *to* $\boldsymbol{\gamma} = \prod_{k=1}^{l} \gamma_k^{m_k}$. *If the negative ends of* u *partition* m_k *into* $(m_{k1}, m_{k2}, \dots)$, *then*

$$\text{ind}_{W_+}(u) + 2\delta(u) \leq I_{W_+}(u).$$

Moreover, if equality holds, then $P^{in}_{\gamma_k}(m_k) = (m_{k1}, m_{k2}, \dots)$ *for all* k. *Similarly, if* $\overline{u}$ *is a* $\overline{W}_-$*-curve from* $\boldsymbol{\gamma} = \prod_{k=1}^{l} \gamma_k^{m_k}$ *to* $\mathbf{y}$ *and the positive ends of* u *partition* m_k *into* $(m_{k1}, m_{k2}, \dots)$, *then*

$$\text{ind}_{W_-}(\overline{u}) + 2\delta(\overline{u}) \leq I_{W_-}(\overline{u}).$$

Moreover, if equality holds, then $P^{out}_{\gamma_k}(m_k) = (m_{k1}, m_{k2}, \dots)$ *for all* k.

Here $\delta(u)$ is the signed count of interior singularities of u, where each singularity contributes positively to $\delta(u)$. There are no boundary singularities because $\pi_{B_+} \circ u$ has no branch point at the boundary.

The following are immediate: (i) if $\text{ind}_{W_+}(u) = I_{W_+}(u)$, then u is embedded; and (ii) if $I_{W_+}(u) = 0$ or 1, then u is embedded.

Proof. — If we plug Equation (5.5.1), Equation (5.6.4), and the relative adjunction formula (5.6.1) into $I_{W_+}(u) - \text{ind}_{W_+}(u)$, we obtain:

$$I_{W_+}(u) - \text{ind}_{W_+}(u) = \mu_\tau(\boldsymbol{\gamma}, u) + w^-_\tau(u) - \widetilde{\mu}_\tau(\boldsymbol{\gamma}) + 2\delta(u).$$

The statement for W_+ then follows from the writhe inequality

$$w^-_\tau(u) \geq \widetilde{\mu}_\tau(\boldsymbol{\gamma}) - \mu_\tau(\boldsymbol{\gamma}, u),$$

where equality holds if and only if the partition $(m_{k1}, m_{k2}, \dots)$ of the negative end of u coincides with the incoming partition $P^{in}_{\gamma_k}(m_k)$. (See [Hu2, Lemma 4.20].) The proof for $\overline{W}_-$ is similar. □

5.7. *Holomorphic curves with ends at z_∞.* — In this subsection we explain how to extend the definitions of the Fredholm and ECH indices to holomorphic curves which have ends at multiples of z_∞. The novelty is that the Lagrangian boundary condition is singular at z_∞ and that the chord over z_∞ can be used more than once. We will treat in detail the case of a curve $\overline{u}: \dot{F} \to \overline{W}$ which is a multisection of $\overline{W} \to \mathbf{R} \times [0,1]$; multisections of $\overline{W}_+$ and $\overline{W}_-$ can be treated similarly.

5.7.1. *Data at z_∞^p.* — We define the *data* $\overrightarrow{\mathcal{D}}$ *at* z_∞^p as a p-tuple of matchings

$$\{(i'_1, j'_1) \to (i_1, j_1), \dots, (i'_p, j'_p) \to (i_p, j_p)\},$$

where $i_\ell, i'_\ell \in \{1, \dots, 2g\}, j_\ell, j'_\ell \in \{0,1\}$ for $\ell = 1, \dots, p$ and $i_\ell \neq i_{\ell'}$, $i'_\ell \neq i'_{\ell'}$ for $\ell \neq \ell'$. To the data $\overrightarrow{\mathcal{D}}$ we associate its set of *initial points* $\mathcal{D}^{from} = \{(i'_1, j'_1), \dots, (i'_p, j'_p)\}$ and its set of *terminal points* $\mathcal{D}^{to} = \{(i_1, j_1), \dots, (i_p, j_p)\}$, and define $\mathcal{D} = (\mathcal{D}^{to}, \mathcal{D}^{from})$.

We write $\mathbf{z} = \{z_\infty^p(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$ for a tuple of points of $\overline{\mathbf{a}} \cap \overline{h}(\overline{\mathbf{a}})$, where $\mathbf{y} \subset S$, z_∞ has multiplicity p, $\overrightarrow{\mathcal{D}}$ is the data at z_∞^p, and each arc of $\{\overline{a}_i, \overline{h}(\overline{a}_i)\}_{i=1}^{2g}$ is used at most once. Here $z_\infty^p(\overrightarrow{\mathcal{D}})$ is viewed as a collection of chords z_∞ from $\overline{h}(\overline{a}_{i'_\ell, j'_\ell})$ to $\overline{a}_{i_\ell, j_\ell}$, where $\ell = 1, \dots, p$.

5.7.2. *Multisections.* — In this subsection we define multisections $\overline{u}: (\dot{F}, j) \to (\overline{W}, \overline{J})$ with irreducible components which branched cover σ_∞. The notation is complicated by the fact that there may be branch points of $\overline{\pi}_B \circ \overline{u}$ along ∂B.

For the rest of Section 5.7, (F, j) will be a compact *nodal* Riemann surface, possibly disconnected, with two sets of punctures $\mathbf{q}^+ = \{q_1^+, \dots, q_\ell^+\}$ and $\mathbf{q}^- = \{q_1^-, \dots, q_\ell^-\}$ on ∂F and a set $\mathbf{r} = \{r_1, \dots, r_l\}$ of nodes on ∂F, such that

(i) the nodes $\mathbf{r}$ are disjoint from $\mathbf{q}^+$ and $\mathbf{q}^-$;
(ii) each connected component of F has nonempty boundary;
(iii) on each oriented loop of ∂F whose orientation agrees with that of ∂F, there is at least one puncture from each of $\mathbf{q}^+$ and $\mathbf{q}^-$; and
(iv) the punctures on $\mathbf{q}^+$ and $\mathbf{q}^-$ alternate along each oriented loop of ∂F.

Here, by a *loop or path on* ∂F we mean a loop or path on ∂F which is a concatenation of subarcs of ∂F with endpoints on $\mathbf{q}^+ \cup \mathbf{q}^- \cup \mathbf{r}$. An oriented loop (or path) is a loop (or path) formed from subarcs of ∂F which all point in the same direction, when oriented as the boundary of F.

We write $\dot{F} = F - \mathbf{q}^+ - \mathbf{q}^-$. Given a holomorphic multisection $\overline{u}: (\dot{F}, j) \to (\overline{W}, \overline{J})$, we decompose $F = F' \sqcup F''$ such that

– $\overline{u}' = \overline{u}|_{\dot{F}'}$ is a possibly disconnected branched cover of σ_∞ and
– $\overline{u}'' = \overline{u}|_{\dot{F}''}$ is the union of irreducible components which do not branch cover σ_∞.

By abuse of notation, we write

$$\overline{u} = \overline{u}' \cup \overline{u}''. \qquad (\mathbf{5.7.1})$$

We additionally assume the following:

(v) all the nodes of ∂F lie on $\partial F'$.

Observe that the nodes of $\partial F'$ correspond to branch points of $\overline{\pi}_B \circ \overline{u}$ along ∂B. If $\dot{F}'$ is a Riemann surface that branched covers B with some branch points along ∂B, then let $\dot{F}'_{ext}$ be the Riemann surface that branched covers B, such that the branch points on ∂B have been pushed into $int(B)$. The cases of $\overline{W}_+$ and $\overline{W}_-$ are analogous.

The following is the local model near a branch point along the boundary: Consider the map $\mathbf{C} \to \mathbf{C}$, $z \mapsto z^n$. If we write $z = re^{i\theta}$, then the preimage of the upper half plane is a union of sectors $\theta \in \left[\frac{2k\pi}{n}, \frac{(2k+1)\pi}{n}\right]$ for $k = 0, \dots, n-1$. The local model of the modification from F' to $\dot{F}'_{ext}$ is obtained by going from $z \mapsto z^n$ to $z \mapsto z^n + i\varepsilon$ for $\varepsilon > 0$ small, and taking the preimage of the upper half plane.

Definition **5.7.1** *(Data $\mathcal{C}$). — A* maximal path d of $\partial\dot{F}'$ *is an oriented connected path from* $\mathbf{q}^+$ *to* $\mathbf{q}^-$ *along* $\partial\dot{F}'$*, where the orientation either agrees with that of* ∂F *on all the subarcs of* d *or is opposite that of* ∂F *on all the subarcs of* d*. A set* $\{d_1, \dots, d_r\}$ *is a* decomposition of $\partial\dot{F}'$ into maximal paths, *if* $\partial\dot{F}' = \cup_{i=1}^r d_i$*,* d_i *is a maximal path of* $\partial\dot{F}'$*, and* d_i*,* d_j*,* $i \neq j$*, intersect only at points of* $\mathbf{q}^+ \cup \mathbf{q}^- \cup \mathbf{r}$*. The* canonical decomposition $\Delta = \{d_1, \dots, d_r\}$ of $\partial\dot{F}'$ *is the decomposition which corresponds to* $\partial\dot{F}'_{ext}$*. A* data $\mathcal{C}$ *is a map*

$$\Delta \to \{\overline{a}_{i,j}, \overline{h}(\overline{a}_{i,j})\}_{i,j}.$$

In words, $\overline{u}'$ can be viewed as mapping Δ to the set of Lagrangians $L_{\overline{a}_{i,j}}$ or $L_{\overline{h}(\overline{a}_{i,j})}$.

We then make the following definition which agrees with Definition 4.3.1 when $\mathbf{z}_+ = \mathbf{y}_+$ and $\mathbf{z}_- = \mathbf{y}_-$:

Definition **5.7.2.** — *Let* $\mathbf{z}_+ = \{z_\infty^{p_+}(\overrightarrow{\mathcal{D}}_+)\} \cup \mathbf{y}_+$ *and* $\mathbf{z}_- = \{z_\infty^{p_-}(\overrightarrow{\mathcal{D}}_-)\} \cup \mathbf{y}_-$ *be k-tuples of points of* $\overline{a} \cap \overline{h}(\overline{a})$*, where* z_∞ *is counted with multiplicities* p_+*,* p_-*. A* degree k multisection of $(\overline{W}, \overline{J})$ from $\mathbf{z}_+$ to $\mathbf{z}_-$ *is a pair* $(\overline{u}, \mathcal{C})$ *consisting of a holomorphic map*

$$\overline{u} : (\dot{F}, j) \to (\overline{W}, \overline{J}),$$

where $\overline{u}$ *is a degree k multisection of* $\overline{\pi}_B : \overline{W} \to B$ *and there are decompositions* $\dot{F} = \dot{F}' \sqcup \dot{F}''$*,* $\overline{u} = \overline{u}' \cup \overline{u}''$ *as above, and the data* $\mathcal{C}$ *for* $\overline{u}'$*, and which additionally satisfies the following:*

(1) $\overline{u}''(\partial\dot{F}'') \subset L_{\widehat{\mathbf{a}}} \cup L_{\overline{h}(\widehat{\mathbf{a}})}$;
(2) *there is a canonical decomposition* Δ *of* $\partial\dot{F}'$ *such that* $\overline{u}$ *maps each connected component of* $\partial\dot{F}''$ *and each irreducible component of* Δ *to a different* $L_{\overline{a}_i}$ *or* $L_{\overline{h}(\overline{a}_i)}$ *(here we are using* $\mathcal{C}$ *to assign some* $L_{\overline{a}_i}$ *or* $L_{\overline{h}(\overline{a}_i)}$ *to each irreducible component of* Δ*);*

(3) $\lim_{w\to q_i^+} \overline{\pi}_{\mathbf{R}} \circ \overline{u}(w) = +\infty$ *and* $\lim_{w\to q_i^-} \overline{\pi}_{\mathbf{R}} \circ \overline{u}(w) = -\infty$*;*

(4) $\overline{u}$ *converges to a trivial strip over* $[0,1]\times \mathbf{z}_+$ *near* $\mathbf{q}^+$ *and to a trivial strip over* $[0,1]\times \mathbf{z}_-$ *near* $\mathbf{q}^-$*;*

(5) *the positive and negative ends of* $\overline{u}$ *which limit to* z_∞ *are described by* $\overrightarrow{\mathcal{D}}_+$ *and* $\overrightarrow{\mathcal{D}}_-$*;*

(6) *the energy of* $\overline{u}$ *is finite.*

Remark **5.7.3.** — It will always be assumed that a multisection $\overline{u}$ of $\overline{W}$ from $\{z_\infty^{p_+}\}\cup \mathbf{y}_+$ to $\{z_\infty^{p_-}\}\cup \mathbf{y}_-$ comes with data $\mathcal{C}$. Indeed, in this paper, such a curve $\overline{u}$ only appears as the SFT limit of curves $\overline{u}_i$ without components which branch cover σ_∞ and hence naturally inherits $\mathcal{C}$ and $\overrightarrow{\mathcal{D}}_\pm$.

5.7.3. *Multivalued trivializations.* — We define a class of trivializations $\tau=\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ which will be used in the definition of the Fredholm index of a map $\overline{u}$ from $\{z_\infty^{p_+}(\overrightarrow{\mathcal{D}}_+)\}\cup \mathbf{y}_+$ to $\{z_\infty^{p_-}(\overrightarrow{\mathcal{D}}_-)\}\cup \mathbf{y}_-$. The trivialization $\tau=\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ is *multivalued* near z_∞ and depends on $\overrightarrow{\mathcal{D}}_+$ and $\overrightarrow{\mathcal{D}}_-$.

Let $\tau'=\partial_\rho$ be a trivialization of $T\overline{S}$ on $D^2_\delta-\{z_\infty\}=\{0<\rho\le\delta\}\subset\overline{S}$, where $\delta>0$ is small. Let $\check{\overline{W}}=[-1,1]\times[0,1]\times\overline{S}$. We extend τ' arbitrarily to a trivialization of $T\overline{S}$ along $\overline{\mathbf{a}}$ and pull it back to $\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ on $T\overline{S}_{\check{\overline{W}}}$ along $[-1,1]\times\{1\}\times\overline{\mathbf{a}}$; similarly we extend τ' arbitrarily to a trivialization of $T\overline{S}$ along $\overline{h}(\overline{\mathbf{a}})$ and pull it back to $\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ on $T\overline{S}_{\check{\overline{W}}}$ along $[-1,1]\times\{0\}\times\overline{h}(\overline{\mathbf{a}})$. Then we extend $\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ along $(\{1\}\times[0,1]\times\mathbf{y}_+)\cup(\{-1\}\times[0,1]\times\mathbf{y}_-)$, while keeping the same notation. Finally, for each $(i'_{\pm,\ell},j'_{\pm,\ell})\to(i_{\pm,\ell},j_{\pm,\ell})$ in $\overrightarrow{\mathcal{D}}_\pm$, we choose an extension $\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ to $\{\pm1\}\times[0,1]\times\{z_\infty\}$ in an arbitrary manner.

A multivalued trivialization $\tau=\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ as constructed in the previous paragraph is said to be *compatible with* $(\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-)$. We also say that τ is *compatible with* $(\mathcal{D}_+,\mathcal{D}_-)$ if it is compatible with some $(\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-)$ whose initial/terminal point sets are $(\mathcal{D}_+,\mathcal{D}_-)$.

Remark **5.7.4.** — Note that the extensions to $\overline{\mathbf{a}}$ and to $\overline{h}(\overline{\mathbf{a}})$ might conflict, but it does not matter here. In the cobordism cases (i.e., for $\overline{W}_+$ and $\overline{W}_-$), when $\overline{\mathbf{a}}$ and $\overline{h}(\overline{\mathbf{a}})$ are connected by the Lagrangian $L^\pm_{\overline{\mathbf{a}}}$, we extend the trivialization τ' to $\overline{\mathbf{a}}$ in an arbitrary manner and then sweep it around using the symplectic connection along the boundary of the cobordism.

5.7.4. *Groomed multivalued trivializations.* — Let τ be a multivalued trivialization. Let

$$A_\varepsilon=[0,1]\times\partial D^2_\varepsilon\subset[0,1]\times\overline{S},$$

where $\varepsilon > 0$ is small. We use coordinates (t, ϕ) on

$$A_\varepsilon \simeq ([0,1] \times [0, 2\pi])/(t,0) \sim (t, 2\pi).$$

The branches of the trivialization τ along $[0,1] \times \{z_\infty\}$ can be used to push off $[0,1] \times \{z_\infty\}$ into a collection of arcs $c_\ell^\pm$, $\ell = 1, \dots, p_\pm$, in A_ε, subject to some conditions: If we denote the initial point of $c_\ell^\pm$ by $p_{\ell,0}^\pm$ and the terminal point by $p_{\ell,1}^\pm$, then $p_{\ell,0}^\pm$ lies on $\{0\} \times \overline{h}(\overline{a}_{i'_{\pm,\ell} j'_{\pm,\ell}})$ and $p_{\ell,1}^\pm$ lies on $\{1\} \times \overline{a}_{i_{\pm,\ell} j_{\pm,\ell}}$. We also assume that each $c_\ell^\pm$ is linear with respect to the identification of the universal cover of A_ε with $[0,1] \times \mathbf{R}$.

We then write

$$P_0^\pm = \{p_{\ell,0}^\pm\}_{\ell=1}^{p_\pm} \quad \text{and} \quad P_1^\pm = \{p_{\ell,1}^\pm\}_{\ell=1}^{p_\pm}.$$

Definition **5.7.5.** — *The multivalued trivialization* τ *is* groomed *if the arcs* $\{c_\ell^*\}_{\ell=1}^{p_*}$ *are pairwise disjoint for both* $* = +$ *and* $* = -$. *The collections* $\mathfrak{c}^+ = \{c_\ell^+\}_{\ell=1}^{p_+}$ *and* $\mathfrak{c}^- = \{c_\ell^-\}_{\ell=1}^{p_-}$ *are the* groomings *at the positive and negative ends.*

Note that every groomed multivalued trivialization induces data $\overrightarrow{\mathcal{D}}_\pm$, but not every $\overrightarrow{\mathcal{D}}_\pm$ is compatible with a groomed multivalued trivialization.

5.7.5. *Index formulas.* — We first give the Fredholm index of a multisection.

Proposition **5.7.6.** — *Let* $\overline{u} : \dot{F} \to \overline{W}$ *be a degree* k *multisection with data* $\overrightarrow{\mathcal{D}}_\pm$ *at* z_∞*, and let* $\tau = \tau_{\overrightarrow{\mathcal{D}}_+, \overrightarrow{\mathcal{D}}_-}$. *Then the Fredholm index of* $\overline{u}$ *is given by the formula:*

$$\text{ind}(\overline{u}) = -\chi(F) + k + \mu_\tau(\overline{u}) + 2c_1(\overline{u}^* T\overline{S}, \tau). \tag{5.7.2}$$

The proof is a computation in the pullback bundle $\overline{u}^* T\overline{W}$, which is completely analogous to that yielding Equation (4.4.4).

Next we discuss the ECH index of a multisection $(\overline{u}, \mathcal{C})$ with ends

$$\mathbf{z}_+ = \{z_\infty^{p_+}(\overrightarrow{\mathcal{D}}_+)\} \cup \mathbf{y}_+, \quad \mathbf{z}_- = \{z_\infty^{p_-}(\overrightarrow{\mathcal{D}}_-)\} \cup \mathbf{y}_-.$$

Let $Z_{\mathbf{z}_+, \mathbf{z}_-}$ be the subset of $\check{\overline{W}}$ given by

$$Z_{\mathbf{z}_+, \mathbf{z}_-} = \check{L}_{\widehat{\mathbf{a}}} \cup \check{L}_{\overline{h}(\widehat{\mathbf{a}})} \cup (\{1\} \times [0,1] \times \mathbf{z}_+) \cup (\{-1\} \times [0,1] \times \mathbf{z}_-),$$

where we are viewing $\mathbf{z}_\pm$ as sets of points (i.e., we are forgetting the data $\overrightarrow{\mathcal{D}}_\pm$),

$$\check{L}_{\widehat{\mathbf{a}}} = [-1,1] \times \{1\} \times \widehat{\mathbf{a}}, \quad \text{and} \quad \check{L}_{\overline{h}(\widehat{\mathbf{a}})} = [-1,1] \times \{0\} \times \overline{h}(\widehat{\mathbf{a}}).$$

We then define $H_2(\overline{W}, \mathbf{z}_+, \mathbf{z}_-) \subset H_2(\check{\overline{W}}, Z_{\mathbf{z}_+, \mathbf{z}_-})$ in a manner analogous to that of $H_2(X, \mathbf{y}, \mathbf{y}')$ in Section 4.3. Observe that $H_2(\overline{W}, \mathbf{z}_+, \mathbf{z}_-)$ and $Z_{\mathbf{z}_+, \mathbf{z}_-}$ depend on the endpoints $\mathcal{D}_\pm$ but not on the matchings in $\overrightarrow{\mathcal{D}}_\pm$.

τ-trivial representative. Let τ be a groomed multivalued trivialization which is compatible with $(\mathcal{D}_+, \mathcal{D}_-)$ and let $\mathfrak{c}^\pm = \{c_\ell^\pm\}$ be the groomings of τ. *The matchings defined by $\mathfrak{c}^\pm$ do not need to coincide with the matchings in $\overrightarrow{\mathcal{D}}_\pm$; only their endpoints do.* The construction of a τ-trivial representative $\check{C} \subset \check{\overline{W}}$ of $[\overline{u}] \in H_2(\overline{W}, \mathbf{z}_+, \mathbf{z}_-)$ is as follows:
Step 1. Replace $\overline{u}' : \dot{F}' \to \overline{W}$ by $\overline{u}'_{ext} : \dot{F}'_{ext} \to \overline{W}$ so that there are no nodes along $\partial \dot{F}_{ext}$ and each component of $\partial \dot{F}_{ext}$ is mapped to some $L_{\overline{a}_{i,j}}$ or $L_{\overline{h}(\overline{a}_{i,j})}$ in a manner consistent with the data $\mathcal{C} : \Delta \to \{\overline{a}_{i,j}, \overline{h}(\overline{a}_{i,j})\}_{i,j}$.
Step 2. Compactify the ends of $\overline{u}_{ext} = \overline{u}'_{ext} \cup \overline{u}''$ to $\check{\overline{u}}_{ext}$ as in Section 4.3.
Step 3. Perturb $\check{\overline{u}}_{ext}$ so that the resulting representative $\check{C}$ is immersed, $\partial \check{C} \subset Z_{\mathbf{z}_+, \mathbf{z}_-, \tau}$, each component of $\partial \check{C} \cap \{t = 0, 1\}$ is mapped to $\check{L}_{\widehat{a}_{i,j}}$ or $\check{L}_{\overline{h}(\widehat{a}_{i,j})}$ as specified by $\mathcal{C}$, and $\check{C}$ satisfies

$$\pi(\check{C}|_{s=\pm 1}) \cap ([0,1] \times int(D^2)) = \mathfrak{c}^\pm \subset A_\varepsilon$$

and the conditions in Definition 4.5.2 at all the other ends. Here

$$\pi : [-1,1] \times [0,1] \times \overline{S} \to [0,1] \times \overline{S}$$

is the projection onto the second and third factors. Then resolve the self-intersections to make $\check{C}$ embedded.
The quadratic form Q_τ. Let $\check{C}$ be a τ-trivial representative of $\overline{u}$, where τ is viewed as a nonsingular vector field. Let $c_\ell^\pm(\delta)$ be the $\phi = \delta$ translate of $c_\ell^\pm$, where $\delta > 0$ is small. We then take a pushoff $\check{C}'$ of $\check{C}$ such that $\partial \check{C}$ is pushed in the direction of $J\tau + Y$, where Y is C^0-small, and

$$\pi(\check{C}'|_{s=\pm 1}) \cap ([0,1] \times int(D^2)) = \cup_{\ell=1}^{p_\pm} c_\ell^\pm(\delta) \subset A_\varepsilon.$$

Then $Q_\tau([u]) = \langle \check{C}, \check{C}' \rangle$ as in Definition 4.5.5.

Definition **5.7.7** *(ECH index). — Given $A \in H_2(\overline{W}, \mathbf{z}_+, \mathbf{z}_-)$ and a groomed multivalued trivialization τ compatible with $(\mathcal{D}_+, \mathcal{D}_-)$, we define*

$$I_\tau(A) = Q_\tau(A) + \widetilde{\mu}_\tau(\partial A) + c_1(T\overline{S}|_A, \tau), \quad (\mathbf{5.7.3})$$

where the analog $\widetilde{\mu}_\tau(\partial A)$ of the symmetric Conley-Zehnder index will be given in Section 5.7.6.

Sometimes we will write $I_\tau(u)$ to mean $I_\tau(A)$, where A is the relative homology class defined by u.

5.7.6. *Definition of $\widetilde{\mu}_\tau(\partial A)$.* — We first define the Conley-Zehnder index $\mu_\tau(\partial A)$. The groomed multivalued trivialization τ determines the matchings $\mathcal{D}_\pm^{from} \to \mathcal{D}_\pm^{to}$. Pick a cycle ζ which represents ∂A and respects the matchings along $\{\pm 1\} \times [0,1] \times \{z_\infty\}$. The pullback bundle $\zeta^* T\overline{S}$ is trivialized by the pullback of τ. We now define a multivalued real rank one subbundle $\mathcal{L}_0$ of $T\overline{S}$ along $Z_{\overline{\mathbf{a}},\overline{h}(\overline{\mathbf{a}})}$ by setting $\mathcal{L}_0 = T\check{L}_{\overline{\mathbf{a}}} \cap T\overline{S}$ on $\check{L}_{\overline{\mathbf{a}}}$ and $T\check{L}_{\overline{h}(\overline{\mathbf{a}})} \cap T\overline{S}$ on $\check{L}_{\overline{h}(\overline{\mathbf{a}})}$ and extending $\mathcal{L}_0$ across $\{\pm 1\} \times [0,1] \times \mathbf{z}_\pm$ by rotating in the counterclockwise direction from $T\overline{h}(\overline{\mathbf{a}})$ to $T\overline{\mathbf{a}}$ in $T\overline{S}$ by the minimum amount possible. We then define $\mu_\tau(\partial A)$ as the Maslov index of $\mathcal{L} = \zeta^*\mathcal{L}_0$ with respect to τ.

Now we define the corrections to add to $\mu_\tau(\partial A)$ to obtain the symmetric Conley-Zehnder index $\widetilde{\mu}_\tau(\partial A)$. Let τ be a groomed multivalued trivialization which is compatible with $(\mathcal{D}_+, \mathcal{D}_-)$ and which induces the groomings $\mathfrak{c}^+ = \{c_\ell^+\}_{\ell=1}^{p_+}$ and $\mathfrak{c}^- = \{c_\ell^-\}_{\ell=1}^{p_-}$.

Definition **5.7.8.** — *Given a grooming* $\mathfrak{c} = \{c_\ell\}_{\ell=1}^{p}$ *from* P_0 *to* P_1*, its* winding number *is given by:*

$$w(\mathfrak{c}) = \sum_\ell w(c_\ell), \quad w(c_\ell) = \langle c_\ell, \{\phi = \pi\}\rangle, \tag{5.7.4}$$

where $\langle\cdot,\cdot\rangle$ *is the algebraic intersection number in* A_ε*, the arcs* c_ℓ *and* $\{\phi=\pi\}$ *are oriented from* $t=0$ *to* $t=1$*, and* A_ε *is oriented by* $(\partial_\phi, \partial_t)$.

Remark **5.7.9.** — A grooming $\mathfrak{c}$ is determined by its endpoints P_0 and P_1 and its winding number $w(\mathfrak{c})$.

Given P_0, P_1, and $q \in \mathbf{Z}$, choose a grooming $\mathfrak{c} = \{c_\ell\}_{\ell=1}^{p} = \mathfrak{c}(P_0, P_1, q)$ from P_0 to P_1 such that $w(\mathfrak{c}) = q$. Let $c_\ell^\flat$ be the linear arc in A_ε which is disjoint from $\{\phi=\pi\}$ and has the same endpoints as c_ℓ. (Note that the collection $\{c_\ell^\flat\}$ is usually not groomed.)

Definition **5.7.10.** — $\alpha_{(P_0,P_1,q)}$ *(or simply* α_q *if* P_0, P_1 *are understood) is the number of arcs in* $\{c_\ell^\flat\}$ *whose* ϕ*-coordinate* decreases *as* t *increases (in the universal cover).*

If P_0, P_1 are fixed, then the number α_q only depends on q (modulo p), through the bijection $P_0 \xrightarrow{\sim} P_1$.

Let $q_\pm = w(\mathfrak{c}^\pm)$ and $P_0^\pm, P_1^\pm$ be the endpoints of $\mathfrak{c}^\pm$.

Definition **5.7.11.** — *The* discrepancies $d^\pm$ *at the positive and negative ends are:*

$$d^\pm = -(\alpha_{(P_0^\pm,P_1^\pm,q_\pm)} - \alpha_{(P_0^\pm,P_1^\pm,0)}) - q_\pm(p_\pm - 1). \tag{5.7.5}$$

Remark **5.7.12.** — Observe that $d^\pm = 0$ if the grooming $\mathfrak{c}^\pm$ satisfies $w(\mathfrak{c}^\pm) = q_\pm = 0$; in other words, we are viewing $q_\pm = 0$ as a reference point and comparing the discrepancy that arises from choosing another grooming $\mathfrak{c}^\pm$.

Definition **5.7.13.** — *The* symmetric Conley-Zehnder index $\widetilde{\mu}_\tau(\partial A)$ *is given by:*

$$\widetilde{\mu}_\tau(\partial A) = \mu_\tau(\partial A) + d^+ - d^-. \tag{5.7.6}$$

The discrepancy term $d^+ - d^-$ was (somewhat artificially) added to make Lemma 5.7.15 hold.

Remark **5.7.14.** — Let us view P_0^*, P_1^* as points on $(-\pi,\pi)$, where $*=\pm$. In the special case where the points of P_0^* and P_1^* alternate along $(-\pi,\pi)$, we can write:

$$d^* = \begin{cases} -p_*(q_* - \llfloor \frac{q_*}{p_*} \rrfloor), & \text{if } \min_{x\in P_0} x < \min_{x\in P_1} x; \\ -p_*(q_* - \llceil \frac{q_*}{p_*} \rrceil), & \text{if } \min_{x\in P_0} x > \min_{x\in P_1} x. \end{cases} \tag{5.7.7}$$

Here $\llfloor x \rrfloor$ is the greatest integer $\leq x$ and $\llceil x \rrceil$ is the smallest integer $\geq x$.

5.7.7. *ECH indices of branched covers of sections at infinity.* —

Lemma **5.7.15.** — *If* $A \in H_2(\overline{W}, \{z_\infty^p(\overrightarrow{\mathcal{D}}_+), \{z_\infty^p(\overrightarrow{\mathcal{D}}_-)\})$ *is the relative homology class of a p-fold branched cover of* σ_∞ *with possibly empty branch locus, then* $I_\tau(A)=0$ *for all groomed multivalued trivializations* τ *compatible with* $(\mathcal{D}_+,\mathcal{D}_-)$.

Proof. — Let $\mathfrak{c}^+=\{c_\ell^+\}$ and $\mathfrak{c}^-=\{c_\ell^-\}$ be the groomings of τ and let $q_\pm = w(\mathfrak{c}^\pm)$. Here $P_i = P_i^+ = P_i^-$, $i=0,1$. Let us also assume that τ' which appears in the definition of $\tau_{\overrightarrow{\mathcal{D}}_+,\overrightarrow{\mathcal{D}}_-}$ in Section 5.7.3 satisfies $\tau' = \partial_\rho$ on $D^2_{2\varepsilon} - \{z_\infty\}$.

In order to compute $Q_\tau(A)$, we choose a τ-trivial representative $\check{C}$ of A such that:

- $\check{C} \cap ([0,1]\times[0,1]\times\overline{S}) = [0,1]\times\mathfrak{c}^+$;
- $\check{C} \cap ((-1,0)\times[0,1]\times\overline{S}) \subset (-1,0)\times[0,1]\times int(D^2_\varepsilon)$;
- $\check{C} \cap (\{-1\}\times[0,1]\times\overline{S}) = \{-1\}\times\mathfrak{c}^-$;

and a representative $\check{C}'$ such that:

- $\check{C}' \cap (\{1\}\times[0,1]\times\overline{S}) = \{1\}\times\mathfrak{c}^+(\delta)$;
- $\check{C}' \cap ((0,1)\times[0,1]\times\overline{S}) \subset (0,1)\times[0,1]\times int(D^2_\varepsilon)$;
- $\check{C}' \cap ([-1,0]\times[0,1]\times\overline{S}) = [-1,0]\times\mathfrak{c}^-(\delta)$.

Recall that $\mathfrak{c}^\pm \subset A_\varepsilon = [0,1]\times\partial D^2_\varepsilon$. Since all the intersections between $\check{C}$ and $\check{C}'$ are contained in the level $s=0$,

$$Q_\tau(A) = \mathfrak{c}^+\cdot\mathfrak{c}^-(\delta) = p(q_+ - q_-).$$

Next we claim that

$$\mu_\tau(\partial A) = (\alpha_{q_+} - 2q_+) - (\alpha_{q_-} - 2q_-).$$

Indeed, given the end which corresponds to the strand $c_\ell^\pm$, the Maslov index of the end is given by $-2w(c_\ell^\pm)$ if the endpoints of $c_\ell^\pm$ satisfy $0 < p^\pm_{\ell,0} < p^\pm_{\ell,1} < 2\pi$, and is given by $1-2w(c_\ell^\pm)$ if $0 < p^\pm_{\ell,1} < p^\pm_{\ell,0} < 2\pi$. On the other hand, the number of strands for which

$p^{\pm}_{\ell,0} > p^{\pm}_{\ell,1}$ holds is exactly $\alpha_{q_\pm}$. Since the Maslov indices of the portions of ∂A that map to $[-1,1]\times\{1\}\times\overline{\mathbf{a}}$ and $[-1,1]\times\{0\}\times\overline{h}(\overline{\mathbf{a}})$ are zero, the claim follows.

Hence,

$$\begin{aligned}\widetilde{\mu}_\tau(\partial A) &= (\alpha_{q_+} - 2q_+) - (\alpha_{q_+} - \alpha_0) - q_+(p-1)\\ &\quad - (\alpha_{q_-} - 2q_-) + (\alpha_{q_-} - \alpha_0) + q_-(p-1)\\ &= -(q_+ - q_-)(p+1).\end{aligned}$$

We also have $c_1(T\overline{S}|_A, \tau) = q_+ - q_-$. Putting everything together, we obtain:

$$I_\tau(A) = p(q_+ - q_-) - (q_+ - q_-)(p+1) + (q_+ - q_-) = 0.$$

This proves the lemma. □

We also state the following lemmas without proof:

Lemma **5.7.16.** — *If $\overline{u}: \dot{F} \to \overline{W}_-$ is a degree $p \leq 2g$ multisection which branch covers σ^-_∞ with possibly empty branch locus, then $I_\tau(\overline{u}) = 0$ for all groomed multivalued trivializations τ that are compatible with $\mathcal{D}_-$ for the negative ends of $\overline{u}$.*

Lemma **5.7.17.** — *If $\overline{u}: \dot{F} \to \overline{W}_- - int(W_-)$ is a degree $p \leq 2g$ multisection with positive ends at δ_0 with total multiplicity p and negative ends at a p-element subset of $\{x_1, \dots, x_{2g}, x'_1, \dots, x'_{2g}\}$, then $I(\overline{u}) = p$.*

5.7.8. *Additivity of indices and independence of the trivialization.* — The Fredholm and the ECH index are additive with respect to concatenation. The proofs of the following lemmas are straightforward.

Lemma **5.7.18.** — *Let $\overline{u}_1$ be a multisection with positive end $\{z^p_\infty(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$, and let $\overline{u}_2$ be a multisection with negative end $\{z^p_\infty(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$. If $\overline{u}_1 \# \overline{u}_2$ is the multisection obtained by gluing $\overline{u}_1$ and $\overline{u}_2$ along their common end, then*

$$\mathrm{ind}(\overline{u}_1 \# \overline{u}_2) = \mathrm{ind}(\overline{u}_1) + \mathrm{ind}(\overline{u}_2).$$

Lemma **5.7.19.** — *Let τ_2 and τ_1 be groomed multivalued trivializations compatible with $(\mathcal{D}_2, \mathcal{D}_1)$ and $(\mathcal{D}_1, \mathcal{D}_0)$, respectively, and let τ be obtained by concatenating τ_2 and τ_1. Given relative homology classes*

$$A_2 \in H_2(\overline{W}, \{z^{p_2}_\infty(\overrightarrow{\mathcal{D}}_2)\} \cup \mathbf{y}_2, \{z^{p_1}_\infty(\overrightarrow{\mathcal{D}}_1)\} \cup \mathbf{y}_1),$$

$$A_1 \in H_2(\overline{W}, \{z^{p_1}_\infty(\overrightarrow{\mathcal{D}}_1)\} \cup \mathbf{y}_1, \{z^{p_0}_\infty(\overrightarrow{\mathcal{D}}_0)\} \cup \mathbf{y}_0),$$

we can form the concatenation

$$A_2\#A_1 \in H_2(\overline{W}, \{z_\infty^{p_2}(\overrightarrow{\mathcal{D}}_2)\} \cup \mathbf{y}_2, \{z_\infty^{p_0}(\overrightarrow{\mathcal{D}}_0)\} \cup \mathbf{y}_0).$$

Then we have:

$$I_\tau(A_2\#A_1) = I_{\tau_2}(A_2) + I_{\tau_1}(A_1).$$

In view of the following lemma, we can suppress τ from I_τ.

Lemma **5.7.20.** — $I_\tau(A)$ *is independent of the choice of groomed multivalued trivialization* τ.

Proof. — Let τ and τ' be two groomed multivalued trivializations adapted to the same data $(\mathcal{D}_+, \mathcal{D}_-)$. It suffices to consider the particular cases when τ and τ' differ only either at some y_i or at z_∞. In the first case, the argument is analogous to the proof of Lemma 4.5.6. In the second case, we can glue branched covers of σ_∞ to switch groomings. Then the statement follows from Lemma 5.7.15 and the additivity of the ECH index. □

5.7.9. *The ECH index inequality.* — Let $\overline{u}$ be a degree k multisection of $\overline{W}$ from $\mathbf{z}_+ = \{z_\infty^{p_+}(\overrightarrow{\mathcal{D}}_+)\} \cup \mathbf{y}_+$ to $\mathbf{z}_- = \{z_\infty^{p_-}(\overrightarrow{\mathcal{D}}_-)\} \cup \mathbf{y}_-$ such that $\overline{u} = \overline{u}''$. Let $c_\ell^\pm$, $\ell = 1, \dots, p_\pm$ be the intersections of A_ε with the π-projections of the $\pm$ ends of $\overline{u}$ that limit to z_∞. Here $\varepsilon > 0$ is small and depends on $\overline{u}$, and the map π projects out the s-direction. Let $\mathfrak{c}^\pm = \{c_\ell^\pm\}_{\ell=1,\dots,p_\pm}$ and let $P_0^\pm$ (resp. $P_1^\pm$) be the set of the initial (resp. terminal) points of the arcs $c_\ell^\pm$.

Lemma **5.7.21.** — *If* $\mathfrak{c}^\pm$ *are groomings and* $A \in H_2(\overline{W}, \mathbf{z}_+, \mathbf{z}_-)$ *is the relative homology class of* $\overline{u}$, *then*

$$I(A) \geq \operatorname{ind}(\overline{u}) + (d^+ - d^-). \tag{5.7.8}$$

In particular, if the points of P_0^* *and* P_1^* *alternate along* $(0, 2\pi)$ *for both* $* = +$ *and* $-$, *then* $I(A) \geq \operatorname{ind}(\overline{u})$.

Proof. — Let τ be a groomed multivalued trivialization which is compatible with $(\mathcal{D}_+, \mathcal{D}_-)$ and $\mathfrak{c}^\pm$. If we use τ to compute both the Fredholm and ECH indices, then the proof of the relative adjunction formula (Lemma 4.5.9) goes through unmodified. Hence Equation (5.7.2) and Lemma 4.5.9 imply:

$$\begin{aligned}\operatorname{ind}(\overline{u}) &= -\chi(F) + k + \mu_\tau(\overline{u}) + 2c_1(\overline{u}^*T\overline{S}, \tau)\\ &= Q_\tau(A) + \mu_\tau(\overline{u}) + c_1(\overline{u}^*T\overline{S}, \tau) - 2\delta(\overline{u}).\end{aligned}$$

Since τ is compatible with $\mathfrak{c}^{\pm}$, we have $\mu_\tau(\overline{u}) = \mu_\tau(\partial A)$. Comparing with the definition of $I(\overline{u})$, we have:

$$(\mathbf{5.7.9}) \qquad \begin{aligned} I(\overline{u}) &= \operatorname{ind}(\overline{u}) + (\widetilde{\mu}_\tau(\partial A) - \mu_\tau(\partial A)) + 2\delta(\overline{u}) \\ &= \operatorname{ind}(\overline{u}) + (d^+ - d^-) + 2\delta(\overline{u}). \end{aligned}$$

(Note that $\overline{u}$ has no boundary singularities because $\overline{\pi}_B \circ \overline{u}$ has no branch point at the boundary: in fact $\overline{u}$ has boundary values in $L_{\hat{\mathbf{a}}} \cup L_{\overline{h}(\hat{\mathbf{a}})}$, and therefore the images of different boundary components of $\dot{F}$ are mutually disjoint.) Equation (5.7.8) follows from observing that $\delta(\overline{u}) \geq 0$ since $\overline{u}$ is J-holomorphic.

Next suppose that the points of P_0^* and P_1^* alternate along $(0, 2\pi)$ for both $* = +$ and $-$. We claim that $d^+ \geq 0$. Let $\pi : \mathbf{R} \times [0,1] \times \overline{S} \to [0,1] \times \overline{S}$ be the projection onto the second and third factors. By the positivity of intersections, $\pi(\overline{u})$ is positively transverse to the Hamiltonian vector field ∂_t; hence $q_+ \leq 0$. By Equation (5.7.7), if $q_+ \leq 0$, then $d^+ \geq 0$ (in both cases). $d^- \leq 0$ is proved similarly. □

5.7.10. *ECH index calculation.* — In this subsection we compute the ECH index of a multisection of $\overline{W}$ which is the union of a branched cover of the section at infinity σ_∞ and a curve which limits to a multiple of z_∞. This calculation will be used in Section II.3.7.2. For simplicity we consider the cases of positive ends and negative ends at z_∞ separately. The general statement can easily be deduced from the two separate cases.

Let $\overline{u} = \overline{u}' \cup \overline{u}''$ be a degree $p_1 + p_2 + l$ multisection of $\overline{W}$, where $\deg \overline{u}' = p_1$, $\overline{u}''$ is a multisection from $\mathbf{y}$ to $\{z_\infty^{p_2}(\overrightarrow{\mathcal{D}}_2)\} \cup \mathbf{y}'$, and l is the cardinality of $\mathbf{y}'$. Suppose that the data $\overrightarrow{\mathcal{D}}_1$ and $\overrightarrow{\mathcal{D}}_2$ for $\overline{u}'$ and $\overline{u}''$ at the negative ends $z_\infty^{p_1}$ and $z_\infty^{p_2}$ satisfy:

(D_1) $\overrightarrow{\mathcal{D}}_1 = \{(i_1, j_1) \to (i_1, j_1), \dots, (i_{p_1}, j_{p_1}) \to (i_{p_1}, j_{p_1})\}$;
(D_2) $\overrightarrow{\mathcal{D}}_2 = \{(i_1', j_1') \to (i_1'', j_1''), \dots, (i_{p_2}', j_{p_2}') \to (i_{p_2}'', j_{p_2}'')\}$;
(D_3) $\mathcal{D}_1^{from} = \mathcal{D}_1^{to}$ and the sets $\mathcal{D}_2^{from}$ and $\mathcal{D}_2^{to}$ are disjoint from $\mathcal{D}_1^{from} = \mathcal{D}_1^{to}$.

Let $\mathcal{E}_{-,\ell}$ be the end of $\overline{u}''$ at z_∞ corresponding to $(i_\ell', j_\ell') \to (i_\ell'', j_\ell'')$.

Suppose there exist groomings $\mathfrak{c}_1^- \subset A_{\varepsilon/2} = \partial D^2_{\varepsilon/2} \times [0,1]$[7] and $\mathfrak{c}_2^- \subset A_\varepsilon = \partial D^2_\varepsilon \times [0,1]$ which correspond to $\overrightarrow{\mathcal{D}}_1$ and $\overrightarrow{\mathcal{D}}_2$ and satisfy the following:

(G_1) $\mathfrak{c}_1^-$ has winding number $q_1 := w(\mathfrak{c}_1^-) = 0$;
(G_2) $\mathfrak{c}_2^- = \pi(\cup_{i=1}^{p_2} \mathcal{E}_{-,\ell}) \cap A_\varepsilon$ and $q_2 := w(\mathfrak{c}_2^-) = 0$ or 1;
(G_3) the points of P_0^2 and P_1^2 alternate along $(0, 2\pi)$, i.e., the projection of $\mathfrak{c}_2^-$ to ∂D^2_ε is injective.

Here P_0^i (resp. P_1^i) is the set of initial (resp. terminal) points of $\mathfrak{c}_i^-$.

[7] Here we are writing $A_{\varepsilon/2} = \partial D^2_{\varepsilon/2} \times [0,1]$ instead of $[0,1] \times \partial D^2_{\varepsilon/2}$ to indicate the orientation of $A_{\varepsilon/2}$.

Let π be the projection of $\check{\overline{W}} = [-1,1]\times[0,1]\times\overline{S}$ to $[0,1]\times\overline{S}$ and let $\check{C}'$ and $\check{C}''$ be representatives of $\overline{u}'$ and $\overline{u}''$ in $\check{\overline{W}}$ such that $\pi(\check{C}'|_{s=-1}) = \mathfrak{c}_1^-$ and $\pi(\check{C}''|_{s=-1}) = \mathfrak{c}_2^-$.

Let $\pi_\rho : [\frac{\varepsilon}{2},\varepsilon]\times\partial D^2\times[0,1] \to \partial D^2\times[0,1]$ be the projection along the ρ-direction. Let $\mathfrak{w}$ be the signed number of crossings of $\pi_\rho(\mathfrak{c}_1^-\cup\mathfrak{c}_2^-)$, i.e., the writhe. Observe that all the crossings of $\pi_\rho(\mathfrak{c}_1^-\cup\mathfrak{c}_2^-)$ are positive as a consequence of (G1) and (G_2), and $p_1 \geq \mathfrak{w}$ as a consequence of (G_3).

Lemma **5.7.22.** — *Let* $\overline{u} = \overline{u}'\cup\overline{u}''$ *be a degree* p_1+p_2+l *multisection of* $\overline{W}$, *where* $\deg\overline{u}' = p_1$, $\overline{u}''$ *is a multisection from* $\mathbf{y}$ *to* $\{z_\infty^{p_2}(\overrightarrow{\mathcal{D}}_2)\}\cup\mathbf{y}'$, *and* l *is the cardinality of* $\mathbf{y}'$. *If the data* $\overrightarrow{\mathcal{D}}_1$, $\overrightarrow{\mathcal{D}}_2$ *at the negative end satisfy (D_1)–(D_3) and (G_1)–(G_3), then*

$$(\mathbf{5.7.10})\qquad I(\overline{u}) \geq I(\overline{u}') + I(\overline{u}'') + \begin{cases} 2\mathfrak{w}, & \text{if } q_2 = 0;\\ p_1+\mathfrak{w}, & \text{if } q_2 = 1.\end{cases}$$

If $\overline{u}'\cap\overline{u}'' = \varnothing$ *in addition, then equality holds.*

Proof. — It suffices to prove the equality, assuming $\overline{u}'\cap\overline{u}'' = \varnothing$, since the extra intersections contribute positively towards the ECH index. The representatives $\check{C}'$ and $\check{C}''$ can be taken to be disjoint; in fact we can assume that $\check{C}''$ is disjoint from the $\frac{\varepsilon}{2}$-neighborhood $\mathbf{R}\times[0,1]\times D^2_{\varepsilon/2}$ of σ_∞.

If $q_2 = 0$, then we resolve the positive crossings of $\pi_\rho(\mathfrak{c}_1^-\cup\mathfrak{c}_2^-)$. This is equivalent to appending a union of disks $\check{D}\subset[-2,-1]\times[0,1]\times\overline{S}$ to $\check{C}'\cup\check{C}''$ such that $\check{D}|_{s=-1} = (\check{C}'\cup\check{C}'')|_{s=-1}$ and $\pi(\check{D}|_{s=-2})$[8] is a grooming on A_ε which satisfies $w(\pi(\check{D}|_{s=-2})) = 0$. A quick calculation shows that $\check{D}$ contributes $\mathfrak{w}$ to Q, 0 to c_1, and $\mathfrak{w}$ to μ. Since the discrepancy for $\pi(\check{D}|_{s=-2})$ is zero, Equation (5.7.10) follows for $q_2 = 0$.

If $q_2 = 1$, then we first switch the crossings of $\pi_\rho(\mathfrak{c}_1^-\cup\mathfrak{c}_2^-)$ by appending a union of disks $\check{D}_{-1}\subset[-2,-1]\times[0,1]\times\overline{S}$ such that $\pi(\check{D}_{-1}|_{s=-2})$ consists of $\mathfrak{c}_1^-$ on A_ε and $\mathfrak{c}_2^-$ on $A_{\varepsilon/2}$. This contributes $2\mathfrak{w}$ to Q and 0 to c_1 and μ. Next append $\check{D}_{-2}\subset[-3,-2]\times[0,1]\times\overline{S}$ so that $\pi(\check{D}_{-2}|_{s=-3})$ consists of $\mathfrak{c}_1^-$ on A_ε and $\mathfrak{d}_2^-$ on $A_{\varepsilon/2}$, where $\mathfrak{d}_2^-$ is groomed and $w(\mathfrak{d}_2^-) = 0$. The surface $\check{D}_{-2}$ is similar to that used in the proof of Lemma 5.7.15 and has zero ECH index. Finally, we resolve the positive crossings of $\pi_\rho(\mathfrak{c}_1^-\cup\mathfrak{d}_2^-)$ by appending $\check{D}_{-3}\subset[-4,-3]\times[0,1]\times\overline{S}$. Since there are $p_1-\mathfrak{w}$ crossings, $\check{D}_{-3}$ contributes $p_1-\mathfrak{w}$ to Q, 0 to c_1, and 0 to μ. Equation (5.7.10) then follows for $q_2 = 1$. □

Next we consider the variant where the multiple of z_∞ is at the positive end. Let $\overline{u} = \overline{u}'\cup\overline{u}''$ be a degree p_1+p_2+l multisection of $\overline{W}$, where $\deg\overline{u}' = p_1$, $\overline{u}''$ is a multisection from $\{z_\infty^{p_2}(\overrightarrow{\mathcal{D}}_2)\}\cup\mathbf{y}'$ to $\mathbf{y}$, and l is the cardinality of $\mathbf{y}'$. We use the same notation as above, with $-$ replaced by $+$.

We make the following assumptions:

[8] Here we are taking π to be the projection of the appropriate space to $[0,1]\times\overline{S}$.

(G'_1) $\mathfrak{c}_1^+$ has winding number $q_1 := w(\mathfrak{c}_1^+) = 0$;
(G'_2) $\mathfrak{c}_2^+ = \pi(\cup_{i=1}^{p_2} \mathcal{E}_{+,\ell}) \cap A_\varepsilon$ and $q_2 := w(\mathfrak{c}_2^+) = 0$ or -1; and
(G'_3) the projection of $\mathfrak{c}_2^+$ to ∂D_ε^2 is injective except on $\kappa \geq 0$ short intervals of ∂D_ε^2 which correspond to thin sectors between $\overline{a}_{i,j}$ and $\overline{h}(\overline{a}_{i,j})$.

Let $\mathfrak{w}$ be the signed number of crossings of $\pi_\rho(\mathfrak{c}_1^+ \cup \mathfrak{c}_2^+)$. In this case, all the crossings of $\pi_\rho(\mathfrak{c}_1^+ \cup \mathfrak{c}_2^+)$ are negative. The analog of Lemma 5.7.22, stated without proof, is:

Lemma **5.7.23.** — *Let $\overline{u} = \overline{u}' \cup \overline{u}''$ be a degree $p_1 + p_2 + l$ multisection of $\overline{W}$, where $\deg \overline{u}' = p_1$, $\overline{u}''$ is a multisection from $\{z_\infty^{p_2}(\overrightarrow{\mathcal{D}}_2)\} \cup \mathbf{y}'$ to $\mathbf{y}$, and l is the cardinality of $\mathbf{y}'$. If the data $\overrightarrow{\mathcal{D}}_1$, $\overrightarrow{\mathcal{D}}_2$ at the positive end satisfy (D_1)–(D_3) and (G'_1)–(G'_3), then*

$$\text{(5.7.11)}\qquad I(\overline{u}) \geq I(\overline{u}') + I(\overline{u}'') + \begin{cases} -\mathfrak{w}, & \text{if } q_2 = 0; \\ -2\mathfrak{w}, & \text{if } q_2 = -1. \end{cases}$$

If $\overline{u}' \cap \overline{u}'' = \varnothing$ in addition, then equality holds.

We also compute the discrepancy d_+ of $\mathfrak{c}_2^+$ when $q_2 = -1$. Since $\alpha_{-1} = p_2 - 1$ and $\alpha_0 = \kappa$,

$$\text{(5.7.12)}\qquad \begin{aligned} d_+ &= -(\alpha_{-1} - \alpha_0) - q_2(p_2 - 1) \\ &= -((p_2 - 1) - \kappa) - (-1)(p_2 - 1) = \kappa. \end{aligned}$$

By Lemma 5.7.21,

$$\text{(5.7.13)}\qquad I(\overline{u}'') \geq \operatorname{ind}(\overline{u}'') + \kappa.$$

5.7.11. *Extended moduli spaces.* — We now describe the extended moduli spaces which involve multiples of z_∞ at the ends. Details will be given for $\overline{W}$; the $\overline{W}_+$ and $\overline{W}_-$ cases are analogous.

Let $\mathcal{M} = \mathcal{M}_{\overline{J}}(\mathbf{z}_+, \mathbf{z}_-)$ be the moduli space of multisections of $(\overline{W}, \overline{J})$ from $\mathbf{z}_+ = \{z_\infty^{p_+}(\overrightarrow{\mathcal{D}}_+)\} \cup \mathbf{y}_+$ to $\mathbf{z}_- = \{z_\infty^{p_-}(\overrightarrow{\mathcal{D}}_-)\} \cup \mathbf{y}_-$. A map $\overline{u} \in \mathcal{M}$ has boundary conditions on $L_{\overline{\mathbf{a}}} \cup L_{\overline{h}(\overline{\mathbf{a}})}$. Let $\dagger$ be the modifier "$\overline{u}' = \varnothing$". We now describe an enlargement of the moduli space $\mathcal{M}^\dagger$.

Recall that $\vec{a}_{i,j} \subset D^2$ is of the form $\{-1 < \rho \leq 1, \phi = \phi_{i,j}\}$ for some constant $\phi_{i,j}$.

Definition **5.7.24.** — *An* extended $\overline{W}$-curve *$\overline{u}$ from $\mathbf{z}_+$ to $\mathbf{z}_-$ is a multisection of $(\overline{W}, \overline{J})$ which satisfies the conditions of Definition 5.7.2 with $\dot{F}' = \varnothing$ and (1) and (2) replaced by the following:*

($1'$) *There exist either positive or negative ends (or both) $\mathcal{E}_{+,i}$ and $\mathcal{E}_{-,i}$ of $\dot{F}$ that limit to z_∞ such that:*

– $\overline{u}(\partial \dot{F} - \cup_i \mathcal{E}_{+,i} - \cup_i \mathcal{E}_{-,i}) \subset L_{\widehat{\mathbf{a}}} \cup L_{\overline{h}(\widehat{\mathbf{a}})}$;

– $\overline{u}$ *maps one component of* $\partial\dot{F}\cap\mathcal{E}_{+,i}$ *(resp.* $\partial\dot{F}\cap\mathcal{E}_{-,i}$*) to* $L_{\vec{a}_{i_\ell,j_\ell}}$ *and the other to* $L_{\overline{h}(\vec{a}_{i'_\ell,j'_\ell})}$*, where* $(i'_\ell,j'_\ell)\to(i_\ell,j_\ell)\in\overrightarrow{\mathcal{D}}_+$ *(resp.* $\in\overrightarrow{\mathcal{D}}_-$*).*

(2′) $\overline{u}$ *maps each connected component of* $\partial\dot{F}-\cup_i\mathcal{E}_{+,i}-\cup_i\mathcal{E}_{-,i}$ *to a different* $L_{\widehat{a}_i}$ *or* $L_{\overline{h}(\widehat{a}_i)}$.

The moduli space of extended $\overline{W}$*-curves from* $\mathbf{z}_+$ *to* $\mathbf{z}_-$ *is denoted by* $\mathcal{M}^{\dagger,ext}_{\overline{J}}(\mathbf{z}_+,\mathbf{z}_-)$.

The extended moduli spaces $\mathcal{M}^{\dagger,ext}_{\overline{J}_+}(\mathbf{z}_+,\delta_0^r\boldsymbol{\gamma}')$ and $\mathcal{M}^{\dagger,ext}_{\overline{J}_-}(\delta_0^r\boldsymbol{\gamma}',\mathbf{z}_-)$ are defined similarly.

Terminology. A *sector* $\mathfrak{S}$ *of* $D^2_{\rho_0}$, $0<\rho_0\le 1$, *from* ϕ_0 *to* ϕ_1 is the map

$$[0,\rho_0]\times[\phi_0,\phi_1]\to D^2_{\rho_0},\quad (\rho,\phi)\mapsto\rho e^{i\phi}.$$

By abuse of notation, we also refer to the image of $\mathfrak{S}$ as a sector.

If $R,R'\subset D^2_{\rho_0}$ are distinct radial rays and $I\subset\mathbf{R}$ is an interval, then $\mathfrak{S}(R,R';I)$ is a sector of $D^2_{\rho_0}$ from ϕ_0 to ϕ_1, where R and R' can be written as $\{\phi=\phi_0,\rho\ge 0\}$ and $R'=\{\phi=\phi_1,\rho\ge 0\}$ and $[\phi_0,\phi_1]\subset I$. If we do not specify I, then we write $\mathfrak{S}(R,R')$ and assume that $\phi_1-\phi_0$ is the smallest positive angle.

A sector is *large* if it has angle $\phi_1-\phi_0>\pi$ and *small* if it has angle $0<\phi_1-\phi_0<\pi$.

Let $\mathcal{M}^*=\mathcal{M}^*(\mathbf{z}_+,\mathbf{z}_-)$, where $*$ denotes any set of modifiers, and let

$$\pi_{D^2}:D^2\times[0,1]\to D^2$$

be the projection onto the first factor. If $\overline{u}\in\mathcal{M}^\dagger$, then $(i'_\ell,j'_\ell)\to(i_\ell,j_\ell)$ in $\overrightarrow{\mathcal{D}}_+$ corresponds to an end $\mathcal{E}_{+,\ell}$ of $\overline{u}$ such that the restriction of $\pi_{D^2}\circ\overline{u}(\mathcal{E}_{+,\ell})$ to D^2_ε for $\varepsilon>0$ small is a sector $\mathfrak{S}=\mathfrak{S}(\overline{a}_{i_\ell,j_\ell},\overline{h}(\overline{a}_{i'_\ell,j'_\ell}))$. If $\mathfrak{S}$ is large, then $\pi_{D^2}\circ\overline{u}(\mathcal{E}_{+,\ell})$ has a slit of length 0 by the definition of $\overline{u}\in\mathcal{M}^\dagger$ (i.e., $\pi_{D^2}\circ\overline{u}$ maps no boundary point of $\mathcal{E}_{+,i}$ to $\vec{a}_{i_\ell,j_\ell}-\overline{a}_{i_\ell,j_\ell}$ or $\overline{h}(\vec{a}_{i'_\ell,j'_\ell}-\overline{a}_{i'_\ell,j'_\ell})$). The ends of $\overline{u}$ can cover at most one large sector because $n(\overline{u})=m$, and therefore $\overline{u}$ is an extended curve. The neighborhood of $\overline{u}$ in $\mathcal{M}^\dagger$ is generically a codimension ≥ 1 submanifold of an extended moduli space $\mathcal{M}^{\dagger,ext}$.

5.8. *Transversality.* — We first discuss the regularity of almost complex structures on $\overline{W}$, $\overline{W'}$, $\overline{W}_+$ and $\overline{W}_-$.

5.8.1. *Transversality for* $\overline{W}$ *and* $\overline{W'}$. —

Definition **5.8.1.** — *The almost complex structure* $J\in\mathcal{J}_W$ *is* regular *if* $\mathcal{M}_J(\mathbf{y},\mathbf{y}')$ *is transversely cut out for all tuples* $\mathbf{y}$ *and* $\mathbf{y}'$ *in* $\mathbf{a}\cap h(\mathbf{a})$. *The almost complex structure* $\overline{J}\in\mathcal{J}_{\overline{W}}$ *is* regular *if* $\mathcal{M}^{\dagger,ext}_{\overline{J}}(\mathbf{z},\mathbf{z}')$ *is transversely cut out for all tuples* $\mathbf{z}=\{z^p_\infty(\mathcal{D})\}\cup\mathbf{y}$ *and* $\mathbf{z}'=\{z^q_\infty(\mathcal{D}')\}\cup\mathbf{y}'$.

Note that $\mathcal{M}_{\mathrm{J}}(\mathbf{y},\mathbf{y}') = \mathcal{M}^s_{\mathrm{J}}(\mathbf{y},\mathbf{y}')$ and $\mathcal{M}^{\dagger,ext}_{\overline{\mathrm{J}}}(\mathbf{z},\mathbf{z}') = \mathcal{M}^{\dagger,ext,s}_{\overline{\mathrm{J}}}(\mathbf{z},\mathbf{z}')$ because the positive ends have boundaries on distinct Lagrangian submanifolds. This implies that a generic $\mathrm{J} \in \mathcal{J}_{\mathrm{W}}$ is regular; see Lemma 4.7.2. The same proof also gives:

Lemma **5.8.2.** — *A generic* $\overline{\mathrm{J}} \in \mathcal{J}_{\overline{\mathrm{W}}}$ *is regular.*

We write $\mathcal{J}^{reg}_{\mathrm{W}} \subset \mathcal{J}_{\mathrm{W}}$ for the subset of all regular J and $\mathcal{J}^{reg}_{\overline{\mathrm{W}}} \subset \mathcal{J}_{\overline{\mathrm{W}}}$ for the subset of all regular $\overline{\mathrm{J}}$.

Definition **5.8.3.** — *The almost complex structure* $\mathrm{J}' \in \mathcal{J}_{\mathrm{W}'}$ *is* regular *if* $\mathcal{M}^s_{\mathrm{J}'}(\boldsymbol{\gamma},\boldsymbol{\gamma}')$ *is transversely cut out (in the Morse-Bott sense) for all* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\gamma}'$ *in* $\widehat{\mathcal{O}}_k$ *and all* $k \leq 2g$*. The almost complex structure* $\overline{\mathrm{J}}' \in \mathcal{J}_{\overline{\mathrm{W}'}}$ *is* regular *if* $\mathcal{M}^s_{\overline{\mathrm{J}'}}(\delta^p\boldsymbol{\gamma},\delta^q\boldsymbol{\gamma}')$ *is transversely cut out for all* $\delta^p\boldsymbol{\gamma}, \delta^p\boldsymbol{\gamma}' \in \overline{\mathcal{O}}_k$ *and for all* $k \leq 2g$*.*

Recall that a generic $\mathrm{J}' \in \mathcal{J}_{\mathrm{W}'}$ is regular by Lemma 3.5.2. The same proof also gives:

Lemma **5.8.4.** — *A generic* $\overline{\mathrm{J}'} \in \mathcal{J}_{\overline{\mathrm{W}'}}$ *is regular.*

We write $\mathcal{J}^{reg}_{\mathrm{W}'} \subset \mathcal{J}_{\mathrm{W}'}$ for the subset of all regular J' and $\mathcal{J}^{reg}_{\overline{\mathrm{W}'}} \subset \mathcal{J}_{\overline{\mathrm{W}'}}$ for the subset of all regular $\overline{\mathrm{J}'}$.

5.8.2. *Transversality for* W_+, $\overline{\mathrm{W}}_+$ *and* $\overline{\mathrm{W}}_-$. —

Definition **5.8.5.** — *The almost complex structure* $\mathrm{J}_+ \in \mathcal{J}_{\mathrm{W}_+}$ *is* regular *if the following hold:*

(1) *all moduli spaces* $\mathcal{M}_{\mathrm{J}_+}(\mathbf{y},\boldsymbol{\gamma})$ *with* $\mathbf{y}$ *a* k*-tuple of* $\mathbf{a} \cap h(\mathbf{a})$ *and* $\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_k$*, for all* $k \leq 2g$*, are transversely cut out (in the Morse-Bott sense in the case of a Morse-Bott building); and*
(2) *the restrictions* J *and* J' *of* J_+ *to the positive and negative ends belong to* $\mathcal{J}^{reg}_{\mathrm{W}}$ *and* $\mathcal{J}^{reg}_{\mathrm{W}'}$*, respectively.*

Here every $u \in \mathcal{M}_{\mathrm{J}_+}(\mathbf{y},\boldsymbol{\gamma})$ is somewhere injective due to the presence of the HF end. In other words, $\mathcal{M}_{\mathrm{J}_+}(\mathbf{y},\boldsymbol{\gamma}) = \mathcal{M}^s_{\mathrm{J}_+}(\mathbf{y},\boldsymbol{\gamma})$.

Definition **5.8.6.** — *The almost complex structure* $\overline{\mathrm{J}}_+ \in \mathcal{J}_{\overline{\mathrm{W}}_+}$ *is* regular *if the following hold:*

(1) *all moduli spaces* $\mathcal{M}^{\dagger,ext}_{\overline{\mathrm{J}}_+}(\mathbf{z},\delta^r_0\boldsymbol{\gamma}')$ *with* $\mathbf{z} = \{z^p_\infty(\mathcal{D})\} \cup \mathbf{y}$*,* $\boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_k$ *for some* $k \leq 2g$*, and* $\mathbf{y}$ *a tuple of* $\mathbf{a} \cap h(\mathbf{a})$*, are transversely cut out; and*
(2) *the restrictions* $\overline{\mathrm{J}}$ *and* $\overline{\mathrm{J}'}$ *of* $\overline{\mathrm{J}}_+$ *to the positive and negative ends belong to* $\mathcal{J}^{reg}_{\overline{\mathrm{W}}}$ *and* $\mathcal{J}^{reg}_{\overline{\mathrm{W}'}}$*, respectively.*

Note that $\mathcal{M}^{\dagger,ext}_{\overline{\mathrm{J}}_+}(\mathbf{z},\delta^r_0\boldsymbol{\gamma}') = \mathcal{M}^{\dagger,ext,s}_{\overline{\mathrm{J}}_+}(\mathbf{z},\delta^r_0\boldsymbol{\gamma}')$. The regularity of $\overline{\mathrm{J}}_- \in \mathcal{J}_{\overline{\mathrm{W}}_-}$ is defined similarly.

We write $\mathcal{J}^{reg}_{W_+}$ for the space of regular $J_+ \in \mathcal{J}_{W_+}$ and $\mathcal{J}^{reg}_{\overline{W}_\pm}$ for the space of regular $\overline{J}_\pm \in \mathcal{J}_{\overline{W}_\pm}$. Without loss of generality we may assume that the regular J_+ of interest are the restrictions of regular $\overline{J}_+$.

Remark **5.8.7.** — The vertical fibers $\{(s,t)\} \times S$ and $\{(s,t)\} \times \overline{S}$ are holomorphic, but are not transversely cut out.

Proposition **5.8.8.** — *A generic admissible* J_+ *(resp.* $\overline{J}_\pm$*) is regular.*

Proof. — We first treat the W_+ case. The proposition follows from a standard transversality argument along the lines of [MS, Theorem 3.1.5], with some modifications. The necessary modifications for almost complex structures $J' \in \mathcal{J}_{W'}$, defined on $\mathbf{R} \times N$, were described in [Hu1, Lemma 9.12(b)], and our situation is almost identical since $J_+ \in \mathcal{J}_{W_+}$ is the restriction to W_+ of some $J' \in \mathcal{J}_{W'}$.

The key observation is that each irreducible component of a W_+-curve $u : \dot{F} \to W_+$ is somewhere injective, since each $[0,1] \times \{y_i\}$, $y_i \in \mathbf{y}$, is used exactly once as a positive asymptotic limit. Let $\pi_N : W_+ \to N$ be the restriction of the projection $\pi_N : \mathbf{R} \times N \to N$ onto the second factor. We then claim that there is a dense open set of points $p \in \dot{F}$ which are π_N*-injective*, i.e.,

(i) $d(\pi_N \circ u)(p)$ has rank 2; and
(ii) $(\pi_N \circ u)(p) = (\pi_N \circ u)(q)$ implies $p = q$.

First note that if the claim does not hold, then there exist open sets $U, U' \subset \dot{F}$ such that $u(U)$ and $u(U')$ differ by some translation T_{s_0} by s_0 in the s-direction. By repeated application of T_{s_0} or T_{-s_0}, there is an infinite sequence $U, U'', U''', \dots \subset \dot{F}$ that have the same $\pi_N \circ u$ image. This contradicts the finite energy condition, and hence proves the claim. The perturbations to J_+ can then be carried out in a neighborhood of a π_N-injective point $p \in \dot{F}$ as in [Hu1, Lemma 9.12(b)].

The regularity of the almost complex structures J and J' at the ends was already treated, i.e., $\mathcal{J}^{reg}_{W} \subset \mathcal{J}_W$ and $\mathcal{J}^{reg}_{W'} \subset \mathcal{J}_{W'}$ are dense by Lemmas 3.5.2 and 4.7.2.

In the $\overline{W}_+$ case, the perturbations of $\overline{J}_+$ are allowed on the subset $U = \overline{W}_+ \cap ((\mathbf{R} \times \overline{N}) - \{\rho \leq \varepsilon\})$ for some small $\varepsilon > 0$. We simply observe that all the curves $\overline{u}$ in the moduli spaces $\mathcal{M}^{\dagger,ext}_{\overline{J}_+}(\mathbf{z}, \delta_0^r \boldsymbol{\gamma}')$ in Definition 5.8.6 pass through U, and pick a π_N-injective point $p \in \dot{F}$ such that $\overline{u}(p) \in U$. The $\overline{W}_-$ case is similar. □

5.8.3. *Some automatic transversality results.* — We collect some automatic transversality results.

Lemma **5.8.9.** — *The curve* $\sigma^-_\infty \subset \overline{W}_-$*, viewed as having Lagrangian boundary* $L^-_{\overline{a}_{i,j}}$*, is a regular holomorphic curve with* $\mathrm{ind}(\sigma^-_\infty) = 0$.

Proof. — We first calculate the Fredholm index of the holomorphic embedding $\overline{u}: \dot{F} \to \overline{W}_-$ with image $\sigma_\infty^- \subset \overline{W}_-$. Here $\dot{F}$ is a disk with a boundary puncture and an interior puncture. Let τ be a groomed trivialization whose grooming $\mathfrak{c}$ corresponds to the matching $(i,j) \to (i,j)$ and satisfies $w(\mathfrak{c}) = 0$. Then we compute that $-\chi(\dot{F}) = 0$, $\mu_\tau(\delta_0, \overline{u}) = 1$, $\mu_\tau(z_\infty) = 1$, $c_1(\overline{u}^* T\overline{S}, \tau) = 0$. Hence $\operatorname{ind}(\overline{u}) = 0$ by Equation (5.5.3), which is still valid in the current situation.

We now use the doubling technique from Theorem 5.5.1. The double of $\dot{F}$ — a sphere with three punctures — is denoted by $2\dot{F}$ and the double of $\overline{u}$ is denoted by $2\overline{u}$. The index of the doubled operator $2D_{\overline{u}}$ is $\operatorname{ind}(2\overline{u}) = 2\operatorname{ind}(\overline{u}) = 0$ and $D_{\overline{u}}$ is surjective if and only if $2D_{\overline{u}}$ is surjective.

Now, by Wendl's automatic transversality theorem [We2, Theorem 1], $2D_{\overline{u}}$ is surjective if

$$\text{(5.8.1)} \qquad \operatorname{ind}(2\overline{u}) \geq 2g + \#\Gamma_0 - 1,$$

where g is the genus of $2\dot{F}$ and $\#\Gamma_0$ is the count of punctures with even Conley-Zehnder index. We have $g = 0$. Recall that doubling a chord gives a closed orbit with odd Conley-Zehnder index by Lemma 5.5.3. Hence $\#\Gamma_0 = 0$ and Equation (5.8.1) becomes $\operatorname{ind}(2\overline{u}) \geq -1$, which is satisfied since $\operatorname{ind}(2\overline{u}) = 0$. □

Lemma **5.8.10.** — *Let $\overline{J}$ and $\overline{J}_0$ be almost complex structures as in Definition 5.3.2. If $\overline{J}$ is sufficiently close to $\overline{J}_0$, then $\mathcal{M}_{\overline{J}}^{\dagger, n^*=1}(\{z_\infty\} \cup \mathbf{y}, \{y_0\} \cup \mathbf{y})/\mathbf{R}$ is transversely cut out and consists of a unique curve which is represented by a thin strip in D^2 from z_∞ to $y_0 = x_i$ or x_i', together with trivial cylinders.*

Proof. — The proof of transversality is similar to but easier than that of Lemma 5.8.9 and is omitted. The uniqueness of the curve (modulo $\mathbf{R}$-translation in the target) from z_∞ to y_0 is immediate from projecting the curve to D^2 and using the fact that $\overline{J}$ is sufficiently close to $\overline{J}_0$. □

5.8.4. *Marked points and transversality.* — In the definition of the Ψ-map in Section 7, we consider multisections of $\overline{W}_-$ which pass through the marked point $\overline{\mathfrak{m}} = ((0, \frac{3}{2}), z_\infty)$. The marked point $\overline{\mathfrak{m}}$, however, is *nongeneric*. In order to ensure the regularity of such moduli spaces with respect to $\overline{\mathfrak{m}}$, we need to enlarge the class of $\overline{J}_- \in \mathcal{J}_{\overline{W}_-}^{reg}$ to the class of $\overline{J}_-^\diamond$, which we now define.

Definition **5.8.11.** — *Let $\varepsilon > 0$ and let $U \not\ni \overline{\mathfrak{m}}$ be an open set of $\overline{W}_-$. Then an almost complex structure $\overline{J}_-^\diamond$ on $\overline{W}_-$ is* (ε, U)-close *to $\overline{J}_-$ if:*

- $\overline{J}_-^\diamond = \overline{J}_-$ *on* $\overline{W}_- - U$;
- $\overline{J}_-^\diamond$ *is ε-close to $\overline{J}_-$ on U; and*

– $\nabla \overline{J}_-^\diamondsuit$ *is* ε*-close to* $\nabla \overline{J}_-$ *on* U.

Here the ε-closeness is measured with respect to a metric g on $\overline{W}_-$ which is the restriction of an s-invariant metric on $\mathbf{R} \times \overline{N}$ and ∇ is the Levi-Civita connection of g.

Convention **5.8.12.** — *Unless stated otherwise:*

– $U = U_{p,2\delta} = \overline{\pi}_{B_-}^{-1}(B_\delta(p)) - \{\rho \leq \delta\}$ *is an open neighborhood of* $K_{p,2\delta} = \overline{\pi}_{B_-}^{-1}(p) - \{\rho < 2\delta\}$, *where* $\delta > 0$ *is arbitrarily small,* $\overline{\mathfrak{m}}^b \neq p \in B_-$, *and* $B_\delta(p) \subset B_-$ *is an open ball of radius* δ *about* p.
– $\overline{J}_-^\diamondsuit$ *is* (ε, U)*-close to* $\overline{J}_- \in \mathcal{J}^{reg}_{\overline{W}_-}$, *where* $\varepsilon > 0$ *is arbitrarily small.*

Observe that U *is disjoint from the section at infinity. When we want to emphasize* (ε, U) *or* (ε, δ, p), *we write* $\overline{J}_-^\diamondsuit(\varepsilon, U)$ *or* $\overline{J}_-^\diamondsuit(\varepsilon, \delta, p)$.

Definition **5.8.13.** — *A* degree k almost multisection of $(\overline{W}_-, \overline{J}_-^\diamondsuit)$ from $\delta_0^r \boldsymbol{\gamma}$ to $\mathbf{z}' = \{z_\infty^q(\mathcal{D}')\} \cup \mathbf{y}'$ *is a pair* $(\overline{u}, \mathcal{C})$ *which is defined in the same way as a degree k multisection of* $(\overline{W}_-, \overline{J}_-)$, *except that* $\overline{u}$ *is a degree k multisection of*

$$\overline{\pi}_{B_-} : \overline{W}_- - \overline{\pi}_{B_-}^{-1}(B_\delta(p)) \to B_- - B_\delta(p).$$

A $(\overline{W}_-, \overline{J}_-^\diamondsuit)$-curve *is a degree* $2g$ *almost multisection of* $(\overline{W}_-, \overline{J}_-^\diamondsuit)$ *satisfying* $n^-(\overline{u}) = m$ *if* $r = q = 0$ *and* $0 < n^-(\overline{u}) < m$ *if* $r > 0$ *or* $q > 0$.

Let $\mathcal{M}_{\overline{J}_-^\diamondsuit}(\delta_0^r \boldsymbol{\gamma}, \mathbf{z}')$ be the moduli space of almost multisections of $(\overline{W}_-, \overline{J}_-^\diamondsuit)$ from $\delta_0^r \boldsymbol{\gamma}$ to $\mathbf{z}'$. The regularity of $\overline{J}_-$ and the closeness of $\overline{J}_-^\diamondsuit$ to $\overline{J}_-$ imply:

(i) $\mathcal{M}^\dagger_{\overline{J}_-^\diamondsuit}(\delta_0^r \boldsymbol{\gamma}, \mathbf{z}')$ is regular;
(ii) $\mathcal{M}^\dagger_{\overline{J}_-^\diamondsuit}(\delta_0^r \boldsymbol{\gamma}, \mathbf{z}')$ is close to $\mathcal{M}^\dagger_{\overline{J}_-}(\delta_0^r \boldsymbol{\gamma}, \mathbf{z}')$; and
(iii) all the boundary strata of $\mathcal{M}^\dagger_{\overline{J}_-^\diamondsuit}(\delta_0^r \boldsymbol{\gamma}, \mathbf{z}')$ are close to the corresponding boundary strata of $\mathcal{M}^\dagger_{\overline{J}_-}(\delta_0^r \boldsymbol{\gamma}, \mathbf{z}')$.

Note that we can still refer to $n^*(\overline{u})$ since it is a homological quantity.

Let $K \not\ni \overline{\mathfrak{m}}$ be a compact set of $\overline{W}_-$. We define the modifier K to mean that $\overline{u}$ passes through K.

Definition **5.8.14.** — *The almost complex structure* $\overline{J}_-^\diamondsuit$ *on* $\overline{W}_-$ *is* K-regular with respect to $\overline{\mathfrak{m}}$ *if all the moduli spaces* $\mathcal{M}^{\dagger,ext,K}_{\overline{J}_-^\diamondsuit}(\delta_0^r \boldsymbol{\gamma}, \mathbf{z}'; \overline{\mathfrak{m}})$ *are transversely cut out.*

Lemma **5.8.15.** — *A generic* $\overline{J}_-^\diamondsuit$ *is* $K_{p,2\delta}$*-regular with respect to* $\overline{\mathfrak{m}}$.

Proof. — The proof is similar to that of [MS, Theorem 3.4.1], with modifications as in Proposition 5.8.8. □

6. The chain map from $\widehat{\mathrm{HF}}$ to PFH

6.1. *Compactness for* W_+*-curves.* — In this subsection we treat the compactness of holomorphic curves in W_+ which will be used to establish the chain map Φ in Section 6.2.

Suppose $J_+ \in \mathcal{J}_{W_+}$ and J, J' are the restrictions of J_+ to the positive and negative ends. Let $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ and $\boldsymbol{\gamma} = \prod_{k=1}^{l} \gamma_k^{m_k} \in \widehat{\mathcal{O}}_{2g}$.

In this section, we may pass to a subsequence of a sequence of holomorphic curves without specific mention.

6.1.1. *Euler characteristic bounds.* — We first state a preliminary lemma:

Lemma **6.1.1.** — *Let* $u_i : (\dot{F}_i, j_i) \to (W_+, J_+)$, $i \in \mathbf{N}$, *be a sequence of* W_+*-curves from* $\mathbf{y}$ *to* $\boldsymbol{\gamma}$ *with index* $I_{W_+}(u_i) = n$ *for some integer* n*. Then there is a subsequence such that all the* $\dot{F}_i$ *are diffeomorphic to a fixed* $\dot{F}$.

Proof. — The proof is given in two steps.

Step 1 (ω-area bounds). We define the ω*-area* of u_i as

$$E_\omega(u_i) = \int_{\dot{F}_i} u_i^* \omega,$$

where ω is the 2-form as in Section 5.1. The boundedness of $E_\omega(u_i)$ is a consequence of the vanishing of the flux F_h of h (cf. Section 3.3). View the broken closed string $\gamma_{\mathbf{y}}$ corresponding to $\mathbf{y}$ as a collection of curves in

$$(L_{\mathbf{a}}^+ \cap \check{W}_+) \cup (\{3\} \times [0,1] \times \mathbf{y}) \subset \check{W}_+.$$

Then $\gamma_{\mathbf{y}}$ is uniquely determined up to a homotopy which is supported on $L_{\mathbf{a}}^+ \cap \check{W}_+$. Let $u_i : \dot{F}_i \to W_+$, $i = 1, 2$, be two W_+-curves from $\mathbf{y}$ to $\boldsymbol{\gamma}$ and let $\check{u}_i : \check{F}_i \to \check{W}_+$ be their compactifications. Then $\check{u}_i(\partial_+ \check{F}_i)$ is homotopic to $\gamma_{\mathbf{y}}$, $i = 1, 2$, where $\partial_+ \check{F}_i$ is the union of boundary components of $\check{F}_i$ which map to the positive ($s > 0$) part of $\check{W}_+$. Hence $\check{u}_1 - \check{u}_2$ can be viewed as a closed surface $Z \in H_2(W_+) \simeq H_2(\check{W}_+) \simeq H_2(N)$. Since the flux F_h vanishes, the integral of ω over Z vanishes, and therefore the ω-area of a W_+-curve u only depends on $\mathbf{y}$ and $\boldsymbol{\gamma}$.

Step 2 (Genus bounds). The ω-area bound on the sequence $\{u_i\}$ and the fact that all u_i have the same asymptotics imply an energy bound on the u_i (for the energy defined as in Equation (4.3.1)).

We then apply the Gromov-Taubes compactness theorem [T3, Proposition 3.3], whose proof carries over to the symplectic cobordism (W_+, Ω_+) with little change: The energy bound implies a local Ω_+-area bound, and we note the statement of [T3, Proposition 3.3] is local, i.e., works on a fixed ball in W_+. To take care of the Lagrangian boundary condition, we take local doubles of u_i using Conditions (SR) that appear in Definitions 5.3.2 and 5.3.14.

More precisely, given any point $p \in L^+_{\mathbf{a}}$, there is a sufficiently small neighborhood of p in $\overline{W_+}$ of the form $\mathcal{U} \times D$ and a homeomorphism

$$\phi = (\phi_1, \phi_2) \colon H \times D^2 \stackrel{\sim}{\to} \mathcal{U} \times D,$$

where:

(1) $H = \{(x_1, y_1) \in \mathbf{R}^2 \mid x_1^2 + y_1^2 \leq 1, y_1 \geq 0\}$ and $D^2 = \{(x_2, y_2) \in \mathbf{R}^2 \mid x_2^2 + y_2^2 \leq 1\}$ with the standard complex structures, $\mathcal{U} \subset B_+$ is a closed neighborhood of $\pi_{W_+}(p)$, and $D \subset \overline{S}$ is a closed disk;
(2) $\phi_1 : H \stackrel{\sim}{\to} \mathcal{U}$ is a homeomorphism which is holomorphic on int(H) and is a diffeomorphism away from the two corner points of H;
(3) $\phi_2 : D^2 \stackrel{\sim}{\to} D$ is a biholomorphism; and
(4) $\phi^{-1}(p) = (0, 0, 0, 0)$ and $\phi^{-1}(L^+_{\mathbf{a}} \cap (\mathcal{U} \times D)) = \{y_1 = y_2 = 0\} \subset H \times D^2$.

The existence of the map ϕ_1 is proved as follows: Let $\mathcal{U}$ be the intersection of B_+ with a closed ball in $\mathbf{R} \times S^1$ centered at $\pi_{W_+}(p)$. If the radius of the ball is sufficiently small, then $\mathcal{U}$ is a closed half-disk whose boundary is continuous and piecewise smooth with two corner points only at $\mathcal{U} \cap \partial B_+$. By the Riemann mapping theorem there exists a biholomorphism $\phi_1' : \mathrm{int}(H) \stackrel{\sim}{\to} \mathrm{int}(\mathcal{U})$, which, by Carathéodory's theorem, extends to a homeomorphism $\phi_1 : H \stackrel{\sim}{\to} \mathcal{U}$; see [Ru, Theorem 14.18]. Away from the two corner points, the extension is smooth by [Wa1, Theorem IV] and has nonvanishing derivative by [Wa2, Theorem 1]. Note that we cannot assume that ϕ_1 is holomorphic up to the boundary because ∂B_+ was not defined to be real analytic. The existence of the map ϕ_2 is similar but easier. Condition (SR) from Definitions 5.3.2 implies that the pullback of J_+ to $\{y_1 < 0\} \times D$ by ϕ is the standard split complex structure.

Denote $G_i := (u_i)^{-1}(\mathcal{U} \times D) \subset \dot{F}$ and $v_i = \phi^{-1} \circ u_i|_{G_i}$. By the Schwarz reflection principle we can double G_i and H along $G_i \cap \partial\dot{F}$ and $\partial H \cap \{y_1 = 0\}$ and obtain maps $\tilde{v}_i : \widetilde{G}_i \to \widetilde{H} \times D^2$ which extend the maps v_i and are holomorphic on $\widetilde{G}_i \setminus \partial\dot{F}$. Then the $\tilde{v}_i$ are holomorphic on $\widetilde{G}_i$ by an application of Morera's theorem; see [Ru, Theorem 16.8]. The maps $\tilde{v}_i$ also satisfy energy bounds and [T3, Proposition 3.3] carries over to $\widetilde{H} \times D^2$, hence yielding a holomorphic limit $\tilde{v}_\infty$. It remains to note that half of $\tilde{v}_\infty$ corresponding to the limit of v_i will have boundary on $L^+_{\mathbf{a}}$ since during the construction of $\mathrm{Im}\, \tilde{v}_\infty$ as a set in the proof of [T3, Lemma 3.5], we take small balls centered at points of $\mathrm{Im}\, \tilde{v}_i$ and covering $\mathrm{Im}\, \tilde{v}_i$ and we can choose a subset of them to be centered at points of $v_i(\partial\dot{F}_i)$ and to cover $v_i(\partial\dot{F}_i)$.

The Gromov-Taubes compactness theorem and the argument of [Hu1, Lemma 9.8] then imply the weak convergence of u_i as currents to a holomorphic building u_∞. In particular, we may assume that $[u_i] \in H_2(W_+, \mathbf{y}, \boldsymbol{\gamma})$ is fixed for all i.

We now use the fact that $[u_i]$ is fixed to bound the genus of $\dot{F}_i$. The relative adjunction formula (Lemma 5.6.3) gives:

$$c_1(\check{u}_i^*TW_+,(\tau,\partial_t)) = \chi(\dot{F}_i) - w_\tau^-(u_i) + Q_\tau(u_i) - 2\delta(u_i).$$

In view of the writhe bound

$$w_\tau^-(u_i) \geq \widetilde{\mu}_\tau(\boldsymbol{\gamma}) - \mu_\tau^-(u_i)$$

from [Hu2, Lemma 4.20] and the nonnegativity of $\delta(u_i)$, we obtain:

$$\chi(\dot{F}_i) \geq c_1(\check{u}_i^*TW_+,(\tau,\partial_t)) + \widetilde{\mu}_\tau(\boldsymbol{\gamma}) - \mu_\tau^-(u_i) - Q_\tau(u_i). \tag{6.1.1}$$

This bounds $\chi(\dot{F}_i)$ from below, since all the terms on the right-hand side either depend on the homology class of u_i or the data of the ends. Hence we may assume that all the $\dot{F}_i$ are diffeomorphic to a fixed $\dot{F}$. □

6.1.2. *SFT compactness.*

Definition **6.1.2.** — *A* holomorphic W_+-building

$$u_\infty = v_{-b} \cup \cdots \cup v_a, \quad a, b \in \mathbf{Z}^{\geq 0}$$

consists of the following data:

- (B1) *For each $j = -b, \dots, a$, a compact* nodal *Riemann surface G_j (possibly with boundary), a nodal set $\mathfrak{n}_j$, disjoint sets of interior punctures $\mathbf{p}_j^+$ and $\mathbf{p}_j^-$ if $j \leq 0$, and disjoint sets of boundary punctures $\mathbf{q}_j^+$ and $\mathbf{q}_j^-$ if $j \geq 0$. The nodes may be interior or boundary nodes and are disjoint from $\mathbf{p}_j^\pm$, $\mathbf{q}_j^\pm$, and the nodal set $\mathfrak{n}_j$ may be empty.*
- (B2) *For each $j = -b, \dots, a$, a holomorphic map $v_j : \dot{G}_j \to W_j$, where $W_j = W$ for $0 < j \leq a$, $W_0 = W_+$ for $j = 0$, $W_j = W'$ for $-b \leq j < 0$, and $\dot{G}_j = G_j - \mathbf{p}_j^+ - \mathbf{p}_j^- - \mathbf{q}_j^+ - \mathbf{q}_j^-$ for all j. (Here some sets of punctures may be empty.)*
- (B3) *For each $j = 0, \dots, a$, $\partial \dot{G}_j$ is mapped to the appropriate Lagrangian submanifold $L_{\mathbf{a}} \sqcup L_{h(\mathbf{a})}$ or $L_{\mathbf{a}}^+$.*
- (B4) *For each $j = -b, \dots, a$, v_j converges to a strip over a "Reeb chord" at the positive (resp. negative) end near each boundary puncture of $\mathbf{q}_j^+$ (resp. $\mathbf{q}_j^-$) and to a cylinder over a closed orbit at the positive (resp. negative) end near each interior puncture of $\mathbf{p}_j^+$ (resp. $\mathbf{p}_j^-$).*
- (B5) *for each $j = -b, \dots, a-1$, there is an identification between $\mathbf{p}_j^+$ and $\mathbf{p}_{j+1}^-$ and an identification between $\mathbf{q}_j^+$ and $\mathbf{q}_{j+1}^-$ such that the pairs that are identified are asymptotic to the same Reeb chord or closed orbit.*
- (B6) *No level v_j is a union of trivial cylinders or trivial strips and no constant sphere has fewer than three nodes.*

The levels v_j are arranged in order from lowest to highest.

Proposition **6.1.3.** — *Let $u_i : (\dot{F}_i, j_i) \to (W_+, J_+)$, $i \in \mathbf{N}$, be a sequence of W_+-curves from $\mathbf{y}$ to $\boldsymbol{\gamma}$ with index $I_{W_+}(u_i) = n$ for some integer n. Then there is a subsequence which converges in the sense of SFT to a level $a+b+1$ holomorphic W_+-building $u_\infty = v_{-b} \cup \cdots \cup v_a$.*

Here "convergence in the SFT sense" means convergence with respect to the topology described in [BEHWZ].

Proof. — By Lemma 6.1.1, we may assume that $\dot{F}_i = \dot{F}$ as smooth surfaces. We can then apply the SFT compactness theorem from [BEHWZ]; note that SFT compactness in the presence of Lagrangian boundary conditions is sketched in [BEHWZ, Section 11.3] and [Abb2, Theorem 3.20]. More details can be extracted from the proof of the less standard compactness theorem for $\overline{W}_-$-curves; see Proposition 7.3.2. □

The limiting curve u_∞ can be written as a level $a+b+1$ holomorphic building $v_{-b} \cup \cdots \cup v_a$, where each v_j is not necessarily irreducible and may have nodes. Here (i) a, b are nonnegative integers, (ii) v_j is a holomorphic map to W_j, and (iii) the levels v_j are ordered from the negative end to the positive end as j increases. As usual, if the level v_j is just a union of trivial cylinders, then it will be elided.

6.1.3. *Main theorem.* — Suppose $J_+ \in \mathcal{J}^{reg}_{W_+}$ and J, J' are the restrictions of J_+ to the positive and negative ends.

The following is the main theorem of this subsection:

Theorem **6.1.4.** — *Let $u_i : (\dot{F}_i, j_i) \to (W_+, J_+)$, $i \in \mathbf{N}$, be a sequence of W_+-curves from $\mathbf{y}$ to $\boldsymbol{\gamma}$. If $I_{W_+}(u_i) = 1$ for all i, then a subsequence of u_i converges in the sense of SFT to one of the following:*

(1) *an $I_{W_+} = 1$ curve;*
(2) *a building with two levels consisting of an $I_{HF} = 1$ curve and an $I_{W_+} = 0$ curve; or*
(3) *a building with multiple levels consisting of an $I_{W_+} = 0$ curve, an $I_{ECH} = 1$ curve, and possible $I_{ECH} = 0$ connectors in between.*

Similarly, if $I_{W_+}(u_i) = 0$ for all i, then a subsequence of u_i converges to an $I_{W_+} = 0$ curve.

We write "an $I_\# = i$ curve" as shorthand for "a #-curve with ECH index $I_\# = i$".

We postpone the proof of the main theorem until after a more detailed discussion of the structure of the SFT limit. Let u_∞ be the SFT limit of the sequence $\{u_i\}$, given by Proposition 6.1.3. A *ghost component* of u_∞ is an irreducible component of u_∞ which maps to a point. By the SFT compactness theorem, the domain of a ghost component is necessarily a stable Riemann surface. We recall that a Riemann surface is stable if the following holds for each component F with k_{int} interior marked points and k_{bdr} boundary marked points:

– $-\chi(F) + k_{int} \geq 1$ when $\partial F = \varnothing$, or

– $-2\chi(F)+2k_{int}+k_{bdr}\geq 1$ if $\partial F\neq\varnothing$.

Let us write $u_\infty = v_{-b}\cup\cdots\cup v_a\cup u^g_\infty$, where v_j has no ghost components and u^g_∞ is the union of ghost components. We will also write $u^{ng}_\infty = v_{-b}\cup\cdots\cup v_a$.

Remark **6.1.5.** — The ECH index of a curve depends only on its relative homology class and therefore ghost components do not contribute to it. Hence, by the additivity of ECH indices (Lemma 5.6.8), if u_i is a sequence of J_+-holomorphic maps with constant ECH index, then:

$$\sum_{j=1}^{a} I_{HF}(v_j) + I_{W_+}(v_0) + \sum_{j=1}^{b} I_{ECH}(v_{-j}) = I_{W_+}(u_i). \tag{6.1.2}$$

Lemma **6.1.6.** — *Each level v_j, $j=-b,\dots,a$, is a degree $2g$ multisection of $\pi_j: W_j\to B_j$ with no branch points along ∂B_j.*

Proof. — Since u_i is a degree $2g$ multisection of $\pi_{B_+}: W_+\to B_+$ for all i, it follows that, with the exception of finitely many $p\in B_j$, every level v_j intersects a fiber $\pi_j^{-1}(p)$ exactly $2g$ times.

We show that on any level v_j there are no irreducible components which lie in a fiber $\pi_j^{-1}(p)$. Arguing by contradiction, suppose $\widetilde{v}:\widetilde{F}\to W_j$ is an irreducible component which maps to a fiber $\pi_j^{-1}(p)$. If $p\in int(B_j)$, then $\widetilde{v}$ is a holomorphic map from a closed Riemann surface $\widetilde{F}$ to $\pi_j^{-1}(p)$. Since $\pi_j^{-1}(p)$ is a Riemann surface with nonempty boundary, $\widetilde{v}$ must be constant. On the other hand, if $p\in\partial B_j$, it is also possible that $\widetilde{F}$ is a compact Riemann surface with nonempty boundary and $\widetilde{v}(\partial\widetilde{F})\subset\mathbf{a}$. However, since $S-\mathbf{a}$ is connected and nontrivially intersects ∂S, $\widetilde{v}$ must also be constant. Since ghost components are excluded from v_j by definition, we have a contradiction.

Finally, if $j\geq 0$, then we claim that $\pi_j\circ v_j$ has no branch points along ∂B_j. This is due to the fact that v_0 uses each component of $L^+_{\mathbf{a}}$ exactly once and v_j, $j>0$, uses each component of $\mathbf{R}\times\{1\}\times\mathbf{a}$ and each component of $\mathbf{R}\times\{0\}\times h(\mathbf{a})$ exactly once. □

Lemma **6.1.7.** — *Let u_∞ be the SFT limit of a sequence of J_+-holomorphic multisections u_i with constant ECH index. If J_+ is regular, then:*

– $I_{HF}(v_j)>0$ *for* $j>0$,
– $I_{W_+}(v_0)\geq 0$, *and*
– $I_{ECH}(v_j)\geq 0$ *for* $j<0$.

Moreover, all the v_j, $j\geq 0$, are somewhere injective and satisfy $\operatorname{ind}(v_j)\geq 0$. *If* $I_{W_+}(u_i)\leq 1$ *in addition, then v_j, $j<0$, is somewhere injective and satisfies* $\operatorname{ind}(v_j)\geq 0$.

Proof. — Since v_j, $j\geq 0$, is a degree $2g$ multisection by Lemma 6.1.6 and uses each connected component of the boundary Lagrangian exactly once, it is somewhere injective. Also since J and J_+ are regular, it follows that the curves v_j, $j \geq 0$, are regular. Hence $\mathrm{ind}_W(v_j) \geq 0$ for $j > 0$ and $\mathrm{ind}_{W_+}(v_0) \geq 0$. Moreover, since $\mathrm{ind}_W(v_j) = 0$, $j > 0$, if and only if v_j is a union of trivial strips, we may assume that $\mathrm{ind}_W(v_j) > 0$ for all $j > 0$. By the index inequality (Theorems 4.5.13 and 5.6.9) we have $I_{HF}(v_j) > 0$ for $j > 0$ and $I_{W_+}(v_0) \geq 0$. On the other hand, if $j < 0$, then $I_{ECH}(v_j) \geq 0$ by [HT1, Proposition 7.15(a)].

If $I_{W_+}(u_i) \leq 1$ in addition, then $I_{ECH}(v_j) \leq 1$ by the additivity of the ECH index. Hence [Hu1, Lemma 9.5] implies that v_j is somewhere injective and $\mathrm{ind}(v_j) \geq 0$ since J' is regular. □

Lemma **6.1.8.** — *Let u_∞ be the SFT limit of a sequence of J_+-holomorphic multisections u_i with $I_{W_+}(u_i) \leq 1$ for all i. If J_+ is regular, then $u^g_\infty = \varnothing$.*

Proof. — Let $\mathrm{ind}(u^g_\infty)$ and $\mathrm{ind}(u^{ng}_\infty)$ be the sum of the Fredholm indices of the irreducible components of u^g_∞ and u^{ng}_∞, respectively. Also let F^g and F^{ng} be the domains of u^g_∞ and u^{ng}_∞, respectively. Then $\mathrm{ind}(u^g_\infty) = -\chi(F^g)$. If u^g_∞ and u^{ng}_∞ are attached along k_{int} interior nodes and k_{bdr} boundary nodes, then

$$\mathrm{ind}(u_i) = \mathrm{ind}(u^{ng}_\infty) + \mathrm{ind}(u^g_\infty) + 2k_{int} + k_{bdr} \tag{6.1.3}$$

$$= \mathrm{ind}(u^{ng}_\infty) - \chi(F^g) + 2k_{int} + k_{bdr} \leq 1. \tag{6.1.4}$$

Next, if $F^g \neq \varnothing$, then the stability of the Riemann surface and the observation that each component of ∂F^g has a marked point (which in turn follows from Definition 5.4.3(1)) imply that

$$-\chi(F^g) + 2k_{int} + k_{bdr} \geq 2. \tag{6.1.5}$$

Equations (6.1.4) and (6.1.5) together imply that $\mathrm{ind}(u^{ng}_\infty) \leq -1$. Hence we have $\mathrm{ind}(v_j) \leq -1$ for some v_j, which is a contradiction of Lemma 6.1.7 for a regular J_+. The contradiction came from assuming that $u^g_\infty \neq \varnothing$. □

We now finish the proof of Theorem 6.1.4.

Proof of Theorem 6.1.4. The proof is based on a classification of the types of allowable buildings $u_\infty = v_{-b} \cup \cdots \cup v_a \cup u^g_\infty$. We consider the situation of $I_{W_+}(u_i) = 1$, leaving the easier $I_{W_+}(u_i) = 0$ case to the reader.

By Lemmas 6.1.6 and 6.1.8, the limit u_∞ consists of a building of degree $2g$ multisections v_j. Moreover, by Lemma 6.1.7, $I_{W_j}(v_j) \geq 0$ for all j. The additivity of ECH indices gives three possibilities for the limit:

(1) $u_\infty = v_0$, where v_0 is a multisection of W_+ and $I_{W_+}(v_0) = 1$;
(2) $u_\infty = v_0 \cup v_1$, where v_0 is a multisection of W_+ with $I_{W_+}(v_0) = 0$, and v_1 is a multisection of W with $I_{HF}(v_1) = 1$;

(3) $u_\infty = v_{-b} \cup \cdots \cup v_0$, where $v_{-b}, \dots, v_{-1}$ are multisections of W', v_0 is a multisection of W_+, all but one of $v_{-b}, \dots, v_0$ have $I = 0$, and the remaining level has $I = 1$.

What is left to prove is that $I_{ECH}(v_{-b}) = 1$ in Case (3). This follows from considerations of incoming partitions as in [HT1, Proof of Lemma 7.23]. Let m_k be the multiplicity of the elliptic orbit γ_k in the orbit set $\boldsymbol{\gamma}$ and let θ_k be the rotation of the first return map of γ_k. By [HT1, Definition 1.8], there is a partial order $\geq_{\theta_k}$ on the set of partitions of m_k, where

$$(a_1, \dots, a_{l_1}) \geq_{\theta_k} (b_1, \dots, b_{l_2})$$

if there is a Fredholm index zero branched cover of $\mathbf{R} \times \gamma_k$ with positive ends which partition m_k into $(a_1, \dots, a_{l_1})$ and negative ends which partition m_k into $(b_1, \dots, b_{l_2})$. By [HT1, Lemma 7.5], the incoming partition $P^{out}_{\gamma_k}(m_k)$ — the partition which corresponds to $\gamma_k^{m_k}$ at the negative end of v_{-b} — is maximal. Hence $I_{ECH}(v_{-b}) = 0$ implies that v_{-b} is a trivial cylinder, which we have already eliminated. □

6.2. *Definition of* Φ. — Suppose $J_+ \in \mathcal{J}^{reg}_{W_+}$ and J, J' are the restrictions of J_+ to the positive and negative ends. In this subsection we define the chain map

$$\Phi_{J_+} : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), J) \to PFC_{2g}(N, J'). \tag{6.2.1}$$

We will usually suppress J, J', and J_+ from the notation.

We first define an approximation of Φ:

Definition **6.2.1.** — *Let $\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$ be the chain complex generated by $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$, before quotienting by the equivalence relation $\sim$ given in Section 4.9.3. We define the map*

$$\Phi' : \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N),$$

$$\Phi'(\mathbf{y}) = \sum_{\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_{2g}} \langle \Phi'(\mathbf{y}), \boldsymbol{\gamma} \rangle \cdot \boldsymbol{\gamma},$$

where $\langle \Phi'(\mathbf{y}), \boldsymbol{\gamma} \rangle$ is the mod 2 count of $\mathcal{M}^{I=0}_{J_+}(\mathbf{y}, \boldsymbol{\gamma})$. The count is meaningful since $\mathcal{M}^{I=0}_{J_+}(\mathbf{y}, \boldsymbol{\gamma})$ is compact by Theorem 6.1.4.

Proposition **6.2.2.** — *The map*

$$\Phi' : \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N),$$

is a chain map.

Proof. — By Theorem 6.1.4 and Lemma 5.4.9, $\partial\mathcal{M}^{I=1}_{J_+}(\mathbf{y},\boldsymbol{\gamma}) = A\sqcup B$, where

$$A=\coprod_{\mathbf{y}'\in\mathcal{S}_{\mathbf{a},h(\mathbf{a})}}\left(\left(\mathcal{M}^{I=1}_{J}(\mathbf{y},\mathbf{y}')/\mathbf{R}\right)\times\mathcal{M}^{I=0}_{J_+}(\mathbf{y}',\boldsymbol{\gamma})\right),$$

$$B=\coprod_{\boldsymbol{\gamma}\in\widehat{\mathcal{O}}_{2g}}\left(\mathcal{M}^{I=0}_{J_+}(\mathbf{y}',\boldsymbol{\gamma}')\times\left(\mathcal{M}^{I=1}_{J'}(\boldsymbol{\gamma}',\boldsymbol{\gamma})/\mathbf{R}\right)\right).$$

Here we have omitted the connector components for simplicity.

We examine the corresponding gluings of the holomorphic buildings. The first gluing is that of $(v_1,v_0)\in A$. This type of gluing was treated by Lipshitz; see Propositions A.1 and A.2 in [Li, Appendix A]. Observe that there are no multiply-covered curves to glue, since each Reeb chord of a $2g$-tuple is used exactly once.

The second type of gluing is that of $(v_0,v_{-b})\in B$, with $I_{ECH}=0$ connectors $v_{-1},\dots,v_{-b+1}$ in between. The curve v_0 is simply-covered since it has an HF end and the curve v_{-b} is simply-covered since $I_{ECH}=1$ (see [HT1, Proposition 7.15]). This type of gluing was treated carefully in [HT1, HT2]. Although the setting there was the gluing for $\partial^2=0$, in fact most of the work goes towards properly counting $I_{ECH}=0$ connectors, i.e., branched covers of trivial cylinders. Their treatment of gluing/counting the $I_{ECH}=0$ connectors carries over with little modification to our case. See Section 6.5 for more details. □

Let $\delta_x=([0,2]\times\{x\})/(2,x)\sim(0,x)$, where $x\in\partial S$; it is a Reeb orbit in the negative Morse-Bott family $\mathcal{N}$ which foliates ∂N. Also let $(\mathbf{R}\times\delta_x)^+$ be the restriction of $\mathbf{R}\times\delta_x$ to W_+.

Lemma **6.2.3.** — *A* W_+*-curve* u *which has* x_i *or* x_i' *at the positive end must have* $(\mathbf{R}\times\delta_{x_i})^+$ *or* $(\mathbf{R}\times\delta_{x_i'})^+$ *as an irreducible component.*

Recall that the intersection points x_i, x_i' are components of the Heegaard Floer contact class of $\xi_{(S,h)}$.

Proof. — Let $v:\dot{F}\to W_+$ be the irreducible component of u which has x_i at the positive end. The component v may *a priori* have other positive ends besides x_i. We will show that $v=(\mathbf{R}\times\delta_{x_i})^+$. Let $\pi_S:\{s\geq 3\}\cap W_+\to S$ be defined by identifying $\{s\geq 3\}\cap W_+=[3,\infty)\times[0,1]\times S$ and projecting to the third factor.

Let (r,θ) be polar coordinates on a small neighborhood $\nu(x_i)\subset S$ of x_i so that $h(a_i)=\{\theta=-\frac{\pi}{4}\}$, $a_i=\{\theta=0\}$, and $\nu(x_i)=\{-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}, r>0\}$. By Condition (1) in Definition 5.3.2, the composition $\pi_S\circ v$ is holomorphic when restricted to the end that limits to x_i. If v is not the restriction of a trivial cylinder, then $\pi_S\circ v$ must map a neighborhood of the puncture of $\dot{F}$ corresponding to x_i to a sector $\{0\leq\theta\leq k\pi-\frac{\pi}{4}, r>0\}$ or $\{\pi\leq\theta\leq(k+1)\pi-\frac{\pi}{4}, r>0\}$, where $k\geq 1$. Since such a sector cannot be contained in S, the map $\pi_S \circ v|_{s\geq C}$ must map identically to x_i, where $C \gg 0$. By the unique continuation property, $v = (\mathbf{R} \times \delta_{x_i})^+$. □

Theorem **6.2.4.** — *The map* Φ' *descends to a chain map*

$$\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N)$$

which maps the Heegaard Floer contact invariant for $\xi_{(S,h)}$ *to the ECH contact invariant for* $\xi_{(S,h)}$.

Proof. — Let $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$. Assume without loss of generality that $y_1 = x_1$. Then $\mathbf{y}$ is equivalent to $\mathbf{y}' = \{x_1', y_2, \dots, y_{2g}\}$. Let u be an $I_{W_+} = 0$ Morse-Bott building from $\mathbf{y}$ to a generator $\boldsymbol{\gamma}$ of $PFC_{2g}(N)$. Since $(\mathbf{R} \times \delta_{x_1})^+$ is an irreducible component of u by Lemma 6.2.3 and is automatically transverse in the Morse-Bott sense, $\boldsymbol{\gamma}$ can be written as $e\boldsymbol{\gamma}'$, where e is the elliptic orbit of the negative Morse-Bott family $\mathcal{N}$. Now, by replacing $(\mathbf{R} \times \delta_{x_1})^+$ by $(\mathbf{R} \times \delta_{x_1'})^+$ and the augmenting gradient trajectory from δ_{x_1} to e by the augmenting trajectory from $\delta_{x_1'}$ to e, we obtain an $I_{W_+} = 0$ Morse-Bott building from $\mathbf{y}'$ to $\boldsymbol{\gamma} = e\boldsymbol{\gamma}'$. Hence there is a one-to-one correspondence between W_+-curves from $\mathbf{y}$ to $\boldsymbol{\gamma}$ and W_+-curves from $\mathbf{y}'$ to $\boldsymbol{\gamma}$. Since $\Phi'(\mathbf{y}) = \Phi'(\mathbf{y}')$, the map Φ' descends to $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$.

The Heegaard Floer contact invariant is given by the equivalence class of $\mathbf{x} = \{x_1, \dots, x_{2g}\}$ and the ECH contact invariant is given by e^{2g}. By Lemma 6.2.3 and some ECH index considerations, the only $I_{W_+} = 0$ Morse-Bott building from $\mathbf{x}$ consists of $(\mathbf{R} \times \delta_{x_i})^+$, $i = 1, \dots, 2g$, augmented by connecting gradient trajectories from δ_{x_i} to e in the Morse-Bott family $\mathcal{N}$. Hence $\Phi'(\mathbf{x}) = e^{2g}$. Then Φ maps the equivalence class of $\mathbf{x}$ to e^{2g}. □

6.3. Spin^c-*structures.* — Let $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ be the set of $2g$-tuples of $\mathbf{a} \cap h(\mathbf{a})$ and let $\overline{\mathcal{S}}_{\mathbf{a},h(\mathbf{a})} = \mathcal{S}_{\mathbf{a},h(\mathbf{a})}/\sim$, where $\{x_i\} \cup \mathbf{y}' \sim \{x_i'\} \cup \mathbf{y}'$ for all $(2g-1)$-tuples $\mathbf{y}'$.

We define a map

$$\mathfrak{h}_+' : \mathcal{S}_{\mathbf{a},h(\mathbf{a})} \to H_1(W_+, \partial_h W_+)$$

as follows: Given $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$, where $y_i \in a_i \cap h(a_{\sigma(i)})$ for some permutation $\sigma \in \mathfrak{S}_{2g}$, we define $\mathfrak{h}_+'(\mathbf{y})$ as the homology class of the broken closed string obtained by concatenating the following oriented arcs:

- for each $i \in \{1, \dots, 2g\}$, the arc $\{3\} \times [0,1] \times \{y_i\}$, with orientation given by ∂_t;
- for each $i \in \{1, \dots, 2g\}$, an arc from $\{3\} \times \{1\} \times \{y_i\}$ to $\{3\} \times \{0\} \times \{y_{\sigma^{-1}(i)}\}$ contained in $L_{a_i}^+$.

The homology class $\mathfrak{h}_+'(\mathbf{y})$ is well-defined since the Lagrangians $L_{a_i}^+$ are simply-connected.

The map $\mathfrak{h}'_+$ descends to a map

$$\mathfrak{h}_+ : \overline{\mathcal{S}}_{\mathbf{a},h(\mathbf{a})} \to H_1(W_+, \partial_h W_+),$$

since the components x_i and x'_i of the contact class are both converted into broken closed strings which are nullhomologous in $H_1(W_+, \partial_h W_+)$.

The following lemma is an immediate consequence of the definition of W_+-curves.

Lemma **6.3.1.** — *Let* $\boldsymbol{\gamma}$ *be an orbit set at the negative end of* W_+ *with total homology class* $[\boldsymbol{\gamma}] \in H_1(W_+, \partial_h W_+)$. *Then* $\mathcal{M}_{J_+}(\mathbf{y}, \boldsymbol{\gamma}) \neq \varnothing$ *implies that* $\mathfrak{h}_+(\mathbf{y}) = [\boldsymbol{\gamma}]$.

There are natural isomorphisms

$$H_1(W_+, \partial_h W_+) \cong H_1(N, \partial N) \cong H_1(M).$$

With respect to these isomorphisms, the total homology class in $H_1(W_+, \partial_h W_+)$ of an orbit set $\boldsymbol{\gamma}$ at the negative end of W_+ corresponds to the usual total homology class of $\boldsymbol{\gamma}$ in $H_1(M)$. Moreover, $\mathfrak{h}_+(\mathbf{y}) = \mathfrak{h}(\mathbf{y})$, where $\mathfrak{h}$ is as defined in Section 4.10.

Combining Proposition 4.10.1 and Lemma 6.3.1, we obtain the following theorem:

Theorem **6.3.2.** — *The chain map* Φ *respects the splitting according to* Spin^c*-structures, i.e.,* Φ *is the direct sum of maps*

$$\Phi_A : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}_\xi + PD(A)) \to PFC_{2g}(N, A),$$

with $A \in H_1(M; \mathbf{Z})$.

6.4. *Twisted coefficients.* — Let $\underline{PFH}_{2g}(N, A)$ be the twisted coefficient version of $PFH_{2g}(N, A)$, defined as in Section 2.4. For any homology class $A \in H_1(M)$ we can define a map

$$\underline{\Phi}_A : \underline{\widehat{CF}}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}_\xi + PD(A)) \to \underline{PFC}_{2g}(N, A),$$

given in the following paragraphs. Choose a $2g$-tuple of intersection points $\mathbf{y}_0$ such that $s_z(\mathbf{y}_0) = \mathfrak{s}_\xi + PD(A)$ and a complete set of paths $\{C_\mathbf{y}\}$ for $\mathfrak{s}_\xi + PD(A)$ based at $\mathbf{y}_0$.

Let $\pi_N : W_+ \to N$ be the restriction of the projection $\mathbf{R} \times N \to N$, $(s, x) \mapsto x$, and let $\Gamma \subset N$ be the projection by π_N of a broken closed string associated to $\mathbf{y}_0$. By Lemma 6.3.1, $[\Gamma] = A$. We choose a complete set of paths $\{C_\boldsymbol{\gamma}\}$ for A based at Γ.

The projection π_N associates a well-defined homology class in $H_2(N) \cong H_2(M)$ to any 2-chain in $\check{W}_+$ whose boundary consists of a broken closed string corresponding to $\mathbf{y}_0$ on one side and Γ on the other side. Then we can define maps

$$\mathfrak{A}_+ : H_2(W_+, \mathbf{y}, \boldsymbol{\gamma}) \to H_2(M)$$

for all $\mathbf{y}$ such that $s_z(\mathbf{y_0}) = \mathfrak{s}_\xi + PD(A)$ and $\boldsymbol{\gamma}$ such that $[\boldsymbol{\gamma}] = A$ by

$$\mathfrak{A}_+(C) = (\pi_N)_*[C_\boldsymbol{\gamma} \cup C \cup -C_\mathbf{y}].$$

Definition **6.4.1.** — *We define the* $\mathbf{F}[H_2(M;\mathbf{Z})]$*-linear map*

$$\underline{\Phi}_A : \underline{\widehat{CF}}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}_\xi + PD(A)) \to \underline{PFC}_{2g}(N, A)$$

by

$$\underline{\Phi}_A(\mathbf{y}) = \sum_{\boldsymbol{\gamma}\in\widehat{\mathcal{O}}_{2g}} \sum_{C\in H_2(W_+,\mathbf{y},\boldsymbol{\gamma})} \#\mathcal{M}_{J_+}^{I=0}(\mathbf{y},\boldsymbol{\gamma},C)e^{\mathfrak{A}_+(C)}\boldsymbol{\gamma}.$$

Theorem **6.4.2.** — *The map* $\underline{\Phi}_A$ *is a chain map.*

Proof. — The proof is the same as that of Theorem 6.2.4, plus some bookkeeping of the homology classes of the holomorphic maps involved. □

6.5. *Gluing.* — We explain here how to glue the pair (v_0, v_{-1}) consisting of a W_+-curve v_0 with $I(v_0) = 0$ and an ECH curve v_{-1} with $I(v_{-1}) = 1$, by inserting a branched cover of a trivial cylinder (a *connector*). The connector is needed because, while the negative ends of v_0 and the positive ends of v_{-1} converge to the same orbit sets, they usually do not match; in fact the ends of v_0 must satisfy the outgoing partition, while the ends of v_{-1} must satisfy the incoming partition. The same situation appears in the definition of ECH and the procedure of gluing two $I_{ECH} = 1$ curves in the symplectization, as explained in Hutchings-Taubes [HT1, HT2], applies with very little modification.

6.5.1. *Close to breaking.* — We first make precise what we mean by a holomorphic curve which is "close to breaking". We rephrase the definition for W' (which appears in [HT1]), leaving the analogous definitions for W, W_+, and W_- to the reader.

Choose an s-invariant Riemannian metric g on W' and denote the induced distance by d_g.

Definition **6.5.1.** — *Given* $\kappa, \nu > 0$*, two curves* $u_i : (F_i, j_i) \to W'$*,* $i = 1, 2$*, are* (κ,ν)-close *if there exists a* $(1+\nu)$*-quasiconformal homeomorphism* $\phi : F_1 \xrightarrow{\sim} F_2$ *with respect to* (j_1, j_2) *such that*

$$\sup_{x\in F_1} d_g(u_1(x), u_2\circ\phi(x)) \leq \kappa.$$

Given a J-holomorphic curve u in $\mathbf{R}\times Y$, by an abuse of notation we will denote the restriction of u to the preimage of $[C, +\infty)\times M$ by $u|_{s\leq C}$. (Self-explanatory variations of this notations will also be used). The following is similar to [HT1, Definition 1.10], although the phrasing is slightly different.

Definition **6.5.2.** — *Let* $\kappa > 0$ *and* $\nu \geq 0$*. A curve* u *in* W' *is* (κ,ν)-close *to an* $(a+b+1)$*-level building* $u_\infty = v_{-b}\cup\cdots\cup v_a$ *if there exist* $R^{(-2b+1)}, \dots, R^{(2a)} > \frac{1}{\kappa}$ *such that, after a suitable translation of* u *in the* s*-direction which we still call* u*, each of the pairs below is* (κ,ν)*-close:*

– $u|_{-R^{(0)}\leq s\leq R^{(1)}}$ *and* $v_0|_{-R^{(0)}\leq s\leq R^{(1)}}$*;*
– $u|_{R^{(1)}\leq s\leq R^{(1)}+R^{(2)}}$ *and the restriction of a collection of branched covers of trivial cylinders to* $R^{(1)}\leq s\leq R^{(1)}+R^{(2)}$*;*
– $u|_{R^{(1)}+R^{(2)}\leq s\leq R^{(1)}+2R^{(2)}+R^{(3)}}$ *and an* $s\mapsto s+R^{(1)}+2R^{(2)}$ *translate of* $v_1|_{-R^{(2)}\leq s\leq R^{(3)}}$*;*
– $u|_{R^{(1)}+2R^{(2)}+R^{(3)}\leq s\leq R^{(1)}+2R^{(2)}+R^{(3)}+R^{(4)}}$ *and the restriction of a collection of branched covers of trivial cylinders to* $R^{(1)}+2R^{(2)}+R^{(3)}\leq s\leq R^{(1)}+2R^{(2)}+R^{(3)}+R^{(4)}$*;*

and so on.

Note that, in the case of u in W_+ or W_-, we do not need to (and indeed we cannot) translate u in the s-direction.

6.5.2. *Review of [HT1, HT2].* — We now summarize the Hutchings-Taubes proof of $\partial^2=0$ and discuss the small changes that need to be carried out.

Let $(Y,\xi=\ker\lambda)$ be a closed 3-manifold with a nondegenerate contact form λ and corresponding Reeb vector field $R=R_\lambda$. Let $(\mathbf{R}\times Y, d(e^s\lambda))$ be the symplectization of Y, where s denotes the $\mathbf{R}$-coordinate, and let J be an adapted almost complex structure on $\mathbf{R}\times Y$.

Let $\alpha=(\alpha_1,\dots,\alpha_k)$ and $\beta=(\beta_1,\dots,\beta_l)$ be ordered sets of Reeb orbits of R. *This notation is confined to Section 6.5 and therefore should not create confusion with the similar notation denoting the Heegaard Floer multicurve.* Here the Reeb orbits may not be embedded and may also be repeated. Then let $\mathcal{M}_J(\alpha,\beta)$ be the moduli space of finite energy J-holomorphic curves in $(\mathbf{R}\times Y, J)$ from α to β. Let $\{\gamma_1,\gamma_2,\dots\}$ be the list of simple orbits of R. For each simple orbit γ_i, we tally the total multiplicity $m_i(\alpha)$ of γ_i in α. In this way we can assign an orbit set $\boldsymbol{\gamma}(\alpha)=\prod_i\gamma_i^{m_i(\alpha)}$ to α.

We want to glue the pair (u_+,u_-), where $u_+\in\mathcal{M}_J^{I=1}(\alpha_+,\beta_+)$ and $u_-\in\mathcal{M}_J^{I=1}(\beta_-,\alpha_-)$, provided $\boldsymbol{\gamma}(\beta_+)=\boldsymbol{\gamma}(\beta_-)$. Since $I(u_+)=I(u_-)=1$, the curves u_+ and u_- satisfy the *partition condition* at β_+ and β_- for a generic J (cf. Definition 7.11 and Proposition 7.14 of [HT1]). In particular, for each simple orbit γ_i, the total multiplicity $m_i(\beta_+)=m_i(\beta_-)$ completely determines the number of ends of u_+ or u_- going to a cover of γ_i, together with their individual multiplicities. For u_+ (resp. u_-), this is encoded by the *incoming partition* (resp. *outgoing partition*) and is denoted by $P^{in}_{\gamma_i}(m_i(\beta_+))$ (resp. $P^{out}_{\gamma_i}(m_i(\beta_-))$). If $P^{in}_{\gamma_i}(m_i(\beta_+))=(a_1,\dots,a_r)$, then u_+ has r negative ends which go to a cover of γ_i with covering multiplicities $a_1,\dots,a_r$. A similar statement holds for the positive ends of u_-.

Since $P^{in}_{\gamma_i}(m_i(\beta_+))\neq P^{out}_{\gamma_i}(m_i(\beta_-))$ in general, we need to insert branched covers of cylinders $\mathbf{R}\times\gamma_i$ in order to be able to glue u_+ to u_-. Such branched covers $\pi:\Sigma\to\mathbf{R}\times\gamma_i$ must have β_+ at the positive end and β_- at the negative end. A Fredholm index count ([HT1, Lemma 1.7]) implies that Σ must have genus zero; moreover, the partition condition is equivalent to saying that the Fredholm index of the composition $u_\Sigma=\iota\circ\pi:\Sigma\to\mathbf{R}\times Y$ is zero, where $\iota:\mathbf{R}\times\gamma_i\to\mathbf{R}\times Y$ is the inclusion of the trivial cylinder $\mathbf{R}\times\gamma_i$.

Given $u_+ \in \mathcal{M}_J^{I=1}(\alpha_+, \beta_+)$ and $u_- \in \mathcal{M}_J^{I=1}(\beta_-, \alpha_-)$ with $\boldsymbol{\gamma}(\beta_+) = \boldsymbol{\gamma}(\beta_-)$, we define $\mathcal{G}_{\kappa,\nu}(u_+, u_-)$ as the set of curves in $\mathcal{M}_J^{I=2}(\alpha_+, \alpha_-)$ which are (κ, ν)-close to breaking to (u_+, u_-). This is a rephrasing of [HT1, Definition 1.10]. We define $G(u_+, u_-)$ as the modulo 2 count of the boundary points of $\mathcal{G}_{\kappa,\nu}(u_+, u_-)/\mathbf{R}$ for κ and ν sufficiently small.

The main result of [HT1, HT2] implies the following.

Theorem **6.5.3.** — *Let $u_+ \in \mathcal{M}_J^{I=1}(\alpha_+, \beta_+)$ and $u_- \in \mathcal{M}_J^{I=1}(\beta_-, \alpha_-)$ be J-holomorphic curves with $\boldsymbol{\gamma}(\beta_+) = \boldsymbol{\gamma}(\beta_-)$. Then $G(u_+, u_-) = 1$.*

It is clear that Theorem 6.5.3 is sufficient to define ECH; the original Hutchings-Taubes gluing theorem is more general, but for simplicity we have stated only the relevant part.

We now give the steps of the Hutchings-Taubes gluing construction. For simplicity we assume that $\boldsymbol{\gamma}(\beta_+) = \boldsymbol{\gamma}(\beta_-) = \gamma^m$ and γ is elliptic. The case of more than one orbit γ_i only differs in notation and the case of hyperbolic orbits is simpler since it can be treated with standard techniques; see [HT1, Section 1.5].

Step 1: Form a preglued curve (cf. [HT2, Section 5.2]). First we fix some notation. Let $\mathcal{M}$ be the moduli space of connected, genus zero branched covers $\pi : \Sigma \to \mathbf{R} \times \gamma$ which have positive and negative ends determined by $P_\gamma^{in}(m)$ and $P_\gamma^{out}(m)$. Abusing the notation, we will often write only Σ. The preglued curve is constructed as follows:

(1) Choose gluing constants $0 < h < 1$ and $r \gg 1/h$.

(2) With r, h fixed, choose the "gluing parameters" which consist of $\Sigma \in \mathcal{M}$ and real numbers $T_+, T_- \geq 5r$.

(3) Given $\pi : \Sigma \to \mathbf{R} \times \gamma$, let $\mathfrak{b} \subset \mathbf{R} \times \gamma$ be the set of branch points of π. Then let $s_+ = \max_{b\in\mathfrak{b}} s(b) + 1$ and $s_- = \min_{b\in\mathfrak{b}} s(b) - 1$. Let Σ' be the preimage of $[s_- - T_-, s_+ + T_+] \times \gamma$. The preimages of $[s_+, s_+ + T_+] \times \gamma$ are cylinders of "height" T_+ and the preimages of $[s_- - T_-, s_-]$ are cylinders of "height" T_-.

(4) Fix $\kappa > 0$ sufficiently small. We choose a representative u_+ in $[u_+] \in \mathcal{M}_J^{I=1}(\alpha_+, \beta_+)/\mathbf{R}$ such that each component of $u_+|_{s\leq 0}$ is $(\kappa, 0)$-close to a cylinder over some multiple cover of γ; here the multiplicities are given by $P_\gamma^{in}(m)$. Similarly, choose u_- so that each component of $u_-|_{s\geq 0}$ is $(\kappa, 0)$-close to a cylinder over some multiple cover of γ.

(5) Let u_{+T} be the $(s_+ + T_+)$-translate of u_+ in the $\mathbf{R}$-direction and let $u'_{+T} = u_{+T}|_{s\geq s_+ + T_+}$. Similarly, let u_{-T} be the $(s_- - T_-)$-translate of u_- in the $\mathbf{R}$-direction and let $u'_{-T} = u_{-T}|_{s\leq s_- - T_-}$. The domain C_* of the preglued curve is $C'_{+T} \cup \Sigma' \cup C'_{-T}$, where $C'_{\pm T}$ are domains of $u'_{\pm T}$, modulo the identifications along their boundary components. (To do the identifications correctly, we need asymptotic markers at the ends.)

(6) The preglued map u_* is defined explicitly (see [HT2, Equations (5.5) and (5.6)]) via a cutoff functions which allow us to interpolate between $u_{\pm T}$ and Σ in the preimages of the regions $s_+ \leq s \leq s_+ + T_+$ and $s_- - T_- \leq s \leq s_-$. These cutoff functions involve the constants h and r.

Step 2: Deform the preglued curve (cf. [HT2, Section 5.3]). We choose "exponential maps" e_-, e, e_+ which are obtained by flowing in the directions "normal" to u_-, u_Σ, u_+ in $\mathbf{R} \times Y$.

The exponential maps can be glued to give the exponential map e_* which maps to a small tubular neighborhood of u_* in $\mathbf{R}\times Y$. Also choose the cutoff function $\widetilde{\beta}_+ : C_* \to [0,1]$[9] which equals 1 on C'_{+T}, interpolates between 0 and 1 on the cylinders $s_+ \leq s \leq s_+ + T_+$ in Σ' and equals 0 elsewhere. Similarly define $\widetilde{\beta}_\Sigma$ and $\widetilde{\beta}_-$.

Let $(\psi_+, \psi_\Sigma, \psi_-)$ be a triple, where $\psi_\pm$ is a section of the normal bundle of $u_{\pm T}$ and ψ_Σ is a section of the normal bundle of u_Σ (which is trivial, so that ψ_Σ can be identified with a function $\Sigma \to \mathbf{C}$). The deformation of u_* with respect to $(\psi_+, \psi_\Sigma, \psi_-)$ is a map $C_* \to \mathbf{R}\times Y$ given by

$$x \mapsto e_*(x, \widetilde{\beta}_-\psi_- + \widetilde{\beta}_\Sigma\psi_\Sigma + \widetilde{\beta}_+\psi_+).$$

Step 3: We now consider the equation for the deformation to be J-holomorphic. This equation has the form:

$$\widetilde{\beta}_-\Theta_-(\psi_-, \psi_\Sigma) + \widetilde{\beta}_\Sigma\Theta_\Sigma(\psi_-, \psi_\Sigma, \psi_+) + \widetilde{\beta}_+\Theta_+(\psi_\Sigma, \psi_+) = 0. \tag{6.5.1}$$

See Equations (5.11), (5.12) and (5.13) of [HT2] for explicit expressions of Θ_-, Θ_+ and Θ_Σ.

The strategy is to solve three equations separately:

$$\Theta_-(\psi_-, \psi_\Sigma) = 0 \quad \text{on all of } u_{-T}; \tag{6.5.2}$$

$$\Theta_+(\psi_\Sigma, \psi_+) = 0 \quad \text{on all of } u_{+T}; \tag{6.5.3}$$

$$\Theta_\Sigma(\psi_-, \psi_\Sigma, \psi_+) = 0 \quad \text{on all of } \Sigma. \tag{6.5.4}$$

In [HT2, Proposition 5.6] it is shown that for sufficiently small ψ_Σ (in a suitable Banach space) there exist maps $\psi_\pm$ such that $\psi_\pm = \psi_\pm(\psi_\Sigma)$ solves Equations (6.5.2) and (6.5.3).

We can then view Equation (6.5.4) as an equation in the variable ψ_Σ on all of Σ:

$$\Theta_\Sigma(\psi_-(\psi_\Sigma), \psi_\Sigma, \psi_+(\psi_\Sigma)) = 0. \tag{6.5.5}$$

Equation (6.5.5) can be written as

$$D_\Sigma\psi_\Sigma + \mathcal{F}_\Sigma(\psi_\Sigma) = 0, \tag{6.5.6}$$

where D_Σ is the normal component of the linearized Cauchy-Riemann operator at u_Σ and $\mathcal{F}_\Sigma$ is a nonlinear term which is small in some suitable sense.

Step 4: If N denotes the normal bundle to $\mathbf{R}\times\gamma$ in $\mathbf{R}\times M$, then

$$D_\Sigma : L^2_1(\pi^*N) \to L^2(\pi^*N \otimes T^{0,1}\Sigma).$$

[9] The notation in [HT2] is β_+. Here write $\widetilde{\beta}_+$ to distinguish it from the sets of orbits in Section 6.5.2.

The operator D_Σ is injective and moreover $\dim \operatorname{coker}(D_\Sigma) = \dim \mathcal{M}$; see [HT1, Lemma 2.14] and [HT1, Lemma 2.15]. The explicit form of D_Σ is given in [HT1, Definition 2.3].

We introduce the orthogonal projection

$$\Pi : L^2(\pi^* N \otimes T^{0,1}\Sigma) \to \ker(D_\Sigma^*) \cong \operatorname{coker}(D_\Sigma)$$

and decompose Equation (6.5.6) into two equations (cf. Equations (5.37) and (5.38) in [HT2]):

$$D_\Sigma \psi_\Sigma + (1-\Pi)\mathcal{F}_\Sigma(\psi_\Sigma) = 0, \tag{6.5.7}$$

$$\Pi \mathcal{F}_\Sigma(\psi_\Sigma) = 0 \tag{6.5.8}$$

Equation (6.5.7) has a unique solution by [HT2, Proposition 5.7]. Hence the problem reduces to solving Equation (6.5.8).

This is shown to be equivalent to finding the zeros of a section of the associated obstruction bundle, defined in [HT1, Section 2]. Briefly, the obstruction bundle [10] $\mathcal{O}' \to [5r, \infty)^2 \times \mathcal{M}$ has fiber

$$\mathcal{O}'_{(T_+, T_-, \Sigma)} = \operatorname{Hom}(\operatorname{Coker}(D_\Sigma), \mathbf{R}),$$

and the section $\mathfrak{s} : [5r, \infty)^2 \times \mathcal{M} \to \mathcal{O}'$ is given by:

$$\mathfrak{s}(T_+, T_-, \Sigma)(\sigma) = \langle \sigma, \mathcal{F}_\Sigma(\psi_\Sigma) \rangle,$$

where $\sigma \in \operatorname{Coker}(D_\Sigma)$ and ψ_Σ is the solution to Equation (6.5.7) corresponding to the gluing parameters (T_+, T_-, Σ).

Step 5: Recall that $\mathcal{G}_{\kappa,\nu}(u_+, u_-)$ is the set of J-holomorphic maps of index $I = 2$ which are (κ, ν)-close to breaking into (u_+, u_-) (see [HT2, Section 7.1]). Let $\mathcal{U}_{\kappa,\nu} \subset [5r, \infty)^2 \times \mathcal{M}$ be the set of (T_+, T_-, Σ) such that the corresponding pre-glued curve $u_*(T_+, T_-, \Sigma)$ (as given in Step 1) is (κ, ν)-close to breaking. It remains to show that $\mathcal{G}_{\kappa,\nu}(u_+, u_-)$ is homeomorphic to $\mathfrak{s}^{-1}(0) \cap \mathcal{U}_{\kappa,\nu}$ for $r > 0$ sufficiently large and κ, ν sufficiently small. This is the content of [HT2, Theorem 7.3].

Step 6: We define the linearized section $\mathfrak{s}_0$ which depends only on the asymptotic eigenfunctions of the negative ends of u_+ and the positive ends of u_- but not on the actual choices of u_+ and u_-. The explicit formula for $\mathfrak{s}_0$ is given in [HT1, Definition 8.1]. We consider the linear deformation $\mathfrak{s}_t = t\mathfrak{s} + (1-t)\mathfrak{s}_0$.

Step 7: Consider the action of $\mathbf{R}$ on $[5r, +\infty)^2 \times \mathcal{M}$ that fixes the $[5r, +\infty)$ factors and translates the s-coordinate on $\mathcal{M}$. This action extends to the obstruction bundle and the sections $\mathfrak{s}_t$ are invariant under this action. Hence the obstruction bundle can be viewed

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[10] In [HT2] the obstruction bundle is denoted by $\mathcal{O}$. However, in [HT1], $\mathcal{O}$ denotes a related, but different, bundle and this causes confusion. We reserve the notation $\mathcal{O}$ for the latter bundle.

as a bundle over $[5r, +\infty)^2 \times \mathcal{M}/\mathbf{R}$ which we still denote by $\mathcal{O}'$ by abuse of notation, and each $\mathfrak{s}_t$ can be viewed as a section of it.

Given $R \geq 10r$, we define the "slice" $\mathcal{V}_R$ which consists of triples

$$(T_-, T_+, \Sigma) \in [5r, +\infty)^2 \times \mathcal{M}, \quad T_+ + s_+ - s_- + T_- = R.$$

The action of $\mathbf{R}$ on $[5r, +\infty)^2 \times \mathcal{M}$ preserves $\mathcal{V}_R$ and the boundary of $\mathcal{G}_{\kappa,\nu}(u_+, u_-)$ is in bijection with $\mathfrak{s}^{-1}(0) \cap \mathcal{V}_R/\mathbf{R}$; in other words

$$G(u_+, u_-) = \#\left(\mathfrak{s}^{-1}(0) \cap \mathcal{V}_R/\mathbf{R}\right).$$

It turns out that the section $\mathfrak{s}_0$ has the same number of zeroes as $\mathfrak{s}$ on $\mathcal{V}_R/\mathbf{R}$. The key point to check is that, during the deformation $(\mathfrak{s}_t)_{t\in[0,1]}$ from $\mathfrak{s}_1 = \mathfrak{s}$ to $\mathfrak{s}_0$, the zeros of $\mathfrak{s}_t$ do not cross the boundary of $\mathcal{V}_R/\mathbf{R}$. This is guaranteed by [HT2, Proposition 8.2].

Step 8: Define

$$\mathcal{M}_R = \{\Sigma \in \mathcal{M} \mid -\tfrac{R}{2} + 5r \leq s_-, s_+ \leq \tfrac{R}{2} - 5r\}.$$

Then $\mathcal{V}_R/\mathbf{R} \simeq \mathcal{M}_R$ and the bundle $\mathcal{O} \to \mathcal{M}_R$ with fiber $\mathcal{O}_\Sigma = \mathrm{Hom}(\mathrm{Coker}(D_\Sigma), \mathbf{R})$ is isomorphic to the bundle $\mathcal{O}' \to \mathcal{V}_R/\mathbf{R}$. The explicit formula for $\mathfrak{s}_0$, regarded as a section of $\mathcal{O} \to \mathcal{M}_R$, is given in [HT1, Definition 3.2]. See [HT2, Remark 8.5] for more details about the correspondence between $\mathcal{O}'$ and $\mathcal{O}$. Finally, the combinatorial formula for the algebraic count of zeros of $\mathfrak{s}_0$ (as a section of $\mathcal{O}$) is given by [HT1, Theorem 1.13].

6.5.3. *The Φ-map.* — We now turn to gluing the pair (v_0, v_{-1}), where v_0 is a W_+-curve with $I(v_0) = 0$ and v_{-1} is an ECH curve with $I(v_{-1}) = 1$. Suppose the negative end of v_0 is given by β_+, the positive end of v_{-1} is given by β_-, and $\boldsymbol{\gamma}(\beta_+) = \boldsymbol{\gamma}(\beta_-) = \gamma^m$ for some elliptic orbit γ. Again, the case of more than one orbit γ_i only differs in notation and the case of hyperbolic orbits can be treated by standard techniques. In our case, there are a few things to check:

(1) The curves v_0 and v_{-1} must satisfy the partition conditions at their negative and positive ends, respectively. This is a consequence of the ECH index inequality, i.e., Theorem 5.6.9.

(2) Since the W_+-curve v_0 is not s-translation invariant, we pick s_0 so that each component of $v_0|_{s\leq s_0}$ is $(\kappa, 0)$-close to a cylinder over some multiple cover of γ. (We may still assume that v_{-1} satisfies the condition that $v_{-1}|_{s\geq 0}$ is $(\kappa, 0)$-close to a cylinder.) Given $(T_+, T_-, \Sigma) \in [5r, +\infty)^2 \times \mathcal{M}$, we take

- $v_0|_{s\geq s_0}$;
- Σ' shifted by $s_0 - (s_+ + T_+)$; and
- $v_{-1}|_{s\leq 0}$ shifted by $(s_- - T_-) + s_0 - (s_+ + T_+)$;

and preglue. The preglued curve only depends on $(T_+, T_-, [\Sigma])$, where $[\Sigma] \in \mathcal{M}/\mathbf{R}$.

(3) [HT2, Proposition 5.6] allows us to solve Equations (6.5.2) and (6.5.3) in terms of ψ_Σ. The inputs for [HT2, Proposition 5.6] are [HT2, Lemmas 5.3 and 5.4], which are consequences of the fact that the linearized $\overline{\partial}$-operators corresponding to v_0 and v_{-1} are Fredholm and surjective; this also holds in our case with Lagrangian boundary conditions. Hence Step 3 extends easily to our setting, yielding a section $\mathfrak{s}$ of the obstruction bundle $\mathcal{O}' \to [5r, +\infty)^2 \times \mathcal{M}/\mathbf{R}$. The linearized section $\mathfrak{s}_0$ and the "slice" $\mathcal{V}_R$ are defined similarly and, after identifying $\mathcal{O}' \to \mathcal{V}_R/\mathbf{R}$ with the bundle $\mathcal{O} \to \mathcal{M}_R$, the remaining steps carry over without change.

6.6. *The variant* $\widetilde{\Phi}$. — In this subsection we define a complex $\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$ which is closely related to $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ and a variant

$$\widetilde{\Phi} : \widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N)$$

of Φ. This will be useful in the proof of Theorem II.3.3.1. As a vector space, $\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$ will also be used in Section 7.

6.6.1. *The chain complex* $\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$. — Let $\overline{J} \in \mathcal{J}^{reg}_{\overline{W}}$. We recall some notation introduced in Section 4.9.3: Let $\mathcal{I} \subset \{1, \dots, 2g\}$ be a subset, $\mathcal{I}^c$ its complement, and $\mathfrak{S}_{\mathcal{I}^c}$ the group of permutations of $\mathcal{I}^c$.

Definition **6.6.1.** — *We define the chain complex* $(\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}), \overline{J}), \widetilde{\partial})$ *generated by the* $2g$*-tuples* $\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}'$, *where* $\mathcal{I} \subset \{1, \dots, 2g\}$, $\mathbf{y}' = \{y_i'\}_{i\in\mathcal{I}^c}$, *and the following hold:*

- $z_{\infty,i}$ *is viewed as an intersection point of* $\overline{a}_i$ *and* $\overline{h}(\overline{a}_i)$ *and*
- $y_i' \in a_i \cap h(a_{\sigma(i)})$ *for some* $\sigma \in \mathfrak{S}_{\mathcal{I}^c}$.

We denote the set of intersection points $\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}'$ *as above by* $\mathcal{S}_{\overline{\mathbf{a}},\overline{h}(\overline{\mathbf{a}})}$.

The datum $z_{\infty,i}$ *is equivalent to "*z_∞ *with either of the two matchings* $(i, 0) \to (i, 0)$ *or* $(i, 1) \to (i, 1)$*" and therefore, using the notation of Section 5.7,* $\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}'$ *is equivalent to the equivalence class of elements* $z_\infty^p(\overrightarrow{\mathcal{D}}) \cup \mathbf{y}'$, *where* $\mathcal{I} = \{i_1, \dots, i_p\}$ *and* $\overrightarrow{\mathcal{D}} = \{(i_1, j_1) \to (i_1, j_1), \dots, (i_1, j_1) \to (i_p, j_p)\}$ *for any choice of* $(j_1, \dots, j_p) \in \{0, 1\}^p$.

The differential $\widetilde{\partial}$ *counts* $I = 1$ *multisections* $\overline{u}$ *in* $\overline{W}$ *with* $n(\overline{u}) \leq 1$, *which satisfy one extra condition, i.e., if we write* $\overline{u} = \overline{u}' \cup \overline{u}''$ *(according to the notation introduced in Section 5.7.2), then* $\overline{u}'$ *has empty branch locus. The homology of* $(\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})), \widetilde{\partial})$ *is denoted by* $\widetilde{HF}(S, \mathbf{a}, h(\mathbf{a}))$.

In the differential $\widetilde{\partial}$, with the exception of trivial strips, we are counting the following curves:

(1) thin strips from $z_{\infty,i}$ to either x_i or x_i'; and
(2) $I = 1$ curves whose projections to $\overline{S}$ have image in S.

Lemma **6.6.2.** — *There is an isomorphism of chain complexes:*

$$\kappa : (\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z), \widehat{\partial}) \xrightarrow{\sim} (\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})), \widetilde{\partial}),$$

$$\{x_i''\}_{i\in\mathcal{I}} \cup \mathbf{y}' \mapsto \{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}',$$

where $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ *is as given in Section 4.9.1.*

Proof. — This is immediate from the definitions. □

Recall from Section 4.9.3 that

$$(\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), \widehat{\partial'}) = (\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})), \partial')/\sim$$

is the E^1-term of the spectral sequence for $(\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z), \widehat{\partial})$ (viewed as a double complex) in Theorem 4.9.4. (Here we are writing $\widehat{\partial'}$ for the differential of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ to distinguish it from the differential $\widehat{\partial}$ of $\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$.)

In this paragraph $\mathbf{y}$ and $\mathbf{y}_i'$ denote linear combinations of tuples. By tracing the zigzags in the double complex we obtain the isomorphism

$$\nu : \widehat{HF}(S, \mathbf{a}, h(\mathbf{a})) \xrightarrow{\sim} \widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z),$$

which is defined as follows: Let $\mathbf{y} \in \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$ be a cycle in $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$, i.e.,

$$\partial'\mathbf{y} = \sum_i (\{x_i\} \cup \mathbf{y}_i' + \{x_i'\} \cup \mathbf{y}_i').$$

Then ν maps the equivalence class $[\mathbf{y}]$ to the equivalence class of

$$\mathbf{y} + \sum_i \{x_i''\} \cup \mathbf{y}_i' + \text{ h.o.},$$

where "h.o." means terms with more x_i'' components. Composing with the map induced by κ in homology, we obtain an isomorphism

$$\tilde{\nu} = \kappa_* \circ \nu : \widehat{HF}(S, \mathbf{a}, h(\mathbf{a})) \xrightarrow{\sim} \widetilde{HF}(S, \mathbf{a}, h(\mathbf{a})).$$

6.6.2. *The map* $\widetilde{\Phi}$. — Let

$$\mathcal{M}_{\bar{J}_+}(\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}', \boldsymbol{\gamma}) = \coprod_{\vec{\mathcal{D}}} \mathcal{M}_{\bar{J}_+}(\{z_\infty^{\#\mathcal{I}}(\vec{\mathcal{D}})\} \cup \mathbf{y}', \boldsymbol{\gamma}),$$

where the moduli spaces on the right-hand side are defined in a manner analogous to Definition 5.7.2 and $\vec{\mathcal{D}}$ ranges over all matchings $\{(i, j_i) \to (i, j_i)\}_{i\in\mathcal{I}, j_i\in\{0,1\}}$.

Definition **6.6.3.** — *Let* $\bar{J}_+ \in \mathcal{J}^{reg}_{W_+}$ *which restricts to* $\bar{J} \in \mathcal{J}^{reg}_{W}$ *and* $\bar{J}' \in \mathcal{J}^{reg}_{W'}$. *We define the map* $\widetilde{\Phi} : \widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N)$ *as follows:*

$$\langle \widetilde{\Phi}(\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}'), \boldsymbol{\gamma}\rangle = \#\mathcal{M}^{I=0, n^*\le|\mathcal{I}|}_{\bar{J}_+}(\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}', \boldsymbol{\gamma}).$$

In our analysis of $\widetilde{\Phi}$ we will use balanced coordinates (cf. Section 5.1.2) for $\overline{\mathrm{N}} - int(\mathrm{N}) = \mathrm{D}^2 \times (\mathbf{R}/2\mathbf{Z})$. The Morse-Bott family $\mathcal{N}$ can be identified with $\partial \mathrm{D}^2$ and we write γ_ϕ for the orbit in $\mathcal{N}$ corresponding to $e^{i\phi}$. So far in this paper $h \in \mathcal{N}$ was an arbitrary point.

Convention **6.6.4.** — *From now on we specialize h so that $h = \gamma_{\phi_h}$ is generic and ϕ_h is close to $-\frac{2\pi}{m}$, where m is as defined in Section 5.2.2. In particular, the radial ray corresponding to ϕ_h does not lie on the thin wedges from $\overline{a}_i$ to $\overline{h}(\overline{a}_i)$ for all i. There are no restrictions on the orbit e except that $e \neq h$.*

Lemma **6.6.5.** — $\widetilde{\Phi}(\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}') = 0$ *if* $\mathcal{I} \neq \varnothing$ *and* $\widetilde{\Phi}(\mathbf{y}') = \Phi(\mathbf{y})$ *if* $\mathcal{I} = \varnothing$.

Proof. — Let us fix $\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}'$ with $|\mathcal{I}| \geq 1$ and write

$$\mathcal{M}^0(\boldsymbol{\gamma}) := \mathcal{M}^{\mathrm{I}=0, n^* \leq |\mathcal{I}|}_{\bar{\mathrm{J}}_+}(\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}', \boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_{2g}$.

We claim that $\mathcal{M}^0(\boldsymbol{\gamma}) = \varnothing$. Arguing by contradiction, suppose there exists $\overline{u} \in \mathcal{M}^0(\boldsymbol{\gamma})$. By our assumptions on $\bar{\mathrm{J}}_+$, $\overline{u}$ is close to breaking into a Morse-Bott building. We will show that such a Morse-Bott building cannot exist using the positivity of intersections, thus establishing the contradiction. For the rest of the proof, $\overline{u}$ will denote the holomorphic part of the Morse-Bott building instead of the original curve.

We consider the projection $\pi_{\overline{\mathrm{N}}} : \overline{\mathrm{W}}_+ \to \overline{\mathrm{N}}$, obtained by restricting the projection $\overline{\mathrm{W}'} = \mathbf{R} \times \overline{\mathrm{N}} \to \overline{\mathrm{N}}$ to $\overline{\mathrm{W}}_+$. Let T_ρ be the boundary of the neighborhood $\mathrm{V}_\rho = \mathrm{S}^1 \times \mathrm{D}^2_\rho$ of the orbit δ_0. We choose $\rho > 1$, but arbitrarily close to 1, so that T_ρ is contained in N and parallel to $\partial \mathrm{N}$. Then T_ρ is a pre-Lagrangian torus and we can assume without loss of generality that it is foliated by closed orbits.

We identify $\mathrm{H}_1(\mathrm{T}_\rho) \cong \mathbf{Z}^2$ such that — writing vectors as rows — $(1,0)$ corresponds to the homology class of the meridian (i.e., the closed curve which bounds a disk in V_ρ intersecting δ_0 once positively) and $(0,1)$ corresponds to the class of the orbits in $\partial \mathrm{N}$ (i.e., those which are perturbed into e and h) under the obvious identification $\mathrm{H}_1(\mathrm{T}_\rho) \cong \mathrm{H}_1(\partial \mathrm{N})$. Then the Hamiltonian orbits on T_ρ represent a homology class $(-p, q)$ with $p, q \geq 1$.

By abuse of notation, we do not distinguish $\overline{u}$ from its image, e.g., we write $\pi_{\overline{\mathrm{N}}}(\overline{u})$ to denote the image of the composition $\pi_{\overline{\mathrm{N}}} \circ \overline{u}$. The intersection $\pi_{\overline{\mathrm{N}}}(\overline{u}) \cap \mathrm{T}_\rho$, oriented as the boundary of $\pi_{\overline{\mathrm{N}}}(\overline{u}) \cap (\overline{\mathrm{N}} - \mathrm{V}_\rho)$, represents a class $\mathrm{C} \in \mathrm{H}_1(\mathrm{T}_\rho, \pi_{\overline{\mathrm{N}}}(\mathrm{L}^+_{\mathbf{a}}) \cap \mathrm{T}_\rho)$. We recall that $\pi_{\overline{\mathrm{N}}}(\mathrm{L}^+_{\mathbf{a}}) \cap \mathrm{T}_\rho$ consists of $2g$ pairwise disjoint segments parallel to the Hamiltonian flow. The condition $n^*(\overline{u}) \leq |\mathcal{I}|$ implies that C consists of at most $|\mathcal{I}|$ arcs; moreover each arc has both endpoints on the same connected component of $\pi_{\overline{\mathrm{N}}}(\mathrm{L}^+_{\mathbf{a}}) \cap \mathrm{T}_\rho$ and represents the image of the class $(0,1) \in \mathrm{H}_1(\mathrm{T}_\rho)$ under the map $\mathrm{H}_1(\mathrm{T}_\rho) \xrightarrow{\sim} \mathrm{H}_1(\mathrm{T}_\rho, \pi_{\overline{\mathrm{N}}}(\mathrm{L}^+_{\mathbf{a}}))$. This can be seen by considering the relation of the asymptotic eigenfunctions of $\overline{u}$ at $z_{\infty,i}$ with $n^*(\overline{u})$ and the possible ends of $\overline{u}$ in V_ρ.

The algebraic intersection between a Hamiltonian orbit of T_ρ and the image of $\overline{u}$ is the same as the algebraic intersection in T_ρ of a Hamiltonian orbit with C, which is negative. This contradicts the positivity of intersections, unless $\pi_{\overline{N}}(\overline{u})$ is disjoint from T_ρ. Since this holds for $\rho > 1$ and arbitrarily close to 1, there is a decomposition $\overline{u} = \overline{u}'' \cup \overline{u}'''$, where $I(\overline{u}'') = I(\overline{u}''') = 0$, $\overline{u}''$ has image in $\overline{W}_+ - int(W_+)$, and $\overline{u}'''$ has image in W_+.

Then $\overline{u}''$ is a curve from z_∞ to h since $I(\overline{u}'') = 0$. The proof of the nonexistence of $\overline{u}''$ is modeled on the proof of [0, Lemma 8.4.8]. By [We1, Section 4.2], $\mathbf{R} \times (\overline{N} - N - \delta_0)$ is foliated by finite energy cylinders Z_{s_0,ϕ_0}, $(s_0, \phi_0) \in \mathbf{R} \times (\mathbf{R}/2\pi\mathbf{Z})$, from δ_0 to γ_{ϕ_0} such that:

- the image of Z_{s_0,ϕ_0} under the projection $\pi_{\overline{N}} : \mathbf{R} \times \overline{N} \to \overline{N}$ is the open annulus $\{\phi = \phi_0, 0 < \rho < 1\}$;
- $Z_{s_0+s_1,\phi_0}$ is the s_1-translate of Z_{s_0,ϕ_0}.

We then set $Z^+_{s,\phi} = Z_{s,\phi} \cap \overline{W}_+$ and examine the intersections $Z^+_{s,\phi} \cap \overline{u}''$. Observe that $K = \pi_{\overline{N}}(Z^+_{s,\phi}) \cap \pi_{\overline{N}}(\overline{u}'') \neq \varnothing$ for a suitable choice of ϕ which is close to but not equal to ϕ_h; this is possible by the positioning of h given by Convention 6.6.4. Hence $Z^+_{s_0,\phi} \cap \overline{u}'' \neq \varnothing$ for some s_0. On the other hand, since K is compact, $Z^+_{s_0+s_1,\phi} \cap \overline{u}'' = \varnothing$ for a sufficiently large s_1. Finally, since $Z^+_{s,\phi}$ and $\overline{u}''$ have no boundary intersections and no intersections near their ends for all $s \in \mathbf{R}$, we have a contradiction. □

Theorem **6.6.6.** — *$\widetilde{\Phi}$ is a chain map.*

Proof. — Since Φ is a chain map by Theorem 6.2.4, it suffices to verify that

$$\partial_{PFH} \circ \widetilde{\Phi}(\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}') = \widetilde{\Phi} \circ \widetilde{\partial}(\{z_{\infty,i}\}_{\mathcal{I}} \cup \mathbf{y}'),$$

whenever $\mathcal{I} \neq \varnothing$. On the one hand, $\widetilde{\Phi}(\{z_{\infty,i}\}_{\mathcal{I}} \cup \mathbf{y}') = 0$ by Lemma 6.6.5. On the other hand, if $|\mathcal{I}| = 1$, then

$$\widetilde{\Phi} \circ \widetilde{\partial}(\{z_{\infty,i_1}\} \cup \mathbf{y}') = \widetilde{\Phi}(\{x_{i_1}\} \cup \mathbf{y}' + \{x'_{i_1}\} \cup \mathbf{y}') = 0;$$

and if $|\mathcal{I}| > 1$, then each term of $\widetilde{\partial}(\{z_{\infty,i}\}_{i\in\mathcal{I}} \cup \mathbf{y}')$ contains some copy of z_∞, and $\widetilde{\Phi}$ maps the term to zero by Lemma 6.6.5. This proves the theorem. □

A corollary of Lemma 6.6.5 is the following:

Corollary **6.6.7.** — *$\Phi = \widetilde{\Phi} \circ \kappa \circ \nu$ on the level of homology.*

7. The chain map from PFH to $\widehat{HF}$

7.1. *Definition of Ψ.* — Fix the integer $m \gg 0$ which appears in the definition of the monodromy map $\overline{h} = \overline{h}_m$ from Section 5.1.2. The condition $m \gg 0$ will be useful when applying limiting arguments in Sections 7.8 and 7.10.

Suppose $\overline{J}_- \in \mathcal{J}^{reg}_{\overline{W}_-}$, $\overline{J}'$ and $\overline{J}$ are restrictions of $\overline{J}_-$ to the positive and negative ends, and $\overline{J}^{\diamond}_-$ is $K_{p,2\delta}$-regular with respect to $\overline{\mathfrak{m}} = ((0, \frac{3}{2}), z_\infty)$. We assume that the constants $\varepsilon, \delta > 0$ that go into the definition of $\overline{J}^{\diamond}_-$ (cf. Section 5.8.4) are arbitrarily small.

Definition **7.1.1** *(Definitions of Ψ' and Ψ).*

(1) *We define the map*

$$\Psi' = \Psi'_{\overline{J}^{\diamond}_-}(m, \overline{\mathfrak{m}}) : PFC_{2g}(N) \to \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$$
$$\boldsymbol{\gamma} \mapsto \sum_{\mathbf{y} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}} \langle \Psi'(\boldsymbol{\gamma}), \mathbf{y} \rangle \cdot \mathbf{y},$$

where $\langle \Psi'(\boldsymbol{\gamma}), \mathbf{y} \rangle$ is the mod 2 count of $\mathcal{M}^{I=2, n^*=m}_{\overline{J}^{\diamond}_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}})$.

(2) *We define the map $\Psi = \Psi_{\overline{J}^{\diamond}_-}(m, \overline{\mathfrak{m}})$ as the composition*

$$PFC_{2g}(N) \stackrel{\Psi'}{\to} \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})) \stackrel{q}{\to} \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})),$$

where q is the quotient map of chain complexes.

The count $\langle \Psi'(\boldsymbol{\gamma}), \mathbf{y} \rangle$ is meaningful because of the following theorem:

Theorem **7.1.2.** — $\mathcal{M}^{I=2, n^*=m}_{\overline{J}^{\diamond}_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}})$ *is compact.*

Proof. — This follows from Theorems 7.6.3 and 7.10.1(i) and Corollary 7.11.1. □

Now we define the maps $\widetilde{U}_{m-1}$, $\widetilde{\Psi}_0$, and $\widetilde{\partial}_1$ which measure the failure of Ψ' to be a chain map. The subscripts in $\widetilde{U}_{m-1}$, $\widetilde{\Psi}_0$, and $\widetilde{\partial}_1$ indicate that we are counting curves $\overline{u}$ satisfying $n^*(\overline{u}) = m-1$, 0, and 1.

First we define $\widetilde{U}_{m-1} : ECC_{2g}(N) \to ECC_{2g}(\overline{N})$: Let $\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_{2g}$ and $\boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_{2g-p}$, $p = 0, \dots, 2g$. Then

$$\langle \widetilde{U}_{m-1}(\boldsymbol{\gamma}), \delta_0^p \boldsymbol{\gamma}' \rangle = \begin{cases} A, & \text{if } p = 1; \\ 0, & \text{if } p \neq 1, \end{cases}$$

where A is the count of $I = 2$ multisections $\overline{u}$ of $\overline{W'}$ from $\boldsymbol{\gamma}$ to $\delta_0 \boldsymbol{\gamma}'$ which satisfy $n^*(\overline{u}) = m-1$ and a certain asymptotic condition near δ_0 (the precise definition will be given in Section 7.7.3).

Next we define $\widetilde{\Psi}_0 : ECC_{2g}(\overline{N}) \to \widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$: Let $\boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_{2g-p}$, $p = 0, \dots, 2g$, and $\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}' \in \mathcal{S}_{\overline{\mathbf{a}}, \overline{h}(\overline{\mathbf{a}})}$. Then

$$\langle \widetilde{\Psi}_0(\delta_0^p \boldsymbol{\gamma}'), \{z_{\infty,i}\}_{\mathcal{I}} \cup \mathbf{y}' \rangle = \begin{cases} B, & \text{if } p = 1 \text{ and } |\mathcal{I}| = 1; \\ 0, & \text{if } p \neq 1 \text{ or } |\mathcal{I}| \neq 1, \end{cases}$$

where B is the count of degree $2g-1$, $I=0$ almost multisections $\overline{u}$ of $\overline{W}_-$ from $\boldsymbol{\gamma}'$ to $\mathbf{y}'$ which satisfy $n^*(\overline{u})=0$, together with the section at infinity σ_∞^- from δ_0 to z_∞.

Finally we define $\widetilde{\partial}_1 : \widetilde{CF}(S,\mathbf{a},h(\mathbf{a})) \to \widehat{CF}(S,\mathbf{a},h(\mathbf{a}))$. If $\{z_{\infty,i}\}_{\mathcal{I}} \cup \mathbf{y}' \in \mathcal{S}_{\overline{\mathbf{a}},\overline{h}(\overline{\mathbf{a}})}$, then we define

$$\widetilde{\partial}_1(\{z_{\infty,i}\}_{\mathcal{I}} \cup \mathbf{y}') = \begin{cases} \{x_i\}\cup \mathbf{y}' + \{x_i'\}\cup\mathbf{y}', & \text{if } \mathcal{I}=\{i\}; \\ 0, & \text{if } |\mathcal{I}|\neq 1. \end{cases} \tag{7.1.1}$$

We can see $\widetilde{\partial}_1$ as the composition of two maps: a map from $\widetilde{CF}(S,\mathbf{a},h(\mathbf{a}))$ to itself induced by the count of $n^*(\overline{u})=1$ curves in $\overline{W}$ (which are necessarily thin wedges), followed by the projection $\widetilde{CF}(S,\mathbf{a},h(\mathbf{a})) \to \widehat{CF}(S,\mathbf{a},h(\mathbf{a}))$ sending every $2g$-tuple of chords involving z_∞ to zero.

Let ∂'_{HF} be the differential for $\widehat{CF'}(S,\mathbf{a},h(\mathbf{a}))$ and let ∂_{PFH} be the differential for $PFC_{2g}(N)$. The proof of the following theorem will occupy the rest of Section 7.

Theorem **7.1.3.** — *If $m\gg 0$, then*

$$\partial'_{HF}\circ\Psi' + \Psi'\circ\partial_{PFH} = \widetilde{\partial}_1\circ\widetilde{\Psi}_0\circ\widetilde{U}_{m-1}. \tag{7.1.2}$$

Assuming Theorem 7.1.3 for the moment, we have the following:

Corollary **7.1.4.** — *For all $m\gg 0$, Ψ is a chain map.*

Proof. — Similar to the proof of Theorem 6.2.4 and based on Equation (7.1.2). There is one major difference: Ψ' *is not a chain map*. However, since the image of $\widetilde{\partial}_i$ is contained in $\ker q$, by composing Equation (7.1.2) with q we obtain:

$$\widehat{\partial}_{HF}\circ\Psi + \Psi\circ\partial_{PFH} = 0, \tag{7.1.3}$$

where $\widehat{\partial}_{HF}$ is the differential for $\widehat{CF}(S,\mathbf{a},h(\mathbf{a}))$. ☐

We can also define a map

$$\underline{\Psi} : \underline{PFH}_{2g}(N) \to \underline{\widehat{CF}}(S,\mathbf{a},h(\mathbf{a})) \tag{7.1.4}$$

which is $\mathbf{F}[H_2(M;\mathbf{Z})]$-linear. As for $\underline{\Phi}$ in Section 6.4, the key point is the definition of maps

$$\mathfrak{A}_- : H_2(\overline{W}_-,\boldsymbol{\gamma},\mathbf{y}) \to H_2(M).$$

The main difference is that $H_2(\overline{W}_-)\cong H_2(\overline{N})\cong H_2(M)\oplus\langle\overline{S}\rangle$. Hence we define

$$\mathfrak{A}_-(C) = p((\pi_{\overline{N}})_*[C_{\mathbf{y}}\cup C\cup -C_{\boldsymbol{\gamma}}]),$$

where $\pi_{\overline{N}}$ is the projection $\overline{W}_-\to\overline{N}$ and $p: H_2(\overline{N})\to H_2(M)$ is the projection given by $p = id - \langle\delta_0,\cdot\rangle[\overline{S}]$.

7.2. *Outline of proof of Theorem 7.1.3.* — In this subsection we outline the proof of Theorem 7.1.3.

Let $\overline{J}_- \in \mathcal{J}^{reg}_{\overline{W}_-}$, with restrictions $\overline{J}'$ and $\overline{J}$ to the positive and negative ends, and let $\overline{J}^\diamondsuit_-$ be $K_{p,2\delta}$-regular with respect to $\overline{\mathfrak{m}}$. Suppose that the constants $\varepsilon, \delta > 0$ that go into the definition of $\overline{J}^\diamondsuit_-$ are arbitrarily small.

We abbreviate:

$$\mathcal{M}^i := \mathcal{M}^{I=i,n^*=m}_{\overline{J}^\diamondsuit_-}(\boldsymbol{\gamma}, \mathbf{y}), \quad \mathcal{M}^i_{\overline{\mathfrak{m}}} := \mathcal{M}^{I=i,n^*=m}_{\overline{J}^\diamondsuit_-}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}}).$$

Let $\overline{\mathcal{M}^i_{\overline{\mathfrak{m}}}}$ be the SFT compactification of $\mathcal{M}^i_{\overline{\mathfrak{m}}}$, obtained by adding holomorphic $\overline{W}_-$-buildings as in Definition 7.3.1, and let $\partial\mathcal{M}^i_{\overline{\mathfrak{m}}} = \overline{\mathcal{M}^i_{\overline{\mathfrak{m}}}} - \mathcal{M}^i_{\overline{\mathfrak{m}}}$.

Step 1. By SFT compactness for $\overline{W}_-$-curves (Proposition 7.3.2), a sequence $\overline{u}_i \in \mathcal{M}^3_{\overline{\mathfrak{m}}}$ admits a subsequence which converges to a holomorphic $\overline{W}_-$-building

$$\overline{u}_\infty = \overline{v}_{-b} \cup \cdots \cup \overline{v}_a.$$

As before, we write $\overline{v}_j = \overline{v}'_j \cup \overline{v}''_j$, where $\overline{v}'_j$ branch covers the section at infinity σ^*_∞ and $\overline{v}''_j$ is the union of irreducible components which do not branch cover σ^*_∞. Here $* = \varnothing, {}'$, or $-$.

There are two cases: $\overline{v}'_0 = \varnothing$ or $\overline{v}'_0 \neq \varnothing$. The latter case is harder, and will be treated first. The former one will be treated in Step 3. By analyzing the two constraints $n^-(\overline{u}_i) = m$ and $I(\overline{u}_i) = 3$, we obtain Theorem 7.6.1, which gives a preliminary list of possibilities for $\overline{u}_\infty$ when $\overline{v}'_0 \neq \varnothing$. Many of the possibilities in Theorem 7.6.1 actually do not occur. Theorem 7.10.1 eliminates Cases (2)–(6) from the list. This is done by a finer analysis of the behavior of $\overline{u}_i$ in the vicinity of the component σ^-_∞ and is similar in spirit to the *layer structures* of Ionel-Parker [IP, Section 7].

Summarizing, we have:

Lemma **7.2.1.** — *If $\overline{u}_\infty \in \partial\mathcal{M}^3_{\overline{\mathfrak{m}}}$ and $\overline{v}'_0 \neq \varnothing$, then $\overline{u}_\infty \in A_1$, where*

$$A_1 = \coprod_{\delta_0\boldsymbol{\gamma}', \{z_\infty\}\cup\mathbf{y}'} \left(\mathcal{M}^{I=2,n^*=m-1}_{\overline{J}'}(\boldsymbol{\gamma}, \delta_0\boldsymbol{\gamma}') \times \mathcal{M}^{I=0,n^*=0}_{\overline{J}^\diamondsuit_-}(\delta_0\boldsymbol{\gamma}', \{z_\infty\} \cup \mathbf{y}') \right.$$
$$\left. \times \left(\mathcal{M}^{I=1,n^*=1}_{\overline{J}}(\{z_\infty\} \cup \mathbf{y}', \mathbf{y})/\mathbf{R} \right) \right),$$

if $\mathbf{y} = \{x_i\} \times \mathbf{y}'$ or $\{x'_i\} \times \mathbf{y}'$ and $A_1 = \varnothing$ otherwise. Here we have omitted the potential contributions of connector components for simplicity.

Case (1) in Theorem 7.6.1 corresponds to A_1.

Step 2. We now glue the triples $(\overline{v}_1, \overline{v}_0, \overline{v}_{-1})$ in A_1, subject to the constraint $\overline{\mathfrak{m}}$. This gluing accounts for the term $\widetilde{\partial}_1 \circ \widetilde{\Psi}_0 \circ \widetilde{U}_{m-1}$ and is a bit involved. Let us abbreviate

$$\mathcal{M}_1 := \mathcal{M}^{I=2,n^*=m-1}_{\overline{J}'}(\boldsymbol{\gamma}, \delta_0\boldsymbol{\gamma}');$$

$$\mathcal{M}_1' := \mathcal{M}_{\overline{J}'}^{I=2,n^*=m-1,f_{\delta_0}}(\boldsymbol{\gamma},\delta_0\boldsymbol{\gamma}');$$

$$\mathcal{M}_0 := \mathcal{M}_{\overline{J}_-^{\diamondsuit}}^{I=0,n^*=0}(\delta_0\boldsymbol{\gamma}',\{z_\infty\}\cup\mathbf{y}');$$

$$\mathcal{M}_{-1} := \mathcal{M}_{\overline{J}}^{I=1,n^*=1}(\{z_\infty\}\cup\mathbf{y}',\mathbf{y}),$$

where $\mathbf{y}=\{x_i\}\cup\mathbf{y}''$ or $\{x_i'\}\cup\mathbf{y}''$. Here f_{δ_0} is a nonzero normalized asymptotic eigenfunction of δ_0 at the negative end which, used as a modifier, stands for "the normalized asymptotic eigenfunction at the negative end δ_0 is f_{δ_0}." See Section 7.7 for more details on asymptotic eigenfunctions.

We first observe that $\overline{v}_0=\overline{v}_0'\cup\overline{v}_0''\in\mathcal{M}_0$ is regular: $\overline{v}_0'$ is regular by Lemma 5.8.9 and $\overline{v}_0''$ is regular since $\overline{J}_-\in\mathcal{J}^{reg}_{\overline{W}_-}$ and $\overline{J}_-^{\diamondsuit}$ is (ε,U)-close to $\overline{J}_-$. The moduli spaces $\mathcal{M}_1$ and $\mathcal{M}_{-1}$ are also regular since $\overline{J}_-$ is regular. Hence we can glue triples $([\overline{v}_1],\overline{v}_0,[\overline{v}_{-1}])$ in $(\mathcal{M}_1/\mathbf{R})\times\mathcal{M}_0\times(\mathcal{M}_{-1}/\mathbf{R})$. More precisely, consider the gluing parameter space

$$\mathfrak{P} := \coprod_{\delta_0\boldsymbol{\gamma}',\{z_\infty\}\cup\mathbf{y}'}\mathfrak{P}_{\delta_0\boldsymbol{\gamma}',\{z_\infty\}\cup\mathbf{y}'}, \tag{7.2.1}$$

where

$$\mathfrak{P}_{\delta_0\boldsymbol{\gamma}',\{z_\infty\}\cup\mathbf{y}'}=(5r,\infty)^2\times(\mathcal{M}_1/\mathbf{R})\times\mathcal{M}_0\times(\mathcal{M}_{-1}/\mathbf{R}), \tag{7.2.2}$$

$0<h<1$ and $r\gg 1/h$ are gluing constants, and $\mathcal{M}_1,\mathcal{M}_0,\mathcal{M}_{-1}$ correspond to the pair $(\delta_0\boldsymbol{\gamma}',\{z_\infty\}\cup\mathbf{y}')$. We may assume that all the multiplicities of $\boldsymbol{\gamma}'$ are 1, since the Hutchings-Taubes gluing of branched covers is essentially independent of the present gluing problem.

We recall the extended moduli space

$$\mathcal{M}^{ext} := \mathcal{M}_{\overline{J}_-}^{I=3,n^*=m,ext}(\boldsymbol{\gamma},\mathbf{y})$$

from Definition 5.7.24, where $\mathbf{y}=\{x_i\}\cup\mathbf{y}''$ or $\{x_i'\}\cup\mathbf{y}''$. Here the modifier *ext* means that $\overline{u}:(\dot{F},j)\to(\overline{W}_-,\overline{J}_-)$ is a multisection which maps all the connected components of $\partial\dot{F}$ but one to a different $L^-_{\vec{a}_i}$ and the last connected component to some $L^-_{\vec{a}_i\cup\vec{a}_{i,j}}$.

There is a gluing map

$$G:\mathfrak{P}\to\mathcal{M}^{ext},\quad \mathfrak{d}=(T_\pm,\overline{v}_1,\overline{v}_0,\overline{v}_{-1})\mapsto\overline{u}(\mathfrak{d}),$$

which is a diffeomorphism onto its image for $r\gg 0$. Here $T_\pm$ is shorthand for the pair T_+,T_-. The map G is defined in a manner similar to that of Section 6.5.2 and is described in more detail in Section 7.12.3. Let $r_0\gg 0$, let $\mathfrak{P}_{(r_0)}\subset\mathfrak{P}$ be the subset

$\{T_+ \geq r_0\}$[11] and let

$$\mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}} := \mathcal{M}^{3}_{\overline{\mathfrak{m}}} - G(\mathfrak{P}_{(r_0)})$$

be the truncated moduli space. For generic $r_0 \gg 0$,

$$\partial_0 \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}} := G(\partial \mathfrak{P}_{(r_0)}) \cap \mathcal{M}^{3}_{\overline{\mathfrak{m}}}$$

is a transverse intersection.

The following theorem is proved in Section 7.12:

Theorem **7.2.2.** — *For generic $r_0 \gg 0$,*

$$\#(\partial_0 \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}} \cap G(\mathfrak{P}_{\delta_0 \boldsymbol{\gamma}', \{z_\infty\} \cup \mathbf{y}'})) \equiv \#(\mathcal{M}'_1/\mathbf{R}) \cdot \#\mathcal{M}_0 \cdot \#(\mathcal{M}_{-1}/\mathbf{R}) \; mod\; 2. \tag{7.2.3}$$

Hence the contributions from A_1 account for the term $\widetilde{\partial}_1 \circ \widetilde{\Psi}_0 \circ \widetilde{U}_{m-1}$.

Step 3. Let $\partial_1 \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}} = \partial \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}} - \partial_0 \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}}$. The following lemma is proved in Section 7.11.

Lemma **7.2.3.** — $\partial_1 \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}} \subset A_2 \sqcup A_3$, *where*

$$A_2 = \coprod_{\mathbf{y}'' \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}} \left(\mathcal{M}^{I=2, n^*=m}_{\overline{J}^{\diamond}_{-}}(\boldsymbol{\gamma}, \mathbf{y}''; \overline{\mathfrak{m}}) \times \left(\mathcal{M}^{I=1, n^*=0}_{\overline{J}}(\mathbf{y}'', \mathbf{y})/\mathbf{R} \right) \right);$$

$$A_3 = \coprod_{\boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_{2g}} \left(\left(\mathcal{M}^{I=1, n^*=0}_{\overline{J'}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}')/\mathbf{R} \right) \times \mathcal{M}^{I=2, n^*=m}_{\overline{J}^{\diamond}_{-}}(\boldsymbol{\gamma}', \mathbf{y}; \overline{\mathfrak{m}}) \right).$$

Here we have omitted the potential contributions of connector components for simplicity.

On the other hand, gluing the pairs in A_2 (resp. A_3) using the usual (resp. Hutchings-Taubes) gluing theorem implies that $A_2 \cup A_3 \subset \partial_1 \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}}$ and accounts for the term $\partial'_{HF} \circ \Psi'$ (resp. $\Psi' \circ \partial_{PFH}$). This proves Theorem 7.1.3.

Organization of Section 7. In Sections 7.3 we describe the necessary modifications to apply the proof of SFT compactness to the nonstandard setting of $\overline{W}_-$-curves. The signed intersection numbers $n^*(\overline{u})$ are analyzed in Sections 7.4 and 7.5. Theorem 7.6.1 is then proved in Section 7.6. We then discuss asymptotic eigenfunctions in Section 7.7, rescaling in Section 7.8, and the "involution lemmas" in Section 7.9, on our way to proving Theorem 7.10.1 in Section 7.10. Lemma 7.2.3 is proved in Section 7.11. Finally, the gluing is discussed in Section 7.12.

7.3. *SFT compactness.*

[11] Restricting to $\{T_+ \geq r_0\}$ seems a little asymmetrical, but it will become clear in Section 7.12 that T_- depends on $(T_+, \overline{v}_1, \overline{v}_0, \overline{v}_{-1})$ if $\overline{u}(\mathfrak{d}) \in \mathcal{M}^{3}_{\overline{\mathfrak{m}}}$.

7.3.1. *Statement of the compactness theorem.*

Definition **7.3.1.** — *A* holomorphic $\overline{W}_-$-building

$$\overline{u}_\infty = \overline{v}_{-b} \cup \cdots \cup \overline{v}_a, \quad a, b \in \mathbf{Z}^{\geq 0},$$

consists of the following data:

(B1′) *For each $j = -b, \dots, a$, a compact* nodal *Riemann surface G_j, possibly with boundary and possibly disconnected. The nodal Riemann surface G_j is obtained from a Riemann surface $\widetilde{G}_j$ by identifying pairs of interior points or pairs of boundary points. The set of identified points is the* nodal set *$\mathfrak{n}_j$, which may be empty. Let $\mathbf{p}_j^+, \mathbf{p}_j^- \subset G_j$ be sets of interior punctures for $j \geq 0$ and let $\mathbf{q}_j^+, \mathbf{q}_j^- \subset G_j$ be sets of boundary punctures for $j \leq 0$, subject to $\mathbf{p}_0^-, \mathbf{q}_0^+ = \varnothing$. The sets $\mathfrak{n}_j$, $\mathbf{p}_j^\pm$, $\mathbf{q}_j^\pm$ are mutually disjoint.*

(B2′) *For each $j = -b, \dots, a$, a holomorphic map $\overline{v}_j : \dot{G}_j \to \overline{W}_j$, where $\overline{W}_j = \overline{W}'$ for $0 < j \leq a$, $\overline{W}_0 = \overline{W}_-$ for $j = 0$, $\overline{W}_j = \overline{W}$ for $-b \leq j < 0$, and $\dot{G}_j = G_j - \mathbf{p}_j^+ - \mathbf{p}_j^- - \mathbf{q}_j^+ - \mathbf{q}_j^-$ for all j. (Here some sets of punctures may be empty.)*

(B3′) *For each $j = -b, \dots, 0$, $\partial \dot{G}_j$ is mapped to the appropriate Lagrangian submanifold $L_{\overline{\mathbf{a}}} \sqcup L_{\overline{h}(\overline{\mathbf{a}})}$ or $L^-_{\overline{\mathbf{a}}}$.*

(B4′) *For each $j = -b, \dots, a$, $\overline{v}_j$ converges to a trivial strip over a "Reeb chord" at the positive (resp. negative) end near each boundary puncture of $\mathbf{q}_j^+$ (resp. $\mathbf{q}_j^-$) and to a trivial cylinder over a closed orbit at the positive (resp. negative) end near each interior puncture of $\mathbf{p}_j^+$ (resp. $\mathbf{p}_j^-$).*

(B5′) *For each $j = -b, \dots, a-1$, there is an identification between $\mathbf{p}_j^+$ and $\mathbf{p}_{j+1}^-$ and an identification between $\mathbf{q}_j^+$ and $\mathbf{q}_{j+1}^-$ such that the pairs that are identified are asymptotic to the same Reeb chord or closed orbit.*

(B6′) *No level $\overline{v}_j$ is a union of trivial cylinders or trivial strips and no constant sphere has fewer than three nodes.*

The levels $\overline{v}_j$ are arranged in order from lowest to highest.

Proposition **7.3.2.** — *Let $\overline{J}_- \in \mathcal{J}_{\overline{W}_-}$ and let $\overline{u}_i : (\dot{F}_i, j_i) \to (\overline{W}_-, \overline{J}_-)$, $i \in \mathbf{N}$, be a sequence of $\overline{W}_-$-curves from $\boldsymbol{\gamma}$ to $\mathbf{y}$. Then there is a subsequence which converges in the sense of SFT to a level $a + b + 1$ holomorphic $\overline{W}_-$-building*

$$\overline{u}_\infty = \overline{v}_{-b} \cup \cdots \cup \overline{v}_a.$$

The same also holds when $\{\overline{u}_i\}$ is a sequence of $(\overline{W}_-, \overline{J}_-^\diamond)$-curves (see Definition 5.8.13).

Remark **7.3.3.** — Our setting is more general than that of [BEHWZ] in some ways, e.g., the presence of a singular Lagrangian boundary condition. However, the nonexistence of contractible Hamiltonian orbits or chords and the existence of a holomorphic projection $\pi_{\overline{W}_-} : \overline{W}_- \to B_-$ simplify the proofs considerably to the point that they are often simpler than the corresponding proofs in [BEHWZ]. Other important simplifying features are the absence of “teardrops” (i.e., punctured holomorphic disks with boundary in $L^-_{\overline{\mathbf{a}}}$ which, at the puncture, are asymptotic to a double point of $L^-_{\overline{\mathbf{a}}}$) and the fact that different connected components of the boundary of the domain are mapped to different smooth components of $L^-_{\overline{\mathbf{a}}}$.

7.3.2. *Preliminary lemmas.* — Here we prove some lemmas which will be used in the proof of Proposition 7.3.2. First we recall some terminology. Let $\overline{W}_*$ denote one of $\overline{W}$, $\overline{W'}$, $\overline{W}_\pm$ and $\overline{\omega}$ the 2-form on $\overline{W}_*$ defined in Section 5.1.2. We recall that the $\overline{\omega}$-area of a map $w : \dot{F} \to \overline{W}_*$ is defined as

$$E_{\overline{\omega}}(w) = \int_{\dot{F}} w^*\overline{\omega}.$$

We also introduce the *energy* $E(w)$ in the spirit of Definition 4.3.2. More precisely, let $\mathcal{C}$ be the set of nondecreasing smooth functions $\phi : \mathbf{R} \to [0, 1]$; for all $\phi \in \mathcal{C}$ we define the 1-form $\lambda_\phi = \overline{\pi}^*_{B_*}(\phi(s)dt)$, where as usual B_* can stand for B, B' or $B_\pm$. Then the energy of a map $w : \dot{F} \to \overline{W}_*$ is

$$E(w) = \int_{\dot{F}} w^*\overline{\omega} + \sup_{\phi\in\mathcal{C}} \int_{\dot{F}} w^* d\lambda_\phi = E_{\overline{\omega}}(w) + \sup_{\phi\in\mathcal{C}} \int_{\dot{F}} w^* d\lambda_\phi. \tag{7.3.1}$$

If w is a holomorphic map, then $E_{\overline{\omega}}(w) \geq 0$ and moreover a finite energy holomorphic cylinder $w : [-R, R] \times S^1 \to \overline{W}_*$ (or strip $w : [-R, R] \times [0, 1] \to \overline{W}_*$) with $E_{\overline{\omega}}(w) = 0$ is either a constant map or is part of a trivial cylinder (or strip) over a Hamiltonian orbit (or chord). We recall also that the *action* of a Hamiltonian orbit or chord is the integral, over that orbit or chord, of the 1-form $\overline{\alpha}_0$ defined in Section 5.1.2. By Equation (7.3.1), the energy of w is determined by its $\overline{\omega}$-area and the actions of the orbits and chords at its ends.

In what follows we measure distances with respect to fixed Riemannian metrics g on $\overline{N}$ and $ds^2 + g$ on $\overline{W'}$, as well as their restrictions to N and $\overline{W}_*$.

The following lemma is a by-now standard observation originally due to Hofer.

Lemma **7.3.4.** — *Let* F *be a finite type Riemann surface with boundary and interior and boundary punctures. Fix a compatible metric on* F *which is cylindrical near the punctures. Let* $F_1 \subset F_2 \subset \cdots \subset F$ *be a sequence of Riemann surfaces with corners of the same topological type, where* ∂F_n *admits a decomposition* $\partial_0 F_n \cup \partial_1 F_n$ *into codimension* 0 *submanifolds that intersect at the corners,* $\partial_0 F_n \subset \partial F$*, the number of components of* $\partial_i F_n$*,* $i = 0, 1$*, are independent of* n*, each component of* $\partial_0 F_n$ *is contained in a different component of* $\partial_0 F_{n+1}$*, and* $\mathrm{int}(\partial_1 F_n) \subset \mathrm{int}\, F$*. Let* $w_n : F_n \to \overline{W}_*$ *be a sequence of holomorphic maps such that:*

- (A) $\partial_0 F_n$ *is mapped to the appropriate Lagrangian boundary* $L_{\overline{\mathbf{a}}} \sqcup L_{\overline{h}(\overline{\mathbf{a}})}$ *or* $L^-_{\overline{\mathbf{a}}}$*,*
- (B) $E(w_n) \leq C_0$ *for some constant* C_0*, and*

(C) $\lim_{n \to +\infty} E_{\overline{\omega}}(w_n) = 0.$

Then for any $\epsilon > 0$ *there is a constant* $C > 0$ *such that* $\|\nabla w_n|_{F_n^\circ}\|_{C^0} \leq C$ *for all n, where* $F_n^\circ = F_n - N_\epsilon(\partial_1 F_n)$ *and* $N_\epsilon(\partial_1 F_n)$ *is an* ϵ*-neighborhood of* $\partial_1 F_n$*.*

Sketch of proof. — The lemma follows from the standard bubbling argument: if there exists a sequence of points $p_n \in F_n^\circ$ such that $\lim_{n \to +\infty} |\nabla w_n(p_n)| = +\infty$, then, by the usual rescaling argument (we may apply a translation in the target if the sequence w_n is not bounded), we have bubbling of a *nonconstant,* $E_{\overline{\omega}}(\tilde{w}) = 0$ holomorphic plane or half-plane $\tilde{w}$ with Lagrangian boundary.

The $E_{\overline{\omega}}(\tilde{w}) = 0$ condition implies that $\tilde{w}$ maps to a trivial cylinder or strip over a (not necessarily closed) Hamiltonian orbit or chord. This key fact does not depend on whether the Lagrangian boundary condition is singular. The function $f = \overline{\pi}_* \circ \tilde{w}$ is holomorphic and has finite energy. We explain two cases where the existence of a nontrivial f leads to a contradiction. First consider the case $f : \mathbf{C} \to B'$, which can be lifted to the universal covering map $\tilde{f} : \mathbf{C} \to \mathbf{C}$. By the Little Picard Theorem, $\tilde{f}$ misses at most one point of $\mathbf{C}$, which contradicts the fact that f has finite energy. Next consider the case $f : \mathbf{H}_+ \to B_-$ (here $\mathbf{H}_+$ is the upper half-plane), which can be lifted to the universal covering map $\tilde{f} : \mathbf{H}_+ \to \mathbf{H}_+$ such that $\tilde{f}(\partial \mathbf{H}_+) \subset \partial \mathbf{H}_+$. Doubling $\tilde{f}$ to a map $\mathbf{C} \to \mathbf{C}$ using the Schwarz reflection principle and applying the Little Picard Theorem again gives a contradiction. □

Lemma **7.3.5** *(Long Cylinder Lemma, Version 1).* — *Let* γ_+ *and* γ_- *be closed Hamiltonian orbits in* $\overline{N}$*. Fix* E_0 *larger than the actions of both* γ_+ *and* γ_-*, and* $\eta_0 > 0$ *such that every closed Hamiltonian orbit* γ *of action at most* E_0 *is at distance at least* $2\eta_0$ *from* γ_+*, if* $\gamma \neq \gamma_+$*, and from* γ_-*, if* $\gamma \neq \gamma_-$*. Then for all* $\eta \in (0, \eta_0)$ *there exist* $c_0 > 0$*,* $R_0 > 0$*,* $\eta' > 0$ *and* $\delta > 0$ *such that, for all* $c \geq c_0$ *and all* $R > R_0$*, the following holds: If*

$$w : [-R - c, R + c] \times S^1 \to \overline{W'}$$

is a $\overline{J}'$*-holomorphic map which satisfies:*

(1) $w|_{[-R-c,-R]\times S^1}$ *and* $w|_{[R,R+c]\times S^1}$ *are* η'*-close in the* C^1*-metric to portions of trivial cylinders over* γ_- *and* γ_+*, respectively, and*
(2) $E_{\overline{\omega}}(w) < \delta$*,*

then $\gamma_- = \gamma_+$ *and* w *is* η*-close in the* C^1*-metric to a portion of the trivial cylinder over* γ_-*.*

Lemma **7.3.6** *(Long Strip Lemma, Version 1).* — *Let* y_+ *and* y_- *be chords corresponding to intersection points in* $\overline{a}_i \cap \overline{h}(\overline{a}_{i'})$*. Fix* $\eta_0 > 0$ *such that all other chords are a distance at least* $2\eta_0$ *from* y_+ *and* y_-*. Then for all* $\eta \in (0, \eta_0)$ *there exist* $c_0 > 0$*,* $R_0 > 0$*,* $\eta' > 0$ *and* $\delta > 0$ *such that, for all* $c \geq c_0$ *and all* $R > R_0$*, the following holds: If*

$$w : [-R - c, R + c] \times [0, 1] \to \overline{W}$$

is a $\overline{\mathrm{J}}$*-holomorphic map which satisfies:*

(1) w *maps* $[-\mathrm{R}-c, \mathrm{R}+c] \times \{1\}$ *to* $\mathrm{L}_{\overline{a}_i}$ *and* $[-\mathrm{R}-c, \mathrm{R}+c] \times \{0\}$ *to* $\mathrm{L}_{\overline{a}_{i'}}$*;*
(2) $w|_{[-\mathrm{R}-c,-\mathrm{R}]\times[0,1]}$ *and* $w|_{[\mathrm{R},\mathrm{R}+c]\times[0,1]}$ *are* η'*-close in the* C^1*-metric to portions of trivial strips over* y_- *and* y_+*, respectively, and*
(3) $\mathrm{E}_{\overline{\omega}}(w) < \delta$*,*

then $y_- = y_+$ *and* w *is* η*-close in the* C^1*-metric to a portion of the trivial strip over* y_-*.*

Proof. — The proof will not depend on whether the Lagrangian boundary condition is singular because we are only concerned with one smooth component at a time in the lemma.

First we claim that, if the hypotheses of the lemma hold, then the image of w is contained in an η-neighborhood of the strip over y_-. Arguing by contradiction, assume there exist $\eta < \eta_0$, sequences R_n, c_n, η'_n, δ_n, $p_n \in [-\mathrm{R}, \mathrm{R}] \times [0, 1]$, and $\overline{\mathrm{J}}$-holomorphic maps

$$w_n : [-\mathrm{R}_n - c_n, \mathrm{R}_n + c_n] \times [0, 1] \to \overline{\mathrm{W}}$$

such that:

- $\mathrm{R}_n \to +\infty$, $c_n \to +\infty$, $\eta'_n \to 0$ and $\delta_n \to 0$,
- w_n satisfy (1)–(3) with R_n, c_n, η'_n and δ_n in place of R, c, η' and δ, and
- $w_n(p_n)$ is at a distance of η from the trivial strip over y_-.

Conditions (1)–(3) imply that $\mathrm{E}_{\overline{\omega}}(w_n) \to 0$ and $\mathrm{E}(w_n) < \mathrm{E}_0$ for n sufficiently large. Hence by Lemma 7.3.4 there is a constant C such that $\|\nabla w_n\|_{\mathrm{C}^0} \leq \mathrm{C}$ for all n. Then, after translating the maps w_n so that the points $w_n(p_n) \in \overline{\mathrm{W}} = \mathbf{R} \times [0, 1] \times \overline{\mathrm{S}}$ all have the same $\mathbf{R}$-coordinate and taking a subsequence, the maps w_n converge in C^∞_{loc} to a $\overline{\mathrm{J}}$-holomorphic map $w_\infty : \mathbf{R} \times [0, 1] \to \overline{\mathrm{W}}$ and the points p_n converge to a point $p_\infty = (s_\infty, t_\infty) \in \mathbf{R} \times [0, 1]$ such that $\mathrm{E}(w_\infty) < \mathrm{E}_0$, $\mathrm{E}_{\overline{\omega}}(w_\infty) = 0$, and $w_\infty(p_\infty)$ is at η-distance from the trivial strip over y_-. Since there is no chord at η-distance from y_-, the existence of the map w_∞ is a contradiction. This proves the claim, which in turn implies that $y_- = y_+$.

Next we claim that w is η-close in the C^0-metric to a trivial strip over y_-. The composition $\overline{\pi}_{\mathrm{B}} \circ w$ of w with the projection $\overline{\pi}_{\mathrm{B}} : \overline{\mathrm{W}} \to \mathbf{R} \times [0, 1]$ is holomorphic. Since $\overline{\pi}_{\mathrm{B}} \circ w$ maps $[-\mathrm{R}-c, \mathrm{R}+c] \times \{i\}$ to $\mathbf{R} \times \{i\}$, for $i = 0, 1$, and $\{\pm(\mathrm{R}+c)\} \times [0, 1]$ to arcs that are η' close to $s = s_\pm$, where $s_\pm$ are constants, it follows that $\overline{\pi}_{\mathrm{B}} \circ w$ must be a biholomorphism onto the image and moreover must be C^0-close to a translation of the map $(s, t) \mapsto (s, t)$, provided η' is small. Therefore, after possibly reducing η' in the statement of the theorem, w is close to a trivial strip over y_-.

Finally, the C^0-closeness implies C^1-closeness (and in fact C^k-closeness for all k) by elliptic regularity. This concludes the proof of the lemma. □

Now we consider long cylinders and strips whose extremities are close to constant maps. First we develop a bound on their $\overline{\omega}$-area in terms of the behavior of the extremities, which is a consequence of the topology of $\overline{W}_-$.

Lemma **7.3.7.** — *For every $\delta > 0$ there is $\eta' > 0$ such that, for every $R > 0$, if either*

- $w : [-R, R] \times S^1 \to \overline{W}_-$ *is a $\overline{J}_-$-holomorphic map, or*
- $w : [-R, R] \times [0, 1] \to \overline{W}_-$ *is a $\overline{J}_-$-holomorphic map mapping $[-R, R] \times \{0, 1\}$ to $L^-_{\overline{\mathbf{a}}}$,*

and $w|_{\{\pm R\} \times S^1}$ (or $w|_{\{\pm R\} \times [0,1]}$) are η'-close to constant maps in the C^1-metric, then $E_{\overline{\omega}}(w) < \delta$.

Proof. — We consider strips first since cylinders are easier. We have two cases: either the boundary of $[-R, R] \times [0, 1]$ is mapped to the same component $L^-_{\overline{a}_i}$ or the two boundary components are mapped to different components $L^-_{\overline{a}_i}$ and $L^-_{\overline{a}_j}$.

We embed $[-R, R] \times [0, 1]$ holomorphically in $D^2 = \{|z| \leq 1 \mid z \in \mathbf{C}\}$ as the complement of an open neighborhood of $\{1, -1\}$ and extend w to a smooth map $\widetilde{w} : D^2 \to \overline{W}_-$ with boundary on $L^-_{\overline{a}_i}$ or $L^-_{\overline{a}_i} \cup L^-_{\overline{a}_j}$, depending on the situation. We can make the extension so that, in the neighborhood of $\{1, -1\}$, $\widetilde{w}$ remains η'-close in the C^1-metric to constant maps, and therefore there is a positive constant C depending only on $\overline{\omega}$ such that

$$|E_{\overline{\omega}}(w) - E_{\overline{\omega}}(\widetilde{w})| \leq C\eta'. \qquad (\mathbf{7.3.2})$$

In the first case, $\widetilde{w}$ defines an element in $\pi_2(\overline{W}_-, L^-_{\overline{a}_i})$, and, since the latter group is trivial, $E_{\overline{\omega}}(\widetilde{w}) = 0$ and then we can chose η' so that Equation (7.3.2) implies that $E_{\overline{\omega}}(w) < \delta$.

The second case is similar, but the map $\widetilde{w}$ defines an element in the relative homotopy group $\pi_2(\overline{W}_-, L^-_{\overline{a}_i} \cup L^-_{\overline{a}_j})$. If $g(\overline{S}) > 1$, then the group is trivial and we can conclude as before.

If $\overline{S}$ is a torus, the first observation is that $\widetilde{w}$ is homotopic to a map (denoted by the same symbol) $\widetilde{w} : (D^2, \partial D^2) \to (\overline{S}, \overline{a}_i \cup \overline{a}_j)$ such that the preimage of an η'-neighborhood of z_∞ consists of the two fixed neighborhoods of $+1$ and -1, respectively. The homotopy is obtained by taking the universal cover $\widetilde{B}_-$ of B_-, identifying the pullback of the fibration $\overline{W}_- \to B_-$ with $\widetilde{B}_- \times \overline{S}$, lifting $\widetilde{w}$ to $\widetilde{B}_- \times \overline{S}$, and composing the lift with the projection onto $\overline{S}$.

The universal cover of $\overline{S}$ is diffeomorphic to $\mathbf{R}^2$ and we can assume that the preimage of $\overline{a}_i \cup \overline{a}_j$ is a grid with vertices in the integer points of $\mathbf{R}^2$. If we lift $\widetilde{w}$ to the universal cover of $\overline{S}$, we obtain a map from D^2 to $\mathbf{R}^2$ with boundary on the grid. If $\widetilde{w}$ represents a non-trivial element in $\pi_2(\overline{S}, \overline{a}_i \cup \overline{a}_j)$, then its lift maps ∂D^2 surjectively to some square of the grid, and therefore the preimage of an η'-neighborhood of the grid points consists of at least four disjoint open sets in D^2, which is a contradiction.

The case of cylinders is proved in the same way, using the fact that $\pi_2(\overline{W}_-) = 0$. □

Remark **7.3.8.** — Since we are free to chose the genus of the open book decomposition in Theorem 1.2.1, the simpler case of $g(\overline{S}) > 1$ would have sufficed for the main result of the paper. However we also treated the case of $g(\overline{S}) = 1$ for completeness.

Lemma **7.3.9** *(Long Cylinder Lemma, Version 2). — For all $\eta > 0$ there exists $\eta' > 0$ such that, for all $R > 0$, if*

$$w : [-R, R] \times S^1 \to \overline{W}_-$$

is a $\overline{J}_-$-holomorphic such that $w|_{\{-R\}\times S^1}$ and $w|_{\{R\}\times S^1}$ are η'-close in the C^1-metric to constant maps to points p_- and p_+, respectively, then w is η-close to a constant map in the C^1-metric.

Lemma **7.3.10** *(Long Strip Lemma, Version 2). — For all $\eta > 0$ there exists $\eta' > 0$ such that for all $R > 0$ the following holds: If*

$$w : [-R, R] \times [0, 1] \to \overline{W}_-$$

is a $\overline{J}_-$-holomorphic map which satisfies:

(1) *w maps $[-R, R] \times \{1\}$ to $L^-_{\overline{a}_i}$ and $[-R, R] \times \{0\}$ to $L^-_{\overline{a}_{i'}}$,*
(2) *$w|_{\{R\}\times[0,1]}$ and $w|_{\{-R\}\times[0,1]}$ are η'-close in the C^1-norm to constant maps to p_+ and p_-, respectively,*

then w is η-close to a constant map in the C^1-metric.

Proof. — First we show that, for η' sufficiently small (and at least smaller than $\eta/2$), the map $\overline{\pi}_{B_-} \circ w$ is η-close to a constant map. We embed $[-R, R] \times [0, 1]$ holomorphically in D^2 as the complement of an open neighborhood of $\{-1, 1\}$ as in the proof of Lemma 7.3.7. Since η' is sufficiently small, we can

- assume without loss of generality that $p_+, p_- \in L^-_{\overline{a}_i}$;
- extend w to a smooth map $\widetilde{w} : D^2 \to \overline{W}_-$ such that ∂D^2 is mapped to $L^-_{\overline{a}_i} \cup L^-_{\overline{a}_{i'}}$; and
- assume that the extension occurs inside η'-neighborhoods of p_+ and p_-.

The composition $\overline{\pi}_{B_-} \circ w : [-R, R] \times [0, 1] \to B_-$ is holomorphic and its extension $\overline{\pi}_{B_-} \circ \widetilde{w} : D^2 \to B_-$, viewed as an element of $\pi_2(B_-, \partial B_-)$, is contractible. This implies that the image of $\overline{\pi}_{B_-} \circ \widetilde{w}$ is contained in the union of the balls of radius η' centered at $\overline{\pi}_{B_-}(p_+)$ and $\overline{\pi}_{B_-}(p_-)$, as follows: Assume by contradiction that this is not the case. Then there is a regular value $q \in B_- - \overline{\pi}_{B_-}(p_+) - \overline{\pi}_{B_-}(p_-)$ of $\overline{\pi}_{B_-} \circ \widetilde{w}$ such that $(\overline{\pi}_{B_-} \circ \widetilde{w})^{-1}(q)$ is contained in an open subset of D^2 on which $\overline{\pi}_{B_-} \circ \widetilde{w}$ coincides with $\overline{\pi}_{B_-} \circ w$, and therefore is holomorphic. This is a contradiction since all preimages of q contribute positively to the degree of $\overline{\pi}_{B_-} \circ \widetilde{w}$. The balls of radius η' centered at $\overline{\pi}_{B_-}(p_+)$ and $\overline{\pi}_{B_-}(p_-)$ have a nonempty intersection because $[-R, R] \times [0, 1]$ is connected, and therefore the image of $\overline{\pi}_{B_-} \circ w$ is contained in a ball or radius $2\eta' < \eta$ around some point $y \in B_-$.

Let $\overline{S}_y = \overline{\pi}_{B_-}^{-1}(y)$ be the fiber over y, $N_\eta(\overline{S}_y) \subset \overline{W}_-$ the η-neighborhood of $\overline{S}_y$, and $p : N(\overline{S}_y) \to \overline{S}_y$ the map obtained by projecting out the directions transverse to the fibers after straightening out using the symplectic connection. Since the image of $\widetilde{w}$ is contained in $N_\eta(\overline{S}_y)$, Lemma 7.3.7 and its proof imply that the map $p \circ \widetilde{w} : D^2 \to \overline{S}_y$, viewed as an element of $\pi_2(\overline{S}_y, \overline{a}_i \cup \overline{a}_{i'})$, is contractible and $E_{\overline{\omega}}(\widetilde{w}) \leq \delta$ for some δ which depends only on $\overline{\omega}$.

The map p is not holomorphic, but its differential $dp(x)$ at any point $x \in N_\eta(\overline{S}_y)$ is the composition of the $\mathbf{C}$-linear projection $T_x\overline{W}_- \to T_x\overline{S}_{\overline{\pi}_{B_-}(x)}$ with the differential of the orientation-preserving diffeomorphism $\overline{S}_{\overline{\pi}_{B_-}(x)} \to \overline{S}_y$ induced by parallel transporting using the symplectic connection. This implies that all the preimages of a regular value of $p \circ w$ come with positive signs for the purposes of computing the degree. We can therefore repeat for $p \circ w$ the argument we used for $\overline{\pi}_{B_-} \circ w$ and this proves the lemma. □

7.3.3. *Proof of the compactness theorem.* — We follow the lines of the proof of Proposition 6.1.3. First we state the analog of Lemma 6.1.1 and sketch its proof.

Lemma **7.3.11.** — *Let* $\overline{u}_i : (\dot{F}_i, j_i) \to (\overline{W}_-, \overline{J}_-)$, $i \in \mathbf{N}$, *be a sequence of* $\overline{W}_-$*-curves from* $\boldsymbol{\gamma}$ *to* $\mathbf{y}$ *with index* $I_{\overline{W}_-}(\overline{u}_i) = n$ *for some integer* n. *Then there is a subsequence such that all the* $\dot{F}_i$ *are diffeomorphic to a fixed* $\dot{F}$.

Proof. — The $\overline{\omega}$-area bound for the $\overline{W}_-$ case is similar to that of the W_+ case. We take the difference of two $\overline{W}_-$-curves $\overline{u}_1$ and $\overline{u}_2$ from $\boldsymbol{\gamma}$ to $\mathbf{y}$ to obtain $Z \in H_2(\overline{W}_-)$. Since they both intersect σ_∞^- once, Z can be represented by a surface in W_-, and hence can be viewed as a class in $H_2(N)$. The zero flux condition implies that the $\overline{\omega}$-area of Z is zero.

We then use the Gromov-Taubes compactness theorem as in Step 2 of Lemma 6.1.1 to extract a subsequence $\overline{u}_i$ which converges in the sense of currents to a holomorphic building. In order to apply Schwarz reflection, we need to slightly extend $\overline{a}$ past z_∞, but the rest of the proof is the same. After passing to a subsequence, we may assume that the homology class $[\overline{u}_i] \in H_2(\check{\overline{W}}_-, Z_{\boldsymbol{\gamma},\mathbf{y}})$ is fixed for all i. Once the homology class $[\overline{u}_i]$ is fixed, we use the relative adjunction formula (Lemma 5.6.3) in the same way as in Lemma 6.1.1 to obtain a bound on $\chi(\dot{F}_i)$. □

The geometric setup for the $\overline{W}_-$-curves deviates from the standard SFT setup because the Lagrangian submanifold $L_{\overline{\mathbf{a}}}$ is a union of finitely many immersed non-compact Lagrangians with boundary, with the property that each is an embedding when restricted to the interior, and the boundary components all lie in a properly embedded noncompact one-manifold along which the various branches of $L_{\overline{\mathbf{a}}}$ have a clean intersection.

The noncompactness of the singular set means that the resulting "Legendrian" boundary condition at the negative end is singular. For this reason we will sketch the proof of Proposition 7.3.2.

Proof of Proposition 7.3.2. — By Lemma 7.3.11 we may assume that $\dot{F}_i = \dot{F}$ as smooth surfaces. We recall the main steps of the proof of SFT compactness.

(1) gradient bound,
(2) degeneration of the conformal structure,
(3) convergence on the thick part, and
(4) convergence on the thin part.

Gradient bound. Let h be a Riemannian metric on $\overline{W}_-$ which is compatible with $\bar{J}_-$ and is cylindrical at both ends. After passing to a subsequence, we can add finite sets of marked points Z_i to $\dot{F}$, all of the same cardinality, so that the following holds:

Lemma **7.3.12.** — *Let g_i be a complete hyperbolic metric on $\dot{F} - Z_i$ which is compatible with j_i and which has geodesic boundary and cusps at the boundary/interior punctures. Then there is a uniform bound*

$$\rho_i \|\nabla \overline{u}_i\|_{C^0} \leq C, \tag{7.3.3}$$

the so-called "gradient bound", where ρ_i is the injectivity radius of $(\dot{F} - Z_i, g_i)$ doubled along the boundary, and the norm $\|\nabla \overline{u}_i\|_{C^0}$ is computed with respect to the metrics g_i and h.

Lemma 7.3.12 is proved in [BEHWZ, Lemma 10.7] for the case without boundary and the appropriate modifications are outlined in [Abb2, Remark 3.11]. Summarizing the argument, if there is a sequence $p_i \in \dot{F}$ for which $\rho_i(p_i)\|\nabla \overline{u}_i(p_i)\|_{C^0} \to \infty$, then by the usual rescaling argument as in Lemma 7.3.4, we have the bubbling of a nonconstant holomorphic plane or half-plane. The Long Cylinder Lemmas control the ends of the holomorphic plane or half-plane. We can place a pair of points of Z_i in the middle of this bubbling region, which effectively bounds $\rho_i \|\nabla \overline{u}_i\|_{C^0}$ on the bubbling region with respect to the modified hyperbolic metric. Each bubble uses up a quantum of E_{ϖ} (i.e., there exists $\hbar > 0$ such that $E_{\varpi} > \hbar$ on the bubble) — the key point is that this also holds for the singular Lagrangian boundary condition as we explained in the proof of Lemma 7.3.4 — which bounds the number of bubbles and hence the number of marked points.

Degeneration of the conformal structure. If g_i degenerates as $i \to \infty$, then after passing to a subsequence there is a finite collection of homotopy classes of properly embedded arcs and closed curves on $\dot{F} - Z_i$, whose geodesic representatives are mutually disjoint, and which are pinched as $i \to \infty$, i.e., the lengths of the geodesic representatives go to zero. Let $\dot{F}_\infty$ be the nodal surface obtained by pinching all these geodesics. The complex structures j_i converge in the C^∞_{loc} topology outside the pinching geodesics to a complex structure j_∞ on $\dot{F}_\infty$. Then in the limit we obtain a nodal punctured Riemann surface with boundary $(\dot{F}_\infty, j_\infty)$. Let F'_∞ be $\dot{F}_\infty$ with the nodes removed.

For any $\varepsilon > 0$ we define $\mathrm{Thick}_\varepsilon(\dot{F}_i, g_i)$ as the set of points $x \in \dot{F}_i$ with injectivity radius $\rho_i(x) \geq \varepsilon$ and $\mathrm{Thin}_\varepsilon(\dot{F}_i, g_i)$ as the closure of the set of points $x \in \dot{F}_i$ with injectivity radius $\rho_i(x) < \varepsilon$. If i is sufficiently large and ε is sufficiently small, then $\mathrm{Thin}_\varepsilon(\dot{F}_i, g_i)$ is a disjoint union of cusps and tubular neighborhoods of pinching geodesics. In this case, let υ be any pinching geodesic; we denote $\mathrm{Thin}_\varepsilon(\dot{F}_i, g_i, \upsilon)$ the connected component of $\mathrm{Thin}_\varepsilon(\dot{F}_i, g_i)$ containing υ.

Convergence on the thick part. Fix $\varepsilon > 0$. Then the gradient bound (Equation (7.3.3)) implies that the sequence $\overline{u}_i$ converges in the C^∞_{loc} topology on the ε-thick part, after possibly passing to a subsequence and translating in the s-direction. In fact an elliptic bootstrapping argument propagates the gradient bound to bounds on the derivatives of any order and the Arzelà-Ascoli theorem produces a convergent subsequence. Now we take a sequence $\varepsilon_i \to 0$ and use a diagonal argument to find a limit finite energy holomorphic map $\overline{u}'_\infty$ defined on F'_∞ (i.e., $\dot{F}_\infty$ minus the double points). Depending on the connected component of F'_∞, the restriction of $\overline{u}'_\infty$ can take values in $\overline{W}_-$, $\overline{W}$ or $\overline{W}'$. We observe that, although the maps $\overline{u}_i$ have boundary conditions in $L^-_{\widehat{\mathbf{a}}}$, the levels of the limit $\overline{u}'_\infty$ can have boundary conditions in $L^-_{\overline{\mathbf{a}}}$ or in $L_{\overline{\mathbf{a}}} \cup L_{\overline{h}(\overline{\mathbf{a}})}$, depending on the connected component of F'_∞.

We analyze the convergence of $\overline{u}'_\infty$ near the punctures in detail because the singularity in $L^-_{\overline{\mathbf{a}}}$ is an unusual situation. We describe only the case of components mapping to $\overline{W}_-$; the behavior of components mapping to $\overline{W}$ or $\overline{W}'$ is analogous. The first step is to replace $\overline{\mathbf{a}}$ with $\vec{\mathbf{a}}$ (this is temporary notation for the union of $\overline{a}_i$ which have been extended by adding the $\vec{a}_{i,j}$). This replacement has no practical effect on the curve $\overline{u}'_\infty$, but makes the geometric setting slightly less exotic: in fact the Lagrangian $L^-_{\vec{\mathbf{a}}}$ has a *clean self-intersection* along $\partial B_- \times \{z_\infty\}$. We recall that a clean intersection is an intersection between two submanifolds which are nontransverse in a uniform way. Clean intersections in Lagrangian Floer homology — which represent the Morse-Bott case in that theory — have been treated in [Po] and [Schm].

There are four possibilities, depending whether the puncture is in the interior or on the boundary of F'_∞ and whether $\overline{u}'_\infty$ is bounded or unbounded near the puncture. Interior punctures are treated in the standard way: if $\overline{u}'_\infty$ is bounded near an interior puncture, then the singularity is removable (i.e., $\overline{u}'_\infty$ can be extended as a holomorphic map across the puncture) by [MS, Theorem 4.1.2], and if $\overline{u}'_\infty$ is unbounded, then it is asymptotic to a closed orbit in the positive end of $\overline{W}_-$ by [HWZ1, Theorem 1.2]. (Note that this orbit can either be an orbit in N or a multiple cover of the extra orbit δ_0 over the point z_∞.)

Next we consider boundary punctures of $\overline{u}'_\infty$. We restrict attention to the neighborhood $Z_- = (-\infty, 0] \times [0, 1] \subset \dot{F}_\infty$ of a boundary puncture and prove the following lemma.

Lemma **7.3.13.** — *Let* $w : Z_- \to \overline{W}_-$ *be a finite energy* $\overline{J}_-$*-holomorphic map which maps* $(-\infty, 0] \times \{0, 1\}$ *to* $L^-_{\vec{\mathbf{a}}}$. *Then there exists either*

(1) *a point* $p \in L^-_{\vec{\mathbf{a}}}$ *such that* $\lim_{s \to -\infty} w(s, t) = p$ *(uniformly in t), or*

(2) *an intersection point* $y \in \overline{\mathbf{a}} \cap \overline{h}(\overline{\mathbf{a}})$ *such that* w *is asymptotic to the trivial strip over* y *for* $s \to -\infty$.

The removal of boundary singularities in the presence of clean intersections between Lagrangian submanifolds has been treated in [Frau, Section 3.7] and [Schm, Section 3]. For completeness we reprove it in a weak form (i.e., without reproving exponential convergence estimates) which suffices for our purposes, using ideas from [Ho].

Proof. — Consider the holomorphic map $f = \overline{\pi}_{B_-} \circ w : Z_- \to B_-$. Since f has finite energy, it either converges to a point $q \in \partial B_-$ (uniformly in t) as $s \to -\infty$ or for any $C \gg 0$ there exists $s_0 \gg 0$ such that $f|_{\{s \geq s_0\}}$ has image in the negative end $(-\infty, -C] \times [0, 1] \subset B_-$. Note that, in the latter case, the two components of $\partial Z_-|_{\{s \geq s_0\}}$ are mapped to the two boundary components of the end.

Fix a sequence $s_n \to +\infty$ and define $w_n : Z_- \to \overline{W}_-$ as $w_n(s,t) = w(s - s_n, t)$. The finiteness of the energy $E(w)$ implies the finiteness of the $\overline{\omega}$-area $E_{\overline{\omega}}(w)$. Therefore the sequence w_n satisfies

– $E(w_n) < E(w)$, and
– $\lim_{n \to \infty} E_{\overline{\omega}}(w_n) = 0$.

(1) If f converges to a point $q \in \partial B_-$, then the image of w is contained in a compact subset of $\overline{W}_-$. Then by Lemma 7.3.4 there is a constant C such that $\|\nabla w_n\|_{C^0} \leq C$ for all n and therefore we can extract a subsequence w_{k_n} which converges in C^∞_{loc} to a $\overline{J}_-$-holomorphic map $w_\infty : Z_- \to \overline{W}_-$. The $\overline{\Omega}_-$-area

$$E_{\overline{\Omega}_-}(w) = \int_{Z_-} w^* \overline{\Omega}_-$$

is finite because the energy of w is finite and the image of w is contained in a compact subset of $\overline{W}_-$. This implies that

$$E_{\overline{\Omega}_-}(w_\infty) = \lim_{n \to \infty} E_{\overline{\Omega}_-}(w_{k_n}) = 0$$

and thus w_∞ is constant. Hence, for any sequence $s_n \to +\infty$, there is a subsequence s_{k_n} and a point $p \in L^-_{\overline{\mathbf{a}}}$ such that

$$\lim_{n \to \infty} w(s_{k_n}, t) = p$$

and the limit is uniform in t. Moreover, $\nabla w(s_{k_n}, t)$ also converges to 0 uniformly in t.

A different sequence $s'_n \to +\infty$ might a priori have a subsequence s'_{l_n} such that $w(s_{l_n}, t)$ converges to a different $p' \in L^-_{\hat{\mathbf{a}}}$. We are then in Case (1) if we prove that $p = p'$. Up to extracting further subsequences we can assume that $s_{k_n} < s'_{l_n}$. We fix ε which is strictly smaller than the distance between p and p'. Then the Long Strip Lemma, Version

2 (Lemma 7.3.10) implies that, for n sufficiently large, $w|_{[s_{k_n},s'_{l_n}]\times[0,1]}$ is ε-close to a constant map, which is a contradiction.
(2) If $f=\overline{\pi}_{B_-}\circ w$ has image in the negative end of B_- for s sufficiently large, the sequence w_n converges, after a translation of the image, to a trivial strip in $\overline{W}$ over some intersection point $y\in\overline{\mathbf{a}}\cap\overline{h}(\overline{\mathbf{a}})$. The uniqueness of the limit is proved as in the previous case using the Long Strip Lemma, Version 1 (Lemma 7.3.6). □

Convergence on the thin part. The map $\overline{u}'_\infty$ defined above is not the SFT limit of the sequence $\overline{u}_i$ in general. In fact the $\overline{\omega}$-area of $\overline{u}'_\infty$ can be smaller than the $\overline{\omega}$-area of $\overline{u}_i$ and, when this happens, we have to look for the missing area in the thin part.

We analyze the behavior of the limit near a node of $\dot{F}_\infty$. Let υ be the pinching geodesics corresponding to a node n. We slightly change perspectives by identifying the neighborhoods $\mathrm{Thin}_\varepsilon(\dot{F}_i,g_i,\upsilon)$ with standard Euclidean cylinders $[-R_i^\varepsilon,R_i^\varepsilon]\times S^1$ or strips $[-R_i^\varepsilon,R_i^\varepsilon]\times[0,1]$.

Suppose that n is an interior node. For i large and ε small, $u_i(\{\pm R_i^\varepsilon\}\times S^1)$ is close to closed orbits $\gamma_\pm$ or points $p_\pm$ by the asymptotic properties of u'_∞. The first observation is that $u_i([-R_i^\varepsilon,R_i^\varepsilon]\times S^1)$ cannot be close to a closed orbit on one side and to a point on the other side because there is no homotopically trivial closed Hamiltonian orbit in N. Similarly, if n is a boundary node, then $u_i([-R_i^\varepsilon,R_i^\varepsilon]\times[0,1])$ cannot be close to a chord on one side and to a point on the other side because all chords define nontrivial classes in $H_1(\overline{W}_-,L^-_{\overline{\mathbf{a}}})$.

Lemma **7.3.14.** — *Let $n\in\dot{F}_\infty$ be a node and let $e_\pm:[0,+\infty)\times S^1\to F'_\infty$ or $e_\pm:[0,+\infty)\times[0,1]\to F'_\infty$ be the two ends corresponding to n. If*

$$\lim_{s\to+\infty}(\overline{u}'_\infty\circ e_\pm)(s,t)=p_\pm\in\overline{W}_-,$$

then $p_+=p_-$.

Proof. — Suppose that n is a boundary node; the proof for an interior node is completely analogous. Arguing by contradiction, suppose that $p_-\neq p_+$. We choose $\eta<\frac{1}{2}d(p_-,p_+)$, where $d(p_-,p_+)$ is the distance between p_- and p_+. Let $\eta'>0$ be the constant which depends on η and appears in the Long Strip Lemma, Version 2 and let $\overline{s}$ be a constant such that, for all $s>\overline{s}$, the distance from $p_\pm$ to $(\overline{u}'_\infty\circ e_\pm)(s,t)$ is less than $\eta'/2$. Let υ be the pinching geodesic that gives rise to n and let $\mathrm{Thin}_\varepsilon(F'_\infty,g_\infty,\upsilon)$ be the union of the two connected components of $\mathrm{Thin}_\varepsilon(F'_\infty,g_\infty)$ corresponding to the node n. Then there is a constant $\overline{\varepsilon}>0$ such that, for all $\varepsilon<\overline{\varepsilon}$, the portions of the two ends of F'_∞ corresponding to n which are parametrized by $[0,\overline{s}]\times[0,1]$ are contained in $\mathrm{Thick}_\varepsilon(F'_\infty,g_\infty,\upsilon)$. We then identify $\mathrm{Thin}_\varepsilon(\dot{F}_i,g_i,\upsilon)\cong[-R_i^\varepsilon,R_i^\varepsilon]\times[0,1]$ with its standard complex structure. Then the C^∞_{loc}-convergence of $\overline{u}_i$ to $\overline{u}'_\infty$ implies that, for $\varepsilon<\overline{\varepsilon}$ and $i\gg 0$, $\overline{u}_i(\{\pm R_i^\varepsilon\}\times[0,1])$ are contained in η'-neighborhoods of $p_\pm$. The Long Strip Lemma, Version 2 then implies that $p_+=p_-$, a contradiction. □

Lemma **7.3.15.** — *Let $n \in \dot{F}_\infty$ be a node arising from pinching a geodesic in the homotopy class υ. Assume that the two ends of F'_∞ corresponding to n are asymptotic to closed orbits γ_+ and γ_- or to chords y_+ and y_- over intersection points in $\overline{a}_i \cap \overline{h}(\overline{a}_j)$. Then, for all ε sufficiently small, the restrictions $\overline{u}_i|_{\mathrm{Thin}_\varepsilon(\dot{F}_i,g_i,\upsilon)}$ converge (in the SFT sense), as $i \to \infty$, to a stack of holomorphic cylinders or strips whose ends are asymptotic to closed orbits or chords, and whose successive components have matching ends, as in [BEHWZ, Figure 14].*

Proof. — We consider the case of boundary nodes first. For $\varepsilon > 0$ sufficiently small and i sufficiently large (depending on ε), the maps $\overline{u}_i|_{\mathrm{Thin}_\varepsilon(\dot{F}_i,g_i,\upsilon)}$ define a homology class $A \in H_2([0,1] \times \overline{S}, (\{1\} \times \overline{\mathbf{a}}) \cup (\{0\} \times \overline{h}(\overline{\mathbf{a}})) \cup y_+ \cup y_-)$. There are two cases:

(1) $E_{\overline{\omega}}(A) = 0$, or
(2) $E_{\overline{\omega}}(A) > 0$.

We will identify $\mathrm{Thin}_\varepsilon(\dot{F}_i, g_i, \upsilon) \cong [-R_i^\varepsilon, R_i^\varepsilon] \times [0,1]$.
Case (1). We verify that $\overline{u}_i|_{\mathrm{Thin}_\varepsilon(\dot{F}_i,g_i,\upsilon)}$ satisfies the conditions of the Long Strip Lemma, Version 1, for $i \gg 0$: Fix η arbitrarily small, and consider c_0, R_0, η' and δ as in the Long Strip Lemma, Version 1. Let $\varepsilon > 0$ be sufficiently small such that $\mathrm{Thin}_\varepsilon(F'_\infty, g_\infty, \upsilon)$ is mapped by $\overline{u}'_\infty$ to $\frac{\eta'}{2}$-neighborhoods of trivial strips over y_- and y_+. Then there exists $c > c_0$ such that for all $i \gg 0$,

– $R_i^\varepsilon - c > R_0$,
– $\overline{u}_i|_{[-R_i^\varepsilon, -R_i^\varepsilon + c] \times [0,1]}$ is η'-close to a portion of the trivial strip over y_-,
– $\overline{u}_i|_{[R_i^\varepsilon - c, R_i^\varepsilon] \times [0,1]}$ is η'-close to a portion of the trivial strip over y_+, and
– $E_{\overline{\omega}}(\overline{u}_i|_{[-R_i^\varepsilon, R_i^\varepsilon] \times [0,1]}) < \delta$.

This follows from observing that every compact subset $K \subset \mathrm{Thin}_\varepsilon(\dot{F}'_\infty, g_\infty, \upsilon)$ can be identified with some compact subset $K_i \subset \mathrm{Thin}_\varepsilon(\dot{F}'_i, g_i, \upsilon)$ provided $i \gg 0$, and the sequence $\overline{u}_i|_{K_i}$ converges uniformly to $\overline{u}'_\infty|_K$ with respect to all derivatives.

We then apply the Long Strip Lemma, Version 1, to $\overline{u}_i|_{[-R_i^\varepsilon, R_i^\varepsilon] \times [0,1]}$ and conclude that $y_+ = y_-$ and $\overline{u}_i|_{[-R_i^\varepsilon, R_i^\varepsilon] \times [0,1]}$ is C^1-close to a portion of the trivial strip over $y_+ = y_-$.
Case (2). Fix a constant $\hbar > 0$ such that, for every $A \in H_2([0,1] \times \overline{S}, (\{1\} \times \overline{\mathbf{a}}) \cup (\{0\} \times \overline{h}(\overline{\mathbf{a}})) \cup y' \cup y'')$ with y' and y'' of action less than $E = E(\overline{u}_i)$, we have $\int_A \overline{\omega} > \hbar$. Such a constant exists by [BEHWZ, Lemma 10.9].

Fix $\varepsilon > 0$ sufficiently small. Since $\overline{u}_i([-R_i^\varepsilon, R_i^\varepsilon] \times [0,1])$ is contained in the strip end of $\overline{W}_-$, we may assume that the maps $\overline{u}_i$ take values in $\overline{W}$. Using the argument from the proof of the Long Strip Lemma, Version 1, if the image of $\overline{u}_i|_{[-R_i^\varepsilon, R_i^\varepsilon] \times [0,1]}$ is contained in an η-neighborhood of the strip over y_-, then $\overline{u}_i|_{[-R_i^\varepsilon, R_i^\varepsilon] \times [0,1]}$ is η-close in the C^0-metric to a trivial strip, which contradicts (2). Hence there is a sequence of points $p_i \in [-R_i^\varepsilon, R_i^\varepsilon] \times [0,1]$ such that (after passage to a subsequence) $\overline{u}_i(p_i)$ limits to a point q_∞ which is a distance $> \eta$ from the strips over $y_\pm$, after suitable translations of $\overline{u}_i$ in the target.

Next consider the domain translations $\overline{u}_i^1(s,t)=\overline{u}_i(s-s(p_i),t)$. After passage to a subsequence the maps $\overline{u}_i^1$ converge in C^∞_{loc} to a $\overline{J}'$-holomorphic map $\overline{u}_\infty^1:\mathbf{R}\times[0,1]\to\overline{W}$, after translations in the target. By construction, $E_{\overline{\omega}}(\overline{u}_\infty^1)>0$. We repeat this process on $\overline{u}_i$ defined on $[-R_i^\varepsilon,R_i^\varepsilon]\times[0,1]$ minus a suitable annulus $[K_i',K_i'']\times[0,1]$ until we obtain $\overline{J}'$-holomorphic maps $\overline{u}_\infty^1,\dots,\overline{u}_\infty^l$ such that $E_{\overline{\omega}}(\overline{u}_\infty^1)+\dots+E_{\overline{\omega}}(\overline{u}_\infty^l)=E_{\overline{\omega}}(A)$. An argument similar to that of Case (1) and based on the Long Strip Lemma, Version 1, shows that the ends of $\overline{u}'_\infty,\overline{u}_\infty^1,\dots,\overline{u}_\infty^l$ match up with those of their adjacent curves. (Here we may need to reorder the $\overline{u}_\infty^j$ so they are adjacent.)

The case of interior nodes is completely analogous: the only difference is that we have to replace the Long Strip Lemma by the Long Cylinder Lemma. □

This completes the proof of Proposition 7.3.2. □

Remark **7.3.16.** — Proposition 7.3.2 is only a preliminary result. We are still left with the task of analyzing the limit more precisely.

7.4. *Intersection numbers.* — In order to analyze the SFT limit $\overline{u}_\infty$, we use the intersection numbers $n^*(\overline{u}_i)$ and $n^*(\overline{v}_j)$ to constrain the behavior of holomorphic maps which are asymptotic to δ_0 or to $[0,1]\times\{z_\infty\}$.

We briefly recall the definition of the intersection numbers $n^*(\overline{u})$ given in Section 5. Let $\rho_0>0$ be sufficiently small. Consider the torus $T_{\rho_0}=\{\rho=\rho_0\}\subset\overline{N}$, oriented as the boundary of $\{\rho\le\rho_0\}$. We take an oriented identification $T_{\rho_0}\simeq\mathbf{R}^2/\mathbf{Z}^2$ such that the meridian has slope 0 and the closed orbits of $\overline{R}_0$ on T_{ρ_0} have slope m. We pick a closed orbit $\delta_0^\dagger\subset T_{\rho_0}$ and consider the parallel sections $(\sigma_\infty^*)^\dagger$ determined by $\delta_0^\dagger$. We assume we have chosen $\delta_0^\dagger$ so that $(\sigma_\infty^*)^\dagger$ is disjoint from the relevant Lagrangian submanifold. Then $n^*(\overline{u})=\langle\overline{u},(\sigma_\infty^*)^\dagger\rangle$.

Since $n^-(\overline{u}_i)=m\gg 2g$ for an $\overline{W}_-$-curve $\overline{u}_i$ by Lemma 5.4.14, we have

$$\sum_{j=-b}^{a} n^*(\overline{v}_j)=m. \tag{7.4.1}$$

For each level $\overline{v}_j$ of $\overline{u}_\infty$ we have a decomposition

$$\overline{v}_j=\overline{v}_j'\cup\overline{v}_j''$$

as in Equation (5.7.1), where $\overline{v}_j'$ is the union of the irreducible components of $\overline{v}_j$ which branch cover the section at infinity σ_∞^* and $\overline{v}_j''$ is the union of all other irreducible components.

Lemma **7.4.1** *(Intersection with $\overline{v}_j$, $j>0$). — Suppose $j>0$. Then the following hold for $\rho_0>0$ sufficiently small:*

(1) *If* $\overline{v}''_j$ *has a positive end* $\mathcal{E}_+$ *which converges to* δ_0^p*, then*

$$\langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle \geq p. \tag{7.4.2}$$

(2) *If* $\overline{v}''_j$ *has a negative end* $\mathcal{E}_-$ *which converges to* δ_0^p*, then*

$$\langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle \geq m - p. \tag{7.4.3}$$

If $\overline{v}''_j$ *has multiple ends at covers of* δ_0*, then their contributions to* $n'(\overline{v}_j)$ *are summed.*

Proof. — Let $\mathcal{E}_+$ be a positive end which converges to δ_0^p. Let $\pi_{\overline{N}} : \mathbf{R} \times \overline{N} \to \overline{N}$ be the projection onto the second factor. Provided ρ_0 is sufficiently small, $\pi_{\overline{N}}(\mathcal{E}_+) \cap T_{\rho_0}$ determines a homology class $(q, p) \in H_1(T_{\rho_0}) \simeq \mathbf{R}^2/\mathbf{Z}^2$. One can easily check that

$$\langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle = \det \begin{pmatrix} 1 & q \\ m & p \end{pmatrix} = p - qm.$$

Since $\langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle > 0$ by the positivity of intersections in dimension four and $m \gg p$, we must have $q \leq 0$. We then obtain:

$$\langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle \geq p.$$

Let $\mathcal{E}_-$ be a negative end which converges to δ_0^p. As above, $\pi_{\overline{N}}(\mathcal{E}_-) \cap T_{\rho_0}$ determines a homology class $(q, -p) \in H_1(T_{\rho_0}) \simeq \mathbf{R}^2/\mathbf{Z}^2$ such that

$$\langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle = det \begin{pmatrix} 1 & q \\ m & -p \end{pmatrix} = -p - qm.$$

Since $\langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle > 0$ by the positivity of intersections in dimension four and $m \gg p$, we must have $q \leq -1$. We then obtain:

$$\langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle \geq m - p.$$

Finally, if $\overline{v}''_j$ has multiple ends at covers of δ_0, then the total intersection of $\overline{v}''_j$ with $(\sigma'_\infty)^\dagger$ is bounded below by the sum of the contributions of each end. □

Next we consider $\overline{v}_j$ when $j < 0$. Let $\pi_{\overline{S}} : \mathbf{R} \times [0, 1] \times \overline{S} \to \overline{S}$ be the projection along the Hamiltonian vector field ∂_t. Also let $\mathcal{R}_{\phi_0}$ be the radial ray $\{\phi = \phi_0\} \subset D^2_\varepsilon = \{\rho \leq \varepsilon\}$, where $\varepsilon > 0$ is small. Under the projection $\pi_{\overline{S}}$, each positive end $\mathcal{E}_+$ of $\overline{v}''_j$ which limits to $[0, 1] \times \{z_\infty\}$ maps to a sector $\mathfrak{S}(\mathcal{E}_+)$ of $D^2_\varepsilon = \{\rho \leq \varepsilon\}$ going in the counterclockwise direction from $\mathcal{R}_{\phi_1} \subset \overline{a}_{i_1}$ to $\mathcal{R}_{\phi_2} \subset \overline{h}(\overline{a}_{i_2})$. The sector $\mathfrak{S}(\mathcal{E}_+)$ is a *thin wedge* if $i_1 = i_2$ and the angle is $\frac{2\pi}{m}$. Similarly, each negative end $\mathcal{E}_-$ of $\overline{v}''_j$ which limits to $[0, 1] \times \{z_\infty\}$ maps to a sector $\mathfrak{S}(\mathcal{E}_-)$ going counterclockwise from $\mathcal{R}_{\phi_1} \subset \overline{h}(\overline{a}_{i_1})$ to $\mathcal{R}_{\phi_2} \subset \overline{a}_{i_2}$.

Lemma **7.4.2** *(Intersection with $\overline{v}_j$, $j<0$). — Suppose $j<0$. Then the following hold for $\rho_0>0$ sufficiently small:*

(1) *If $\overline{v}''_j$ has a positive end $\mathcal{E}_+$ which converges to $[0,1]\times\{z_\infty\}$, then*

$$\langle \mathcal{E}_+, \sigma^\dagger_\infty\rangle \geq 1, \tag{7.4.4}$$

and the relevant sector is a thin wedge if and only if equality holds. Moreover, if the sector is not a thin wedge, then

$$\langle \mathcal{E}_+, \sigma^\dagger_\infty\rangle > 2g.$$

(2) *If $\overline{v}''_j$ has a negative end $\mathcal{E}_-$ which converges to $[0,1]\times\{z_\infty\}$, then*

$$\langle \mathcal{E}_-, \sigma^\dagger_\infty\rangle > 2g. \tag{7.4.5}$$

Proof. — Suppose $\rho_0<\varepsilon$. The restriction of δ_{ρ_0} to $[0,1]\times\overline{S}$ consists of m Reeb arcs $[0,1]\times\{\phi_l\}$ on $\{\rho=\rho_0\}$, where $0\leq l<m$ and $\phi_l=\phi_0+l(2\pi/m)$. Let $\mathcal{E}_+$ (resp. $\mathcal{E}_-$) be a positive (resp. negative) end of $\overline{v}''_j$ which converges to $[0,1]\times\{z_\infty\}$ and let $\mathfrak{S}(\mathcal{E}_\pm)$ be the corresponding sector in D^2_ε.

By the assumptions on the $\overline{a}_i$ given in Section 5.2.2, the angles of the thin wedges are $\frac{2\pi}{m}$ and the other sectors of $D^2_\varepsilon - \cup_i\overline{a}_i - \cup_i\overline{h}(\overline{a}_i)$ have angles greater than $\frac{2\pi(2g)}{m}$. This implies that:

(i) $\langle \mathcal{E}_+, \sigma^\dagger_\infty\rangle \geq 1$;
(ii) $\langle \mathcal{E}_+, \sigma^\dagger_\infty\rangle = 1$ if and only if $\mathfrak{S}(\mathcal{E}_+)$ is a thin wedge; and
(iii) $\langle \mathcal{E}_-, \sigma^\dagger_\infty\rangle > 2g$.

The lemma follows. □

Similarly, we have the following lemma, whose proof is the same as those of Lemmas 7.4.1 and 7.4.2 and will be omitted.

Lemma **7.4.3** *(Intersection with $\overline{v}_0$). — The following hold for $\rho_0>0$ sufficiently small:*

(1) *If $\overline{v}''_0$ has a positive end $\mathcal{E}_+$ which converges to δ^p, then*

$$\langle \mathcal{E}_+, (\sigma^-_\infty)^\dagger\rangle \geq p. \tag{7.4.6}$$

(2) *If $\overline{v}''_0$ has a negative end $\mathcal{E}_-$ which converges to $[0,1]\times\{z_\infty\}$, then*

$$\langle \mathcal{E}_-, (\sigma^-_\infty)^\dagger\rangle > 2g. \tag{7.4.7}$$

Definition **7.4.4** *(Boundary point at z_∞). — Consider $\overline{v}''_j:\dot{F}''_j\to\overline{W}_j$ for $j\leq 0$. A point $p\in\partial\dot{F}''_j$ will be called a* boundary point at z_∞ *if $\overline{v}''_j(p)\in L^-_{\overline{\mathbf{a}}}-L^-_{\hat{\mathbf{a}}}$ for $j=0$ and $\overline{v}''_j(p)\in (L_{\overline{h}(\overline{\mathbf{a}})}-L_{\overline{h}(\hat{\mathbf{a}})})\cup(L_{\overline{\mathbf{a}}}-L_{\hat{\mathbf{a}}})=\mathbf{R}\times\{0,1\}\times\{z_\infty\}$ for $j<0$.*

Lemma **7.4.5.** — *Consider* $\overline{v}''_j : \dot{F}''_j \to \overline{W}_j$, $j \le 0$. *If* $p \in \partial \dot{F}''_j$ *is a boundary point at* z_∞*, then, for any sufficiently small neighborhood* $\nu(p) \subset F''_j$ *of* p*, there exists* $\rho_0 > 0$ *such that* $\langle \overline{v}''_j(\nu(p)), (\sigma^*_\infty)^\dagger \rangle \ge k_0 - 1 > 2g$.

Here the constant k_0 is as given in Section 5.2.2.

Proof. — We will prove the case $j < 0$, leaving $j = 0$ for the reader. Let p be a boundary point at z_∞ of $\overline{v}''_j$ which maps to a point on $\mathbf{R} \times \{1\} \times \{z_\infty\}$ (without loss of generality). We consider the projection $\pi_{\overline{S}} : \mathbf{R} \times [0,1] \times \overline{S} \to \overline{S}$. By Definition 5.3.2, the projection $\pi_{\overline{S}}$ is holomorphic when restricted to $\pi_{\overline{S}}^{-1}(D^2_\varepsilon)$, where $\varepsilon > 0$ is sufficiently small. Hence $\pi_{\overline{S}} \circ \overline{v}''_j$ is holomorphic when restricted to $(\pi_{\overline{S}} \circ \overline{v}''_j)^{-1}(D^2_\varepsilon)$ and some sector of $D^2_\varepsilon - \overline{\mathbf{a}}$ must be contained in $\mathrm{Im}(\pi_{\overline{S}} \circ \overline{v}''_j)$ because holomorphic maps are open. This in turn shows that

$$\langle \overline{v}''_j(\nu(p)), (\sigma^*_\infty)^\dagger \rangle \ge k_0 - 1 > 2g$$

by assumption, provided ρ_0 is sufficiently small. □

7.5. *Some restrictions on* $\overline{u}_\infty$. — We now present some lemmas in preparation for Theorems 7.6.1 and 7.6.3. Lemmas 7.5.1–7.5.3 give restrictions on $\overline{u}_\infty$, which arise from intersection number calculations from Section 7.4, and Lemma 7.5.5 gives a lower bound on the ECH index of the levels $\overline{v}_j$.

In what follows, "component" is shorthand for "irreducible component". We first discuss the "fiber components", i.e., components $\widetilde{v} : F \to \overline{W}_j$ of $\overline{v}_j$ which map to fibers of $\overline{W}_j$. There are three types of fiber components:

(i) ghosts, i.e., $\widetilde{v}$ is constant;
(ii) "closed fiber components", i.e., F is closed and $\widetilde{v}$ is nonconstant;
(iii) "boundary fiber components", i.e., $\partial F \neq \varnothing$ and $\widetilde{v}$ is nonconstant.

If $\widetilde{v}$ is a closed fiber component, then $\widetilde{v}$ is a branched cover of a fiber of $\overline{W}_j$. Next let $\widetilde{v} : F \to \overline{W}_j$ be a boundary fiber component. If $j < 0$ and $\widetilde{v}$ maps to $\overline{\pi}_B^{-1}(p)$, $p \in \partial B$, then $\widetilde{v}(F) \supset \overline{\pi}_B^{-1}(p) - A$, where $A = L_{\overline{\mathbf{a}}}$ or $L_{\overline{h}(\mathbf{a})}$. Similarly, if $j = 0$ and $\widetilde{v}$ maps to $\overline{\pi}_{B_-}^{-1}(p)$, $p \in \partial B_-$, then $\widetilde{v}(F) \supset \overline{\pi}_{B_-}^{-1}(p) - L^-_{\overline{\mathbf{a}}}$.

Lemma **7.5.1.** — *The only possible non-ghost fiber component of* $\overline{u}_\infty$ *is a closed component* $\overline{\pi}_{B_-}^{-1}(p)$ *which passes through* $\overline{\mathfrak{m}}$. *There is at most one such component.*

Proof. — Let $\widetilde{v}$ be a non-ghost fiber component. Then $\widetilde{v}$ is a d-fold branched cover of a fiber if it is a closed fiber component and a d-fold branched cover of a fiber cut along $\overline{\mathbf{a}}$ (or $\overline{h}(\overline{\mathbf{a}})$) if it is a boundary fiber component. In either case, $n^*(\widetilde{v}) = d \cdot m$. This implies that $d = 1$.

We now prove that $\widetilde{v}$ is a closed fiber component passing through $\overline{\mathfrak{m}}$. Arguing by contradiction, if $\overline{\mathfrak{m}} \notin \mathrm{Im}(\widetilde{v})$, then there is a component $\hat{v}$ of $\overline{u}_\infty$ such that $\overline{\mathfrak{m}} \in \mathrm{Im}(\hat{v})$. If $\hat{v}$ is a cover of σ_∞^-, then there must be some component of $\overline{v}_j''$ for $j > 0$, which has a negative end at some δ_0^p. Therefore $n^-(\overline{v}_j'') > 0$ by Lemma 7.4.1(2). If $\hat{v}$ is not a cover of σ_∞^-, then $\hat{v}$ has a nonzero intersection with σ_∞^- and hence $n^-(\hat{v}) > 0$. In either case we have $n^*(\overline{u}_\infty) > m$, which is a contradiction. This proves that $\overline{\mathfrak{m}} \in \mathrm{Im}(\widetilde{v})$.

Finally boundary fiber components are eliminated because they project to a point in $\partial \mathrm{B}_-$, and therefore cannot pass through $\overline{\mathfrak{m}}$. □

We recall the notation $\overline{v}_j = \overline{v}_j' \cup \overline{v}_j''$, where $\overline{v}_j'$ denotes the union of all components of $\overline{v}_j$ which cover the section at infinity σ_∞^* and $\overline{v}_j''$ denotes the union of all other components of $\overline{v}_j$. The covering degree of $\overline{v}_j'$ will always be denoted by p_j. We also define $\overline{v}_j^\sharp$ to be the union of the components of $\overline{v}_j''$ which are asymptotic to a multiple of δ_0 or z_∞ at either end and are not trivial cylinders or strips, $\overline{v}_j^\flat$ to be the union of the remaining non-fiber components of $\overline{v}_j''$, and $\overline{v}_j^f$ to be the union of the fiber components of $\overline{v}_j''$.

Lemma **7.5.2.** — *If $\overline{v}_0' = \varnothing$, then, with the exception of ghost components:*

(1) *$\overline{v}_j' = \varnothing$ and $\overline{v}_j^\sharp = \varnothing$ for all j;*
(2) *no $\overline{v}_j''$, $j \le 0$, has a boundary point at z_∞;*
(3) *every level $\overline{v}_j$, $j \neq 0$, has image inside W' or W; and*
(4) *$\overline{v}_0$ is a $\overline{\mathrm{W}}_-$-curve or a* degenerate $\overline{\mathrm{W}}_-$-curve, *i.e., $\overline{v}_0$ is the union of a W_--curve and a fiber $\pi_{\mathrm{B}_-}^{-1}(p)$ which passes through $\overline{\mathfrak{m}}$.*

Proof. — If $\overline{v}_0' = \varnothing$, then $\overline{v}_0 = \overline{v}_0''$. Since $\overline{u}_i$ passes through $\overline{\mathfrak{m}}$ for all i, the level $\overline{v}_0 : \dot{\mathrm{F}}_0 \to \overline{\mathrm{W}}_-$ must also pass through $\overline{\mathfrak{m}}$. By Lemma 5.4.14 and the proof of Lemma 5.4.8, $n^-(\overline{v}_0) \ge m$ because $\overline{\mathfrak{m}} \in \sigma_\infty^-$. Equation (7.4.1) and the nonnegativity of n^* then imply that $n^-(\overline{v}_0) = m$ and $n^*(\overline{v}_j) = 0$ for $j \neq 0$.

(1) If $\overline{v}_j' \neq \varnothing$ or $\overline{v}_j^\sharp \neq \varnothing$, then at least one of Equations (7.4.3)–(7.4.6) applies and $\sum_{j\neq 0} n^*(\overline{v}_j^\sharp) > 0$.

(2) This follows from $n^-(\overline{v}_0) = m$ and Lemma 7.4.5.

(3) Since $n^*(\overline{v}_j) = 0$ for $j \neq 0$, (1), combined with Lemma 5.3.20, implies that $\mathrm{Im}(\overline{v}_j) \subset \mathrm{W}'$ if $j > 0$, whereas (1) and (2), combined with Lemma 5.3.9, implies that $\mathrm{Im}(\overline{v}_j) \subset \mathrm{W}$ if $j < 0$.

(4) Since $n^-(\overline{v}_0) = m$, $\overline{v}_0$ intersects σ_∞^- only at $\overline{\mathfrak{m}}$ and the intersection is transverse by Lemma 5.4.14(2). By (1) and (2) and Lemmas 7.5.1 and 5.4.16, if $\overline{v}_0$ is not a $\overline{\mathrm{W}}_-$-curve, then it must be a degenerate $\overline{\mathrm{W}}_-$-curve.

This completes the proof of the lemma. □

Lemma **7.5.3.** — *If $\overline{v}_0' \neq \varnothing$, then, with the exception of ghost components:*

(1) *there is only one negative end of* $\overline{v}_j^\sharp, j>0$, *(say* $\overline{v}_{a'}^\sharp$*) which is asymptotic to a multiple of* δ_0*;*
(2) $\overline{u}_\infty$ *has no fiber components;*
(3) *no* $\overline{v}_j'', j \le 0$, *has a boundary point at* z_∞*;*
(4) $\overline{v}_j^\flat$ *has image inside* W, W_-, *or* W′*;*
(5) $\overline{v}_j^\sharp, j<0$, *is a union of thin strips from* z_∞ *to some* x_i *or* x_i'*;*
(6) $\overline{v}_0^\sharp$ *has image inside* $\overline{W_-} - int(W_-)$ *and has* $\delta_0^{r_0}$, *for some* $r_0>0$, *at the positive end and some subset of* $\{x_1,\dots,x_{2g},x_1',\dots,x_{2g}'\}$ *of cardinality* r_0 *at the negative end;*
(7) $\overline{v}_j^\sharp$, $0<j\le a'$, *has image inside* $\overline{W'} - int(W')$ *and has* $\delta_0^{r_j}$, *for some* $r_j>0$, *at the positive end and* $h^{r_j'}e^{r_j''}$ *with* $r_j'+r_j''=r_j$, *at the negative end.*

Proof. — (1) By Lemma 7.4.1, each negative end of $\overline{v}_j^\sharp$, for $j>0$, which is asymptotic to a multiple of δ_0 contributes at least $m-2g$ to $\sum_{j=1}^a n^*(\overline{v}_j)$, where $m \gg 2g$, because the total multiplicity of δ_0 is $\le 2g$. If there are at least two such negative ends of $\overline{v}_j^\sharp$, then the total contribution to $\sum_{j=1}^a n^*(\overline{v}_j)$ is at least $2(m-2g)>m$, which is a contradiction. Suppose $\overline{v}_{a'}^\sharp$, $0<a'\le a$, has a negative end at a multiple at δ_0.

(2) By Lemma 7.5.1, a fiber component of $\overline{u}_\infty$ is a fiber $\overline{\pi}_{B_-}^{-1}(p)$ which passes through $\overline{\mathfrak{m}}$ and additionally contributes m to $\sum_{j=-b}^a n^*(\overline{v}_j)$, a contradiction.

(3) By Lemma 7.4.5, a boundary point at z_∞ additionally contributes $\ge 2g$ to $\sum_{j=-b}^a n^*(\overline{v}_j)$. This is again a contradiction.

(4) By (3), $\overline{v}_j^\flat$ does not have any boundary point at z_∞. Since the ends of $\overline{v}_j^\flat$ are contained in N or $[0,1]\times S$ and $n^*(\overline{v}_j^\flat)=0$, we conclude that $\mathrm{Im}(\overline{v}_j^\flat)\subset W, W_-$, or W by Lemmas 5.3.9, 5.4.16 and 5.3.20.

(5), (6) Let $p_j=\deg(\overline{v}_j')$ be the covering degree of $\overline{v}_j'$ over σ_∞'. Then $p_0\le p_1\le\dots\le p_{a'-1}$ and $p_{a'}=0$. This follows from (1) since a negative end of $\overline{v}_{j+1}^\sharp$ at (a multiple of) δ_0 is required for $p_j>p_{j+1}$.

Let $p_{a'}^-$ be the multiplicity of δ_0 at the negative end of $\overline{v}_{a'}^\sharp$. The negative end of $\overline{v}_{a'}^\sharp$ contributes at least $m-p_{a'}^-$ to n^* by Equation (7.4.3) and the positive ends of $\overline{v}_j^\sharp$, for $j=0,\dots,a'-1$, give a total contribution of at least $p_{a'}^- - p_0$ to n^* by Equations (7.4.2) and (7.4.6). Hence,

$$\textbf{(7.5.1)}\qquad \sum_{j=0}^{a} n^*(\overline{v}_j^\sharp)\ge m-p_0.$$

On the other hand, by Equation (7.4.4), the contributions of the positive ends of $\overline{v}_j^\sharp$, $j=-b,\dots,-1$, add up to

$$\textbf{(7.5.2)}\qquad \sum_{j=-b}^{-1} n^*(\overline{v}_j^\sharp)\ge p_0.$$

Equations (7.5.1) and (7.5.2) give:

$$\textbf{(7.5.3)} \qquad \sum_{j=-b}^{a} n^*(\overline{v}_j^\sharp) \geq m.$$

Equality holds by Equation (7.4.1). This in turn implies that:

(i) equality holds in both Equations (7.5.1) and (7.5.2); and
(ii) $\overline{v}_j^\sharp, j \leq 0$, has no negative end which limits to z_∞.

Since $p_0 \leq 2g$, each $\overline{v}_j^\sharp, j < 0$, must be a union of thin strips by Lemma 7.4.2. This gives (5). Moreover, $\overline{v}_0^\sharp$ has $\delta_0^{p_1-p_0}$ at the positive end and no negative ends at z_∞ by (ii) and

$$\textbf{(7.5.4)} \qquad n^-(\overline{v}_0^\sharp) = p_1 - p_0$$

by Lemma 7.4.3 and (i).

The argument to prove (6) is similar to the proof of the blocking lemma in [0], and the presence of the Lagrangian boundary condition for $\overline{v}_0^\sharp$ does not change the proof in any essential way. We consider $C_\rho = \pi_{\overline{N}}(\overline{v}_0^\sharp) \cap T_\rho$, where $0 < \rho < 1$, as an element of $H_1(T_\rho, \pi_{\overline{N}}(L_{\mathbf{a}}^-) \cap T_\rho)$; we are viewing C_ρ as the boundary of the portion of $\pi_{\overline{N}}(\overline{v}_0^\sharp)$ outside the solid torus of radius ρ. We recall that $\pi_{\overline{N}}(L_{\mathbf{a}}^-) \cap T_\rho$ consists of $2g$ parallel segments in T_ρ which are tangent to the Hamiltonian flow. Suppose $\rho_1 > 0$ is small. Let $\cup_i \mathcal{E}_i$ be the union of positive ends of $\overline{v}_0^\sharp$ that limit to multiples of δ_0, and let $C'_{\rho_1} = \pi_{\overline{N}}(\cup_i \mathcal{E}_i) \cap T_{\rho_1}$. We view C'_{ρ_1} as an element of

$$H_1(T_{\rho_1}) \hookrightarrow H_1(T_{\rho_1}, \pi_{\overline{N}}(L_{\mathbf{a}}^-) \cap T_{\rho_1}).$$

Using "balanced coordinates" on $V = \overline{N} - int(N)$ (cf. Section 5.1.2), we obtain an identification $H_1(T_{\rho_1}) \cong \mathbf{Z}^2$ such that — writing vectors as rows — $(1,0)$ corresponds to the homology class of the meridian (i.e., the closed curve which bounds a disk in V) and $(1,m)$ corresponds to the class of a closed Hamiltonian orbit.

With respect to this identification, $C'_{\rho_1} = (q, p_1 - p_0)$ for some $q \leq 0$ and we have

$$p_1 - p_0 - qm = \det \begin{pmatrix} 1 & q \\ m & p_1 - p_0 \end{pmatrix} \geq p_1 - p_0 \geq 0.$$

By Equation (7.5.4), we obtain $q = 0$ and $C_{\rho_1} = C'_{\rho_1}$, since any intersections besides those coming from $\cup_i \mathcal{E}_i$ would give some extra contributions to $n^-(\overline{v}_0^\sharp)$.

Next consider C_{ρ_2}, where $\rho_2 = 1 - \varepsilon$, $\varepsilon > 0$ small. Since there are no ends of $\overline{v}_0^\sharp$ between T_{ρ_1} and T_{ρ_2}, it follows that C_{ρ_2} is the image of $(0, p_1 - p_0) \in H_1(T_{\rho_2})$ in $H_1(T_{\rho_2}, \pi_{\overline{N}}(L_{\mathbf{a}}^-) \cap T_{\rho_2})$. Negative ends can approach x_i or x'_i either from the interior of V or from the exterior of V. A negative end of $\overline{v}_0^\sharp$ approaching x_i or x'_i from the interior traces a segment in T_{ρ_2} with both endpoints in the same connected component of $\pi_{\overline{N}}(L_{\mathbf{a}}^-) \cap T_{\rho_2}$

and whose relative homology class is the image of the class $(0,1) \in H_1(T_{\rho_2})$. This means that at most $p_1 - p_0$ ends of $\overline{v}_0^\sharp$ approach x_i or x_i' from the interior.

Finally let $\rho_3 = 1+\varepsilon$, $\varepsilon > 0$ small. The ends of $\overline{v}_0^\sharp$ between T_{ρ_2} and T_{ρ_3} are negative ends at x_i, x_i' or positive ends at closed orbits on T_1. Let $C'_{\rho_3} = \pi_{\overline{N}}(\widetilde{v}) \cap T_{\rho_3}$, where $\widetilde{v}$ is obtained from $\overline{v}_0^\sharp$ by truncating the negative ends that limit to x_i or x_i' from the exterior of V. There are no positive ends of $\overline{v}_0^\sharp$ that limit to a closed orbit on T_1; this follows from asymptotic winding considerations of the positive ends and a comparison with C_{ρ_2}. Since there are no negative ends of $\overline{v}_0^\sharp$ that limit to a closed orbit on T_1, it follows that $[C'_{\rho_3}] = 0 \in H_1(T_{\rho_3}, \pi_{\overline{N}}(L^-_{\overline{\mathbf{a}}}) \cap T_{\rho_3})$.

Finally, we eliminate the possibility of ends which limit to x_i or x_i' from the outside of V by observing that nothing else can contribute to C_{ρ_2} but the ends at x_i or x_i'. This implies that there are $p_1 - p_0$ of them, each representing the image of the class $(0,1) \in H_1(T_{\rho_2})$, and leaving no room for ends limiting to x_i or x_i' from the outside. Then C_{ρ_3} is homologically trivial, and therefore the positivity of intersections implies that it is empty. Since ρ_3 was arbitrarily close to 1, it follows that the image of $\overline{v}_0^\sharp$ cannot escape $\overline{W}_- - int(W_-)$, which implies (6).

(7) This is similar to (6) and is left to the reader. □

Remark **7.5.4.** — One can easily compute that $I(\overline{v}_0^\sharp) = r_0$ and $I(\overline{v}_j^\sharp) = r_j' + 2r_j''$ when $j > 0$.

Lemma **7.5.5.** — *The ECH index of each level $\overline{v}_j$, $j = -b, \dots, a$, is nonnegative if $\overline{J}_- \in \mathcal{J}^{reg}_{\overline{W}_-}$. Moreover, if $j \neq 0$, the ECH index of $\overline{v}_j$ is strictly positive.*

Proof. — Recall that the restrictions $\overline{J}$ and $\overline{J}'$ are also regular by definition. *Case $j > 0$.* By [HT1, Proposition 7.15], $I_{ECH}(\overline{v}_j) \geq 1$ for $j > 0$ since $\overline{J}'$ is regular and no level $\overline{v}_j$ with $j > 0$ consists uniquely of trivial cylinders.

Case $j < 0$. We write $\overline{v}_j = \overline{v}_j' \cup \overline{v}_j^\sharp \cup \overline{v}_j^\flat$.

Suppose that $\overline{v}_0' = \varnothing$. Then $\overline{v}_j' = \varnothing$ and $\overline{v}_j^\sharp = \varnothing$ for all $j < 0$ by Lemma 7.5.2, and we are left with $\overline{v}_j^\flat$, which is simply-covered and has positive Fredholm index by regularity and translation invariance. By the index inequality (Theorem 4.5.13), $I(\overline{v}_j) = I(\overline{v}_j^\flat) \geq 1$.

Next suppose that $\overline{v}_0' \neq \varnothing$. Then $\overline{v}_j^\sharp$ is a union of thin strips from z_∞ to some x_i or x_i' by Lemma 7.5.3 and each thin strip has ECH index 1. We also have $I(\overline{v}_j') = 0$ by Lemma 5.7.15 and $I(\overline{v}_j^\flat) \geq 0$ by the previous paragraph.

We claim that

$$I(\overline{v}_j) = I(\overline{v}_j' \cup \overline{v}_j^\sharp \cup \overline{v}_j^\flat) = I(\overline{v}_j') + I(\overline{v}_j^\sharp) + I(\overline{v}_j^\flat). \qquad (\mathbf{7.5.5})$$

Note that, although $\overline{v}_j'$, $\overline{v}_j^\sharp$ and $\overline{v}_j^\flat$ are disjoint, $\overline{v}_j'$ and $\overline{v}_j^\sharp$ are both asymptotic to (a multiple of) z_∞ at the positive end and the additivity of the ECH indices of $\overline{v}_j'$ and $\overline{v}_j^\sharp$ needs to be verified. For that purpose, recall from Section 5.7 that each $\overline{v}'_j$ comes equipped with data $\mathcal{D}'_j = ((\mathcal{D}')^{to}_j, (\mathcal{D}')^{from}_j)$ at the positive and negative ends, since $\overline{u}_\infty$ is the limit of the sequence $\{\overline{u}_i\}$. The key observation here is that $(\mathcal{D}')^{to}_j = (\mathcal{D}')^{from}_j$, since all the components of $\overline{v}^\sharp_{j'}$, $j' < j$, are thin strips whose data $(\mathcal{D}^\sharp_+)_{j'} = ((\mathcal{D}^\sharp_+)^{to}_{j'}, (\mathcal{D}^\sharp_+)^{from}_{j'})$ at the positive end satisfies $(\mathcal{D}^\sharp_+)^{to}_{j'} = (\mathcal{D}^\sharp_+)^{from}_{j'}$. Hence we can choose a simultaneous grooming $\mathfrak{c}^+ = \{c^+_k\}$ for both $\overline{v}'_j$ and $\overline{v}^\sharp_j$ at the positive end z_∞ such that c^+_k has winding number $w(c^+_k) = 0$ (see Equation (5.7.4)) and connects $\overline{h}(\overline{a}_{i_k j_k})$ to $\overline{a}_{i_k j_k}$. If we choose a groomed multivalued trivialization τ compatible with $\mathfrak{c}^+$, then

$$\mathrm{I}_\tau(\overline{v}'_j \cup \overline{v}^\sharp_j) = \mathrm{I}_\tau(\overline{v}'_j) + \mathrm{I}_\tau(\overline{v}^\sharp_j),$$

which immediately implies Equation (7.5.5).

Case $j = 0$. Suppose that $\overline{v}'_0 = \varnothing$. Then, $\overline{v}_0$ is a $\overline{\mathrm{W}}_-$-curve or a degenerate $\overline{\mathrm{W}}_-$-curve by Lemma 7.5.2. If $\overline{v}_0$ is a $\overline{\mathrm{W}}_-$-curve, then it is simply-covered and satisfies $\mathrm{I}(\overline{v}_0) \geq \mathrm{ind}(\overline{v}_0) \geq 0$. If $\overline{v}_0$ is a degenerate $\overline{\mathrm{W}}_-$-curve, then $\overline{v}_0$ can be written as a union of a fiber C and a W_--curve $\overline{v}^\flat_0$. The Fredholm index of $\overline{v}^\flat_0$ is nonnegative since $\overline{v}^\flat_0$ is simply-covered and hence is regular. The Fredholm index of C is given by:

$$\begin{aligned}\mathrm{ind}(\mathrm{C}) &= -\chi(\mathrm{C}) + 2\langle c_1(T\mathrm{W}_-), \mathrm{C}\rangle \\ &= (2g-2) + 2(2-2g) = 2-2g.\end{aligned}$$

The algebraic intersection number $\langle \mathrm{C}, \overline{v}^\flat_0\rangle$ is equal to $2g$ and

$$\begin{aligned}\mathrm{I}(\overline{v}_0) &\geq \mathrm{ind}(\mathrm{C}) + \mathrm{ind}(\overline{v}^\flat_0) + 2\langle \mathrm{C}, \overline{v}^\flat_0\rangle \\ &\geq (2-2g) + 0 + 2(2g) = 2g+2,\end{aligned} \tag{7.5.6}$$

by Theorem 5.6.9.

Next suppose that $\overline{v}'_0 \neq \varnothing$. We have $\mathrm{I}(\overline{v}'_0) = 0$ by Lemma 5.7.16. Next, by Lemma 7.5.3, if $\overline{v}^\sharp_0 \neq \varnothing$, then $\mathrm{Im}(\overline{v}^\sharp_0) \subset \overline{\mathrm{W}}_- - int(\mathrm{W}_-)$ and has $\delta_0^{p_1-p_0}$, $p_1 - p_0 > 0$, at the positive end and some (p_1-p_0)-element subset of $\{x_1, \dots, x_{2g}, x'_1, \dots, x'_{2g}\}$ at the negative end. Hence $\mathrm{I}(\overline{v}^\sharp_0) = p_1 - p_0$ by Lemma 5.7.17. Finally, since $\mathrm{Im}(\overline{v}^\flat_0) \subset \mathrm{W}_-$ by Lemma 7.5.3, $\overline{v}^\flat_0$ is simply-covered and $\mathrm{I}(\overline{v}^\flat_0) \geq 0$. The ECH indices of $\overline{v}'_0$, $\overline{v}^\sharp_0$, and $\overline{v}^\flat_0$ are additive. This completes the proof of the lemma. □

7.6. *Compactness theorem.* — Let $\overline{\mathrm{J}}^\diamondsuit_-(\varepsilon,\delta,p)$ be a generic almost complex structure which is $(\varepsilon, \mathrm{U})$-close to $\overline{\mathrm{J}}_- \in \mathcal{J}^{reg}_{\overline{\mathrm{W}}_-}$, where U and $\mathrm{K}_{p,2\delta}$ are as in Convention 5.8.12. We write:

$$\mathcal{M}^i_{\overline{\mathfrak{m}}}(\varepsilon,\delta,p) := \mathcal{M}^{\mathrm{I}=i, n^*=m}_{\overline{\mathrm{J}}^\diamondsuit_-(\varepsilon,\delta,p)}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}}). \tag{7.6.1}$$

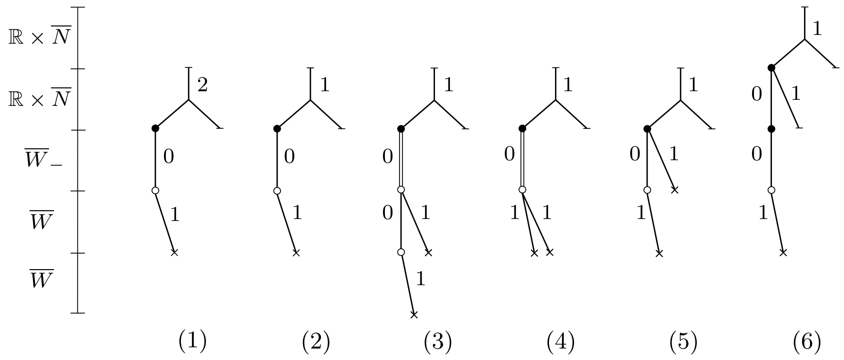

FIG. 4. — Schematic diagrams for the possible types of degenerations. Here • represents δ_0, ∘ represents z_∞, and × represents some x_i or x_i'. A vertical line indicates a trivial cylinder or a restriction of a trivial cylinder, and a double vertical line indicates a branched double cover of a trivial cylinder or a restriction of a trivial cylinder. The labels on the graphs are ECH indices of each component

As before, $\overline{u}_\infty \in \partial\mathcal{M}^3_{\overline{\mathfrak{m}}}(\varepsilon,\delta,p)$ is written as $\overline{u}_\infty = \overline{v}_{-b}\cup\dots\cup\overline{v}_a$ and each $\overline{v}_j$ is written as $\overline{v}_j = \overline{v}_j'\cup\overline{v}_j'' = \overline{v}_j'\cup\overline{v}_j^\sharp\cup\overline{v}_j^\flat\cup\overline{v}_j^f$, where $\overline{v}_j^f=\varnothing$ for $j\neq 0$. There are no ghost components by an index argument as in Lemma 6.1.8.

We now prove the following compactness theorem, which is basically a consequence of two constraints: $I(\overline{u}_i)=3$ and $n^*(\overline{u}_i)=m$. The list of possibilities should be viewed as a preliminary list, since we will subsequently eliminate Cases (2)–(6) in Theorem 7.10.1.

Theorem **7.6.1.** — *Let $\overline{J}_-^\diamondsuit$ be (ε, U)-close to $\overline{J}_-\in\mathcal{J}^{reg}_{\overline{W}_-}$ and $K_{p,2\delta}$-regular with respect to $\overline{\mathfrak{m}}$, and let $\overline{u}_\infty\in\partial\mathcal{M}^3_{\overline{\mathfrak{m}}}(\varepsilon,\delta,p)$. If $\overline{v}_0'\neq\varnothing$, then $\overline{u}_\infty$ contains one of the following subbuildings:*

1. *A 3-level building consisting of $\overline{v}_1$ with $I=2$ from $\boldsymbol{\gamma}$ to $\delta_0\boldsymbol{\gamma}'$; $\overline{v}_0'=\sigma_\infty^-$; and $\overline{v}_{-1}^\sharp$ which is a thin strip from z_∞ to x_i or x_i'.*
2. *A 3-level building consisting of $\overline{v}_1$ with $I=1$ from some $\boldsymbol{\gamma}''$ to $\delta_0\boldsymbol{\gamma}'$; $\overline{v}_0'=\sigma_\infty^-$; and $\overline{v}_{-1}^\sharp$ which is a thin strip.*
3. *A 4-level building consisting of $\overline{v}_1$ with $I=1$ from $\boldsymbol{\gamma}$ to $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}_0'$ which is a branched double cover of σ_∞^-; $\overline{v}_{-1}'=\sigma_\infty$; and $\overline{v}_{-1}^\sharp$ and $\overline{v}_{-2}^\sharp$ which are both thin strips.*
4. *A 3-level building consisting of $\overline{v}_1$ with $I=1$ from $\boldsymbol{\gamma}$ to $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}_0'$ which is a branched double cover of σ_∞^-; and $\overline{v}_{-1}^\sharp$ which is the union of two thin strips.*
5. *A 3-level building consisting of $\overline{v}_1$ with $I=1$ from $\boldsymbol{\gamma}$ to $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}_0'=\sigma_\infty^-$; $\overline{v}_0^\sharp$ with $I=1$ from δ_0 to x_i or x_i'; and $\overline{v}_{-1}^\sharp$ which is a thin strip.*
6. *A 4-level building consisting of $\overline{v}_2$ with $I=1$ from $\boldsymbol{\gamma}$ to $\delta_0^2\boldsymbol{\gamma}'$; $\overline{v}_1'=\sigma_\infty'$; $\overline{v}_1^\sharp$ with $I=1$ which is a cylinder from δ_0 to h; $\overline{v}_0'=\sigma_\infty^-$; and $\overline{v}_{-1}^\sharp$ which is a thin strip.*

Here we are omitting levels which are connectors.

See Figure 4.

Proof. — By Proposition 7.3.2, $\overline{u}_i$ converges in the sense of SFT to a holomorphic building $\overline{u}_\infty = \overline{v}_a \cup \cdots \cup \overline{v}_{-b}$. We have three constraints:

(i) $\sum_{j=-b}^{a} \mathrm{I}(\overline{v}_j) = 3$,
(ii) $\mathrm{I}(\overline{v}_j) \geq 0$ for all j, and
(iii) $\sum_{j=-b}^{a} n^*(\overline{v}_j) = m$.

(i) comes from the additivity of ECH indices, (ii) comes from Lemma 7.5.5, and (iii) comes from Equation (7.4.1).

Suppose that $\overline{v}_0' \neq \varnothing$, i.e., we are in the situation of Lemma 7.5.3. We have the following restrictions:

- the top level $\overline{v}_a$ is nontrivial and satisfies $\mathrm{I}(\overline{v}_a) \geq 1$;
- $\cup_{j<0} \overline{v}_j^\sharp$ consists of $p_0 \geq 1$ thin strips and contributes $\sum_{j<0} \mathrm{I}(\overline{v}_j^\sharp) = p_0 \geq 1$ to the total ECH index;
- $\cup_{j=-b}^{a} \overline{v}_j^\sharp$ consists of $p_{a'}^-$ components and each component has ECH index ≥ 1, with exception of $\overline{v}_0^\sharp$, which has ECH index 0, where a' and $p_{a'}^-$ are as in Lemma 7.5.3.

This immediately implies $p_{a'}^- \leq 2$ because $p_{a'}^-$ is also the number of nontrivial curves with a positive end at δ_0 and each of them has ECH index $\mathrm{I} \geq 1$ by Lemma 7.5.3. We also have $p_0 \leq p_{a'}^-$, and therefore we can divide the proof into three cases:

- Case I: $p_{a'}^- = p_0 = 1$.
- Case II': $p_{a'}^- = 2$ and $p_0 = 1$.
- Case II'': $p_{a'}^- = 2$ and $p_0 = 2$.

Case I. In this case $\overline{v}_0' = \sigma_\infty^-$ and $\overline{v}_{-1}^\sharp$ is a thin strip. This leaves two possibilities for $\overline{v}_1$: either $\mathrm{I}(\overline{v}_1) = 2$ and we are in Case (1), or $\mathrm{I}(\overline{v}_1) = 1$ and we are in Case (2).
Case II'. In this case we have $\overline{v}_{j_0}^\sharp \neq \emptyset$ and with a positive end at δ_0 for some $j_0 \geq 0$. Since $\mathrm{I}(\overline{v}_{j_0}^\sharp) = 1$ by Remark 7.5.4, $\overline{v}_{-1}$ consists of a single thin strip and there are no other levels with $j < 0$. If $j_0 = 0$ we are in Case (5), and if $j_0 > 0$ we are in Case (6). In Case (6), $\overline{v}_1^\sharp$ is a cylinder connecting δ_0 with h by Lemma 7.5.3 and Remark 7.5.4.
Case II''. In this case $\cup_{j<0} \overline{v}_j$ consists of two thin strips. If they are on the same level we are in Case (4) and if they are on different levels we are in Case (3).

This completes the proof of Theorem 7.6.1. ☐

Remark **7.6.2.** — In Cases (3)–(6), the total number of branch points of $\cup_{j=-b}^{a} \overline{v}_j'$ is one, where we are not ignoring connector components that cover σ_∞^*: Assume without loss of generality that the only nontrivial $\overline{v}_j'$ is $\overline{v}_0'$ and that $\overline{v}_0'$ double covers σ_∞^-. Let $\mathfrak{b}$ be the number of branch points of $\overline{v}_0'$. Then $\mathrm{ind}(\overline{v}_0') = \mathfrak{b} - 1$ by Proposition 5.5.5, the Riemann-Hurwitz formula, and the proof of Lemma 5.8.9. The index inequality, the additivity of the indices, and the condition $\mathrm{I}(\overline{u}_i) = 3$ force $\mathfrak{b} = 1$.

The proof of the following theorem is similar and will be omitted.

Theorem **7.6.3.** — *Let* $\overline{J}_-^{\Diamond}$ *be* (ε, U)*-close to* $\overline{J}_- \in \mathcal{J}^{reg}_{\overline{W}_-}$ *and* $K_{p,2\delta}$*-regular with respect to* $\overline{\mathfrak{m}}$*, and let* $\overline{u}_\infty \in \partial\mathcal{M}^2_{\overline{\mathfrak{m}}}(\varepsilon, \delta, p)$*. If* $\overline{v}_0' \neq \varnothing$*, then* $\overline{u}_\infty$ *contains a* 3*-level subbuilding consisting of* $\overline{v}_1$ *with* $I = 1$ *from* $\boldsymbol{\gamma}$ *to* $\delta_0\boldsymbol{\gamma}'$*;* $\overline{v}_0' = \sigma_\infty^-$*; and* $\overline{v}_{-1}^\sharp$ *which is a thin strip. Here it is possible to have* $I_{ECH} = 0$ *connectors in between.*

7.7. *Asymptotic eigenfunctions.* — We now collect some facts about the asymptotic operator and asymptotic eigenfunctions, referring the reader to [HWZ1] and [HT2].

7.7.1. *The asymptotic operator.* — We study the local behavior of a holomorphic half-cylinder which converges to a degree $l \geq 1$ multiple cover of δ_0, denoted by δ_0^l. Let us denote $D^2_{\rho_0} = \{\rho \leq \rho_0\} \subset D^2$ with $\rho_0 > 0$ small (in particular $\leq \frac{1}{2}$) and $K = [C, +\infty) \times (\mathbf{R}/2l\mathbf{Z})$ with $l \in \mathbf{N}$. We will be using balanced coordinates on $(\mathbf{R}/2\mathbf{Z}) \times D^2_{\rho_0}$; see Section 5.1.2. A holomorphic half-cylinder which is asymptotic to δ_0^l at the positive end restricts to a holomorphic map

$$\overline{u} : K \to \mathbf{R} \times (\mathbf{R}/2\mathbf{Z}) \times D^2_{\rho_0},$$

which can be written as:

$$(s, t) \mapsto (s, t, z(s, t)),$$

where $\lim_{s\to+\infty} z(s, t) = 0$.

Lemma **7.7.1.** — *The function* $z : K \to D^2_{\rho_0}$ *satisfies the equation*

$$\partial_s z + i\partial_t z + \varepsilon z = 0, \tag{7.7.1}$$

where $\varepsilon = \frac{\pi}{m}$.

Proof. — The partial derivatives of $\overline{u}$ are: $\partial_s\overline{u} = (1, 0, \partial_s z)$ and $\partial_t\overline{u} = (0, 1, \partial_t z)$. Then $J(\partial_s\overline{u}) = R + (0, 0, i\partial_s z)$, where R is the Hamiltonian vector field. We compute that $R = \partial_t + \varepsilon\partial_\phi$. Hence $J(\partial_s\overline{u}) = (0, 1, i\partial_s z + i\varepsilon z)$. This gives Equation (7.7.1). □

Definition **7.7.2.** — *We define the* asymptotic operator

$$A_l : L^2_1(\mathbf{R}/2l\mathbf{Z}, \mathbf{C}) \to L^2(\mathbf{R}/2l\mathbf{Z}, \mathbf{C}),$$
$$f \mapsto -i\partial_t f - \varepsilon f.$$

We remark that the asymptotic operator which appears in [HWZ1] is A_l, whereas the asymptotic operator in [HT2] is $-A_l$. The eigenfunctions of A_l are the *asymptotic eigenfunctions*, and are given by $ce^{\pi int/l}$, $c \in \mathbf{C}$, $n \in \mathbf{Z}$, with corresponding eigenvalues $\frac{\pi n}{l} - \varepsilon$.

An asymptotic eigenfunction $ce^{\pi int/l}$ is said to be *normalized* if $|c| = 1$. Let $E_{\pi n/l-\varepsilon}$ be the eigenspace of A_l corresponding to the eigenvalue $\frac{\pi n}{l} - \varepsilon$.

For a strip-like end asymptotic to the intersection point z_∞, the asymptotic operator is still $f \mapsto -i\partial_t f - \varepsilon f$, but now acting on functions $f : [0, 1] \to \mathbf{C}$ with boundary conditions $f(0) \in e^{i(c_0+\varepsilon)}\mathbf{R}$ and $f(1) \in e^{ic_0}\mathbf{R}$ for some real constant c_0. The eigenfunctions are $ce^{(\pi n-\varepsilon)it+i(c_0+\varepsilon)}$, $c \in \mathbf{R}$, $n \in \mathbf{Z}$, with corresponding eigenvalues $\pi n - 2\varepsilon$. As before we say that an eigenfunction is *normalized* if $|c| = 1$.

7.7.2. *The asymptotic eigenfunction at an end.* — Let $\boldsymbol{\gamma} \in \widehat{\mathcal{O}}_{2g}$ and $\boldsymbol{\gamma}' \in \widehat{\mathcal{O}}_{2g-l}$. As before, the modifier $*$ is placed as in $\mathcal{M}^*_{\overline{J}_-}(\boldsymbol{\gamma}, \delta_0^l\boldsymbol{\gamma}')$ to denote the subset of $\mathcal{M}_{\overline{J}_-}(\boldsymbol{\gamma}_0, \delta_0^l\boldsymbol{\gamma}')$ satisfying $*$. The modifier $* = (l_1, \dots, l_\lambda)$ means $(l_1, \dots, l_\lambda)$ is a partition of l and we restrict to curves with λ ends at δ_0 with covering multiplicities $l_1, \dots, l_\lambda$.

We consider the asymptotics of $\overline{u} \in \mathcal{M}^{(l_1,\dots,l_\lambda)}_{\overline{J}'}(\boldsymbol{\gamma}, \delta_0^l\boldsymbol{\gamma}')$ near the negative end $\delta_0^{l_j}$. Let

$$\overline{\pi}_{D^2_{\rho_0}} : \mathbf{R} \times (\mathbf{R}/2\mathbf{Z}) \times D^2_{\rho_0} \to D^2_{\rho_0}$$

be the projection to $D^2_{\rho_0}$ with respect to balanced coordinates; we also write $\overline{\pi}$ for $\overline{\pi}_{D^2_{\rho_0}}$. Also let

$$z_j = \overline{\pi}_{D^2_{\rho_0}} \circ \overline{u} : (-\infty, s_0] \times (\mathbf{R}/2l_j\mathbf{Z}) \to D^2_{\rho_0}$$

be the projection of the negative end of $\overline{u}$ which corresponds to $\delta_0^{l_j}$. The following asymptotic result is due to Hofer-Wysocki-Zehnder [HWZ1]:

Lemma **7.7.3.** — *There exist constants* $C_0, C_1 > 0$ *such that the following holds: For any* $\overline{u} \in \mathcal{M}^{(l_1,\dots,l_\lambda)}_{\overline{J}'}(\boldsymbol{\gamma}, \delta_0^l\boldsymbol{\gamma}')$ *with a negative end at* $\delta_0^{l_j}$*, there exists an asymptotic eigenfunction* $f_j : \mathbf{R}/2l_j\mathbf{Z} \to \mathbf{C}$ *given by* $f_j(t) = ce^{\pi it/l_j}$ *such that*

$$|z_j(s,t) - e^{(\pi/l_j-\varepsilon)s}f_j(t)| < C_0e^{(\pi/l_j-\varepsilon+C_1)(s-s_0)}, \tag{7.7.2}$$

$$|\partial_t z_j(s,t) - e^{(\pi/l_j-\varepsilon)s}f_j'(t)| < C_0e^{(\pi/l_j-\varepsilon+C_1)(s-s_0)}. \tag{7.7.3}$$

(We allow $c = 0$*.)*

Note that, because of the particularly simple form of the Hamiltonian flow around δ_0, Lemma 7.7.3 could also be proved directly by standard Fourier series arguments.

Definition **7.7.4.** — *The function* $f_j(t)$ *satisfying Equation* (7.7.2) *is called* the asymptotic eigenfunction for $\overline{u}$ at the negative end $\delta_0^{l_j}$.[12]

[12] This is the terminology from [HT2], which is slightly different from that of the Hofer school.

The following is due to Wendl [We3] and Hutchings-Taubes [HT2, Prop. 3.2]. (The proof of [HT2, Prop. 3.2] essentially proves the following lemma, but is not quite stated in the same way.)

Lemma **7.7.5.** — *There exists an arbitrarily small perturbation* $\widetilde{\mathrm{J}}'$ *of* $\overline{\mathrm{J}}'$ *in* $\mathcal{J}^{reg}_{\overline{\mathrm{W}'}}$, *which is supported on* $\mathbf{R}\times\{\rho_1<\rho<\rho_2\}\subset\mathbf{R}\times\overline{\mathrm{N}}$ *for* $0<\rho_0<\rho_1<\rho_2$ *arbitrarily small, such that the following holds:*

($\star$) *for all* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\gamma}'$, *partitions* $(l_1,\dots,l_\lambda)$ *of* l, $j\in\{1,\dots,\lambda\}$ *and* $r\in\mathbf{N}\cup\{0\}$, *the set of elements of* $\mathcal{M}^{\mathrm{ind}=r,(l_1,\dots,l_\lambda)}_{\widetilde{\mathrm{J}}'}(\boldsymbol{\gamma},\delta_0^l\boldsymbol{\gamma}')/\mathbf{R}$ *with vanishing asymptotic eigenfunction at* $\delta_0^{l_j}$ *is a real codimension 2 submanifold.*

The real codimension 2 condition is due to the fact that $\dim_{\mathbf{R}}\mathrm{E}_{\pi/l_i+\varepsilon}=2$. Let $\mathcal{J}^{\star}_{\overline{\mathrm{W}'}}\subset\mathcal{J}^{reg}_{\overline{\mathrm{W}'}}$ be the (dense) subset of almost complex structures which satisfy ($\star$).

7.7.3. *Definition of* $\widetilde{\mathrm{U}}_{m-1}$. — Let $\boldsymbol{\gamma}\in\widehat{\mathcal{O}}_{2g}$, $\boldsymbol{\gamma}'\in\widehat{\mathcal{O}}_{2g-1}$, and $\widetilde{\mathrm{J}}'\in\mathcal{J}^{\star}_{\overline{\mathrm{W}'}}$. Let $\overline{u}\in\mathcal{M}^{\mathrm{I}=\mathrm{ind}=2,n^*=m-1}_{\widetilde{\mathrm{J}}'}(\boldsymbol{\gamma},\delta_0\boldsymbol{\gamma}')$. Since $\widetilde{\mathrm{J}}'$ satisfies ($\star$), the asymptotic eigenfunction of $\overline{u}$ at δ_0 is nonzero. Hence we can associate a normalized asymptotic eigenfunction at δ_0 to $\overline{u}$.

Now let

$$\langle\widetilde{\mathrm{U}}_{m-1}(\boldsymbol{\gamma}),\delta_0^p\boldsymbol{\gamma}'\rangle=\begin{cases}0, & \text{if } p\neq 1;\\ \#\left(\mathcal{M}^{\mathrm{I}=\mathrm{ind}=2,n^*=m-1,f_{\delta_0}}_{\widetilde{\mathrm{J}}'}(\boldsymbol{\gamma},\delta_0\boldsymbol{\gamma}')/\mathbf{R}\right), & \text{if } p=1,\end{cases}$$

for a generic normalized asymptotic eigenfunction f_{δ_0}. The modifier f_{δ_0} stands for "the normalized asymptotic eigenfunction at the negative end δ_0 is f_{δ_0}".

7.7.4. *The limit* $m\to\infty$. — Suppose $m\gg 0$. Let $\overline{h}_m:\overline{\mathrm{S}}\xrightarrow{\sim}\overline{\mathrm{S}}$ be a smooth extension of $h:\mathrm{S}\xrightarrow{\sim}\mathrm{S}$ via ν_m, as defined in Section 5.1.2, and let $\overline{h}_\infty:\overline{\mathrm{S}}\xrightarrow{\sim}\overline{\mathrm{S}}$ be the smooth extension of h via ν_∞. We recall that $\overline{h}_\infty$ is the identity on the disk $\mathrm{D}^2_{1/2}\subset\overline{\mathrm{S}}$ of radius $\frac{1}{2}$.

We will denote $\overline{\mathrm{N}}_m$ (including $m=\infty$) the mapping torus of $\overline{h}_m$ and define $\overline{\mathrm{W}'}_m=\mathbf{R}\times\overline{\mathrm{N}}_m$. Of course $\overline{\mathrm{W}'}_m$ is diffeomorphic to $\overline{\mathrm{W}'}$ for all m (including $m=\infty$); we will write $\overline{\mathrm{W}'}_m$ instead of $\overline{\mathrm{W}'}$ when it is necessary to keep track of m. This will happen especially when $m=\infty$. The Hamiltonian structure on $\overline{\mathrm{N}}_m$ will be written as $(\alpha_{0,m},\omega)$. We then define $\mathcal{J}_{\overline{\mathrm{W}'}_m}$ as in Definition 5.3.14, with α_0 replaced by $\alpha_{0,m}$. Similarly, we will use the notation $\overline{\mathrm{W}}_{-,m}$ and $\mathcal{J}_{\overline{\mathrm{W}}_{-,m}}$ instead of $\overline{\mathrm{W}}_-$ and $\mathcal{J}_{\overline{\mathrm{W}}_-}$ when we need to specify m.

Although the orbit δ_0 is degenerate when $m=\infty$, we still define the set $\overline{\mathcal{P}}=\widehat{\mathcal{P}}\cup\{\delta_0\}$ of orbits, the set $\widehat{\mathcal{O}}_k$ (resp. $\overline{\mathcal{O}}_k$) of orbit sets constructed from $\widehat{\mathcal{P}}$ (resp. $\overline{\mathcal{P}}$) which intersect $\overline{\mathrm{S}}\times\{0\}$ exactly k times, as in Section 5.3.2.

Let $\overline{\mathrm{J}}'_\infty\in\mathcal{J}_{\overline{\mathrm{W}'}_\infty}$. In analogy with Definition 5.3.16, let $\mathcal{M}_{\overline{\mathrm{J}}'_\infty}(\boldsymbol{\gamma},\delta_0^l\boldsymbol{\gamma}')$ be the moduli space of $\overline{\mathrm{J}}'_\infty$-holomorphic maps in $\overline{\mathrm{W}'}$ from $\boldsymbol{\gamma}$ to $\delta_0^l\boldsymbol{\gamma}'$ without fiber components. (We recall that $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ consist of orbits in N.)

The Hamiltonian vector field of $(\alpha_{0,\infty},\omega)$ is degenerate, but all degenerate orbits are contained in the interior of V, and therefore transversality for the moduli spaces $\mathcal{M}_{\overline{J}'_\infty}(\boldsymbol{\gamma},\delta_0^l\boldsymbol{\gamma}')$ works in the usual way. An almost complex structure $\overline{J}'_\infty\in\mathcal{J}_{\overline{W'}_\infty}$ is *regular* if $\mathcal{M}^s_{\overline{J}'_\infty}(\boldsymbol{\gamma},\delta_0^l\boldsymbol{\gamma}')$ is transversely cut out for all $0<k\leq 2g$, $l\geq 0$, $\boldsymbol{\gamma}\in\widehat{\mathcal{O}}_k$, and $\boldsymbol{\gamma}'\in\widehat{\mathcal{O}}_{k-l}$.

Lemma **7.7.6.** — *A generic $\overline{J}'_\infty\in\mathcal{J}_{\overline{W'}_\infty}$ is regular and satisfies (⋆).*

Proof. — The Fredholm theory for holomorphic curves with Morse-Bott asymptotics uses Sobolev spaces with exponential weights. The regularity of simply-covered moduli spaces in this setting is treated in Wendl [We3]. □

As before, we write $\mathcal{J}^{reg}_{\overline{W'}_\infty}\subset\mathcal{J}_{\overline{W'}_\infty}$ for the dense subset of regular almost complex structures and $\mathcal{J}^\star_{\overline{W'}_\infty}\subset\mathcal{J}^{reg}_{\overline{W'}_\infty}$ for the dense subset of almost complex structures which satisfy (⋆). We claim that if $\overline{J}'_\infty\in\mathcal{J}^\star_{\overline{W'}_\infty}$, then nearby almost complex structures $\overline{J}'_m\in\mathcal{J}_{\overline{W'}_m}$ for $m\gg 0$ are in $\mathcal{J}^\star_{\overline{W'}_m}$. Indeed, using the same exponential weights at δ_0^l for both $\overline{J}'_\infty$ and $\overline{J}'_m$, if $\overline{J}'_\infty\in\mathcal{J}_{\overline{W'}_\infty}$ is regular, then so is a nearby $\overline{J}'_m\in\mathcal{J}_{\overline{W'}_m}$ by continuity. We can then turn off the exponential weights for $0\ll m<\infty$ without affecting the regularity. The condition (⋆) also carries over by continuity.

On the other hand, compactness is more subtle. In fact the Hamiltonian vector field of $(\alpha_{0,\infty},\omega)$ has a Morse-Bott family of orbits corresponding to $int(D^2_{1/2})$ and a circle of degenerate orbits corresponding to $\partial D^2_{1/2}$, and the standard asymptotic convergence theorems do not work for degenerate orbits. For this reason, we will need extra work to be able to apply the SFT compactness theorem; namely we will need to show that a family of degenerating $\overline{J}'_m$-holomorphic curves satisfying the appropriate topological constraints cannot approach an orbit of $D^2_{1/2}$ besides a multiple of δ_0.

Given a $\overline{J}'_\infty$-holomorphic map $u:\dot{F}\to\overline{W'}$, we say that an end of $\dot{F}$ *accumulates* to a Hamiltonian orbit γ if, after parametrizing the end by $(s_0,+\infty)\times S^1$ or $(-\infty,-s_0)\times S^1$, there is a sequence $s_k\to+\infty$ such that $\lim_{k\to\infty}u(\pm s_k,t)=\gamma(t)$ uniformly in t.

Lemma **7.7.7.** — *Let $u:\dot{F}\to\overline{W'}$ be a finite energy $\overline{J}'_\infty$-holomorphic map. Then every end of $\dot{F}$ accumulates to a Hamiltonian orbit γ. If an end of $\dot{F}$ does not accumulate to any degenerate orbit, than it is asymptotic to a (Morse-Bott) nondegenerate orbit.*

Proof. — The accumulation to a Hamiltonian orbit follows from [Ho]. If an end does not accumulate to any degenerate orbit, then there is a neighborhood of the degenerate orbits which does not intersect the image of the end, as the set of degenerate orbits is compact. Then we can repeat the arguments of [HWZ1, HWZ2] in the complement of that neighborhood. □

Let γ_z be the orbit corresponding to $z \in D^2_{1/2}$ for the stable Hamiltonian structure $(\alpha_{0,\infty}, \omega)$ on $\overline{N}$. We will use intersection arguments to constrain the behavior of a sequence of $\overline{J}'_m$-holomorphic maps that accumulate to orbits γ_z for some $z \in D^2_{1/2}$. (Recall that all other closed orbits in $int(V)$ intersect a fiber more than $2g$ times for $m \gg 0$, and therefore are automatically excluded.)

Lemma **7.7.8.** — *Let* F *be a compact Riemann surface with boundary and let*

$$\partial F = B_1^+ \sqcup \ldots \sqcup B_{l_+}^+ \sqcup B_1^- \sqcup \ldots \sqcup B_{l_-}^-, \quad l_\pm \geq 1$$

be a decomposition of ∂F *into connected components. We parametrize the components* B_j^-, $j = 1, \ldots, l_-$, *by* $\beta_j : S^1 \to B_j^-$ *in an orientation-reversing manner. Let* η *be a positive real number and* $v : F \to \overline{W'}$ *a* $\overline{J}'_m$*-holomorphic map with* $m \gg 0$, *such that*

(i) *each* $\pi_{\overline{N}} \circ v \circ \beta_j$ *is* η*-close to a cover of an orbit* γ_{z_j} *with* $z_j \in D^2_{1/2}$ *and* $\eta < \frac{|z_j|}{2}$, *and*
(ii) *each* $\pi_{\overline{N}} \circ v(B_j^+)$ *is contained in a small neighborhood of* $N = \overline{N} - int(V)$.

Then $n'(v) \geq m$.

Proof. — Consider subsets $V_\eta = D^2_\eta \times S^1$ and $V_{1/2+\eta} = D^2_{1/2+\eta} \times S^1$ of $\overline{N}$. Both ∂V_η and $\partial V_{1/2+\eta}$ are foliated by Hamiltonian orbits. We choose coordinates on ∂V_η and $\partial V_{1/2+\eta}$ such that the meridian has slope $(1, 0)$, the orbits on ∂V_η have slope $(0, 1)$ and the orbits on $\partial V_{1/2+\eta}$ have slope $(1, k)$ with $k \gg 0$ (here we are assuming that η has been perturbed so that the foliation on $\partial V_{1/2+\eta}$ has rational slope).

Perturb η if necessary so that $\pi_{\overline{N}} \circ v$ is transverse to ∂V_η and $\partial V_{1/2+\eta}$ and let

$$C_- = \pi_{\overline{N}} \circ v(F) \cap \partial V_\eta \quad \text{and} \quad C_+ = \pi_{\overline{N}} \circ v(F) \cap \partial V_{1/2+\eta},$$

where C_- and C_+ are oriented as the boundaries of $\mathrm{Im}(\pi_{\overline{N}} \circ v) \cap V_\eta$ and $\mathrm{Im}(\pi_{\overline{N}} \circ v) \cap V_{1/2+\eta}$. Since $\pi_{\overline{N}} \circ v$ maps no B_j^+ inside $V_{1/2+\eta}$ and C_+ is positively transverse to the orbits on $\partial V_{1/2+\eta}$, it follows that C_+ represents some homology class (q, p) with $p, q \geq 1$. Then a simple homology computation implies that C_- represents the homology class $(q, 0)$. Hence $n'(v) = qm \geq m$. □

Lemma **7.7.9.** — *Let* $\overline{J}'_{m_i} \in \mathcal{J}^\star_{\overline{W'}_{m_i}}$ *be a sequence such that* $\overline{J}'_{m_i} \to \overline{J}'_\infty \in \mathcal{J}^\star_{\overline{W'}_\infty}$ *as* $i \to \infty$. *Given a partition* (l) *of* l, *a sequence of curves*

$$\overline{u}_i \in \mathcal{M}^{I=1,(l),n'=m_i-l}_{\overline{J}'_{m_i}}(\boldsymbol{\gamma}, \delta_0^l \boldsymbol{\gamma}')$$

with $i \to \infty$ *converges (up to extracting a subsequence) to a curve*

$$\overline{u}_\infty \in \mathcal{M}^{I=1,(l),\widetilde{n}=0}_{\overline{J}'_\infty}(\boldsymbol{\gamma}, \delta_0^l \boldsymbol{\gamma}'),$$

where $\widetilde{n}(\overline{u}_\infty)$ *is the intersection number of* $\overline{u}_\infty$ *with the section at infinity.*

Proof. — The first steps of the compactness theorem are unchanged by the presence of degenerate orbits. Let $\dot{F}_i$ be the domains of the maps $\overline{u}_i$, which we assume to have the same topological type and be hyperbolic. (Otherwise we can remove one point from $\dot{F}_i$, without changing the name.) We claim that $\overline{u}_i$ satisfies the uniform gradient bound (Equation (7.3.3)). Indeed, if there is a sequence of points in $\dot{F}_i$ with gradient blow up, then we obtain a nonconstant holomorphic plane that accumulates to an orbit, contradicting the noncontractibility of all the orbits.

After passing to a subsequence, the punctured surfaces $\dot{F}_i$ converge to a punctured nodal surface $\dot{F}_\infty$ in the Deligne-Mumford moduli space. We denote by F'_∞ the punctured Riemann surface obtained by removing the nodes from $\dot{F}_\infty$. By the gradient bound, we can extract a subsequence from $\overline{u}_i$ which converges in C^∞_{loc} to a $\overline{J}'_\infty$-holomorphic map $\overline{u}'_\infty : F'_\infty \to \overline{W'}$. (The holomorphic map denoted by $\overline{u}'_\infty$ in this proof should not be confused with the holomorphic map $\overline{u}'_\infty$ introduced in Section 5.7.2). We assume that no component of $\overline{u}'_\infty$ is a connector over δ_0. The modifications required by the presence of a connector over δ_0 are easy and left to the reader.

Fix $\varepsilon > 0$ small and suppose that $i \gg 0$. Consider the connected component of $\mathrm{Thin}_\varepsilon(\dot{F}_i)$ with a puncture that limits to δ_0^l; we identify it with $(-\infty, s_i) \times S^1$. We claim that there is no $s < s_i$ such that $\pi_{\overline{N}} \circ \overline{u}_i|_{\{s\}\times S^1}$ is sufficiently close to an orbit γ_z with $z \in D^2_{1/2} - \{0\}$. Arguing by contradiction, if there is such an s, then we consider $C_0 := (-\infty, s] \times S^1$ and $C_1 := \dot{F}_i - (-\infty, s] \times S^1$. Since C_0 limits to δ_0^l, we have $n'(\overline{u}_i|_{C_0}) \geq m_i - l$. On the other hand, there exists a compact surface with boundary $F \subset C_0$ such that $\overline{u}_i|_F$ satisfies the conditions of Lemma 7.7.8 and $n'(\overline{u}_i|_F) \geq m_i$, a contradiction. We can similarly argue that, for any other component $(s_-, s_+) \times S^1$ of $\mathrm{Thin}_\varepsilon(\dot{F}_i)$, there is no $s \in (s_-, s_+)$ such that $\pi_{\overline{N}} \circ \overline{u}_i|_{\{s\}\times S^1}$ is sufficiently close to an orbit γ_z with $z \in D^2_{1/2} - \{0\}$; here $s_- = -\infty$ or $s_+ = +\infty$ are allowed.

We now apply the usual compactness argument (in view of Lemma 7.7.7) to obtain a limit $\overline{J}'_\infty$-holomorphic building $\overline{u}_\infty$ such that no level of the building has an end that accumulates to an orbit γ_z for $z \in D^2_{1/2} - \{0\}$. Finally, since $\overline{J}'_\infty \in \mathcal{J}^\star_{\overline{W'}_\infty}$, every level of $\overline{u}_\infty$ has ind ≥ 1. On the other hand, since $\overline{u}_\infty$ has total Fredholm index 1, it is a single-level building. □

Let $\overline{J}'_\infty \in \mathcal{J}^\star_{\overline{W'}_\infty}$. By the compactness and regularity of moduli spaces, there are only finitely many curves $\mathcal{C}_1, \dots, \mathcal{C}_r$, modulo $\mathbf{R}$-translation, such that

$$\mathcal{C}_i \in \mathcal{M}^{I=1,\widetilde{n}=0,(l_i)}_{\overline{J}'_\infty}(\boldsymbol{\gamma}_i, \delta_0^{l_i}\boldsymbol{\gamma}'_i)$$

for some orbit sets $\boldsymbol{\gamma}_i, \boldsymbol{\gamma}'_i \in \widehat{\mathcal{O}}_*$ and partition (l_i) of l_i. Here if $\overline{u}$ is a curve with an end at δ_0^p, then its ECH index is computed using the Conley-Zehnder index of δ_0^p with respect to $\overline{h}_m$, $m \gg 0$, and $\widetilde{n}(\overline{u})$ is defined as the intersection number of $\overline{u}$ and the section at infinity. Let

$$f_i : \mathbf{R}/2l_i\mathbf{Z} \to \mathbf{C}, \quad t \mapsto c_i e^{\pi i t/l_i},$$

be the asymptotic eigenfunction corresponding the end $\delta_0^{l_i}$ of $\mathcal{C}_i$. The condition $c_i \neq 0$ follows from Lemma 7.7.6. We may therefore assume without loss of generality that all the f_i are normalized.

7.7.5. *Radial rays.*

Definition **7.7.10.** — *A* bad radial ray *is a radial ray* $\mathcal{R}_{\phi_0} = \{\phi = \phi_0, \rho \geq 0\}$ *in* $\mathbf{C}$ *which passes through a point in*

$$\{f_i(t) \mid i = 1, \dots, r; 0 < t < 2l_i; t \equiv 3/2 \text{ mod } 2\}.$$

A radial ray $\mathcal{R}_{\phi_0}$ *which is not bad is said to be* good.

A good radial ray must exist and, as it is explained in the following remark, we can assume it is $\mathcal{R}_\pi$.

Remark **7.7.11.** — The set of bad radial rays is determined by $(\overline{W'}_\infty, \overline{J'}_\infty)$. Strictly speaking, we should choose the set of endpoints $E \subset \partial D^2$ as in Section 5.2.2 such that $\mathcal{R}_{\phi_0+\pi}$ is a good radial ray and $\phi_0 < \phi(y_j(m)) < \phi_0 + c(m)$ where $c(m) \to 0$ as $m \to \infty$. After a rotational coordinate change of D^2, we may assume that $\mathcal{R}_\pi$ is a good radial ray and $0 < \phi(y_j(m)) < c(m)$.

7.8. *The rescaled function.* — In this subsection we will use limiting arguments in which $m \to \infty$ and $\overline{h}_m \to \overline{h}_\infty$; see Section 7.7.4. Hence many of the almost complex structures and moduli spaces will have an additional subscript m, where $m = \infty$ is also a possibility. Let $\overline{J'}_\infty \in \mathcal{J}^\star_{\overline{W'}_\infty}$ and let $\overline{J'}_m \in \mathcal{J}^\star_{\overline{W'}_m}$ be a nearby almost complex structure with respect to the integer $m \gg 0$. Let $\overline{J}_{-,m} \in \mathcal{J}^{reg}_{\overline{W}_{-,m}}$ be an almost complex structure which restricts to $\overline{J'}_m$ and let $\overline{J}^\diamondsuit_{-,m}$ be (ε, U)-close to $\overline{J}_{-,m}$.

Let m_i, $i \in \mathbf{N}$, and $\overline{u}_{ij}$, $i, j \in \mathbf{N}$, be sequences satisfying the following properties:

(S1) $\lim_{i\to\infty} m_i = \infty$;

(S2) $\overline{u}_{ij} \in \mathcal{M}^{I=3, n^*=m_i}_{\overline{J}^\diamondsuit_{-,m_i}}(\boldsymbol{\gamma}, \mathbf{y}; \overline{\mathfrak{m}})$, where $\mathbf{y} = \{x_l\} \cup \mathbf{y}'$ or $\{x'_l\} \cup \mathbf{y}'$ for some l; and

(S3) for all $i \in \mathbf{N}$ and $\kappa, \nu > 0$, there exists $j_{i,\kappa,\nu}$ such that, if $j \geq j_{i,\kappa,\nu}$, then $\overline{u}_{ij}$ is (κ, ν)-close (cf. Definition 6.5.2) to a $\overline{J}^\diamondsuit_{-,m_i}$-holomorphic building $\overline{u}_{i\infty} = \cup_l \overline{v}_{l,i}$ with $\overline{v}'_{0,i} \neq \varnothing$.

Moreover, assume that one of the following holds:

(S4') the first negative end at δ_0 in the building $\overline{u}_{i\infty}$ has multiplicity one, or

(S4") the first negative end at δ_0 in the building $\overline{u}_{i\infty}$ has multiplicity two.

Property (S3) is a consequence of the fact that the sequence $\overline{u}_{ij}$ converges in the SFT sense to the building $\overline{u}_{i\infty}$ for each fixed i. Property (S4') corresponds to Cases (1) and (2) in Theorem 7.6.1 and Property (S4") corresponds to the remaining cases.

7.8.1. *Truncation.* — Recall the projection $\overline{\pi} = \overline{\pi}_{D^2_{\rho_0}}$ using balanced coordinates.

Lemma **7.8.1.** — *Let $\overline{v}$ be a $\overline{J}'_m$-holomorphic curve in $\overline{W'}_m$ with ECH index $I \leq 2$ for $\overline{J}'_m \in \mathcal{J}^{\star}_{\overline{W'}_m}$ and let $\widetilde{v} : (-\infty, s_0] \to \overline{W'}$ be a negative end of $\overline{v}$ which is asymptotic to a multiple cover of δ_0. Then for every $\eta, \kappa > 0$ there exist positive constants $\overline{R}$ and $\kappa' < \eta$ such that $\frac{\kappa'}{\kappa} > \frac{1}{\eta}$ and*

$$\left|\overline{\pi} \circ \widetilde{v}(-\overline{R}, t) - \kappa' f(t)\right| \leq \frac{\kappa}{2},$$

$$\left|\partial_t(\overline{\pi} \circ \widetilde{v})(-\overline{R}, t) - \kappa' f'(t)\right| \leq \frac{\kappa}{2}$$

for all t, where f is the normalized asymptotic eigenfunction for $\widetilde{v}$.

Proof. — This is a slightly weaker rephrasing of Lemma 7.7.3, in view of Lemma 7.7.5. □

Let F be a Riemann surface and $\overline{u} : F \to (\overline{W}_-, \overline{J}_-)$ a holomorphic map. We denote by $p : F \to B_-$ the map $\overline{\pi}_{B_-} \circ \overline{u}$ and by $s : F \to \mathbf{R}$, $t : F \to S^1$ the functions obtained by composing p with the coordinates (s, t) on B_-. The functions (s, t) give local coordinates on F outside the critical points of p.

Let $\overline{g}$ be the restriction of an s-invariant Riemannian metric on $\overline{W'}$ to $\overline{W}_-$ and let d be the distance induced by $\overline{g}$ on $\overline{W}_-$.

Definition **7.8.2** *(Truncation).* — *A* truncation $(\widetilde{u}, \widetilde{F}, \{R^{(l)}\}_{l=1}^{l_0}, \widetilde{\mathfrak{m}}, \mathfrak{e}^{\pm}, \kappa')$ *of a holomorphic map $\overline{u} : F \to (\overline{W}_-, \overline{J}_-)$ is a tuple where:*

- *$\widetilde{F}$ is a subsurface of F and $\widetilde{u}$ is the restriction of $\overline{u}$ to $\widetilde{F}$;*
- *$p(\widetilde{F}) = B_- \cap \{R^{(1)} \leq s \leq R^{(l_0)}\}$;*
- *for each $l = 1, \ldots, l_0 - 1$, the restriction*

  $$p : \widetilde{F} \cap p^{-1}(\{R^{(l)} \leq s \leq R^{(l+1)}\}) \to B_- \cap \{R^{(l)} \leq s \leq R^{(l+1)}\}$$

  is a branched cover;
- *$\sup_{x \in \widetilde{u}(\widetilde{F})} d(x, \sigma_\infty^-) \leq 2\kappa'$;*
- *if $d(\overline{u}(y), \sigma_\infty^-) \leq \frac{\kappa'}{2}$, then $y \in \widetilde{F}$;*
- *$\widetilde{\mathfrak{m}} \in \widetilde{F}$ is the unique point such that $\widetilde{u}(\widetilde{\mathfrak{m}}) = \overline{\mathfrak{m}}$;*
- *$\mathfrak{e}^+$ (resp. $\mathfrak{e}^-$) is the union of components of $\partial\widetilde{F} - p^{-1}(\partial B_-)$ for which $ds(\mathfrak{n}) > 0$ (resp. < 0), where $\mathfrak{n}$ is the outward normal vector field along $\partial\widetilde{F}$.*

The map p is a branched cover only restricted to portions of $\widetilde{F}$, and not globally, because there can be portions of $\partial\widetilde{F}$ which are mapped to the interior of $B_- \cap \{R^{(1)} \leq s \leq R^{(l_0)}\}$.

Definition **7.8.3.** — *A* good truncation

$$(\widetilde{u}, \widetilde{\mathrm{F}}, \{\mathrm{R}^{(l)}\}_{l=1}^{l_0}, \widetilde{\mathfrak{m}}, \mathfrak{e}^{\pm}, \kappa, \kappa')$$

is a truncation $(\widetilde{u}, \widetilde{\mathrm{F}}, \{\mathrm{R}^{(l)}\}_{l=1}^{l_0}, \widetilde{\mathfrak{m}}, \mathfrak{e}^{\pm}, \kappa')$, *together with a constant* $\kappa > 0$, *such that* $\frac{\kappa'}{\kappa} > 2$ *and:*

(G) $\left\|\overline{\pi} \circ \widetilde{u}|_{\mathfrak{e}^{\pm}} - f\right\|_{C^1} \leq \kappa$, *where* f *is an asymptotic eigenfunction of* δ_0 *or* z_∞ *(as appropriate) on each component of* $\mathfrak{e}^{\pm}$ *and* $\|f\|_{C^0} \geq \kappa'$.

Lemma **7.8.4.** — *Let* m_i *and* $\overline{u}_{ij}$ *be sequences satisfying (S1)–(S3) and either (S4') or (S4"). Then there is a sequence* $j(i)$ *such that the following hold for all* $j \geq j(i)$*:*

(1) $\overline{u}_{ij}$ *admits a good truncation*

$$(\widetilde{u}_{ij}, \widetilde{\mathrm{F}}_{ij}, \{\mathrm{R}_{ij}^{(l)}\}_{l=1}^{l_0}, \widetilde{\mathfrak{m}}_{ij}, \mathfrak{e}_{ij}^{\pm}, \kappa_i, \kappa_i');$$

(2) $\lim_{i\to\infty} \kappa_i' = \lim_{i\to\infty} \kappa_i = 0$ *and* $\lim_{i\to\infty} \frac{\kappa_i'}{\kappa_i} = +\infty$*;*

(3) $\lim_{i\to\infty} \mathrm{R}_{ij(i)}^{(l_0)} = +\infty$ *and* $\lim_{i\to\infty} \mathrm{R}_{ij(i)}^{(1)} = -\infty$.

Proof. — We prove the lemma in Case (S4'), where the notation is simpler, i.e., we can use (s, t) as global coordinates on $\widetilde{\mathrm{F}}_{ij}$. In this case $l_0 = 2$. Case (S4") is conceptually the same.

Fix a sequence $\eta_i \to 0$. Then, by Lemma 7.8.1, for each i there exist κ_i', κ_i and $\overline{\mathrm{R}}_i$ which satisfy:

- $\kappa_i', \kappa_i < \eta_i$ and $\frac{\kappa_i'}{\kappa_i} > \frac{1}{\eta_i}$; and
- $\left|\overline{\pi} \circ \widetilde{v}_i(-\overline{\mathrm{R}}_i, t) - \kappa_i' f_i(t)\right| \leq \frac{\kappa_i}{2}$ for all t, where f_i is the normalized asymptotic eigenfunction for a negative end $\widetilde{v}_i$ of a $\overline{\mathrm{J}}'_{m_i}$-holomorphic curve $\widehat{v}_i$ of ECH index $\mathrm{I} \leq 2$ that limits to δ_0. The curve $\widehat{v}_i$ is a component of the SFT limit of the sequence $\{\overline{u}_{ij}\}_j$.

On the other hand, by (S3) and Definition 6.5.2, for all i there exists $j(i)$ such that if $j > j(i)$ then there exists R'_{ij} such that

$$\left|\overline{\pi} \circ \overline{u}_{ij}(\mathrm{R}'_{ij} - \overline{\mathrm{R}}_i, t) - \overline{\pi} \circ \widetilde{v}_i(-\overline{\mathrm{R}}_i, t)\right| \leq \frac{\kappa_i}{2}$$

for all t. Setting $\mathrm{R}_{ij}^{(1)} = \mathrm{R}'_{ij(i)} - \overline{\mathrm{R}}_i$, we obtain:

$$\left|\overline{\pi} \circ \overline{u}_{ij}(\mathrm{R}_{ij}^{(1)}, t) - \kappa_i' f_i(t)\right| \leq \left|\overline{\pi} \circ \widetilde{u}_{ij}(\mathrm{R}_{ij}^{(1)}, t) - \overline{\pi} \circ \widetilde{v}_i(-\overline{\mathrm{R}}_i, t)\right| + \left|\overline{\pi} \circ \widetilde{v}_i(-\overline{\mathrm{R}}_i, t) - \kappa_i' f_i(t)\right| \leq \kappa_i.$$

Similar considerations also hold for the t-derivative and at the negative truncated end. □

Remark **7.8.5.** — As $i \to \infty$, the curves $\widehat{v}_i$ approach one of the $\mathcal{C}_{i_0}$ from Section 7.7.4, modulo $\mathbf{R}$-translation, by Lemma 7.7.9. Hence the normalized asymptotic eigenfunctions f_i limit to the normalized eigenfunction $f_{i_0 j_0}$ for $\mathcal{C}_{i_0}$ at the negative end as $i \to \infty$.

7.8.2. *Ansatz.* — We define $\widetilde{z}_i = \overline{\pi} \circ \widetilde{u}_{ij(i)}$ and, to simplify computations, we will make the ansatz

$$(\mathbf{7.8.1}) \qquad \widetilde{z}_i = e^{-\varepsilon_i s} \widetilde{w}_i,$$

where $\varepsilon_i = \frac{\pi}{m_i}$.

Lemma **7.8.6.** — *The functions $\widetilde{w}_i : \widetilde{F}_i \to \mathbf{C}$ defined by Equation (7.8.1) are holomorphic with respect to the standard complex structure on $\mathbf{C}$.*

Proof. — We give the proof for Case (S4"); Case (S4') is similar but simpler. The functions (s, t) give local conformal coordinates on $\widetilde{F}_i$ outside the branch locus because $p_i : \widetilde{F}_i \to B_-$ is holomorphic. Then $\widetilde{z}_i : \widetilde{F}_i \to \mathbf{C}$ satisfies Equation (7.7.1):

$$\partial_s \widetilde{z}_i + i \partial_t \widetilde{z}_i + \varepsilon_i \widetilde{z}_i = 0.$$

If we plug in Equation (7.8.1) into Equation (7.7.1) we obtain:

$$-\varepsilon_i e^{-\varepsilon_i s} \widetilde{w}_i + e^{-\varepsilon_i s} \partial_s \widetilde{w}_i + i e^{-\varepsilon_i s} \partial_t \widetilde{w}_i + \varepsilon_i e^{-\varepsilon_i s} \widetilde{w}_i = 0.$$

Hence $\partial_s \widetilde{w} + i \partial_t \widetilde{w}_i = 0$, so $\widetilde{w}_i$ is a holomorphic map to the standard complex line $(\mathbf{C}, i)$ in the complement of the branch point. Then it is holomorphic everywhere by the removal of singularities. □

7.8.3. *Rescaling.* — Consider the diagonal subsequence $\overline{u}_i = \overline{u}_{ij(i)}$, $i = 1, 2, \dots$. We abbreviate $\widetilde{F}_i = \widetilde{F}_{ij(i)}$, $p_i = p_{ij(i)}|_{\widetilde{F}_{ij(i)}}$, $\widetilde{\mathfrak{m}}_i = \widetilde{\mathfrak{m}}_{ij(i)}$, etc. Fix $R_0', R_0 > 0$ and let K_i be the connected component of $p_i^{-1}(B_- \cap \{-R_0' \leq s \leq R_0\})$ containing $\widetilde{\mathfrak{m}}_i$. By passing to a subsequence we may assume that the topologies of K_i and $\widetilde{F}_i - K_i$ are independent of i. In order to simplify the exposition, we will assume that this is the case for all i.

Definition **7.8.7.** — *We define* $C_i = \sup_{z \in K_i} |\widetilde{w}_i(z)|$ *and* $w_i = \widetilde{w}_i / C_i$.

The holomorphic maps $w_i : \widetilde{F}_i \to \mathbf{C}$ satisfy the following properties:

(1) $\sup_{z \in K_i} |w_i(z)| = 1$;
(2) there is a unique point $\widetilde{\mathfrak{m}}_i \in \widetilde{F}_i$ such that $w_i(\widetilde{\mathfrak{m}}_i) = 0$; the zero $\widetilde{\mathfrak{m}}_i$ is a simple zero;

(3) if $x \in \partial \widetilde{F}_i$ and $p_i(x) = (s,t) \in \partial B_-$, then $w_i(x) \in e^{i(\varepsilon_i t + \phi_i)}\mathbf{R}^+$ with $\varepsilon_i = \frac{\pi}{m_i}$ and $\lim_{i\to\infty} \phi_i = 0$ for some ϕ_i depending on the boundary component of $\partial \widetilde{F}_i$ containing x;

(4) for every truncated end $\mathfrak{c}$ (i.e., a connected component of $\mathfrak{e}^+$ or $\mathfrak{e}^-$) there are constants $\widetilde{\kappa}_i$ and $\widetilde{\kappa}'_i$ such that the inequalities

$$\left\| w_i|_{\mathfrak{c}} - f_i \right\|_{C^1} \leq \widetilde{\kappa}_i, \quad \|f_i\|_{C^0} \geq \widetilde{\kappa}'_i \tag{7.8.2}$$

holds, where f_i is an asymptotic eigenfunction of δ or z_∞ (as appropriated) on $\mathfrak{c}$, and $\widetilde{\kappa}_i$ and the constants $\widetilde{\kappa}'_i$ satisfy $\widetilde{\kappa}'_i/\widetilde{\kappa}_i \to \infty$ as $i \to \infty$.

Remark **7.8.8.** — The loops of $w_i|_{\mathfrak{e}_i^\pm}$ have winding number 1 around the origin when i is sufficiently large: in fact $\widetilde{\kappa}'_i > \widetilde{\kappa}_i$, so the linear homotopy between $w_i|_{\mathfrak{e}_i^\pm}$ and $f_i(t)$ is contained in $\mathbf{C}^\times$.

Lemma **7.8.9.** — *After passing to a subsequence, $w_i|_{K_i}$ converges to a nonconstant holomorphic function $w_\infty|_{K_\infty}$.*

Proof. — After passing to a subsequence we may assume that K_i converges to a Riemann surface K_∞ with boundary. Since $w_i|_{K_i}$ is uniformly bounded, the lemma follows from Montel's theorem. □

From now on we assume that we have passed to a subsequence so that Lemma 7.8.9 holds. In the following subsections we analyze the convergence of w_i, not just on the uniformly bounded part K_i. This involves techniques of SFT compactness.

7.8.4. *Energy bound.* — Following Hofer [Ho], we define an *energy* for holomorphic functions on Riemann surfaces with boundary and punctures.

Definition **7.8.10** *(Energy).* — *Let $\mathcal{C}$ be the set of smooth functions $\varphi : [0,\infty) \to [0,1]$ such that $\varphi(0) = 0$ and $\varphi'(r) \geq 0$ for all $r \in [0,\infty)$. If F is a Riemann surface and $u : F \to \mathbf{C}$ a holomorphic function, we define the* energy *of u as*

$$E(u) = \sup_{\varphi \in \mathcal{C}} \int_F u^* d(\varphi(r) d\theta),$$

where (r,θ) are polar coordinates on $\mathbf{C}$.

Remark **7.8.11.** — If we identify $\mathbf{C}^\times \xrightarrow{\sim} \mathbf{R} \times S^1$ using the log map, then

$$\sup_{\varphi \in \mathcal{C}} \int_{F - u^{-1}(0)} u^* d(\varphi(r) d\theta)$$

agrees with the usual expression for the Hofer energy of

$$\log\circ u : \mathrm{F} - u^{-1}(0) \to \mathbf{R}\times \mathrm{S}^1.$$

Lemma **7.8.12.** — *The sequence* w_i *has uniformly bounded energy.*

Proof. — By Stokes' theorem,

$$\int_{\widetilde{\mathrm{F}}_i} w_i^*(d\varphi(r)d\theta) = \int_{\partial\widetilde{\mathrm{F}}_i} w_i^*(\varphi(r)d\theta).$$

The boundary $\partial\widetilde{\mathrm{F}}_i$ is the union of $\mathfrak{e}_i^\pm$, $\mathfrak{f}_{ik} = p_i^{-1}(\{t=k\})\cap\partial\widetilde{\mathrm{F}}_i$, $k=0,1$, and $\mathfrak{f}_{i2} = p_i^{-1}(\{1 < t < 2\})\cap\partial\widetilde{\mathrm{F}}_i$, all oriented using the boundary orientation. If $\mathfrak{c}$ is a component of $\mathfrak{e}_i^\pm$, then $\int_{\mathfrak{c}} w_i^* d\theta = 2\pi\deg(w_i|_{\mathfrak{c}})$ if $\mathfrak{c}$ is a circle and $\int_{\mathfrak{c}} w_i^*d\theta \approx \frac{\pi}{m}$ if $\mathfrak{c}$ is an arc because $w_i|_{\mathfrak{c}}$ is C^l-close to an asymptotic eigenfunction with $l\geq 1$ (compared to the C^0 norm of the eigenfunction). This follows from the exponential decay estimates of [HWZ1]; also see [HT2, Lemma 2.3]. Moreover, for $k=0,1$, $\int_{\mathfrak{f}_{ik}} w_i^*(\varphi(r)d\theta)=0$ since $w|_{\mathfrak{f}_{ik}}$, $k=0,1$, projects to a radial ray. Finally, $\int_{\mathfrak{f}_{i2}} w_i^*(\varphi(r)d\theta) < 0$ since $w|_{\mathfrak{f}_{i2}}$ always has a component in the negative θ-direction; here it is important to remember that we are projecting using balanced coordinates. This proves the lemma. □

Remark **7.8.13.** — Observe that $\mathrm{E}(cu)=\mathrm{E}(u)$ where $c\in\mathbf{C}^\times$.

7.8.5. *Bubbling.* — The goal of this subsection is to eliminate certain types of bubbling.

Let $(\widetilde{\mathrm{F}}^i, \widetilde{g}_i)$, $i\in\mathbf{N}$, be a sequence of Riemannian surfaces which are compatible with the complex structures and have injectivity radii which are uniformly bounded below. For example, one could obtain the metrics $\widetilde{g}_i$ by scaling the compatible hyperbolic metrics by a conformal factor so that the thick parts remain hyperbolic, while the thin parts become flat cylinders. We will use these metrics to compute the pointwise norm of the differential of the functions w_i which are denoted by $|w_i'|$.

Lemma **7.8.14.** — *For every compact set* $\mathrm{K}\subset\mathrm{B}_-$, *there is a constant* $\mathrm{C_K}>0$ *such that* $|w_i(z)| < \mathrm{C_K}$ *and* $|w_i'(z)| < \mathrm{C_K}$ *for all* $z\in p_i^{-1}(\mathrm{K})$ *and all* $i\in\mathbf{N}$.

Proof. — The lemma holds for $\mathrm{K}\cap\mathrm{K}_i$ by Lemma 7.8.9. Hence it suffices to consider $\mathrm{K}\cap\mathrm{K}_i^c$, where $\mathrm{K}_i^c := \widetilde{\mathrm{F}}_i - int(\mathrm{K}_i)$. Since $w_i(\mathrm{K}_i^c)\subset\mathbf{C}^\times$, we compose with $\log:\mathbf{C}^\times\to\mathbf{R}\times(\mathbf{R}/2\pi\mathbf{Z})$ and consider $\zeta_i = \log\circ w_i|_{\mathrm{K}_i^c}$. The sequence ζ_i has uniformly bounded energy by Lemma 7.8.12. If $|\zeta_i'|$, $i\in\mathbf{N}$, is not uniformly bounded on K_i^c, then the usual bubbling analysis yields a nonconstant finite energy plane $\widetilde{v}_\infty^+ : \mathbf{C}\to\mathbf{R}\times(\mathbf{R}/2\pi\mathbf{Z})$ or half-plane $\widetilde{v}_\infty^+:\mathbf{H}\to\mathbf{R}\times(\mathbf{R}/2\pi\mathbf{Z})$; see the proof of [Ho, Theorem 31]. If $\widetilde{v}_\infty^+$ is a half-plane, then $\widetilde{v}_\infty^+(\partial\mathbf{H})$ is contained in a line $\mathbf{R}\times\{pt\}\subset\mathbf{R}\times(\mathbf{R}/2\pi\mathbf{Z})$ as a consequence of the boundary conditions for the maps w_i, and we can double $\widetilde{v}^+_\infty$ to obtain a nonconstant finite energy plane by the Schwarz reflection principle. On the other hand, by [Ho, Lemma 28], there are no nonconstant finite energy planes in $\mathbf{R}\times(\mathbf{R}/2\pi\mathbf{Z})$, a contradiction. Hence $|\zeta_i'|$, $i\in\mathbf{N}$, is uniformly bounded on K_i^c. This in turn implies uniform bounds on $|w_i|$ and $|w_i'|$ on $K\cap K_i^c$. □

7.8.6. *Case (S4'). — In Cases (S4') and (S4") we often pass to a subsequence without explicit mention.*

Suppose we are in Case (S4'). Recall the compactification $cl(B_-)$ of B_- defined in Section 5.1.1, obtained by adjoining the points at infinity $\mathfrak{p}_+,\mathfrak{p}_-$. Here $cl(B_-)$ is isomorphic to the closed unit disk, $\mathfrak{p}_+$ is a marked point in the interior and $\mathfrak{p}_-$ is a marked point on the boundary under this identification.

Theorem **7.8.15.** — *The sequence $(w_i)_{i\in\mathbf{N}}$ converges uniformly on compact subsets to a holomorphic map $w_\infty: B_-\to\mathbf{C}$ such that:*

1. $w_\infty(\partial B_-)\subset\mathbf{R}^+$ *and* $w_\infty(\overline{\mathfrak{m}}^b)=0$;
2. $\lim_{s\to+\infty}|w_\infty(s,t)|=+\infty$ *and* $\lim_{s\to+\infty}\dfrac{w_\infty(s,t)}{|w_\infty(s,t)|}=f_{i_0j_0}(t)$, *for some* i_0,j_0;
3. $\lim_{s\to-\infty}w_\infty(s,t)=c\in\mathbf{R}^+$;
4. $w_\infty|_{int(B_-)}$ *is a biholomorphism onto its image. In particular* $\overline{\mathfrak{m}}^b$ *is the unique zero of* $w_\infty|_{cl(B_-)}$ *and is simple;*
5. w_∞ *extends to a holomorphic map* $cl(B_-)\to\mathbf{CP}^1$, *still called* w_∞, *such that* $w_\infty(\partial cl(B_-))=[a_1,a_2]\subset\mathbf{R}^+$ *and* $w_\infty(\mathfrak{p}_+)=\infty$.

Recall that $f_{i_0j_0}(t)$ is a normalized asymptotic eigenfunction of the curve $\mathcal{C}_{i_0}$ with a negative end asymptotic to a multiple of δ_0.

Proof. — The uniform convergence on compact subsets is a consequence of the bounds from Lemma 7.8.14. (1) is immediate from the convergence, together with the observation that the angles between the arcs in $\overline{\mathbf{a}}$ tend to 0 as $m_i\to\infty$.

(2) We can either use SFT compactness and analyze the levels, or argue directly as follows: Let K_i^+ (resp. K_i^-) be the component of $\widetilde{F}_i-int(K_i)$ with positive (resp. negative) s-coordinates. We expand w_i and w_∞ in Fourier series on K_i^+:

$$w_i(s,t)=\sum_{n=-\infty}^{+\infty}a_n^ie^{\pi n(s+it)},\qquad w_\infty(s,t)=\sum_{n=-\infty}^{+\infty}a_n^\infty e^{\pi n(s+it)}.$$

By the uniform convergence of w_i to w_∞, $\lim_{i\to\infty}a_n^i=a_n^\infty$ for all n. Then (2) is equivalent to the conditions:

(i) $\lim_{i\to\infty}\frac{a_1^i}{|a_1^i|}e^{i\pi t}=f_{i_0j_0}(t)$; and

(ii) $\lim_{i\to\infty} a_n^i = 0$ when $n \geq 2$.

Since we are in Case (S4'), $\{\mathrm{R}_i^{(l)}\}_{l=1}^{l_0} = \{\mathrm{R}_i^{(1)}, \mathrm{R}_i^{(2)}\}$, i.e., $l_0 = 2$. By Equation (7.8.2),

$$\left|\frac{1}{2}\int_0^2 \left(w_i(\mathrm{R}_i^{(2)}, t) - f_i(t)\right) e^{-\pi int} dt\right| \leq \widetilde{\kappa}_i,$$

for all $n \in \mathbf{Z}$. On the other hand, if we write $f_i(t) = c_i e^{\pi it}$, where $f_i(t)$ is not necessarily normalized, then

$$\frac{1}{2}\int_0^2 \left(w_i(\mathrm{R}_i^{(2)}, t) - f_i(t)\right) e^{-\pi int} dt = a_n^i e^{\pi n \mathrm{R}_i^{(2)}} - c_i \delta_{1n},$$

where δ_{1n} is the Kronecker delta. Hence

(a) $|a_n^i| \cdot e^{\pi n \mathrm{R}_i^{(2)}} \leq \widetilde{\kappa}_i$ for all $n \neq 1$; and
(b) $|a_1^i e^{\pi \mathrm{R}_i^{(2)}} - c_i| \leq \widetilde{\kappa}_i$.

We prove (ii). Arguing by contradiction, suppose that $\lim_{i\to\infty} |a_n^i| \neq 0$ for some $n \geq 2$. By (a), there exists $\mathrm{C} > 0$ such that $\mathrm{C}e^{2\pi \mathrm{R}_i^{(2)}} < \widetilde{\kappa}_i$ for all i. Since $\mathrm{R}_i^{(2)} \to \infty$ and $\lim_{i\to\infty} a_1^i$ exists, we obtain $|a_1^i e^{\pi \mathrm{R}_i^{(2)}}| < \widetilde{\kappa}_i$ for all i. On the other hand, $|c_i| \geq \widetilde{\kappa}'_i$ and $\lim_{i\to\infty} \frac{\widetilde{\kappa}'_i}{\widetilde{\kappa}_i} = +\infty$, which contradicts (b).

Next we prove (i). We claim that $\lim_{i\to\infty} a_1^i \neq 0$ for topological reasons. Indeed, if $\lim_{i\to\infty} a_1^i = 0$, then $\lim_{i\to\infty} a_n^i = 0$ for all $n \geq 1$ by (ii) and the curve $w_i|_{s=\mathrm{R}_i^{(2)}}$ has nonpositive winding number around 0 when $i \gg 0$, a contradiction. This proves the claim. Finally, $\lim_{i\to\infty} \left| a_1^i \cdot \frac{e^{\pi \mathrm{R}_i^{(2)}}}{|c_i|} - \frac{c_i}{|c_i|} \right| = 0$, since $\lim_{i\to\infty} \frac{\widetilde{\kappa}_i}{\widetilde{\kappa}'_i} = \lim_{i\to\infty} \frac{\widetilde{\kappa}_i}{|c_i|} = 0$. Hence

$$\lim_{i\to\infty} \frac{a_1^i}{|a_1^i|} e^{\pi it} = f_{i_0 j_0}(t),$$

which proves (i).

(3) This is similar to (2). We expand w_i in Fourier series on K_i^-:

$$w_i(s,t) = \sum_{-\infty}^{+\infty} a_n^i e^{\varepsilon_i i} e^{(\pi n - \varepsilon_i)(s+it)}.$$

By the uniform convergence, $\lim_{s\to-\infty} a_n^i = a_n^\infty$ and we can similarly prove that $a_n^\infty = 0$ for all $n < 0$. This implies (3) because the normalized eigenfunctions converge to a constant as $i \to +\infty$.

(4) Since $H_2(cl(B_-), \partial cl(B_-)) \cong H_2(\mathbf{CP}^1, [a_1, a_2]) \cong \mathbf{Z}$, we have a well-defined notion of degree for w_∞. Moreover, as in the closed case, the degree is equal to the cardinality of the inverse image of a regular value in $\mathbf{CP}^1 - [a_1, a_2]$. Hence $\deg w_\infty|_{int(B_-)} = 1$ because $w_\infty^{-1}(\infty) = \{0\}$ and 0 is a simple pole. This implies that $w_\infty : int(B_-) \to \mathbf{C} - [a_1, a_2]$ is a biholomorphism.

(5) follows from (2) and (3). □

7.8.7. *Case (S4").* — We give a brisk treatment of the construction of the limit, mostly pointing out the differences with (S4'). The main difference is that the holomorphic maps $\overline{u}_i$ admit good truncations with a component which is a branched double cover over its image with a single branch point $b_i \in B_-$, and we must analyze different cases depending on the behavior of the branch point as $i \to \infty$.

There are five cases:

(a) $\lim_{i\to\infty} b_i = b_\infty \in int(B_-)$,
(b) $\lim_{i\to\infty} b_i = b_\infty \in \partial B_-$,
(c) $\lim_{i\to\infty} s(b_i) = +\infty$,
(d') $\lim_{i\to\infty} s(b_i) = -\infty$ and $d(b_i, \partial B_-) \geq C$, where $C > 0$ is a constant, or
(d") $\lim_{i\to\infty} s(b_i) = -\infty$ and $d(b_i, \partial B_-) \to 0$.

The sequence $p_i : \widetilde{F}_i \to B_-$ converges in the SFT sense to a 1- or 2-level holomorphic building $p_\infty = p^0_\infty, p^0_\infty \cup p^1_\infty$, or $p^{-1}_\infty \cup p^0_\infty$, where $p^{-1}_\infty, p^0_\infty, p^1_\infty$ are branched covers of B, B_-, B', respectively. We have a 1-level building in Cases (a) and (b) and a 2-level building in Cases (c), (d'), and (d"). Let $p^{\mathfrak{m}}_\infty : \widetilde{F}^{\mathfrak{m}}_\infty \to B_-$ be the restriction of p^0_∞ to the component containing the limit $\widetilde{\mathfrak{m}}_\infty$ of $\widetilde{\mathfrak{m}}_i$ and let $p^b_\infty : \widetilde{F}^b_\infty \to B$ or B' be the restriction of $p^{-1}_\infty, p^0_\infty$, or p^1_∞ to the component containing the limit b_∞ of b_i. If $p^{\mathfrak{m}}_\infty = p^b_\infty$ (i.e., in Cases (a) and (b)) we drop the superscripts.

Near all the punctures of $\widetilde{F}^\star_\infty$, $\star \in \{\mathfrak{m}, b\}$, we use cylindrical or rectangular coordinates (s, t) induced by the coordinates (s, t) on B_- by pullback, after a possible translation in the s-direction.

For $* \in \{\varnothing, ', -\}$, we denote the compactification of B^*, obtained by adjoining the points at infinity $\mathfrak{p}^*_\pm$, by $cl(B^*)$. Similarly, let $cl(\widetilde{F}^\star_\infty)$, $\star \in \{\mathfrak{m}, b\}$, be the compactification of $\widetilde{F}^\star_\infty$, obtained by adjoining $\mathfrak{q}^\star_{\pm,j}$, where $j \in \{1\}$ or $\{1, 2\}$, depending on the number of ends. (If there is only one end, we suppress the index.) The maps $p^\star_\infty : \widetilde{F}^\star_\infty \to B^*$, can be compactified to $p^\star_\infty : cl(\widetilde{F}^\star_\infty) \to cl(B^*)$, where $\mathfrak{q}^\star_{\pm,j}$ is mapped to $\mathfrak{p}^*_\pm$.

In Case (a), $\widetilde{F}_\infty$ is an annulus with a puncture $\mathfrak{q}_+$ in the interior and a puncture $\mathfrak{q}_{-,j}$ on each boundary component. In Case (b), $\widetilde{F}_\infty$ is a disk with one puncture $\mathfrak{q}_+$ in the interior, two punctures $\mathfrak{q}_{-,j}, j = 1, 2$, on the boundary, and two boundary points $b_{\infty,j}$, $j = 1, 2$, which are glued together to give b_∞. The points $\mathfrak{q}_{-,j}$ and $b_{\infty,j}$ alternate along the boundary. In Case (c), $\widetilde{F}^{\mathfrak{m}}_\infty$ is a disk with a puncture $\mathfrak{q}^{\mathfrak{m}}_+$ in the interior and a puncture $\mathfrak{q}^{\mathfrak{m}}_-$

on the boundary, and $\widetilde{F}^b_\infty$ is a sphere with three punctures $\mathfrak{q}^b_+$, $\mathfrak{q}^b_{-,j}$, $j=1,2$. In Case (d'), $\widetilde{F}^{\mathfrak{m}}_\infty$ is a disk with a puncture $\mathfrak{q}^{\mathfrak{m}}_+$ in the interior and two punctures $\mathfrak{q}^{\mathfrak{m}}_{-,j}$ on the boundary and $\widetilde{F}^b_\infty$ is a disk with four punctures $\mathfrak{q}^b_{\pm,j}$ on the boundary. In Case (d"), $\widetilde{F}^{\mathfrak{m}}_\infty$ is a disk with a puncture $\mathfrak{q}^{\mathfrak{m}}_+$ in the interior and two punctures $\mathfrak{q}^{\mathfrak{m}}_{-,j}$ on the boundary and $\widetilde{F}^b_\infty$ is a disk with four punctures $\mathfrak{q}^b_{\pm,j}$ on the boundary and two boundary points $b_{\infty,j}$, $j=1,2$, identified.

Cases (a) and (b) are similar to Case (S4'), while the situation in Cases (c), (d') and (d") is complicated by the fact that the limit is a 2-level holomorphic building.

Cases (a) and (b).

Theorem **7.8.16.** — *Suppose* $\lim_{i\to\infty} b_i = b_\infty \in B_-$. *Then* $(w_i)_{i\in\mathbf{N}}$ *converges to a holomorphic map* $w_\infty : \widetilde{F}_\infty \to \mathbf{C}$ *such that:*

(1) $w_\infty(\partial\widetilde{F}_\infty) \subset \mathbf{R}^+$;

(2) *at the positive puncture* $\mathfrak{q}_+$,

$$\lim_{s\to+\infty} |w_\infty(s,t)| = +\infty, \qquad \lim_{s\to+\infty} \frac{w_\infty(s,t)}{|w_\infty(s,t)|} = f_{i_0 j_0}(t);$$

(3) *at the negative punctures* $\mathfrak{q}_{-,j}$, $j=1,2$, $\lim_{s\to-\infty} w_\infty(s,t) = c_j \in \mathbf{R}^+$;

(4a) *in Case (a),* w_∞ *extends to a holomorphic map* $w_\infty : cl(\widetilde{F}_\infty) \to \mathbf{CP}^1$ *such that* $w_\infty : int(cl(\widetilde{F}_\infty)) \to \mathbf{CP}^1 - ([a_1,a_2] \sqcup [a_3,a_4])$ *is a biholomorphism;*

(4b) *in Case (b),* w_∞ *extends to a holomorphic map* $w_\infty : cl(\widetilde{F}_\infty) \to \mathbf{CP}^1$ *such that* $w_\infty : int(cl(\widetilde{F}_\infty)) \to \mathbf{CP}^1 - [a_1,a_2]$ *is a biholomorphism;*

(5) $\widetilde{\mathfrak{m}}_\infty$ *is the unique zero of* $w_\infty|_{int(cl(\widetilde{F}_\infty))}$ *and is simple;* $p_\infty(\widetilde{\mathfrak{m}}_\infty) = \overline{\mathfrak{m}}^b$.

Proof. — The proof of Theorem 7.8.15 goes through without modification to give (1)–(3).

(4a) As in the proof of Theorem 7.8.15(4), we can define the degree for maps of pairs $(cl(\widetilde{F}_\infty), \partial cl(\widetilde{F}_\infty)) \to (\mathbf{CP}^1, \mathbf{R}^+)$. The degree of w_∞ is 1 because it has a unique pole of order 1. The order of the pole at the positive puncture can be computed from the winding number of $f_{i_0 j_0}$, which is the smallest one for a positive eigenvalue by Lemma 7.7.5. Then w_∞ can have no branch points in the interior of $cl(\widetilde{F}_\infty)$. (4b) is similar. (5) follows from (4a) and (4b). □

Cases (c), (d') and (d"). When the sequence $\{b_i\}$ is unbounded, there are surfaces $\widetilde{F}^\star_i$, $\star \in \{\mathfrak{m}, b\}$, with embeddings $\iota^\star_i : \widetilde{F}^\star_i \to \widetilde{F}_i$, such that $\widetilde{F}^\star_i$ converges to $\widetilde{F}^\star_\infty$. Let $p^{\mathfrak{m}}_i : \widetilde{F}^{\mathfrak{m}}_i \to B_-$ be the restriction of p_i and let $p^b_i : \widetilde{F}^b_i \to B^*$, $* = \varnothing$ or $'$, be the composition of an s-translation and $p_i|_{\widetilde{F}^b_i}$ so that s-coordinate of $p^b_i(b_i)$ is zero.

Let w_i be as in Definition 7.8.7. Let

$$K^b_i = (p_i \circ \iota^b_i)^{-1}(B_- \cap \{s(b_i) - 1 \le s \le s(b_i) + 1\}) \subset \widetilde{F}^b_i$$

and $C_i^b = \sup_{z\in K_i^b} |w_i(\iota_i^b(z))|$. Then we set $w_i^{\mathfrak{m}} = w_i \circ \iota_i^{\mathfrak{m}}$ and $w_i^b = (w_i \circ \iota_i^b)/C_i^b$.

Lemma **7.8.17.** — *For every compact set $K \subset B^*$, there is a constant $C_K > 0$ such that $|w_i^\star(z)| < C_K$, and $|(w_i^\star)'(z)| < C_K$ for all $z \in (p_i^\star)^{-1}(K)$ and all $i \in \mathbf{N}$. Here $\star \in \{\mathfrak{m}, b\}$ and $* \in \{\varnothing, ', -\}$, as appropriate.*

Proof. — Similar to Lemma 7.8.14. □

Lemma 7.8.17 implies that the limits $w_\infty^{\mathfrak{m}}$ and w_∞^b exist. The following lemma gives the behavior of $w_\infty^{\mathfrak{m}}$ and w_∞^b near the punctures.

Lemma **7.8.18.** —

(1) *Let $u : \mathbf{R}^+ \times (\mathbf{R}/\pi\mathbf{Z}) \to \mathbf{C}^\times$ be a finite energy holomorphic map. If the map $t \mapsto u(s,t)$ has degree one for some (and therefore all) $s \in \mathbf{R}^+$, then $\lim_{s\to+\infty} |u_\infty(s,t)| = +\infty$ and $\lim_{s\to+\infty} \frac{u_\infty(s,t)}{|u_\infty(s,t)|} = ce^{\pi i t}$ with $c \neq 0$.*

(2) *Let $u : \mathbf{R}^- \times (\mathbf{R}/\pi\mathbf{Z}) \to \mathbf{C}^\times$ be a finite energy holomorphic map. If the map $t \mapsto u(s,t)$ has degree one for some (and therefore all) $s \in \mathbf{R}^-$, then $\lim_{s\to-\infty} |u_\infty(s,t)| = 0$.*

Proof. — (1) Let us view u as a map $\mathbf{R}^+ \times (\mathbf{R}/\pi\mathbf{Z}) \to \mathbf{R} \times (\mathbf{R}/\pi\mathbf{Z})$. As in the proof of Lemma 7.8.14, since u has finite energy, it has bounded derivative. Let $u_n(s,t) = u(s+k_n, t)$, where $k_n \in \mathbf{R}^+$ and $\lim_{n\to+\infty} k_n = +\infty$. The sequence u_n has uniformly bounded derivative and converges to a finite energy holomorphic map

$$u_\infty : \mathbf{R} \times (\mathbf{R}/\pi\mathbf{Z}) \to \mathbf{R} \times (\mathbf{R}/\pi\mathbf{Z}).$$

Such a holomorphic map is of the form $u_\infty(s,t) = (s+a, t+b)$, where a, b are constants. This implies (1). (2) is similar and is left to the reader. □

Case (c).

Theorem **7.8.19.** — *Suppose $\lim_{i\to\infty} s(b_i) = +\infty$. Then $(w_i^{\mathfrak{m}})_{i\in\mathbf{N}}$ converges to a holomorphic map $w_\infty^{\mathfrak{m}} : \widetilde{F}_\infty^{\mathfrak{m}} \to \mathbf{C}$ such that:*

(1) *$w_\infty^{\mathfrak{m}}(\partial\widetilde{F}_\infty^{\mathfrak{m}}) \subset \mathbf{R}^+$;*

(2) *at the positive puncture $\mathfrak{q}_+^{\mathfrak{m}}$,*

$$\lim_{s\to+\infty} |w_\infty^{\mathfrak{m}}(s,t)| = +\infty, \qquad \lim_{s\to+\infty} \frac{w_\infty^{\mathfrak{m}}(s,t)}{|w_\infty^{\mathfrak{m}}(s,t)|} = f(t),$$

where f is a normalized eigenfunction of the asymptotic operator at δ_0 with winding number one;

(3) *at the negative puncture* $\mathfrak{q}_-^{\mathfrak{m}}$, $\lim_{s\to-\infty} w_\infty^{\mathfrak{m}}(s,t) = c \in \mathbf{R}^+$;

(4) $w_\infty^{\mathfrak{m}}$ *extends to a holomorphic map* $w_\infty^{\mathfrak{m}} : cl(\widetilde{F}_\infty^{\mathfrak{m}}) \to \mathbf{CP}^1$ *such that* $w_\infty^{\mathfrak{m}} : int(cl(\widetilde{F}_\infty^{\mathfrak{m}})) \to \mathbf{CP}^1 - [a_1, a_2]$ *is a biholomorphism; and*

(5) $\overline{\mathfrak{m}}^b$ *is the unique zero of* $w_\infty^{\mathfrak{m}}|_{int(cl(\widetilde{F}_\infty^{\mathfrak{m}}))}$ *and is simple.*

Also $(w_i^b)_{i\in\mathbf{N}}$ *converges to a holomorphic map* $w_\infty^b : \widetilde{F}_\infty^b \to \mathbf{C}$ *such that:*

(6) *at the positive puncture* $\mathfrak{q}_+^b$,

$$\lim_{s\to+\infty} |w_\infty^b(s,t)| = +\infty, \qquad \lim_{s\to+\infty} \frac{w_\infty^b(s,t)}{|w_\infty^b(s,t)|} = f_{i_0 j_0}(t);$$

(7) *at the negative puncture* $\mathfrak{q}_{-,1}^b$ *that connects to* $\mathfrak{q}_+^{\mathfrak{m}}$, $\lim_{s\to-\infty} w_\infty^b(s,t) = 0$;

(8) *at the other negative puncture* $\mathfrak{q}_{-,2}^b$, $\lim_{s\to-\infty} w_\infty^b(s,t) = c \in \mathbf{R}^+$;

(9) *at the punctures* $\mathfrak{q}_+^{\mathfrak{m}}$ *and* $\mathfrak{q}_{-,1}^b$, $\lim_{s\to+\infty} \frac{w_\infty^{\mathfrak{m}}(s,t)}{|w_\infty^{\mathfrak{m}}(s,t)|} = \lim_{s\to-\infty} \frac{w_\infty^b(s,t)}{|w_\infty^b(s,t)|}$;

(10) w_∞^b *extends to a biholomorphism* $w_\infty^b : \mathbf{CP}^1 \to \mathbf{CP}^1$*; and*

(11) $\mathfrak{q}_{-,1}^b$ *is the unique zero of* w_∞^b *and is simple.*

Proof. — The proof of Theorem 7.8.15 goes through with little modification, in view of Lemma 7.8.18. We remark that (8) is a consequence of Convention 6.6.4 and the proof technique of Lemma 6.6.5. □

Cases (d') and (d").

Theorem **7.8.20.** — *Suppose* $\lim_{i\to\infty} s(b_i) = -\infty$. *Then* $(w_i^{\mathfrak{m}})_{i\in\mathbf{N}}$ *converges to a holomorphic map* $w_\infty^{\mathfrak{m}} : \widetilde{F}_\infty^{\mathfrak{m}} \to \mathbf{C}$ *such that:*

(1) $w_\infty^{\mathfrak{m}}(\partial\widetilde{F}_\infty^{\mathfrak{m}}) \subset \mathbf{R}^+$;

(2) *at the positive puncture* $\mathfrak{q}_+^{\mathfrak{m}}$,

$$\lim_{s\to+\infty} |w_\infty^{\mathfrak{m}}(s,t)| = +\infty, \qquad \lim_{s\to+\infty} \frac{w_\infty^{\mathfrak{m}}(s,t)}{|w_\infty^{\mathfrak{m}}(s,t)|} = f_{i_0 j_0}(t);$$

(3) *at the negative punctures* $\mathfrak{q}_{-,j}^{\mathfrak{m}}$, $j = 1, 2$, $\lim_{s\to-\infty} w_\infty^{\mathfrak{m}}(s,t) = c_j \in \mathbf{R}^+$;

(4) $w_\infty^{\mathfrak{m}}$ *extends to a holomorphic map* $w_\infty^{\mathfrak{m}} : cl(\widetilde{F}_\infty^{\mathfrak{m}}) \to \mathbf{CP}^1$ *such that* $w_\infty^{\mathfrak{m}} : int(cl(\widetilde{F}_\infty^{\mathfrak{m}})) \to \mathbf{CP}^1 - [a_1, a_2]$ *is a biholomorphism;*

(5) $\widetilde{\mathfrak{m}}_\infty$ *is the unique zero of* $w_\infty^{\mathfrak{m}}|_{int(cl(\widetilde{F}_\infty^{\mathfrak{m}}))}$ *and is simple.*

Also $(w_i^b)_{i\in\mathbf{N}}$ *converges to a constant map* $w_\infty^b : \widetilde{F}_\infty^b \to \mathbf{C}$.

7.9. *Involution lemmas.* — In this subsection we collect some lemmas on holomorphic maps between Riemann surfaces with anti-holomorphic involutions. These lemmas, collectively referred to as the *involution lemmas*, will play an important role in Section 7.10 and in the sequel [II].

Our starting point is the following observation, whose proof is straightforward.

Observation **7.9.1.** — *Let Σ_1, Σ_2 be Riemann surfaces with anti-holomorphic involutions ι_1, ι_2, respectively. If $f : \Sigma_1 \to \Sigma_2$ is a holomorphic map, then $\widetilde{f} := \iota_2 \circ f \circ \iota_1$ is also holomorphic. Moreover, if $f = \widetilde{f}$, then $f(\mathrm{Fix}(\iota_1)) \subset \mathrm{Fix}(\iota_2)$, where $\mathrm{Fix}(\iota_i)$ is the fixed point set of ι_i.*

There are four versions of the involution lemma; the first two will be used in this paper and the last two only in the sequel [II]. We start by introducing some common notation which will be used in all four versions: For $i = 1, 2$, the Riemann surface Σ_i is an open subset of $\mathbf{CP}^1$ which is invariant under complex conjugation and has finitely generated fundamental group; moreover no component of $\mathbf{CP}^1 - \Sigma_i$ is a single point. The complex conjugation on $\mathbf{CP}^1$ restricts to an anti-holomorphic involution $\iota_i : \Sigma_i \to \Sigma_i$. On each Σ_i we fix "radial rays"

$$\mathcal{R}_i = \Sigma_i \cap (\mathbf{R}^{\leq 0} \cup \{\infty\}).$$

The *asymptotic marker* $\dot{\mathcal{R}}_i(0)$ is the connected component of $T_0\mathbf{RP}^1 - \{0\}$ (i.e., a tangent half-line) consisting of vectors with negative ∂_x-component; similarly, the asymptotic marker $\dot{\mathcal{R}}_i(\infty)$ is the component of $T_\infty\mathbf{RP}^1 - \{0\}$ that is mapped to $\dot{\mathcal{R}}_i(0)$ under the inversion $z \mapsto \frac{1}{z}$. The radial rays $\mathcal{R}_i$ and their related asymptotic markers are invariant under the involution ι_i. In this section we will use the notation $\mathbf{D}$ for the open unit disk in $\mathbf{C}$, considered as a Riemann surface.

Lemma **7.9.2.** — *Given Σ_i as above, there is a compact Riemann surface with boundary $\overline{\Sigma}_i$ with a biholomorphism $\Sigma_i \stackrel{\sim}{\to} int(\overline{\Sigma}_i)$. Moreover there is an anti-holomorphic involution $\iota_i : \overline{\Sigma}_i \to \overline{\Sigma}_i$ such that the diagram*

$$\begin{array}{ccc} \Sigma_i & \hookrightarrow & \overline{\Sigma}_i \\ \downarrow{\scriptstyle \iota_i} & & \downarrow{\scriptstyle \iota_i} \\ \Sigma_i & \hookrightarrow & \overline{\Sigma}_i \end{array}$$

commutes.

Sketch of proof. We outline the proof of the first statement. Use the uniformization theorem to identify the universal cover of $\Sigma = \Sigma_1$ or Σ_2 with the open upper half space $\mathbf{H}$. Let $G \subset PSL(2, \mathbf{R})$ be the deck transformation group of $\mathbf{H}$ such that $\mathbf{H}/G = \Sigma$. If G is finitely generated, then $(\partial\mathbf{H} - L)/G$ is a collection of boundary circles of Σ, where $L \subset \partial\mathbf{H}$ is the limit set of G. Hence $\overline{\Sigma} = (\overline{\mathbf{H}} - L)/G$. □

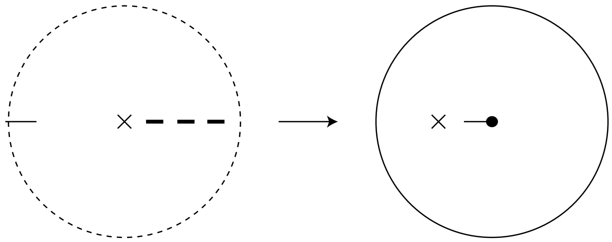

FIG. 5. — The map $f|_{\Sigma_1}$ in Lemma 7.9.3. The left-hand side is Σ_1, where the point at infinity is represented by the dotted circle, and the right-hand side is Σ_2. The thicker lines on left-hand side are the slits $[a_{2i-1}, a_{2i}]$ and the thinner lines are the asymptotic markers. The cross on the left is the origin and the cross on the right is $f(0) \in \mathcal{R}_2$. Lemma 7.9.3 states that f maps the asymptotic marker $\dot{\mathcal{R}}_1(\infty)$ at $\infty \in \Sigma_1$ to the asymptotic marker $\dot{\mathcal{R}}_2(0)$ at $0 \in \Sigma_2$

Note that $\overline{\Sigma}_i$ will not necessarily be the closure of Σ_i in $\mathbf{CP}^1$. However, when referring to points in the interior of $\overline{\Sigma}_i$, we denote them by the corresponding point in Σ_i. In the same way we view the radial rays $\mathcal{R}_i$ as subsets of $\overline{\Sigma}_i$, and the asymptotic markers as tangent half-lines to $\overline{\Sigma}_i$ (provided they make sense). Lemma 7.9.2 implies that they are invariant by the involution on $\overline{\Sigma}_i$.

The image of an asymptotic marker by a holomorphic function is defined by the differential at the regular points. At the singular points the local behavior of a holomorphic map still allows us to define the image of a tangent ray.

For the first version of the involution lemma, let

$$\Sigma_1 = \mathbf{CP}^1 - ([a_1, a_2] \cup \cdots \cup [a_{2p-1}, a_{2p}]),$$

where $a_i \in \mathbf{R}^+$ and $a_1 < \cdots < a_{2p}$, and let $\Sigma_2 = \mathbf{D}$. We write $\partial\overline{\Sigma}_1 = \partial_1\overline{\Sigma}_1 \sqcup \cdots \sqcup \partial_p\overline{\Sigma}_1$, where $\partial_i\overline{\Sigma}_1$, $i = 1, \dots, p$, corresponds to the slit $[a_{2i-1}, a_{2i}]$.

Lemma **7.9.3** *(Involution Lemma, Version 1). — Let $f : \overline{\Sigma}_1 \to \overline{\Sigma}_2$ be a holomorphic map which is a q-fold branched cover with $q \geq p$, such that:*

(i) $f(\partial_i\overline{\Sigma}_1) = \partial\overline{\Sigma}_2$, $i = 1, \dots, p$;
(ii) $f^{-1}(0) = \{\infty\}$; *and*
(iii) $f(0) \in \mathcal{R}_2$.

Then f maps $\mathrm{Fix}(\iota_1)$ *to* $\mathrm{Fix}(\iota_2)$ *and* $\dot{\mathcal{R}}_1(\infty)$ *to* $\dot{\mathcal{R}}_2(0)$. *(See Figure 5.)*

Proof. — We claim that $f = \widetilde{f} = \iota_2 \circ f \circ \iota_1$. To compare f and $\widetilde{f}$, we consider their quotient $Q = f/\widetilde{f}$. We observe three facts:

(1) Q has no poles, since f and $\widetilde{f}$ have zeros only at ∞ and their orders agree.
(2) $|Q(z)| = 1$ for all $z \in \partial\overline{\Sigma}_1$, so the maximum modulus theorem implies that $Q(\overline{\Sigma}_1) \subset \mathbf{D} = \{|z| \leq 1\}$.
(3) The degree of $Q|_{\partial_i\overline{\Sigma}_1}$, viewed as a map to $S^1 = \partial\mathbf{D}$, is zero, since

$$\deg(f|_{\partial_i\overline{\Sigma}_i}) = \deg(\widetilde{f}|_{\partial_i\overline{\Sigma}_i}).$$

If Q is not constant, then by (3) there must be a branch point of Q along $\partial_i\overline{\Sigma}_1$. In particular, $Q(\overline{\Sigma}_1)\not\subset \mathbf{D}$, which contradicts (2). Hence Q is a constant map and $f=c\widetilde{f}$ for some $c\in\mathbf{C}-\{0\}$. Now (iii) implies that $c=1$, and we have $f=\widetilde{f}$.

Finally we apply Observation 7.9.1 to conclude that f maps $\mathrm{Fix}(\iota_1)$ to $\mathrm{Fix}(\iota_2)$ and, by (iii), maps $\dot{\mathcal{R}}_1(\infty)$ to $\dot{\mathcal{R}}_2(0)$. □

The proofs of the other versions of the involution lemma are similar, and will be omitted.

Lemma **7.9.4** *(Involution Lemma, Version 2). — Let* $\Sigma_1=\Sigma_2=\mathbf{CP}^1$ *and* $a_i\in\mathbf{R}^{\geq 0}$ *with* $a_1=0<a_2<\cdots<a_p$. *Assume* $f:\Sigma_1\to\Sigma_2$ *is a holomorphic map which is a q-fold branched cover with* $q\geq p$, *such that:*

(i) $f^{-1}(\infty)=\{a_1,\dots,a_p\}$;
(ii) $f^{-1}(0)=\{\infty\}$; *and*
(iii) f *maps* $\dot{\mathcal{R}}_1(0)$ *to* $\dot{\mathcal{R}}_2(\infty)$.

Then f *maps* $\mathrm{Fix}(\iota_1)$ *to* $\mathrm{Fix}(\iota_2)$ *and* $\dot{\mathcal{R}}_1(\infty)$ *to* $\dot{\mathcal{R}}_2(0)$.

Lemma **7.9.5** *(Involution Lemma, Version 3). — Let* $\Sigma_1=\Sigma_2=\mathbf{D}$ *and* $a_i\in\mathbf{R}^{\geq 0}$ *with* $a_1=0<a_2<\cdots<a_p<1$. *Assume* $f:\overline{\Sigma}_1\to\overline{\Sigma}_2$ *is a holomorphic map which is a q-fold branched cover with* $q\geq p$, *such that:*

(i) $f^{-1}(0)=\{a_1,\dots,a_p\}$;
(ii) $f^{-1}(\partial\overline{\Sigma}_1)=\partial\overline{\Sigma}_2$; *and*
(iii) f *maps* $\dot{\mathcal{R}}_1(0)$ *to* $\dot{\mathcal{R}}_2(0)$.

Then f *maps* $\mathrm{Fix}(\iota_1)$ *to* $\mathrm{Fix}(\iota_2)$. *Moreover* $f(-1)=-1$.

For the fourth version of the involution lemma, we consider $\Sigma_1=\mathbf{D}-([a_1,a_2]\cup\cdots\cup[a_{2p-1},a_{2p}])$, where $a_i\in(0,1)$ with $a_1<\cdots<a_{2p}$, and $\Sigma_2=\{R<|z|<1\}$, where $0<R<1$. We write $\partial\overline{\Sigma}_1=\partial_0\overline{\Sigma}_1\sqcup\cdots\sqcup\partial_p\overline{\Sigma}_1$, where $\partial_0\overline{\Sigma}_1$ is the boundary component which can identified with $\{|z|=1\}$ and $\partial_i\overline{\Sigma}_1$, $i=1,\dots,p$, corresponds to the slit $[a_{2i-1},a_{2i}]$.

Lemma **7.9.6** *(Involution Lemma, Version 4). — Let* $f:\overline{\Sigma}_1\to\overline{\Sigma}_2$ *be a holomorphic map which is a q-fold branched cover with* $q\geq p$, *and* $\mathcal{I}\subset\{1,\dots,p\}$, *such that:*

(i) $f(\partial_i\overline{\Sigma}_1)=\{|z|=1\}$ *when* $i\in\mathcal{I}$;
(ii) $f(\partial_i\overline{\Sigma}_1)=\{|z|=R\}$ *when* $i=0$ *or* $i\notin\mathcal{I}$; *and*
(iii) $f(0)\in\mathcal{R}_2$.

Then f *maps* $\mathrm{Fix}(\iota_1)$ *to* $\mathrm{Fix}(\iota_2)$. *Moreover* $f(-1)=-R$.

See Figure 6.

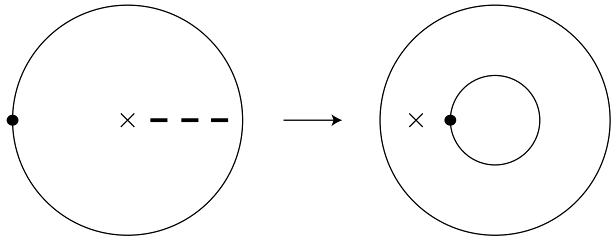

FIG. 6. — The map $f|_{\Sigma_1}$ in Lemma 7.9.6. The left-hand side is Σ_1 and the right-hand side is Σ_2. The thick lines on the left-hand side are the slits $[a_{2i-1}, a_{2i}]$. The cross on the left is the origin and the cross on the right is $f(0) \in \mathcal{R}_2$. The dots represent $-1 \in \partial\Sigma_1$ and $f(-1) \in \partial\Sigma_2$. Lemma 7.9.6 states that $f(-1) = -R$

7.10. *Elimination of some cases.* — We are now in a position to eliminate some of possibilities that appear in Theorem 7.6.1:

Theorem **7.10.1.** — *Suppose $m \gg 0$, $\varepsilon, \delta > 0$ are sufficiently small, $\overline{u}_\infty \in \partial\mathcal{M}^i_{\overline{\mathfrak{m}}}(\varepsilon, \delta, p)$, and $\overline{v}'_0 \neq \varnothing$.*

(i) *If $i = 2$, then the 3-level subbuilding described in Theorem 7.6.3 does not occur.*
(ii) *If $i = 3$, then Cases (2)–(6) of Theorem 7.6.1 do not occur.*

We briefly sketch the idea of the proof. In all the cases that are eliminated by Theorem 7.10.1, the (unique) component of $\cup_{j=1}^a \overline{v}_j^\sharp \subset \overline{u}_\infty$ which is asymptotic to a multiple of δ_0 at the negative end has ECH index $\mathrm{I} = 1$.

Suppose that, for $m \gg 0$, there is a sequence of holomorphic curves which converges to a configuration $\overline{u}_\infty$ that we want to exclude. In Section 7.8, we applied a rescaling argument to construct a holomorphic building which keeps track of how the limit $\overline{u}_\infty$ is approached; this is similar to the layer structures of Ionel-Parker [IP, Section 7]. In the simplest case, this building is a holomorphic map $w_\infty : B_- \to \mathbf{C}$ which satisfies the asymptotic condition

$$\lim_{s\to+\infty} \frac{w_\infty(s,t)}{|w_\infty(s,t)|} = f_{i_0 j_0}(t),$$

where $f_{i_0 j_0}(t)$ is a normalized asymptotic eigenfunction of an $\mathrm{I} = 1$ curve with a negative end asymptotic to δ_0. The condition $\mathrm{I} = 1$ is used as follows: since there are only finitely many $\mathrm{I} = 1$ curves with negative ends asymptotic to δ_0, we may assume that $-1 \notin \{f_{i_0 j_0}(\frac{3}{2})\}$ as explained in Section 7.7.5 and Remark 7.7.11.

Remark **7.10.2.** — In this subsection we identify $cl(B_-) \simeq \overline{\mathbf{D}}$ so that $\mathfrak{p}_+$ corresponds to 0 and $\mathfrak{p}_-$ corresponds to 1. There is an anti-holomorphic involution ι on B_- that fixes the half-line $\{t = \frac{3}{2}\}$, and $\{t = \frac{3}{2}\}$ corresponds to the radial ray $\mathcal{R} = \overline{\mathbf{D}} \cap \mathbf{R}^{\leq 0}$ by Observation 7.9.1. In particular, $\overline{\mathfrak{m}}^b$ is mapped to a point on $\mathcal{R}$.

Similarly we identify $cl(B') \simeq \mathbf{CP}^1$ so that $\mathfrak{p}_+$ corresponds to 0, $\mathfrak{p}_-$ corresponds to ∞, and $\{t = \frac{3}{2}\}$ corresponds to the radial ray $\mathbf{R}^{\leq 0}$, We now use the involutions lemmas from Section 7.9 to obtain a contradiction. By the involution lemmas and the symmetric placement of the basepoint $\overline{\mathfrak{m}}^b$, we obtain that $w_\infty \circ \iota = i \circ w_\infty$. Hence $\lim_{s\to+\infty} \frac{w_\infty(s,\frac{3}{2})}{|w_\infty(s,\frac{3}{2})|} = -1$, which contradicts $-1 \notin \{f_{i_0j_0}(\frac{3}{2})\}$.

7.10.1. *Proof of Theorem 7.10.1.* — In this subsection we use limiting arguments in which $m \to \infty$ and $\overline{h}_m \to \overline{h}_\infty$; see Section 7.7.4. Hence many of the almost complex structures and moduli spaces will have an additional subscript m, where $m = \infty$ is also a possibility. For example, we will use the notation $\mathcal{J}'_{\overline{W'}_m}$ and $\mathcal{J}_{\overline{W}_{-,m}}$ introduced in Section 7.7.4 to refer to $\mathcal{J}_{\overline{W'}}$ and $\mathcal{J}_{\overline{W}_-}$ with respect to m.

Let $\overline{J'}_\infty \in \mathcal{J}^\star_{\overline{W'}_\infty}$ and let $\overline{J'}_m \in \mathcal{J}^\star_{\overline{W'}_m}$ be a nearby almost complex structure with respect to the integer $m \gg 0$. Let $\overline{J}_{-,m} \in \mathcal{J}^{reg}_{\overline{W}_{-,m}}$ be an almost complex structure which restricts to $\overline{J'}_m$ and let $\overline{J}^\Diamond_{-,m}$ be (ε, U)-close to $\overline{J}_{-,m}$.

We will treat Theorem 7.6.1 in detail and leave Theorem 7.6.3 to the reader. Suppose that, for a sequence $m_i \to \infty$, there sequences $\overline{u}_{ij}$ of $\overline{J}^\Diamond_{-,m_i}$-holomorphic curves which converge to a $\overline{J}^\Diamond_{-,m_i}$-holomorphic building $\overline{u}_{i\infty}$ falling into one of Cases (2)–(6).

Elimination of Case (2). Suppose for each i the sequence $\overline{u}_{ij}$ converges to a building $\overline{u}_{i\infty}$ satisfying Case (2). By Theorem 7.8.15, we obtain a holomorphic map $w_\infty : cl(B_-) \to \mathbf{CP}^1$, whose restriction to $int(cl(B_-))$ is a biholomorphism onto its image.

We apply the Involution Lemma 7.9.3 to obtain a contradiction: Let $\overline{\Sigma}_1$ be the compactification of $\Sigma_1 = \mathbf{CP}^1 - [a_1, a_2]$ and let $\overline{\Sigma}_2 = cl(B_-)$ be identified with $\overline{\mathbf{D}}$ as in Remark 7.10.2. Let $f : \overline{\Sigma}_1 \to \overline{\Sigma}_2$ be the extension of $(w_\infty|_{int(cl(B_-))})^{-1}$. Such an extension exists because Σ_1 is biholomorphic to the open unit disk and biholomorphisms of the open unit disk extend continuously to the boundary.

By Lemma 7.9.3, f maps $\dot{\mathcal{R}}_1(\infty)$ to $\dot{\mathcal{R}}_2(0)$ and, conversely, w_∞ maps $\dot{\mathcal{R}}_2(0)$ to $\dot{\mathcal{R}}_1(\infty)$. Since the asymptotic marker $\dot{\mathcal{R}}_2(0)$ in $\overline{\Sigma}_2$ corresponds to the asymptotic marker $\{t = \frac{3}{2}\}$ for $\mathfrak{p}_+ \in cl(B_-)$ by Remark 7.10.2, $\dot{\mathcal{R}}_1(\infty)$ is a bad radial ray (in the sense of Definition 7.7.10) by Theorem 7.8.15(2). This contradicts Remark 7.7.11, so we have eliminated Case (2).

Elimination of Cases (3) and (4). We will treat Case (4); Case (3) is almost identical. Suppose for each i the sequence $\overline{u}_{ij}$ converges to a building $\overline{u}_{i\infty}$ satisfying Case (4). By Remark 7.6.2, for each i, the total number of branched points of $\cup_{j=-b}^{a} \overline{v}'_{j,i}$ is one. If we exercise some care in choosing the diagonal sequence in Lemma 7.8.4, we can divide the argument for Case (4) further into Subcases (a), (b), (c), (d') and (d'') as in Section 7.8.7, depending on the behavior of the branch points of the maps $\overline{\pi}_{B_-} \circ \overline{v}'_{0,i}$.

Subcases (a) and (b). By Theorem 7.8.16, we obtain holomorphic maps

$$w_\infty : cl(\widetilde{F}_\infty) \to \mathbf{CP}^1, \quad p_\infty : \widetilde{F}_\infty \to B_-,$$

where $w_\infty|_{int(cl(\widetilde{F}_\infty))}$ is a biholomorphism onto its image $\Sigma_1 = \mathbf{CP}^1 - ([a_1, a_2] \cup [a_3, a_4])$ and p_∞ is a branched double cover with one branch point. Let $\overline{\Sigma}_1$ be the compactification of Σ_1 as in Lemma 7.9.2, and let $\overline{\Sigma}_2 = cl(B_-)$ be identified with $\overline{\mathbf{D}}$ as in Remark 7.10.2. We define $f : \overline{\Sigma}_1 \to \overline{\Sigma}_2$ as the extension of $p_\infty \circ (w_\infty|_{int(cl(\widetilde{F}_\infty))})^{-1}$. Such an extension exists because Σ_1 is biholomorphic to an open annulus, and biholomorphisms of the annulus always extend to the boundary. At this point we apply Lemma 7.9.3 as in Case (2) to obtain a contradiction. Case (b) is completely analogous and can be excluded in the same way.

Subcase (c). By Theorem 7.8.19, we obtain pairs of holomorphic maps

$$w_\infty^b : cl(\widetilde{F}_\infty^b) \to \mathbf{CP}^1, \quad p_\infty^b : \widetilde{F}_\infty^b \to B',$$
$$w_\infty^{\mathfrak{m}} : cl(\widetilde{F}_\infty^{\mathfrak{m}}) \to \mathbf{CP}^1, \quad p_\infty^{\mathfrak{m}} : \widetilde{F}_\infty^{\mathfrak{m}} \to B.$$

The map $w_\infty^{\mathfrak{m}}$ restricts to a biholomorphism of $int(cl(\widetilde{F}_\infty^{\mathfrak{m}}))$ with $\Sigma_1^{\mathfrak{m}} = \mathbf{CP}^1 - [a_1, a_2]$ and the map $p_\infty^{\mathfrak{m}}$ is a biholomorphism because it has degree 1. Let $\overline{\Sigma}_1^{\mathfrak{m}}$ be the compactification of $\Sigma_1^{\mathfrak{m}}$. Identify $\overline{\Sigma}_2^{\mathfrak{m}} = cl(B_-)$ with $\overline{\mathbf{D}}$ as in Remark 7.10.2. Let $f^{\mathfrak{m}} : \overline{\Sigma}_1^{\mathfrak{m}} \to \overline{\Sigma}_2^{\mathfrak{m}}$ be the extension of $p_\infty^{\mathfrak{m}} \circ (w_\infty^{\mathfrak{m}}|_{int(cl(\widetilde{F}_\infty^{\mathfrak{m}}))})^{-1}$. As in Case (2), Lemma 7.9.3 implies that $f^{\mathfrak{m}}$ maps $\dot{\mathcal{R}}_1^{\mathfrak{m}}(\infty)$ to $\dot{\mathcal{R}}_2^{\mathfrak{m}}(0)$ and, conversely, $w_\infty^{\mathfrak{m}}$ maps $\dot{\mathcal{R}}_2^{\mathfrak{m}}(0)$ to $\dot{\mathcal{R}}_1^{\mathfrak{m}}(\infty)$.

Next we consider the "upper level" (w_∞^b, p_∞^b). The map $w_\infty^b : cl(\widetilde{F}_\infty^b) \to \mathbf{CP}^1$ is a biholomorphism and the map $p_\infty^b : \widetilde{F}_\infty^b \to B'$ is a branched double cover with one branch point. We define $f^b = p_\infty^b \circ (w_\infty^b)^{-1} : \mathbf{CP}^1 \to \mathbf{CP}^1$, using the identification $cl(B') \simeq \mathbf{CP}^1$ from Remark 7.10.2. By Theorem 7.8.19(9) and the previous paragraph, f^b maps $\dot{\mathcal{R}}_1(0)$ to $\dot{\mathcal{R}}_2(\infty)$. Then, by Lemma 7.9.4 and Theorem 7.8.19(7),(8), f^b maps $\dot{\mathcal{R}}_1(\infty)$ to $\dot{\mathcal{R}}_2(0)$. As in Case (2), this is a contradiction because $\dot{\mathcal{R}}_1(\infty)$ is a good radial ray.

Subcases (d') and (d"). By Theorem 7.8.20, we obtain a holomorphic map $w_\infty^{\mathfrak{m}} : cl(\widetilde{F}_\infty^{\mathfrak{m}}) \to \mathbf{CP}^1$ which restricts to a biholomorphism of $int(cl(\widetilde{F}_\infty^{\mathfrak{m}}))$ with $\mathbf{CP}^1 - [a_1, a_2]$. Then the proof proceeds as in Case (2).

Elimination of Cases (5) and (6). The limit configurations of Cases (5) and (6) must contain a connector over δ_0. This implies that Theorem 7.8.19 applies, and we obtain pairs of holomorphic maps

$$w_\infty^b : cl(\widetilde{F}_\infty^b) \to \mathbf{CP}^1, \quad p_\infty^b : \widetilde{F}_\infty^b \to B',$$
$$w_\infty^{\mathfrak{m}} : cl(\widetilde{F}_\infty^{\mathfrak{m}}) \to \mathbf{CP}^1, \quad p_\infty^{\mathfrak{m}} : \widetilde{F}_\infty^{\mathfrak{m}} \to B.$$

Then the argument of Case (4c) applies.

This completes the proof of Theorem 7.10.1.

7.11. *Proof of Lemma 7.2.3*. — We begin with the following corollary of Theorem 7.10.1. Recall the notation from Convention 5.8.12, Step 2 of Section 7.2 and Equation (7.6.1).

Corollary **7.11.1.** — *Suppose $m \gg 0$ and $\varepsilon, \delta > 0$ are sufficiently small constants. If $\overline{u} \in \mathcal{M}_{\overline{\mathfrak{m}}}^2(\varepsilon, \delta, p)$ or $\mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)}(\varepsilon, \delta, p)$, then $\mathrm{Im}(\overline{u}) \cap K_{p,2\delta} \neq \varnothing$. In the latter case, $r_0 \gg 0$ and $\varepsilon, \delta > 0$ are sufficiently small constants which depend on r_0.*

Proof. — Arguing by contradiction, suppose there are sequences $\varepsilon_i, \delta_i \to 0$ and $\overline{u}_i \in \mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}}(\varepsilon_i, \delta_i, p)$ such that $\mathrm{Im}(\overline{u}_i) \cap \mathrm{K}_{p,2\delta_i} = \varnothing$. Then the limit $\overline{u}_\infty \in \partial\mathcal{M}^3_{\overline{\mathfrak{m}}}(0,0,p)$ of $\overline{u}_i$ has a nontrivial $\overline{v}'_0$ component. By Theorems 7.6.1 and 7.10.1(ii), $\overline{u}_\infty$ satisfies Case (1) of Theorem 7.6.1. Hence, for $i \gg 0$, $\overline{u}_i \in \mathrm{G}(\mathfrak{P}_{(r_0)})$, which is a contradiction. The case of $\overline{u} \in \mathcal{M}^2_{\overline{\mathfrak{m}}}(\varepsilon, \delta, p)$ is easier and is a consequence of Theorems 7.6.3 and 7.10.1(i). □

Proof of Lemma 7.2.3. — Suppose that $\overline{u}_\infty \in \partial_1\mathcal{M}^{3,(r_0)}_{\overline{\mathfrak{m}}}$. We are in the situation of Lemma 7.5.2. If $\overline{v}_0$ is a degenerate $\overline{\mathrm{W}}_-$-curve, then $\mathrm{I}(\overline{v}_0) \geq 4$ by Equation (7.5.6) since $g \geq 1$. Therefore degenerate $\overline{\mathrm{W}}_-$-curves are ruled out by Constraint (i) in the proof of Theorem 7.6.1. Hence $\overline{v}_0$ is a $\overline{\mathrm{W}}_-$-curve and the other levels $\overline{v}_j, j \neq 0$, are multisections of W′ or W. In this case, the curve $\overline{v}_0$ is simply-covered. Corollary 7.11.1 implies that the component through $\overline{\mathfrak{m}}$ also intersects $\mathrm{K}_{p,2\delta}$. Hence passing through $\overline{\mathfrak{m}}$ is a generic condition and we must have $\mathrm{ind}(\overline{v}_0) \geq 2$ and $\mathrm{I}(\overline{v}_0) \geq 2$. This gives us two options for $\overline{u}_\infty = \overline{v}_{-b} \cup \cdots \cup \overline{v}_a$: either

(α) $b = 0$, $\mathrm{I}(\overline{v}_a) = 1$, and $\mathrm{I}(\overline{v}_j) = 0, j = 1, \dots, a-1$; or
(β) $a = 0$, $b = 1$, and $\mathrm{I}(\overline{v}_{-1}) = 1$.

Ghost components are not possible since each ghost component takes up ind ≥ 2 by Lemma 6.1.8. Hence $\overline{u}_\infty \in \mathrm{A}_1 \cup \mathrm{A}_2$. □

7.12. *Gluing.* — The goal of this subsection is to prove Theorem 7.2.2.

7.12.1. *The moduli space $\mathcal{N}$.* — Let $\varepsilon = \frac{\pi}{m}$, where m is a sufficiently large integer and let $\eta_\varepsilon : [-\pi, \pi] \to \mathbf{R}$ be a smooth function such that:

- $\eta_\varepsilon(\theta) = \varepsilon$ for $-\pi \leq \theta \leq \theta_1$;
- $\eta_\varepsilon(\theta) = 0$ for $\theta_2 \leq \theta \leq \pi$; and
- η_ε is monotonically decreasing for $\theta_1 \leq \theta \leq \theta_2$;

for some $-\pi < \theta_1 < \theta_2 < \pi$.

We use two coordinate systems on B_-: (i) the usual coordinates $(s,t) \in \mathbf{R} \times [0,2]/(0 \sim 2)$ and (ii) complex coordinates via an identification of $cl(\mathrm{B}_-)$ with $\overline{\mathbf{D}}$ such that $\mathfrak{p}_+$ is mapped to 0 and $\mathfrak{p}_-$ is mapped to -1. Then $\partial\mathrm{B}_-$ is parametrized in an orientation-preserving manner by a coordinate $e^{i\theta}$, $\theta \in (-\pi, \pi)$.

Definition **7.12.1.** — *Let $\mathcal{N} = \mathcal{N}_{\eta_\varepsilon}$ be the space of holomorphic maps $w : \mathrm{B}_- \to \mathbf{C}$ such that the following properties hold:*

(N_1) $w(e^{i\theta}) \in \mathbf{R}^+ \cdot e^{i\eta_\varepsilon(\theta)}$ *for all* $\theta \in (-\pi, \pi)$;
(N_2) $\lim_{s\to-\infty} \left| w(s,t) - c_1 e^{-\varepsilon(s+it-i)} \right| < \infty$ *for some* $c_1 \in \mathbf{R}^+$; *and*
(N_3) $\lim_{s\to+\infty} \left| w(s,t) - c_2 e^{\pi(s+it)} \right| < \infty$ *for some* $c_2 \in \mathbf{C}^\times$.

In particular:

(N_4) $\deg(w) = 1$ away from the sector $\{0 \le \phi \le \varepsilon\} \subset \mathbf{CP}^1$; and

(N_5) after composing with the chosen identification $cl(B_-) \cong \overline{\mathbf{D}}$, w extends continuously to $\overline{\mathbf{D}}$ so that $w(0) = w(-1) = \infty$.

Multiplication by a real constant gives an $\mathbf{R}^+$-action on $\mathcal{N}$.

Even though $\mathcal{N}$ is the space we are interested in, it will be convenient for technical reasons to regard $\mathcal{N}$ as an open subset of a vector space $\widetilde{\mathcal{N}}$ obtained by relaxing properties (N_1)–(N_3).

Definition **7.12.2.** — *Let $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_{\eta_\varepsilon}$ be the space of holomorphic maps $w : B_- \to \mathbf{C}$ such that the following properties hold:*

($\widetilde{N}_1$) $w(e^{i\theta}) \in \mathbf{R} \cdot e^{i\eta_\varepsilon(\theta)}$ *for all* $\theta \in (-\pi, \pi)$*;*

($\widetilde{N}_2$) $\lim_{s \to -\infty} \left| w(s,t) - c_1 e^{-\varepsilon(s+it-i)} \right| < \infty$ *for some* $c_1 \in \mathbf{R}$*; and*

($\widetilde{N}_3$) $\lim_{s \to +\infty} \left| w(s,t) - c_2 e^{\pi(s+it)} \right| < \infty$ *for some* $c_2 \in \mathbf{C}$.

In order to compute the dimension of $\widetilde{\mathcal{N}}$, we identify B_- with $\overline{\mathbf{D}} - \{-1, 0\}$ and $\widetilde{\mathcal{N}}$ with the space of the holomorphic sections of a holomorphic line bundle $E \to \overline{\mathbf{D}}$ with values in a real rank one subbundle F along $\partial\overline{\mathbf{D}} - \{-1\}$.

We construct the bundles E and F as follows. Consider a cover of $\overline{\mathbf{D}}$ by three open sets

$$U_0 = \overline{\mathbf{D}} - \{0, 1\},\ U_1 = \{z \in \mathbf{D} \mid |z| < 1/3\},\ U_2 = \{z \in \overline{\mathbf{D}} \mid |z+1| < 1/3\}.$$

Over each open set we take a trivial line bundle $E_i = \mathbf{C} \times U_i \to U_i$ and define the bundle E by gluing the bundles E_i via the transition maps

$$\psi_1 : E_0|_{U_0 \cap U_1} \to E_1|_{U_0 \cap U_1}, \quad \psi_1(z, v) = (z, zv),$$

$$\psi_2 : E_0|_{U_0 \cap U_2} \to E_2|_{U_0 \cap U_2}, \quad \psi_2(z, v) = \left(z, i\left(\frac{z+1}{-z+1}\right) v\right).$$

Let $\pi_{E_i} : E_i = \mathbf{C} \times U_i \to \mathbf{C}$ be the projections corresponding to the trivializations. If we parametrize $\partial\overline{\mathbf{D}} - \{-1\}$ by $\theta \in (-\pi, \pi)$, the subbundle F is given, as a subbundle of E_0, by $F(\theta) = \mathbf{R} \cdot e^{i\eta_\varepsilon(\theta)}$.

It is convenient to view $\Sigma = \overline{\mathbf{D}} - \{-1, 0\}$ as a surface with a negative strip-like end and a positive cylindrical end and E as a line bundle over Σ. Let $(-\infty, -R) \times [0, 1]$ be a strip with coordinates (s, t). We identify the strip-like end Z of Σ with $(-\infty, -R) \times [0, 1]$ via the map

$$\phi : (-\infty, -R) \times [0, 1] \to \Sigma, \quad \phi(s, t) = \frac{e^{\pi(s+it)} - i}{e^{\pi(s+it)} + i}.$$

Here the fractional linear transformation $B(\zeta) = \frac{\zeta - i}{\zeta + i}$ maps the upper half plane $\mathbf{H}$ to the unit disk $\mathbf{D}$ and 0 to -1. Observe that $A(\zeta) = i(\frac{\zeta+1}{-\zeta+1})$ which appears in the definition of

ψ_2 is the inverse of $B(\zeta)$. Hence the gluing map ψ_2 becomes

$$\psi_2((s,t),v) = ((s,t), e^{\pi(s+it)}),$$

with respect to coordinates (s,t).

The linear Cauchy-Riemann operator

$$D : W^{1,p}(E,F) \to L^p(T^{0,1}\Sigma \otimes_{\mathbf{C}} E)$$

is Fredholm for $p > 2$ and its kernel consists of smooth holomorphic functions. We denote by $H^0(E,F)$ its kernel and by $H^1(E,F)$ its cokernel.

Lemma **7.12.3.** — *There is an identification* $H^0(E,F) \cong \widetilde{\mathcal{N}}$ *for every choice of* η_ε.

Proof. — The isomorphism $H^0(E,F) \cong \widetilde{\mathcal{N}}$ associates to a holomorphic section $\xi \in H^0(E,F)$ the holomorphic function $\pi_{E_0} \circ \xi : \Sigma \to \mathbf{C}$, i.e., $\pi_{E_0} \circ \xi$ is obtained by writing $\xi|_{U_0}$ with respect to the trivialization of E_0. On the negative end Z we can write

$$\pi_{E_2} \circ \xi(s,t) = \sum_{n \geq 1} c_n e^{(n\pi - \varepsilon)(s+it) + i\varepsilon},$$

since $\pi_{E_2} \circ \xi$ is holomorphic. By applying the transition function ψ_2^{-1}, we obtain that the leading term of $\pi_{E_0} \circ \xi$ on Z is $e^{-\varepsilon(s+it)+i\varepsilon}$, which is condition $(\widetilde{N}_2)$ in Definition 7.12.2. For a similar reason $\pi_{E_0} \circ \xi$ has a pole at 0 of order at most 1. ☐

We will consider also the compactified surface $\check{\Sigma}$ obtained by adding the "segment at infinity" to the strip-like end Z. Alternatively, $\check{\Sigma}$ admits an identification with the truncated surface $\Sigma - Z$. Let $\check{E} \to \check{\Sigma}$ be the line bundle obtained by extending $E \to \Sigma$.

Lemma **7.12.4.** — $\operatorname{ind} D = 3$ *for every choice of* η_ε.

Proof. — We decompose $\partial\check{\Sigma} = cl(\partial\Sigma) \cup (\partial\check{\Sigma} - \partial\Sigma)$, where $\partial\check{\Sigma} - \partial\Sigma$ corresponds to the "segment at infinity" of the negative strip-like end Z and $cl(\partial\Sigma)$ is the closure of $\partial\Sigma$ in $\partial\check{\Sigma}$. We parametrize $cl(\partial\Sigma)$ by $\theta \in [-\pi,\pi]$ and $\partial\check{\Sigma} - \partial\Sigma$ by $\theta' \in [0,1]$ in a manner compatible with the orientation of $\check{\Sigma}$ induced by the complex structure on Σ. In particular, $\theta = -\pi$ is identified with $\theta' = 1$ and $\theta = \pi$ is identified with $\theta' = 0$.

We define a trivialization τ of $\check{E}|_{\partial\check{\Sigma}}$ by:

$$\begin{cases} \tau(\theta) = e^{i\eta_\varepsilon(\theta)} \\ \quad \text{along } cl(\partial\Sigma), \text{ with respect to the trivialization of } E_0, \\ \tau(\theta') = -e^{i(\pi+\varepsilon)\theta'} \\ \quad \text{along } \partial\check{\Sigma} - \partial\Sigma, \text{ with respect to the trivialization of } E_2. \end{cases}$$

We also define a Lagrangian subbundle $\check{F} \subset \check{E}|_{\partial\check{\Sigma}}$ by $\check{F}|_{\partial\Sigma} = F$ and rotating it in the counterclockwise direction by the minimal amount on $\partial\check{\Sigma} - \partial\Sigma$. This means:

$$\begin{cases} F(\theta) = \mathbf{R} \cdot e^{i\eta_\varepsilon(\theta)} \\ \quad \text{along } cl(\partial\Sigma), \text{ with respect to the trivialization of } E_0, \\ F(\theta') = \mathbf{R} \cdot e^{i\varepsilon\theta'} \\ \quad \text{along } \partial\check{\Sigma} - \partial\Sigma, \text{ with respect to the trivialization of } E_2. \end{cases}$$

By the doubling argument of Theorem 5.5.1, Lemma 5.5.3 (or, rather, its proof) and the formula for the index of the Cauchy–Riemann operator on line bundles over punctured surfaces (see for example [We2, Formula 2.1]), the index of D is

$$\operatorname{ind} D = \chi(\mathbf{D}) + \mu_\tau(\check{F}) + 2c_1(\check{E}, \tau) - 1.$$

From the explicit definitions of τ and $\check{F}$ one computes that $\mu_\tau(\check{F}) = -1$ and $c_1(\check{E}, \tau) = 2$. Hence $\operatorname{ind} D = 3$. □

Lemma **7.12.5.** — *The operator* D *is surjective for every choice of* η_ε.

Proof. — In view of Theorem 5.5.1, the surjectivity of D follows from [We2, Formula 2.5] and [We2, Proposition 2.2(2)] applied to the double of D. □

Corollary **7.12.6.** — *The real dimension of* $\widetilde{\mathcal{N}}$ *is* 3 *for every choice of* η_ε.

7.12.2. *The maps* $\mathfrak{E}$ *and* $\mathfrak{F}$. — We define an **R**-linear map:

$$\mathfrak{E} : \widetilde{\mathcal{N}}_{\eta_\varepsilon} \to \mathbf{R} \times \mathbf{C},$$
$$w \mapsto (c_1, c_2),$$

where c_1, c_2 are the coefficients from ($\widetilde{N}_1$) and ($\widetilde{N}_2$) of Definition 7.12.2.

Lemma **7.12.7.** — *The map* $\mathfrak{E}$ *is an isomorphism.*

Hence $\mathfrak{E}(\mathcal{N})$ is an open positive cone contained in $\mathbf{R}^+ \times \mathbf{C}^\times$.

Proof. — Since $\mathfrak{E}$ is a linear map, it suffices to check that $\mathfrak{E}$ is injective. First observe that $\ker \mathfrak{E}$ consists of continuous functions $w : \overline{\mathbf{D}} \to \mathbf{C}$ such that $w(-1) = 0$, $w(e^{i\theta}) \in \mathbf{R} \cdot e^{\eta_\varepsilon(\theta)}$ for all $\theta \in (-\pi, \pi)$, and w is holomorphic on $\mathbf{D}$.

If $w \neq 0$, then we can apply [Se2, Lemma 11.5]; referring to the equation on the second-to-last line of [Se2, p.156], the left-hand side of the equation is positive while the right-hand side is zero, a contradiction. □

Next we prove that $\mathcal{N}$ is nonempty for any choice of η_ε.

Lemma **7.12.8.** — *Let* $w_0 = \mathfrak{E}^{-1}(1,0)$. *Then* $w_0 : \overline{\mathbf{D}} - \{-1\} \to \mathbf{C}$ *is holomorphic,* $w_0(e^{i\theta}) \in \mathbf{R}^+ \cdot e^{i\eta_\varepsilon(\theta)}$, w_0 *has no zeros, and* $\lim_{s\to-\infty} e^{\varepsilon(s+it-i)} u_0(s,t) = 1$ *for coordinates* $(s,t) \in (-\infty, 0) \times [0,1]$ *on a negative strip-like end around* -1.

Proof. — The only nontrivial part of the statement is that w_0 has no zeros in the interior and along the boundary — this follows from the equation on the second-to-last line of [Se2, p.156], where the right-hand side of the equation is zero. □

Corollary **7.12.9.** — $\mathcal{N}$ *is nonempty for any choice of* η_ε.

Proof. — Take any $w \in \widetilde{\mathcal{N}}$ with $c_2 \neq 0$. Then $w + cw_0 \in \mathcal{N}$ for any c sufficiently large. □

Lemma **7.12.10.** — *If* $w \in \mathcal{N}$, *then* w *has a unique (simple) zero in* $\mathbf{D} - \{0\}$.

Proof. — The lemma follows from a straightforward winding number argument, since w maps ∂B_- to the sector $\{0 \le \theta \le \varepsilon, \rho > 0\}$ and has a pole at 0. □

We denote $\mathbf{P}\mathcal{N} = \mathcal{N}/\mathbf{R}^+$, where $\mathbf{R}^+$ acts on $\mathcal{N}$ by multiplication and we define the maps

$$\widehat{\mathfrak{F}} : \mathcal{N} \to \mathbf{D} - \{0\}, \qquad \mathfrak{F} : \mathbf{P}\mathcal{N} \to \mathbf{D} - \{0\}$$

by $\widehat{\mathfrak{F}}(w) = w^{-1}(0)$ and $\mathfrak{F}([w]) = \widehat{\mathfrak{F}}(w)$.

Lemma **7.12.11.** — *The map* $\mathfrak{F} : \mathbf{P}\mathcal{N} \to \mathbf{D} - \{0\}$ *is a diffeomorphism.*

Proof. — We first prove the injectivity of $\mathfrak{F}$. Let w_0 and w_1 be maps in $\mathcal{N}$ such that $w_0^{-1}(0) = w_1^{-1}(0)$. Then $\omega = \frac{w_0}{w_1}$ is a holomorphic map on $\overline{\mathbf{D}}$ such that $\omega(\partial\overline{\mathbf{D}}) \subset \mathbf{R}$, so it is constant. Then w_0 and w_1 represent the same element in $\mathbf{P}\mathcal{N}$.

Next we prove surjectivity. Fix an element $w_0 \in \mathcal{N}$ and let $z_0 \in \mathbf{D}$ such that $w_0(z_0) = 0$. For any $z_1 \in \mathbf{D} - \{0, z_0\}$ we look for a holomorphic function $\omega_{z_1} : \overline{\mathbf{D}} \to \mathbf{CP}^1$ such that $\omega_{z_1}(z_1) = 0$, $\omega_{z_1}(z_0) = \infty$, $\omega_{z_1}(\partial\overline{\mathbf{D}}) \subset \mathbf{R}^+$, and $\omega_{z_1}|_{\mathbf{D}}$ is a biholomorphism onto its image. By the argument of the previous paragraph, if such ω_{z_1} exists, it is unique up to multiplication by a positive real constant. If we set $w = \omega_{z_1} w_0$, then $w \in \mathcal{N}$, $w(z_1) = 0$, and $\mathfrak{F}(w) = z_1$.

The function ω_{z_1} with the desired properties is given by

$$\omega_{z_1}(\zeta) = f(g(\zeta)^2),$$

where $g : \overline{\mathbf{D}} \xrightarrow{\sim} \overline{\mathbf{H}}$ is a fractional linear transformation such that $g(z_0) = i$, $\mathrm{Re}(g(z_1)) = 0$, and $0 < \mathrm{Im}(g(z_1)) < 1$, and $f : \mathbf{CP}^1 \xrightarrow{\sim} \mathbf{CP}^1$ is an element of $\mathrm{PSL}(2, \mathbf{R})$ (i.e., f fixes the real axis) such that $f((g(z_1))^2)=0, f((g(z_0))^2)=f(-1)=\infty$, and $f(0)=1$. Note that f and g are uniquely determined by the above conditions. This proves the surjectivity of $\mathfrak{F}$.

Finally, we claim that the $d\mathfrak{F}([w])$ is an isomorphism for all $[w]\in \mathbf{P}\mathcal{N}$. To this end, let us define $\Xi(z)=\omega_z w_0\in\mathcal{N}$. Then $\mathfrak{F}^{-1}(z)$ is the class of $\Xi(z)$ in $\mathbf{P}\mathcal{N}$. The map Ξ is smooth because the maps f, g, and hence ω_z are rational maps whose coefficients depend smoothly on z. In order to prove the claim it suffices to prove that $d\Xi(z)$ is injective for any $z\in\mathbf{D}-\{0\}$. Here, in order to distinguish the differential of Ξ at $z\in\mathbf{D}-\{0\}$ from the differential of the function $\Xi(z):\overline{\mathbf{D}}-\{0,-1\}\to\mathbf{C}$, we will use the notation $d\Xi(z)$ for the former and $\frac{\partial\Xi(z)}{\partial\zeta}$ for the latter. We define the map

$$\mathrm{F}:\mathcal{N}\times(\mathbf{D}-\{0\})\to\mathbf{C},\quad \mathrm{F}(w,z)=w(z).$$

By differentiating the identity $\mathrm{F}(\Xi(z),z)=0$ we obtain

$$\frac{\partial\mathrm{F}}{\partial w}d\Xi(z)+\frac{\partial\mathrm{F}}{\partial z}=0.$$

We observe that $\frac{\partial\mathrm{F}}{\partial z}\Big|_{(\Xi(z),z)}=\frac{\partial\Xi(z)}{\partial\zeta}\Big|_{\zeta=z}$, which is invertible because z is a simple zero of $\Xi(z)$. Hence $d\Xi(z)$ is injective. □

7.12.3. *Proof of Theorem 7.2.2.* — We refer to Section 7.2 for the definition of the moduli spaces $\mathcal{M}_1$, $\mathcal{M}_0$, and $\mathcal{M}_{-1}$, and of the gluing parameter space $\mathfrak{P}$. In order to simplify the exposition we will assume (without loss of generality) that all the multiplicities of $\boldsymbol{\gamma}'$ are 1 and that each of $\mathcal{M}_1/\mathbf{R}$, $\mathcal{M}_0$, and $\mathcal{M}_{-1}/\mathbf{R}$ is connected; in particular, both $\mathcal{M}_0$ and $\mathcal{M}_{-1}/\mathbf{R}$ are single points.

We choose a smooth slice $\widetilde{\mathcal{M}}_1$ of the $\mathbf{R}$-action on $\mathcal{M}_1$ such that the following hold for some $\kappa_0',\kappa_0>0$, $\mathrm{K}_0>\pi-\varepsilon$, and for all $\overline{v}_1\in\widetilde{\mathcal{M}}_1$:

- each component of $\overline{v}_1|_{s\le 0}$ is $(\kappa_0'+\kappa_0,0)$-close to a cylinder over a component of $\delta_0\boldsymbol{\gamma}'$; and
- the component $\widetilde{v}_1$ of $\overline{v}_1|_{s\le 0}$ which is close to σ_∞' satisfies

$$\left|\overline{\pi}\circ\widetilde{v}_1-\kappa_0' e^{(\pi-\varepsilon)s}(ce^{\pi it})\right|_{C^0}\le\kappa_0 e^{\mathrm{K}_0 s}, \tag{7.12.1}$$

where $ce^{\pi it}$ is the normalized asymptotic eigenfunction corresponding to the negative end δ_0 of $\overline{v}_1$ and $\overline{\pi}=\overline{\pi}_{\mathrm{D}^2_{\rho_0}}$ is the projection to $\mathrm{D}^2_{\rho_0}$ with respect to balanced coordinates. By a slight abuse of notation, we will refer to $\kappa_0' e^{(\pi-\varepsilon)s}(ce^{\pi it})|_{s=0}=\kappa_0' ce^{\pi it}$ as *the* asymptotic eigenfunction of $\overline{v}_1$ at δ_0.

Similarly, we choose a smooth slice $\widetilde{\mathcal{M}}_{-1}$ of the $\mathbf{R}$-action on $\mathcal{M}_{-1}$ such that the following hold for some $\kappa_1',\kappa_1>0$, $\mathrm{K}_1>2\varepsilon$:

- each component of $\overline{v}_{-1}|_{s\ge 0}$ is $(\kappa_1'+\kappa_1,0)$-close to a strip over a component of $\{z_\infty\}\cup\mathbf{y}'$; and

– the component $\widetilde{v}_{-1}$ of $\overline{v}_{-1}|_{s\geq 0}$ which is close to σ_∞ satisfies

$$\left|\overline{\pi}\circ\widetilde{v}_{-1}-\kappa_1' e^{-2\varepsilon s}(de^{-\varepsilon it})\right|_{C^0}\leq \kappa_1 e^{-K_1 s}, \tag{7.12.2}$$

where $de^{-\varepsilon it}$ is a normalized asymptotic eigenfunction corresponding to the positive end z_∞ of $\overline{v}_{-1}$. Note that d is completely determined by the data $\{(i,j)\to(i,j)\}$ at the positive end z_∞. Without loss of generality, we assume that $\overline{a}_{i,j}=\mathbf{R}^+$, $\overline{h}(\overline{a}_{i,j})=\mathbf{R}^+\cdot e^{i\varepsilon}$, so that $d=e^{i\varepsilon}$. Similarly, we refer to $\kappa_1' e^{-2\varepsilon s}de^{-\varepsilon it}|_{s=0}=\kappa_1' de^{-\varepsilon it}$ as *the* asymptotic eigenfunction of $\overline{v}_{-1}$ at z_∞.

In the rest of this section, when we write $\overline{v}_i$, $i=1,-1$, we will assume that $\overline{v}_i$ is in the slice $\widetilde{\mathcal{M}}_i$. Also, for $T\in\mathbf{R}$ and $i=\pm 1$, let $\overline{v}_{i,T}$ be the T-translates of $\overline{v}_i$ in the $\mathbf{R}$-direction; i.e., if $s:\overline{W}^*\to\mathbf{R}$, $*=\varnothing,'$, is the $\mathbf{R}$-coordinate, then $s\circ\overline{v}_{i,T}=s\circ\overline{v}_i+T$. Let $f(t)=\beta_0 e^{\pi it}$ be the asymptotic eigenfunction of $\overline{v}_1$ at δ_0 with eigenvalue $\pi-\varepsilon$. Then the asymptotic eigenfunction of $\overline{v}_{1,T}$ at δ_0 is $e^{-(\pi-\varepsilon)T}f$. Similarly, let $g(t)=\alpha_0 e^{i\varepsilon(1-t)}$ be the asymptotic eigenfunction of $\overline{v}_{-1}$ at z_∞ with eigenvalue 2ε. Then the asymptotic eigenfunction of $\overline{v}_{-1,T}$ at z_∞ is $e^{2\varepsilon T}g$.

Also, for the rest of this section, we fix $\boldsymbol{\gamma}\in\widetilde{\mathcal{O}}_{2g}$ and $\mathbf{y}\in\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ once and for all. Since there is no risk of confusion, we will write

$$\mathcal{M}=\mathcal{M}^{I=3,n^-=m}_{\overline{J}_-}(\boldsymbol{\gamma},\mathbf{y}),\quad \mathcal{M}^{ext}=\mathcal{M}^{I=3,n^-=m,ext}_{\overline{J}_-}(\boldsymbol{\gamma},\mathbf{y}).$$

After identifying the quotient $\mathcal{M}_i/\mathbf{R}$ with the slice $\widetilde{\mathcal{M}}_i$ for $i=\pm1$ we write

$$\mathfrak{P}=(5r,\infty)^2\times\widetilde{\mathcal{M}}_1\times\mathcal{M}_0\times\widetilde{\mathcal{M}}_{-1};$$

compare with Equation (7.2.2).

We describe the gluing map

$$G:\mathfrak{P}\to\mathcal{M}^{ext},\quad \mathfrak{d}=(T_\pm,\overline{v}_1,\overline{v}_0,\overline{v}_{-1})\mapsto\overline{u}(\mathfrak{d}),$$

defined in a manner similar to that of Section 6.5.2. First we form the preglued curve $\overline{u}^\#(\mathfrak{d})$ from the data $\mathfrak{d}=(T_\pm,\overline{v}_1,\overline{v}_0,\overline{v}_{-1})$ by patching together

$$(\overline{v}_{1,2T_+})|_{s\geq T_+},\quad \overline{v}_0|_{-T_-\leq s\leq T_+},\quad (\overline{v}_{-1,-2T_-})|_{s\leq -T_-}.$$

Then we choose cutoff functions $\widetilde{\beta}_i$, $i=-1,0,1$, on the domains of $\overline{v}_i$. If ψ_{-1}, ψ_0, and ψ_1 are deformations of $\overline{v}_{-1,-2T_-}$, $\overline{v}_0$, and $\overline{v}_{1,2T_+}$, the condition for the deformation $\widetilde{\beta}_{-1}\psi_{-1}+\widetilde{\beta}_0\psi_0+\widetilde{\beta}_1\psi_1$ of $\overline{u}^\#(\mathfrak{d})$ to be holomorphic is given by the following equation:

$$\widetilde{\beta}_{-1}\Theta_{-1}(\psi_{-1},\psi_0)+\widetilde{\beta}_0\Theta_0(\psi_{-1},\psi_0,\psi_1)+\widetilde{\beta}_1\Theta_1(\psi_0,\psi_1)=0, \tag{7.12.3}$$

Equation (7.12.3) is analogous to Equation (6.5.1) and is solved as in Step 3 of Section 6.5.2. The situation considered here is simpler than that of Section 6.5.2: in fact there is no obstruction bundle because $\overline{v}_0$ is regular. For r sufficiently large, G is a homeomorphism onto its image.

Given $\delta > 0$ small, let $D_\delta(\overline{\mathfrak{m}}^b)$ be the closed ball of radius δ and center $\overline{\mathfrak{m}}^b$ in B_- with respect to some fixed metric. Then let $\mathcal{M}^\delta_{(\kappa,\nu)} \subset \mathcal{M}^{ext}$ be the subset of curves $\overline{u}$ that pass through $D_\delta(\overline{\mathfrak{m}}^b) \times \{z_\infty\} \subset \overline{W}_-$ and are (κ,ν)-close to breaking.

Claim **7.12.12.** — $\mathcal{M}^\delta_{(\kappa,\nu)} \subset \mathcal{M}$.

Proof. — Something stronger is actually true: if the image of $\overline{u} \in \mathcal{M}^{ext}$ intersects the section at infinity σ^-_∞ at a point in the interior of σ^-_∞, then $\overline{u} \in \mathcal{M}$. This is a consequence of the intersection theory developed in Section 7.4: in fact $n^-(\overline{u}) = m$ and the intersection point with σ^-_∞ at the interior already contributes m to $n^-(\overline{u})$. However, if $\overline{u} \in \mathcal{M}^{ext} - \mathcal{M}$, then the image of $\overline{u}$ also intersects σ^-_∞ at some boundary points, which give some extra contribution to $n^-(\overline{u})$. This is a contradiction. □

Let $\mathfrak{d} = (T_\pm, \overline{v}_1, \overline{v}_0, \overline{v}_{-1}) \in \mathfrak{P}$. Writing $\overline{u}^\# = \overline{u}^\#(\mathfrak{d})$ and $\overline{u} = \overline{u}(\mathfrak{d})$, we define $w^\# = w^\#(\mathfrak{d})$ and $w = w(\mathfrak{d})$ as follows:

$$(\mathbf{7.12.4}) \qquad w^\# = e^{\varepsilon s} \cdot (\overline{\pi} \circ \overline{u}^\#|_{-2T_- \le s \le 2T_+}) \quad \text{and} \quad w = e^{\varepsilon s} \cdot (\overline{\pi} \circ \overline{u}|_{-2T_- \le s \le 2T_+}).$$

The map w is holomorphic by Lemma 7.8.6.

Lemma **7.12.13.** — *Let* $\mathfrak{d}_i = (T_{\pm,i}, \overline{v}_{1,i}, \overline{v}_0, \overline{v}_{-1}) \in \mathfrak{P}$ *be a sequence such that* $\lim_{i\to\infty} T_{\pm,i} = +\infty$ *and* $\lim_{i\to\infty} \overline{v}_{1,i} = \overline{v}_{1,\infty} \in \widetilde{\mathcal{M}}_1$. *Then the sequence* $\overline{u}_i = \overline{u}(\mathfrak{d}_i)$ *has a subsequence which converges to* $(\overline{v}_{1,\infty}, \overline{v}_0, \overline{v}_{-1})$.

Proof. — The lemma is a consequence of the following standard gluing result (cf. [HT2, Theorem 7.3(a)]): Given (κ,ν), there exists $r \gg 0$ such that $\overline{u}(\mathfrak{d})$ is (κ,ν)-close to $(\overline{v}_1, \overline{v}_0, \overline{v}_{-1})$ for each $\mathfrak{d} = (T_\pm, \overline{v}_1, \overline{v}_0, \overline{v}_{-1}) \in \mathfrak{P} = (5r,\infty)^2 \times \widetilde{\mathcal{M}}_1 \times \mathcal{M}_0 \times \widetilde{\mathcal{M}}_{-1}$. □

Recall that $f(t) = \beta_0 e^{\pi i t}$ is the asymptotic eigenfunction of $\overline{v}_1$ at the negative end δ_0 and $g(t) = \alpha_0 e^{i\varepsilon(1-t)}$ is the asymptotic eigenfunction of $\overline{v}_{-1}$ at the positive end z_∞. We define the function

$$\Pi : \mathfrak{P} \to \mathbf{R}^+ \times \mathbf{C}^\times,$$
$$\Pi(T_\pm, \overline{v}_1, \overline{v}_0, \overline{v}_{-1}) = (\alpha, \beta),$$

where $\alpha e^{i\varepsilon(1-t)}$ is the asymptotic eigenfunction of $\overline{v}_{-1,-2T_-}$ at z_∞ and $\beta e^{i\pi t}$ is the asymptotic eigenfunction of $\overline{v}_{1,2T_+}$ at δ_0.

Lemma **7.12.14.** — *Let* $\mathfrak{d}_i = (T_{\pm,i}, \overline{v}_{1,i}, \overline{v}_0, \overline{v}_{-1}) \in \mathfrak{P}$ *be a sequence such that* $\lim_{i\to+\infty} T_{\pm,i} = +\infty$. *Let* $\overline{\mathfrak{m}}_i = \overline{u}(\mathfrak{d}_i)^{-1}(\sigma^-_\infty)$ *and let* $f_i(t) = \beta_i e^{\pi i t}$ *be the asymptotic eigenfunction of* $\overline{v}_{1,i}$ *at the negative end* δ_0. *Then, after passing to a subsequence,* $\lim_{i\to+\infty} \beta_i = \beta_\infty$, *and, after rescaling* *by positive real numbers in addition, the sequence* $w_i = w(\mathfrak{d}_i)$ *of holomorphic functions defined as in Equation* (7.12.4) *converges in the* C^∞_{loc}*-topology to a holomorphic map* $w_\infty \in \widetilde{\mathcal{N}}$ *which satisfies the following:*

1. $\mathfrak{E}(w_\infty) = (\lambda_1\alpha_0, \lambda_2\beta_\infty)$ *for some* $\lambda_1, \lambda_2 \geq 0$, λ_1, λ_2 *not both zero;*
2. $\Pi(\mathfrak{d}_i)$, *after rescaling by positive constants, limits to* $\mathfrak{E}(w_\infty)$*;*
3. *if* $\lim_{i\to+\infty} \overline{\mathfrak{m}}_i = \overline{\mathfrak{m}}_\infty \in int(B_-)$, *then* $w_\infty(\overline{\mathfrak{m}}_\infty) = 0$ *and* $w_\infty \in \mathcal{N}$*; and*
4. *if* $w_\infty(\overline{\mathfrak{m}}_\infty) = 0$ *with* $\overline{\mathfrak{m}}_\infty \in int(B_-)$, *then* $\lim_{i\to+\infty} \overline{\mathfrak{m}}_i = \overline{\mathfrak{m}}_\infty$.

Proof. — By Lemma 7.12.13 we can extract a subsequence (which we still call $\mathfrak{d}_i$) such that $\overline{u}(\mathfrak{d}_i)$ converges to $\overline{u}_\infty = (\overline{v}_{1,\infty}, \overline{v}_0, \overline{v}_{-1})$. By the SFT convergence of $\overline{u}(\mathfrak{d}_i)$ to $\overline{u}_\infty$ we obtain a sequence of good truncations satisfying the estimates of Lemma 7.8.4. The proof of Theorem 7.8.15 then goes through essentially unchanged to yield the limit w_∞. (1) is a consequence of the nontriviality of w_∞ and (2)–(4) are consequences of the convergence in C^∞_{loc}. □

Let $w_0 \in \mathcal{N}$ be a map such that $w_0(\overline{\mathfrak{m}}^b) = 0$ and let $\mathcal{V}_\delta \subset \mathcal{N}$ be the subset consisting of maps w such that $w(\overline{\mathfrak{m}}) = 0$ for some $\overline{\mathfrak{m}} \in D_\delta(\overline{\mathfrak{m}}^b)$. Note that $\mathcal{V}_\delta$ is an $\mathbf{R}^+$-invariant neighborhood of w_0, whose slice under the $\mathbf{R}^+$-action is compact in view of Lemma 7.12.11. We now have the following diagram:

$$\begin{array}{ccc} \mathfrak{P} & \xrightarrow{\ G\ } & \mathcal{M}^{ext} \supset \mathcal{M}^{\delta}_{(\kappa,\nu)} \\ \Pi \big\downarrow & & \\ \mathbf{R}^+ \times \mathbf{C}^\times & \xleftarrow{\ \mathfrak{E}\ } & \mathcal{N} \supset \mathcal{V}_\delta. \end{array}$$

Let us write $\mathfrak{P}_\delta := \Pi^{-1}(\mathfrak{E}(\mathcal{V}_\delta))$. Recall that $\mathfrak{E}|_{\widetilde{\mathcal{N}}}$ is a diffeomorphism.

Lemma **7.12.15.** — *If r is sufficiently large, then the following hold:*

1. $\mathcal{M}^{\delta/3}_{(\kappa,\nu)} \cap \mathrm{Im}(G) \subset G(\mathfrak{P}_{\delta/2})$.
2. $G(\mathfrak{P}_{2\delta/3}) \subset int(\mathcal{M}^{\delta}_{(\kappa,\nu)})$.

Proof. — (1) We show that, for r sufficiently large, if

$$\overline{u}(\mathfrak{d})^{-1}(\sigma^-_\infty) \in D_{\delta/3}(\overline{\mathfrak{m}}^b),$$

then $\mathfrak{d} \in \mathfrak{P}_{\delta/2}$. Arguing by contradiction, suppose there exist sequences $r_i \to \infty$, $\mathfrak{d}_i = (T_{\pm,i}, \overline{v}_{1,i}, \overline{v}_0, \overline{v}_{-1}) \notin \mathfrak{P}_{\delta/2}$, and $\overline{\mathfrak{m}}_i \in D_{\delta/3}(\overline{\mathfrak{m}}^b)$ such that $T_{\pm,i} > 5r_i$ and $\overline{u}(\mathfrak{d}_i)^{-1}(\sigma^-_\infty) = \overline{\mathfrak{m}}_i$. By passing to a subsequence, we may assume that $\lim_{i\to\infty} \overline{\mathfrak{m}}_i = \overline{\mathfrak{m}}_\infty \in D_{5\delta/12}(\overline{\mathfrak{m}}^b)$ and that the asymptotic eigenfunctions of $\overline{v}_{1,i}$ converge to $\beta_\infty e^{\pi i t}$ for $i \to \infty$. We now apply

Lemma 7.12.14 to the sequence $\overline{u}(\mathfrak{d}_i)$ to obtain a holomorphic map $w_\infty \in \mathcal{N}$ satisfying Lemma 7.12.14(1)–(3). In particular, $\Pi(\mathfrak{d}_i)$, after rescaling, limits to $\mathfrak{E}(w_\infty)$ by Lemma 7.12.14(2) and $w_\infty(\overline{\mathfrak{m}}_\infty) = 0$ by Lemma 7.12.14(3). Since $\overline{\mathfrak{m}}_\infty \in \mathrm{D}_{5\delta/12}(\overline{\mathfrak{m}}^b)$, this contradicts $\mathfrak{d}_i \notin \mathfrak{P}_{\delta/2}$ (or equivalently $\Pi(\mathfrak{d}_i) \in \mathfrak{E}(\mathcal{V}_{\delta/2})$).

(2) Suppose there exist sequences $r_i \to \infty$ and $\mathfrak{d}_i = (\mathrm{T}_{\pm,i}, \overline{v}_{1,i}, \overline{v}_0, \overline{v}_{-1}) \in \mathfrak{P}_{2\delta/3}$ such that $\mathrm{T}_{\pm,i} > 5r_i$ and $\overline{u}(\mathfrak{d}_i) \notin int(\mathcal{M}^\delta_{(\kappa,\nu)})$. By passing to a subsequence and applying Lemma 7.12.14 to $\overline{u}(\mathfrak{d}_i)$, we obtain a holomorphic map $w_\infty \in \widetilde{\mathcal{N}}$ such that $\mathfrak{E}(w_\infty) \in \mathfrak{E}(\mathcal{V}_{2\delta/3})$ by Lemma 7.12.14(2). On the other hand, $w_\infty^{-1}(0) \notin \mathrm{D}_\delta(\overline{\mathfrak{m}}^b)$ since $\overline{u}(\mathfrak{d}_i)^{-1}(\sigma_\infty^-) \notin \mathrm{D}_\delta(\overline{\mathfrak{m}}^b)$ by Lemma 7.12.14(4). This is a contradiction. □

Fix $\delta > 0$ and take $r_0 > 5r$ sufficiently large. We write

$$\partial\mathfrak{P}_{2\delta/3,(r_0)} := \mathfrak{P}_{2\delta/3} \cap \partial\mathfrak{P}_{(r_0)},$$

where $\mathfrak{P}_{(r_0)} = \{\mathrm{T}_+ \geq r_0\} \subset \mathfrak{P}$ and $\partial\mathfrak{P}_{(r_0)} = \{\mathrm{T}_+ = r_0\}$. By Lemma 7.12.15(2), we can define

$$\Upsilon' : \partial\mathfrak{P}_{2\delta/3,(r_0)} \to \mathrm{D}_\delta(\overline{\mathfrak{m}}^b), \tag{7.12.5}$$

which maps $\mathfrak{d}$ to $\mathrm{G}(\mathfrak{d})^{-1}(\sigma_\infty^-)$. Since $\partial\mathfrak{P}_{2\delta/3,(r_0)}$ is compact (note that restricting to $\{\mathrm{T}_+ = r_0\}$ is the same as taking a slice under the $\mathbf{R}^+$-action on $\mathcal{V}_{2\delta/3}$), the map Υ' is proper and the local degree is well-defined.

Also define

$$\Upsilon'' : \partial\mathfrak{P}_{2\delta/3,(r_0)} \to \mathrm{D}_\delta(\overline{\mathfrak{m}}^b), \tag{7.12.6}$$

which maps $\mathfrak{d}$ to the zero of $\mathfrak{E}^{-1} \circ \Pi(\mathfrak{d})$. The map Υ'' is well-defined by the definitions of $\mathfrak{P}_{2\delta/3}$ and $\mathcal{V}_\delta$. It is also proper and the local degree is well-defined.

The maps Υ' and Υ'' are sufficiently close in the following sense.

Lemma **7.12.16.** — *For any k, Υ' and Υ'' can be made arbitrarily* C^0*-close by choosing r_0 sufficiently large.*

Proof. — This follows from Lemma 7.12.14. □

In particular, the local degrees of Υ' and Υ'' near $\overline{\mathfrak{m}}^b$ agree. The local degree of Υ' is equal to the left-hand side of Equation (7.2.3), while the local degree of Υ'' is equal to the right-hand side of Equation (7.2.3). (We were assuming that each of $\mathcal{M}_1/\mathbf{R}$, $\mathcal{M}_0$, and $\mathcal{M}_{-1}/\mathbf{R}$ is connected.) This completes the proof of Theorem 7.2.2.

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Competing Interests

The authors declare no competing interests.

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