THE GLOBAL GAN-GROSS-PRASAD CONJECTURE FOR UNITARY GROUPS: THE ENDOSCOPIC CASE

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ABSTRACT

In this paper, we prove the Gan-Gross-Prasad conjecture and the Ichino-Ikeda conjecture for unitary groups $U_n \times U_{n+1}$ in all the endoscopic cases. Our main technical innovation is the computation of the contributions of certain cuspidal data, called $*$-regular, to the Jacquet-Rallis trace formula for linear groups. We offer two different computations of these contributions: one, based on truncation, is expressed in terms of regularized Rankin-Selberg periods of Eisenstein series and Flicker-Rallis intertwining periods introduced by Jacquet-Lapid-Rogawski. The other, built upon Zeta integrals, is expressed in terms of functionals on the Whittaker model. A direct proof of the equality between the two expressions is also given. Finally several useful auxiliary results about the spectral expansion of the Jacquet-Rallis trace formula are provided.

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1. Introduction

1.1. *The endoscopic cases of the Gan-Gross-Prasad conjecture*

1.1.1. One of the main motivation of the paper is the obtention of the remaining cases, the so-called "endoscopic cases", of the Gan-Gross-Prasad and the Ichino-Ikeda conjectures for unitary groups. To begin with, we shall give the main statements we prove.

1.1.2. Let E/F be a quadratic extension of number fields and c be the non-trivial element of the Galois group Gal(E/F). Let $\mathbf{A}$ be the ring of adèles of F. Let η be the quadratic idele class character associated to the extension E/F. Let $n \geqslant 1$ be an integer. Let $\mathcal{H}_n$ be the set of isomorphism classes of non-degenerate c-Hermitian spaces h over E of rank n. For any $h_n \in \mathcal{H}_n$, we identify h_n with a representative and we shall denote by $\mathrm{U}(h_n)$ its automorphisms group. Let $h_0 \in \mathcal{H}_1$ be the element of rank 1 given by the norm $\mathrm{N}_{\mathrm{E/F}}$.

We attach to any $h \in \mathcal{H}_n$ the following algebraic groups over F:

- the unitary group U'_h of automorphisms of h;
- the product of unitary groups $\mathrm{U}_h = \mathrm{U}(h) \times \mathrm{U}(h \oplus h_0)$ where $h \oplus h_0$ denoted the orthogonal sum.

We have an obvious diagonal embedding $\mathrm{U}'_h \hookrightarrow \mathrm{U}_h$.

1.1.3. *Arthur parameter.* — Let G_n be the group of automorphisms of the E-vector space E^n. We view G_n as an F-group by Weil restriction. By a *Hermitian Arthur parameter*[1] of G_n, we mean an irreducible automorphic representation Π for which there exists a partition $n_1 + \cdots + n_r = n$ of n and for any $1 \leqslant i \leqslant r$ a cuspidal automorphic representation Π_i of $\mathrm{G}_{n_i}(\mathbf{A})$ such that

[1] Strictly speaking, it is a *generic discrete* Arthur parameter. By simplicity, we shall omit the adjectives *generic discrete*.

1. each Π_i is conjugate self-dual and the Asai L-function $L(s, \Pi_i, As^{(-1)^{n+1}})$ has a pole at $s = 1$;
2. the representations Π_i are mutually non-isomorphic for $1 \leqslant i \leqslant r$;
3. the representation Π is isomorphic to the full induced representation $\mathrm{Ind}_P^{G_n}(\Pi_1 \boxtimes \ldots \boxtimes \Pi_r)$ where P is a parabolic subgroup of G_n of Levi factor $G_{n_1} \times \cdots \times G_{n_r}$.

Remark **1.1.3.1.** — It is well-known (see [Fli88]) that condition 1 above is equivalent to the fact that Π_i is $(GL_{n_i,F}, \eta^{n+1})$-distinguished in the sense of Section 4.1.2 below.

The integer r and the representations $(\Pi_i)_{1\leqslant i\leqslant r}$ are unique (up to a permutation). We set $S_\Pi = (\mathbf{Z}/2\mathbf{Z})^r$.

Let $G = G_n \times G_{n+1}$. By a *Hermitian Arthur parameter* of G, we mean an automorphic representation of the form $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ where Π_i is a Hermitian Arthur parameter of G_i for $i = n, n+1$. For such a Hermitian Arthur parameter, we set $S_\Pi = S_{\Pi_n} \times S_{\Pi_{n+1}}$.

1.1.4. Let $h \in \mathcal{H}_n$ and σ be a cuspidal automorphic representation of $U_h(\mathbf{A})$. We say that a Hermitian Arthur parameter Π of G is a *weak base-change* of σ if for almost all places of F that split in E, the local component Π_v is the split local base change of σ_v. If this is the case, we write $\Pi = BC(\sigma)$.

Remark **1.1.4.1.** — By the work of Mok [Mok15] and Kaletha-Minguez-Shin-White [KMSW], we know that if σ admits a weak base-change then it admits a *strong base-change* that is a Hermitian Arthur parameter Π of G_n such that Π_v is the base-change of σ_v for every place v of F (where the local base-change of ramified representations is also constructed in *loc. cit.* and characterized by certain local character relations). Besides, a result of Ramakrishnan [Ram18] implies that a weak base-change is automatically a strong base-change. Therefore, we could have used the notion of strong base-change instead. However, we prefer to stick with the terminology of weak base-change in order to keep the statement of the next theorem independent of [Mok15] and [KMSW].

1.1.5. *Gan-Gross-Prasad conjecture.* — Our first main result is the global Gan-Gross-Prasad conjecture [GGP12, Conjecture 24.1] in the case of $U(n) \times U(n+1)$ and can be stated as follows. In the following, for a reductive group H over F, we denote by [H] the quotient $H(F)\backslash H(\mathbf{A})$ equipped with the quotient of a Haar measure on $H(\mathbf{A})$ by the counting measure on H(F).

Theorem **1.1.5.1.** — *Let* Π *be a Hermitian Arthur parameter of* G. *The following two statements are equivalent:*

1. *The complete*[2] *Rankin-Selberg* L*-function of* Π *satisfies*

$$\mathrm{L}(\frac{1}{2}, \Pi) \neq 0;$$

2. *There exists* $h \in \mathcal{H}_n$ *and an irreducible cuspidal automorphic subrepresentation* σ *of* U_h *such that* Π *is a weak base change of* σ *and the period integral* $\mathcal{P}_h$ *defined by*

$$\mathcal{P}_h(\varphi) = \int_{[\mathrm{U}'_h]} \varphi(h)\, dh$$

induces a non-zero linear form on the space of σ.

Remark **1.1.5.2.** — If the Arthur parameter is moreover simple (that is if Π is cuspidal), the theorem is proved by Beuzart-Plessis-Liu-Zhang-Zhu (cf. [BPLZZ21, Theorem 1.7]). Previous works had to assume extra local hypothesis on Π, which implied that Π was also simple (see [Zha14b], [Xue19], [BP21a] and [BP21c]) or only proved the direction 2. $\Rightarrow$ 1. of the theorem ([GJR09], [IY19], [JZ20]).

As observed in [Zha14b, Theorem 1.2] and [BPLZZ21, Theorem 1.8] we can deduce from Theorem 1.1.5.1 the following statement (whose proof is word for word that of [Zha14b]):

Theorem **1.1.5.3.** — *Let* Π_{n+1} *be a Hermitian Arthur parameter of* G_{n+1}. *Then there exists a simple Hermitian Arthur parameter* Π_n *of* G_n *such that the Rankin-Selberg* L*-function satisfies:*

$$\mathrm{L}(\frac{1}{2}, \Pi_n \times \Pi_{n+1}) \neq 0.$$

1.1.6. *Ichino-Ikeda conjecture.* — Let $\sigma = \bigotimes'_v \sigma_v$ be an irreducible cuspidal automorphic representation of U_h that is tempered everywhere in the following sense: for every place v, the local representation σ_v is tempered. By [Mok15] and [KMSW], σ admits a weak (hence a strong) base-change Π to G. Set

$$\mathcal{L}(s, \sigma) = \prod_{i=1}^{n+1} \mathrm{L}(s + i - 1/2, \eta^i) \frac{\mathrm{L}(s, \Pi)}{\mathrm{L}(s + 1/2, \sigma, \mathrm{Ad})}$$

where $\mathrm{L}(s, \eta^i)$ is the completed Hecke L-function associated to η^i and $\mathrm{L}(s, \sigma, \mathrm{Ad})$ is the completed adjoint L-function of σ (defined using the local Langlands correspondence for

[2] Currently known bounds towards the Ramanujan conjecture do not exclude the possibility of certain local Rankin-Selberg L-factors of Π to have a pole at $s = 1/2$. Therefore, the non-vanishing of the central value could a priori be affected if we replace $\mathrm{L}(s, \Pi)$ by a partial L-function and here we have to include all places (including Archimedean ones).

G from [Mok15], [KMSW]). We denote by $\mathcal{L}(s,\sigma_v)$ the corresponding quotient of local L-factors. For each place v of F, we define a *local normalized period* $\mathcal{P}^{\natural}_{h,\sigma_v}:\sigma_v\times\sigma_v\to\mathbf{C}$ as follows. It depends on the choice of a Haar measure on $\mathrm{U}'_h(\mathrm{F}_v)$ as well as an invariant inner product $(.,.)_v$ on σ_v and is given by

$$\mathcal{P}^{\natural}_{h,\sigma_v}(\varphi_v,\varphi'_v)=\mathcal{L}(\frac{1}{2},\sigma_v)^{-1}\int_{\mathrm{U}'_h(\mathrm{F}_v)}(\sigma_v(h_v)\varphi_v,\varphi'_v)_v dh_v,\ \ \varphi_v,\varphi'_v\in\sigma_v,$$

where, thanks to the temperedness assumption, the integral is absolutely convergent [Har14, Proposition 2.1] and the local factor $\mathcal{L}(s,\sigma_v)$ has no zero (nor pole) at $s=\frac{1}{2}$. Moreover, by [Har14, Theorem 2.12], if $\varphi=\otimes'_v\varphi_v\in\sigma$, then for almost all places v we have

$$\mathcal{P}^{\natural}_{h,\sigma_v}(\varphi_v,\varphi_v)=\mathrm{vol}(\mathrm{U}'_h(\mathcal{O}_v))(\varphi_v,\varphi_v)_v. \tag{1.1.6.1}$$

We also recall that the global representation σ has a natural invariant inner product given by

$$(\varphi,\varphi)_{\mathrm{Pet}}=\int_{[\mathrm{U}_h]}|\varphi(g)|^2dg,\ \ \varphi\in\sigma.$$

Our second main result is the global Ichino-Ikeda conjecture for unitary groups formulated in [Har14, Conjecture 1.3] and can be stated as follows (this result can be seen as a refinement of Theorem 1.1.5.1, the precise relation requiring the local Gan-Gross-Prasad conjecture and Arthur's multiplicity formula for unitary groups will not be discussed here).

Theorem **1.1.6.1.** — *Assume that σ is a cuspidal automorphic representation of U_h that is tempered everywhere and let $\Pi=\Pi_n\boxtimes\Pi_{n+1}$ be the weak (hence the strong) base-change of σ to G. Suppose that we normalize the period integral $\mathcal{P}_h$ and the Petersson inner product $(.,.)_{\mathrm{Pet}}$ by choosing the invariant Tamagawa measures*[3] *$d_{\mathrm{Tam}}h$ and $d_{\mathrm{Tam}}g$ on $\mathrm{U}'_h(\mathbf{A})$ and $\mathrm{U}_h(\mathbf{A})$ respectively. Assume also that the local Haar measures dh_v on $\mathrm{U}'_h(\mathrm{F}_v)$ factorize the Tamagawa measure: $d_{\mathrm{Tam}}h=\prod_v dh_v$. Then, for every nonzero factorizable vector $\varphi=\otimes'_v\varphi_v\in\sigma$, we have*

$$\frac{|\mathcal{P}_h(\varphi)|^2}{(\varphi,\varphi)_{\mathrm{Pet}}}=|\mathrm{S}_\Pi|^{-1}\mathcal{L}(\frac{1}{2},\sigma)\prod_v\frac{\mathcal{P}^{\natural}_{h,\sigma_v}(\varphi_v,\varphi_v)}{(\varphi_v,\varphi_v)_v}$$

where we recall that S_Π denotes the finite group $\mathrm{S}_{\Pi_n}\times\mathrm{S}_{\Pi_{n+1}}$.

Note that the product over all places in the theorem is well-defined by (1.1.6.1). Moreover, once again, this theorem is proved in [BPLZZ21] under the extra assumption

[3] We warn the reader that our convention is to include the global normalizing L-values in the definition of Tamagawa measures, cf. Section 2.3 for precise definitions.

that Π is cuspidal (in which case $|S_\Pi| = 4$). Previous results in that direction includes [Zha14a], [BP21a], [BP21c] where some varying local assumptions on σ entailing the cuspidality of Π were imposed. In a slightly different direction, the paper [GL] establishes the above identity up to an unspecified algebraic number under some arithmetic assumptions on σ.

1.2. *The spectral expansion of the Jacquet-Rallis trace formula for the linear groups*

1.2.1. *Motivations.* — As in [Zha14b], [Zha14a], [Xue19], [BP21a], [BP21c] and [BPLZZ21], our proofs of Theorems 1.1.5.1 and 1.1.6.1 follow the strategy of Jacquet and Rallis [JR11] and are thus based on a comparison of *relative trace formulas* on unitary groups U_h for $h \in \mathcal{H}_n$ and the group G. Let's recall that these trace formulas have two different expansions: one, called the geometric side, in terms of distributions indexed by geometric classes and the other, called the spectral side, in terms of distributions indexed by cuspidal data. As usual, the point is to get enough test functions to first compare the geometric sides which gives a comparison of spectral sides.

For specific test functions, the trace formula boils down to a simple and quite easy equality between a sum of relative regular orbital integrals and a sum of relative characters attached to cuspidal representations. This is the simple trace formula used by Zhang in [Zha14b] and [Zha14a] to prove special cases of Theorems 1.1.5.1 and 1.1.6.1. In return one has to impose restrictive local conditions on the representations one considers.

In [Zyd16], [Zyd18], [Zyd20], Zydor established general Jacquet-Rallis trace formulas. Besides, in [CZ21], Chaudouard-Zydor proved the comparison of all the geometric terms for matching test functions, that is functions with matching local orbital integrals. Using these results, Beuzart-Plessis-Liu-Zhang-Zhu in [BPLZZ21] proved 1.1.5.1 and 1.1.6.1 when Π is cuspidal. Their main innovation is a construction of Schwartz test functions only detecting certain cuspidal data. In this way, they were able to construct matching test functions for which the spectral expansions reduce to some relative characters attached to cuspidal representations.

1.2.2. In this paper, we also want to use the construction of Beuzart-Plessis-Liu-Zhang-Zhu. But for this, we need two extra ingredients. First we need the slight extension of Zydor's work to the space of Schwartz test functions. For the geometric sides, this was done in [CZ21]. For the test functions we need, the spectral side of the trace formulas for unitary groups still reduces to relative characters attached to cuspidal representations and we need nothing more. But, for the group G, we shall extend the spectral side of the trace formula to the space of Schwartz functions. Second there is an even more serious question: since the representation Π is no longer assumed to be cuspidal, the spectral contribution associated to Π is much more involved. In this section, we shall explain alternative and somewhat more tractable expressions for the spectral contributions in the trace formula for G. For the specific cuspidal datum attached to Π, we get a precise result as we shall see in Section 1.3 below.

1.2.3. *The spectral expansion for the Schwartz space.* — Let $\mathfrak{X}(G)$ be the set of cuspidal data of G (see Section 2.9.1). To χ is associated a direct invariant factor $L^2_\chi([G])$ of $L^2([G])$ (see [MW89, Chap. II] or Section 2.9 for a review). Let f be a function in the Schwartz space $\mathcal{S}(G(\mathbf{A}))$ (cf. Section 2.5.2 for a definition). Let K_f, resp. $K_{f,\chi}$, be the kernel associated to the action by right convolution of f on $L^2([G])$, resp. $L^2_\chi([G])$.

Following [Zyd20] (see Section 3.2.3), we introduce the modified kernel $K^T_{f,\chi}$ depending on a parameter T in a certain real vector space. Set $H = G_n$ and $G' = GL_{n,F} \times GL_{n+1,F}$ both seen as subgroups of G (the embedding $H \hookrightarrow G$ being the "diagonal" one where the inclusion $G_n \hookrightarrow G_{n+1}$ is induced by the identification of E^n with the hyperplane of E^{n+1} of vanishing last coordinate). The following theorem is an extension to Schwartz functions of [Zyd20, théorème 0.1].

Theorem **1.2.3.1.** — *(see Theorem 3.2.4.1)*

1. *For any* T *in a certain positive Weyl chamber, we have*

$$\sum_{\chi\in\mathfrak{X}(G)} \int_{[H]} \int_{[G']} |K^T_{f,\chi}(h,g')|\, dg'\, dh < \infty$$

2. *Let* $\eta_{G'}$ *be the quadratic character of* $G'(\mathbf{A})$ *defined in Section 3.1.6. For each* $\chi \in \mathfrak{X}(G)$, *the integral*

$$\int_{[H]} \int_{[G']} K^T_{f,\chi}(h,g')\eta_{G'}(g')\, dg'\, dh \qquad (\mathbf{1.2.3.1})$$

   coincides with a polynomial-exponential function in T *whose purely polynomial part is constant and denoted by* $I_\chi(f)$.
3. *The distributions* I_χ *are continuous, left* $H(\mathbf{A})$*-equivariant and right* $(G'(\mathbf{A}), \eta_{G'})$*-equivariant. Moreover the sum*

$$I(f) = \sum_\chi I_\chi(f) \qquad (\mathbf{1.2.3.2})$$

   is absolutely convergent and defines a continuous distribution.

The (coarse) spectral expansion of the trace formula for G is precisely the expression (1.2.3.2).

1.2.4. The definition of I_χ given in Theorem 1.2.3.1 is convenient to relate the spectral expansion to the geometric expansion. However, to get more explicit forms of the distributions I_χ, we shall use the following three expressions:

$$\int_{[H]} \int_{[G']} (\Lambda^T_r K_{f,\chi})(h,g')\, \eta_{G'}(g')\, dg'\, dh \qquad (\mathbf{1.2.4.3})$$

$$(\mathbf{1.2.4.4}) \qquad \int_{[\mathrm{H}]} \mathrm{F}^{\mathrm{G}_{n+1}}(h, \mathrm{T}) \int_{[\mathrm{G}']} \mathrm{K}_{f,\chi}(h, g')\, \eta_{\mathrm{G}'}(g')\, dg'\, dh$$

$$(\mathbf{1.2.4.5}) \qquad \int_{[\mathrm{G}']} \mathrm{F}^{\mathrm{G}_{n+1}}(g'_n, \mathrm{T}) \int_{[\mathrm{H}]} \mathrm{K}_{f,\chi}(h, g')\, dh\, \eta_{\mathrm{G}'}(g')\, dg'$$

Essentially they are given by integration of the kernel $\mathrm{K}_{f,\chi}$ along $[\mathrm{H}] \times [\mathrm{G}']$. However, to have a convergent expression for a general χ, one needs to use some truncation depending on the same parameter T as above. We introduce the Ichino-Yamana truncation operator, denoted by Λ_r^{T}, whose definition is recalled in Section 3.3.2. In (1.2.4.3), we apply it to the left-variable of $\mathrm{K}_{f,\chi}$. But one can also use the Arthur characteristic function $\mathrm{F}^{\mathrm{G}_{n+1}}(\cdot, \mathrm{T})$ whose definition is recalled in 3.3.4. In (1.2.4.4), this function is evaluated at $h \in \mathrm{H}(\mathbf{A})$ through the embedding $\mathrm{H} = \mathrm{G}_n \hookrightarrow \mathrm{G}_{n+1}$. In (1.2.4.5), it is evaluated at the component g'_n of the variable $g' = (g'_n, g'_{n+1}) \in \mathrm{G}'(\mathbf{A}) = \mathrm{GL}_n(\mathbf{A}) \times \mathrm{GL}_{n+1}(\mathbf{A})$.

The link with the distribution I_χ is provided by the following theorem (which is a combination of Propositions 3.3.3.1 and 3.3.5.1 and Theorem 3.3.9.1). Note that we shall not need the full strength of the theorem in this paper. However it will be used in a greater generality in a subsequent paper.

Theorem **1.2.4.1.** — *Let $f \in \mathcal{S}(\mathrm{G}(\mathbf{A}))$ and $\chi \in \mathfrak{X}(\mathrm{G})$.*

1. *For any* T *in some positive Weyl chamber, the expressions* (1.2.4.3), (1.2.4.4) *and* (1.2.4.5) *are absolutely convergent.*
2. *Each of the three expressions is asymptotically equal (in the technical sense of assertion 2 of Theorem 3.3.9.1)) to a polynomial-exponential function of* T *whose purely polynomial term is constant and equal to* $\mathrm{I}_\chi(f)$.

1.2.5. Note that powerful estimates for modified kernels are introduced and used in the proofs of Theorems 1.2.3.1 and 1.2.4.1. We refer the reader to Theorem 3.3.7.1 for a precise statement.

1.3. *On the $*$-regular contribution for the Jacquet-Rallis trace formula for the linear groups*

1.3.1. From now on we assume that the cuspidal datum χ is relevant $*$-regular that is χ is the class of a pair (M, π) with the property that the normalized induction $\Pi := \mathrm{Ind}_{\mathrm{P}(\mathbf{A})}^{\mathrm{G}(\mathbf{A})}(\pi)$, where we have fixed a parabolic subgroup P with Levi component M, is a Hermitian Arthur parameter of G. To Π we associate, following [Zha14a, §3.4], a *relative character* I_Π. The precise definition of this object is recalled in Section 8.1.3. Let us just say here that it is associated to two functionals λ and β_η on the Whittaker model $\mathcal{W}(\Pi, \psi_{\mathrm{N}})$ of Π, where ψ_{N} is a certain generic automorphic character of the standard maximal unipotent subgroup N of G, that naturally show up in integrals of Rankin-Selberg type. More precisely, λ is the value at $s = \frac{1}{2}$ of a family of Zeta integrals, studied by Jacquet-Piatetski-Shapiro-Shalika [JPSS83], representing the Rankin-Selberg

L-function $L(s, \Pi)$ whereas β_η is essentially the pole at $s = 1$ of another family of Zeta integrals, first introduced by Flicker [Fli88], representing the (product of) Asai L-functions $L(s, \Pi, As_G) := L(s, \Pi_n, As^{(-1)^{n+1}})L(s, \Pi_{n+1}, As^{(-1)^n})$. The relative character I_Π is then given in terms of these functionals by

$$(\mathbf{1.3.1.1}) \qquad I_\Pi(f) = \sum_{\varphi \in \Pi} \lambda(\Pi(f)W_\varphi)\overline{\beta_\eta(W_\varphi)}, \; f \in \mathcal{S}(G(\mathbf{A})),$$

where the sum runs over an orthonormal basis of Π (for the Petersson inner product) and W_φ denotes the Whittaker function associated to the Eisenstein series $E(\varphi)$ (obtained, as usual, by integrating $E(\varphi)$ against ψ_N^{-1} over $[N]$).

The following is our main technical result whose proof occupies most part of the paper.

Theorem **1.3.1.1.** — *Let* χ *be a cuspidal datum associated to a Hermitian Arthur parameter* Π *as above. Then, for every function* $f \in \mathcal{S}(G(\mathbf{A}))$ *we have*

$$I_\chi(f) = 2^{-\dim(A_M)} I_\Pi(f)$$

where A_M *denotes the maximal central split torus of* M.

Remark **1.3.1.2.** — It is perhaps worth emphasizing that the contribution of χ is *purely discrete* in the Jacquet-Rallis trace formula. Such a phenomenon happens in Jacquet relative trace formula, see [Lap06]. By contrast, the contribution of the same kind of cuspidal datum χ to the Arthur-Selberg trace formula is purely continuous (unless, of course, if Π is cuspidal).

We shall provide two different proofs of Theorem 1.3.1.1, one based on truncations, the other using integral representations of Asai and Rankin-Selberg L-functions. Let's explain separately the main steps of each approach.

1.3.2. *A journey through truncations.* — We first begin with the approach based on truncations. The first step is to get a spectral decomposition of the function

$$(\mathbf{1.3.2.2}) \qquad \int_{[G']} K_{f,\chi}(g, g')\eta_{G'}(g')dg',$$

of the variable $g \in [G]$.

The kernel itself $K_{f,\chi}$ has a well-known spectral decomposition based on the Langlands decomposition. Then the problem is basically to invert an adelic integral and a complex integral. It is solved by Lapid in [Lap06] (up to some non-explicit constants) but we will use a slightly different method avoiding delicate Lapid's contour moving. Instead we replace the integral (1.3.2.2) by its truncated version

$$(\mathbf{1.3.2.3}) \qquad \int_{[G']} (K_{f,\chi}\Lambda_m^T)(g,g')\eta_{G'}(g')dg',$$

where the *mixed truncation operator* Λ_m^T defined by Jacquet-Lapid-Rogawski [JLR99] is applied to the right variable of the kernel. We can recover (1.3.2.2) by taking the limit when $T \to +\infty$. It is easy to get the spectral decomposition of (1.3.2.3) (see Proposition 4.2.3.3). Using an analog of the famous Maaß-Selberg relations due to Jacquet-Lapid-Rogawski (see [JLR99] and Lemma 4.3.6.2 below), we get in Proposition 4.3.6.1 that (1.3.2.3) is equal to a finite sum of contributions (up to an explicit constant) of the following type

$$(\mathbf{1.3.2.4}) \qquad \int_{i\mathfrak{a}_P^{G,*}} \sum_{Q\in\mathcal{P}(M)} J_{Q,\chi}(g,\lambda,f)\frac{\exp(-\langle\lambda,T_Q\rangle)}{\theta_Q(-\lambda)}\,d\lambda.$$

Here it suffices to say that $i\mathfrak{a}_P^{G,*}$ is some space of unramified unitary characters and that $J_{Q,\chi}(g,\lambda,f)$ is a certain relative character built upon Flicker-Rallis intertwining periods (introduced by Jacquet-Lapid-Rogawski). The integrand is a familiar expression of Arthur's theory of (G,M)-families with quite standard notations. It turns out that the family $(J_{Q,\chi}(g,\lambda,f))_{Q\in\mathcal{P}(M)}$ is indeed an Arthur (G,M)-family of Schwartz functions in the parameter λ. Let's emphasize that this Schwartz property relies in fact on deep estimates introduced by Lapid in [Lap06] and [Lap13]. By a standard argument, it is then easy to get the limit of (1.3.2.4) when $T \to +\infty$ which gives the spectral decomposition of (1.3.2.2) (see Theorem 4.3.3.1). Note that the spectral decomposition we get is already discrete at this stage.

From this result, one gets the equality

$$(\mathbf{1.3.2.5}) \qquad \int_{[H]}\int_{[G']} (\Lambda_r^T K_{f,\chi})(h,g')\,\eta_{G'}(g')\,dhdg' = 2^{-\dim(A_M)} I_{P,\pi}(f).$$

The left-hand side has been defined in Section 1.2.4 and the relative character $I_{P,\pi}$ is defined as follows:

$$\sum_{\varphi\in\Pi} I_{RS}(\Pi(f)\varphi)\cdot\overline{J_\eta(\varphi)}$$

where the sum is over an orthonormal basis, $I_{RS}(\varphi)$ is the regularized Rankin-Selberg period of the Eisenstein series $E(\varphi)$ defined by Ichino-Yamana and $J_\eta(\varphi)$ is a Flicker-Rallis intertwining period (for more detail we refer to Section 5.1.5).

In particular, the left-hand side of (1.3.2.5) does not depend on T. So Theorem 1.2.4.1 implies

Theorem **1.3.2.1.** — *(see Theorem 5.2.1.1 for a slightly more precise statement)*

$$I_\chi(f) = 2^{-\dim(A_M)} I_{P,\pi}(f).$$

Remark **1.3.2.2.** — As the reading of Section 10.2 should make it clear, this statement suffices to prove the Gan-Gross-Prasad conjecture namely Theorem 1.1.5.1. However, to get the Ichino-Ikeda conjecture, namely Theorem 1.1.6.1, we will want to use statements about comparison of local relative characters written in terms of Whittaker functions. For this purpose, Theorem 1.3.1.1 will be more convenient.

The link between regularized Rankin-Selberg period of Eisenstein series and Whittaker functionals has been investigated by Ichino-Yamana (see [IY15]). The following theorem relates the Flicker-Rallis intertwining periods to the functional $\beta_\eta(W_\varphi)$ in (1.3.1.1). It uses a local unfolding method inspired from [FLO12, Appendix A] (see Section 9).

Theorem **1.3.2.3.** — *For all* $\varphi \in \Pi$, *we have*

$$J_\eta(\varphi) = \beta_\eta(W_\varphi)$$

In this way, one proves the following theorem which implies Theorem 1.3.1.1.

Theorem **1.3.2.4.**

$$I_{P,\pi} = I_\Pi.$$

1.3.3. *Second proof: the use of Zeta integrals.* — The spectral decomposition of (1.3.2.2) essentially boils down to a spectral expansion of the period integral

$$P_{G',\eta}(\varphi) := \int_{[G']} \varphi(g')\eta_{G'}(g')dg'$$

for test functions $\varphi \in \mathcal{S}_\chi([G])$, where $\mathcal{S}_\chi([G])$ denotes the *Schwartz space* of $[G]$ consisting of smooth functions rapidly decaying with all their derivatives that are "supported on χ" (see Section 2.5 for a precise definition). Choose a parabolic subgroup $P = MN_P$ with Levi component M. By Langlands L^2 spectral decomposition and of the special form of χ, any $\varphi \in \mathcal{S}_\chi([G])$ admits a spectral decomposition

$$\textbf{(1.3.3.6)} \qquad \varphi = \int_{i\mathfrak{a}_M^*} E(\varphi_\lambda)d\lambda$$

where $i\mathfrak{a}_M^*$ denotes the real vector space of unramified unitary characters of $M(\mathbf{A})$ and φ_λ belongs to the normalized induction space $\mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi \otimes \lambda)$ and $E(\varphi_\lambda)$ is the associated Eisenstein series.

Theorem **1.3.3.1.** — *For every* $\varphi \in \mathcal{S}_\chi([G])$, *we have*

$$P_{G',\eta}(\varphi) = 2^{-\dim(A_M)}\beta_\eta(W_{\varphi_0}),$$

where W_{φ_0} *stands for the Whittaker function of the Eisenstein series* $E(\varphi_0)$.

The proof of Theorem 1.3.3.1 is close to the computation by Flicker [Fli88] of the Flicker-Rallis period of cusp forms in terms of an Asai L-function and local Zeta integrals. More precisely, we first realize $P_{G',\eta}(\varphi)$ as the residue at $s=1$ of the inner product of the restriction $\varphi_{|[G']}$ with some Eisenstein series $E(s,\phi)$ (where ϕ is an auxiliary Schwartz function on $\mathbf{A}^n \oplus \mathbf{A}^{n+1}$). Mimicking the unfolding of *loc. cit.* we connect this inner product with an Eulerian Zeta integral $Z^{FR}(s, W_\varphi, \phi)$ involving the Whittaker function W_φ of φ (obtained as before by integration against ψ_N^{-1}). We should emphasize here that, since φ is not a cusp form, the unfolding gives us more terms but using the special nature of the cuspidal datum χ we are able to show that these extra terms do not contribute to the residue at $s=1$. The formation of $Z^{FR}(s, W_\varphi, \phi)$ commutes with the spectral expansion (1.3.3.6) when $\Re(s) \gg 1$ and, as follows from the local theory, the Zeta integrals $Z^{FR}(s, W_{\varphi_\lambda}, \phi)$ for $\lambda \in i\mathfrak{a}_M^*$ are essentially Asai L-functions whose meromorphic continuations, poles and growths in vertical strips are known. Combining this with an application of the Phragmen-Lindelöf principle, we are then able to deduce Theorem 1.3.3.1.

Let us mention here that, as in the proof of (1.3.2.5), a key point is the fact (due to Lapid ([Lap13] or [Lap06])) that the spectral transform $\lambda \mapsto \varphi_\lambda$ is, in a suitable technical sense, "Schwartz" that is rapidly decreasing together with all its derivatives.

The second step is to integrate (1.3.2.2) over $g \in [H]$. To do so, we define a regularization of the integral over [H] that doesn't require truncation. More precisely, denoting by $\mathcal{T}([G])$ the space of functions of *uniform moderate growth* on [G], we can define the "χ-part" $\mathcal{T}_\chi([G])$ of $\mathcal{T}([G])$ (see Section 2.5) of which $\mathcal{S}_\chi([G])$ is a dense subspace. Moreover, starting with $\varphi \in \mathcal{T}([G])$ we can also form its Whittaker function W_φ and consider the usual Rankin-Selberg integral $Z^{RS}(s, W_\varphi)$ that converges for $\Re(s) \gg 1$ and represents, when φ is an automorphic form, the Rankin-Selberg L-function for $G_n \times G_{n+1}$.

Theorem **1.3.3.2.** — *(see Theorem 7.1.3.1) The functional*

$$\varphi \in \mathcal{S}_\chi([G]) \mapsto \int_{[H]} \varphi(h)dh$$

extends by continuity to a functional on $\mathcal{T}_\chi([G])$ denoted by $\varphi \in \mathcal{T}_\chi([G]) \mapsto \int_{[H]}^* \varphi(h)dh$. Moreover, for every $\varphi \in \mathcal{T}_\chi([G])$, the Zeta function $s \mapsto Z^{RS}(s, W_\varphi)$ extends to an entire function on $\mathbf{C}$ and we have

$$\int_{[H]}^* \varphi(h)dh = Z^{RS}(\frac{1}{2}, W_\varphi).$$

The proof of this theorem is similar to that of Theorem 1.3.3.1: we first show that, for $\varphi \in \mathcal{S}_\chi([G])$ and $\Re(s) \gg 1$, we have

$$\int_{[H]} \varphi(h)|\det h|^s dh = Z^{RS}(s+\frac{1}{2}, W_\varphi)$$

by mimicking the usual unfolding for the Rankin-Selberg integral. Once again, as φ is not necessarily a cusp form, we get extra terms in the course of the unfolding but, thanks to the special nature of the cuspidal datum χ, we are able to show that they all vanish. At this point, we use the spectral decomposition (1.3.3.6) to express $Z^{RS}(s, W_\varphi)$ as the integral of $Z^{RS}(s, W_{\varphi_\lambda})$ when $\Re(s) \gg 1$. By Rankin-Selberg theory, $Z^{RS}(s, W_{\varphi_\lambda})$ is essentially a Rankin-Selberg L-function whose meromorphic continuation, location of the poles, control in vertical strips and functional equation are known. Combining this with another application of the Phragmen-Lindelöf principle, we are able to bound $Z^{RS}(\frac{1}{2}, W_\varphi) = \int_{[H]} \varphi(h)dh$ in terms of $Z^{RS}(s, W_\varphi)$ for $\Re(s) \gg 1$ and this readily gives the theorem.

One direct consequence of Theorem 1.3.3.2 is that the regularized period $\int_{[H]}^* E(h, \Pi(f)\varphi)dh$ coincides with $Z^{RS}(\frac{1}{2}, \Pi(f)W_\varphi) = \lambda(\Pi(f)W_\varphi)$. Thus by a combination of Theorems 1.3.3.1 and 1.3.3.2, we get

$$\text{(1.3.3.7)} \qquad \int_{[H]}^* \int_{[G']} K_{f,\chi}(h, g')\eta_{G'}(g')dg'\, dh = 2^{-\dim(A_M)} I_\Pi(f).$$

Finally we have to show that the left-hand side is equal to $I_\chi(f)$. In fact, we show (see Theorem 8.1.4.1 and Section 8.2.3) that we have

$$\int_{[H]}^* \int_{[G']} K_{f,\chi}(h, g')\eta_{G'}(g')dg'\, dh = \int_{[G']} \int_{[H]} K_{f,\chi}(h, g')\, dh\, \eta_{G'}(g')dg'$$

where the right-hand side is (conditionally) convergent. We can conclude that it is equal to $I_\chi(f)$ by applying Theorem 1.2.4.1 to the expression (1.2.4.5).

1.4. *Outline of the paper*

We now give a quick outline of the content of the paper. Section 2 contains preliminary material. Notably, we fix most notation to be used in the paper, we explain our convention on normalization of measures, we introduce the various spaces of functions we need and we discuss several properties of Langlands decomposition along cuspidal data as well as kernel functions that are important for us. Section 3 contains the statements and proofs concerning the spectral expansion of the Jacquet-Rallis trace formula for G that were discussed in Section 1.2 above.

In Section 4, we introduce the Flicker-Rallis intertwining periods and prove the spectral expansion of the Flicker-Rallis period of the kernel associated to $*$-regular cuspidal datum. In Section 5 we deduce from it Theorem 1.3.2.1 namely the spectral expansion for $I_\chi(f)$. Sections 6 and 7 are devoted to the proofs of Theorems 1.3.3.1 and 1.3.3.2 respectively. These two theorems are combined in Section 8 to give another proof of the spectral expansion of $I_\chi(f)$ (Theorem 1.3.1.1). In Section 9, we relate the Flicker-Rallis intertwining periods to the functional β_η. From this, we deduce Theorem 1.3.2.3

and Theorem 1.3.2.4. The final Section 10 explain the deduction of Theorems 1.1.5.1 and 1.1.6.1 from Theorem 1.3.1.1. Finally, we have gathered in Appendix A some useful facts on topological vector spaces and holomorphic functions valued in them. It contains in particular some variations on the theme of the Phragmen-Lindelöf principle for such functions that will be crucial for the proofs of Theorems 1.3.3.1 and 1.3.3.2.

2. Preliminaries

2.1. *General notation*

2.1.1. For f and g two positive functions on a set X, we way that f is *essentially bounded* by g and we write

$$f(x) \ll g(x),\ x \in \mathrm{X},$$

if there exists a constant $\mathrm{C} > 0$ such that $f(x) \leqslant \mathrm{C}g(x)$ for every $x \in \mathrm{X}$. If we want to emphasize that the constant C depends on auxiliary parameters $y_1, \ldots, y_k$, we will write $f(x) \ll_{y_1,\ldots,y_k} g(x)$. We say that the functions f and g are *equivalent* and we write

$$f(x) \sim g(x),\ x \in \mathrm{X},$$

if $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

2.1.2. For every $\mathrm{C}, \mathrm{D} \in \mathbf{R} \cup \{-\infty\}$ with $\mathrm{D} > \mathrm{C}$, we set $\mathcal{H}_{>\mathrm{C}} = \{z \in \mathbf{C} \mid \Re(z) > \mathrm{C}\}$ and $\mathcal{H}_{]\mathrm{C},\mathrm{D}[} = \{z \in \mathbf{C} \mid \mathrm{C} < \Re(z) < \mathrm{D}\}$. A *vertical strip* is a subset of $\mathbf{C}$ which is the closure of $\mathcal{H}_{]\mathrm{C},\mathrm{D}[}$ for some $\mathrm{C}, \mathrm{D} \in \mathbf{R}$ with $\mathrm{D} > \mathrm{C}$.

When f is a meromorphic function on some open subset U of $\mathbf{C}$ and $s_0 \in \mathrm{U}$, we denote by $f^*(s_0)$ the leading term in the Laurent expansion of f at s_0.

2.1.3. When G is a group and we have a space of functions on it invariant by right (resp. left) translation, we denote by R (resp. L) the corresponding representation of G. If G is a Lie group and the representation is differentiable, we will also denote by the same letter the induced action of the Lie algebra or of its associated enveloping algebra. If G is a topological group equipped with a bi-invariant Haar measure, we denote by $*$ the convolution product (whenever it is well-defined).

2.1.4. We refer the reader to Appendix A for reminders on relevant notions of functional analysis that will be used without further comments in the core of the paper. In particular, we will use the notation $\widehat{\otimes}$ to denote the projective completed tensor product between two locally convex topological vector spaces (over $\mathbf{C}$). Moreover, most of the topological vector spaces we will consider are Banach, Hilbert, Fréchet or LF.

2.2. *Algebraic groups and adelic points*

2.2.1. Let F be a number field and $\mathbf{A}$ its adele ring. We write $\mathbf{A}_f$ for the ring of finite adeles and $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R}$ for the product of Archimedean completions of F so that $\mathbf{A} = F_\infty \times \mathbf{A}_f$. Let V_F be the set of places of F and $V_{F,\infty} \subset V_F$ be the subset of Archimedean places. For every $v \in V_F$, we let F_v be the local field obtained by completion of F at v. We denote by $|\cdot|$ the morphism $\mathbf{A}^\times \to \mathbf{R}_+^\times$ given by the product of normalized absolute values $|\cdot|_v$ on each F_v. For any finite subset $S \subset V_F \setminus V_{F,\infty}$, we denote by $\mathcal{O}_F^S$ the ring of S-integers in F.

2.2.2. Let G be an algebraic group defined over F. We denote by N_G the unipotent radical of G. Let $X^*(G)$ be the group of characters of G defined over F. Let $\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\mathfrak{a}_G = \mathrm{Hom}_{\mathbf{Z}}(X^*(G), \mathbf{R})$. We have a canonical pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{a}_G^* \times \mathfrak{a}_G \to \mathbf{R}. \tag{2.2.2.1}$$

We have also a canonical homomorphism

$$H_G : G(\mathbf{A}) \to \mathfrak{a}_G \tag{2.2.2.2}$$

such that $\langle \chi, H_G(g) \rangle = \log|\chi(g)|$ for any $g \in G(\mathbf{A})$. The kernel of H_G is denoted by $G(\mathbf{A})^1$. We define $[G] = G(F)\backslash G(\mathbf{A})$ and $[G]^1 = G(F)\backslash G(\mathbf{A})^1$.

We let $\mathfrak{g}_\infty$ be the Lie algebra of $G(F_\infty)$ and $\mathcal{U}(\mathfrak{g}_\infty)$ be the enveloping algebra of its complexification and $\mathcal{Z}(\mathfrak{g}_\infty) \subset \mathcal{U}(\mathfrak{g}_\infty)$ be its center.

2.2.3. From now on we assume that G is also reductive. We will mainly use the notations of Arthur's works. For the convenience of the reader, we recall some of them. Let P_0 be a parabolic subgroup of G defined over F and minimal for these properties. Let M_0 be a Levi factor of P_0 defined over F.

We call a parabolic (resp. and semi-standard, resp. and standard) subgroup of G a parabolic subgroup of G defined over F (resp. which contains M_0, resp. which contains P_0). For any semi-standard parabolic subgroup P, we have a Levi decomposition $P = M_P N_P$ where M_P contains M_0 and we define $[G]_P = M_P(F)N_P(\mathbf{A})\backslash G(\mathbf{A})$. We call a Levi subgroup of G (resp. semi-standard, resp. standard) a Levi factor defined over F of a parabolic subgroup of G defined over F (resp. semi-standard, resp. standard).

2.2.4. Let $K = \prod_{v \in V_F} K_v \subset G(\mathbf{A})$ be a "good" maximal compact subgroup in good position relative to M_0. We write

$$K = K_\infty K^\infty$$

where $K_\infty = \prod_{v \in V_{F,\infty}} K_v$ and $K^\infty = \prod_{v \in V_F \setminus V_{F,\infty}} K_v$. We let $\mathfrak{k}_\infty$ be the Lie algebra of K_∞ and $\mathcal{U}(\mathfrak{k}_\infty)$ be the enveloping algebra of its complexification and $\mathcal{Z}(\mathfrak{k}_\infty) \subset \mathcal{U}(\mathfrak{k}_\infty)$ be its center.

2.2.5. Let P be a semi-standard parabolic subgroup. We extend the homomorphism $H_P : P(\mathbf{A}) \to \mathfrak{a}_P$ (see (2.2.2.2)) into the Harish-Chandra map

$$H_P : G(\mathbf{A}) \to \mathfrak{a}_P$$

in such a way that for every $g \in G(\mathbf{A})$ we have $H_P(g) = H_P(p)$ where $p \in P(\mathbf{A})$ is given by the Iwasawa decomposition namely $g \in pK$. Let $H_0 = H_{P_0}$.

2.2.6. Let A be a split torus over F. Then, A admits an unique split model over $\mathbf{Q}$ (which is also the maximal $\mathbf{Q}$-split subtorus of $\mathrm{Res}_{F/\mathbf{Q}}(A)$) and by abuse of notation we denote by $A(\mathbf{R})$ the group of $\mathbf{R}$-points of this model. In the particular case of the multiplicative group $\mathbf{G}_{m,F}$, we get an embedding $\mathbf{R}^\times \subset F_\infty^\times \subset \mathbf{A}^\times$. We also write A^∞ for the neutral component of $A(\mathbf{R})$. Let A_G be the maximal central F-split torus of G. We define $[G]_0 = A_G^\infty G(F)\backslash G(\mathbf{A})$.

Let P be a semi-standard parabolic subgroup of G. We define $A_P = A_{M_P}$, $A_P^\infty = A_{M_P}^\infty$ and $[G]_{P,0} = A_P^\infty M_P(F) N_P(\mathbf{A})\backslash G(\mathbf{A})$. The restrictions maps $X^*(P) \to X^*(M_P) \to X^*(A_P)$ induce isomorphisms $\mathfrak{a}_P^* \simeq \mathfrak{a}_{M_P}^* \simeq \mathfrak{a}_{A_P}^*$. Let $\mathfrak{a}_0^* = \mathfrak{a}_{P_0}^*$, $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$, $A_0 = A_{P_0}$ and $A_0^\infty = A_{P_0}^\infty$.

2.2.7. For any semi-standard parabolic subgroups $P \subset Q$ of G, the restriction map $X^*(Q) \to X^*(P)$ induces maps $\mathfrak{a}_Q^* \to \mathfrak{a}_P^*$ and $\mathfrak{a}_P \to \mathfrak{a}_Q$. The first one is injective whereas the kernel of the second one is denoted by $\mathfrak{a}_P^Q$. The restriction map $X^*(A_P) \to X^*(A_Q)$ gives a surjective map $\mathfrak{a}_P^* \to \mathfrak{a}_Q^*$ whose kernel is denoted by $\mathfrak{a}_P^{Q,*}$. We get also an injective map $\mathfrak{a}_Q \to \mathfrak{a}_P$. In this way, we get dual decompositions $\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q$ and $\mathfrak{a}_P^* = \mathfrak{a}_P^{Q,*} \oplus \mathfrak{a}_Q^*$. Thus we have projections $\mathfrak{a}_0 \to \mathfrak{a}_P^Q$ and $\mathfrak{a}_0^* \to \mathfrak{a}_P^{Q,*}$ which we will denote by $X \mapsto X_P^Q$.

We denote by $\mathfrak{a}_{P,\mathbf{C}}^{Q,*}$ and $\mathfrak{a}_{P,\mathbf{C}}^Q$ the $\mathbf{C}$-vector spaces obtained by extension of scalars from $\mathfrak{a}_P^{Q,*}$ and $\mathfrak{a}_P^Q$. We still denote by $\langle \cdot, \cdot \rangle$ the pairing (2.2.2.1) we get by extension of the scalars to $\mathbf{C}$. We have a decomposition

$$\mathfrak{a}_{P,\mathbf{C}}^{Q,*} = \mathfrak{a}_P^{Q,*} \oplus i\mathfrak{a}_P^{Q,*}$$

where $i^2 = -1$. We shall denote by $\Re$ and $\Im$ the associated projections and call them real and imaginary parts. The same holds for the dual spaces $\mathfrak{a}_{P,\mathbf{C}}^Q$. In the obvious way, we define the complex conjugate denoted by $\bar{\lambda}$ of $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{Q,*}$.

2.2.8. Let Ad_P^Q be the adjoint action of M_P on the Lie algebra of $M_Q \cap N_P$. Let ρ_P^Q be the unique element of $\mathfrak{a}_P^{Q,*}$ such that for every $m \in M_P(\mathbf{A})$

$$|\det(\mathrm{Ad}_P^Q(m))| = \exp(\langle 2\rho_P^Q, H_P(m)\rangle).$$

For every $g \in G(\mathbf{A})$, we set

$$\delta_P^Q(g) = \exp(\langle 2\rho_P^Q, H_P(g)\rangle)$$

so that, in particular, the restriction of δ_P^Q to $P(\mathbf{A}) \cap M_Q(\mathbf{A})$ coincides with the modular character of the latter. For $Q = G$, the exponent G is omitted. Finally, we set $\rho_0 = \rho_{P_0}$.

2.2.9. Let $P'_0 = M_0 N_{P'_0}$ be a minimal semi-standard parabolic subgroup such that $P'_0 \subset P$. Let $\Delta_{P'_0}^P$ be the set of simple roots of A_0 in $M_P \cap P'_0$. We denote this set by Δ_0^P if $P'_0 = P_0$. Let Δ_P be the image of $\Delta_{P'_0} \setminus \Delta_{P'_0}^P$ (viewed as a subset of $\mathfrak{a}_0^*$) by the projection $\mathfrak{a}_0^* \to \mathfrak{a}_P^*$. It does not depend on the choice of P'_0. More generally one defines Δ_P^Q. We have also the set of coroots $\Delta_P^{Q,\vee} \subset \mathfrak{a}_P^Q$. By duality, we get a set of simple weights $\hat{\Delta}_P^Q$. The sets Δ_P^Q and $\hat{\Delta}_P^Q$ determine open cones in $\mathfrak{a}_0$ whose characteristic functions are denoted respectively by τ_P^Q and $\hat{\tau}_P^Q$. We set

$$A_P^{\infty,Q+} = \left\{ a \in A_P^\infty \mid \langle \alpha, H_P(a)\rangle \geqslant 0,\ \forall \alpha \in \Delta_P^Q \right\},$$

$$\mathfrak{a}_P^{*,Q+} = \left\{ \lambda \in \mathfrak{a}_P^* \mid \langle \lambda, \alpha^\vee\rangle \geqslant 0,\ \forall \alpha^\vee \in \Delta_P^{Q,\vee} \right\}.$$

We define similarly $\mathfrak{a}_P^{Q+}$ using roots instead of coroots. If $Q = G$, the exponent G is omitted and if $P = P_0$, we replace the subscript P_0 by 0.

For $\lambda, \mu \in \mathfrak{a}_0^*$, we will write $\lambda \prec_P \mu$ to indicate that $\mu - \lambda$ is a nonnegative linear combination of the simple roots Δ_0^P.

2.2.10. *Weyl group.* — Let W be the Weyl group of (G, A_0) that is the quotient by M_0 of the normalizer of A_0 in $G(F)$. For $P = M_P N_P$ and $Q = M_Q N_Q$ two standard parabolic subgroups of G, we denote by $W(P, Q)$ the set of $w \in W$ such that $w\Delta_0^P = \Delta_0^Q$. For $w \in W(P, Q)$, we have $wM_P w^{-1} = M_Q$. When $P = Q$, the group $W(P, P)$ is simply denoted by $W(P)$. Sometimes, we shall also denote $W(P, Q)$ by $W(M_P, M_Q)$ if we want to emphasize the Levi components (and $W(M_P) = W(P)$). We will also write W^{M_P} for the Weyl group of (M_P, A_0).

2.2.11. Let M be a standard Levi subgroup of G. We denote by $\mathcal{P}(M)$ the set of semi-standard parabolic subgroups P of G such that $M_P = M$. There is an unique element $P \in \mathcal{P}(M)$ which is standard and the map

(**2.2.11.3**) $$(Q, w) \mapsto w^{-1} Q w$$

induces a bijection from the disjoint union $\bigcup_Q W(P, Q)$ where Q runs over the set of standard parabolic subgroups of G onto $\mathcal{P}(M)$.

2.2.12. *Truncation parameter.* — We shall denote by T a point of $\mathfrak{a}_0$ such that $\langle\alpha, T\rangle$ is large enough for every $\alpha \in \Delta_0$. We do not want to be precise here. We just need that Arthur's formulas about truncation functions hold for the T′*s* we consider (see [Art78] §§5, 6). The point T plays the role of a truncation parameter.

For any semi-standard parabolic subgroup P, we define a point $T_P \in \mathfrak{a}_P$ such that for any $w \in W$ such that $wP_0w^{-1} \subset P$, the point T_P is the projection of $w \cdot T$ on $\mathfrak{a}_P$ (this does not depend on the choice of w). The reader should be warned that it is not consistent with the notation of Section 2.2.7 since there T_P denotes instead the projection of T onto $\mathfrak{a}_P$ (of course, the two conventions coincide when P is standard).

2.2.13. Let $\omega_0 \subset P_0(\mathbf{A})^1$ be a compact subset such that $P_0(\mathbf{A})^1 = P_0(F)\omega_0$. Let P be a standard parabolic subgroup. By a *Siegel domain* of $[G]_P$ we mean a subset of $G(\mathbf{A})$ of the form

$$\mathfrak{s}_P = \omega_0 \left\{ a \in A_0^\infty \mid \langle\alpha, H_0(a)\rangle \geqslant \langle\alpha, T_-\rangle,\ \forall\alpha \in \Delta_0^P \right\} K$$

where $T_- \in \mathfrak{a}_0$ and such that $G(\mathbf{A}) = M_P(F)N_P(\mathbf{A})\mathfrak{s}_P$. We henceforth fix a Siegel domain $\mathfrak{s}_P$ of $[G]_P$ for every standard parabolic subgroup P and we assume that these Siegel domains are all associated to the same $T_- \in \mathfrak{a}_0$. In particular, for $P \subset Q$ we have $\mathfrak{s}_Q \subset \mathfrak{s}_P$. Moreover, there exists a compact subset $\mathcal{K} \subset G(\mathbf{A})$ such that

$$\mathfrak{s}_P \subset N_P(\mathbf{A})A_0^{\infty,P+}\mathcal{K}, \quad \text{for every } P \subset G.$$

2.3. *Haar measures*

2.3.1. We equip $\mathfrak{a}_P$ with the Haar measure that gives a covolume 1 to the lattice $\mathrm{Hom}(X^*(P), \mathbf{Z})$. The space $i\mathfrak{a}_P^*$ is then equipped with the dual Haar measure so that we have

$$\int_{i\mathfrak{a}_P^*}\int_{\mathfrak{a}_P} \phi(H)\exp(-\langle\lambda, H\rangle)\, dH d\lambda = \phi(0)$$

for all $\phi \in C_c^\infty(\mathfrak{a}_P)$. Note that this implies that the covolume of $iX^*(P)$ in $i\mathfrak{a}_P^*$ is given by

(**2.3.1.1**) $$\mathrm{vol}(i\mathfrak{a}_P^*/iX^*(P)) = (2\pi)^{-\dim(\mathfrak{a}_P)}.$$

The group A_P^∞ is equipped with the Haar measure compatible with the isomorphism $A_P^\infty \simeq \mathfrak{a}_P$ induced by the map H_P. The groups $\mathfrak{a}_P^G \simeq \mathfrak{a}_P/\mathfrak{a}_G$ and $i\mathfrak{a}_P^{G,*} \simeq i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ are provided with the quotient Haar measures. For any basis B of $\mathfrak{a}_P^G$ we denote by $\mathbf{Z}(B)$ the lattice generated by B and by $\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(B))$ the covolume of this lattice. We have on $\mathfrak{a}_0^*$ the polynomial function:

$$\theta_P(\lambda) = \mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))^{-1} \prod_{\alpha\in\Delta_P} \langle\lambda, \alpha^\vee\rangle.$$

2.3.2. Let H be a connected linear algebraic group over F (not necessarily reductive). In this paper, we will always equip $H(\mathbf{A})$ with its right-invariant Tamagawa measure simply denoted dh. Let us recall how it is defined in order to fix some notation. We choose a right-invariant rational volume form ω_H on H as well as a non-trivial continuous additive character $\psi' : \mathbf{A}/F \to \mathbf{C}^\times$. For each place $v \in V_F$, the local component ψ'_v of ψ' induces an additive measure on F_v which is the unique Haar measure autodual with respect to ψ'_v. Then using local F-analytic charts, we associate to ω_H a right Haar measure $dh_v = |\omega_H|_{\psi'_v}$ on $H(F_v)$ as in [Wei82, §2.2]. By [Gro97], there exists an Artin-Tate L-function $L_H(s)$ such that, denoting by $L_{H,v}(s)$ the corresponding local L-factor and setting $\Delta_{H,v} = L_{H,v}(0)$, for any model of H over $\mathcal{O}_F^S$ for some finite set $S \subseteq V_F \setminus V_{F,\infty}$, we have

$$(\mathbf{2.3.2.2}) \qquad \mathrm{vol}(H(\mathcal{O}_v)) = \Delta_{H,v}^{-1}$$

for almost all $v \in V_F$. Setting $\Delta_H^* = L_H^*(0)$, where we recall that $L_H^*(0)$ stands for the leading coefficient in the Laurent expansion of $L_H(s)$ at 0, the Tamagawa measure on $H(\mathbf{A})$ is defined as the product

$$(\mathbf{2.3.2.3}) \qquad dh = (\Delta_H^*)^{-1} \prod_v \Delta_{H,v} dh_v.$$

Although the local measures dh_v depend on choices, the global measure dh doesn't (by the product formula).

2.3.3. For $S \subseteq V_F$ a finite subset, we put $\Delta_H^{S,*} = L_H^{S,*}(0)$ where $L_H^S(s)$ stands for the corresponding partial L-function and we equip $H(F_S)$, $H(\mathbf{A}^S)$ with the right Haar measures $dh_S = \prod_{v\in S} dh_v$ and $dh^S = (\Delta_H^{S,*})^{-1} \prod_{v\notin S} \Delta_{H,v} dh_v$ respectively. Note that we have the decomposition

$$(\mathbf{2.3.3.4}) \qquad dh = dh_S \times dh^S.$$

In particular, setting $S = \{v\}$, this means that $H(F_v)$ is equipped with the right Haar measure dh_v for every $v \in V_F$.

2.3.4. We have $L_H(s) = L_{H_{\mathrm{red}}}(s)$ where $H_{\mathrm{red}} = H/N_H$ denotes the quotient of H by its unipotent radical. When $H = N$ is unipotent we have $\mathrm{vol}([N]) = 1$. For $H = GL_n$, the L-function $L_H(s)$ is given by

$$L_H(s) = \zeta_F(s+1)\dots\zeta_F(s+n)$$

where ζ_F stands for the (completed) zeta function of the number field F. In this case, we will take

$$\omega_H = (\det h)^{-1} \bigwedge_{1\leqslant i,j\leqslant n} dh_{i,j}$$

so that (2.3.2.2) is satisfied for every non-Archimedean place v where ψ_v is unramified.

2.3.5. The homogeneous space $[G]$ (resp. $[G]^1 \simeq [G]_0$) is equipped with the quotient of the Tamagawa measure on $G(\mathbf{A})$ by the counting measure on $G(F)$ (resp. by the product of the counting measure on $G(F)$ with the Haar measure we fixed on A_G^∞). For P a standard parabolic subgroup, we equip similarly $[G]_P$ with the quotient of the Tamagawa measure on $G(\mathbf{A})$ by the product of the counting measure on $M_P(F)$ with the Tamagawa measure on $N_P(\mathbf{A})$. Since the action by left translation of $a \in A_P^\infty$ on $[G]_P$ multiplies the measure by $\delta_P(a)^{-1}$, taking the quotient by the Haar measure on A_P^∞ induces a "semi-invariant" measure on $[G]_{P,0} = A_P^\infty \backslash [G]_P$ that is a positive linear form on the space of continuous functions $\varphi : [G]_P \to \mathbf{C}$ satisfying $\varphi(ag) = \delta_P(a)\varphi(g)$ for $a \in A_P^\infty$ and compactly supported modulo A_P^∞.

2.4. *Heights, weights and Harish-Chandra Ξ function*

2.4.1. *Height on* G. — We fix an embedding $\iota : G \hookrightarrow GL_N$ for some integer $N > 0$. Using ι, we define a *height* on $G(\mathbf{A})$ by

$$\|g\| = \prod_v \max_{1 \leqslant i,j \leqslant N} (|\iota(g)_{i,j}|_v, |\iota(g^{-1})_{i,j}|_v).$$

Note that for another choice of embedding ι' yielding a height $\|.\|'$, there exists $r_0 > 0$ such that $\|g\|^{1/r_0} \ll \|g\|' \ll \|g\|^{r_0}$ for $g \in G(\mathbf{A})$. We have

$$(\mathbf{2.4.1.1}) \qquad 1 \ll \|g\|, \quad \|gh\| \ll \|g\| \, \|h\| \text{ and } \|g\| = \|g^{-1}\|$$

for $g, h \in G(\mathbf{A})$. Let $P \subset G$ be a standard parabolic subgroup. We set

$$\|x\|_P = \inf_{\gamma \in M_P(F)N_P(\mathbf{A})} \|\gamma x\| \sim \inf_{\gamma \in P(F)} \|\gamma x\|, \quad \sigma_P(x) = 1 + \log \|x\|_P, \quad \text{for } x \in [G]_P.$$

Note that, for $P \subset Q$, we have

$$(\mathbf{2.4.1.2}) \qquad \|x\|_Q \ll \|x\|_P, \quad \text{for } x \in P(F)\backslash G(\mathbf{A}).$$

Letting $\mathfrak{s}_P$ be a Siegel domain as in Section 2.2.13, we have (see [MW94, Eq. I.2.2(vii)])

$$(\mathbf{2.4.1.3}) \qquad \|g\|_P \sim \|g\|, \quad \text{for } g \in \mathfrak{s}_P.$$

If $H \subset G$ is a reductive subgroup, we equip $H(\mathbf{A})$ with the restriction of the height $\|.\|$ from which we deduce as above a function $\|.\|_{P_H}$ on $[H]_{P_H}$ for every parabolic subgroup $P_H \subset H$. For $P \subset G$ a standard parabolic subgroup, we have

$$(\mathbf{2.4.1.4}) \qquad \|m\|_{M_P} \sim \|m\|_P, \quad \text{for } m \in [M_P].$$

More generally, if $P_H = P \cap H$ happens to be a parabolic subgroup of H with unipotent radical $N_{P_H} = H \cap N_P$ (so that in particular $[H]_{P_H} \subset [G]_P$), we have

$$\|h\|_{P_H} \sim \|h\|_P, \text{ for } h \in [H]_{P_H}. \tag{2.4.1.5}$$

Indeed, using a Siegel domain for $[H]_{P_H}$ we are reduced to show the above equivalence for $h = a \in A_{0,H}^\infty$ where $A_{0,H} \subset H$ is a maximal split torus. Up to conjugation, we may assume that $A_{0,H} \subset A_0$ and then it readily follows from (2.4.1.3) that $\|a\|_P \sim \|a\| \sim \|a\|_{P_H}$ for $a \in A_{0,H}^\infty$.

We also have

$$\|a\| \ll \|ax\|_P, \quad \text{for } (a,x) \in A_P^\infty \times [G]_P \text{ such that } H_P(x) = 0. \tag{2.4.1.6}$$

Indeed, by (2.4.1.4) we may assume that $P = G$. Then, up to conjugation, we may assume that there exists distinct characters $\lambda_1, \dots, \lambda_k \in X^*(A_G)$ and integers $N_1, \dots, N_k \geqslant 1$ such that

$$\iota(a) = \begin{pmatrix} \lambda_1(a) I_{N_1} & & \\ & \ddots & \\ & & \lambda_k(a) I_{N_k} \end{pmatrix}, \quad \text{for } a \in A_G.$$

Then, there exist homomorphisms $\iota_i : G \to GL_{N_i}$, for $1 \leqslant i \leqslant k$, such that $\iota(g) = \begin{pmatrix} \iota_1(g) & & \\ & \ddots & \\ & & \iota_k(g) \end{pmatrix}$ for $g \in G$. As $|\det \iota_i(g)|_{\mathbf{A}} = 1$ for every $g \in G(\mathbf{A})^1$, we can now deduce (2.4.1.6) from the simple inequality

$$|\det(g)|_{\mathbf{A}}^{1/n} \ll \prod_v \max(|g_{i,j}|_v), \quad \text{for } g \in GL_{N_i}(\mathbf{A}).$$

2.4.2. *Heights on vector spaces.* — Let V be a vector space over F of finite dimension and let $v_1, \dots, v_d$ be a basis of V. For $v = x_1 v_1 + \cdots + x_d v_d \in V_{\mathbf{A}} = V \otimes_F \mathbf{A}$, we set

$$\|v\|_V = \prod_v \max(1, |x_1|_v, \dots, |x_d|_v).$$

Note that another choice of basis would yield equivalent functions. A *height* on $V_{\mathbf{A}}$ will be for us any positive function equivalent to $\|.\|_V$.

The above construction applies in particular to $V = F^n$ with its standard basis $e_1, \dots, e_n$ and we will denote by $\|.\|_{\mathbf{A}^n}$ the resulting norm. Note that we have

$$\|x\|_{\mathbf{A}} = \prod_v \max(1, |x|_v), \quad \text{for } x \in \mathbf{A}.$$

2.4.3. *Weights.* — Let $P \subset G$ be a standard parabolic subgroup. Following [Ber88, §3.1] and [Fra98, §2.1], by a *weight* on $[G]_P$ we mean a positive measurable function w on $[G]_P$ such that for every compact subset $\mathcal{U} \subset G(\mathbf{A})$ we have

$$w(xk) \sim w(x), \quad \text{for } x \in [G]_P \text{ and } k \in \mathcal{U}.$$

We say that two weights w_1, w_2 are *equivalent*, and we will write $w_1 \sim w_2$, if $w_1(x) \sim w_2(x)$ for $x \in [G]_P$. By (2.4.1.1), $\|.\|_P$ is a weight on $[G]_P$.

If w is a weight on $[G]_P$, we denote by w_A its restriction to A_P^∞. It is a weight on the latter group i.e. for every compact subset $\mathcal{U} \subset A_P^\infty$ we have $w_A(ak) \ll w_A(a)$ for $a \in A_P^\infty$ and $k \in \mathcal{U}$. Conversely, if w_A is a weight on A_P^∞ then we can view it as a weight on $[G]_P$ through composition with $[G]_P \xrightarrow{H_P} \mathfrak{a}_P \xrightarrow{\exp} A_P^\infty$.

Lemma **2.4.3.1.** — *Let w be a weight on $[G]_P$. Then, there exists $N_0 > 0$ such that*

$$w(xg) \ll w(x)\|g\|^{N_0}, \quad \textit{for } (x,g) \in [G]_P \times G(\mathbf{A}). \qquad (\mathbf{2.4.3.7})$$

Proof. — First we prove the existence of $N_0' > 0$ such that

$$w(xa) \ll w(x)\|a\|^{N_0'}, \quad \text{for } (x,a) \in [G]_P \times A_0^\infty. \qquad (\mathbf{2.4.3.8})$$

Indeed, let $\mathcal{K}_A \subset A_0^\infty$ be a compact neighborhood of 1. Then, $\mathcal{K}_A$ generates A_0^∞ and we have

$$1 + \log\|a\| \sim \min\{n \geqslant 1 \mid a \in \mathcal{K}_A^n\}. \qquad (\mathbf{2.4.3.9})$$

Moreover, as w is a weight, there exists a constant $C > 0$ such that

$$w(xk_A) \leqslant C w(x), \quad \text{for } (x, k_A) \in [G]_P \times \mathcal{K}_A.$$

By induction, this gives, for any $n \geqslant 1$,

$$w(xk_A) \leqslant C^n w(x), \quad \text{for } (x, k_A) \in [G]_P \times \mathcal{K}_A^n.$$

Using (2.4.3.9), this readily gives (2.4.3.8) for some $N_0' > 0$.

We now prove the lemma. By the existence of Siegel domain, it suffices to show the existence of $N_0 > 0$ such that the estimate (2.4.3.7) holds for $(x,g) \in A_0^{\infty,P+} \times G(\mathbf{A})$. By the Iwasawa decomposition $G(\mathbf{A}) = P_0(\mathbf{A})K$, any $g \in G(\mathbf{A})$ can be written $g = p_0^1(g)a_0(g)k(g)$ where $p_0^1(g) \in P_0(\mathbf{A})^1$, $a_0(g) \in A_0^\infty$ and $k(g) \in K$. Moreover, there exists a compact $\mathcal{K} \subset G(\mathbf{A})$ such that

$$xP_0(\mathbf{A})^1 \subset P_0(F)N_P(\mathbf{A})x\mathcal{K}$$

for every $x \in A_0^{\infty,P+}$. Therefore, by (2.4.3.9), we have

$$w(xg) = w(xp_0^1(g)a_0(g)k(g)) \ll w(xp_0^1(g))\|a_0(g)\|^{N_0'} \ll w(x)\|a_0(g)\|^{N_0'}$$

for every $(x, g) \in A_0^{\infty,P+} \times G(\mathbf{A})$. To conclude, it suffices to notice that there exists $N_0 > 0$ such that

$$\|a_0(g)\|^{N_0'} \ll \|g\|^{N_0}$$

for every $g \in G(\mathbf{A})$. □

According to Franke [Fra98, Proposition 2.1], for every $\lambda \in \mathfrak{a}_0^*$ there exists a weight $d_{P,\lambda}$ on $[G]_P$ such that

$$d_{P,\lambda}(x) \sim \exp(\langle \lambda, H_0(x)\rangle), \text{ for } x \in \mathfrak{s}_P.$$

These weights have the following elementary properties: for $t \in \mathbf{R}$ and $\lambda, \mu \in \mathfrak{a}_0^*$, we have

(**2.4.3.10**) $$d_{P,t\lambda} \sim (d_{P,\lambda})^t \text{ and } d_{P,\lambda+\mu} \sim d_{P,\lambda} d_{P,\mu}.$$

Moreover,

(**2.4.3.11**) If $\lambda \prec_P \mu$ (see Section 2.2.9), then $d_{P,\lambda} \ll d_{P,\mu}$.

Also, if $\Lambda \subset \mathfrak{a}_0^*$ is a $W^P = \mathrm{Norm}_{M_P(F)}(A_0)/M_0(F)$-invariant subset then

(**2.4.3.12**) $$\max_{\lambda\in\Lambda} d_{P,\lambda}(a) \sim \max_{\lambda\in\Lambda} \exp(\langle \lambda, H_0(a)\rangle), \text{ for } a \in A_0^\infty.$$

Let $\Lambda_\iota^P \subset \mathfrak{a}_0^{*,P+}$ is the set of maximal A_0-weights of the representation $g \mapsto diag(\iota(g), {}^t\iota(g)^{-1})$ for the partial order $\prec_P$. It follows from (2.4.1.3) that

(**2.4.3.13**) $$\|x\|_P \sim \max_{\lambda\in\Lambda_\iota^P} d_{P,\lambda}(x), \text{ for } x \in [G]_P.$$

Lemma **2.4.3.2.** — *Let* $\lambda \in \mathfrak{a}_0^{*,P+}$. *Then, we have*

(**2.4.3.14**) $$d_{P,\lambda}(x) \sim \sup_{\gamma\in P(F)} e^{\langle\lambda, H_0(\gamma x)\rangle}, \textit{ for } x \in [G]_P.$$

Proof. — As $d_{P,\lambda}(m) \sim d_{M_P,\lambda}(m)$ for $m \in [M_P]$, up to replacing G by M_P, we may assume that $P = G$. Moreover, we can restrict to $x \in \mathfrak{s}_G$ in which case the left-hand side is clearly essentially bounded by the right-hand side (by definition of the weight $d_{G,\lambda}$). Thus, the lemma boils down to the inequality, where $\lambda \in \mathfrak{a}_0^{*,+}$,

$$e^{\langle\lambda, H_0(\gamma x)\rangle} \ll e^{\langle\lambda, H_0(x)\rangle}, \text{ for } (\gamma, x) \in G(F) \times \mathfrak{s}_G,$$

which is a simple reformulation of [LW13, lemme 3.5.4]. □

We note the immediate corollary.

Corollary **2.4.3.3.** — *Let* $Q \subset P$ *be standard parabolic subgroups of* G *and* $\lambda \in \mathfrak{a}_0^{*,P+}$. *Then, we have*

$$d_{Q,\lambda}(x) \ll d_{P,\lambda}(x), \text{ for } x \in Q(F)\backslash G(\mathbf{A}). \tag{2.4.3.15}$$

2.4.4. *Neighborhoods of infinity.* — Let $P \subset Q$ be standard parabolic subgroups of G and Σ_P^Q be the set of roots of A_0 in $\mathfrak{n}_P/\mathfrak{n}_Q$. We define the following weights:

$$d_P^Q(x) = \min_{\alpha \in \Sigma_P^Q} d_{P,\alpha}(x), \quad x \in [G]_P, \tag{2.4.4.16}$$

and

$$d_Q^P(x) = \min_{\alpha \in \Sigma_P^Q} d_{Q,\alpha}(x), \quad x \in [G]_Q. \tag{2.4.4.17}$$

Since for every $\beta \in \Sigma_P^Q$, there exist a family of nonnegative integers $(n_\alpha)_{\alpha \in \Delta_0^Q \backslash \Delta_0^P}$, not all zero, such that $\sum_{\alpha \in \Delta_0^Q \backslash \Delta_0^P} n_\alpha \alpha \prec_P \beta$, we see that there exists $n > 0$ such that

$$d_P^Q(x) \ll \min_{\alpha \in \Delta_0^Q \backslash \Delta_0^P} d_{P,\alpha}(x) \ll \max(d_P^Q(x), d_P^Q(x)^{1/n}), \quad \text{for } x \in [G]_P. \tag{2.4.4.18}$$

For every $C > 0$, we set

$$\omega_P^Q[> C] := \{x \in P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A}) \mid d_P^Q(x) > C\}.$$

Let π_P^Q and π_Q^P be the two natural projections

$$\begin{array}{ccccc} & & P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A}) & & \\ & \swarrow^{\pi_P^Q} & & \searrow^{\pi_Q^P} & \\ [G]_P & & & & [G]_Q \end{array}$$

The next lemma summarizes some classical results from reduction theory.

Lemma **2.4.4.1.**

1. *There exists* $\epsilon > 0$ *such that* π_Q^P *sends* $\omega_P^Q[> \epsilon]$ *onto* $[G]_Q$.
2. *Let* $\epsilon > 0$. *Then, for every* $\lambda \in \mathfrak{a}_0^*$ *we have* $d_{P,\lambda}(x) \sim d_{Q,\lambda}(x)$ *for* $x \in \omega_P^Q[> \epsilon]$. *In particular,*

$$d_P^Q(x) \sim d_Q^P(x) \text{ and } \|x\|_P \sim \|x\|_Q, \text{ for } x \in \omega_P^Q[> \epsilon].$$

3. *For every* $\epsilon > 0$, *the restriction of* π_Q^P *to* $\omega_P^Q[> \epsilon]$ *has uniformly bounded fibers. More precisely, there exist* $c > 0$ *and* $r > 0$ *such that for every* $x \in [G]_Q$ *and* $1 \geqslant \epsilon > 0$, *we have*

$$\#((\pi_Q^P)^{-1}(x) \cap \omega_P^Q[> \epsilon]) \leqslant c\epsilon^{-r}$$

4. *For every* $\epsilon > 0$, *we can find* $C > 0$ *such that for all* $(x, y) \in \omega_P^Q[> \epsilon] \times \omega_P^Q[> C]$, $\pi_Q^P(x) = \pi_Q^P(y)$ *implies* $x = y$.
5. *The map* π_Q^P *is a local homeomorphism which locally preserves measures. The map* π_P^Q *is proper and the pushforward of the invariant measure on* $P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A})$ *by it is the invariant measure on* $[G]_P$.

Proof. — 1. is just the existence of Siegel domains noting that if $\mathfrak{s}_Q$ and $\mathfrak{s}_P$ are Siegel domains as in Section 2.2.13 for $[G]_Q$ and $[G]_P$ respectively, there exists $\epsilon > 0$ such that the inverse image of $\omega_P^Q[> \epsilon]$ in $\mathfrak{s}_P$ contains $\mathfrak{s}_Q$. Similarly, for any $\epsilon > 0$, this inverse image is contained in another, perhaps bigger, Siegel domain $\mathfrak{s}'_Q$ for $[G]_Q$ and this immediately gives 2. (where for the last equivalence we have used (2.4.3.13)). Properties 3. and 4. are Lemma 2.11 and Lemma 2.12 of [Lan76] respectively. Finally, 5. is clear. □

Note that, from points 1. and 2. of the above lemma, we immediately deduce that

$$d_P^Q(x) \ll d_Q^P(x), \quad \text{for } x \in P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A}). \tag{2.4.4.19}$$

Lemma **2.4.4.2.** — *Let* $H \subset G$ *be a reductive subgroup such that* $P_H = P \cap H$ *and* $Q_H = Q \cap H$ *are parabolic subgroups of* H *with unipotent radicals* $N_{P_H} = N_P \cap H$ *and* $N_{Q_H} = N_Q \cap H$ *respectively. Then, we have*

$$d_P^Q(h) \ll d_{P_H}^{Q_H}(h), \ \textit{for } h \in [H]_{P_H}. \tag{2.4.4.20}$$

If moreover, $G = \mathrm{Res}_{K/F} H_K$, $P = \mathrm{Res}_{K/F}(P_H)_K$ *and* $Q = \mathrm{Res}_{K/F}(Q_H)_K$ *for some finite extension* K/F, *then*

$$d_P^Q(h) \sim d_{P_H}^{Q_H}(h), \ \textit{for } h \in [H]_{P_H}. \tag{2.4.4.21}$$

Proof. — Let $A_{0,H} \subset P_H$ be a maximal split torus. Up to conjugation by $P(F)$, we may assume that $A_{0,H} \subset A_0$. By the existence of Siegel domains, and since d_P^Q, $d_{P_H}^{Q_H}$ are weights, we just need to show (2.4.4.20) for $x = a \in A_{0,H}^\infty$. Since Σ_P^Q and $\Sigma_{P_H}^{Q_H}$ are invariant by the Weyl groups $W^P = \mathrm{Norm}_{M_P(F)}(A_0)/M_0(F)$ and W^{P_H} respectively, by (2.4.3.12), we have

$$\begin{aligned} d_P^Q(a) &\sim \min_{\alpha \in \Sigma_P^Q} \exp(\langle \alpha, H_0(a)\rangle) \ \text{ and} \\ d_{P_H}^{Q_H}(a) &\sim \min_{\alpha \in \Sigma_{P_H}^{Q_H}} \exp(\langle \alpha, H_0(a)\rangle), \ \text{ for } a \in A_{0,H}^\infty. \end{aligned} \tag{2.4.4.22}$$

By the assumption, we have $\mathfrak{n}_{P_H}/\mathfrak{n}_{Q_H} \subset \mathfrak{n}_P/\mathfrak{n}_Q$ so that the restriction of Σ_P^Q to $A_{0,H}$ contains $\Sigma_{P_H}^{Q_H}$. With (2.4.4.22), this gives the inequality (2.4.4.20). We can deduce (2.4.4.21) similarly noting that, in the case where $G = \mathrm{Res}_{K/F} H_K$, $P = \mathrm{Res}_{K/F}(P_H)_K$ and $Q = \mathrm{Res}_{K/F}(Q_H)_K$, the restriction of Σ_P^Q to $A_{0,H}$ is *equal to* $\Sigma_{P_H}^{Q_H}$. □

2.4.5. *Xi function.* — Let $\mathcal{U} \subseteq G(\mathbf{A})$ be a compact neighborhood of 1. We set

$$\Xi^P(x) = \mathrm{vol}_{[G]_P}(x\mathcal{U})^{-1/2}, \quad x \in [G]_P.$$

Replacing $\mathcal{U}$ by another compact neighborhood of 1 would yield an equivalent function which is why we dropped the subset $\mathcal{U}$ from the notation. We have

$$(\mathbf{2.4.5.23}) \qquad \Xi^P(x) \sim d_{P,\rho_0}(x), \text{ for } x \in [G]_P.$$

Indeed, there exists $\epsilon > 0$ such that, with the notation of Section 2.4.4, $P_0(F)N_P(\mathbf{A})\mathfrak{s}_P\mathcal{U} \subset \omega_{P_0}^P[> \epsilon]$. Thus, by Lemma 2.4.4.1 3. and 5., we have

$$\mathrm{vol}_{[G]_P}(g\mathcal{U}) \sim \mathrm{vol}_{P_0(F)N_P(\mathbf{A})\backslash G(\mathbf{A})}(g\mathcal{U}), \text{ for } g \in \mathfrak{s}_P.$$

On the other hand, there exists a compact neighborhood $\mathcal{U}'$ of 1 in $G(\mathbf{A})$ such that

$$P_0(F)N_0(\mathbf{A})g\mathcal{U} \subseteq P_0(F)N_P(\mathbf{A})g\mathcal{U}' \text{ for every } g \in \mathfrak{s}_P.$$

By Lemma 2.4.4.1 5., this gives

$$\mathrm{vol}_{[G]_{P_0}}(g\mathcal{U}) = \mathrm{vol}_{P_0(F)\backslash G(\mathbf{A})}(N_0(\mathbf{A})g\mathcal{U}) \sim \mathrm{vol}_{P_0(F)N_P(\mathbf{A})\backslash G(\mathbf{A})}(g\mathcal{U}), \text{ for } g \in \mathfrak{s}_P.$$

Finally, as the left action of A_0^∞ multiplies the invariant measure on $[G]_{P_0}$ by the inverse modular character $\delta_{P_0}^{-1}$, we obtain

$$\mathrm{vol}_{[G]_P}(g\mathcal{U}) \sim \mathrm{vol}_{[G]_{P_0}}(g\mathcal{U}) \sim \exp(-\langle 2\rho_0, H_0(g)\rangle) \text{ for } g \in \mathfrak{s}_P$$

which is another way to state (2.4.5.23).

By [Lap13, §2, (9)], we also have[4]

$$(\mathbf{2.4.5.24}) \text{ There exists } d_0 > 0 \text{ such that } \int_{[G]_P} \Xi^P(g)^2 \sigma_P(g)^{-d_0} dg < \infty;$$

From (2.4.5.23), we also deduce the existence of $N_0 \geqslant 1$ such that

$$(\mathbf{2.4.5.25}) \qquad \Xi^P(g) \ll \|g\|_P^{N_0}, \quad g \in [G]_P.$$

[4] Note that the definition of the Ξ function in *loc. cit.* coincides, up to equivalence, with ours by (2.4.5.23).

2.5. *Spaces of functions*

2.5.1. Let V be a Fréchet space. We say of a function $f : G(\mathbf{A}) \to V$ that it is *smooth* if it is right invariant by a compact-open subgroup J of $G(\mathbf{A}_f)$ and for every $g_f \in G(\mathbf{A}_f)$ the function $g_\infty \in G(F_\infty) \mapsto f(g_f g_\infty) \in V$ is C^∞. This definition applies in particular for $V = \mathbf{C}$ and, for $P \subset G$ a parabolic subgroup, we denote by $C^\infty([G]_P)$ the space of smooth complex-valued functions on $[G]_P$.

We also denote by $C_c([G]_P)$ (resp. $L^1_{loc}([G]_P)$) the spaces of complex-valued continuous compactly supported (resp. locally integrable) functions on $[G]_P$.

2.5.2. Let C be a compact subset of $G(\mathbf{A}_f)$ and let $J \subset K^\infty$ be a compact-open subgroup. Let $\mathcal{S}(G(\mathbf{A}), C, J)$ be the space of smooth functions $f : G(\mathbf{A}) \to \mathbf{C}$ which are biinvariant by J, supported in the subset $G(F_\infty) \times C$ and such that the semi-norms

$$\|f\|_{r,X,Y} = \sup_{g \in G(\mathbf{A})} \|g\|^r |(R(X)L(Y)f)(g)|$$

are finite for every integer $r \geqslant 1$ and $X, Y \in \mathcal{U}(\mathfrak{g}_\infty)$. This family of semi-norms defines a topology on $\mathcal{S}(G(\mathbf{A}), C, J)$ making it into a Fréchet space. The global *Schwartz space* $\mathcal{S}(G(\mathbf{A}))$ is the locally convex topological direct limit over all pairs (C, J) of the spaces $\mathcal{S}(G(\mathbf{A}), C, J)$. It is a strict LF space. Moreover, the Schwartz space $\mathcal{S}(G(\mathbf{A}))$ is an algebra for the convolution product denoted by $*$. It contains the dense subspace $C_c^\infty(G(\mathbf{A}))$ of smooth and compactly supported functions. For an integer $r \geqslant 0$, we will also consider the space $C_c^r(G(\mathbf{A}))$ generated by products $f_\infty f^\infty$ where f_∞ is a compactly supported function on $G(F \otimes_{\mathbf{Q}} \mathbf{R})$ which admits derivatives up to the order r and f^∞ is a smooth compactly supported function on $G(\mathbf{A}_f)$.

For every integer $n \geqslant 1$, we define similarly the global Schwartz space $\mathcal{S}(\mathbf{A}^n)$ using the norm $\|.\|_{\mathbf{A}^n}$ (see Section 2.4.2).

2.5.3. In order to organize in some uniform way the different spaces of functions that we are about to define, we now introduce, mostly following the terminology of [BK14], some nice categories of complex linear representations of $G(\mathbf{A})$. Recall from [BK14, §2.3] that a Fréchet representation V of $G(F_\infty)$ is said to be a F-*representation* if there exists a family of semi-norms $(\|.\|_{V,n})_n$ defining the topology of V such that the $G(F_\infty)$-action is continuous with respect to each of them. Moreover, a F-representation V of $G(F_\infty)$ is called a *SF-representation* if for every vector $v \in V$, the map $g \in G(F_\infty) \mapsto g \cdot v$ is smooth (that is C^∞) and for every $X \in \mathcal{U}(\mathfrak{g}_\infty)$, the resulting linear operator $v \in V \mapsto X \cdot v$ is continuous (cf. [BK14, §2.4.3]).

In a similar fashion, we say that a representation of $G(\mathbf{A})$ on a Fréchet space V is a F-*representation* if there exists a family of semi-norms $(\|.\|_{V,n})_n$ defining the topology of V such that the $G(\mathbf{A})$-action is continuous with respect to each of them. In particular, continuous representations of $G(\mathbf{A})$ on Banach spaces are automatically F-representations.

Let V be a F-representation of $G(\mathbf{A})$. A vector $v \in V$ is *smooth* if the orbit map $g \in G(\mathbf{A}) \mapsto g \cdot v$ is smooth. We denote by V^∞ the subspace of smooth vectors. It is a $G(\mathbf{A})$-invariant subspace and for every compact-open subgroup $J \subseteq G(\mathbf{A}_f)$, we equip the subspace $(V^\infty)^J$ of J-fixed vectors with the topology associated to the semi-norms $v \mapsto \|Xv\|_V$ where $\|.\|_V$ runs over a family of semi-norms defining the topology of V, X runs over (a basis of) $\mathcal{U}(\mathfrak{g}_\infty)$ and $Xv := R(X)(g_\infty \mapsto g_\infty v)_{|g_\infty=1}$. With this topology, $(V^\infty)^J$ becomes a Fréchet space and even a SF representation of $G(F_\infty)$ in the sense recalled above. Moreover, the inclusions $(V^\infty)^J \subset (V^\infty)^{J'}$ for $J' \subseteq J \subseteq G(\mathbf{A}_f)$ are closed embeddings and so $V^\infty = \bigcup_J (V^\infty)^J$ has a structure of strict LF space.

For every $v \in V$ and $f \in \mathcal{S}(G(\mathbf{A}))$, the integral

$$(\mathbf{2.5.3.1}) \qquad f \cdot v = \int_{G(\mathbf{A})} f(g) g \cdot v dg$$

converges in V and this defines an action of the algebra $(\mathcal{S}(G(\mathbf{A})), *)$ on V. Moreover, by the Dixmier-Malliavin theorem [DM78], we have

$$(\mathbf{2.5.3.2}) \qquad V^\infty = \mathcal{S}(G(\mathbf{A})) \cdot V$$

where the right-hand side stands for the set of finite linear combinations $\sum_i f_i \cdot v_i$ with $v_i \in V$ and $f_i \in \mathcal{S}(G(\mathbf{A}))$.

2.5.4. In this paper, by a *SLF representation* of $G(\mathbf{A})$ we mean a $\mathbf{C}$-vector space V equipped with a linear action of $G(\mathbf{A})$ and, for every compact-open subgroup $J \subseteq G(\mathbf{A}_f)$, the structure of a Fréchet space on V^J such that the following conditions are satisfied:

- For every $J \subseteq G(\mathbf{A}_f)$, V^J is a SF representation of $G(F_\infty)$ in the sense of [BK14, §2.4.3];
- $V = \bigcup_J V^J$ where J runs over all compact-open subgroups of $G(\mathbf{A}_f)$ (i.e. the action of $G(\mathbf{A}_f)$ on V is smooth);
- For every $J' \subseteq J \subseteq G(\mathbf{A}_f)$, the inclusion $V^J \subseteq V^{J'}$ is a closed embedding.

By the second and third points above, a SLF representation of $G(\mathbf{A})$ has a natural structure of strict LF space. If V is a F-representation of $G(\mathbf{A})$, the subspace of smooth vectors V^∞ has a natural structure of SLF representation of $G(\mathbf{A})$ by the previous paragraph. Most of the function spaces that we are going to introduce carry natural structures of F- or SLF representations of $G(\mathbf{A})$.

Let V be a SLF representation of $G(\mathbf{A})$. Then, (2.5.3.1) still defines an action of $(\mathcal{S}(G(\mathbf{A})), *)$ on V and the equality (2.5.3.2) holds with V instead of V^∞.

The following lemma implies, by the open mapping theorem for LF spaces, that equivariant morphisms of SLF representations with closed image are necessarily topological embeddings.

Lemma **2.5.4.1.** — *Let* V *be a SLF representation of* $G(\mathbf{A})$ *and* $W \subset V$ *be a closed* $G(\mathbf{A})$-*invariant subspace. Then, equipping* W^J *with the topology induced from* V^J *for every compact-open subgroup* $J \subseteq G(\mathbf{A}_f)$ *gives* W *the structure of a SLF representation of* $G(\mathbf{A})$ *and the corresponding LF topology on* W *(given as the locally convex direct limit of the topologies on the* W^J*'s) coincides with the topology induced from* V.

Proof. — It is immediate that W is a SLF representation of $G(\mathbf{A})$ (as a closed invariant subspace of a SF representation is itself a SF representation). On the other hand, to check that the LF topology on W is induced from V, it suffices to notice the decompositions

$$V = \bigoplus_{\rho \in \widehat{K_f}} V_\rho, \ W = \bigoplus_{\rho \in \widehat{K_f}} W_\rho$$

where $\widehat{K_f}$ denotes the set of isomorphism classes of smooth irreducible representations of K_f and for $\delta \in \widehat{K_f}$, V_ρ and W_ρ stand for the corresponding isotypical subspaces. Indeed, V_ρ and W_ρ have structures of Fréchet spaces (these are closed subspaces of V^J and W^J for some compact-open subgroup $J \subset G(\mathbf{A}_f)$ respectively) and these decompositions identify the LF spaces V, W with the topological direct sums of the families $(V_\rho)_{\rho \in \widehat{K_f}}$, $(W_\rho)_{\rho \in \widehat{K_f}}$ respectively: this is because for every ρ, V_ρ maps continuously into some V^J and conversely for every J, V^J maps continuously into $V_{\rho_1} \oplus \cdots \oplus V_{\rho_n}$ where $\rho_1, \ldots, \rho_n$ are the, finitely many, irreducible representations of K_f with a nonzero J-fixed vector (and similarly for W). Therefore, the end of the lemma is a consequence of the following general property of topological direct sums:

(**2.5.4.3**) Let $(V_i)_{i \in I}$ be a family of locally convex topological vector spaces and for each $i \in I$, let W_i be a subspace of V_i equipped with the induced topology. Then, the locally convex topological direct sum $\bigoplus_{i \in I} V_i$ induces on its subspace $\bigoplus_{i \in I} W_i$ the locally convex direct sum topology. □

2.5.5. Let P be a semi-standard parabolic subgroup of G. We denote by $L^2([G]_P)$ the space of L^2-measurable functions on $[G]_P$. It is a Hilbert space when equipped with the scalar product

$$\langle \varphi_1, \varphi_2 \rangle_P = \int_{[G]_P} \varphi_1(g) \overline{\varphi_2(g)} dg$$

associated to the Tamagawa invariant measure on $[G]_P$. We denote similarly by $L^2([G]_{P,0})$ the Hilbert space of measurable functions φ on $[G]_P$ satisfying $\varphi(ag) = \delta_P(a)^{1/2} \varphi(g)$ for almost all $a \in A_P^\infty$ and such that $\int_{[G]_{P,0}} |\varphi(g)|^2 dg$ is convergent.

More generally, if w is a weight on $[G]_P$, we write $L^2_w([G]_P)$ for the Hilbert space of functions that are square-integrable with respect to the measure $w(g)dg$. This space is equipped with a continuous (non-unitary) representation of $G(\mathbf{A})$ by right-translation. Its subspace $L^2_w([G]_P)^\infty$ of smooth vectors consists of smooth functions $\varphi : [G]_P \to \mathbf{C}$ such that $R(X)\varphi \in L^2_w([G]_P)$ for every $X \in \mathcal{U}(\mathfrak{g}_\infty)$. By the Sobolev inequality, see [Ber88, §3.4, Key Lemma], for every $\varphi \in L^2_w([G]_P)^\infty$ we have

$$(\mathbf{2.5.5.4}) \qquad |\varphi(g)| \ll \Xi^P(g)w(g)^{-1/2}, \quad g \in [G]_P.$$

Moreover, from Riesz representation theorem, we have:

(**2.5.5.5**) For every continuous linear form $T \in (L^2_w([G]_P)^\infty)'$ and $f \in \mathcal{S}(G(\mathbf{A}))$, there exists $\varphi \in L^2_{w^{-1}}([G]_P)^\infty$ such that $T(R(f)\psi) = \langle \psi, \varphi \rangle_P$ for every $\psi \in L^2_w([G]_P)$.

To save some space, for $N \in \mathbf{R}$, $d \geqslant 0$ and any weight w on $[G]_P$, we will adopt the following notation

$$L^2_{N,w}([G]_P) = L^2_{\|.\|_P^N w}([G]_P) \text{ and } L^2_{\sigma,d,w}([G]_P) = L^2_{\sigma_P^d w}([G]_P).$$

Moreover, for $w = 1$ we will simply drop the index w. The following result is a consequence of [Ber88, Theorem p. 688] as the composition of two Hilbert-Schmidt operators is nuclear.

Proposition **2.5.5.1.** — *There exists $d_0 \geqslant 0$ such that for every compact-open subgroup $J \subset G(\mathbf{A}_f)$ and every weight w on $[G]_P$ the inclusion of Fréchet spaces*

$$L^2_{\sigma,d_0,w}([G]_P)^{\infty,J} \subset L^2_w([G]_P)^{\infty,J}$$

is nuclear. In particular, any summable family in $L^2_{\sigma,d_0,w}([G]_P)^{\infty,J}$ becomes absolutely summable in $L^2_w([G]_P)^{\infty,J}$ (see Lemma A.0.6.1).

2.5.6. We let $\mathcal{S}^0([G]_P)$ be the space of measurable complex-valued functions φ on $[G]_P$ such that

$$\|\varphi\|_{\infty,N} = \sup_{x\in[G]_P} \|x\|_P^N |\varphi(x)| < \infty$$

for every $N > 0$. We equip $\mathcal{S}^0([G]_P)$ with the topology associated to the family of semi-norms $(\|.\|_{\infty,N})_{N>0}$. It is a Fréchet space. Note that $\mathcal{S}^0([G]_P)$ is *not* a F-representation of $G(\mathbf{A})$ (because the action by right translation is not continuous) but the closed subspace $\mathcal{S}^{00}([G]_P) \subset \mathcal{S}^0([G]_P)$ of *continuous functions* is so.

2.5.7. The *Schwartz space* $\mathcal{S}([G]_P)$ of $[G]_P$ is defined as the space of smooth functions $\varphi : [G]_P \to \mathbf{C}$ such that for every $N > 0$ and $X \in \mathcal{U}(\mathfrak{g}_\infty)$ we have

$$\|\varphi\|_{\infty,N,X} = \sup_{x\in[G]_P} \|x\|_P^N |(R(X)\varphi)(x)| < \infty.$$

Then, $\mathcal{S}([\mathrm{G}]_{\mathrm{P}})$ is a SLF representation of $\mathrm{G}(\mathbf{A})$ which is equal to $\mathcal{S}^{00}([\mathrm{G}]_{\mathrm{P}})^{\infty}$. Moreover, for every weight w on $[\mathrm{G}]_{\mathrm{P}}$, $\mathcal{S}([\mathrm{G}]_{\mathrm{P}})$ is dense in $\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^{\infty}$ (as $\mathcal{S}^{00}([\mathrm{G}]_{\mathrm{P}})$ is dense in $\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})$). From (2.5.5.4) and the open mapping theorem, we have an equality of SLF representations (where the right-hand side is equipped with the locally convex projective limit topology)

(**2.5.7.6**) $$\mathcal{S}([\mathrm{G}]_{\mathrm{P}}) = \bigcap_{\mathrm{N}>0} \mathrm{L}^2_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})^{\infty}.$$

2.5.8. The *Harish-Chandra Schwartz space* $\mathcal{C}([\mathrm{G}]_{\mathrm{P}})$ of $[\mathrm{G}]_{\mathrm{P}}$ is defined as the space of smooth functions $\varphi : [\mathrm{G}]_{\mathrm{P}} \to \mathbf{C}$ such that for every $d > 0$ and $\mathrm{X} \in \mathcal{U}(\mathfrak{g}_{\infty})$ we have

$$\|\varphi\|_{\infty,d,\sigma,\mathrm{X}} = \sup_{x\in[\mathrm{G}]_{\mathrm{P}}} \Xi^{\mathrm{P}}(x)^{-1}\sigma_{\mathrm{P}}(x)^{d}|(\mathrm{R}(\mathrm{X})\varphi)(x)| < \infty.$$

For J a compact-open subgroup of $\mathrm{G}(\mathbf{A}_f)$, we equip $\mathcal{C}([\mathrm{G}]_{\mathrm{P}})^{\mathrm{J}}$ with the topology induced from the family of semi-norms $(\|.\|_{\infty,d,\sigma,\mathrm{X}})_{d,\mathrm{X}}$. This makes $\mathcal{C}([\mathrm{G}]_{\mathrm{P}})$ into a SLF representation of $\mathrm{G}(\mathbf{A})$ when equipped with the action by right translation. The Schwartz space $\mathcal{S}([\mathrm{G}]_{\mathrm{P}})$ is dense in $\mathcal{C}([\mathrm{G}]_{\mathrm{P}})$. Moreover, by (2.5.5.4), we have the alternative description

(**2.5.8.7**) $$\mathcal{C}([\mathrm{G}]_{\mathrm{P}}) = \bigcap_{d>0} \mathrm{L}^2_{\sigma,d}([\mathrm{G}]_{\mathrm{P}})^{\infty}.$$

2.5.9. For every weight w on $[\mathrm{G}]_{\mathrm{P}}$, we let $\mathcal{T}^0_w([\mathrm{G}]_{\mathrm{P}})$ be the space of complex Radon measure φ on $[\mathrm{G}]_{\mathrm{P}}$ such that

$$\|\varphi\|_{1,w^{-1}} = \int_{[\mathrm{G}]_{\mathrm{P}}} w(x)^{-1}|\varphi(x)| < \infty.$$

We equip $\mathcal{T}^0_w([\mathrm{G}]_{\mathrm{P}})$ with the topology associated to the norm $\|.\|_{1,w^{-1}}$. It is a continuous Banach representation of $\mathrm{G}(\mathbf{A})$. For $\mathrm{N} > 0$, we write $\mathcal{T}^0_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})$ for $\mathcal{T}^0_{\|\cdot\|^{\mathrm{N}}_{\mathrm{P}}}([\mathrm{G}]_{\mathrm{P}})$ and we set

$$\mathcal{T}^0([\mathrm{G}]_{\mathrm{P}}) = \bigcup_{\mathrm{N}>0} \mathcal{T}^0_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}}).$$

We equip this space with the corresponding (non-strict) LF topology. We have a sesquilinear pairing

$$(\varphi, \psi) \in \mathcal{T}^0([\mathrm{G}]_{\mathrm{P}}) \times \mathcal{S}^0([\mathrm{G}]_{\mathrm{P}}) \mapsto \langle \varphi, \psi \rangle_{\mathrm{P}} = \int_{[\mathrm{G}]_{\mathrm{P}}} \overline{\psi(x)}\varphi(x)$$

which identifies $\mathcal{T}^0([\mathrm{G}]_{\mathrm{P}})$ with the topological dual of $\mathcal{S}^{00}([\mathrm{G}]_{\mathrm{P}})$.

2.5.10. For every weight w on $[G]_P$, we define $\mathcal{T}_w([G]_P)$ as the space of smooth functions $\varphi : [G]_P \to \mathbf{C}$ such that for every $X \in \mathcal{U}(\mathfrak{g}_\infty)$ we have

$$\|\varphi\|_{\infty,w^{-1},X} = \sup_{x\in[G]_P} w(x)^{-1}|(R(X)\varphi)(x)| < \infty.$$

We equip $\mathcal{T}_w([G]_P)^J$, for J a compact-open subgroup of $G(\mathbf{A}_f)$, with the topology induced from the family of semi-norms $(\|.\|_{\infty,w^{-1},X})_X$. This makes $\mathcal{T}_w([G]_P)$ a SLF representation of $G(\mathbf{A})$. For $N > 0$, we write $\mathcal{T}_N([G]_P)$ and $\mathcal{T}_{w,N}([G]_P)$ for $\mathcal{T}_{\|.\|_P^N}([G]_P)$ and $\mathcal{T}_{w\|.\|_P^N}([G]_P)$ respectively.

The *space of functions of uniform moderate growth* on $[G]_P$ is defined as

$$\mathcal{T}([G]_P) = \bigcup_{N>0} \mathcal{T}_N([G]_P).$$

We equip this space with the corresponding (non-strict) LF topology. The Schwartz space $\mathcal{S}([G]_P)$ is dense in $\mathcal{T}([G]_P)$ but we warn the reader that usually $\mathcal{S}([G]_P)$ is not dense in $\mathcal{T}_N([G]_P)$ (unless $[G]_P$ is compact). By (2.5.5.4), we have the alternative descriptions (as a LF space)

$$(\mathbf{2.5.10.8}) \qquad \mathcal{T}([G]_P) = \bigcup_{N>0} L^2_{-N}([G]_P)^\infty = \bigcup_{N>0} \mathcal{T}^0_N([G]_P)^\infty.$$

2.5.11. More generally, for a weight w on $[G]_P$ and $N > 0$, we define $\mathcal{S}_{w,N}([G]_P)$ as the space of smooth functions $\varphi : [G]_P \to \mathbf{C}$ such that for every $r \geqslant 0$ and $X \in \mathcal{U}(\mathfrak{g}_\infty)$ we have

$$\|\varphi\|_{\infty,-N,w^r,X} = \sup_{x\in[G]_P} \|x\|_P^{-N} w(x)^r |(R(X)\varphi)(x)| < \infty.$$

We equip $\mathcal{S}_{w,N}([G]_P)^J$, for J a compact-open subgroup of $G(\mathbf{A}_f)$, with the topology induced from the family of semi-norms $(\|.\|_{\infty,-N,w^r,X})_{X,r\geqslant 0}$ and this makes $\mathcal{S}_{w,N}([G]_P)$ into a SLF representation of $G(\mathbf{A})$. We set

$$\mathcal{S}_w([G]_P) = \bigcup_{N>0} \mathcal{S}_{w,N}([G]_P)$$

and we equip this space with the corresponding (non-strict) LF topology. Once again, $\mathcal{S}([G]_P)$ is dense in $\mathcal{S}_w([G]_P)$ but in general not in $\mathcal{S}_{w,N}([G]_P)$. Moreover, by (2.5.5.4), we have the alternative description (as a LF space)

$$(\mathbf{2.5.11.9}) \qquad \mathcal{S}_w([G]_P) = \bigcup_{N>0} \bigcap_{r\geqslant 0} L^2_{-N,w^r}([G]_P)^\infty.$$

We note the following lemma.

Lemma **2.5.11.1.** — *Let* $\varphi \in \mathcal{T}^0([G]_P)$. *If* w *is bounded from above on the support of* φ, *then* $R(f)\varphi \in \mathcal{S}_w([G]_P)$ *for every* $f \in \mathcal{S}(G(\mathbf{A}))$.

Proof. — Indeed, if w is bounded from above on the support of φ, then, as w is a weight, it is also bounded from above on the support of $R(f)\varphi$ for $f \in C_c^\infty(G(\mathbf{A}))$. As $R(f)\varphi \in \mathcal{T}([G]_P)$, it readily follows that $R(C_c^\infty(G(\mathbf{A})))\varphi \subset \mathcal{S}_w([G]_P)$. Moreover, by Dixmier-Malliavin we have $\mathcal{S}(G(\mathbf{A})) = \mathcal{S}(G(\mathbf{A})) * C_c^\infty(G(\mathbf{A}))$ and $\mathcal{S}_w([G]_P)$ is stable by right convolution by $\mathcal{S}(G(\mathbf{A}))$. The lemma follows. □

2.5.12. By [Ber88, end of Section 3.5] (see also [Cas89b, Corollary 2.6]) we have

(**2.5.12.10**) For every compact-open subgroup $J \subset G(\mathbf{A}_f)$, the Fréchet spaces $\mathcal{S}([G]_P)^J$ and $\mathcal{C}([G]_P)^J$ are nuclear.

Assume that $G = G_1 \times G_2$ where G_1 and G_2 are two connected reductive groups over F. Let $J_1 \subset G_1(\mathbf{A}_f)$, $J_2 \subseteq G_2(\mathbf{A}_f)$ be two compact open subgroups and set $J = J_1 \times J_2$. By (2.5.12.10), (A.0.7.8) and a reasoning similar to (the proof of) [BP20, Proposition 4.4.1 (v)] we obtain:

(**2.5.12.11**) There are topological isomorphisms

$$\mathcal{S}([G_1])^{J_1} \widehat{\otimes} \mathcal{S}([G_2])^{J_2} \simeq \mathcal{S}([G])^J \quad \text{and} \quad \mathcal{C}([G_1])^{J_1} \widehat{\otimes} \mathcal{C}([G_2])^{J_2} \simeq \mathcal{C}([G])^J$$

sending a pure tensor $\varphi_1 \otimes \varphi_2$ to the function $(g_1, g_2) \mapsto \varphi_1(g_1)\varphi_2(g_2)$.

By the above, given two continuous linear forms L_1, L_2 on $\mathcal{C}([G_1])$, $\mathcal{C}([G_2])$ respectively, the linear form $L_1 \otimes L_2$ on $\mathcal{C}([G_1]) \otimes \mathcal{C}([G_2])$ extends by continuity to a linear form on $\mathcal{C}([G])$ that we shall denote by $L_1 \widehat{\otimes} L_2$.

2.5.13. *Constant terms and pseudo-Eisenstein series.* — Let $Q \supset P$ be another standard parabolic subgroup. We have two continuous $G(\mathbf{A})$-equivariant linear maps

$$\mathcal{S}^0([G]_P) \to \mathcal{S}^0([G]_Q), \quad \varphi \mapsto E_P^Q(\varphi) \quad \text{and}$$
$$\mathcal{T}^0([G]_Q) \to \mathcal{T}^0([G]_P), \quad \varphi \mapsto \varphi_P$$

defined as the following compositions of pullbacks and pushforwards

$$E_P^Q(\varphi) = \pi^P_{Q*}\pi_P^{Q*}(\varphi), \quad \varphi_P = \pi^Q_{P*}\pi_Q^{P*}(\varphi)$$

where π_Q^P and π_P^Q are as in Section 2.4.4. More concretely, we have

$$E_P^Q(\varphi, x) = \sum_{\gamma \in P(F)\backslash Q(F)} \varphi(\gamma x), \quad \text{for } \varphi \in \mathcal{S}^0([G]_P) \text{ and } x \in [G]_Q$$

whereas the map $\varphi \mapsto \varphi_P$ sends $\mathcal{T}([G]_Q)$ into $\mathcal{T}([G]_P)$ and is given on this subspace by the familiar formula

$$\varphi_P(x) = \int_{[N_P]} \varphi(ux)du, \quad \text{for } \varphi \in \mathcal{T}([G]_Q) \text{ and } x \in [G]_P.$$

The pseudo-Eisenstein map $\varphi \mapsto E_P^Q(\varphi)$ sends $\mathcal{S}([G]_P)$ continuously into $\mathcal{S}([G]_Q)$ and the constant term map $\varphi \mapsto \varphi_P$ sends $\mathcal{T}_N([G]_Q)$ continuously into $\mathcal{T}_N([G]_P)$ for every $N > 0$ (as follows from (2.4.1.2)). We also have the adjunction

$$(\mathbf{2.5.13.12}) \qquad \langle \varphi_P, \psi \rangle_P = \langle \varphi, E_P^Q(\psi) \rangle_Q \text{ for } \varphi \in \mathcal{T}^0([G]_Q),\ \psi \in \mathcal{S}^0([G]_P).$$

Lemma **2.5.13.1.** — *There is a constant $c > 0$ such that for every $N \geqslant 0$,*

$$\varphi \mapsto \sup_{x \in [G]_P} \delta_P^Q(x)^{cN} \|x\|_P^N |\varphi_P(x)|$$

is a continuous semi-norm on $\mathcal{S}^0([G]_Q)$.

Proof. — Indeed, this follows from Corollary 2.4.3.3 and (2.4.3.13) noting that for every $\lambda \in \mathfrak{a}_0^{*,P+}$ there exists $c > 0$ such that $\lambda + 2c\rho_P^Q \in \mathfrak{a}_0^{*,Q+}$. □

2.5.14. *Approximation by the constant term.* — Recall that in Section 2.4.4, we have introduced a weight d_P^Q on $[G]_P$. The next proposition is a reformulation of the well-known approximation property of the constant term (see [MW89, Lemma I.2.10]).

Proposition **2.5.14.1.**

1. *Let $N > 0$, $r \geqslant 0$ and $X \in \mathcal{U}(\mathfrak{g}_\infty)$. Then, there exists a continuous semi-norm $\|.\|_{N,X,r}$ on $\mathcal{T}_N([G]_Q)$ such that*

$$(\mathbf{2.5.14.13}) \qquad |R(X)\varphi(x) - R(X)\varphi_P(x)| \leqslant \|x\|_P^N d_P^Q(x)^{-r} \|\varphi\|_{N,X,r}$$

   for $\varphi \in \mathcal{T}_N([G]_Q)$ and $x \in P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A})$.

2. *Let w be a weight on $[G]_Q$ and $\varphi \in L^2_w([G]_Q)^\infty$. Then, there exists $N > 0$ such that for every $r \geqslant 0$, we have*

$$(\mathbf{2.5.14.14}) \qquad \int_{A_Q^\infty} |\varphi(ax) - \varphi_P(ax)|^2 w(a) \delta_Q(a)^{-1} da \ll_r \|x\|_P^N d_P^Q(x)^{-r}$$

   for $x \in P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A})$.

Proof. — 1. Since $\varphi \in \mathcal{T}_N([G]_Q) \mapsto \varphi_P \in \mathcal{T}_N([G]_P)$ is continuous (as follows from (2.4.1.2)), there exists at least a continuous semi-norm $\|.\|_{N,X}$ on $\mathcal{T}_N([G]_Q)$ such that

$$|R(X)\varphi(x) - R(X)\varphi_P(x)| \leqslant \|x\|_P^N \|\varphi\|_{N,X},$$

for $\varphi \in \mathcal{T}_{\mathrm{N}}([\mathrm{G}]_{\mathrm{Q}})$ and $x \in \mathrm{P}(\mathrm{F})\mathrm{N}_{\mathrm{Q}}(\mathbf{A})\backslash\mathrm{G}(\mathbf{A})$.

By the above estimate, and with the notation of Section 2.4.4, it suffices to find a continuous semi-norm $\|.\|_{\mathrm{N},\mathrm{X},r}$ such that (2.5.14.13) is satisfied for $x \in \omega_{\mathrm{P}}^{\mathrm{Q}}[>\mathrm{C}]$ where $\mathrm{C} > 0$ is some fixed but arbitrary constant. We can choose $\mathrm{C} > 0$ such that $\omega_{\mathrm{P}}^{\mathrm{Q}}[>\mathrm{C}] \subset \mathrm{P}(\mathrm{F})\mathfrak{s}_{\mathrm{Q}}$ and therefore, it suffices to show (2.5.14.13) for $x \in \mathfrak{s}_{\mathrm{Q}}$ only.

If P is a maximal parabolic subgroup of Q i.e. $\Delta_0^{\mathrm{Q}} \setminus \Delta_0^{\mathrm{P}} = \{\alpha\}$ for some root α, the result is then a direct consequence of [MW89, Lemma I.2.10]. To deduce the general case, we choose a tower of parabolic subgroups $\mathrm{P} = \mathrm{P}_0 \subset \mathrm{P}_1 \subset \cdots \subset \mathrm{P}_n = \mathrm{Q}$ with $\Delta_0^{\mathrm{P}_i} \setminus \Delta_0^{\mathrm{P}_{i-1}} = \{\alpha_i\}$ for every $1 \leqslant i \leqslant n$. Then, we have $\Delta_0^{\mathrm{Q}} \setminus \Delta_0^{\mathrm{P}} = \{\alpha_1, \dots, \alpha_n\}$ and, for $1 \leqslant i \leqslant n$, $\mathfrak{s}_{\mathrm{Q}}$ is included in $\mathfrak{s}_{\mathrm{P}_i}$. Therefore, by [MW89, Lemma I.2.10], for every $1 \leqslant i \leqslant n$, we can find a continuous semi-norm $\|.\|_{i,\mathrm{N},\mathrm{X},r}$ on $\mathcal{T}_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}_i})$ such that

$$|\mathrm{R}(\mathrm{X})\varphi_{\mathrm{P}_i}(x) - \mathrm{R}(\mathrm{X})\varphi_{\mathrm{P}_{i-1}}(x)| \leqslant \|x\|_{\mathrm{P}}^{\mathrm{N}} \exp(-r\langle\alpha_i, \mathrm{H}_0(x)\rangle)\|\varphi_{\mathrm{P}_i}\|_{i,\mathrm{N},\mathrm{X},r},$$

for $\varphi \in \mathcal{T}_{\mathrm{N}}([\mathrm{G}]_{\mathrm{Q}})$ and $x \in \mathfrak{s}_{\mathrm{Q}}$.

Then, we get

$$\begin{aligned}|\mathrm{R}(\mathrm{X})\varphi(x) - \mathrm{R}(\mathrm{X})\varphi_{\mathrm{P}}(x)| &\leqslant \sum_{i=1}^{n}|\mathrm{R}(\mathrm{X})\varphi_{\mathrm{P}_i}(x) - \mathrm{R}(\mathrm{X})\varphi_{\mathrm{P}_{i-1}}(x)| \\ &\leqslant \|x\|_{\mathrm{P}}^{\mathrm{N}} \max_{1\leqslant i\leqslant n} \exp(-r\langle\alpha_i, \mathrm{H}_0(x)\rangle)\|\varphi\|_{\mathrm{N},\mathrm{X},r} \\ &\ll \|x\|_{\mathrm{P}}^{\mathrm{N}} d_{\mathrm{P}}^{\mathrm{Q}}(x)^{-r}\|\varphi\|_{\mathrm{N},\mathrm{X},r}\end{aligned}$$

for $\varphi \in \mathcal{T}_{\mathrm{N}}([\mathrm{G}]_{\mathrm{Q}})$ and $x \in \mathfrak{s}_{\mathrm{Q}}$ where $\|\varphi\|_{\mathrm{N},\mathrm{X},r} = \sum_i \|\varphi_{\mathrm{P}_i}\|_{i,\mathrm{N},\mathrm{X},r}$ is a continuous semi-norm on $\mathcal{T}_{\mathrm{N}}([\mathrm{G}]_{\mathrm{Q}})$.

2. Let $\mathrm{N}' > 0$ that we will assume large enough. Then, for some $\mathrm{N} > 0$, we have continuous inclusions $\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{Q}})^\infty \subset \mathrm{L}^2_{-\mathrm{N}'}([\mathrm{G}]_{\mathrm{Q}})^\infty \subset \mathcal{T}_{\mathrm{N}/2}([\mathrm{G}]_{\mathrm{Q}})$. Thus, by 1., for every $r \geqslant 0$ we can find elements $\mathrm{X}_1, \dots, \mathrm{X}_k \in \mathcal{U}(\mathfrak{g}_\infty)$ such that

$$|\psi(x) - \psi_{\mathrm{P}}(x)|^2 \leqslant \|x\|_{\mathrm{P}}^{\mathrm{N}} d_{\mathrm{P}}^{\mathrm{Q}}(x)^{-r} \sum_{1\leqslant i\leqslant \ell} \int_{[\mathrm{G}]_{\mathrm{Q}}} |\mathrm{R}(\mathrm{X}_i)\psi(y)|^2 \|y\|_{\mathrm{Q}}^{-\mathrm{N}'} dy$$

for $\psi \in \mathrm{L}^2_w([\mathrm{G}]_{\mathrm{Q}})^\infty$ and $x \in \mathrm{P}(\mathrm{F})\mathrm{N}_{\mathrm{Q}}(\mathbf{A})\backslash\mathrm{G}(\mathbf{A})$. Applying this inequality to the function $\mathcal{L}_a\varphi(x) = \varphi(ax)$ where $a \in \mathrm{A}_{\mathrm{Q}}^\infty$, we are reduced to show the convergence of

$$\sum_{1\leqslant i\leqslant k} \int_{[\mathrm{G}]_{\mathrm{Q}}} \int_{\mathrm{A}_{\mathrm{Q}}^\infty} |\mathrm{R}(\mathrm{X}_i)\varphi(ay)|^2 w(a)\delta_{\mathrm{Q}}(a)^{-1} da \|y\|_{\mathrm{Q}}^{-\mathrm{N}'} dy$$

for N' large enough. After the change of variable $y \mapsto a^{-1}y$, and since $\varphi \in L^2_w([G]_Q)^\infty$, this boils down to showing that

$$\int_{A_Q^\infty} w(a)\|a^{-1}y\|_Q^{-N'}\,da \ll w(y), \quad \text{for } y \in [G]_Q.$$

From Lemma 2.4.3.1 (applied to the weight w^{-1}), there exists $N_0 > 0$ such that $w(a) \ll w(ax)\|x\|_Q^{N_0}$ for $(a, x) \in A_Q^\infty \times [G]_Q$ and there exists $N_1 > 0$ such that $\int_{A_Q^\infty} \|a^{-1}y\|_Q^{-N_1}\,da \ll 1$ for $y \in [G]_Q$. Then, by (2.4.1.6), any $N' \geqslant N_0 + N_1$ works. □

2.5.15. The proof of the next result is partly inspired from [Fra98, §4, p. 204].

Proposition **2.5.15.1.** — *Let $P \subset Q$ be standard parabolic subgroups of G. For every weight w on $[G]_Q$, there exist weights w_P^-, w_P^+ on $[G]_P$ such that:*

- *The constant term $\varphi \mapsto \varphi_P$ maps $L^2_w([G]_Q)$ continuously into $L^2_{w_P^-}([G]_P)$;*
- *The pseudo-Eisenstein map E_P^Q extends to a continuous linear map from $L^2_{w_P^+}([G]_P)$ into $L^2_w([G]_Q)$;*
- *For every $\epsilon > 0$, we have $w_P^+(x) \sim w(x) \sim w_P^-(x)$ for $x \in \omega_P^Q[> \epsilon]$.*

Proof. — First, we observe that it suffices to prove that for every weight w on $[G]_Q$, there exists a weight w_P^- on $[G]_P$ such that:

- The constant term $\varphi \mapsto \varphi_P$ maps $L^2_w([G]_Q)$ continuously into $L^2_{w_P^-}([G]_P)$;
- For every $\epsilon > 0$, we have $w_P^-(x) \sim w(x)$ for $x \in \omega_P^Q[> \epsilon]$.

Indeed, if it were the case, by the adjunction (2.5.13.12), for every weight w on $[G]_Q$, E_P^Q would extend to a continuous linear map $L^2_{w_P^+}([G]_P) \to L^2_w([G]_Q)$ where

$$w_P^+ = \left((w^{-1})_P^-\right)^{-1}$$

is a weight satisfying, for any given $\epsilon > 0$,

$$w_P^+(x) \sim (w(x)^{-1})^{-1} = w(x), \text{ for } x \in \omega_P^Q[> \epsilon].$$

Let $r > 0$ and set

$$w_P^-(x) = \min(1, d_P^Q(x))^r \inf_{u \in [N_P]} w(ux), \quad x \in [G]_P.$$

Clearly, w_P^- is a weight on $[G]_P$ and we will check that it has the desired properties when r is sufficiently large. First, for $\epsilon > 0$, there exists a compact $\mathcal{K}_N \subset N_P(\mathbf{A})$ such that

$P(F)N_P(\mathbf{A})x \subset P(F)N_Q(\mathbf{A})x\mathcal{K}_N$ for every $x \in \omega_P^Q[>\epsilon]$. As w is a weight, this shows that

$$w_P^-(x) \sim w(x), \quad \text{for } x \in \omega_P^Q[>\epsilon].$$

On the other hand, for $\varphi \in L^2_w([G]_Q)$ and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}(\mathbf{2.5.15.15}) \qquad \int_{[G]_P} |\varphi_P(x)|^2 w_P^-(x)dx &\leqslant \int_{[G]_Q} |\varphi(x)|^2 \sum_{\gamma\in P(F)\backslash Q(F)} w_P^-(\gamma x)dx \\ &\leqslant \int_{[G]_Q} |\varphi(x)|^2 w(x) \sum_{\gamma\in P(F)\backslash Q(F)} \min(1, d_P^Q(\gamma x))^r dx.\end{aligned}$$

By Lemma 2.4.4.1 3., if r is sufficiently large we have

$$\sum_{\gamma\in P(F)\backslash Q(F)} \min(1, d_P^Q(\gamma x))^r \ll 1, \text{ for } x \in [G]_Q.$$

For such a choice, (2.5.15.15) gives

$$\int_{[G]_P} |\varphi_P(x)|^2 w_P^-(x)dx \ll \int_{[G]_Q} |\varphi(x)|^2 w(x)dx, \text{ for } \varphi \in L^2_w([G]_Q)$$

which is exactly saying that the constant term $\varphi \mapsto \varphi_P$ maps $L^2_w([G]_Q)$ continuously into $L^2_{w_P^-}([G]_P)$. □

Recall that if w is a weight on $[G]_Q$, we denote by w_A its restriction to A_Q^∞.

Corollary **2.5.15.2.** — *Let w and w' be weights on $[G]_Q$ such that $w_A \sim w'_A$. Then,*

1. *If $w' \ll w$, the linear map*

$$(\mathbf{2.5.15.16}) \qquad \begin{aligned} L^2_w([G]_Q)^\infty &\to \prod_{P\subsetneq Q} L^2_{w_P^-}([G]_P)^\infty \times L^2_{w'}([G]_Q)^\infty, \\ \varphi &\mapsto ((\varphi_P)_P, \varphi)\end{aligned}$$

   is a closed embedding.

2. *Dually, if $w' \gg w$, the linear map*

$$(\mathbf{2.5.15.17}) \qquad \begin{aligned} \prod_{P\subsetneq Q} L^2_{w_P^+}([G]_P)^\infty \times L^2_{w'}([G]_Q)^\infty &\to L^2_w([G]_Q)^\infty \\ \left((\varphi^P)_P, \varphi\right) &\mapsto \varphi + \sum_{P\subsetneq Q} E_P^Q(\varphi^P)\end{aligned}$$

   is an open surjection.

Proof. — 1. By the open mapping theorem, it suffices to show that the image of (2.5.15.16) is the closed subspace of tuples $((\varphi^P)_P, \varphi) \in \prod_{P \subsetneq Q} L^2_{w_P^-}([G]_P)^\infty \times L^2_{w'}([G]_Q)^\infty$ such that $\varphi^P = \varphi_P$ for every $P \subsetneq Q$. That the image is included in this subspace follows from Proposition 2.5.15.1. Conversely, let $\varphi \in L^2_{w'}([G]_Q)^\infty$ be such that $\varphi_P \in L^2_{w_P^-}([G]_P)^\infty$ for every $P \subsetneq Q$. Then, we need to show that $\varphi \in L^2_w([G]_Q)$ (as, applying the same reasoning to the derivatives of φ, we can actually deduce $\varphi \in L^2_w([G]_Q)^\infty$).

Let $G(\mathbf{A})^1_Q \subset G(\mathbf{A})$ be the inverse image of $0 \in \mathfrak{a}_Q$ by H_Q. We equip $G(\mathbf{A})^1_Q$ with the unique measure dx such that, through the identification $G(\mathbf{A}) = A_Q^\infty \times G(\mathbf{A})^1_Q$, the invariant measure on $G(\mathbf{A})$ decomposes as $dadx$. For every $x \in G(\mathbf{A})$, we set $x^1 = \exp(-H_Q(x))x \in G(\mathbf{A})^1_Q$. Let $\epsilon > 0$. For $P \subsetneq Q$, we introduce the set

$$\omega^Q_{P,\epsilon} = \{x \in P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A}) \mid d^Q_P(x) > \|x^1\|^\epsilon_P\}.$$

First we show that

$$\textbf{(2.5.15.18)} \qquad \int_{\pi^P_Q(\omega^Q_{P,\epsilon})} |\varphi(x)|^2 w(x)dx < \infty.$$

Indeed, as $\omega^Q_{P,\epsilon} \subseteq \omega^Q_P[> C]$ for some $C > 0$, by Lemma 2.4.4.1 3. and 5. we can replace the domain of integration by $\omega^Q_{P,\epsilon}$. Moreover, since $\varphi_P \in L^2_{w_P^-}([G]_P)$ and $w_P^-(x) \sim w(x)$ for $x \in \omega^Q_P[> C]$ (cf. Proposition 2.5.15.1), the function $x \in \omega^Q_{P,\epsilon} \mapsto \varphi_P(x)$ is square-integrable with respect to the measure $w(x)dx$. Thus, it only remains to show that

$$\int_{\omega^Q_{P,\epsilon}} |\varphi(x) - \varphi_P(x)|^2\, w(x)dx < \infty.$$

But this follows from Proposition 2.5.14.1 2. as $d^Q_P(x) \sim d^Q_P(x^1)$ and there exists $N > 0$ such that $\int_{P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A})^1_Q} \|x\|^{-N} dx$ converges.

From (2.5.15.18), it only remains to show that $x \mapsto |\varphi(x)|^2 w(x)$ is integrable over the complement

$$[G]_Q \setminus \bigcup_{P \subsetneq Q} \pi^P_Q(\omega^Q_{P,\epsilon}).$$

However, by Lemma 2.4.4.1 2., there exists $c > 0$ such that this subset is contained in the set of $x \in [G]_Q$ such that

$$d_{Q,\alpha}(x) \leqslant c\|x^1\|^\epsilon$$

for every $\alpha \in \Delta^Q_0$. We readily check that for ϵ sufficiently small, the resulting domain is compact modulo A_Q^∞. Thus, as $\varphi \in L^2_{w'}([G]_Q)$ and $w'_A \sim w_A$, the claim follows.

2. Again by the open mapping theorem, it suffices to show that the map (2.5.15.17) is surjective. But this follows by duality from 1., (2.5.5.5) and the Dixmier-Malliavin theorem. □

2.6. *Estimates on Fourier coefficients*

2.6.1. Let $P \subset G$ be a standard parabolic subgroup, $\psi : \mathbf{A}/F \to \mathbf{C}^\times$ be a non-trivial character and V_P be the vector space of additive algebraic characters $N_P \to \mathbf{G}_a$. Let $\ell \in V_P(F)$ and set $\psi_\ell := \psi \circ \ell_{\mathbf{A}} : [N_P] \to \mathbf{C}^\times$ where $\ell_{\mathbf{A}}$ denotes the homomorphism between adelic points $N_P(\mathbf{A}) \to \mathbf{A}$. For $\varphi \in C^\infty([G])$, we set

$$\varphi_{N_P,\psi_\ell}(g) = \int_{[N_P]} \varphi(ug)\psi_\ell(u)^{-1} du, \quad g \in G(\mathbf{A}).$$

The adjoint action of M_P on N_P induces one on V_P that we denote by Ad^*. We fix a height $\|.\|_{V_P}$ on $V_P(\mathbf{A})$ as in Section 2.4.2.

Lemma **2.6.1.1.**

1. *There exists* $c > 0$ *such that for every* $N_1, N_2 \geqslant 0$,

$$\varphi \mapsto \sup_{m \in M_P(\mathbf{A})} \|\mathrm{Ad}^*(m^{-1})\ell\|_{V_P}^{N_1} \|m\|_{M_P}^{N_2} \delta_P(m)^{cN_2} |\varphi_{N_P,\psi_\ell}(m)|$$

is a continuous semi-norm on $\mathcal{S}([G])$.

2. *Let* $N > 0$. *Then, for every* $N_1 \geqslant 0$,

$$\varphi \mapsto \sup_{m \in M_P(\mathbf{A})} \|\mathrm{Ad}^*(m^{-1})\ell\|_{V_P}^{N_1} \|m\|_{M_P}^{-N} |\varphi_{N_P,\psi_\ell}(m)|$$

is a continuous semi-norm on $\mathcal{T}_N([G])$.

Proof. — Bounding brutally under the integral sign, we have

$$|\varphi_{N_P,\psi_\ell}(g)| \leqslant |\varphi|_P(g)$$

for $\varphi \in C^\infty([G])$ and $g \in G(\mathbf{A})$. Let $N \geqslant 0$ and $J \subseteq G(\mathbf{A}_f)$ be a compact-open subgroup. By Lemma 2.5.13.1 and (2.4.1.4), it suffices to show the existence of elements $X_1, \dots, X_k \in \mathcal{U}(\mathfrak{g}_\infty)$ such that

$$|\varphi_{N_P,\psi_\ell}(m)| \leqslant \|\mathrm{Ad}^*(m^{-1})\ell\|_{V_P}^{-1} \sum_i \left|(R(X_i)\varphi)_{N_P,\psi_\ell}(m)\right| \tag{2.6.1.1}$$

for every $\varphi \in C^\infty([G])^J$ and $m \in M_P(\mathbf{A})$. Let $u \in N_P(\mathbf{A})$. By definition of $\|.\|_{V_P}$, we just need to show the existence of $X_1, \dots, X_k \in \mathcal{U}(\mathfrak{g}_\infty)$ such that

$$|\varphi_{N_P,\psi_\ell}(m)| \leqslant \|\ell(\mathrm{Ad}(m)u)\|_{\mathbf{A}}^{-1} \sum_i |(R(X_i)\varphi)_{N_P,\psi_\ell}(m)| \tag{2.6.1.2}$$

for every $\varphi \in C^\infty([G])^J$ and $m \in M_P(\mathbf{A})$. This last claim is a consequence of the two following facts whose proofs are elementary and left to the reader:

(**2.6.1.3**) For every non-Archimedean place v, there exists a constant $C_v \geqslant 1$, with $C_v = 1$ for almost all v, such that $|\ell(\mathrm{Ad}(m_v)u_v)|_v > C_v$ implies $\varphi_{N_P,\psi_\ell}(m) = 0$ for every $\varphi \in C^\infty([G])^J$ and $m \in M_P(\mathbf{A})$.

(**2.6.1.4**) Let v be an Archimedean place and let $X \in \mathfrak{g}_v$ be such that $u_v = e^X$. Then, we have $(R(X)\varphi)_{N_P,\psi_\ell}(m) = d\psi_v(\ell(\mathrm{Ad}(m)u)_v)\varphi_{N_P,\psi_\ell}(m)$ for all $\varphi \in C^\infty([G])$ and $m \in M_P(\mathbf{A})$ where $d\psi_v : F_v \to i\mathbf{R}$ is the differential of ψ_v at the origin. □

2.6.2. Let $n \geqslant 1$ be a positive integer. We let GL_n acts on F^n by right multiplication and we denote by $e_n = (0, \dots, 0, 1)$ the last element of the standard basis of F^n. We also denote by $\mathcal{P}_n$ the *mirabolic* subgroup of GL_n, that is the stabilizer of e_n in GL_n. We identify A_{GL_n} with $\mathbf{G}_m$, and thus $A^\infty_{GL_n}$ with $\mathbf{R}_{>0}$, in the usual way. The next lemma will be used in conjunction with Lemma 2.6.1.1 to show the convergence of various Zeta integrals.

Lemma **2.6.2.1.** — *Let* $C > 1$. *Then, for* $N_1 \gg_C 1$ *and* $N_2 \gg_C 1$ *the integral*

$$\int_{\mathcal{P}_n(F)\backslash GL_n(\mathbf{A})\times\mathbf{R}_{>0}} \|ag\|_{GL_n}^{-N_1} \|e_n g\|_{\mathbf{A}^n}^{-N_2} |\det g|^s \, da\, dg$$

converges for $s \in \mathcal{H}_{]1,C[}$ *uniformly on every (closed) vertical strip.*

Proof. — The integral of the lemma can be rewritten as

$$(\mathbf{2.6.2.5})\qquad \int_{[GL_n]} \|g\|_{GL_n}^{-N_1} \int_{\mathbf{R}_{>0}} \sum_{\xi \in F^n\backslash\{0\}} \|\xi ag\|_{\mathbf{A}^n}^{-N_2} |\det ag|^s \, da\, dg.$$

There exists $N_3 > 0$ such that $\|v\|_{\mathbf{A}^n} \ll \|vg\|_{\mathbf{A}^n}^{N_3} \|g\|^{N_3}$ for $(v, g) \in \mathbf{A}^n \times GL_n(\mathbf{A})$. Therefore, the inner integral above is essentially bounded by

$$|\det g|^{\Re(s)} \|g\|^{N_2} \int_{\mathbf{R}_{>0}} \sum_{\xi \in F^n\backslash\{0\}} \|a\xi\|_{\mathbf{A}^n}^{-N_2/N_3} |a|^{n\Re(s)} \, da$$

hence, for $1 < \Re(s) < C$, by

$$\|g\|^{N_2+N_4} \int_{\mathbf{R}_{>0}} \sum_{\xi \in F^n\backslash\{0\}} \|a\xi\|_{\mathbf{A}^n}^{-N_2/N_3} |a|^{n\Re(s)} \, da$$

for some $N_4 > 0$. However, since the inner integral in (2.6.2.5) is left invariant by $GL_n(F)$, as a function of g, we may replace $\|g\|$ in the estimate above by $\|g\|_{GL_n}$. As for $N \gg 1$ we have

$$\int_{[\mathrm{GL}_n]} \|g\|_{\mathrm{GL}_n}^{-\mathrm{N}} dg < \infty$$

[BP21a, Proposition A.1.1 (vi)], it only remains to show that for $\mathrm{N} \gg 1$ the integral

$$\int_{\mathbf{R}_{>0}} \sum_{\xi \in \mathrm{F}^n \setminus \{0\}} \|a\xi\|_{\mathbf{A}^n}^{-\mathrm{N}} |a|^{ns} da$$

converges for $1 < \Re(s) < \mathrm{C}$ uniformly in vertical strips. This is a consequence of the following claim:

(**2.6.2.6**) For every $k \geqslant n$, if N is sufficiently large we have

$$\sum_{\xi \in \mathrm{F}^n \setminus \{0\}} \|a\xi\|_{\mathbf{A}^n}^{-\mathrm{N}} \ll |a|^{-k}, \ a \in \mathbf{R}_{>0}.$$

Indeed, we have

$$|a| = \max_{1 \leqslant i \leqslant n} (|a\xi_i|) \ll \|a\xi\|_{\mathbf{A}^n}$$

for $(a, \xi) \in \mathbf{R}_{>0} \times (\mathrm{F}^n \setminus \{0\})$. Therefore, we just need to prove (2.6.2.6) when $k = n$. Let $\mathrm{C} \subset \mathbf{A}^n$ be a compact subset which surjects onto $\mathbf{A}^n/\mathrm{F}^n$. We have

$$\|a\xi + av\|_{\mathbf{A}^n} \ll \|a\xi\|_{\mathbf{A}^n} \max(1, |a|), \quad \max(1, |a|) \ll \|a\xi\|_{\mathbf{A}^n}$$

for $(a, \xi, v) \in \mathbf{R}_{>0} \times (\mathrm{F}^n \setminus \{0\}) \times \mathrm{C}$. Hence,

$$\sum_{\xi \in \mathrm{F}^n \setminus \{0\}} \|a\xi\|_{\mathbf{A}^n}^{-\mathrm{N}} \ll \int_{\mathrm{C}} \sum_{\xi \in \mathrm{F}^n \setminus \{0\}} \|a\xi + av\|_{\mathbf{A}^n}^{-\mathrm{N}/2} dv$$
$$\ll \int_{\mathbf{A}^n/\mathrm{F}^n} \sum_{\xi \in \mathrm{F}^n} \|a\xi + av\|_{\mathbf{A}^n}^{-\mathrm{N}/2} dv = |a|^{-n} \int_{\mathbf{A}^n} \|v\|_{\mathbf{A}^n}^{-\mathrm{N}/2} dv$$

for $a \in \mathbf{R}_{>0}$. The last integral above is absolutely convergent when $\mathrm{N} \gg 1$ [BP21a, Proposition A.1.1 (vi)] and the claim (2.6.2.6) follows. □

2.7. *Automorphic forms and representations*

2.7.1. Let P be a standard parabolic subgroup of G. We define the space $\mathcal{A}_{\mathrm{P}}(\mathrm{G})$ of *automorphic forms* on $[\mathrm{G}]_{\mathrm{P}}$ as the subspace of $\mathcal{Z}(\mathfrak{g}_\infty)$-finite functions in $\mathcal{T}([\mathrm{G}]_{\mathrm{P}})$. We let $\mathcal{A}_{\mathrm{P},\mathrm{cusp}}(\mathrm{G})$ (resp. $\mathcal{A}_{\mathrm{P},\mathrm{disc}}(\mathrm{G})$) be the subspace of cuspidal (resp. square-integrable) automorphic forms i.e. the space of forms $\varphi \in \mathcal{A}_{\mathrm{P}}(\mathrm{G})$ such that $\varphi_{\mathrm{Q}} = 0$ for every proper parabolic subgroup $\mathrm{Q} \subsetneq \mathrm{P}$ (resp. such that $|\varphi| \in \mathrm{L}^2([\mathrm{G}]_{\mathrm{P},0})$).

For $\mathcal{J} \subset \mathcal{Z}(\mathfrak{g}_\infty)$ an ideal of finite codimension, we denote by $\mathcal{A}_{P,\mathcal{J}}(G)$ the subspace of automorphic forms $\varphi \in \mathcal{A}_P(G)$ such that $R(z)\varphi = 0$ for every $z \in \mathcal{J}$ and we set

$$\mathcal{A}_{P,\mathrm{cusp},\mathcal{J}}(G) = \mathcal{A}_{P,\mathcal{J}}(G) \cap \mathcal{A}_{P,\mathrm{cusp}}(G),$$
$$\mathcal{A}_{P,\mathrm{disc},\mathcal{J}}(G) = \mathcal{A}_{P,\mathcal{J}}(G) \cap \mathcal{A}_{P,\mathrm{disc}}(G).$$

There exists $N \geqslant 1$ such that $\mathcal{A}_{P,\mathcal{J}}(G)$ is a closed subspace of $\mathcal{T}_N([G]_P)$ and we equip $\mathcal{A}_{P,\mathcal{J}}(G)$ with the induced topology from $\mathcal{T}_N([G]_P)$. This topology does not depend on the choice of N by Lemma 2.5.4.1 and the open mapping theorem. The subspaces $\mathcal{A}_{P,\mathrm{cusp},\mathcal{J}}(G)$ and $\mathcal{A}_{P,\mathrm{disc},\mathcal{J}}(G)$ of $\mathcal{A}_{P,\mathcal{J}}(G)$ are closed and we equip them with the induced topologies. Then, by Lemma 2.5.4.1, $\mathcal{A}_{P,\mathrm{cusp},\mathcal{J}}(G)$, $\mathcal{A}_{P,\mathrm{disc},\mathcal{J}}(G)$ and $\mathcal{A}_{P,\mathcal{J}}(G)$ are all SLF representations of $G(\mathbf{A})$ (in the sense of Section 2.5.4) for the action by right translation.

For convenience, we also endow $\mathcal{A}_P(G) = \bigcup_{\mathcal{J}} \mathcal{A}_{P,\mathcal{J}}(G)$, $\mathcal{A}_{P,\mathrm{cusp}}(G) = \bigcup_{\mathcal{J}} \mathcal{A}_{P,\mathrm{cusp},\mathcal{J}}(G)$ and $\mathcal{A}_{P,\mathrm{disc}}(G) = \bigcup_{\mathcal{J}} \mathcal{A}_{P,\mathrm{disc},\mathcal{J}}(G)$ with the corresponding locally convex direct limit topologies. These spaces are not LF because the poset of ideals of finite codimension in $\mathcal{Z}(\mathfrak{g}_\infty)$ does not admit a countable cofinal subset. However, for every maximal ideal $\mathfrak{m} \subset \mathcal{Z}(\mathfrak{g}_\infty)$, the subspaces

$$\mathcal{A}_P(G)_{\mathfrak{m}} = \bigcup_n \mathcal{A}_{P,\mathfrak{m}^n}(G), \ \mathcal{A}_{P,\mathrm{cusp}}(G)_{\mathfrak{m}} = \bigcup_n \mathcal{A}_{P,\mathrm{cusp},\mathfrak{m}^n}(G) \text{ and}$$
$$\mathcal{A}_{P,\mathrm{disc}}(G)_{\mathfrak{m}} = \bigcup_n \mathcal{A}_{P,\mathrm{disc},\mathfrak{m}^n}(G)$$

are strict LF spaces and we have decompositions as locally convex topological direct sums

$$\mathcal{A}_P(G) = \bigoplus_{\mathfrak{m}} \mathcal{A}_P(G)_{\mathfrak{m}}, \ \mathcal{A}_{P,\mathrm{cusp}}(G) = \bigoplus_{\mathfrak{m}} \mathcal{A}_{P,\mathrm{cusp}}(G)_{\mathfrak{m}},$$
$$\mathcal{A}_{P,\mathrm{disc}}(G) = \bigoplus_{\mathfrak{m}} \mathcal{A}_{P,\mathrm{disc}}(G)_{\mathfrak{m}}$$

where $\mathfrak{m}$ runs over all maximal ideals of $\mathcal{Z}(\mathfrak{g}_\infty)$.

For $P = G$, we simply set $\mathcal{A}(G) = \mathcal{A}_G(G)$, $\mathcal{A}_{\mathrm{disc}}(G) = \mathcal{A}_{G,\mathrm{disc}}(G)$ and $\mathcal{A}_{\mathrm{cusp}}(G) = \mathcal{A}_{G,\mathrm{cusp}}(G)$.

2.7.2. By a *cuspidal* (resp. *discrete*) *automorphic representation* π of $M_P(\mathbf{A})$ we mean a topologically irreducible subrepresentation of $\mathcal{A}_{\mathrm{cusp}}(M_P)$ (resp. $\mathcal{A}_{\mathrm{disc}}(M_P)$). Let π be a cuspidal or discrete automorphic representation of $M_P(\mathbf{A})$. We endow π with the topology induced from $\mathcal{A}_{\mathrm{cusp}}(M_P)$ or $\mathcal{A}_{\mathrm{disc}}(M_P)$. With this topology, it becomes a SLF representation of $M_P(\mathbf{A})$. Moreover, for every compact-open subgroup $J \subset G(\mathbf{A}_f)$, the subspace $\pi^J_{(K_\infty)}$ of J-fixed and K_∞-finite vectors is a Harish-Chandra $(\mathfrak{g}_\infty, K_\infty)$-module whose smooth Fréchet globalization of moderate growth (which is unique by the Casselman-Wallach globalization theorem [Cas89a], [Wal92, Chapter 11], [BK14]) is isomorphic to π^J (the subspace of J-fixed vectors).

For every $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$, we define the twist $\pi_\lambda = \pi \otimes \lambda$ as the space of functions of the form

$$m \in [M_P] \mapsto \exp(\langle \lambda, H_P(m)\rangle)\varphi(m), \quad \text{for } \varphi \in \pi.$$

If π is cuspidal (resp. discrete and $\lambda \in i\mathfrak{a}^*_P$), π_λ is again a cuspidal (resp. discrete) automorphic representation.

We denote by $\mathcal{A}_{\pi,\mathrm{cusp}}(M_P)$ (resp. $\mathcal{A}_{\pi,\mathrm{disc}}(M_P)$) the π-isotypic component of $\mathcal{A}_{\mathrm{cusp}}(M_P)$ (resp. $\mathcal{A}_{\mathrm{disc}}(M_P)$) i.e. the sum of all cuspidal (resp. discrete) automorphic representations of $M_P(\mathbf{A})$ that are isomorphic to π. Let $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi)$ (resp. $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G) = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\mathcal{A}_{\pi,\mathrm{cusp}}(M_P))$, $\mathcal{A}_{P,\pi,\mathrm{disc}}(G) = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\mathcal{A}_{\pi,\mathrm{disc}}(M_P))$) be the *normalized smooth induction* of π (resp. $\mathcal{A}_{\pi,\mathrm{cusp}}(M_P)$, $\mathcal{A}_{\pi,\mathrm{disc}}(M_P)$) that we identify with the space of forms $\varphi \in \mathcal{A}_P(G)$ such that

$$m \in [M_P] \mapsto \exp(-\langle \rho_P, H_P(m)\rangle)\varphi(mg)$$

belongs to π (resp. $\mathcal{A}_{\pi,\mathrm{cusp}}(M_P)$, $\mathcal{A}_{\pi,\mathrm{disc}}(M_P)$) for every $g \in G(\mathbf{A})$. Then, Π and $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ (resp. $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$) are closed subspaces of $\mathcal{A}_{P,\mathrm{cusp}}(G)$ (resp. $\mathcal{A}_{P,\mathrm{disc}}(G)$) if π is cuspidal (resp. discrete) and with the induced topologies these become SLF representations of $G(\mathbf{A})$. In particular, the algebra $\mathcal{S}(G(\mathbf{A}))$ acts on $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ (resp. $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$) by right convolution. When the context is clear (that is when the automorphic representation π is fixed), for every $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$, we will denote by $I(\lambda)$ the action on $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ we get by transport from the action of $\mathcal{S}(G(\mathbf{A}))$ on $\mathcal{A}_{P,\pi_\lambda,\mathrm{cusp}}$ and the identification $\mathcal{A}_{P,\pi,\mathrm{cusp}} \to \mathcal{A}_{P,\pi_\lambda,\mathrm{cusp}}$ given by $\varphi \mapsto \exp(\langle \lambda, H_P(.)\rangle)\varphi$. In the same way, we get an action on $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$ also denoted by $I(\lambda)$.

If the central character of π is unitary, we equip $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi)$ and $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ (resp. $\mathcal{A}_{P,\pi,\mathrm{disc}}(G)$) with the *Petersson inner product*

$$\|\varphi\|^2_{\mathrm{Pet}} = \langle \varphi, \varphi\rangle_{\mathrm{Pet}} = \int_{[G]_{P,0}} |\varphi(g)|^2 dg, \ \varphi \in \Pi.$$

2.7.3. *Eisenstein series.* — Let P be a standard parabolic subgroup of G. For every $\varphi \in \mathcal{A}_{P,\mathrm{disc}}(G)$, $g \in G(\mathbf{A})$ and $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$, we denote by

$$E(g, \varphi, \lambda) = \sum_{\delta \in P(F)\backslash G(F)} \exp(\langle \lambda, H_P(\delta g)\rangle \varphi(\delta g)$$

the corresponding Eisenstein series which is absolutely convergent for $\Re(\lambda)$ in a suitable cone. By [Lan76], [BL19] it admits a meromorphic continuation to $\mathfrak{a}^*_{P,\mathbf{C}}$ whenever φ is K_∞-finite and this still holds without this assumption by [Lap08]. Let π be a discrete automorphic representation of $M_P(\mathbf{A})$ and $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi) \subseteq \mathcal{A}_{P,\mathrm{disc}}(G)$ be the induced representation of $G(\mathbf{A})$. Then, for $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^*$ where $E(\varphi,\lambda)$ is regular for every $\varphi \in \Pi$, $\varphi \mapsto E(\varphi,\lambda)$ induces a continuous linear map $\Pi \to \mathcal{T}([G])$ by [Lap08, Theorem 2.2] that actually factors through $\mathcal{T}_N([G])$ for some $N > 0$ giving a map $\Pi \to \mathcal{T}_N([G])$ that is also continuous (by the closed graph theorem).

2.7.4. Let P and Q be standard parabolic subgroups of G. For any $w \in W(P,Q)$ and $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^*$, we have the intertwining operator

$$M(w,\lambda) : \mathcal{A}_{P,\mathrm{disc}}(G) \to \mathcal{A}_{Q,\mathrm{disc}}(G)$$

defined by analytic continuation from the integral

$$(M(w,\lambda)\varphi)(g) = \exp(-\langle w\lambda, H_P(g)\rangle) \times \int_{(N_Q \cap wN_Pw^{-1})(\mathbf{A})\backslash N_Q(\mathbf{A})} \exp(\langle \lambda, H_P(w^{-1}ng)\rangle)\varphi(w^{-1}ng)\,dn.$$

Once again, the K_∞-finite case follows from [Lan76], [BL19] whereas the extension to general smooth discrete automorphic forms is proved in [Lap08].

2.7.5. Assume that $G = G_1 \times G_2$ where G_1 and G_2 are connected reductive groups over F. We have corresponding decompositions $P = P_1 \times P_2$ and $M_P = M_{P_1} \times M_{P_2}$. Let π be a discrete automorphic representation of $M_P(\mathbf{A})$ and set as before $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi)$. Then, there exist two, uniquely determined, cuspidal automorphic representations π_1, π_2 of $M_{P_1}(\mathbf{A})$ and $M_{P_2}(\mathbf{A})$ respectively such that, setting $\Pi_1 = \mathrm{Ind}_{P_1(\mathbf{A})}^{G_1(\mathbf{A})}(\pi_1)$ and $\Pi_2 = \mathrm{Ind}_{P_2(\mathbf{A})}^{G_2(\mathbf{A})}(\pi_2)$, for every compact-open subgroups $J_1 \subseteq G_1(\mathbf{A}_f)$, $J_2 \subseteq G_2(\mathbf{A}_f)$ (resp. $J_1 \subseteq M_{P_1}(\mathbf{A}_f)$, $J_2 \subseteq M_{P_2}(\mathbf{A}_f)$), setting $J = J_1 \times J_2$, there is a topological isomorphism

$$\Pi_1^{J_1} \widehat{\otimes} \Pi_2^{J_2} \simeq \Pi^J \quad (\text{resp. } \pi_1^{J_1} \widehat{\otimes} \pi_2^{J_2} \simeq \pi^J) \tag{2.7.5.1}$$

sending $\varphi_1 \otimes \varphi_2 \in \Pi_1^{J_1} \otimes \Pi_2^{J_2}$ (resp. $\varphi_1 \otimes \varphi_2 \in \pi_1^{J_1} \otimes \pi_2^{J_2}$) to the function $(g_1,g_2) \mapsto \varphi_1(g_1)\varphi_2(g_2)$. We will then write $\Pi = \Pi_1 \boxtimes \Pi_2$ and $\pi = \pi_1 \boxtimes \pi_2$ respectively.

2.7.6. Assume now that G is quasi-split. Let $\psi_N : N_0(\mathbf{A}) \to \mathbf{C}^\times$ be a continuous non-degenerate character which is trivial on $N_0(F)$. If the representation Π is ψ_N-generic, i.e. if it admits a continuous nonzero linear form $\ell : \Pi \to \mathbf{C}$ such that $\ell \circ \Pi(u) = \psi_N(u)\ell$ for every $u \in N(\mathbf{A})$, it is (abstractly) isomorphic to its *Whittaker model*

$$\mathcal{W}(\Pi,\psi_N) = \{g \in G(\mathbf{A}) \mapsto \ell(\Pi(g)\varphi) \mid \varphi \in \Pi\}.$$

We equip this last space with the topology coming from Π (thus it is a SLF representation of $G(\mathbf{A})$).

If we are moreover in the situation of Section 2.7.5, there are decompositions $N_0 = N_{0,1} \times N_{0,2}$, $\psi_N = \psi_1 \boxtimes \psi_2$ and the isomorphism (2.7.5.1) induces one between Whittaker models

$$\mathcal{W}(\Pi_1, \psi_1)^{J_1} \widehat{\otimes} \mathcal{W}(\Pi_2, \psi_2)^{J_2} \simeq \mathcal{W}(\Pi, \psi_N)^J.$$

2.8. *Relative characters*

2.8.1. Let B a $G(F_\infty)$-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}_\infty$. We assume that the restriction of B to $\mathfrak{k}_\infty$ is negative and the restriction of B to the orthogonal complement of $\mathfrak{k}_\infty$ is positive. Let $(X_i)_{i\in I}$ be an orthonormal basis of $\mathfrak{k}_\infty$ relative to $-B$. Let $C_K = -\sum_{i\in I} X_i^2$: this is a "Casimir element" of $\mathcal{U}(\mathfrak{k}_\infty)$.

2.8.2. Let $\hat{K}_\infty$ and $\hat{K}$ be respectively the sets of isomorphism classes of irreducible unitary representations of K_∞ and of K.

2.8.3. Let π be a discrete automorphic representation of M_P. In the following, we denote by $\mathcal{A}_{P,\pi}$ either $\mathcal{A}_{P,\pi,\mathrm{cusp}}$ or $\mathcal{A}_{P,\pi,\mathrm{disc}}$. For any $\tau \in \hat{K}$, let $\mathcal{A}_{P,\pi}(G,\tau)$ be the (finite dimensional) subspace of functions in $\mathcal{A}_{P,\pi}(G)$ which transform under K according to τ. A K-basis $\mathcal{B}_{P,\pi}$ of $\mathcal{A}_{P,\pi}(G)$ is by definition the union over of $\tau \in \hat{K}$ of orthonormal bases $\mathcal{B}_{P,\pi,\tau}$ of $\mathcal{A}_{P,\pi}(G,\tau)$ for the Petersson inner product.

2.8.4. Let

$$B : \mathcal{A}_{P,\pi}(G) \times \mathcal{A}_{P,\pi}(G) \to \mathbf{C}$$

be a continuous sesquilinear form.

Proposition **2.8.4.1.** — *Let ω be a compact subset of $\mathfrak{a}_P^{G,*}$.*

1. *Let $f \in \mathcal{S}(G(\mathbf{A}))$ and $\mathcal{B}_{P,\pi}$ be a K-basis of $\mathcal{A}_{P,\pi}(G)$. The sum*

$$\sum_{\varphi\in\mathcal{B}_{P,\pi}} I_P(\lambda, f)\varphi \otimes \overline{\varphi} \qquad (\mathbf{2.8.4.1})$$

converges absolutely in the completed projective tensor product $\mathcal{A}_{P,\pi}(G)\widehat{\otimes}\overline{\mathcal{A}_{P,\pi}(G)}$ uniformly for $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*}$ such that $\Re(\lambda) \in \omega$. In particular, the sum

$$J_B(\lambda, f) = \sum_{\varphi\in\mathcal{B}_{P,\pi}} B(I_P(\lambda, f)\varphi, \varphi) \qquad (\mathbf{2.8.4.2})$$

is absolutely convergent uniformly for $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*}$ such that $\Re(\lambda) \in \omega$. Moreover these sums do not depend on the choice of $\mathcal{B}_{P,\pi}$.

2. *The map*

$$f \mapsto J_B(\lambda, f)$$

is a continuous linear form on $\mathcal{S}(G(\mathbf{A}))$. *More precisely for* $C \subset G(\mathbf{A}_f)$ *a compact subset and* $K_0 \subset K^\infty$ *a compact-open subgroup, there exists a continuous semi-norm* $\|\cdot\|$ *on* $\mathcal{S}(G(\mathbf{A}), C, K_0)$ *such that for all* $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*}$ *such that* $\Re(\lambda) \in \omega$ *and* $f \in \mathcal{S}(G(\mathbf{A}), C, K_0)$ *we have*

$$|J_B(\lambda, f)| \leqslant \|f\|.$$

Remark **2.8.4.2.** — An examination of the proof below show that the assertion 2 also holds *mutatis mutandis* if $f \in C_c^r(G(\mathbf{A}))$ with r large enough. The semi-norm is then taken among the norms $\|\cdot\|_{r,X,Y}$ for which the sum of the degrees of X and Y is less than r.

Proof. — By definition of the projective tensor product topology, it suffices to show the following: for every continuous semi-norm p on $\mathcal{A}_{P,\pi}(G)$, the series

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}} p(I_P(\lambda, f)\varphi) p(\varphi)$$

is absolutely convergent uniformly for $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*}$ such that $\Re(\lambda) \in \omega$. Let $K_0 \subset K^\infty$ be a normal compact-open subgroup by which f is biinvariant. The series above can be rewritten as

$$\sum_{\tau \in \hat{K}} \sum_{\varphi \in \mathcal{B}_{P,\pi,\tau}} p(I_P(\lambda, f)\varphi) p(\varphi), \tag{2.8.4.3}$$

where only the representations τ admitting K_0-invariant vectors actually contribute to the sum. Note that, since K_0 is normal in K^∞, for such representation τ all the elements $\varphi \in \mathcal{B}_{P,\pi,\tau}$ are automatically K_0-fixed. Moreover, by [Wal92] §10.1, there exist $c > 0$ and an integer r such that for every $\varphi \in \mathcal{A}_{P,\pi}(G)^{K_0}$ we have

$$p(\varphi) \leqslant c\|R(1 + C_K)^r \varphi\|_{Pet}.$$

For any $\tau \in \hat{K}_\infty$ or $\hat{K}$, let $\lambda_\tau \geqslant 0$ be the eigenvalue of C_K acting on τ. Let us fix a large enough $N > 0$. For every $f \in \mathcal{S}(G(\mathbf{A}))$, $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^*$, $\tau \in \widehat{K}$ and $\varphi \in \mathcal{B}_{P,\pi,\tau}$, we have

$$\begin{aligned} \|R(1 + C_K)^r I_P(\lambda, f)\varphi\|_{Pet} &= \|I_P(\lambda, L((1 + C_K)^r) f)\varphi\|_{Pet} \\ &= (1 + \lambda_\tau)^{-N} \|I_P(\lambda, f_{r,N})\varphi\|_{Pet} \end{aligned}$$

where $f_{r,N} = R((1 + C_K)^N) L((1 + C_K))^r f$. Let $C \subset G(\mathbf{A}_f)$ be a compact subset. Then, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbf{A}), C, K_0)$ (among those of Section

2.5.2) such that for any $f \in \mathcal{S}(G(\mathbf{A}), C, K_0)$, $\varphi \in \mathcal{A}_{P,\pi}(G)$ and $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*}$ such that $\Re(\lambda) \in \omega$ we have

$$\|I_P(\lambda, f_{r,N})\varphi\|_{P,\pi} \leqslant \|f\| \|\varphi\|_{Pet}.$$

Thereby we are reduced to prove for large enough N the convergence of

$$\sum_{\tau \in \hat{K}_\infty} (1+\lambda_\tau)^{r-N} \dim(\mathcal{A}_{P,\pi}^\infty(G, K_0, \tau)) \tag{2.8.4.4}$$

where $\mathcal{A}_{P,\pi}^\infty(G, K_0, \tau) \subset \mathcal{A}_{P,\pi}(G)$ denotes the subspace of functions that transform under K_∞ according to τ. However, there exist $c_2 > 0$ and $m \geqslant 1$ such that $\dim(\mathcal{A}_{P,\pi}^\infty(G, K_0, \tau)) \leqslant c_2(1+\lambda_\tau)^m$ (see e.g. the proof of [Mül00] Lemma 6.1). So the convergence of (2.8.4.4) is reduced to that of $\sum_{\tau \in \hat{K}_\infty} (1+\lambda_\tau)^{-N}$ which is well-known. □

Proposition **2.8.4.3.** — *Let* $K_0 \subset K^\infty$ *be a normal open compact subgroup. For any integer* $m \geqslant 1$ *there exist* $Z \in \mathcal{U}(\mathfrak{g}_\infty)$, $g_1 \in C_c^\infty(G(\mathbf{A}))$ *and* $g_2 \in C_c^m(G(\mathbf{A}))$ *such that*

- Z, g_1 *and* g_2 *are invariant under* K_∞*-conjugation;*
- g_1 *and* g_2 *are* K_0*-biinvariant;*
- *for any* $f \in \mathcal{S}(G(\mathbf{A}))$ *that is* K_0*-biinvariant we have:*

$$f = f * g_1 + (f * Z) * g_2.$$

For large enough m, we have

$$J_B(\lambda, f) = \sum_{\varphi \in \mathcal{B}_{P,\pi}} B(I_P(\lambda, f)\varphi, I_P(\bar{\lambda}, g_1^\vee)\varphi) + \sum_{\varphi \in \mathcal{B}_{P,\pi}} B(I_P(\lambda, f * Z)\varphi, I_P(\bar{\lambda}, g_2^\vee)\varphi)$$

where the sums are absolutely convergent and $g_i^\vee(x) = \overline{g_i(x^{-1})}$.

Proof. — The first part of the proposition is lemma 4.1 and corollary 4.2 of [Art78]. Once we have noticed that the operators $I_P(\lambda, g_i)$ preserve the spaces $\mathcal{A}_{P,\pi}(G, \tau)$, the second part results from an easy computation in a finite dimensional space. □

2.9. *Cuspidal data and coarse Langlands decomposition*

2.9.1. *Cuspidal data.* — Let $\underline{\mathfrak{X}}(G)$ be the set of pairs (M_P, π) where

- P is a standard parabolic subgroup of G;
- π is the isomorphism class of a cuspidal automorphic representations of $M_P(\mathbf{A})$ with central character trivial on A_P^∞.

The set of *cuspidal data* $\mathfrak{X}(\mathrm{G})$ is the quotient of $\underline{\mathfrak{X}}(\mathrm{G})$ by the equivalence relation defined as follows: $(\mathrm{M_P}, \pi) \sim (\mathrm{M_Q}, \tau)$ if there exists $w \in \mathrm{W(P, Q)}$ such that $w\pi w^{-1} \simeq \tau$. Note that for every standard parabolic subgroup P of G, the inclusion $\underline{\mathfrak{X}}(\mathrm{M_P}) \subset \underline{\mathfrak{X}}(\mathrm{G})$ descends to a finite-to-one map $\mathfrak{X}(\mathrm{M_P}) \to \mathfrak{X}(\mathrm{G})$. For $\chi \in \mathfrak{X}(\mathrm{G})$ represented by a pair $(\mathrm{M_P}, \pi)$, we denote by $\chi^\vee$ the cuspidal datum associated to $(\mathrm{M_P}, \pi^\vee)$ where $\pi^\vee$ stands for the complex conjugate of π.

2.9.2. *Langlands decomposition.* — For $(\mathrm{M_P}, \pi) \in \underline{\mathfrak{X}}(\mathrm{G})$, we let $\mathcal{S}_\pi([\mathrm{G}]_\mathrm{P})$ be the space of Schwartz functions $\varphi \in \mathcal{S}([\mathrm{G}]_\mathrm{P})$ such that

$$\varphi_\lambda(x) := \int_{\mathrm{A_P^\infty}} \exp(-\langle \rho_\mathrm{P} + \lambda, \mathrm{H_P}(a)\rangle)\varphi(ax)da, \; x \in [\mathrm{G}]_\mathrm{P},$$

belongs to $\mathcal{A}_{\mathrm{P},\pi_\lambda,\mathrm{cusp}}(\mathrm{G})$ for every $\lambda \in \mathfrak{a}^*_{\mathrm{P},\mathbf{C}}$.

Let $\mathrm{P} \subset \mathrm{G}$ a standard parabolic subgroup, $\chi \in \mathfrak{X}(\mathrm{G})$ be a cuspidal datum and

$$\{(\mathrm{M}_{\mathrm{Q}_i}, \pi_i) \mid i \in \mathrm{I}\}$$

be the (possibly empty but finite) inverse image of χ in $\underline{\mathfrak{X}}(\mathrm{M_P})$. Denote by $\mathrm{L}^2_\chi([\mathrm{G}]_\mathrm{P})$ the closure in $\mathrm{L}^2([\mathrm{G}]_\mathrm{P})$ of the subspace

$$\mathfrak{O}^\mathrm{P}_\chi := \sum_{i\in\mathrm{I}} \mathrm{E}^\mathrm{P}_{\mathrm{Q}_i}(\mathcal{S}_{\pi_i}([\mathrm{G}]_{\mathrm{Q}_i})).$$

More generally, for w a weight on $\mathrm{A_P^\infty}$ (see Section 2.4.3), we let $\mathrm{L}^2_{w,\chi}([\mathrm{G}]_\mathrm{P})$ be the closure of $\mathfrak{O}^\mathrm{P}_\chi$ in $\mathrm{L}^2_w([\mathrm{G}]_\mathrm{P})$ and we define similarly a subspace $\mathrm{L}^2_\chi([\mathrm{G}]_{\mathrm{P},0}) \subset \mathrm{L}^2([\mathrm{G}]_{\mathrm{P},0})$. By Langlands (see e.g. [MW94, Proposition II.2.4]), we have decompositions in orthogonal direct sums

$$(\mathbf{2.9.2.1}) \qquad \mathrm{L}^2_w([\mathrm{G}]_\mathrm{P}) = \widehat{\bigoplus_{\chi\in\mathfrak{X}(\mathrm{G})}} \mathrm{L}^2_{w,\chi}([\mathrm{G}]_\mathrm{P}) \quad \text{and} \quad \mathrm{L}^2([\mathrm{G}]_{\mathrm{P},0}) = \widehat{\bigoplus_{\chi\in\mathfrak{X}(\mathrm{G})}} \mathrm{L}^2_\chi([\mathrm{G}]_{\mathrm{P},0}).$$

For every subset $\mathfrak{X} \subseteq \mathfrak{X}(\mathrm{G})$, we set

$$\mathrm{L}^2_{w,\mathfrak{X}}([\mathrm{G}]_\mathrm{P}) := \widehat{\bigoplus_{\chi\in\mathfrak{X}}} \mathrm{L}^2_{w,\chi}([\mathrm{G}]_\mathrm{P}), \; \mathrm{L}^{2,\mathfrak{X}}_w([\mathrm{G}]_\mathrm{P}) := \widehat{\bigoplus_{\chi\in\mathfrak{X}^c}} \mathrm{L}^2_{w,\chi}([\mathrm{G}]_\mathrm{P})$$

where $\mathfrak{X}^c$ denotes the complement of $\mathfrak{X}$ in $\mathfrak{X}(\mathrm{G})$. When $w = 1$, we will drop the index w. We have

(**2.9.2.2**) For two weights w and w' on $\mathrm{A_P^\infty}$, the orthogonal projections $\mathrm{L}^2_w([\mathrm{G}]_\mathrm{P}) \to \mathrm{L}^2_{w,\mathfrak{X}}([\mathrm{G}]_\mathrm{P})$ and $\mathrm{L}^2_{w'}([\mathrm{G}]_\mathrm{P}) \to \mathrm{L}^2_{w',\mathfrak{X}}([\mathrm{G}]_\mathrm{P})$ coincide on the intersection $\mathrm{L}^2_w([\mathrm{G}]_\mathrm{P}) \cap \mathrm{L}^2_{w'}([\mathrm{G}]_\mathrm{P})$.

Indeed, both of these projections coincide on $L^2_w([G]_P) \cap L^2_{w'}([G]_P) = L^2_{w''}([G]_P)$, where $w'' = \max(w, w')$ is again a weight on A_P^∞, with the orthogonal projection $L^2_{w''}([G]_P) \to L^2_{w'',\mathfrak{X}}([G]_P)$ as follows readily by looking at their restrictions to the dense subspace

$$\sum_{\chi \in \mathfrak{X}(G)} \mathfrak{O}^P_\chi.$$

We will denote by $\varphi \mapsto \varphi_{\mathfrak{X}}$ the orthogonal projection $L^2_w([G]_P) \to L^2_{w,\mathfrak{X}}([G]_P)$. By (2.9.2.2), this notation shouldn't lead to any confusion. These projections are $G(\mathbf{A})$-equivariant and so preserve the subspaces of smooth vectors.

2.9.3. Let $\mathfrak{X} \subseteq \mathfrak{X}(G)$ be a subset. We set

$$\mathcal{S}_{\mathfrak{X}}([G]_P) = \mathcal{S}([G]_P) \cap L^2_{\mathfrak{X}}([G]_P) \quad \text{and}$$
$$\mathcal{S}^{\mathfrak{X}}([G]_P) = \mathcal{S}([G]_P) \cap L^{2,\mathfrak{X}}([G]_P).$$

By definition of $L^2_{\mathfrak{X}}([G]_P)$, $\mathcal{S}_{\mathfrak{X}}([G]_P)$ is the orthogonal to

$$\mathfrak{O}^P_{\mathfrak{X}^c} := \sum_{\chi \in \mathfrak{X}^c} \mathfrak{O}^P_\chi$$

in $\mathcal{S}([G]_P)$. In particular, for every weight w on A_P^∞, we also have

$$\mathcal{S}_{\mathfrak{X}}([G]_P) = \mathcal{S}([G]_P) \cap L^2_{w,\mathfrak{X}}([G]_P).$$

Let w be a weight on $[G]_P$ (not necessarily factoring through a weight of A_P^∞). For $\mathcal{F} \in \{L^2_w, \mathcal{C}, \mathcal{T}_N, \mathcal{T}, \mathcal{S}_{w,N}, \mathcal{S}_w\}$ we define $\mathcal{F}_{\mathfrak{X}}([G]_P)$ to be the orthogonal of $\mathcal{S}_{\mathfrak{X}^c}([G]_P)$ in $\mathcal{F}([G]_P)$. We will also write $\mathcal{F}^{\mathfrak{X}}([G]_P)$ for $\mathcal{F}_{\mathfrak{X}^c}([G]_P)$.

Lemma **2.9.3.1.** — *Let $\mathfrak{X} \subseteq \mathfrak{X}(G)$ be a subset and $Q \subseteq P$ be standard parabolic subgroups. Then, we have*

$$(\mathbf{2.9.3.3}) \qquad E^P_Q(\mathcal{S}_{\mathfrak{X}}([G]_Q)) \subseteq \mathcal{S}_{\mathfrak{X}}([G]_P) \ \textit{and} \ \mathcal{T}_{\mathfrak{X}}([G]_P)_Q \subseteq \mathcal{T}_{\mathfrak{X}}([G]_Q).$$

Proof. — The second inclusion follows from the first applied to $\mathfrak{X}^c$ by adjunction. It remains to prove the first inclusion. Since $\mathcal{S}_{\mathfrak{X}}([G]_P)$ is also the orthogonal of $\mathcal{S}^{\mathfrak{X}}([G]_P)$ in $\mathcal{S}([G]_P)$, by adjunction again it suffices to establish that $\mathcal{S}^{\mathfrak{X}}([G]_P)_Q$ is orthogonal to $\mathcal{S}_{\mathfrak{X}}([G]_Q)$. From the definition, it is clear that $\mathcal{S}^{\mathfrak{X}}([G]_P)_Q$ is orthogonal to $\mathfrak{O}^Q_{\mathfrak{X}}$. Let $\kappa \in C_c^\infty(\mathfrak{a}_Q)$. Then, we readily check that $\mathfrak{O}^Q_{\mathfrak{X}}$ is stable by multiplication by $(\kappa \circ H_Q)$. Consequently, $(\kappa \circ H_Q)\mathcal{S}^{\mathfrak{X}}([G]_P)_Q$ is also orthogonal to $\mathfrak{O}^Q_{\mathfrak{X}}$ but, by Lemma 2.5.13.1, we have $(\kappa \circ H_Q)\mathcal{S}^{\mathfrak{X}}([G]_P)_Q \subseteq \mathcal{S}([G]_Q)$. Therefore, by definition of $L^2_{\mathfrak{X}}([G]_Q)$, $(\kappa \circ H_Q)\mathcal{S}^{\mathfrak{X}}([G]_P)_Q$ is orthogonal to $L^2_{\mathfrak{X}}([G]_Q)$ and a fortiori to $\mathcal{S}_{\mathfrak{X}}([G]_Q)$. Letting $(\kappa_n)_n$

be an increasing sequence of positive functions in $C_c^\infty(\mathfrak{a}_Q)$ converging to 1 pointwise we get, by dominated convergence,

$$\langle \varphi_Q, \psi\rangle_Q = \lim_{n\to\infty} \langle (\kappa_n \circ H_Q)\varphi_Q, \psi\rangle_Q = 0$$

for every $\varphi \in \mathcal{S}^{\mathfrak{X}}([G]_P)$ and $\psi \in \mathcal{S}_{\mathfrak{X}}([G]_Q)$. This shows that $\mathcal{S}^{\mathfrak{X}}([G]_P)_Q$ is indeed orthogonal to $\mathcal{S}_{\mathfrak{X}}([G]_Q)$ and this ends the proof of the lemma. □

2.9.4. The following theorem is a variation on the well-known theme of "decomposition along the cuspidal support" (see [MW94, III, 2.6] and [FS98] for similar result on the space of automorphic forms). We refer the reader to Section A.0.2 for a reminder on summable and absolutely summable families in locally convex topological vector spaces.

Theorem **2.9.4.1.** — *Let w be a weight on $[G]_P$. Then*

1. *For every subset $\mathfrak{X} \subset \mathfrak{X}(G)$, the orthogonal projection $\mathcal{S}([G]_P) \to L^2_{\mathfrak{X}}([G]_P)$ extends by continuity to a projection*

$$L^2_w([G]_P)^\infty \to L^2_{w,\mathfrak{X}}([G]_P)^\infty, \ \ \varphi \mapsto \varphi_{\mathfrak{X}}.$$

Moreover, for every $\varphi \in L^2_w([G]_P)^\infty$ the family $(\varphi_\chi)_{\chi\in\mathfrak{X}(G)}$ is summable in $L^2_w([G]_P)^\infty$ with sum φ.

2. *For every subset $\mathfrak{X} \subset \mathfrak{X}(G)$, the orthogonal projection $L^2([G]_P) \to L^2_{\mathfrak{X}}([G]_P)$ restricts to a continuous projection*

$$\mathcal{S}([G]_P) \to \mathcal{S}_{\mathfrak{X}}([G]_P), \ \ \varphi \mapsto \varphi_{\mathfrak{X}}.$$

Moreover, for every $\varphi \in \mathcal{S}([G]_P)$, the family $(\varphi_\chi)_{\chi\in\mathfrak{X}(G)}$ is absolutely summable in $\mathcal{S}([G])$ with sum φ.

3. *For every subset $\mathfrak{X} \subset \mathfrak{X}(G)$ and $\mathcal{F} \in \{\mathcal{C}, \mathcal{T}, \mathcal{S}_w\}$, the projection $\mathcal{S}([G]_P) \to \mathcal{S}_{\mathfrak{X}}([G]_P)$ extends by continuity to a projection*

$$\mathcal{F}([G]_P) \to \mathcal{F}_{\mathfrak{X}}([G]_P), \ \ \varphi \mapsto \varphi_{\mathfrak{X}},$$

satisfying the adjunction

$$\langle \varphi_{\mathfrak{X}}, \psi\rangle_P = \langle \varphi, \psi_{\mathfrak{X}}\rangle_P, \quad \text{for } (\varphi,\psi) \in \mathcal{T}([G]_P) \times \mathcal{S}([G]_P). \tag{2.9.4.4}$$

Moreover, there exists $N_0 > 0$ such that for every $N > 0$ and every function $\varphi \in \mathcal{C}([G]_P)$ (resp. $\varphi \in \mathcal{T}_w([G]_P)$, resp. $\varphi \in \mathcal{S}_{w,N}([G]_P)$) the family $(\varphi_\chi)_{\chi\in\mathfrak{X}(G)}$ is absolutely summable in $\mathcal{C}([G]_P)$ (resp. $\mathcal{T}_{w,N_0}([G]_P)$, resp. $\mathcal{S}_{w,N+N_0}([G]_P)$) with sum φ.

Proof. — First, we note that point 1. implies points 2. and 3. Indeed, that the orthogonal projection $\mathcal{S}([G]_P) \to L^2_{\mathfrak{X}}([G]_P)$ induces continuous linear projections

$\mathcal{F}([\mathrm{G}]_{\mathrm{P}}) \to \mathcal{F}_{\mathfrak{X}}([\mathrm{G}]_{\mathrm{P}})$ for $\mathcal{F} \in \{\mathcal{S}, \mathcal{C}, \mathcal{T}, \mathcal{S}_w\}$ follows from the alternative descriptions (2.5.7.6), (2.5.8.7), (2.5.10.8) and (2.5.11.9) in terms of weighted L^2 spaces. Let $\mathrm{J} \subset \mathrm{G}(\mathbf{A}_f)$ be a compact-open subgroup. Let $\varphi \in \mathcal{S}([\mathrm{G}]_{\mathrm{P}})^{\mathrm{J}}$. Then, the first point of the theorem implies that $(\varphi_\chi)_{\chi \in \mathfrak{X}(\mathrm{G})}$ is a summable family in $\mathrm{L}^2_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}}$ for every $\mathrm{N} > 0$. On the other hand, by Proposition 2.5.5.1, for every $\mathrm{N} > 0$ and $\epsilon > 0$ the inclusion $\mathrm{L}^2_{\mathrm{N}+\epsilon}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}} \subset \mathrm{L}^2_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}}$ is nuclear (note that $\sigma_{\mathrm{P}}^{d_0} \ll \|.\|_{\mathrm{P}}^{\epsilon}$). Therefore, by Lemma A.0.6.1, for every $\mathrm{N} > 0$, the family $(\varphi_\chi)_{\chi \in \mathfrak{X}(\mathrm{G})}$ is actually absolutely summable $\mathrm{L}^2_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}}$ i.e. it is absolutely summable in $\mathcal{S}([\mathrm{G}]_{\mathrm{P}})^{\mathrm{J}}$ (by the presentation (2.5.7.6)). That it sums to φ is clear. The statements on absolute summability in point 3. can be similarly deduced from point 1. noting that, by (2.5.5.4), there exists $\mathrm{N}_1 > 0$ such that for every $\mathrm{N} > 0$

$$\mathrm{L}^2_{w^{-1},-\mathrm{N}+\mathrm{N}_1}([\mathrm{G}]_{\mathrm{P}})^{\infty} \subset \mathcal{T}_{w,\mathrm{N}}([\mathrm{G}]_{\mathrm{P}}) \subset \mathrm{L}^2_{w^{-1},-\mathrm{N}-\mathrm{N}_1}([\mathrm{G}]_{\mathrm{P}})^{\infty}$$

$$(\text{resp. } \bigcap_{r \geqslant 0} \mathrm{L}^2_{-\mathrm{N}+\mathrm{N}_1, w^r}([\mathrm{G}]_{\mathrm{P}})^{\infty} \subset \mathcal{S}_{w,\mathrm{N}}([\mathrm{G}]_{\mathrm{P}}) \subset \bigcap_{r \geqslant 0} \mathrm{L}^2_{-\mathrm{N}-\mathrm{N}_1, w^r}([\mathrm{G}]_{\mathrm{P}})^{\infty})$$

so that for $\mathrm{N}_0 > 2\mathrm{N}_1$, the inclusion $\mathcal{T}_{w,\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})^{\mathrm{J}} \subset \mathcal{T}_{w,\mathrm{N}+\mathrm{N}_0}([\mathrm{G}]_{\mathrm{P}})^{\mathrm{J}}$ (resp. $\mathcal{S}_{w,\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})^{\mathrm{J}} \subset \mathcal{S}_{w,\mathrm{N}+\mathrm{N}_0}([\mathrm{G}]_{\mathrm{P}})^{\mathrm{J}}$) factors through the nuclear inclusion $\mathrm{L}^2_{w^{-1},-\mathrm{N}-\mathrm{N}_1}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}} \subset \mathrm{L}^2_{w^{-1},-\mathrm{N}-\mathrm{N}_1-\epsilon}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}}$ (resp. $\bigcap_{r\geqslant 0} \mathrm{L}^2_{-\mathrm{N}-\mathrm{N}_1,w^r}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}} \subset \bigcap_{r\geqslant 0} \mathrm{L}^2_{-\mathrm{N}-\mathrm{N}_1-\epsilon,w^r}([\mathrm{G}]_{\mathrm{P}})^{\infty\mathrm{J}}$) for some $\epsilon > 0$. Finally, by density of $\mathcal{S}([\mathrm{G}]_{\mathrm{P}})$ in $\mathcal{T}([\mathrm{G}]_{\mathrm{P}})$, the adjunction (2.9.4.4) can be deduced from a similar adjunction for Schwartz functions.

We prove 1. by induction on $a_0 - a_{\mathrm{P}}$. For $\mathrm{P} = \mathrm{P}_0$, we have $w \sim w_{\mathrm{A}}$ and the result follows from (2.9.2.2). Assume now that 1. holds for every parabolic subgroup $\mathrm{Q} \subsetneq \mathrm{P}$.

First we assume that $w \gg w_{\mathrm{A}}$ (recall that w_{A} stands for the restriction of w to $\mathrm{A}_{\mathrm{P}}^{\infty}$). By Corollary 2.5.15.2, in this case we have a closed embedding

$$\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^{\infty} \to \prod_{\mathrm{Q} \subsetneq \mathrm{P}} \mathrm{L}^2_{w_{\mathrm{Q}}^-}([\mathrm{G}]_{\mathrm{Q}})^{\infty} \times \mathrm{L}^2_{w_{\mathrm{A}}}([\mathrm{G}]_{\mathrm{P}})^{\infty}, \;\; \varphi \mapsto \left((\varphi_{\mathrm{Q}})_{\mathrm{Q}}, \varphi\right). \tag{2.9.4.5}$$

Moreover, by the induction hypothesis, for $\mathrm{Q} \subsetneq \mathrm{P}$, we have projections $\mathrm{L}^2_{w_{\mathrm{Q}}^-}([\mathrm{G}]_{\mathrm{Q}})^{\infty} \to \mathrm{L}^2_{w_{\mathrm{Q}}^-,\mathfrak{X}}([\mathrm{G}]_{\mathrm{Q}})^{\infty}$, $\varphi^{\mathrm{Q}} \mapsto \varphi^{\mathrm{Q}}_{\mathfrak{X}}$ satisfying 1. To prove the existence of the continuous projection $\varphi \in \mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^{\infty} \mapsto \varphi_{\mathfrak{X}} \in \mathrm{L}^2_{w,\mathfrak{X}}([\mathrm{G}]_{\mathrm{P}})^{\infty}$, it suffices to check that the continuous projection $((\varphi^{\mathrm{Q}})_{\mathrm{Q}}, \varphi) \mapsto ((\varphi^{\mathrm{Q}}_{\mathfrak{X}})_{\mathrm{Q}}, \varphi_{\mathfrak{X}})$ of $\prod_{\mathrm{Q}\subsetneq \mathrm{P}} \mathrm{L}^2_{w_{\mathrm{Q}}^-}([\mathrm{G}]_{\mathrm{Q}})^{\infty} \times \mathrm{L}^2_{w_{\mathrm{A}}}([\mathrm{G}]_{\mathrm{P}})^{\infty}$ preserves the image of (2.9.4.5). This readily follows from the identity

$$(\varphi_{\mathfrak{X}})_{\mathrm{Q}} = (\varphi_{\mathrm{Q}})_{\mathfrak{X}}, \quad \text{for every } \varphi \in \mathrm{L}^2_{w_{\mathrm{A}}}([\mathrm{G}]_{\mathrm{P}})^{\infty} \text{ and } \mathrm{Q} \subsetneq \mathrm{P}. \tag{2.9.4.6}$$

We emphasize that in the above equation, $\varphi_{\mathfrak{X}}$ is defined through the orthogonal projection $\mathrm{L}^2_{w_{\mathrm{A}}}([\mathrm{G}]_{\mathrm{P}})^{\infty} \to \mathrm{L}^2_{w_{\mathrm{A}},\mathfrak{X}}([\mathrm{G}]_{\mathrm{P}})^{\infty}$ whereas $(\varphi_{\mathrm{Q}})_{\mathfrak{X}}$ is given by the projection $\mathcal{T}([\mathrm{G}]_{\mathrm{Q}}) \to \mathcal{T}_{\mathfrak{X}}([\mathrm{G}]_{\mathrm{Q}})$ from the third part of the theorem (and which exists by the induction hypothesis).

To show (2.9.4.6), we check that $\langle(\varphi_{\mathfrak{X}})_{\mathrm{Q}},\psi\rangle_{\mathrm{Q}}=\langle(\varphi_{\mathrm{Q}})_{\mathfrak{X}},\psi\rangle_{\mathrm{Q}}$ for every $\psi\in\mathcal{S}([\mathrm{G}]_{\mathrm{Q}})$. By the induction hypothesis again, we have $\psi=\psi_{\mathfrak{X}}+\psi^{\mathfrak{X}}$ where $\psi_{\mathfrak{X}}\in\mathcal{S}_{\mathfrak{X}}([\mathrm{G}]_{\mathrm{Q}})$ and $\psi^{\mathfrak{X}}\in\mathcal{S}^{\mathfrak{X}}([\mathrm{G}]_{\mathrm{Q}})$. Moreover, by Lemma 2.9.3.1, we have $\langle\varphi_{\mathfrak{X}},\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi_{\mathfrak{X}})\rangle_{\mathrm{P}}=\langle\varphi,\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi_{\mathfrak{X}})\rangle_{\mathrm{P}}$ and $\langle\varphi_{\mathfrak{X}},\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi^{\mathfrak{X}})\rangle_{\mathrm{P}}=0$. Therefore,

$$\begin{aligned}\langle(\varphi_{\mathrm{Q}})_{\mathfrak{X}},\psi\rangle_{\mathrm{Q}}&=\langle\varphi_{\mathrm{Q}},\psi_{\mathfrak{X}}\rangle_{\mathrm{Q}}=\langle\varphi,\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi_{\mathfrak{X}})\rangle_{\mathrm{P}}=\langle\varphi_{\mathfrak{X}},\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi)\rangle_{\mathrm{P}}\\&=\langle(\varphi_{\mathfrak{X}})_{\mathrm{Q}},\psi\rangle_{\mathrm{Q}}\end{aligned}$$

and this proves (2.9.4.6).

Let $\varphi\in\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^\infty$. By induction, for every $\mathrm{Q}\subsetneq\mathrm{P}$, $((\varphi_{\mathrm{Q}})_\chi)_{\chi\in\mathfrak{X}(\mathrm{G})}$ is a summable family in $\mathrm{L}^2_{w_{\mathrm{Q}}^-}([\mathrm{G}]_{\mathrm{Q}})^\infty$ with sum φ_{Q}. Moreover, as w_{A} is a weight on $\mathrm{A}_{\mathrm{P}}^\infty$, $(\varphi_\chi)_{\chi\in\mathfrak{X}(\mathrm{G})}$ is a summable family in $\mathrm{L}^2_{w_{\mathrm{A}}}([\mathrm{G}]_{\mathrm{P}})^\infty$ with sum φ. Since (2.9.4.5) is a closed embedding, by (2.9.4.6) we deduce that $(\varphi_\chi)_{\chi\in\mathfrak{X}(\mathrm{G})}$ is a summable family in $\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^\infty$ with sum φ. This ends the proof of 1. whenever $w\gg w_{\mathrm{A}}$.

Note that the weight $w=\|.\|_{\mathrm{P}}^{\mathrm{N}}$ satisfies the condition $w\gg w_{\mathrm{A}}$ (see (2.4.1.6)). Therefore, we have already establish 1. for the spaces $\mathrm{L}^2_{\mathrm{N}}([\mathrm{G}]_{\mathrm{P}})^\infty$ ($\mathrm{N}>0$) and thus, by (2.5.7.6) and the reasoning from the beginning, we can already deduce statement 2. for P.

We now deal with the case of a general weight w. Set $w'=\max(w,w_{\mathrm{A}})$. Then, $w'\gg w'_{\mathrm{A}}=w_{\mathrm{A}}$ and therefore the existence of the projections $\varphi\in\mathrm{L}^2_{w'}([\mathrm{G}]_{\mathrm{P}})^\infty\mapsto\varphi_{\mathfrak{X}}\in\mathrm{L}^2_{w',\mathfrak{X}}([\mathrm{G}]_{\mathrm{P}})^\infty$ has already been established. By Corollary 2.5.15.2 again, we have an open surjection

$$\begin{aligned}&\prod_{\mathrm{Q}\subsetneq\mathrm{P}}\mathrm{L}^2_{w_{\mathrm{Q}}^+}([\mathrm{G}]_{\mathrm{Q}})^\infty\times\mathrm{L}^2_{w'}([\mathrm{G}]_{\mathrm{P}})^\infty\to\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^\infty,\\ &\big((\psi^{\mathrm{Q}})_{\mathrm{Q}},\psi\big)\mapsto\sum_{\mathrm{Q}\subsetneq\mathrm{P}}\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi^{\mathrm{Q}})+\psi.\end{aligned}\tag{2.9.4.7}$$

We now check that the projection $\big((\psi^{\mathrm{Q}})_{\mathrm{Q}},\psi\big)\mapsto\Big((\psi^{\mathrm{Q}}_{\mathfrak{X}})_{\mathrm{Q}},\psi_{\mathfrak{X}}\Big)$ descends to this quotient i.e. that it preserves the kernel of (2.9.4.7). For this, it suffices to show that for every $\mathrm{Q}\subsetneq\mathrm{P}$, we have

$$\langle\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi^{\mathrm{Q}}_{\mathfrak{X}}),\varphi\rangle_{\mathrm{P}}=\langle\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi^{\mathrm{Q}}),\varphi_{\mathfrak{X}}\rangle_{\mathrm{P}},\quad\text{for }(\psi^{\mathrm{Q}},\varphi)\in\mathrm{L}^2_{w_{\mathrm{Q}}^+}([\mathrm{G}]_{\mathrm{Q}})^\infty\times\mathcal{S}([\mathrm{G}]_{\mathrm{P}}).\tag{2.9.4.8}$$

By the density of $\mathcal{S}([\mathrm{G}]_{\mathrm{Q}})$ in $\mathrm{L}^2_{w_{\mathrm{Q}}^+}([\mathrm{G}]_{\mathrm{Q}})^\infty$, we can restrict ourself to prove (2.9.4.8) when $\psi^{\mathrm{Q}}\in\mathcal{S}([\mathrm{G}]_{\mathrm{Q}})$ in which case it readily follows from (2.9.4.6).

Let $\varphi\in\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^\infty\mapsto\varphi_{\mathfrak{X}}\in\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^\infty$ be the continuous projection descended from $\big((\psi^{\mathrm{Q}})_{\mathrm{Q}},\psi\big)\mapsto\Big((\psi^{\mathrm{Q}}_{\mathfrak{X}})_{\mathrm{Q}},\psi_{\mathfrak{X}}\Big)$ via the surjection (2.9.4.7). By (2.9.4.8), its image lands in $\mathrm{L}^2_{w,\mathfrak{X}}([\mathrm{G}]_{\mathrm{P}})^\infty$ and it extends the projection $\varphi\in\mathcal{S}([\mathrm{G}]_{\mathrm{P}})\mapsto\varphi_{\mathfrak{X}}\in\mathcal{S}_{\mathfrak{X}}([\mathrm{G}]_{\mathrm{P}})$. Finally, let $\varphi\in\mathrm{L}^2_w([\mathrm{G}]_{\mathrm{P}})^\infty$ that we write $\varphi=\sum_{\mathrm{Q}\subsetneq\mathrm{P}}\mathrm{E}_{\mathrm{Q}}^{\mathrm{P}}(\psi^{\mathrm{Q}})+\psi$ where $\big((\psi^{\mathrm{Q}})_{\mathrm{Q}},\psi\big)\in$

$\prod_{Q\subsetneq P} L^2_{w_Q^+}([G]_Q)^\infty \times L^2_{w'}([G]_P)^\infty$. Then, for every $Q \subsetneq P$, $(\psi_\chi^Q)_{\chi\in\mathfrak{X}(G)}$ is a summable family in $L^2_{w_Q^+}([G]_Q)^\infty$ with sum ψ^Q by the induction hypothesis whereas $(\psi_\chi)_{\chi\in\mathfrak{X}(G)}$ is a summable family in $L^2_{w'}([G]_P)^\infty$ with sum ψ by the case already treated. Thus, from the continuity of the map (2.9.4.7), we deduce that

$$\chi \in \mathfrak{X}(G) \mapsto \varphi_\chi = \sum_{Q\subsetneq P} E_Q^P(\psi_\chi^Q) + \psi_\chi$$

is a summable family in $L^2_w([G]_P)^\infty$ with sum φ. We have now completed the proof by induction of 1. and hence of the theorem. □

2.9.5. Let $\mathfrak{X} \subseteq \mathfrak{X}(G)$ be a subset. By the previous proposition, we have compatible continuous projections $\varphi \mapsto \varphi_{\mathfrak{X}}$ from $\mathcal{S}([G]_P)$, $\mathcal{C}([G]_P)$ and $L^2_{-N}([G]_P)^\infty$ onto $\mathcal{S}_{\mathfrak{X}}([G]_P)$, $\mathcal{C}_{\mathfrak{X}}([G]_P)$ and $L^2_{-N,\mathfrak{X}}([G]_P)^\infty$ respectively. As $\mathcal{S}([G]_P)$ is dense in both $\mathcal{C}([G]_P)$ and $L^2_{-N}([G]_P)^\infty$ this entails that

(**2.9.5.9**) $\mathcal{S}_{\mathfrak{X}}([G]_P)$ is dense in $\mathcal{C}_{\mathfrak{X}}([G]_P)$ and $L^2_{-N,\mathfrak{X}}([G]_P)^\infty$.

2.9.6. Assume that $G = G_1 \times G_2$ where G_1 and G_2 are connected reductive groups over F and write $P = P_1 \times P_2$ accordingly. Then, we have a natural identification $\mathfrak{X}(G) = \mathfrak{X}(G_1) \times \mathfrak{X}(G_2)$. For subsets $\mathfrak{X}_i \subseteq \mathfrak{X}(G_i)$ and compact-open subgroups $J_i \subseteq G_i(\mathbf{A}_f)$, $i = 1, 2$, setting $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ and $J = J_1 \times J_2$, the projection $\mathcal{S}([G]_P)^J \to \mathcal{S}_{\mathfrak{X}}([G]_P)^J$ (resp. $\mathcal{C}([G]_P)^J \to \mathcal{C}_{\mathfrak{X}}([G]_P)^J$) corresponds via the isomorphism (2.5.12.11) to the (completed) tensor product of the projections space $\mathcal{S}([G_i]_{P_i})^{J_i} \to \mathcal{S}_{\mathfrak{X}_i}([G_i]_{P_i})^{J_i}$ (resp. $\mathcal{C}([G_i]_{P_i})^{J_i} \to \mathcal{C}_{\mathfrak{X}_i}([G_i]_{P_i})^{J_i}$) for $i = 1, 2$ as can readily be seen by looking at pure tensors. It follows that

(**2.9.6.10**)
$$\mathcal{S}_{\mathfrak{X}_1}([G_1])^{J_1}\widehat{\otimes}\mathcal{S}_{\mathfrak{X}_2}([G_2])^{J_2} \simeq \mathcal{S}_{\mathfrak{X}}([G])^J \text{ and}$$
$$\mathcal{C}_{\mathfrak{X}_1}([G_1])^{J_1}\widehat{\otimes}\mathcal{C}_{\mathfrak{X}_2}([G_2])^{J_2} \simeq \mathcal{C}_{\mathfrak{X}}([G])^J$$

by restriction of the isomorphisms (2.5.12.11).

2.9.7. *Regular cuspidal data.* — We say that a cuspidal datum $\chi \in \mathfrak{X}(G)$ is *regular* if it is represented by a pair (M_P, π) such that the only element $w \in W(P)$ satisfying $w\pi \simeq \pi$ is $w = 1$. The next result can be deduced from Langlands spectral decomposition [Lan76] but we prefer to give a direct proof.

Proposition **2.9.7.1.** — *Let* $\chi \in \mathfrak{X}(G)$ *be a regular cuspidal datum and* $P \subset G$ *be a parabolic subgroup. Then, for every* $\varphi \in L^2_\chi([G])$ *we have* $\varphi_P \in L^2_\chi([G]_P)$.

Proof. — By duality, it suffices to show that the pseudo-Eisenstein map E_P^G extends to a continuous application from $L^2_\chi([G]_P)$ into $L^2([G])$. Let $\{\chi_i \mid i \in I\}$ be the inverse image of χ in $\mathfrak{X}(M_P)$. Then, we have an orthogonal decomposition

$$L^2_{\chi}([G]_P) = \bigoplus_{i\in I} L^2_{\chi_i}([G]_P)$$

where $L^2_{\chi_i}([G]_P)$ denotes the subspace of $\varphi \in L^2([G]_P)$ such that $m \mapsto \delta_P(m)^{-1/2}\varphi(mg)$ belongs to $L^2_{\chi_i}([M_P])$ for almost all $g \in G(\mathbf{A})$. Thus, it suffices to show that E_P^G extends to a continuous application from $L^2_{\chi_i}([G]_P)$ into $L^2([G])$ for each $i \in I$. Fix $i \in I$ and let (M_Q, π) be a pair representing χ_i (where $Q \subset P$). Then, the inverse image of χ_i in $\underline{\mathfrak{X}}(M_P)$ consists of the pairs $(wM_Qw^{-1}, w\pi)$ where $w \in W^{M_P}$ is such that wM_Qw^{-1} is the Levi component of some standard parabolic subgroup wQ. Therefore, by definition of $L^2_{\chi_i}([M_P])$, $L^2_{\chi_i}([G]_P)$ is the closure of

$$\sum_w E^P_{wQ}(\mathcal{S}_{w\pi}([G]_{wQ})) \qquad (\mathbf{2.9.7.11})$$

in $L^2([G]_P)$ where w runs over elements $w \in W^{M_P}$ as before. Thus, we just need to check that for every such $w \in W^{M_P}$, $\varphi \in \mathcal{S}_\pi([G]_Q)$ and $\varphi' \in \mathcal{S}_{w\pi}([G]_{wQ})$, we have

$$\langle E^G_Q(\varphi), E^G_{wQ}(\varphi')\rangle_G = \langle E^P_Q(\varphi), E^P_{wQ}(\varphi')\rangle_P, \qquad (\mathbf{2.9.7.12})$$

since it will implies that the restriction of E_P^G to the subspace (2.9.7.11) is an isometry.

By the calculation of the scalar product of two pseudo-Eisenstein series [MW94, Proposition II.2.1],[5] we have

$$\langle E^G_Q(\varphi), E^G_{wQ}(\varphi')\rangle_G = \sum_{w_0\in W(Q)} \int_{i\mathfrak{a}^*_Q+\lambda_0} \langle M(ww_0)\varphi_\lambda, \varphi'_{-ww_0\overline{\lambda}}\rangle_{\mathrm{Pet}}\, d\lambda$$

where $\lambda_0 \in \mathfrak{a}^*_Q$ belongs to the range of absolute convergence of the intertwining operators

$$M(ww_0) : \mathcal{A}_{Q,\pi_\lambda,cusp}(G) \to \mathcal{A}_{wQ,(ww_0\pi)_{ww_0\lambda},cusp}(G).$$

As χ is regular, for every $w_0 \in W(Q)$ different from 1, we have $ww_0\pi \not\simeq w\pi$ and therefore $M(ww_0)\varphi_\lambda$ is orthogonal to $\varphi'_{-ww_0\overline{\lambda}}$ for every $\lambda \in i\mathfrak{a}^*_Q + \lambda_0$. It follows that the above expression reduces to

$$\langle E^G_Q(\varphi), E^G_{wQ}(\varphi')\rangle_G = \int_{i\mathfrak{a}^*_Q+\lambda_0} \langle M(w)\varphi_\lambda, \varphi'_{-w\lambda}\rangle_{\mathrm{Pet}}\, d\lambda.$$

A similar argument shows that $\langle E^P_Q(\varphi), E^P_{wQw^{-1}}(\varphi')\rangle_P$ is also equal to the right-hand side above. This shows (2.9.7.12) hence the proposition. □

[5] Strictly speaking *loc. cit.* only applies to K_∞-finite pseudo-Eisenstein series. However, the proof, which ultimately rests upon the computation of the constant terms of cuspidal Eisenstein series in their range of convergence, extends verbatim to general smooth pseudo-Eisenstein series. The skeptical reader can also assume that both φ and φ' are K_∞-finite since, by density of the respective subspaces of K_∞-finite vectors, it suffices to check (2.9.7.12) for such functions.

Corollary **2.9.7.2.** — *Let* $\chi \in \mathfrak{X}(G)$ *be a regular cuspidal datum,* P *be a standard parabolic subgroup of* G *and* χ_M *be the inverse image of* χ *in* $\mathfrak{X}(M_P)$. *Then, for every* $\varphi \in \mathcal{S}_\chi([G])$ *and* $s \in \mathcal{H}_{>0}$, *the function*

$$\varphi_{P,s} : m \in [M_P] \mapsto \delta_P(m)^{s-1/2}\varphi_P(m)$$

belongs to $\mathcal{C}_{\chi_M}([M_P])$. *Moreover, the family of linear maps*

$$\mathcal{S}_\chi([G]) \to \mathcal{C}_{\chi_M}([M_P]), \varphi \mapsto \varphi_{P,s}$$

for $s \in \mathcal{H}_{>0}$ *is holomorphic.*

Proof. — Let $\varphi \in \mathcal{S}_\chi([G])$. By Lemma 2.9.3.1, $\varphi_{P,s}$ is orthogonal to $\mathcal{S}^{\chi_M}([M_P])$ for all $s \in \mathbf{C}$. Hence, we just need to show that the map $s \mapsto \varphi_{P,s}$ induces a holomorphic function $\mathcal{H}_{>0} \to \mathcal{C}([M_P])$. Note that by Lemma 2.5.13.1, $\varphi_{P,s} \in \mathcal{C}([M_P])$ for $\Re(s) \gg 1$. Thus, by the previous proposition, it suffices to show:

(**2.9.7.13**) If $\psi \in L^2([M_P])^\infty$ is such that $\psi_s := \delta_P^s\psi \in \mathcal{C}([M_P])$ for $\Re(s) \gg 1$ then $s \mapsto \psi_s$ induces a holomorphic function $\mathcal{H}_{>0} \to \mathcal{C}([M_P])$.

For $X \in \mathfrak{m}_\infty$, we have

$$R(X)\psi_s = (2s-1)\langle\rho_P, X\rangle\psi_s + (R(X)\psi)_s$$

(where we consider ρ_P as an element of the dual space $\mathfrak{m}_\infty^*$) and it follows, by the equality (2.5.8.7), that it suffices to check that for every $d > 0$ the map $s \mapsto \psi_s$ induces a holomorphic function $\mathcal{H}_{>0} \to L^2_{\sigma,d}([M_P])$. By Hölder inequality, for $d > 0$, $\Re(s) > 0$ and $t \gg 1$ we have

$$\|\psi_s\|_{L^2_{\sigma,d}} \leqslant \|\psi\|_{L^2}^{1-\Re(s)/t}\|\psi_t\|_{L^2_{\sigma,td/\Re(s)}}^{\Re(s)/t}$$

and this implies $\psi_s \in L^2_{\sigma,d}([M_P])$. The holomorphy of the map $s \in \mathcal{H}_{>0} \mapsto \psi_s \in L^2_{\sigma,d}([M_P])$ is equivalent to the holomorphy of $s \in \mathcal{H}_{>0} \mapsto \langle\psi_s, \phi\rangle_{M_P}$ for every $\phi \in L^2_{\sigma,-d}([M_P])$ but this follows from the usual criterion of analyticity for parameter integral and the domination

$$|\psi_s| \leqslant |\psi_{t_1}| + |\psi_{t_2}|$$

for every $s \in \mathbf{C}$ and $t_2 > \Re(s) > t_1$. □

Let $n \geqslant 1$. For $G = GL_n$, regular cuspidal data admit the following explicit description. Let $\chi \in \mathfrak{X}(GL_n)$ be represented by a pair (M_P, π) where

$$M_P = GL_{n_1} \times \cdots \times GL_{n_k}$$

is a standard Levi subgroup of GL_n and

$$\pi = \pi_1 \boxtimes \ldots \boxtimes \pi_k$$

is a cuspidal automorphic representation of $\mathrm{M_P}(\mathbf{A})$ (with a central character trivial on $\mathrm{A}_\mathrm{P}^\infty$). Then, χ is regular if and only if $\pi_i \neq \pi_j$ for each $1 \leqslant i < j \leqslant k$.

2.9.8. We now assume that G is a product of the form $\mathrm{Res}_{\mathrm{K}_1/\mathrm{F}} \mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{\mathrm{K}_r/\mathrm{F}} \mathrm{GL}_{n_r}$, where $\mathrm{K}_1, \ldots, \mathrm{K}_r$ are finite extensions of F. Let $\chi \in \mathfrak{X}(\mathrm{G})$ be a regular cuspidal datum represented by a pair $(\mathrm{M_P}, \pi) \in \underline{\mathfrak{X}}(\mathrm{G})$. Set $\Pi = \mathrm{Ind}_{\mathrm{P}(\mathbf{A})}^{\mathrm{G}(\mathbf{A})}(\pi) = \mathcal{A}_{\mathrm{P},\pi,\mathrm{cusp}}(\mathrm{G})$ for the normalized smooth induction of π. Let $\mathcal{B}_{\mathrm{P},\pi}$ be a K-basis of Π as in Section 2.8.3. For $\varphi \in \mathcal{S}([\mathrm{G}])$ and $\lambda \in i\mathfrak{a}_\mathrm{P}^*$ the series

$$\varphi_{\Pi_\lambda} = \sum_{\psi \in \mathcal{B}_{\mathrm{P},\pi}} \langle \varphi, \mathrm{E}(\psi, \lambda) \rangle_\mathrm{G} \mathrm{E}(\psi, \lambda) \qquad (\mathbf{2.9.8.14})$$

converges absolutely in $\mathcal{T}_\mathrm{N}([\mathrm{G}])$ for some N (that may a priori depend on λ). Indeed, this follows from the continuity of the linear map $\psi \in \mathcal{A}_{\mathrm{P},\pi}(\mathrm{G}) \mapsto \mathrm{E}(\psi, \lambda) \in \mathcal{T}_\mathrm{N}([\mathrm{G}])$ for some $\mathrm{N} > 0$ (see Section 2.7.3) together with Proposition 2.8.4.1 and the Dixmier-Malliavin theorem. The next theorem is a slight restatement of (part of) the main result of [Lap13].[6] We refer the reader to Section A.0.9 for the notion of Schwartz function valued in a TVS.

Theorem **2.9.8.1** *(Lapid). — There exists* $\mathrm{N} > 0$ *such that for* $\varphi \in \mathcal{C}([\mathrm{G}])$*, the series* (2.9.8.14) *still makes sense (that is the scalar products* $\langle \varphi, \mathrm{E}(\psi, \lambda) \rangle_\mathrm{G}$ *are convergent) and converges in* $\mathcal{T}_\mathrm{N}([\mathrm{G}])$ *for every* $\lambda \in i\mathfrak{a}_\mathrm{P}^*$*. Moreover, the function* $\lambda \in i\mathfrak{a}_\mathrm{P}^* \mapsto \varphi_{\Pi_\lambda} \in \mathcal{T}_\mathrm{N}([\mathrm{G}])$ *is Schwartz, in particular absolutely integrable, and if* $\varphi \in \mathcal{C}_\chi([\mathrm{G}])$ *we have the equality*

$$\varphi = \int_{i\mathfrak{a}_\mathrm{P}^*} \varphi_{\Pi_\lambda} d\lambda.$$

Proof. — Note that G satisfies condition (HP) of [Lap13]: it is proven in *loc. cit.* that general linear groups satisfy (HP) and it is straightforward to check that products of groups satisfying (HP) again satisfy (HP). The first part of the theorem is then a consequence of [Lap13, Proposition 5.1]. Indeed, by Dixmier-Malliavin we may assume that $\varphi = \mathrm{R}(f)\varphi'$ where $\varphi' \in \mathcal{C}([\mathrm{G}])$ and $f \in \mathrm{C}_c^\infty(\mathrm{G}(\mathbf{A}))$. By *loc. cit.* the scalar product $\langle \varphi, \mathrm{E}(\psi, \lambda) \rangle_\mathrm{G}$ converges for every $\psi \in \mathcal{B}_{\mathrm{P},\pi}$ and there exists $\mathrm{N} > 0$ such that $\psi \mapsto \mathrm{E}(\psi, \lambda)$ factors through a continuous linear mapping $\Pi \to \mathcal{T}_\mathrm{N}([\mathrm{G}])$ for every $\lambda \in i\mathfrak{a}_\mathrm{P}^*$. As

$$\langle \varphi, \mathrm{E}(\psi, \lambda) \rangle_\mathrm{G} = \langle \varphi', \mathrm{E}(\mathrm{R}(f^*)\psi, \lambda) \rangle_\mathrm{G}, \ \varphi \in \mathcal{B}_{\mathrm{P},\pi},$$

[6] Note that in *loc. cit.* the Harish-Chandra Schwartz space $\mathcal{C}([\mathrm{G}])$ is denoted by $\mathcal{S}(\mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbf{A}))$

we deduce by Proposition 2.8.4.1 that the series (2.9.8.14) converges absolutely in $\mathcal{T}_N([G])$ for every $\lambda \in i\mathfrak{a}_P^*$. That the function $\lambda \in i\mathfrak{a}_P^* \mapsto \varphi_{\Pi_\lambda} \in \mathcal{T}_N([G])$ is Schwartz follows similarly from [Lap13, Corollary 5.7]. The last part of the theorem is a consequence of [Lap13, Theorem 4.5] since χ regular implies that $L^2_\chi([G])$ is included in the "induced from cuspidal part" $L^2_c([G])$ of $L^2([G])$, with the notation of *loc. cit.* □

2.10. *Automorphic kernels*

2.10.1. Let $P \subset G$ be a standard parabolic subgroup. The right convolution by $f \in \mathcal{S}(G(\mathbf{A}))$ on each space of the decompositions (2.9.2.1) gives integral operators whose kernels are respectively denoted by $K_{f,P}(x,y)$, $K_{f,P,\chi}(x,y)$, $K^0_{f,P}(x,y)$ and $K^0_{f,P,\chi}(x,y)$ where $x,y \in G(\mathbf{A})$. If the context is clear, we shall omit the subscript f in the notation as well as the subscript P when $P = G$. The kernels are related by the following equality for all $x,y \in G(\mathbf{A})$

$$K^0_{f,P,\chi}(x,y) = \int_{A_P^\infty} K_{f,P,\chi}(x, ay)\delta_P(a)^{-1/2}da.$$

Recall that we write $[G]_P^1$ for the preimage of 0 by the map $H_P : [G]_P \to \mathfrak{a}_P$.

Lemma **2.10.1.1.** — *There exists* $N_0 > 0$ *such that for every weight* w *on* $[G]_P$ *(see Section 2.4.3) and every continuous semi-norm* $\|.\|_{w,N_0}$ *on* $\mathcal{T}_{w,N_0}([G]_P)$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{S}}$ *on* $\mathcal{S}(G(\mathbf{A}))$ *such that for* $f \in \mathcal{S}(G(\mathbf{A}))$

$$(\mathbf{2.10.1.1}) \qquad \sum_{\chi \in \mathfrak{X}(G)} |K_{f,P,\chi}(x,y)| \leqslant \|f\|_{\mathcal{S}} \|x\|_P^{N_0} w(x) w(y)^{-1}, \ x,y \in [G]_P,$$

$$(\mathbf{2.10.1.2}) \qquad \sum_{\chi \in \mathfrak{X}(G)} |K^0_{f,P,\chi}(x,y)| \leqslant \|f\|_{\mathcal{S}} \|x\|_P^{N_0} w(x) w(y)^{-1}, \ x,y \in [G]_P^1,$$

and

$$(\mathbf{2.10.1.3}) \qquad \sum_{\chi \in \mathfrak{X}(G)} \|K_{f,P,\chi}(.,y)\|_{w,N_0} \leqslant \|f\|_{\mathcal{S}} w(y)^{-1}, \ y \in [G]_P.$$

Proof. — Obviously, (2.10.1.3) implies (2.10.1.1) and (2.10.1.2) thus we will only prove this last estimate.

First, we note that there exists $N_0' > 0$ such that, for every weight w and every $f \in \mathcal{S}(G(\mathbf{A}))$ the operator $R(f)$ of right convolution by f induces a continuous map $\mathcal{T}^0_w([G]_P) \to \mathcal{T}_{w,N_0'}([G]_P)$. Let $f \in \mathcal{S}(G(\mathbf{A}))$ and for $\chi \in \mathfrak{X}(G)$, let $R_\chi(f)$ be the composition of $R(f)$ with the "χ-projection" defined by Theorem 2.9.4.1. Then, from the third point of this theorem, we deduce the existence of $N_0 > 0$ such that, for every weight w, $R_\chi(f)$ sends $\mathcal{T}^0_w([G]_P)$ continuously into $\mathcal{T}_{w,N_0}([G]_P)$ and for every $\varphi \in \mathcal{T}^0_w([G]_P)$, the family $(\mathrm{R}_\chi(f)\varphi)_{\chi\in\mathfrak{X}(\mathrm{G})}$ is absolutely summable in $\mathcal{T}_{w,\mathrm{N}_0}([\mathrm{G}]_\mathrm{P})$. By the uniform boundedness principle, this implies the existence of a constant $\mathrm{C}>0$ such that

$$(\mathbf{2.10.1.4})\qquad \sum_{\chi\in\mathfrak{X}(\mathrm{G})} \|\mathrm{R}_\chi(f)\varphi\|_{w,\mathrm{N}_0} \leqslant \mathrm{C}\|\varphi\|_{1,w^{-1}},\quad \text{for every } \varphi\in\mathcal{T}^0_w([\mathrm{G}]_\mathrm{P})$$

where we recall that $\|.\|_{1,w^{-1}}$ is the norm defining the Banach space $\mathcal{T}^0_w([\mathrm{G}]_\mathrm{P})$ see Section 2.5.9 (we emphasize that the peculiar appearance of w^{-1} as an index in the right hand side is purely the effect of a slight inconsistency in our notation). Using the fact that $f\mapsto \mathrm{R}_\chi(f)\varphi$ is continuous, we get, once again by the uniform boundedness principle, the existence of a continuous semi-norm $\|.\|_\mathcal{S}$ on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$ such that for $f\in\mathcal{S}(\mathrm{G}(\mathbf{A}))$,

$$(\mathbf{2.10.1.5})\qquad \sum_{\chi\in\mathfrak{X}(\mathrm{G})} \|\mathrm{R}_\chi(f)\varphi\|_{w,\mathrm{N}_0} \leqslant \|f\|_\mathcal{S}\|\varphi\|_{1,w^{-1}},\quad \text{for every } \varphi\in\mathcal{T}^0_w([\mathrm{G}]_\mathrm{P}).$$

Applying (2.10.1.5) to $\varphi=\delta_y$ the Dirac measure at $y\in[\mathrm{G}]_\mathrm{P}$ gives the inequality (2.10.1.3) (note that $\mathrm{R}_\chi(f)\delta_y=\mathrm{K}_{f,\mathrm{P},\chi}(.,y)$). □

2.10.2. Let P be a standard parabolic subgroup of G and let $\mathrm{M}=\mathrm{M}_\mathrm{P}$. Let $\chi\in\mathfrak{X}(\mathrm{G})$ and $\mathcal{A}^0_{\mathrm{P},\chi,\mathrm{disc}}(\mathrm{G})$ be the closed subspace of $\mathcal{A}_{\mathrm{P},\mathrm{disc}}(\mathrm{G})$ generated by left $\mathrm{A}^\infty_\mathrm{M}$-invariant functions whose class belongs to $\mathrm{L}^2_\chi([\mathrm{G}]_{\mathrm{P},0})$. We have a isotypical decomposition

$$\mathcal{A}^0_{\mathrm{P},\chi,\mathrm{disc}}(\mathrm{G})=\hat{\bigoplus}\mathcal{A}_{\mathrm{P},\pi,\mathrm{disc}}(\mathrm{G})$$

indexed by a set of discrete automorphic representations π of $\mathrm{M}(\mathbf{A})$. Let $\mathcal{B}_{\mathrm{P},\chi}$ be a K-basis of $\mathcal{A}^0_{\mathrm{P},\chi,\mathrm{disc}}(\mathrm{G})$ that is the union $\cup_\pi\mathcal{B}_{\mathrm{P},\pi}$ over π as above of K-bases of $\mathcal{A}_{\mathrm{P},\pi,\mathrm{disc}}(\mathrm{G})$ (see Section 2.8.3). In the same way we define $\mathcal{B}_{\mathrm{P},\chi,\tau}=\cup_\pi\mathcal{B}_{\mathrm{P},\pi,\tau}$ for any $\tau\in\hat{\mathrm{K}}$.

In the following, we add a subscript x or y to $\mathrm{R}(\mathrm{X})$ to indicate that this operator is applied to the variable x or y. The next lemma is an extension to Schwartz functions of results of Arthur, see [Art78, §4].

Lemma **2.10.2.1.** — *There exists a continuous semi-norm* $\|\cdot\|$ *on* $\mathcal{S}(\mathrm{G}(\mathbf{A}))$ *and an integer* N *such that for all* $\mathrm{X},\mathrm{Y}\in\mathcal{U}(\mathfrak{g}_\infty)$, *all* $x,y\in\mathrm{G}(\mathbf{A})^1$ *and all* $f\in\mathcal{S}(\mathrm{G}(\mathbf{A}))$ *we have*

$$\begin{aligned}&\sum_{\chi\in\mathfrak{X}(\mathrm{G})}\sum_{\mathrm{P}_0\subset\mathrm{P}}|\mathcal{P}(\mathrm{M}_\mathrm{P})|^{-1}\\ &\qquad\times\int_{i\mathfrak{a}^{\mathrm{G},*}_\mathrm{P}}\sum_{\tau\in\hat{\mathrm{K}}}\Big|\sum_{\varphi\in\mathcal{B}_{\mathrm{P},\chi,\tau}}\mathrm{R}_x(\mathrm{X})\mathrm{E}(x,\mathrm{I}_\mathrm{P}(\lambda,f)\varphi,\lambda)\mathrm{R}_y(\mathrm{Y})\overline{\mathrm{E}(y,\varphi,\lambda)}\Big|\,d\lambda\\ &\qquad\leqslant\|\mathrm{L}(\mathrm{X})\mathrm{R}(\mathrm{Y})f\|\,\|x\|^\mathrm{N}_\mathrm{G}\|y\|^\mathrm{N}_\mathrm{G}.\end{aligned}$$

Moreover for all $x, y \in \mathrm{G}(\mathbf{A})$ *and all* $\chi \in \mathfrak{X}(\mathrm{G})$ *we have*

$$\mathrm{K}^0_{f,\chi}(x,y) = \sum_{\mathrm{P}_0\subset\mathrm{P}} |\mathcal{P}(\mathrm{M_P})|^{-1} \int_{i\mathfrak{a}_\mathrm{P}^{\mathrm{G},*}} \sum_{\varphi\in\mathcal{B}_{\mathrm{P},\chi}} \mathrm{E}(x, \mathrm{I_P}(\lambda,f)\varphi,\lambda)\overline{\mathrm{E}(y,\varphi,\lambda)}\, d\lambda.$$

Proof. — Let $\chi \in \mathfrak{X}(\mathrm{G})$ and $\tau \in \hat{\mathrm{K}}$. Let P be a standard parabolic subgroup. For $f, g \in \mathcal{S}(\mathrm{G}(\mathbf{A}))$, $\lambda \in i\mathfrak{a}_\mathrm{P}^{\mathrm{G},*}$ and $x, y \in \mathrm{G}(\mathbf{A})^1$ we define:

$$\mathcal{B}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f, g) = \sum_{\varphi\in\mathcal{B}_{\mathrm{P},\chi,\tau}} \mathrm{E}(x, \mathrm{I_P}(\lambda,f)\varphi,\lambda)\overline{\mathrm{E}(y,\mathrm{I_P}(\lambda,g)\varphi,\lambda)}$$

and

$$\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f) = \sum_{\varphi\in\mathcal{B}_{\mathrm{P},\chi,\tau}} \mathrm{E}(x, \mathrm{I_P}(\lambda,f)\varphi,\lambda)\overline{\mathrm{E}(y,\varphi,\lambda)}.$$

We denote by e_τ the measure supported on K given by $\deg(\tau)\,\mathrm{trace}(\tau(k))dk$ where dk is the Haar measure on K giving the total volume 1. We have $e_\tau * e_\tau = e_\tau$. Let's define $f^\vee$ by $f^\vee(x) = \overline{f(x^{-1})}$ and let $f_\tau = e_\tau * f * e_\tau$. We shall use the following properties one can readily check:

(**2.10.2.6**) $\mathrm{R}_x(\mathrm{X})\mathrm{R}_y(\mathrm{Y})\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f) = \mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, \mathrm{L}(\mathrm{X})\mathrm{R}(\mathrm{Y})f),\ \mathrm{X},\mathrm{Y}\in\mathcal{U}(\mathfrak{g}_\infty)$

(**2.10.2.7**) $\mathcal{B}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f, g) = \mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f * g^\vee)$

(**2.10.2.8**) $\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, x, f * f^\vee) = \mathcal{B}_{\mathrm{P},\chi,\tau}(\lambda, x, x, f, f) \geqslant 0$

(**2.10.2.9**) $|\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f * g^\vee)| \leqslant \mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, x, f * f^\vee)^{\frac12}\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, y, y, g * g^\vee)^{\frac12}$

(**2.10.2.10**) $\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f) = \mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f_\tau)$

(**2.10.2.11**) $\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f_{\tau'}) = 0,\ \tau' \in \hat{\mathrm{K}},\ \tau' \neq \tau.$

From properties (2.10.2.6), we see that we are reduced to the case $\mathrm{X} = \mathrm{Y} = 1$. It suffices to show that there exists an integer N and that, for any normal open compact subgroup $\mathrm{K}_0 \subset \mathrm{K}^\infty$, there exists a continuous semi-norm $\|\cdot\|$ on the subspace $\mathcal{S}(\mathrm{G}(\mathbf{A}))^{\mathrm{K}_0}$ of K_0-bi-invariant Schwartz functions such that for all $x, y \in \mathrm{G}(\mathbf{A})^1$ and all $f \in \mathcal{S}(\mathrm{G}(\mathbf{A}))^{\mathrm{K}_0}$ we have

$$\mathcal{T}(x,y,f) \leqslant \|f\|\,\|x\|_\mathrm{G}^\mathrm{N}\|y\|_\mathrm{G}^\mathrm{N}$$

where we introduce

$$\mathcal{T}_\tau(x,y,f) = \sum_{\chi\in\mathfrak{X}(\mathrm{G})}\sum_{\mathrm{P}_0\subset\mathrm{P}} |\mathcal{P}(\mathrm{M_P})|^{-1} \int_{i\mathfrak{a}_\mathrm{P}^{\mathrm{G},*}} |\mathcal{L}_{\mathrm{P},\chi,\tau}(\lambda, x, y, f_\tau)|\, d\lambda,$$

and

$$\mathcal{T}(x,y,f)=\sum_{\tau\in\hat{K}}\mathcal{T}_\tau(x,y,f).$$

Indeed by the uniform boundedness principle and by (2.10.2.10) and (2.10.2.11) it is easy to conclude. Note that $\mathcal{T}_\tau(x,y,f)=\mathcal{T}_\tau(x,y,f_\tau)$. Let $m\geqslant 1$ large enough and let $K_0\subset K^\infty$ be a normal open compact subgroup. By a slight variant of Proposition 2.8.4.3, we can find $Z\in\mathcal{U}(\mathfrak{g}_\infty)$, $g_1\in C_c^\infty(G(\mathbf{A}))$ and $g_2\in C_c^m(G(\mathbf{A}))$ such that

- Z is invariant under K_∞-conjugation;
- g_1 and g_2 are invariant under K-conjugation;
- for any $f\in\mathcal{S}(G(\mathbf{A}))^{K_0}$ and any $\tau\in\hat{K}$ we have:

$$f=f*g_1+(f*Z)*g_2$$
$$f_\tau=f_\tau*g_{1,\tau}+(f*Z)_\tau*g_{2,\tau}.$$

Thus the expression $\mathcal{T}(x,y,f)$ is bounded by (the sums below are over $\tau\in\hat{K}$)

$$(\mathbf{2.10.2.12})\qquad (\sum_\tau\mathcal{T}_\tau(x,x,f_\tau*f_\tau^\vee))^{\frac12}(\sum_\tau\mathcal{T}_\tau(y,y,g_{1,\tau}*g_{1,\tau}^\vee)^{\frac12}$$
$$+(\sum_\tau\mathcal{T}_\tau(x,x,(f*Z)_\tau*(f*Z)_\tau^\vee))^{\frac12}(\sum_\tau\mathcal{T}_\tau(y,y,g_{2,\tau}*g_{2,\tau}^\vee)^{\frac12}.$$

Arthur shows in [Art78, p. 931 and corollary 4.6] that for every $N>0$ large enough there exists $C>0$ such that

$$(\mathbf{2.10.2.13})\qquad \sum_\tau\mathcal{T}_\tau(y,y,g_{i,\tau}*g_{i,\tau}^\vee)\leqslant C\|y\|_G^{2N}$$

for $i=1,2$ and all $y\in G(\mathbf{A})^1$. At this point we are reduced to bound $\sum_\tau\mathcal{T}_\tau(x,x,f_\tau*f_\tau^\vee)$ for any $f\in\mathcal{S}(G(\mathbf{A}))$. Following [Art78, p. 931] (the compactness of the support of f plays no essential role there), we get for any K-finite functions $f\in\mathcal{S}(G(\mathbf{A}))$:

$$(\mathbf{2.10.2.14})\qquad \mathcal{T}(x,x,f*f^\vee)\leqslant K^0_{f*f^\vee}(x,x).$$

Let $N'>0$ large enough. Using the weight $w=\|\cdot\|_G^N$, we deduce from Lemma 2.10.1.1 that there exist $N>N'$ and a semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbf{A}))$ such that for all $f\in\mathcal{S}(G(\mathbf{A}))$ and all $x,y\in G(\mathbf{A})^1$ we have

$$K_f^0(x,y)\leqslant\|f\|\,\|x\|_G^N\|y\|_G^{-N'}.$$

In particular, we deduce that for any $N>0$ large enough there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbf{A}))$ such that for all $f\in\mathcal{S}(G(\mathbf{A}))$ and all $x\in G(\mathbf{A})^1$

$$(\mathbf{2.10.2.15})\qquad K^0_{f*f^\vee}(x,x)=\int_{[G]_0}|K_f^0(x,y)|^2\,dy\leqslant\|f\|^2\|x\|_G^{2N}.$$

We fix $N, C > 0$ and a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbf{A}))$ such that (2.10.2.13) and (2.10.2.15) hold. Let's define $\|f\|_0^2 = \sum_{\tau\in\hat{K}} \|f_\tau\|^2$ and

$$\|f\|_1 = \|f * Z\|_0 + \|f\|_0$$

for $f \in \mathcal{S}(G(\mathbf{A}))$. These are again continuous semi-norms on $\mathcal{S}(G(\mathbf{A}))$. Using the majorization (2.10.2.14) for the K-finite function f_τ and the majorizations (2.10.2.12), (2.10.2.13) and (2.10.2.15) we get

$$\mathcal{T}(x,y,f) \leqslant C^{\frac{1}{2}}(\|x\|_G\|y\|_G)^N\|f\|_1$$

for all $x, y \in G(\mathbf{A})^1$ and $f \in \mathcal{S}(G(\mathbf{A}))^{K_0}$. Then we can deduce the first majorization.

To get the last equality, it is enough to observe that both members are defined and continuous on $\mathcal{S}(G(\mathbf{A}))$ and that the equality holds on the dense subset of K-finite and compactly supported functions (see [Art78, lemma 4.8]). □

3. The coarse spectral expansion of the Jacquet-Rallis trace formula for Schwartz functions

This section has two goals. The first, accomplished in Theorem 3.2.4.1, is to extend the coarse spectral expansion $I = \sum_{\chi\in\mathfrak{X}(G)} I_\chi$ of the Jacquet-Rallis trace formula for linear groups G (as proved in [Zyd20]) to the Schwartz space. The second, given in Theorem 3.3.9.1, is to provide spectral expressions more suitable for explicit calculations. An asymptotic estimate of modified automorphic kernels (stated in Theorem 3.3.7.1) plays a central role.

3.1. *Notations*

3.1.1. Let E/F be a quadratic extension of number fields. Let η be the quadratic character of $\mathbf{A}_F^\times$ attached to E/F. Let $n \geqslant 1$ be an integer. Let $G'_n = GL_{n,F}$ be the algebraic group of F-linear automorphisms of F^n. Let $G_n = \mathrm{Res}_{E/F}(G'_n \times_F E)$ be the F-group obtained by restriction of scalars from the algebraic group $GL_{n,E}$ of E-linear automorphisms of E^n. We denote by c the Galois involution. We have a natural inclusion $G'_n \subset G_n$ which induces an inclusion $A_{G'_n} \subset A_{G_n}$ which is in fact an equality. The restriction map $X^*(G_n) \to X^*(G'_n)$ gives an isomorphism $\mathfrak{a}^*_{G_n} \simeq \mathfrak{a}^*_{G'_n}$.

3.1.2. Let (B'_n, T'_n) be a pair where B'_n is the Borel subgroup of G'_n of upper triangular matrices and T'_n is the maximal torus of G'_n of diagonal matrices. Let (B_n, T_n) be the pair deduced from (B'_n, T'_n) by extension of scalars to E and restriction to F: it is a pair of a minimal parabolic subgroup of G_n and its Levi factor.

Let $K_n \subset G_n(\mathbf{A})$ and $K'_n = K_n \cap G'_n(\mathbf{A}) \subset G'_n(\mathbf{A})$ be the "standard" maximal compact subgroups. Notice that we have $K'_n \subset K_n$.

3.1.3. The map $P' \mapsto P = \mathrm{Res}_{E/F}(P' \times_F E)$ induces a bijection between the sets of standard parabolic subgroups of G'_n and G_n whose inverse bijection is given by

$$P \mapsto P' = P \cap G'_n.$$

Let P be a standard parabolic subgroup of G_n. The restriction map $X^*(P) \to X^*(P')$ identifies $X^*(P)$ with a subgroup of $X(P')$ of index $2^{\dim(\mathfrak{a}_P)}$. It also induces an isomorphism $\mathfrak{a}_{P'} \to \mathfrak{a}_P$ which fits into the commutative diagram:

$$\begin{array}{ccc} G'_n(\mathbf{A}) & \xrightarrow{H_{P'}} & \mathfrak{a}_{P'} \\ \downarrow & & \downarrow \\ G_n(\mathbf{A}) & \xrightarrow{H_P} & \mathfrak{a}_P \end{array}$$

For any standard parabolic subgroups $P \subset Q$, the restriction of the function τ_P^Q to $\mathfrak{a}_{P'}$ coincides with the function $\tau_{P'}^{Q'}$. However we have for all $x \in G'(\mathbf{A})$

$$\langle \rho_P^Q, H_P(x) \rangle = 2\langle \rho_{P'}^{Q'}, H_{P'}(x) \rangle.$$

Remark **3.1.3.1.** — The map $\mathfrak{a}_{P'} \to \mathfrak{a}_P$ does not preserve Haar measures. In fact, the pull-back on $\mathfrak{a}_{P'}$ of the Haar measure on $\mathfrak{a}_P$ is $2^{\dim(\mathfrak{a}_P)}$ times the Haar measure on $\mathfrak{a}_{P'}$. In particular, although the groups A_P^∞ and $A_{P'}^\infty$ can be canonically identified, the Haar measure on A_P^∞ is $2^{\dim(\mathfrak{a}_P)}$ times the Haar measure on $A_{P'}^\infty$.

3.1.4. We shall use the natural embeddings $G'_n \subset G'_{n+1}$ and $G_n \subset G_{n+1}$ where the smaller group is identified with the subgroup of the bigger one that fixes e_{n+1} and preserves the space generated by $(e_1, \ldots, e_n)$ where $(e_1, \ldots, e_{n+1})$ denotes the canonical basis of F^{n+1}.

3.1.5. Let $G = G_n \times G_{n+1}$ and $G' = G'_n \times G'_{n+1}$. Thus G' is an F-subgroup of G. Let

$$\iota : G_n \hookrightarrow G_n \times G_{n+1}$$

be the diagonal embedding. Let H be the image of ι (so H is isomorphic to G_n).

For an element $g \in G(\mathbf{A})$ we will always write g_n and g_{n+1} for its components in $G_n(\mathbf{A})$ and $G_{n+1}(\mathbf{A})$ respectively.

3.1.6. Let $\det_n$ (resp. $\det_{n+1}$) be the morphism $G' \mapsto \mathbf{G}_{m,F}$ given by the determinant on the first (resp. second) component. Let $\eta_{G'}$ be the character $G'(\mathbf{A}) \to \{\pm 1\}$ given by

$$\eta_{G'}(g') = \eta(\det_n(g'))^{n+1} \eta(\det_{n+1}(g'))^n \quad g' \in [G'].$$

3.1.7. Let $K = K_n \times K_{n+1}$: it is a maximal compact subgroup of $G(\mathbf{A})$. We define pairs $(P_0, M_0) = (B_n \times B_{n+1}, T_n \times T_{n+1})$ and $(P'_0, M'_0) = (B'_n \times B'_{n+1}, T'_n \times T'_{n+1})$ of minimal parabolic F-subgroups of G and G′ with their Levi components. As in Section 3.1.3, we have a bijection given by

$$P \mapsto P' = P \cap G'$$

between the sets of standard parabolic subgroups of G and G′.

3.1.8. Any parabolic subgroup P of G admits a decomposition $P = P_n \times P_{n+1}$. We introduce the "Rankin-Selberg set" $\mathcal{F}_{RS}$ as the set of F-parabolic subgroups of G of the form $P = P_n \times P_{n+1}$ where P_n is a standard parabolic subgroup of G_n and P_{n+1} is a semi-standard parabolic subgroup of G_{n+1} such that $P_{n+1} \cap G_n = P_n$ (here we use the embedding $G_n \hookrightarrow G_{n+1}$).

For $P \in \mathcal{F}_{RS}$, we set

$$P_H = P \cap H, P' = P \cap G', P'_n = P_n \cap G'_n \text{ and } P'_{n+1} = P_{n+1} \cap G'_{n+1}.$$

Let $P, Q \in \mathcal{F}_{RS}$ be such that $P \subset Q$ (from now on, when we write $P \subset Q \in \mathcal{F}_{RS}$ we always implicitly assume that both P and Q are in $\mathcal{F}_{RS}$). Then, we have $\mathfrak{a}_{P'_{n+1}}^{Q'_{n+1}} = \mathfrak{a}_{P_{n+1}}^{Q_{n+1}}$ and the characteristic functions $\widehat{\tau}_{P'_{n+1}}, \tau_{P'_{n+1}}, \sigma_{P'_{n+1}}^{Q'_{n+1}}$ coincide with $\widehat{\tau}_{P_{n+1}}, \tau_{P_{n+1}}, \sigma_{P_{n+1}}^{Q_{n+1}}$ respectively. We will only use the latter set of functions for convenience. We let

$$\epsilon_P^Q = (-1)^{\dim(a_{P_{n+1}}^{Q_{n+1}})}.$$

We set $\mathfrak{a}_{n+1} = \mathfrak{a}_{B_{n+1}}$ and $\mathfrak{a}_{n+1}^+ = \mathfrak{a}_{B_{n+1}}^+$ (see Section 2.2.9).

3.1.9. For $P, Q \in \mathcal{F}_{RS}$, we define two weights on $[G]_P$ by

$$\Delta_P(g) = \inf_{\gamma \in M_{P_{n+1}}(F)N_{P_{n+1}}(\mathbf{A})} \|g_n^{-1}\gamma g_{n+1}\| \quad \text{and}$$

$$d_P^{Q,\Delta}(g) = \min(d_{P_{n+1}}^{Q_{n+1}}(g_n), d_{P_{n+1}}^{Q_{n+1}}(g_{n+1}))$$

for $g \in [G]_P$. Note that $\Delta_P(h) \sim 1$ for $h \in [H]_{P_H}$. Moreover, by Lemma 2.4.4.2, we have

(**3.1.9.1**) $$d_P^{Q,\Delta}(g) \ll d_P^Q(g), \quad \text{for } g \in [G]_P.$$

From Lemma 2.4.3.1, we also deduce:

(**3.1.9.2**) For every weight w on $[G_{n+1}]_{P_{n+1}}$, there exists $N_0 > 0$ such that

$$w(g_{n+1}) \ll w(g_n)\Delta_P(g)^{N_0}, \quad \text{for } g = (g_n, g_{n+1}) \in [G]_P.$$

3.1.10. We will throughout consider a parameter $T \in \mathfrak{a}_{n+1}$ that we assume most of the time to be "sufficiently positive". More precisely, set

$$d(T) = \inf_{\alpha \in \Delta_{n+1}} \alpha(T)$$

where we write Δ_{n+1} for $\Delta_{B_{n+1}}$. Then, when we write "for T sufficiently positive", we mean "for T such that $d(T) \geqslant \max(\varepsilon\|T\|, C)$" where $\|.\|$ is an arbitrary norm on the real vector space $\mathfrak{a}_{n+1}$, $C > 0$ is a large enough constant and $\varepsilon > 0$ is an arbitrary (but in practice small enough) constant.

3.2. *The coarse spectral expansion for Schwartz functions*

3.2.1. Let $f \in \mathcal{S}(G(\mathbf{A}))$ be a Schwartz test function (see Section 2.5.2).

3.2.2. Let P be a parabolic subgroup of G. The right convolution by f on $L^2([G]_P)$ gives an integral operator whose kernel is denoted by $K_{P,f}$. Let $\chi \in \mathfrak{X}(G)$. Replacing $L^2([G]_P)$ by its closed subspace $L^2_\chi([G]_P)$ (see (2.9.2.1)), we get a kernel denoted by $K_{P,\chi,f}$. We have $K_{P,f} = \sum_{\chi\in\mathfrak{X}(G)} K_{P,\chi,f}$. If $P = G$, we omit the subscript P. If the context is clear, we will also omit the subscript f.

3.2.3. *A modified kernel.* — For $h \in H(\mathbf{A})$, $g' \in G'(\mathbf{A})$, $\chi \in \mathfrak{X}(G)$ and $T \in \mathfrak{a}_{n+1}$ we set

$$(\mathbf{3.2.3.1}) \qquad K^T_{f,\chi}(h, g') = \sum_{P\in\mathcal{F}_{RS}} \epsilon_P^G \sum_{\gamma\in P_H(F)\backslash H(F)} \sum_{\delta\in P'(F)\backslash G'(F)} \hat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\delta_n g'_n) - T_{P_{n+1}}) K_{f,P,\chi}(\gamma h, \delta g'),$$

where

- we recall that $\delta = (\delta_n, \delta_{n+1})$ and $g' = (g'_n, g'_{n+1})$ according to the decomposition $G' = G'_n \times G'_{n+1}$;
- in the notation $H_{P_{n+1}}(\delta_n g'_n)$, we consider $\delta_n g'_n$ as an element of $G'_{n+1}(\mathbf{A})$ (via the embedding $G'_n \hookrightarrow G'_{n+1}$);
- $T_{P_{n+1}}$ is defined as in Section 2.2.12.

Remark **3.2.3.1.** — This is the kernel used in [Zyd20] for compactly supported functions. Since we are considering a Schwartz function f, the sums over γ and δ are not necessarily finite. However, the component δ_n may be taken in a *finite* set depending on g'_n (see [Art78] Lemma 5.1) and we can check that the sum defining $K^T_{f,\chi}$ is also absolutely convergent. To see this, we can use the following majorization: for every $N > 0$ there exists $N' > 0$ such that

$$(\mathbf{3.2.3.2}) \qquad \sum_{\chi\in\mathfrak{X}(G)} \left|K_{f,P,\chi}(h, g')\right| \ll_N \|g'_n\|^{N'}_{P'_n} \|h\|^{-N}_{P_H} \|g'_{n+1}\|^{-N}_{P'_{n+1}}, \quad \text{for } (h, g') \in [H]_{P_H} \times [G']_{P'}.$$

This inequality is a simple consequence of Lemma 2.10.1.1 applied to the weight $w = \Delta_{\mathrm{P}}^{2\mathrm{N}}\|.\|_{\mathrm{P}}^{-\mathrm{N}}$ since $\|g'_{n+1}\|_{\mathrm{P}'_{n+1}} \ll \Delta_{\mathrm{P}}(g')\|g'_n\|_{\mathrm{P}'_n}$ (recall that $\|gg'\| \ll \|g\|\|g'\|$).

3.2.4.

Theorem **3.2.4.1.** — *Let* $\mathrm{T} \in \mathfrak{a}_{n+1}^{+}$.

1. *The map*

$$f \in \mathcal{S}(\mathrm{G}(\mathbf{A})) \mapsto \sum_{\chi \in \mathfrak{X}(\mathrm{G})} \int_{[\mathrm{H}]} \int_{[\mathrm{G}']} |\mathrm{K}_{f,\chi}^{\mathrm{T}}(h, g')|\, dg'\, dh$$

is given by a convergent integral and defines a continuous semi-norm on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$.

2. *As a function of* T, *the integral*

$$\mathrm{I}_{\chi}^{\mathrm{T}}(f) = \int_{[\mathrm{H}]} \int_{[\mathrm{G}']} \mathrm{K}_{f,\chi}^{\mathrm{T}}(h, g')\eta_{\mathrm{G}'}(g')\, dg'\, dh \qquad (\mathbf{3.2.4.3})$$

coincides with an exponential-polynomial function in T *whose purely polynomial part is constant and denoted by* $\mathrm{I}_{\chi}(f)$.

3. *The distributions* I_{χ} *are continuous, left* $\mathrm{H}(\mathbf{A})$*-invariant and right* $(\mathrm{G}'(\mathbf{A}), \eta_{\mathrm{G}'})$*-equivariant.*
4. *The sum*

$$\mathrm{I}(f) = \sum_{\chi} \mathrm{I}_{\chi}(f) \qquad (\mathbf{3.2.4.4})$$

is absolutely convergent and defines a continuous distribution I.

Remark **3.2.4.2.** — The last statement is the "coarse spectral expansion" of the Jacquet-Rallis trace formula for G as introduced by Zydor in [Zyd20].

Proof. — All the statements but the continuity and the extension to Schwartz functions are proved in [Zyd20, Theorems 3.1 and 3.9] for compactly supported functions.

The assertion 1 follows from the combination of majorization (3.3.7.10) of Theorem 3.3.7.1 below and Proposition 3.3.5.1 for the map (3.3.5.6). Note that assertion 1 implies the continuity of $\mathrm{I}_{\chi}^{\mathrm{T}}$. The assertion 2 can be proved as in [Zyd20, proof of Theorems 3.7]. One only needs the slight extension of assertion 1 to modified kernels defined in (3.7.4.2) associated to Levi subgroups (see comment above (3.7.4.2)). Continuity and assertion 4 are then the result of the explicit formula of [Zyd20, Theorem 3.7] which also holds for Schwartz functions. Finally one proves assertion 3 as in [Zyd20, Theorems 3.9]. □

3.3. *Auxiliary expressions for* I_χ

3.3.1. The goal of this section is to provide new expressions for the distribution I_χ defined in Theorem 3.2.4.1. In this paper, we will use these expressions to explicitly compute I_χ. The main results are subsumed in Theorem 3.3.9.1. Before giving the statements, we have to explain the main objects. Note that the proof of Theorem 3.3.9.1 relies on two other results, namely Theorem 3.3.7.1 and Proposition 3.3.8.1 whose proofs will be given in subsequent sections. On the other hand Theorem 3.3.7.1 was also used in the proof of Theorem 3.2.4.1.

3.3.2. *The Ichino-Yamana truncation operator.* — Let $T \in \mathfrak{a}_{n+1}$. In [IY15], Ichino-Yamana defined a truncation operator which transforms functions of uniform moderate growth on $[G_{n+1}]$ into rapidly decreasing functions on $[G_n]$. By applying it to the *right* component of $[G] = [G_n] \times [G_{n+1}]$, we get a truncation operator which we denote by Λ_r^T (the subscript r is for right). It associates to any function φ on $[G]$ the function on $[H]$ defined by the following formula: for any $h \in [H]$:

$$(\Lambda_r^T \varphi)(h) = \sum_{P \in \mathcal{F}_{RS}} \epsilon_P^G \sum_{\delta \in P_H(F)\backslash H(F)} \hat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\delta h) - T_{P_{n+1}}) \varphi_{G_n \times P_{n+1}}(\delta h) \quad (\mathbf{3.3.2.1})$$

where we follow notations of Section 3.2.3. Note that in the expression $H_{P_{n+1}}(\delta h)$, we view δh as an element of $G_{n+1}(\mathbf{A})$ by the composition $H \hookrightarrow G \to G_{n+1}$ where the second map is the second projection. We denote by $\varphi_{G_n \times P_{n+1}}$ the constant term of φ along $G_n \times P_{n+1}$.

For properties of Λ_r^T we shall refer to [IY15]. However for our purposes it is convenient to state the following proposition.

Proposition **3.3.2.1.** — *For any positive integers* N *and* N', *any open compact subgroup* $K_0 \subset G(\mathbf{A}_f)$, *there is an integer* $r > 0$ *and a finite family* $(X_i)_{i \in I}$ *of elements of* $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ *of degree* $\leqslant r$ *such that for any* $\varphi \in C^r(G(F)\backslash G(\mathbf{A})/K_0)$ *we have for all* $h \in [H]$

$$(\Lambda_r^T \varphi)(h) \leqslant \|h\|_H^{-N} \sum_{i \in I} \Big(\sup_{x \in G(\mathbf{A})} \|x\|_G^{-N'} |(R(X_i)\varphi)(x)| \Big)$$

Proof. — The result, a variant of Arthur's Lemma 1.4 of [Art80], is proven in [IY15], Lemma 2.4. □

Let $g \in [G]$. We shall apply the truncation operator Λ_r^T to the map $x \in [G] \mapsto K_{f,\chi}(x, g)$. After evaluating at $h \in [H]$ we get an expression we shall simply denote by $\Lambda_r^T K_{f,\chi}(h, g)$.

Proposition **3.3.2.2.** — *For every* $N \geqslant 0$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{S}}$ *on* $\mathcal{S}(G(\mathbf{A}))$ *such that for* $f \in \mathcal{S}(G(\mathbf{A}))$

$$\sum_{\chi \in \mathfrak{X}(G)} |\Lambda_r^T K_{f,\chi}(h, g)| \leqslant \|f\|_{\mathcal{S}} \|h\|_H^{-N} \|g\|_G^{-N} \quad (\mathbf{3.3.2.2})$$

for $h \in [\mathrm{H}]$ *and* $g \in [\mathrm{G}]$.

Proof. — Using the weight $w = \|\cdot\|_{\mathrm{G}}$ on $\mathrm{G}(\mathbf{A})$ we deduce from Lemma 2.10.1.1 that $\mathrm{N}_0 > 0$ and for all $\mathrm{N} \geqslant 0$ a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$ such that:

$$\sum_{\chi \in \mathfrak{X}(\mathrm{G})} |\mathrm{K}_{f,\chi}(x,y)| \leqslant \|f\| \|x\|_{\mathrm{G}}^{\mathrm{N}_0+\mathrm{N}} \|y\|_{\mathrm{G}}^{-\mathrm{N}}, \tag{3.3.2.3}$$

for all $x, y \in \mathrm{G}(\mathbf{A})$. The right derivatives in the first variable of the kernel $\mathrm{K}_{f,\chi}(x,y)$ can be expressed in terms of the kernel $\mathrm{K}_{f,\chi}$ associated to left derivatives of f. Let K_0 be a compact subgroup of $\mathrm{G}(\mathbf{A}_f)$. We deduce from (3.3.2.3) and Proposition 3.3.2.1 that there is a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$ such that for any $f \in \mathcal{S}(\mathrm{G}(\mathbf{A}))$ that is left-invariant under K_0 we have

$$\sum_{\chi \in \mathfrak{X}(\mathrm{G})} |\Lambda_r^{\mathrm{T}} \mathrm{K}_{f,\chi}(h,g)| \leqslant \|f\| \|h\|_{\mathrm{H}}^{-\mathrm{N}} \|g\|_{\mathrm{G}}^{-\mathrm{N}}$$

for all $h \in [\mathrm{H}]$ and $g \in [\mathrm{G}]$. This gives the proposition (a semi-norm on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$ is continuous if and only if its restriction to $\mathcal{S}(\mathrm{G}(\mathbf{A}))^{\mathrm{K}_0}$ is continuous for every compact-open subgroup $\mathrm{K}_0 \subset \mathrm{G}(\mathbf{A}_f)$). □

3.3.3. *Convergence of a first integral.* — It is given by the following proposition.

Proposition **3.3.3.1.** — *The map*

$$f \in \mathcal{S}(\mathrm{G}(\mathbf{A})) \mapsto \sum_{\chi \in \mathfrak{X}(\mathrm{G})} \int_{[\mathrm{H}] \times [\mathrm{G}']} |\Lambda_r^{\mathrm{T}} \mathrm{K}_{f,\chi}(h,g')| \, dh dg', \tag{3.3.3.4}$$

is given by a convergent integral and defines a continuous semi-norm on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$.

Proof. — It is a straightforward consequence of Proposition 3.3.2.2. □

3.3.4. *Arthur function* $\mathrm{F}^{\mathrm{G}_{n+1}}(\cdot, \mathrm{T})$. — For $\mathrm{T} \in \mathfrak{a}_{n+1}$ sufficiently positive, we shall use Arthur function $\mathrm{F}^{\mathrm{G}_{n+1}}(\cdot, \mathrm{T})$ (see [Art78] §6). It is the characteristic function of the set of $x \in \mathrm{G}_{n+1}(\mathbf{A})$ for which there exists a $\delta \in \mathrm{G}_{n+1}(\mathrm{F})$ such that $\delta x \in \mathfrak{s}_{\mathrm{G}_{n+1}}$ (see Section 2.2.13) and $\langle \varpi, \mathrm{H}_0(\delta x) - \mathrm{T} \rangle \leq 0$ for all $\varpi \in \widehat{\Delta}_{\mathrm{B}_{n+1}}$. Recall also that $\mathrm{F}^{\mathrm{G}_{n+1}}(\cdot, \mathrm{T})$ descends to characteristic function of a compact subset of $\mathrm{Z}_{n+1}(\mathbf{A})\mathrm{G}_{n+1}(\mathrm{F})\backslash \mathrm{G}_{n+1}(\mathbf{A})$. We will also use the function $\mathrm{F}^{\mathrm{G}'_{n+1}}(\cdot, \mathrm{T})$ defined relatively to G'_{n+1}.

3.3.5. *Two other convergent integrals.*

Proposition **3.3.5.1.** — *The maps*

$$f \in \mathcal{S}(\mathrm{G}(\mathbf{A})) \mapsto \sum_{\chi \in \mathfrak{X}(\mathrm{G})} \int_{[\mathrm{H}] \times [\mathrm{G}']} \mathrm{F}^{\mathrm{G}_{n+1}}(h, \mathrm{T}) |\mathrm{K}_{f,\chi}(h,g')| \, dh dg' \tag{3.3.5.5}$$

$$f \in \mathcal{S}(G(\mathbf{A})) \mapsto \sum_{\chi\in\mathfrak{X}(G)} \int_{[H]\times[G']} F^{G'_{n+1}}(g'_n, T)|K_{f,\chi}(h,g')|\, dhdg' \tag{3.3.5.6}$$

are given by convergent integrals and define continuous semi-norms on $\mathcal{S}(G(\mathbf{A}))$, *where as usual* $g' = (g'_n, g'_{n+1}) \in G'_n(\mathbf{A}) \times G'_{n+1}(\mathbf{A})$.

Proof. — Observe that the restriction of $F^{G_{n+1}}(\cdot, T)$ to [H] is compactly supported. Then the convergence and the continuity of the integral (3.3.5.5) follow from the majorization (3.3.2.3).

Using Lemma 2.10.1.1, (see also comments on inequality (3.2.3.2)), we see that for every $N > 0$ there exists $N' > 0$ and a continuous semi-norm $\|.\|$ on $\mathcal{S}(G(\mathbf{A}))$ such that

$$\sum_{\chi\in\mathfrak{X}(G)} \left|K_{f,P,\chi}(h,g')\right| \leqslant \|f\| \|g'_n\|^{N'}_{G'_n} \|h\|^{-N}_{H} \|g'_{n+1}\|^{-N}_{G'_{n+1}}, \tag{3.3.5.7}$$

for $(h, g') \in [H] \times [G']$ and $f \in \mathcal{S}(G(\mathbf{A}))$.

The convergence and continuity of the integral (3.3.5.6) result from the above inequality and the fact that the restriction of $F^{G'_{n+1}}(\cdot, T)$ to $[G'_n]$ is compactly supported. □

3.3.6. *A second modified kernel.* — Let $f \in \mathcal{S}(G(\mathbf{A}))$. For $T \in \mathfrak{a}_{n+1}$, $\chi \in \mathfrak{X}(G)$ and $(h, g') \in [H] \times [G']$, we set

$$\kappa^T_{f,\chi}(h,g') = \sum_{P\in\mathcal{F}_{RS}} \epsilon^G_P \sum_{\substack{\gamma\in P_H(F)\backslash H(F)\\ \delta\in P'(F)\backslash G'(F)}} \widehat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\gamma h) - T_{P_{n+1}}) K_{f,P,\chi}(\gamma h, \delta g'). \tag{3.3.6.8}$$

Remark **3.3.6.1.** — Here the expression $H_{P_{n+1}}(h)$ is understood as the value at $h \in H(\mathbf{A}) = G_n(\mathbf{A}) \subset G_{n+1}(\mathbf{A})$ of the map $H_{P_{n+1}}$. The expression defining $\kappa^T_{f,\chi}$ is absolutely convergent as the sum over $\gamma \in P_H(F)\backslash H(F)$ is finite (see [Art78, Lemma 5.1]) and $K_{f,P,\chi}$ is rapidly decaying in the second variable, see (3.3.2.3).

3.3.7. *Asymptotics of the modified kernels.* — We find that the modified kernels (3.2.3.1) and (3.3.6.8) are asymptotic for large parameters T to the kernels truncated by Arthur's characteristic function. More precisely we have:

Theorem **3.3.7.1.**

1. *For every* $N > 0$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{S},N}$ *on* $\mathcal{S}(G(\mathbf{A}))$ *such that*

$$\sum_{\chi\in\mathfrak{X}(G)} \left|K^T_{f,\chi}(h,g') - F^{G'_{n+1}}(g'_n, T) K_{f,\chi}(h,g')\right| \leqslant e^{-N\|T\|} \|h\|^{-N}_H \|g'\|^{-N}_{G'} \|f\|_{\mathcal{S},N} \tag{3.3.7.9}$$

for $f \in \mathcal{S}(G(\mathbf{A}))$, $(h, g') \in [H] \times [G']$ *and* $T \in \mathfrak{a}_{n+1}$ *sufficiently positive. In particular, for every* $N > 0$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{S},N}$ *on* $\mathcal{S}(G(\mathbf{A}))$ *such that*

$$(\mathbf{3.3.7.10}) \qquad \sum_{\chi \in \mathfrak{X}(G)} \int_{[H]\times[G']} \left| K^T_{f,\chi}(h, g') - F^{G'_{n+1}}(g'_n, T) K_{f,\chi}(h, g') \right| dh dg' \leqslant e^{-N\|T\|} \|f\|_{\mathcal{S},N}$$

for $f \in \mathcal{S}(G(\mathbf{A}))$.

2. *For every* $N > 0$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{S},N}$ *on* $\mathcal{S}(G(\mathbf{A}))$ *such that*

$$(\mathbf{3.3.7.11}) \qquad \sum_{\chi \in \mathfrak{X}(G)} \left| \kappa^T_{f,\chi}(h, g') - F^{G_{n+1}}(h, T) K_{f,\chi}(h, g') \right| \leqslant e^{-N\|T\|} \|h\|_H^{-N} \|g'\|_{G'}^{-N} \|f\|_{\mathcal{S},N}$$

for $f \in \mathcal{S}(G(\mathbf{A}))$, $(h, g') \in [H] \times [G']$ *and* $T \in \mathfrak{a}_{n+1}$ *sufficiently positive. In particular, for every* $N > 0$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{S},N}$ *on* $\mathcal{S}(G(\mathbf{A}))$ *such that*

$$(\mathbf{3.3.7.12}) \qquad \sum_{\chi \in \mathfrak{X}(G)} \int_{[H]\times[G']} \left| \kappa^T_{f,\chi}(h, g') - F^{G_{n+1}}(h, T) K_{f,\chi}(h, g') \right| dh dg' \leqslant e^{-N\|T\|} \|f\|_{\mathcal{S},N}$$

for $f \in \mathcal{S}(G(\mathbf{A}))$.

Proof. — The proof of 3.3.7.1 will be given in Section 3.6 after some preparation provided by Sections 3.4 and 3.5. Note that the asymptotics (3.3.7.10), resp. (3.3.7.12), is an obvious consequence of (3.3.7.9), resp. (3.3.7.11). □

3.3.8. Recall that we have built distributions I_χ in Theorem 3.2.4.1 from the kernel K^T_χ. The following Proposition shows that one could have defined I_χ using the kernel κ^T_χ.

Proposition **3.3.8.1.** — *Let* $\chi \in \mathfrak{X}(G)$

1. *The integral*

$$i^T_\chi(f) = \int_{[H]\times[G']} \kappa^T_{f,\chi}(h, g') \eta_{G'}(g') dg' dh,$$

is absolutely convergent for $T \in \mathfrak{a}_{n+1}$ *sufficiently positive.*

2. *The map* $T \mapsto i^T_\chi(f)$, *when* T *is sufficiently positive, coincides with an exponential-polynomial whose purely polynomial part is constant and equal to* $I_\chi(f)$.

Proof. — The first assertion follows from Proposition 3.3.5.1 and the asymptotics (3.3.7.11) of Theorem 3.3.7.1. The second assertion will be proved in Section 3.7. □

3.3.9. We can now state the final theorem of the section.

Theorem **3.3.9.1.** — *Let* $\chi \in \mathfrak{X}(G)$ *and* $f \in \mathcal{S}(G(\mathbf{A}))$. *For* $T \in \mathfrak{a}_{n+1}$ *sufficiently positive let* $P^T_\chi(f)$ *and* $\mathcal{I}^T_\chi(f)$ *be one of the following pairs of expressions:*

$$(\mathbf{3.3.9.13}) \qquad I^T_\chi(f) \text{ and } \int_{[H]\times[G']} F^{G'_{n+1}}(g'_n, T)K_\chi(h, g')\,\eta_{G'}(g')\,dhdg' \;;$$

$$(\mathbf{3.3.9.14}) \qquad i^T_\chi(f) \text{ and } \int_{[H]\times[G']} F^{G_{n+1}}(h, T)K_\chi(h, g')\,\eta_{G'}(g')\,dhdg' \;;$$

$$(\mathbf{3.3.9.15}) \qquad i^T_\chi(f) \text{ and } \int_{[H]\times[G']} \Lambda^T_r K_\chi(h, g')\,\eta_{G'}(g')dhdg'.$$

1. *The integral defining* $\mathcal{I}^T_\chi(f)$ *is absolutely convergent and the map* $T \mapsto P^T_\chi(f)$ *coincides for* $T \in \mathfrak{a}_{n+1}$ *sufficiently positive with an exponential-polynomial whose constant term equals* $I_\chi(f)$.
2. *For every* $N > 0$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{S},N}$ *on* $\mathcal{S}(G(\mathbf{A}))$ *such that*

$$|\mathcal{I}^T_\chi(f) - P^T_\chi(f)| \leqslant \|f\|_{\mathcal{S},N}e^{-N\|T\|}$$

   for $T \in \mathfrak{a}_{n+1}$ *sufficiently positive and* $f \in \mathcal{S}(G(\mathbf{A}))$.

Proof. — For assertion 1, the statement about $P^T_\chi(f)$ is just the statement about $I^T_\chi(f)$ and $i^T_\chi(f)$ that has been given in Theorem 3.2.4.1 and Proposition 3.3.8.1. The three integrals are absolutely convergent by Propositions 3.3.3.1 and 3.3.5.1. Let's prove assertion 2. The asymptotics for the first pair, resp. second, is a direct consequence of the asymptotics (3.3.7.10), resp. (3.3.7.12), of Theorem 3.3.7.1. So it remains to prove the third asymptotics. In fact, for every $N > 0$, there exists a continuous semi-norm $\|.\|_{\mathcal{S},N}$ on $\mathcal{S}(G(\mathbf{A}))$ such that

$$\int_{[H]}\int_{[G']} \left|\Lambda^T_r K_\chi(h, g') - F^{G_{n+1}}(h, T)K_\chi(h, g')\right| dhdg' \leqslant \|f\|_{\mathcal{S},N}e^{-N\|T\|}$$

for $f \in \mathcal{S}(G(\mathbf{A}))$ and T sufficiently positive. This can be proved as in [IY15, proof of Proposition 3.8] and is left to the reader. So the third asymptotics follows from the second one. □

3.4. *Auxiliary function spaces and smoothed constant terms*

3.4.1. For $\mathbf{G} \in \{G'_n, G'_{n+1}, G_n, G_{n+1}, H, G\}$, we let $\mathcal{T}_{\mathcal{F}_{RS}}(\mathbf{G})$ be the space of tuples

$$({}_P\varphi)_{P\in\mathcal{F}_{RS}} \in \prod_{P\in\mathcal{F}_{RS}} \mathcal{T}([\mathbf{G}]_{\mathbf{P}})$$

such that for every $P \subset Q \in \mathcal{F}_{RS}$, we have

$$_{P}\varphi - ({}_{Q}\varphi)_{\mathbf{P}} \in \mathcal{S}_{\theta_{\mathbf{P}}^{\mathbf{Q}}}([\mathbf{G}]_{\mathbf{P}})$$

where we have set $\mathbf{P} = P \cap \mathbf{G}$, $\mathbf{Q} = Q \cap \mathbf{G}$, the weight $\theta_{\mathbf{P}}^{\mathbf{Q}}$ is defined by

$$\theta_{\mathbf{P}}^{\mathbf{Q}} = \begin{cases} d_{P'_{n+1}}^{Q'_{n+1}} & \text{if } \mathbf{G} \in \{G'_n, G'_{n+1}\}, \\ d_{P_{n+1}}^{Q_{n+1}} & \text{if } \mathbf{G} \in \{G_n, G_{n+1}\}, \\ d_P^Q & \text{if } \mathbf{G} \in \{H, G\}, \end{cases}$$

and we refer the reader to Section 2.5.11 for the definition of the function spaces $\mathcal{S}_{\theta_{\mathbf{P}}^{\mathbf{Q}}}([\mathbf{G}]_{\mathbf{P}})$. By Lemma 2.4.4.2, in every case we have $\theta_{\mathbf{P}}^{\mathbf{Q}} \ll d_{\mathbf{P}}^{\mathbf{Q}}$ and therefore, by Proposition 2.5.14.1, the above condition is equivalent to the following: there exists $N_0 > 0$ such that for every $r \geqslant 0$ and $X \in \mathcal{U}(\mathcal{G}_\infty)$ (where we denote by $\mathcal{G}_\infty$ the Lie algebra of $\mathbf{G}(F \otimes_{\mathbf{Q}} \mathbf{R})$), we have

$$(\mathbf{3.4.1.1}) \qquad \left|(R(X)_{P}\varphi)(g) - (R(X)_{Q}\varphi)(g)\right| \ll_{r,X} \|g\|_{\mathbf{P}}^{N_0} \theta_{\mathbf{P}}^{\mathbf{Q}}(g)^{-r}, \quad \text{for } g \in \mathbf{P}(F)\backslash\mathbf{G}(\mathbf{A}).$$

Note that, by Lemma 2.4.4.2 again, for every $P \subset Q \in \mathcal{F}_{RS}$, we have $d_P^Q(h) \sim d_{P_{n+1}}^{Q_{n+1}}(h)$ for $h \in [H]_{P_H}$ so that under the identification $H = G_n$, we have

$$(\mathbf{3.4.1.2}) \qquad \mathcal{T}_{\mathcal{F}_{RS}}([H]) = \mathcal{T}_{\mathcal{F}_{RS}}([G_n]).$$

We define similarly $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$ as the space of tuples

$$({}_{P}\varphi)_{P \in \mathcal{F}_{RS}} \in \prod_{P \in \mathcal{F}_{RS}} \mathcal{S}_{\Delta_P}([G]_P)$$

such that for every $P \subset Q \in \mathcal{F}_{RS}$, we have

$$_{P}\varphi - ({}_{Q}\varphi)_{P} \in \mathcal{S}_{d_P^{Q,\Delta}}([G]_P)$$

where the weights Δ_P and $d_P^{Q,\Delta}$ have been defined in Section 3.1.9. Similarly, using (3.1.9.1) and Proposition 2.5.14.1, the condition above is equivalent to: there exists $N_0 > 0$ such that for every $r \geqslant 0$ and $X \in \mathcal{U}(\mathfrak{g}_\infty)$, we have

$$(\mathbf{3.4.1.3}) \qquad \left|(R(X)_{P}\varphi)(g) - (R(X)_{Q}\varphi)(g)\right| \ll_{r,X} \|g\|_{P}^{N_0} d_P^{Q,\Delta}(g)^{-r}, \quad \text{for } g \in P(F)\backslash G(\mathbf{A}).$$

3.4.2. Obviously, $\mathcal{T}_{\mathcal{F}_{RS}}([G])$ (resp. $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$) embeds as a closed subspace of

$$\prod_{P \in \mathcal{F}_{RS}} \mathcal{T}([G]_P) \times \prod_{P \subset Q \in \mathcal{F}_{RS}} \mathcal{S}_{d_P^Q}([G]_P)$$

$$(\text{resp. } \prod_{P\in\mathcal{F}_{RS}} \mathcal{S}_{\Delta_P}([G]_P) \times \prod_{P\subset Q\in\mathcal{F}_{RS}} \mathcal{S}_{d_P^{Q,\Delta}}([G]_P))$$

and as such it inherits a LF topology. More precisely, for every $N>0$, the inverse image $\mathcal{T}_{\mathcal{F}_{RS},N}([G])$ (resp. $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS},N}([G])$) of $\prod_{P\in\mathcal{F}_{RS}} \mathcal{T}_N([G]_P) \times \prod_{P\subset Q\in\mathcal{F}_{RS}} \mathcal{S}_{d_P^Q,N}([G]_P)$ (resp. of $\prod_{P\in\mathcal{F}_{RS}} \mathcal{S}_{\Delta_P,N}([G]_P) \times \prod_{P\subset Q\in\mathcal{F}_{RS}} \mathcal{S}_{d_P^{Q,\Delta},N}([G]_P)$)) in $\mathcal{T}_{\mathcal{F}_{RS}}([G])$ (resp. in $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$) inherits a strict LF topology (by Lemma 2.5.4.1) and we endow $\mathcal{T}_{\mathcal{F}_{RS}}([G])$ (resp. in $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$) with the locally convex direct limit of these topologies. By Theorem 2.9.4.1, we have

(**3.4.2.4**) For every $\underline{\varphi} = ({}_P\varphi)_{P\in\mathcal{F}_{RS}} \in \mathcal{T}_{\mathcal{F}_{RS}}([G])$ (resp. $\underline{\varphi} = ({}_P\varphi)_{P\in\mathcal{F}_{RS}} \in \mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$), the family

$$\chi \in \mathfrak{X}(G) \mapsto \underline{\varphi}_{\chi} := ({}_P\varphi_{\chi})_{P\in\mathcal{F}_{RS}}$$

is absolutely summable in $\mathcal{T}_{\mathcal{F}_{RS}}([G])$ (resp. in $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$) with sum $\underline{\varphi}$.

For the purpose of the next proposition, we recall that $\mathcal{T}^0([\mathbf{G}])$ denotes the space of Radon measures of moderate growth on $[\mathbf{G}]$ (see Section 2.5.9).

Proposition **3.4.2.1.**

1. *For every* $\varphi \in \mathcal{T}^0([H])$ *and* $f \in \mathcal{S}(G(\mathbf{A}))$, *the family*

$$P \in \mathcal{F}_{RS} \mapsto R(f)\varphi_{P_H}$$

belongs to $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$.

2. *Let* $\mathbf{G} \in \{G_{n+1}, G\}$. *Then, for every* $\varphi \in \mathcal{T}^0([\mathbf{G}'])$ *and* $f \in \mathcal{S}(\mathbf{G}(\mathbf{A}))$, *the family*

$$P \in \mathcal{F}_{RS} \mapsto R(f)\varphi_{\mathbf{P}'},$$

where we have set $\mathbf{P}' = P \cap \mathbf{G}'$, *belongs to* $\mathcal{T}_{\mathcal{F}_{RS}}([\mathbf{G}])$.

3. *For every* $(\mathbf{G}_1, \mathbf{G}_2) \in \{(G_n, G'_n), (G, H)\}$ *and every* $({}_P\varphi)_{P\in\mathcal{F}_{RS}} \in \mathcal{T}_{\mathcal{F}_{RS}}([\mathbf{G}_1])$, *the family of restrictions*

$$P \in \mathcal{F}_{RS} \mapsto {}_P\varphi\,|_{[\mathbf{G}_2]_{\mathbf{P}_2}},$$

where we have set $\mathbf{P}_2 = P \cap \mathbf{G}_2$, *belongs to* $\mathcal{T}_{\mathcal{F}_{RS}}([\mathbf{G}_2])$.

4. *For every* $({}_P\varphi)_{P\in\mathcal{F}_{RS}} \in \mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$ *and* $({}_P\psi)_{P\in\mathcal{F}_{RS}} \in \mathcal{T}_{\mathcal{F}_{RS}}([G_{n+1}])$, *the family of products*

$$P \in \mathcal{F}_{RS} \mapsto \big((g_n, g_{n+1}) \in [G]_P \mapsto {}_P\varphi(g_n, g_{n+1})\, {}_P\psi(g_{n+1})\big)$$

belongs to $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$.

Proof. — 1. By Lemma 2.5.11.1, it suffices to prove that for every $P \subset Q \in \mathcal{F}_{RS}$, we have:

- Δ_{P} is bounded on the support of $\varphi_{\mathrm{P_H}}$;
- $d_{\mathrm{P}}^{\mathrm{Q},\Delta}$ is bounded on the support of $\varphi_{\mathrm{P_H}} - (\varphi_{\mathrm{Q_H}})_{\mathrm{P}}$.

The first requirement is clear as $\varphi_{\mathrm{P_H}}$ is supported on $[\mathrm{H}]_{\mathrm{P_H}} \subset [\mathrm{G}]_{\mathrm{P}}$ and Δ_{P} is bounded on $[\mathrm{H}]_{\mathrm{P_H}}$. For the second one, we need to show the existence of $\mathrm{C} > 0$ such that $\varphi_{\mathrm{P_H}}$ and $(\varphi_{\mathrm{Q_H}})_{\mathrm{P}}$ coincide on $\{g \in [\mathrm{G}]_{\mathrm{P}} \mid d_{\mathrm{P}}^{\mathrm{Q},\Delta}(g) > \mathrm{C}\}$. By adjunction, for every $\psi \in \mathcal{S}^0([\mathrm{G}]_{\mathrm{P}})$, we have

$$\langle \varphi_{\mathrm{P_H}}, \psi \rangle_{\mathrm{P}} = \langle \varphi_{\mathrm{Q_H}}, \mathrm{E}_{\mathrm{P_H}}^{\mathrm{Q_H}}(\psi) \rangle_{\mathrm{Q_H}} \text{ and } \langle (\varphi_{\mathrm{Q_H}})_{\mathrm{P}}, \psi \rangle_{\mathrm{P}} = \langle \varphi_{\mathrm{Q_H}}, \mathrm{E}_{\mathrm{P}}^{\mathrm{Q}}(\psi) \rangle_{\mathrm{Q_H}}.$$

Thus, it suffices to show that for $\mathrm{C} > 0$ sufficiently large, for every function $\psi \in \mathcal{S}^0([\mathrm{G}]_{\mathrm{P}})$ supported in $\{g \in [\mathrm{G}]_{\mathrm{P}} \mid d_{\mathrm{P}}^{\mathrm{Q},\Delta}(g) > \mathrm{C}\}$ we have

$$\mathrm{E}_{\mathrm{P}}^{\mathrm{Q}}(\psi) \mid_{[\mathrm{H}]_{\mathrm{Q_H}}} = \mathrm{E}_{\mathrm{P_H}}^{\mathrm{Q_H}}(\psi).$$

With the notation of Section 2.4.4, this is in turn equivalent to:

(**3.4.2.5**) There exists $\mathrm{C} > 0$ such that for every $g \in \mathrm{P}(\mathrm{F})\mathrm{N}_{\mathrm{Q}}(\mathbf{A})\backslash \mathrm{G}(\mathbf{A})$ with $d_{\mathrm{P}}^{\mathrm{Q},\Delta}(g) > \mathrm{C}$, $\pi_{\mathrm{Q}}^{\mathrm{P}}(g) \in [\mathrm{H}]_{\mathrm{Q_H}}$ implies $g \in \mathrm{P_H}(\mathrm{F})\mathrm{N}_{\mathrm{Q_H}}(\mathbf{A})\backslash \mathrm{H}(\mathbf{A})$.

Writing $g = (g_n, g_{n+1})$, the condition $d_{\mathrm{P}}^{\mathrm{Q},\Delta}(g) > \mathrm{C}$ is equivalent to $(g_n, g_{n+1}) \in \omega_{\mathrm{P}_{n+1}}^{\mathrm{Q}_{n+1}}[> \mathrm{C}]^2$ whereas $\pi_{\mathrm{Q}}^{\mathrm{P}}(g) \in [\mathrm{H}]_{\mathrm{Q_H}}$ (resp. $g \in \mathrm{P_H}(\mathrm{F})\mathrm{N}_{\mathrm{Q_H}}(\mathbf{A})\backslash \mathrm{H}(\mathbf{A})$) is equivalent to $\pi_{\mathrm{Q}_{n+1}}^{\mathrm{P}_{n+1}}(g_n) = \pi_{\mathrm{Q}_{n+1}}^{\mathrm{P}_{n+1}}(g_{n+1})$ (resp. $g_n = g_{n+1}$). Thus, the claim follows directly from Lemma 2.4.4.1.4.

2. Let $\mathrm{P} \subset \mathrm{Q} \in \mathcal{F}_{\mathrm{RS}}$ and set $\mathbf{P} = \mathrm{P} \cap \mathbf{G}$, $\mathbf{Q} = \mathrm{Q} \cap \mathbf{G}$, $\mathbf{P}' = \mathrm{P} \cap \mathbf{G}'$, $\mathbf{Q}' = \mathrm{Q} \cap \mathbf{G}'$. Since $\mathbf{G} \in \{\mathrm{G}_{n+1}, \mathrm{G}\}$, we have $\theta_{\mathbf{P}}^{\mathbf{Q}} = d_{\mathbf{P}}^{\mathbf{Q}}$. Thus, by a similar argument, we are reduced to show:

(**3.4.2.6**) There exists $\mathrm{C} > 0$ such that for every $g \in \mathbf{P}(\mathrm{F})\mathrm{N}_{\mathbf{Q}}(\mathbf{A})\backslash \mathbf{G}(\mathbf{A})$ with $d_{\mathbf{P}}^{\mathbf{Q}}(g) > \mathrm{C}$, $\pi_{\mathbf{Q}}^{\mathbf{P}}(g) \in [\mathbf{G}']_{\mathbf{Q}'}$ implies $g \in \mathbf{P}'(\mathrm{F})\mathrm{N}_{\mathbf{Q}'}(\mathbf{A})\backslash \mathbf{G}'(\mathbf{A})$.

By Lemma 2.4.4.1 1. and Lemma 2.4.4.2, there exists $\epsilon > 0$ such that the set

$$\{g' \in \mathbf{P}'(\mathrm{F})\mathrm{N}_{\mathbf{Q}'}(\mathbf{A})\backslash \mathbf{G}'(\mathbf{A}) \mid d_{\mathbf{P}}^{\mathbf{Q}}(g') > \epsilon\}$$

surjects onto $[\mathbf{G}']_{\mathbf{Q}'}$. Moreover, by Lemma 2.4.4.1 4., there exists $\mathrm{C} > 0$ such that for $g, g' \in \mathbf{P}(\mathrm{F})\mathrm{N}_{\mathbf{Q}}(\mathbf{A})\backslash \mathbf{G}(\mathbf{A})$ with $d_{\mathbf{P}}^{\mathbf{Q}}(g) > \mathrm{C}$ and $d_{\mathbf{P}}^{\mathbf{Q}}(g') > \epsilon$, $\pi_{\mathbf{Q}}^{\mathbf{P}}(g) = \pi_{\mathbf{Q}}^{\mathbf{P}}(g')$ implies $g = g'$. The claim follows.

3. follows from the characterization (3.4.1.1) since, by Lemma 2.4.4.2, for $\mathrm{P} \subset \mathrm{Q} \in \mathcal{F}_{\mathrm{RS}}$ we have $\theta_{\mathbf{P}_1}^{\mathbf{Q}_1} \sim \theta_{\mathbf{P}_2}^{\mathbf{Q}_2}$. Similarly, 4. follows readily from the characterizations (3.4.1.1), (3.4.1.3) as, by definition of $d_{\mathrm{P}}^{\mathrm{Q},\Delta}$, we have $d_{\mathrm{P}}^{\mathrm{Q},\Delta}(g) \leqslant d_{\mathrm{P}_{n+1}}^{\mathrm{Q}_{n+1}}(g_{n+1})$ for $g \in [\mathrm{G}]_{\mathrm{P}}$. $\square$

3.4.3. For $P \in \mathcal{F}_{RS}$, $\varphi \in \mathcal{S}_{\Delta_P}([G]_P)$ and $\psi \in \mathcal{T}^0([G_{n+1}]_{P_{n+1}})$, we define a function $\langle \varphi, \psi \rangle \in \mathcal{T}([G_n]_{P_n})$ by

$$\langle \varphi, \psi \rangle (g_n) = \int_{[G_{n+1}]_{P_{n+1}}} \varphi(g_n, g_{n+1}) \psi(g_{n+1}), \quad g_n \in [G_n]_{P_n}.$$

Note that the integral is absolutely convergent and the resulting function of uniform moderate growth by (3.1.9.2) applied to the weight $w = \|.\|^N_{P_{n+1}}$ for N sufficiently large.

Proposition **3.4.3.1.** — *Let* $({}_P\varphi)_{P \in \mathcal{F}_{RS}} \in \mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$ *and* $\psi \in \mathcal{T}^0([G'_{n+1}])$. *Then, the family*

$$P \in \mathcal{F}_{RS} \mapsto \langle {}_P\varphi, \psi_{P'_{n+1}} \rangle$$

belongs to $\mathcal{T}_{\mathcal{F}_{RS}}([G_n])$.

Proof. — By Dixmier-Malliavin, we may assume that $({}_P\varphi)_{P \in \mathcal{F}_{RS}} = (R(f)_P\varphi')_{P \in \mathcal{F}_{RS}}$ for some $({}_P\varphi')_{P \in \mathcal{F}_{RS}} \in \mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$ and $f \in C_c^\infty(G_{n+1}(\mathbf{A}))$.[7] Then, we have $\langle {}_P\varphi, \psi_{P'_{n+1}} \rangle = \langle {}_P\varphi', R(f^\vee)\psi_{P'_{n+1}} \rangle$ where $f^\vee(g) = f(g^{-1})$. Therefore, by Proposition 3.4.2.1 2. and 4., up to replacing ${}_P\varphi$ by the product

$$g \in [G]_P \mapsto {}_P\varphi'(g)\, R(f^\vee)\psi_{P'_{n+1}}(g_{n+1}),$$

it suffices to prove that:

(**3.4.3.7**) The family of functions

$$P \in \mathcal{F}_{RS} \mapsto p_{n*}({}_P\varphi) : g_n \in [G_n]_{P_n} \mapsto \int_{[G_{n+1}]_{P_{n+1}}} {}_P\varphi(g_n, g_{n+1}) dg_{n+1}$$

belongs to $\mathcal{T}_{\mathcal{F}_{RS}}([G_n])$.

Let $P \subset Q \in \mathcal{F}_{RS}$. By the characterization (3.4.1.1), we need to show the existence of $N_0 > 0$ such that for every $X \in \mathcal{U}(\mathfrak{g}_{n,\infty})$ and $r \geqslant 0$, we have

$$\left| \int_{[G_{n+1}]_{P_{n+1}}} R(X)_P\varphi(g_n, g_{n+1}) dg_{n+1} - \int_{[G_{n+1}]_{Q_{n+1}}} R(X)_Q\varphi(g_n, g_{n+1}) dg_{n+1} \right|$$
$$\ll_{r,X} \|g_n\|^{N_0}_{P_n} d^{Q_{n+1}}_{P_{n+1}}(g_n)^{-r}$$

for $g_n \in P_n(F)N_{Q_n}(\mathbf{A})\backslash G_n(\mathbf{A})$. For notational simplicity, we will prove this for $X = 1$ but it will be clear from the argument that we can choose the same exponent N_0 for every $X \in \mathcal{U}(\mathfrak{g}_{n,\infty})$.

[7] More precisely, this follows from applying the Dixmier-Malliavin theorem to the continuous smooth Fréchet representation $\mathcal{T}^{\Delta}_{\mathcal{F}_{RS},N}([G])^J$ for suitable $N > 0$ and compact-open subgroup $J \subset G(\mathbf{A}_f)$.

Let $C > 0$ and set $\omega = \omega_{P_{n+1}}^{Q_{n+1}}[> C]$ (see Section 2.4.4 for the notation). Since ${}_P\varphi \in \mathcal{S}_{\Delta_P}([G]_P)$, by (3.1.9.2) applied to the weights $w = (d_{P_{n+1}}^{Q_{n+1}})^{-1}$ and $\|.\|_{P_{n+1}}$, there exists $N_1 > 0$ such that for every $N > 0$ and $r \geqslant 0$, we have

$$(\mathbf{3.4.3.8}) \qquad \left|{}_P\varphi(g)\right| \ll_{N,r} \|g_n\|_{P_n}^{N_1+N} \|g_{n+1}\|_{P_{n+1}}^{-N} d_{P_{n+1}}^{Q_{n+1}}(g_{n+1})^r d_{P_{n+1}}^{Q_{n+1}}(g_n)^{-r}, \quad \text{for } g \in [G]_P.$$

Similarly, up to enlarging N_1 and since $\|.\|_{Q_n} \ll \|.\|_{P_n}$, $d_{P_{n+1}}^{Q_{n+1}} \ll d_{Q_{n+1}}^{P_{n+1}}$ on $P_n(F)N_{Q_n}(\mathbf{A})\backslash G_n(\mathbf{A})$ (see (2.4.1.2), (2.4.4.19)), for every $N > 0$ and $r \geqslant 0$, we have

$$(\mathbf{3.4.3.9}) \qquad \left|{}_Q\varphi(g)\right| \ll_{N,r} \|g_n\|_{P_n}^{N_1+N} \|g_{n+1}\|_{Q_{n+1}}^{-N} d_{Q_{n+1}}^{P_{n+1}}(g_{n+1})^r d_{P_{n+1}}^{Q_{n+1}}(g_n)^{-r},$$
$$\text{for } g \in P(F)N_Q(\mathbf{A})\backslash G(\mathbf{A}).$$

By Lemma 2.4.4.1 1. and 2., $d_{Q_{n+1}}^{P_{n+1}}$ and $d_{P_{n+1}}^{Q_{n+1}}$ are bounded from above outside $\pi_{Q_{n+1}}^{P_{n+1}}(\omega)$ and $\pi_{P_{n+1}}^{Q_{n+1}}(\omega)$ respectively. Thus, from (3.4.3.8) and (3.4.3.9) we deduce that there exists $N_2 > 0$ such that for every $r \geqslant 0$, the two functions

$$g_n \mapsto \int_{[G_{n+1}]_{P_{n+1}}\backslash \pi_{P_{n+1}}^{Q_{n+1}}(\omega)} {}_P\varphi(g_n, g_{n+1})dg_{n+1} \text{ and}$$

$$g_n \mapsto \int_{[G_{n+1}]_{Q_{n+1}}\backslash \pi_{Q_{n+1}}^{P_{n+1}}(\omega)} {}_Q\varphi(g_n, g_{n+1})dg_{n+1}$$

are essentially bounded by $\|g_n\|_{P_n}^{N_2} d_{P_{n+1}}^{Q_{n+1}}(g_n)^{-r}$ for $g_n \in P_n(F)N_{Q_n}(\mathbf{A})\backslash G_n(\mathbf{A})$. Thus, it only remains to estimate the difference

$$\int_{\pi_{P_{n+1}}^{Q_{n+1}}(\omega)} {}_P\varphi(g_n, g_{n+1})dg_{n+1} - \int_{\pi_{Q_{n+1}}^{P_{n+1}}(\omega)} {}_Q\varphi(g_n, g_{n+1})dg_{n+1}$$

which, by Lemma 2.4.4.1 and provided C is sufficiently large, is equal to

$$(\mathbf{3.4.3.10}) \qquad \int_\omega {}_P\varphi(g_n, g_{n+1}) - {}_Q\varphi(g_n, g_{n+1})dg_{n+1}.$$

As $d_{Q_{n+1}}^{P_{n+1}} \sim d_{P_{n+1}}^{Q_{n+1}}$ and $\|.\|_{Q_{n+1}} \sim \|.\|_{P_{n+1}}$ on ω (see Lemma 2.4.4.1 2.), by (3.4.3.8), (3.4.3.9), for every $r \geqslant 0$ and $N > 0$, we have

$$\left|{}_P\varphi(g) - {}_Q\varphi(g)\right| \ll_{N,r} \|g_n\|_{P_n}^{N_1+N} \|g_{n+1}\|_{P_{n+1}}^{-N} d_{P_{n+1}}^{Q_{n+1}}(g_{n+1})^r d_{P_{n+1}}^{Q_{n+1}}(g_n)^{-r}$$

for $g \in P_n(F)N_{Q_n}(\mathbf{A})\backslash G_n(\mathbf{A}) \times \omega$. Combining this with the characterization (3.4.1.3) and the equality

$$d_P^{Q,\Delta}(g) \max(1, d_{P_{n+1}}^{Q_{n+1}}(g_n) d_{P_{n+1}}^{Q_{n+1}}(g_{n+1})^{-1}) = d_{P_{n+1}}^{Q_{n+1}}(g_n) \text{ for } g \in [G]_P,$$

it readily follows that there exists $N_3 > 0$ such that for every $N > 0$ and $r \geqslant 0$, we have

$$\left|{}_P\varphi(g) - {}_Q\varphi(g)\right| \ll_{N,r} \|g_n\|_{P_n}^{N_3+N} \|g_{n+1}\|_{P_{n+1}}^{-N} d_{P_{n+1}}^{Q_{n+1}}(g_n)^{-r}.$$

This in turn implies that for some $N_4 > 0$ the integral (3.4.3.10) is essentially bounded by $\|g_n\|_{P_n}^{N_4} d_{P_{n+1}}^{Q_{n+1}}(g_n)^{-r}$ for every $r \geqslant 0$ and this ends the proof of (3.4.3.7) hence of the proposition. □

3.5. *A relative truncation operator*

3.5.1. Let $\mathbf{G} \in \{H, G'_n, G_n\}$. For $\underline{\varphi} = ({}_P\varphi)_{P\in\mathcal{F}_{RS}} \in \mathcal{T}_{\mathcal{F}_{RS}}([\mathbf{G}])$ and $T \in \mathfrak{a}_{n+1}$, we set

$$\Lambda^{T,\mathbf{G}}\underline{\varphi}(g) = \sum_{P\in\mathcal{F}_{RS}} \epsilon_P^G \sum_{\delta\in\mathbf{P}(F)\backslash\mathbf{G}(F)} \widehat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\delta g) - T_{P_{n+1}})\, {}_P\varphi(\delta g)$$

for $g \in [\mathbf{G}]$. We also set

$$\Pi^{T,\mathbf{G}}\underline{\varphi}(g) = \begin{cases} F^{G'_{n+1}}(g, T)_G\varphi(g) \text{ if } \mathbf{G} = G'_n, \\ F^{G_{n+1}}(g, T)_G\varphi(g) \text{ if } \mathbf{G} \in \{H, G_n\}. \end{cases}$$

Recall that $\mathcal{S}^0([\mathbf{G}])$ stands for the space of functions of rapid decay on $[\mathbf{G}]$ (see Section 2.5.6).

Theorem **3.5.1.1.** — *For* T *sufficiently positive, we have* $\Lambda^{T,\mathbf{G}}\underline{\varphi} \in \mathcal{S}^0([\mathbf{G}])$. *More precisely, for every* $c > 0$ *and* $N > 0$, *there exists a continuous semi-norm* $\|.\|_{\mathcal{F}_{RS},c,N}$ *on* $\mathcal{T}_{\mathcal{F}_{RS}}([\mathbf{G}])$ *such that*

$$(\mathbf{3.5.1.1}) \qquad \|\Lambda^{T,\mathbf{G}}\underline{\varphi} - \Pi^{T,\mathbf{G}}\underline{\varphi}\|_{\infty,N} \leqslant e^{-c\|T\|}\|\underline{\varphi}\|_{\mathcal{F}_{RS},c,N}$$

for $\underline{\varphi} \in \mathcal{T}_{\mathcal{F}_{RS}}([\mathbf{G}])$ *and* $T \in \mathfrak{a}_{n+1}$ *sufficiently positive (where the semi-norms* $(\|.\|_{\infty,N})_{N>0}$ *are as in Section 2.5.6).*

Proof. — Note that, by the identification (3.4.1.2), the statement of the theorem for $\mathbf{G} = H$ is exactly equivalent to the statement for $\mathbf{G} = G_n$ which itself can be obtained from the case $\mathbf{G} = G'_n$ by a change of base-field from F to E. Therefore, we shall content ourself to give the proof when $\mathbf{G} = G_n$.

As the restriction of $F^{G_{n+1}}(., T)$ to $[G_n]$ is compactly supported, we have $\Pi^{T,G_n}\underline{\varphi} \in \mathcal{S}^0([G_n])$ and the first assertion of the theorem follows from the second. Now, as in the beginning of the proof of [Zyd20, proposition 2.8], using [Zyd20, lemme 2.1], for T sufficiently positive we obtain

$$\Lambda^{T,G_n}\underline{\varphi}(g) = \sum_{P\subset Q\in\mathcal{F}_{RS}} \sum_{\delta\in P_n(F)\backslash G_n(F)} F^{P_{n+1}}(\delta g, T) \times \sigma_{P_{n+1}}^{Q_{n+1}}(H_{P_{n+1}}(\delta g) - T_{P_{n+1}})\, {}_{P,Q}\varphi(\delta g)$$

where we have set

$$_{\mathrm{P,Q}}\varphi(g) = \sum_{\mathrm{P}\subset\mathrm{R}\subset\mathrm{Q}} \epsilon_{\mathrm{R}}^{\mathrm{G}}\, {}_{\mathrm{R}}\varphi(g), \text{ for } g \in \mathrm{P}_n(\mathrm{F})\backslash\mathrm{G}_n(\mathbf{A}),$$

the sum running over parabolic subgroups $\mathrm{R} \in \mathcal{F}_{\mathrm{RS}}$ such that $\mathrm{P} \subset \mathrm{R} \subset \mathrm{Q}$. Since $\sigma_{\mathrm{G}_{n+1}}^{\mathrm{G}_{n+1}} = 1$ and $\sigma_{\mathrm{P}_{n+1}}^{\mathrm{P}_{n+1}} = 0$ for $\mathrm{P} \subsetneq \mathrm{G}$ (see [Art78, §6, p. 940]), it follows that

$$\Lambda^{\mathrm{T},\mathrm{G}_n}\underline{\varphi}(g) - \Pi^{\mathrm{T},\mathrm{G}_n}\underline{\varphi}(g)$$
$$= \sum_{\mathrm{P}\subsetneq\mathrm{Q}\in\mathcal{F}_{\mathrm{RS}}} \sum_{\delta\in\mathrm{P}_n(\mathrm{F})\backslash\mathrm{G}_n(\mathrm{F})} \mathrm{F}^{\mathrm{P}_{n+1}}(\delta g, \mathrm{T})\sigma_{\mathrm{P}_{n+1}}^{\mathrm{Q}_{n+1}}(\mathrm{H}_{\mathrm{P}_{n+1}}(\delta g) - \mathrm{T}_{\mathrm{P}_{n+1}})\, {}_{\mathrm{P,Q}}\varphi(\delta g)$$

for $g \in [\mathrm{G}_n]$.

Let us fix $\mathrm{P} \subsetneq \mathrm{Q} \in \mathcal{F}_{\mathrm{RS}}$. Since $\mathrm{E}_{\mathrm{P}_n}^{\mathrm{G}_n}$ sends $\mathcal{S}^0([\mathrm{G}_n]_{\mathrm{P}_n})$ into $\mathcal{S}^0([\mathrm{G}_n])$ continuously, it suffices to show that for every $c > 0$ and $\mathrm{N} > 0$, there exists a continuous semi-norm $\|.\|_{\mathcal{F}_{\mathrm{RS}},c,\mathrm{N}}$ on $\mathcal{T}_{\mathcal{F}_{\mathrm{RS}}}([\mathrm{G}_n])$ such that

$$(\mathbf{3.5.1.2}) \qquad \left|{}_{\mathrm{P,Q}}\varphi(g)\right| \leqslant e^{-c\|\mathrm{T}\|}\|g\|_{\mathrm{P}_n}^{-\mathrm{N}}\|\underline{\varphi}\|_{\mathcal{F}_{\mathrm{RS}},c,\mathrm{N}}$$

for $\underline{\varphi} \in \mathcal{T}_{\mathcal{F}_{\mathrm{RS}}}([\mathrm{G}_n])$, $\mathrm{T} \in \mathfrak{a}_{n+1}$ sufficiently positive and $g \in \mathrm{P}_n(\mathrm{F})\backslash\mathrm{G}_n(\mathbf{A})$ such that

$$\mathrm{F}^{\mathrm{P}_{n+1}}(g, \mathrm{T})\sigma_{\mathrm{P}_{n+1}}^{\mathrm{Q}_{n+1}}(\mathrm{H}_{\mathrm{P}_{n+1}}(g) - \mathrm{T}_{\mathrm{P}_{n+1}}) \neq 0.$$

Up to conjugacy, we may assume that P and Q are standard. We will need the following lemma which summarizes part of the analysis performed in [Art78, §6, §7].

Lemma **3.5.1.2.** — *There exists* $r > 0$ *such that*

$$e^{\|\mathrm{T}\|} \ll \left(\min_{\alpha\in\Delta_0^{\mathrm{Q}_{n+1}}\backslash\Delta_0^{\mathrm{P}_{n+1}}} d_{\mathrm{P}_{n+1},\alpha}(g)\right)^r \quad \textit{and}$$

$$\|g\|_{\mathrm{P}_n} \ll \left(\max_{\alpha\in\Delta_0^{\mathrm{Q}_{n+1}}\backslash\Delta_0^{\mathrm{P}_{n+1}}} d_{\mathrm{P}_{n+1},\alpha}(g)\right)^r$$

for every $\mathrm{T} \in \mathfrak{a}_{n+1}$ *sufficiently positive and* $g \in \mathrm{P}_n(\mathrm{F})\backslash\mathrm{G}_n(\mathbf{A})$ *satisfying*

$$\mathrm{F}^{\mathrm{P}_{n+1}}(g, \mathrm{T})\sigma_{\mathrm{P}_{n+1}}^{\mathrm{Q}_{n+1}}(\mathrm{H}_{\mathrm{P}_{n+1}}(g) - \mathrm{T}_{\mathrm{P}_{n+1}}) \neq 0.$$

Proof. — Writing g as zg^1 where $z \in \mathrm{A}_{\mathrm{G}_{n+1}}^{\infty}$ and $g^1 \in \mathrm{G}_{n+1}(\mathbf{A})^1$ we have $d_{\mathrm{P}_{n+1},\alpha}(g) \sim d_{\mathrm{P}_{n+1},\alpha}(g^1)$ and there exists $r_0 > 0$ such that $\|g\|_{\mathrm{P}_n} \ll \|g^1\|_{\mathrm{P}_{n+1}}^{r_0}$. Thus, it suffices to prove that the same statement holds for $g \in \mathrm{P}_{n+1}(\mathrm{F})\backslash\mathrm{G}_{n+1}(\mathbf{A})^1$. Moreover, we may assume that g belongs to some Siegel domain $\mathfrak{s}_{\mathrm{P}_{n+1}}$ for $[\mathrm{G}_{n+1}]_{\mathrm{P}_{n+1}}$. Then, by [Art78, Eq. (7.7)],

$$\mathrm{F}^{\mathrm{P}_{n+1}}(g, \mathrm{T})\sigma_{\mathrm{P}_{n+1}}^{\mathrm{Q}_{n+1}}(\mathrm{H}_{\mathrm{P}_{n+1}}(g) - \mathrm{T}_{\mathrm{P}_{n+1}}) \neq 0$$

implies $\langle\alpha, H_0(g)\rangle > \langle\alpha, T\rangle$ for every $\alpha \in \Delta_0^{Q_{n+1}} \setminus \Delta_0^{P_{n+1}}$ and therefore, as T is sufficiently positive, also

$$\textbf{(3.5.1.3)}\qquad \min_{\alpha\in\Delta_0^{Q_{n+1}}\setminus\Delta_0^{P_{n+1}}} \langle\alpha, H_0(g)\rangle > \epsilon\|T\|$$

for some constant $\epsilon > 0$. The first inequality of the lemma follows for $r > 0$ large enough. For the second inequality, by [Art78, Corollary 6.2], the condition $\sigma_{P_{n+1}}^{Q_{n+1}}(H_{P_{n+1}}(g) - T_{P_{n+1}}) \neq 0$ implies

$$\textbf{(3.5.1.4)}\qquad \|H_{P_{n+1}}(g) - T_{P_{n+1}}\| \leqslant c_0\left(1 + \max_{\alpha\in\Delta_{P_{n+1}}^{Q_{n+1}}} \langle\alpha, H_0(g) - T\rangle\right)$$

whereas the condition $F^{P_{n+1}}(g, T) \neq 0$ implies

$$\textbf{(3.5.1.5)}\qquad \|H^{P_{n+1}}(g)\| \leqslant c_1\|T\|$$

for suitable constants $c_0, c_1 > 0$ and where $H^{P_{n+1}}(g)$ denotes the projection of $H_0(g)$ to $\mathfrak{a}_0^{P_{n+1}}$. As $\Delta_{P_{n+1}}^{Q_{n+1}}$ is the restriction of $\Delta_0^{Q_{n+1}} \setminus \Delta_0^{P_{n+1}}$ to $\mathfrak{a}_{P_{n+1}}$, from (3.5.1.4) and (3.5.1.5) we get

$$\|H_0(g)\| \leqslant c_3\left(1 + \|T\| + \max_{\alpha\in\Delta_0^{Q_{n+1}}\setminus\Delta_0^{P_{n+1}}} \langle\alpha, H_0(g)\rangle\right)$$

for some constant $c_3 > 0$. Combining this with (3.5.1.3), we finally obtain

$$\|H_0(g)\| \leqslant c_4\left(1 + \max_{\alpha\in\Delta_0^{Q_{n+1}}\setminus\Delta_0^{P_{n+1}}} \langle\alpha, H_0(g)\rangle\right)$$

for some $c_4 > 0$. Exponentiating then gives the second inequality of the lemma. □

We are now in position to prove (3.5.1.2). Let $\alpha \in \Delta_0^{Q_{n+1}} \setminus \Delta_0^{P_{n+1}}$. For every parabolic subgroup $P \subset R \subset Q$ such that $R \in \mathcal{F}_{RS}$ and $\alpha \in \Delta_0^{R_{n+1}}$, there exists a unique parabolic subgroup $P \subset R^\alpha \subset Q$ with $R^\alpha \in \mathcal{F}_{RS}$ and $\Delta_0^{R^\alpha_{n+1}} = \Delta_0^{R_{n+1}} \setminus \{\alpha\}$. As $({}_P\varphi)_{P\in\mathcal{F}_{RS}} \in \mathcal{T}_{\mathcal{F}_{RS}}([G_n])$, by the inequality (3.4.1.1), there exists $N_0 > 0$ such that for every $r \geqslant 0$ we can find a continuous semi-norm $\|.\|_{\mathcal{F}_{RS},r,\alpha}$ on $\mathcal{T}_{\mathcal{F}_{RS}}([G_n])$ with

$$|{}_{P,Q}\varphi(g)| \leqslant \sum_{\substack{P\subset R\subset Q\\ \alpha\in\Delta_0^{R_{n+1}}}} |{}_R\varphi(g) - {}_{R^\alpha}\varphi(g)| \leqslant \|g\|_{P_n}^{N_0} \|\underline{\varphi}\|_{\mathcal{F}_{RS},r,\alpha} \sum_{\substack{P\subset R\subset Q\\ \alpha\in\Delta_0^{R_{n+1}}}} d_{R_{n+1},\alpha}(g)^{-r}$$

for $\underline{\varphi} \in \mathcal{T}_{\mathcal{F}_{RS}}([G_n])$ and $g \in P_n(F)\backslash G_n(\mathbf{A})$. Now, if $F^{P_{n+1}}(g, T)\sigma_{P_{n+1}}^{Q_{n+1}}(H_{P_{n+1}}(g) - T_{P_{n+1}}) \neq 0$ for a $T \in \mathfrak{a}_{n+1}$ that is sufficiently positive, the first inequality of the above lemma implies that $g \in \omega_{P_{n+1}}^{Q_{n+1}}[> C]$ for some fixed constant $C > 0$. In particular, by Lemma 2.4.4.1.2,[8] we have $d_{R_{n+1},\alpha}(g) \sim d_{P_{n+1},\alpha}(g)$ for every $P \subset R \subset Q$ and the above estimate becomes

$$|_{P,Q}\varphi(g)| \ll_r \|g\|_{P_n}^{N_0} d_{P_{n+1},\alpha}(g)^{-r} \|\underline{\varphi}\|_{\mathcal{F}_{RS},r,\alpha}.$$

Taking the minimum of the right hand side over $\alpha \in \Delta_0^{Q_{n+1}} \setminus \Delta_0^{P_{n+1}}$ and using Lemma 3.5.1.2 once again, we deduce the estimates (3.5.1.2) and this ends the proof of the theorem. $\square$

3.6. *Proof of Theorem 3.3.7.1*

3.6.1. We prove 1., the proof of 2. being similar and left to the reader. Set $f^\vee(g) = f(g^{-1})$ for $g \in G(\mathbf{A})$. For $T \in \mathfrak{a}_{n+1}$, we consider the two operators

$$L_f^T, P_f^T : \mathcal{T}^0([H]) \otimes \mathcal{T}^0([G'_{n+1}]) \to \mathcal{S}^0([G'_n])$$

defined as the following compositions

$$\begin{aligned} L_f^T : \mathcal{T}^0([H]) \otimes \mathcal{T}^0([G'_{n+1}]) &\xrightarrow{R^\Delta(f^\vee)\otimes Id} \mathcal{T}^\Delta_{\mathcal{F}_{RS}}([G]) \otimes \mathcal{T}^0([G'_{n+1}]) \\ &\xrightarrow{\langle .,. \rangle} \mathcal{T}_{\mathcal{F}_{RS}}([G_n]) \xrightarrow{Res} \mathcal{T}_{\mathcal{F}_{RS}}([G'_n]) \\ &\xrightarrow{\Lambda^{T,G'_n}} \mathcal{S}^0([G'_n]) \end{aligned}$$

and

$$\begin{aligned} P_f^T : \mathcal{T}^0([H]) \otimes \mathcal{T}^0([G'_{n+1}]) &\xrightarrow{R^\Delta(f^\vee)\otimes Id} \mathcal{T}^\Delta_{\mathcal{F}_{RS}}([G]) \otimes \mathcal{T}^0([G'_{n+1}]) \\ &\xrightarrow{\langle .,. \rangle} \mathcal{T}_{\mathcal{F}_{RS}}([G_n]) \xrightarrow{Res} \mathcal{T}_{\mathcal{F}_{RS}}([G'_n]) \\ &\xrightarrow{\Pi^{T,G'_n}} \mathcal{S}^0([G'_n]) \end{aligned}$$

respectively. Here, $R^\Delta(f^\vee)$, $\langle .,. \rangle$ and Res denote the operators $\varphi \mapsto (R(f^\vee)\varphi_{P_H})_{P\in\mathcal{F}_{RS}}$ (see Proposition 3.4.2.1 1.), $({}_P\varphi)_{P\in\mathcal{F}_{RS}} \otimes \psi \mapsto (\langle {}_P\varphi, \psi_{P'_{n+1}} \rangle)_{P\in\mathcal{F}_{RS}}$ (see Proposition 3.4.3.1) and $({}_P\varphi)_{P\in\mathcal{F}_{RS}} \mapsto ({}_P\varphi\,|_{[G'_n]_{P'_n}})_{P\in\mathcal{F}_{RS}}$ (see Proposition 3.4.2.1 3.) respectively whereas Λ^{T,G'_n} and Π^{T,G'_n} are the truncation operators defined in the previous section.

By the closed graph theorem (which is valid for linear maps between LF spaces see [Gro55, théorème B p. 17]), each of the operators $R^\Delta(f^\vee)$, Res, Λ^{T,G'_n} and Π^{T,G'_n} is readily seen to be continuous: indeed, it suffices to check that the compositions of these operators with the linear maps corresponding to the "pointwise evaluations" are

[8] Note that $\omega_{P_{n+1}}^{Q_{n+1}}[> C] \subset \omega_{P_{n+1}}^{R_{n+1}}[> C]$

continuous which, in each case, is straightforward to check. Similarly, the operator $\langle .,. \rangle$ is separately continuous. In particular, L_f^T and P_f^T are separately continuous bilinear maps.

We claim that the functions K_f^T and $F^{G'_{n+1}}(.,T)K_f$ are the kernels of the operators L_f^T and P_f^T respectively that is: for every $\varphi \otimes \psi \in \mathcal{T}^0([H]) \otimes \mathcal{T}^0([G'_{n+1}])$ and $g'_n \in [G'_n]$, we have

$$L_f^T(\varphi \otimes \psi)(g'_n) = \int_{[H]\times[G'_{n+1}]} K_f^T(h; g'_n, g'_{n+1})\varphi(h)\psi(g'_{n+1}) \tag{3.6.1.1}$$

and

$$P_f^T(\varphi \otimes \psi)(g'_n) = \int_{[H]\times[G'_{n+1}]} F^{G'_{n+1}}(g'_n, T)K_f(h; g'_n, g'_{n+1})\varphi(h)\psi(g'_{n+1}). \tag{3.6.1.2}$$

Let us show (3.6.1.1), the proof of (3.6.1.2) being similar (and actually easier). Unfolding the definitions, for $\varphi \otimes \psi \in \mathcal{T}^0([H]) \otimes \mathcal{T}^0([G'_{n+1}])$ and $g'_n \in [G'_n]$, we have

$$\begin{aligned} L_f^T(\varphi \otimes \psi)(g'_n) = \sum_{P\in\mathcal{F}_{RS}} \epsilon_P^G \sum_{\delta_n \in P'_n(F)\backslash G'_n(F)} \widehat{\tau}_{P_{n+1}}(H_{P_{n+1}}(\delta_n g'_n) - T_{P_{n+1}}) \\ \times \langle R(f^\vee)\varphi_{P_H}, \psi_{P'_{n+1}}\rangle(\delta_n g'_n). \end{aligned} \tag{3.6.1.3}$$

Moreover,

$$\begin{aligned} &\langle R(f^\vee)\varphi_{P_H}, \psi_{P'_{n+1}}\rangle(\delta_n g'_n) \\ &= \int_{[G'_{n+1}]_{P'_{n+1}}} R(f^\vee)\varphi_{P_H}(\delta_n g'_n, g'_{n+1})\psi_{P'_{n+1}}(g'_{n+1}) \\ &= \int_{[G'_{n+1}]_{P'_{n+1}}} \int_{[H]_{P_H}} K_{f^\vee,P}(\delta_n g'_n, g'_{n+1}; h)\varphi_{P_H}(h)\psi_{P'_{n+1}}(g'_{n+1}) \\ &= \int_{[G'_{n+1}]_{P'_{n+1}}\times[H]_{P_H}} K_{f,P}(h; \delta_n g'_n, g'_{n+1})\varphi_{P_H}(h)\psi_{P'_{n+1}}(g'_{n+1}) \\ &= \int_{[G'_{n+1}]\times[H]} \sum_{\substack{\gamma\in P_H(F)\backslash H(F) \\ \delta_{n+1}\in P'_{n+1}(F)\backslash G'_{n+1}(F)}} K_{f,P}(\gamma h; \delta_n g'_n, \delta_{n+1}g'_{n+1})\varphi(h)\psi(g'_{n+1}) \end{aligned}$$

and plugging this back into (3.6.1.3) gives (3.6.1.1).

3.6.2. For $\chi \in \mathfrak{X}(G)$, we introduce variants $L_{f,\chi}^T$, $P_{f,\chi}^T$ of the previous operators by replacing in their definitions the operator $R^\Delta(f^\vee)$ by $R^\Delta_{\chi^\vee}(f^\vee) : \varphi \mapsto (R_{\chi^\vee}(f^\vee)\varphi_{P_H})_{P\in\mathcal{F}_{RS}}$ where $R_{\chi^\vee}(f^\vee)$ denotes the composition of $R(f^\vee)$ with the "projection to the $\chi^\vee$-component" defined in Theorem 2.9.4.1. We show similarly that $L_{f,\chi}^T$, $P_{f,\chi}^T$

are separately continuous and that for $\varphi\otimes\psi\in\mathcal{T}^0([\mathrm{H}])\otimes\mathcal{T}^0([\mathrm{G}'_{n+1}])$ and $g'_n\in[\mathrm{G}'_n]$, we have

$$\mathrm{L}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)(g'_n)=\int_{[\mathrm{H}]\times[\mathrm{G}'_{n+1}]}\mathrm{K}^{\mathrm{T}}_{f,\chi}(h;g'_n,g'_{n+1})\varphi(h)\psi(g'_{n+1}) \tag{3.6.2.4}$$

and

$$\mathrm{P}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)(g'_n)=\int_{[\mathrm{H}]\times[\mathrm{G}'_{n+1}]}\mathrm{F}^{\mathrm{G}'_{n+1}}(g'_n,\mathrm{T})\mathrm{K}_{f,\chi}(h;g'_n,g'_{n+1})\varphi(h)\psi(g'_{n+1}). \tag{3.6.2.5}$$

Let $\mathrm{N}>0$. Note that, by (3.4.2.4), Theorem 3.5.1.1 and the continuity of $\langle .,.\rangle$ and Res, for every $\varphi\otimes\psi\in\mathcal{T}^0([\mathrm{H}])\otimes\mathcal{T}^0([\mathrm{G}'_{n+1}])$, we have

$$\sum_{\chi\in\mathfrak{X}(\mathrm{G})}\|\mathrm{L}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)-\mathrm{P}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)\|_{\infty,\mathrm{N}}\ll_{\mathrm{N},\varphi,\psi}e^{-\mathrm{N}\|\mathrm{T}\|}$$

for $\mathrm{T}\in\mathfrak{a}_{n+1}$ sufficiently positive. Moreover, by continuity of $\mathrm{L}^{\mathrm{T}}_{f,\chi}$ and $\mathrm{P}^{\mathrm{T}}_{f,\chi}$, each of the semi-norms $\varphi\otimes\psi\mapsto\|\mathrm{L}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)-\mathrm{P}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)\|_{\infty,\mathrm{N}}$ is bounded on $\mathcal{T}^0_{\mathrm{N}}([\mathrm{H}])\otimes\mathcal{T}^0_{\mathrm{N}}([\mathrm{G}'_{n+1}])$ by a constant multiple of $\|\varphi\|_{1,-\mathrm{N}}\|\psi\|_{1,-\mathrm{N}}$ where we recall that $\|.\|_{1,-\mathrm{N}}$ is the norm on the Banach spaces $\mathcal{T}^0_{\mathrm{N}}([\mathrm{H}])$, $\mathcal{T}^0_{\mathrm{N}}([\mathrm{G}'_{n+1}])$ (and the peculiar transition from the index N to $-\mathrm{N}$ is again due to a slight inconsistency of notation). Thus, by the uniform boundedness principle, we have

$$\sum_{\chi\in\mathfrak{X}(\mathrm{G})}\|\mathrm{L}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)-\mathrm{P}^{\mathrm{T}}_{f,\chi}(\varphi\otimes\psi)\|_{\infty,\mathrm{N}}\ll_{\mathrm{N}}e^{-\mathrm{N}\|\mathrm{T}\|}\|\varphi\|_{1,-\mathrm{N}}\|\psi\|_{1,-\mathrm{N}}$$

for $\varphi\otimes\psi\in\mathcal{T}^0_{\mathrm{N}}([\mathrm{H}])\otimes\mathcal{T}^0_{\mathrm{N}}([\mathrm{G}'_{n+1}])$ and $\mathrm{T}\in\mathfrak{a}_{n+1}$ sufficiently positive. By (3.6.2.4) and (3.6.2.5), applying the above inequality to the Dirac measures $\varphi=\delta_h$ and $\psi=\delta_{g'_{n+1}}$, where $h\in[\mathrm{H}]$ and $g'_{n+1}\in[\mathrm{G}'_{n+1}]$, gives

$$\sum_{\chi\in\mathfrak{X}(\mathrm{G})}|\mathrm{K}^{\mathrm{T}}_{f,\chi}(h;g'_n,g'_{n+1})-\mathrm{F}^{\mathrm{G}'_{n+1}}(g'_n,\mathrm{T})\mathrm{K}_{f,\chi}(h;g'_n,g'_{n+1})|$$
$$\ll_{\mathrm{N}}e^{-\mathrm{N}\|\mathrm{T}\|}\|h\|^{-\mathrm{N}}_{\mathrm{H}}\|g'_{n+1}\|^{-\mathrm{N}}_{\mathrm{G}'_{n+1}}\|g'_n\|^{-\mathrm{N}}_{\mathrm{G}'_{n+1}}$$

for $(h,g'_n,g'_{n+1})\in[\mathrm{H}]\times[\mathrm{G}'_n]\times[\mathrm{G}'_{n+1}]$. This inequality is precisely the content of Theorem 3.3.7.1 1. except that we still have to argue that the implicit constant can be taken to be a continuous semi-norm on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$. Using the uniform boundedness principle once again, it suffices to check that, for every $(h,g')\in[\mathrm{H}]\times[\mathrm{G}']$, the functional

$$f\in\mathcal{S}(\mathrm{G}(\mathbf{A}))\mapsto\mathrm{K}^{\mathrm{T}}_{f,\chi}(h,g')-\mathrm{F}^{\mathrm{G}'_{n+1}}(g'_n,\mathrm{T})\mathrm{K}_{f,\chi}(h,g')$$
$$=(\mathrm{L}^{\mathrm{T}}_{f,\chi}-\mathrm{P}^{\mathrm{T}}_{f,\chi})(\delta_h\otimes\delta_{g'_{n+1}})(g'_n)$$

is continuous. However, this just follows from the continuity of $f \mapsto R^{\Delta}_{\chi^\vee}(f^\vee)\delta_h \in \mathcal{T}^{\Delta}_{\mathcal{F}_{RS}}([G])$ which can be readily inferred from the continuity of right convolution as well as of the "$\chi^\vee$-projection" of Theorem 2.9.4.1.

3.7. *Proof of Proposition 3.3.8.1*

3.7.1. Set

$$\underline{\rho}_P = \rho_{P_{n+1}} - \rho_{P_n} \in \mathfrak{a}_{n+1}$$

for $P \in \mathcal{F}_{RS}$ and note that $\underline{\rho}_P = 0$ if and only if $P = G$.

3.7.2. By [Art81, §2], there exist functions $\Gamma'_{P_{n+1}}$ on $\mathfrak{a}^{G_{n+1}}_{P_{n+1}} \times \mathfrak{a}^{G_{n+1}}_{P_{n+1}}$, for $P \in \mathcal{F}_{RS}$, that are compactly supported in the first variable when the second variable stays in a compact and such that

$$\widehat{\tau}_{P_{n+1}}(H - X) = \sum_{P \subset Q \in \mathcal{F}_{RS}} \epsilon^G_Q \widehat{\tau}^{Q_{n+1}}_{P_{n+1}}(H)\Gamma'_{Q_{n+1}}(H, X) \tag{3.7.2.1}$$

for every $P \in \mathcal{F}_{RS}$ and $H, X \in \mathfrak{a}_{P_{n+1}}$. Then, by [Zyd20, lemme 3.5] for every $Q \in \mathcal{F}_{RS}$, the function

$$X \in \mathfrak{a}_{Q_{n+1}} \mapsto p_Q(X) := \int_{\mathfrak{a}^{G_{n+1}}_{Q_{n+1}}} e^{\underline{\rho}_Q(H)} \Gamma'_{Q_{n+1}}(H, X) dH$$

is an exponential-polynomial.

3.7.3. We set

$$\widetilde{f}_Q(m) = e^{-\langle \underline{\rho}_Q, H_{Q_{n+1}}(m_n)\rangle} p_Q(H_{Q_{n+1}}(m_n)) f_Q(m)$$

for $Q \in \mathcal{F}_{RS}$ and $m \in [M_Q]$ where

$$f_Q(m) = e^{\langle \rho_Q, H_Q(m)\rangle} \int_{K_H \times K'} \int_{N_Q(\mathbf{A})} f(k_H m u k')\eta_{G'}(k') du dk' dk_H.$$

We note that $\widetilde{f}_Q \in \mathcal{S}(M_Q(\mathbf{A}))$.

3.7.4. *Distributions attached to Levi subgroups.* — Let $Q \in \mathcal{F}_{RS}$. We now recall the variant of [Zyd20, Théorème 3.8] for the Levi M_Q. For $T \in \mathfrak{a}_{n+1}$ and $f' \in \mathcal{S}(M_Q(\mathbf{A}))$, we set

$$K^{M_Q,T}_{f',\chi}(m_H, m') = \sum_{Q \supset P \in \mathcal{F}_{RS}} \epsilon^Q_P \sum_{\substack{\gamma \in (P_H(F) \cap M_{Q_H}(F)) \backslash M_{Q_H}(F) \\ \delta \in (P'(F) \cap M_{Q'}(F)) \backslash M_{Q'}(F)}} \widehat{\tau}^{Q_{n+1}}_{P_{n+1}}(H_{P_{n+1}}(\delta_n m'_n) - T_{P_{n+1}}) K_{f', P \cap M_Q, \chi}(\gamma m_H, \delta m')$$

for $(m_H, m') \in [M_H] \times [M']$. Then, it can be shown in the same way as part 1. of Theorem 3.3.7.1 that, for T sufficiently positive, the following expression converges

$$(\mathbf{3.7.4.2}) \qquad I_\chi^{M_Q,T}(f') = \int_{A_{Q,H}^\infty \backslash [M_H] \times [M']} K_{f',\chi}^{M_Q,T}(m_H, m') \eta_{G'}(m') dm' dm_H,$$

where $A_{Q,H}^\infty = A_{Q_H}^\infty \cap A_Q^\infty$ is embedded diagonally in $M_H(\mathbf{A}) \times M'(\mathbf{A})$. Moreover, by the same proof as [Zyd20, Théorème 3.7] (see also [Zyd20, Remarque 3.8], [Zyd18, Remarque 4.9]), the function $T \mapsto I_\chi^{M_Q,T}(f')$ is an exponential-polynomial with exponents in the set $\{\underline{\rho}_P - \underline{\rho}_Q \mid Q \supset P \in \mathcal{F}_{RS}\}$ (but not necessarily with a constant purely polynomial part). Note that the composition of the embedding $\mathfrak{a}_{Q,H} = \mathfrak{a}_{Q_H} \cap \mathfrak{a}_Q \hookrightarrow \mathfrak{a}_{Q_{n+1}}$ with the projection $\mathfrak{a}_{Q_{n+1}} \twoheadrightarrow \mathfrak{a}_{Q_{n+1}}^{G_{n+1}}$ yields an isomorphism $\mathfrak{a}_{Q,H} \simeq \mathfrak{a}_{Q_{n+1}}^{G_{n+1}}$ whose Jacobian we denote by c_Q.

3.7.5. *Proof of Proposition 3.3.8.1.* — It follows from the next lemma, the fact that $I_\chi^T(\widetilde{f}_G) = I_\chi^T(f)$ and the shape of the exponents of $I_\chi^{M_Q,T}(\widetilde{f}_Q)$ recalled below.

Lemma **3.7.5.1.** — *For* $T \in \mathfrak{a}_{n+1}$ *sufficiently positive, we have*

$$(\mathbf{3.7.5.3}) \qquad i_\chi^T(f) = \sum_{Q \in \mathcal{F}_{RS}} c_Q e^{\underline{\rho}_Q(T)} I_\chi^{M_Q,T}(\widetilde{f}_Q).$$

Proof. — From the definition (3.3.6.8) of $\kappa_{f,\chi}^T$ together with the identity (3.7.2.1) applied to $H = H_{P_{n+1}}(\delta_n g_n') - T_{P_{n+1}}$ and $X = H_{P_{n+1}}(\delta_n g_n') - H_{P_{n+1}}(\gamma h)$, we obtain

$$\kappa_{f,\chi}^T(h, g') = \sum_{Q \in \mathcal{F}_{RS}} \sum_{\substack{\gamma \in Q_H(F) \backslash H(F) \\ \delta \in Q'(F) \backslash G'(F)}} \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(\delta_n g_n') - T_{Q_{n+1}}, H_{Q_{n+1}}(\delta_n g_n') - H_{Q_{n+1}}(\gamma h)) K_{f,\chi}^{Q,T}(\gamma h, \delta g')$$

for $(h, g') \in [H] \times [G']$, where we have set

$$K_{f,\chi}^{Q,T}(x, y) = \sum_{Q \supset P \in \mathcal{F}_{RS}} \epsilon_P^Q \sum_{\substack{\gamma \in P_H(F) \backslash Q_H(F) \\ \delta \in P'(F) \backslash Q'(F)}} \widehat{\tau}_{P_{n+1}}^{Q_{n+1}}(H_{P_{n+1}}(\delta_n y_n) - T_{P_{n+1}}) K_{f,P,\chi}(\gamma x, \delta y)$$

for $(x, y) \in [H]_{Q_H} \times [G']_{Q'}$. It follows that

$$i_\chi^T(f) = \sum_{Q \in \mathcal{F}_{RS}} \int_{[H]_{Q_H} \times [G']_{Q'}} K_{f,\chi}^{Q,T}(h, g') \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(g_n') - T_{Q_{n+1}}, H_{Q_{n+1}}(g_n') - H_{Q_{n+1}}(h)) \eta_{G'}(g') dg' dh.$$

By the Iwasawa decompositions $H(\mathbf{A}) = Q_H(\mathbf{A})K_H$ and $G'(\mathbf{A}) = Q'(\mathbf{A})K'$, the summand corresponding to Q can be rewritten as

$$\int_{[M_{Q_H}]\times[M_{Q'}]}\int_{K_H\times K'} K^{Q,T}_{f,\chi}(m_Hk_H, m'k')\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(m'_n) - T_{Q_{n+1}}, \\ H_{Q_{n+1}}(m'_n) - H_{Q_{n+1}}(m_H)) \\ \eta_{G'}(m'k')e^{-2\langle\rho_{Q'},H_Q(m')\rangle}e^{-2\langle\rho_{Q_H},H_{Q_H}(m_H)\rangle}dk'dk_Hdm'dm_H.$$

We readily check that, for all $(m_H, m') \in [M_H] \times [M']$, we have

$$\int_{K_H\times K'} K^{Q,T}_{f,\chi}(m_Hk_H, m'k')\eta_{G'}(k')dk'dk_H = e^{\langle\rho_Q,H_Q(m')+H_Q(m_H)\rangle}K^{M_Q,T}_{f_Q,\chi}(m_H, m').$$

Using that $\rho_Q = 2\rho_{Q'}$ and $\langle\rho_Q, H_Q(m_H)\rangle - 2\langle\rho_{Q_H}, H_{Q_H}(m_H)\rangle = \langle\underline{\rho}_Q, H_{Q_{n+1}}(m_H)\rangle$, we finally obtain

$$\begin{aligned} i^T_\chi(f) = \sum_{Q\in\mathcal{F}_{RS}} &\int_{[M_{Q_H}]\times[M_{Q'}]} K^{M_Q,T}_{f_Q,\chi}(m_H, m')\eta_{G'}(m')e^{\langle\underline{\rho}_Q,H_{Q_{n+1}}(m_H)\rangle} \\ &\times \Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(m'_n) - T_{Q_{n+1}}, H_{Q_{n+1}}(m'_n) - H_{Q_{n+1}}(m_H))dm'dm_H \\ &= \sum_{Q\in\mathcal{F}_{RS}}\int_{A^\infty_{Q,H}\backslash[M_{Q_H}]\times[M_{Q'}]} K^{M_Q,T}_{f_Q,\chi}(m_H, m')\eta_{G'}(m') \\ &\times e^{\langle\underline{\rho}_Q,H_{Q_{n+1}}(m_H)-H_{Q_{n+1}}(m'_n)\rangle} \\ &\times \int_{A^\infty_{Q,H}} e^{\langle\underline{\rho}_Q,H_{Q_{n+1}}(am'_n)\rangle}\Gamma'_{Q_{n+1}}(H_{Q_{n+1}}(am'_n) - T_{Q_{n+1}}, \\ &H_{Q_{n+1}}(m'_n) - H_{Q_{n+1}}(m_H))da\,dm'dm_H \\ &= \sum_{Q\in\mathcal{F}_{RS}} c_Qe^{\underline{\rho}_Q(T)}\int_{A^\infty_{Q,H}\backslash[M_{Q_H}]\times[M_{Q'}]} K^{M_Q,T}_{f_Q,\chi}(m_H, m')\eta_{G'}(m') \\ &\times e^{\langle\underline{\rho}_Q,H_{Q_{n+1}}(m_H)-H_{Q_{n+1}}(m'_n)\rangle} \\ &\times p_Q(H_{Q_{n+1}}(m'_n) - H_{Q_{n+1}}(m_H))dm'dm_H. \end{aligned}$$

To conclude, it suffices to remark that

$$\begin{aligned} &K^{M_Q,T}_{f_Q,\chi}(m_H, m')e^{\langle\underline{\rho}_Q,H_{Q_{n+1}}(m_H)-H_{Q_{n+1}}(m'_n)\rangle}p_Q(H_{Q_{n+1}}(m'_n) - H_{Q_{n+1}}(m_H)) \\ &\quad = K^{M_Q,T}_{\tilde{f}_Q,\chi}(m_H, m') \end{aligned}$$

for $(m_H, m') \in [M_H] \times [M']$. □

4. Flicker-Rallis period of some spectral kernels

The goal of this section is to get the spectral expansion of the Flicker-Rallis integral of the automorphic kernel attached to a linear group and a specific cuspidal datum (called in Section 4.3.2 $*$-regular). This is achieved in Theorem 4.3.3.1. It turns out that the decomposition is discrete and is expressed in terms of some relative characters.

4.1. *Flicker-Rallis intertwining periods and related distributions*

4.1.1. *Notations.* — In all this section, we will fix an integer $n \geqslant 1$ and we will use notations of Sections 3.1.1 to 3.1.3. Since n will be fixed, we will drop the subscript n from the notation: $G = G_n$, $B = B_n$ etc. So we do not follow notations of Section 3.1.5: we hope that it will cause no confusion.

4.1.2. *Flicker-Rallis periods.* — Let π be a cuspidal automorphic representation of $G(\mathbf{A})$ with central character trivial on A_G^∞. We shall denote by π^* the conjugate-dual representation of $G(\mathbf{A})$. We shall say that π is self conjugate-dual if $\pi \simeq \pi^*$ and that π is G'-distinguished, resp. (G', η)-distinguished, if the linear form (called the Flicker-Rallis period)

$$(\mathbf{4.1.2.1}) \qquad \varphi \mapsto \int_{[G']_0} \varphi(h)\, dh, \text{ resp. } \int_{[G']_0} \varphi(h)\eta(\det(h))\, dh$$

does not vanish identically on $\mathcal{A}_\pi(G)$. Then π is self conjugate-dual if and only if π is either G'-distinguished or (G', η)-distinguished. However it cannot be both. This is related to the well-known factorisation of the Rankin-Selberg factorisation $L(s, \pi \times \pi \circ c)$ where c is the Galois involution of $G(\mathbf{A})$ in terms of Asai L-functions and to the fact that the residue at $s = 1$ of the Asai L-functions is expressed in terms of Flicker-Rallis periods (see [Fli88]).

4.1.3. In this section, we will focus on the period in (4.1.2.1) related to distinction. However it is clear that all the results hold *mutatis mutandis* for the period related to η-distinction.

4.1.4. Let $P = MN_P$ be a standard parabolic subgroup (with its standard decomposition). Let π be an irreducible cuspidal automorphic representation of M with central character trivial on A_M^∞.

It will be convenient to write $M = G_{n_1} \times \cdots \times G_{n_r}$ with $n_1 + \cdots + n_r = n$. Accordingly we have $\pi = \sigma_1 \boxtimes \ldots \boxtimes \sigma_r$ where σ_i is an irreducible cuspidal representation of G_{n_i}.

4.1.5. Let $\varphi \in \mathcal{A}_{P,\pi}(G)$. The parabolic subgroup $P' = P \cap G'$ of G' has the following Levi decomposition $M'N_{P'}$ where $M' = M \cap G'$. We then define the following integral which is a specific example of a Flicker-Rallis intertwining period introduced by Jacquet-Lapid-Rogawski (see [JLR99] section VII, note that our definition of $\mathcal{A}_{P,\pi}(G)$ is slightly different from theirs),

$$J(\varphi) = \int_{A_{M'}^{\infty} M'(F) N_{P'}(\mathbf{A}) \backslash G'(\mathbf{A})} \varphi(g)\, dg$$

Clearly we get a $G'(\mathbf{A})$-invariant continuous linear form on $\mathcal{A}_{P,\pi}(G)$. Note that J does not vanish identically if and only if each component σ_i is $G'_{n_i}(F)$-distinguished. In this case, we have $\pi = \pi^*$.

4.1.6. Let $Q \in \mathcal{P}(M)$. As recalled in Section 2.2.11, there is a unique pair (Q', w) such that the conditions are satisfied:

- $Q' = wQw^{-1}$ is the standard parabolic subgroup in the G-conjugacy class of Q;
- $w \in W(P; Q')$.

Let $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*}$. We have $M(w,\lambda)\varphi \in \mathcal{A}_{M_{Q'},w\pi}(G)$ if λ is outside the singular hyperplanes of the intertwining operator. We shall define

$$J_Q(\varphi, \lambda) = J(M(w, \lambda)\varphi) \tag{4.1.6.2}$$

as a meromorphic function of λ.

4.1.7. Let $g \in G(\mathbf{A})$. Let's define for $\varphi, \psi \in \mathcal{A}_{P,\pi}(G)$

$$B_Q(g, \varphi, \psi, \lambda) = E(g, \varphi, \lambda) \cdot J_Q(\bar{\psi}, -\lambda), \tag{4.1.7.3}$$

as a meromorphic function of $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*}$. In fact, by the basic properties of Eisenstein series and intertwining operators, there exists an open subset $\omega_\pi \subset \mathfrak{a}_{P,\mathbf{C}}^{G,*}$ which is the complement of a union of hyperplanes of $\mathfrak{a}_{P,\mathbf{C}}^{G,*}$ such that:

- ω_π contains $i\mathfrak{a}_P^{G,*}$.
- for all $\varphi, \psi \in \mathcal{A}_{P,\pi}(G)$, the map $\lambda \mapsto B_Q(g, \varphi, \psi, \lambda)$ is holomorphic on ω_π and gives for each $\lambda \in \omega_\pi$ a continuous sesquilinear form in φ and ψ.

4.1.8. Let $f \in \mathcal{S}(G(\mathbf{A}))$, $g \in G(\mathbf{A})$ and $Q \in \mathcal{P}(M)$. Let's introduce the distribution

$$J_{Q,\pi}(g, \lambda, f) = \sum_{\varphi \in \mathcal{B}_{P,\pi}} B_Q(g, I_P(\lambda, f)\varphi, \varphi, \lambda) \tag{4.1.8.4}$$

where $\mathcal{B}_{P,\pi}$ is a K-basis of $\mathcal{A}_{P,\pi}(G)$ (see Section 2.8.3 and $\lambda \in \omega_\pi$ (see Section 4.1.7 for the notation ω_π). It follows from Proposition 2.8.4.1 that $J_{Q,\pi}(g,\lambda)$ is a continuous distribution on $\mathcal{S}(G(\mathbf{A}))$.

4.1.9. *A* (G, M)*-family.*

Proposition **4.1.9.1.** — *The family* $(J_{Q,\pi}(g,\lambda,f))_{Q\in\mathcal{P}(M)}$ *is a* (G, M)*-family in the sense of Arthur (see [Art81]): namely each map*

$$\lambda \in \mathfrak{a}_M^{G,*} \mapsto J_{Q,\pi}(g,\lambda,f)$$

is smooth on $i\mathfrak{a}_M^{G,*}$ *(and even holomorphic on* ω_π*) and for adjacent elements* $Q_1, Q_2 \in \mathcal{P}(M)$ *we have*

$$J_{Q_1,\pi}(g,\lambda,f) = J_{Q_2,\pi}(g,\lambda,f) \tag{4.1.9.5}$$

on the hyperplane of $i\mathfrak{a}_M^{G,*}$ *defined by* $\langle\lambda,\alpha^\vee\rangle = 0$ *where* α *is the unique element in* $\Delta_{Q_1}\cap(-\Delta_{Q_2})$.

Proof. — Let's prove the holomorphy of $J_{Q,\pi}(g,\lambda,f)$ on ω_π. Let $C \subset G(\mathbf{A}_f)$ be a compact subset and let $K_0 \subset K^\infty$ be a compact-open normal subgroup such that $f \in \mathcal{S}(G(\mathbf{A}),C,K_0)$. Let $\omega \subset \omega_\pi$ be a compact subset with a non-empty interior denoted by ω°. According to Proposition 2.8.4.1, there is a continuous semi-norm on $\mathcal{S}(G(\mathbf{A}),C,K_0)$ such that for $\lambda\in\omega$ and any $\phi \in \mathcal{S}(G(\mathbf{A}),C,K_0)$ we have:

$$|J_{Q,\pi}(g,\lambda,\phi)| \leqslant \|\phi\|. \tag{4.1.9.6}$$

On the other hand, we have $J_{Q,\pi}(g,\lambda,f) = \sum_{\tau\in\hat{K}} J_{Q,\pi}(g,\lambda,f_\tau)$ where $f_\tau = e_\tau * f * e_\tau$ as in the proof of Lemma 2.10.2.1. Indeed we have

$$J_{Q,\pi}(g,\lambda,f_\tau) = \sum_{\varphi\in\mathcal{B}_{P,\pi,\tau}} B_Q(g, I_P(\lambda,f)\varphi,\varphi,\lambda).$$

Since the sum on the right-hand side is finite, the map $\lambda\mapsto J_{Q,\pi}(g,\lambda,f_\tau)$ is holomorphic on ω_π. Using (4.1.9.6), we get for all $\lambda\in\omega$

$$\sum_{\tau\in\hat{K}} |J_{Q,\pi}(g,\lambda,f_\tau)| \leqslant \sum_{\tau\in\hat{K}} \|f_\tau\| < \infty.$$

Thus on ω°, we observe that $J_{Q,\pi}(g,\lambda,f)$ is a normally convergent series whose general term, namely $J_{Q,\pi}(g,\lambda,f_\tau)$, is holomorphic. In this way $\lambda\mapsto J_{Q,\pi}(g,\lambda,f)$ itself is holomorphic on ω° and on ω_π.

Then let's prove the second condition. Let $Q_1, Q_2 \in \mathcal{P}(M)$ be such that $\Delta_{Q_1}\cap(-\Delta_{Q_2})$ is a singleton $\{\alpha\}$. Let $\lambda \in i\mathfrak{a}_P^{G,*}$ such $\langle\lambda,\alpha^\vee\rangle=0$. For $i=1,2$ let Q_i' be a standard parabolic subgroup and $w_i \in W(M,Q_i')$ be such that $Q_i' = w_iQ_iw_i^{-1}$. Let $\beta = w_1\alpha \in \Delta_{Q_1'}$

and let s_β the "elementary symmetry" associated to β. Then we have $w_2 = s_\beta w_1$. Let $\varphi, \psi \in \mathcal{A}_{P,\pi}(G)$. Clearly it suffices to check the equality:

$$E(g,\varphi,\lambda)\cdot \overline{(J(M(w_1,\lambda)\psi)} = E(g,\varphi,\lambda)\cdot \overline{J(M(w_2,\lambda)\psi)}.$$

Using the functional equations of intertwining operators and Eisentein series, we have $M(w_2,\lambda) = M(s_\beta, w_1\lambda)M(w_1,\lambda)$ and $E(g,\varphi,\lambda) = E(g, M(w_1,\lambda)\varphi, w_1\lambda)$. Thus up to a change of notations (replace P by Q_1'), we may assume that $Q_1 = P$ and thus $w_1 = 1$ and $\alpha = \beta$. We are reduced to prove

$$E(g,\varphi,\lambda)\cdot \overline{J(\psi)} = E(g,\varphi,\lambda)\cdot \overline{J(M(s_\alpha,\lambda)\psi)} \tag{4.1.9.7}$$

on the hyperplane $\langle\lambda,\alpha^\vee\rangle = 0$. The symmetry s_α acts on M as a transposition of two consecutive blocks of M say G_{n_i} and $G_{n_{i+1}}$. Note that $M(s_\alpha,\lambda) = M(s_\alpha,0)$. Then we have even a stronger property:

$$J(\psi) = J(M(s_\alpha,0)\psi)$$

if $n_i \neq n_{i+1}$ or if $n_i = n_{i+1}$ but $\sigma_i \not\simeq \sigma_{i+1}^*$ (see lemma 8.1 case 1 of [Lap06]). Assume that $n_i = n_{i+1}$ and $\sigma_i \simeq \sigma_{i+1}^*$. The case where $\sigma_i \not\simeq \sigma_i^*$ is trivial (J is zero) so we shall also assume that $\sigma_i \simeq \sigma_i^*$. Then $M(s_\alpha,0)\psi = -\psi$ ([KS88] proposition 6.3) and since $s_\alpha(\lambda) = \lambda$ we have $E(g,\varphi,\lambda) = 0$ so (4.1.9.7) is clear. □

4.1.10. *Majorization.* — We will use the following proposition which results from Lapid's majorization of Eisenstein series (see [Lap06] proposition 6.1 and section 7). For the convenience of the reader, we sketch a proof.

Proposition **4.1.10.1.** — *Let $f \in \mathcal{S}(G(\mathbf{A}))$. The map $\lambda \mapsto J_{Q,\pi}(g,\lambda,f)$ belongs to the Schwartz space $\mathcal{S}(i\mathfrak{a}_P^{G,*})$. Moreover*

$$f \mapsto J_{Q,\pi}(g,\cdot,f)$$

is a continuous map from $\mathcal{S}(G(\mathbf{A}))$ to $\mathcal{S}(i\mathfrak{a}_P^{G,*})$ equipped with its usual topology.

Proof. — Let $C \in G(\mathbf{A}_f)$ be a compact subset and $K_0 \subset K^\infty$ be an open-compact subgroup such that $f \in \mathcal{S}(G(\mathbf{A}), C, K_0)$. For any $\alpha,\beta > 0$, we define an open subset $\omega_{\alpha,\beta}$ of $\mathfrak{a}_{P,\mathbf{C}}^{G,*}$ which contains $i\mathfrak{a}_{P,\mathbf{R}}^{G,*}$ by

$$\omega_{\alpha,\beta} = \{\lambda \in \mathfrak{a}_{P,\mathbf{C}}^{G,*} \mid \|\Re(\lambda)\| < \alpha(1+\|\Im(\lambda)\|)^{-\beta}\}.$$

By the arguments in the proof of proposition 6.1 of [Lap06], one sees that there exist $\alpha,\beta > 0$ such that $\omega_{\alpha,\beta}$ is included in the open set ω_π of Section 4.1.7. In particular, $\lambda \mapsto J_{Q,\pi}(g,\lambda,f)$ is holomorphic on $\omega_{\alpha,\beta}$. Using Cauchy formula to control derivatives, it suffices to prove the following majorization: there exists a continuous semi-norm $\|\cdot\|$

on $\mathcal{S}(G(\mathbf{A}), C, K_0)$ and an open subset $\omega_{\alpha,\beta} \subset \omega_\pi$ such that for any integer $N \geqslant 1$ there exists $c > 0$ so that for all $f \in \mathcal{S}(G(\mathbf{A}), C, K_0)$ and all $\lambda \in \omega_{\alpha,\beta}$

$$|J_{Q,\pi}(g,\lambda,f)| \leqslant c\frac{\|f\|}{(1+\|\lambda\|)^N}. \tag{4.1.10.8}$$

Let $m \geqslant 1$ be a large enough integer. Following the notations of Proposition 2.8.4.3, we can write $f = f * g_1 + (f * Z) * g_2$; we get

$$J_{Q,\pi}(g,\lambda,f) = \sum_{\varphi\in\mathcal{B}_{P,\pi}} E(g, I_P(\lambda,f)\varphi,\lambda)\overline{J_Q(I_P(-\lambda,g_1^\vee)\varphi,\lambda)}$$
$$+ \sum_{\varphi\in\mathcal{B}_{P,\pi}} E(g, I_P(\lambda,f*Z)\varphi,\lambda)\overline{J_Q(I_P(-\lambda,g_2^\vee)\varphi,\lambda)}$$

By a slight extension to Schwartz functions of Lapid's majorization (see [FLO12] remark C.2 about [Lap06] proposition 6.1), the expression

$$\Big(\sum_{\varphi\in\mathcal{B}_{P,\pi}} |E(g, I_P(\lambda,f)\varphi,\lambda)|^2\Big)^{1/2}$$

and the same expression where f is replaced by $f * Z$ satisfy a bound like (4.1.10.8). Using Cauchy-Schwartz inequality, we are reduced to bound in λ (recall that g_i is independent of f)

$$\Big(\sum_{\varphi\in\mathcal{B}_{P,\pi}} |J_Q(I_P(-\lambda,g_i^\vee)\varphi,\lambda)|^2\Big)^{1/2}. \tag{4.1.10.9}$$

Let w be such that wQw^{-1} is standard and $w \in W(P, wQw^{-1})$. At this point we will use the notations of the proof of Proposition 2.8.4.1. There exists $c > 0$ and an integer r such that for all $\varphi' \in \mathcal{A}_{P,\pi}(G)^{K_0}$ we have

$$|J_Q(I_P(-\lambda,g_i^\vee)\varphi,\lambda)| = |J(M(w,\lambda)I_P(-\lambda,g_i^\vee)\varphi)|$$
$$\leqslant c\|M(w,\lambda)I_P(-\lambda,g_i^\vee)\varphi\|_r$$

where $\|\varphi\|_r = \|R(1+C_K)^r\varphi\|_{Pet}$. Then we need to bound the operator norm of the intertwining operator $M(w,\lambda)$. Using the normalization of intertwining operators, the bounds of normalizing factors [Lap06] lemma 5.1 and Müller-Speh's bound on the norm of normalized intertwining operators (see [MS04] proposition 4.2 and the proof of proposition 0.2), we get $c_1 > 0$, $N \in \mathbf{N}$ and $\alpha, \beta > 0$ such that for all $\tau \in \hat{K}_\infty$, $\lambda \in \omega_{\alpha,\beta}$ and $\varphi \in \mathcal{A}_{P,\pi}(G, K_0, \tau)$ we have

$$\|M(w,\lambda)I_P(-\lambda,g_i^\vee)\varphi\|_{K_0,r} \leqslant c_1(1+\lambda_\tau)^N\|I_P(-\lambda,g_i^\vee)\varphi\|_r.$$

Using the same kind of arguments as in the proof of Proposition 2.8.4.1 (see also remark (2.8.4.2)), one shows that there exist $\alpha, \beta > 0$ such that (4.1.10.9) is bounded independently of $\lambda \in \omega_{\alpha,\beta}$. □

4.2. *A spectral expansion of a truncated integral*

4.2.1. Let $\chi \in \mathfrak{X}(\mathrm{G})$ be a cuspidal datum. We shall use the notation of Section 2.10.2. In particular, $f \in \mathcal{S}(\mathrm{G}(\mathbf{A}))$ and K_χ^0 is the attached kernel.

4.2.2. Let's consider a parameter T as in Section 2.2.12. Following Jacquet-Lapid-Rogawski (see [JLR99]), we introduce the truncation operator Λ_m^{T} that associates to a function φ on [G] the following function of the variable $h \in [\mathrm{G}']$:

$$(\Lambda_m^{\mathrm{T}}\varphi)(h) = \sum_{\mathrm{P}} (-1)^{\dim(\mathfrak{a}_{\mathrm{P}}^{\mathrm{G}})} \sum_{\delta \in \mathrm{P}'(\mathrm{F})\backslash \mathrm{G}'(\mathrm{F})} \hat{\tau}_{\mathrm{P}}(\mathrm{H}_{\mathrm{P}}(\delta h) - \mathrm{T}_{\mathrm{P}})\varphi_{\mathrm{P}}(\delta h) \tag{4.2.2.1}$$

where the sum is over standard parabolic subgroup of G (those containing B) and φ_{P} is the constant term along P. Recall that $\mathrm{P}' = \mathrm{G}' \cap \mathrm{P}$.

4.2.3. We shall define the mixed truncated kernel $\mathrm{K}_\chi^0 \Lambda_m^{\mathrm{T}}$: the notation means that the mixed truncation is applied to the second variable. This is a function on $\mathrm{G}(\mathbf{A}) \times \mathrm{G}'(\mathbf{A})$. To begin with we have:

Lemma **4.2.3.1.** — *For* $(x, y) \in \mathrm{G}(\mathbf{A}) \times \mathrm{G}'(\mathbf{A})$, *we have:*

$$\begin{aligned}(\mathrm{K}_\chi^0 \Lambda_m^{\mathrm{T}})(x,y) \\ = \sum_{\mathrm{B}\subset \mathrm{P}} |\mathcal{P}(\mathrm{M}_{\mathrm{P}})|^{-1} \int_{i\mathfrak{a}_{\mathrm{P}}^{\mathrm{G},*}} \sum_{\varphi \in \mathcal{B}_{\mathrm{P},\chi}} \mathrm{E}(x, \mathrm{I}_{\mathrm{P}}(\lambda, f)\varphi, \lambda)\overline{\Lambda_m^{\mathrm{T}}\mathrm{E}(y,\varphi,\lambda)}\, d\lambda.\end{aligned}$$

Proof. — As $y \in \mathrm{G}'(\mathbf{A})$, the mixed truncation is defined by a finite sum of constant terms of $\mathrm{K}_\chi^0(x, \cdot)$ (in the second variable). The only point is to permute the sum over φ and the operator Λ_m^{T}. In fact using the continuity properties of Eisenstein series (see [Lap08, theorem 2.2]) and properties of mixed truncation operator (in particular a variant of lemma 1.4 of [Art80]), we can conclude as in the proof of Proposition 2.8.4.1. □

Lemma **4.2.3.2.** — *For any integer* N, *there exists a continuous semi-norm* $\|\cdot\|$ *on* $\mathcal{S}(\mathrm{G}(\mathbf{A}))$ *and an integer* N′ *such that for all* $\mathrm{X} \in \mathcal{U}(\mathfrak{g}_\infty)$, *all* $x \in \mathrm{G}(\mathbf{A})^1$ *and* $y \in \mathrm{G}'(\mathbf{A})^1$, *all* $f \in \mathcal{S}(\mathrm{G}(\mathbf{A}))$

we have

(**4.2.3.2**)
$$\sum_{\chi\in\mathfrak{X}(G)}\sum_{B\subset P}|\mathcal{P}(M_P)|^{-1} \times \int_{i\mathfrak{a}_P^{G,*}}\sum_{\tau\in\hat{K}}\Big|\sum_{\varphi\in\mathcal{B}_{P,\chi,\tau}}(R(X)E)(x,I_P(\lambda,f)\varphi,\lambda)\overline{\Lambda_m^T E(y,\varphi,\lambda)}\Big|\,d\lambda \leqslant \|L(X)f\|\,\|x\|_{[G]}^{N'}\|y\|_{[G]}^{-N}.$$

Proof. — By the basic properties of the mixed truncation operator (see lemma 1.4 of [Art80] and also [LR03] proof of lemma 8.2.1), for any $N, N' > 0$ there exists a finite family $(Y_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ such the expression (4.2.3.2) is majorized by the sum over $i\in I$ of $\|y\|_{[G]}^{-N}$ times the supremum over $g\in G'(\mathbf{A})^1$ of

$$\|g\|_{[G]}^{-N'}\sum_{\chi\in\mathfrak{X}}\sum_{B\subset P}|\mathcal{P}(M_P)|^{-1}\int_{i\mathfrak{a}_P^{G,*}}\sum_{\tau\in\hat{K}}\Big|\sum_{\varphi\in\mathcal{B}_{P,\chi,\tau}}(R(X)E)(x,I_P(\lambda,f)\varphi,\lambda) \times \overline{R(Y_i)E(g,\varphi,\lambda)}\Big|\,d\lambda.$$

Then the lemma is a straightforward consequence of Lemma 2.10.2.1. □

Proposition **4.2.3.3.** — *For all* $x\in G(\mathbf{A})$ *and* $\chi\in\mathfrak{X}(G)$, *we have*

$$\int_{[G']_0}(K_\chi^0\Lambda_m^T)(x,y)\,dy = \sum_{B\subset P}|\mathcal{P}(M_P)|^{-1}\int_{i\mathfrak{a}_P^{G,*}}\sum_{\varphi\in\mathcal{B}_{P,\chi}}E(x,I_P(\lambda,f)\varphi,\lambda) \times \overline{\int_{[G']_0}\Lambda_m^T E(y,\varphi,\lambda)\,dy}\,d\lambda.$$

Proof. — First one decomposes the sum over $\mathcal{B}_{P,\chi}$ as a sum over $\tau\in\hat{K}$ of finite sums over $\mathcal{B}_{P,\chi,\tau}$. Then, by the majorization of Lemma 4.2.3.2 we can permute the integration over $[G']_0$ (which amounts to integrating over $[G']^1$) and the other sums or integrations in the expression we get in Lemma 4.2.3.1. □

4.3. *The case of ∗-regular cuspidal data*

4.3.1. We shall use the notations of Section 4.2.

4.3.2. *∗-Regular cuspidal datum.* — We shall say that a cuspidal datum $\chi\in\mathfrak{X}(G)$ is ∗-regular if for any representative (M,π) of χ and $w\in W(M)$ such that $w\pi$ is isomorphic to π or π^* we have $w=1$. Let's denote by $\mathfrak{X}^*(G)$ the subset of ∗-regular cuspidal data.

With the notations of 4.1.4, we see that (M, π) is $*$-regular if and only if for all $1 \leqslant i, j \leqslant r$ such that $n_i = n_j$ one of the equalities $\sigma_i = \sigma_j$ or $\sigma_i = \sigma_j^*$ implies that $i = j$.

4.3.3. The next theorem is the main result of the section.

Theorem **4.3.3.1.** — *Let $f \in \mathcal{S}(G(\mathbf{A}))$, let $\chi \in \mathfrak{X}(G)$ and let K_χ be the associated kernel. For any $g \in G(\mathbf{A})$, one has:*

1. *We have*

$$\int_{[G']} K_\chi(g, h)\, dh = \frac{1}{2} \int_{[G']_0} K_\chi^0(g, h)\, dh \tag{4.3.3.1}$$

where both integrals are absolutely convergent.

2. *If moreover $\chi \in \mathfrak{X}^*(G)$, we have, for any representative (M_P, π) of χ (where P is a standard parabolic subgroup of G),*

$$\int_{[G']} K_\chi(g, h)\, dh = 2^{-\dim(\mathfrak{a}_P)} J_{P,\pi}(g, f)$$

where one defines (see (4.1.8.4))

$$J_{P,\pi}(g, f) = J_{P,\pi}(g, 0, f).$$

In particular, the integral vanishes unless π is self conjugate dual and $M_{P'}$-distinguished where $P' = G' \cap P$.

The assertion 1 follows readily from Lemma 2.10.1.1, Fubini's theorem and the fact that the Haar measure on A_G^∞ is twice the Haar measure on $A_{G'}^\infty$ (see Remark 3.1.3.1). The rest of the section is devoted to the proof of assertion 2 of Theorem 4.3.3.1. The main steps are Propositions 4.3.4.1 and 4.3.7.1.

4.3.4. *A limit formula.* — We shall use the notation $\lim_{T \to +\infty} f(T)$ to denote the limit of $f(T)$ when $\langle \alpha, T \rangle \to +\infty$ for all $\alpha \in \Delta_B$.

Proposition **4.3.4.1.** — *Under the assumptions of Theorem 4.3.3.1 (but with no regularity condition on χ), we have*

$$\lim_{T \to +\infty} \int_{[G']_0} (K_\chi^0 \Lambda_m^T)(g, h)\, dh = 2 \int_{[G']} K_\chi(g, h)\, dh.$$

Proof. — Let's denote $F^{G'}(\cdot, T)$ the function defined by Arthur relative to G' and its maximal compact subgroup K' (see [Art78, §6] and [Art85, lemma 2.1]). It is the characteristic function of a compact of $[G']_0$. Using the fact that $h \mapsto K_\chi^0(g, h)$ is of uniform moderate growth (see Lemma 2.10.1.1), we can conclude by a variant of [Art85] theorem 3.1 (see also in the same spirit [IY15] proposition 3.8) that

$$\lim_{T\to+\infty}\int_{[G']_0}\left(F^{G'}(h,T)K_\chi^0(g,h)-(K_\chi^0\Lambda_m^T)(g,h)\right)dh=0.$$

We have $\lim_{T\to+\infty}F^{G'}(h,T)=1$. Thus we deduce by Lebesgue's theorem and the absolute convergence of the right-hand side of (4.3.3.1).

$$\lim_{T\to+\infty}\int_{[G']_0}F^{G'}(h,T)K_\chi^0(g,h)\,dh=\int_{[G']_0}K_\chi^0(g,h)\,dh$$

The proposition follows by (4.3.3.1). □

4.3.5. Let $\chi\in\mathfrak{X}^*(G)$. Let $\mathcal{P}_\chi$ be the set of standard parabolic subgroups such that there exists a cuspidal automorphic representation π of M_P such that (M_P,π) in the equivalence class defined by χ. In Section 2.10.2, we defined the space $\mathcal{A}^0_{P,\chi,\mathrm{disc}}(G)$. Since $\chi\in\mathfrak{X}^*(G)$, it is non-zero only if $P\in\mathcal{P}_\chi$. Let P be a standard parabolic subgroup and let (M_P,π) be a pair in χ. For any $P_1\in\mathcal{P}_\chi$, by multiplicity-one theorem, we have

$$\mathcal{A}^0_{P_1,\chi,\mathrm{disc}}(G)=\bigoplus_{w\in W(P,P_1)}\mathcal{A}_{P_1,w\pi}(G).$$

In the following we set $M_1=M_{P_1}$.

Let $P_1\in\mathcal{P}_\chi$ and $g\in G(\mathbf{A})$. With the notations of Section 4.1 (see eq. (4.1.8.4)), for all $Q\in\mathcal{P}(M_1)$, all $\lambda\in i\mathfrak{a}^{G,*}_{P_1,\mathbf{R}}$ we define

$$J_{Q,\chi}(g,\lambda,f)=\sum_{w\in W(P,P_1)}J_{Q,w\pi}(g,\lambda,f).$$

It's a continuous linear form on $\mathcal{S}(G(\mathbf{A}))$.

4.3.6.

Proposition **4.3.6.1.** — *For all* $\chi\in\mathfrak{X}^*(G)$ *and all* $g\in G(\mathbf{A})$, *we have*

(**4.3.6.2**)
$$\int_{[G']_0}(K_\chi^0\Lambda_m^T)(g,h)\,dh$$
$$=\frac{2^{-\dim(\mathfrak{a}_P^G)}}{|\mathcal{P}(M_P)|}\sum_{P_1\in\mathcal{P}_\chi}\int_{i\mathfrak{a}^{G,*}_{P_1}}\sum_{Q\in\mathcal{P}(M_1)}J_{Q,\chi}(g,\lambda,f)\frac{\exp(-\langle\lambda,T_Q\rangle)}{\theta_Q(-\lambda)}\,d\lambda.$$

Proof. — This is an obvious consequence of the definitions, Proposition 4.2.3.3 and Lemma 4.3.6.2 below. □

Lemma **4.3.6.2.** — *Let* $\chi \in \mathfrak{X}^*(G)$. *Let* $P_1 \in \mathcal{P}_\chi$ *and* $\varphi \in \mathcal{A}_{P_1,\chi}$. *We have for all* $\lambda \in i\mathfrak{a}_{P_1}^G$

$$\int_{[G']_0} \Lambda_m^T E(y, \varphi, \lambda)\, dy = 2^{-\dim(\mathfrak{a}_{P_1}^G)} \sum_{Q \in \mathcal{P}(M_1)} J_Q(\varphi, \lambda) \frac{\exp(\langle \lambda, T_Q \rangle)}{\theta_Q(\lambda)}.$$

Proof. — This is simply a rephrasing in our particular situation of a key result of Jacquet-Lapid-Rogawski (see [JLR99] theorem 40). Indeed, because χ is $*$-regular, Theorem 40 of *ibid.* can be stated as:

$$\int_{[G']_0} \Lambda_m^T E(y, \varphi, \lambda)\, dy = 2^{-\dim(\mathfrak{a}_{P_1}^G)} \sum_{(Q,w)} J(M(w, \lambda)\varphi) \frac{\exp(\langle (w\lambda)_Q, T \rangle)}{\theta_Q(w\lambda)}$$

where the sum is over pair (Q, w) where Q is a standard parabolic subgroup and $w \in W(P_1, Q)$. □

4.3.7.

Proposition **4.3.7.1.** — *Let* $\chi \in \mathfrak{X}^*(G)$ *and let* (M_P, π) *be a representative where* P *is a standard parabolic subgroup of* G. *We have:*

$$\lim_{T \to +\infty} \int_{[G']} (K_\chi^0 \Lambda_m^T)(g, h)\, dh = 2^{-\dim(\mathfrak{a}_P^G)} J_{P,\pi}(g, f)$$

where one defines

$$J_{P,\pi}(g, f) = J_{P,\pi}(g, 0, f). \tag{4.3.7.3}$$

Proof. — We start from the expansion (4.3.6.2) of Proposition 4.3.6.1. For each $P_1 \in \mathcal{P}_\chi$, let $M_1 = M_{P_1}$. The family $(J_{Q,\chi}(g, \lambda, f))_{Q \in \mathcal{P}(M_1)}$ is a (G, M_1)-family of Schwartz functions on $i\mathfrak{a}_{P_1}^{G,*}$: this is a straightforward consequence of Propositions 4.1.9.1 and 4.1.10.1. By [Lap11] Lemma 8, we have:

$$\lim_{T \to +\infty} \int_{i\mathfrak{a}_{P_1}^{G,*}} \sum_{Q \in \mathcal{P}(M_1)} J_{Q,\chi}(g, \lambda, f) \frac{\exp(-\langle \lambda, T_Q \rangle)}{\theta_Q(-\lambda)}\, d\lambda = J_{P_1,\chi}(g, 0, f)$$

By definition and Lemma 4.3.7.2 below, one has:

$$\begin{aligned} J_{P_1,\chi}(g, 0, f) &= \sum_{w \in W(P, P_1)} J_{P_1, w\pi}(g, 0, f) \\ &= |W(P, P_1)| J_{P,\pi}(g, 0, f). \end{aligned}$$

Since $|\mathcal{P}(M_P)| = \sum_{P_1 \in \mathcal{P}_\chi} |W(P, P_1)|$ we get the expected limit. □

Lemma **4.3.7.2** *(Lapid). — For any $w \in \mathrm{W}(\mathrm{P}, \mathrm{P}_1)$, we have*

$$\mathrm{J}_{\mathrm{P}_1, w\pi}(g, 0, f) = \mathrm{J}_{\mathrm{P},\pi}(g, 0, f).$$

Proof. — By definition, we have

$$\mathrm{J}_{\mathrm{P}_1, w\pi}(g, 0, f) = \sum_{\varphi \in \mathcal{B}_{\mathrm{P}_1, w\pi}} \mathrm{E}(g, \mathrm{I}_{\mathrm{P}_1}(0, f)\varphi, 0) \cdot \mathrm{J}_1(\overline{\varphi})$$

where J_1 is the linear form on $\mathcal{A}_{\mathrm{P}_1, w\pi}$ defined in Section 4.1.5 and $\mathcal{B}_{\mathrm{P}_1, w\pi}$ is any K-basis of $\mathcal{A}_{\mathrm{P}_1, w\pi}$. Now, the intertwining operator $\mathrm{M}(w, 0)$ induces a unitary isomorphism from $\mathcal{A}_{\mathrm{P},\pi}$ to $\mathcal{A}_{\mathrm{P}_1, w\pi}$, which sends K-bases to K-bases. Thus one has

$$\begin{aligned}\mathrm{J}_{\mathrm{P}_1, w\pi}(g, 0, f) &= \sum_{\varphi \in \mathcal{B}_{\mathrm{P},\pi}} \mathrm{E}(g, \mathrm{M}(w, 0)\mathrm{I}_{\mathrm{P}}(0, f)\varphi, 0) \cdot \mathrm{J}_1(\mathrm{M}(w, 0)\overline{\varphi}) \\ &= \mathrm{J}_{\mathrm{P},\pi}(g, 0, f).\end{aligned}$$

The last equality results from the two equalities:

- $\mathrm{E}(g, \mathrm{M}(w, 0)\mathrm{I}_{\mathrm{P}}(0, f)\varphi, 0) = \mathrm{E}(g, \mathrm{I}_{\mathrm{P}}(0, f)\varphi, 0)$;
- $\mathrm{J}_1(\mathrm{M}(w, 0)\overline{\varphi}) = \mathrm{J}(\overline{\varphi})$ where J is the linear form on $\mathcal{A}_{\mathrm{P},\pi}$ defined in Section 4.1.5.

The first one is the functional equation of Eisenstein series and the second one is a consequence of case 1 of lemma 8.1 of [Lap06]. □

5. The $*$-regular contribution in the Jacquet-Rallis trace formula

The goal of this section is to compute the contribution I_χ of the Jacquet-Rallis trace formula for $*$-regular cuspidal data χ. This is achieved in Theorem 5.2.1.1 below. It turns out that for such χ the contribution I_χ is discrete and equal (up to an explicit constant) to a relative character define in Section 5.1 built upon Rankin-Selberg periods of Eisenstein series and Flicker-Rallis intertwining periods.

5.1. *Relative characters*

5.1.1. We will use the notations of Section 3.1. We emphasize that unlike Section 4 the group G denotes $\mathrm{G}_n \times \mathrm{G}_{n+1}$ and so on.

5.1.2. Let $\chi \in \mathfrak{X}(\mathrm{G})$ be a cuspidal datum and (M, π) be a representative where M is the standard Levi factor of the standard parabolic subgroup P of G. Recall that we have introduced a character $\eta_{G'}$ of $G'(\mathbf{A})$ (see Section 3.1.6). On $\mathcal{A}_{P,\pi}(G)$, we introduce the linear form J_η defined by

$$(\mathbf{5.1.2.1})\qquad J_\eta(\varphi)=\int_{A^\infty_{M'}M'(F)N_{P'}(\mathbf{A})\backslash G(\mathbf{A})}\varphi(g)\eta_{G'}(g)\,dg,\quad \forall\varphi\in\mathcal{A}_{P,\pi}(G)$$

where $M'=M\cap G'$ and $P'=P\cap G'$. This is a slight variation of that defined in Section 4.1.5.

We shall say that π is $(M',\eta_{G'})$-distinguished if J_η does not vanish identically.

5.1.3. *Relevant and regular cuspidal data.* — We shall say that χ is *relevant* if π is $(M',\eta_{G'})$-distinguished.

Let $\mathfrak{X}^*(G)=\mathfrak{X}^*(G_n)\times\mathfrak{X}^*(G_{n+1})$ (cf. Section 4.3.2). We shall say that χ is $*$-regular if it belongs to the subset $\mathfrak{X}^*(G)$. In particular, if χ is both relevant and regular (see Section 2.9.7) then it is $*$-regular.

5.1.4. *Rankin-Selberg period of certain Eisenstein series.* — Let $T\in\mathfrak{a}^+_{n+1}$. Recall that we have introduced in Section 3.3.2 the truncation operator Λ^T_r.

Proposition **5.1.4.1.** — *Let* Q *be a parabolic subgroup of* G *and* $Q'=Q\cap G'$. *Let* π *be an irreducible cuspidal representation of* M_Q *which is* $(M_{Q'},\eta_{G'})$*-distinguished. Let* $\varphi\in\mathcal{A}_{Q,\pi}(G)$. *Then for a regular point* $\lambda\in\mathfrak{a}^{G,*}_Q$ *of the Eisenstein series* $E(g,\varphi,\lambda)$ *(see Section 2.7.3), the integral*

$$(\mathbf{5.1.4.2})\qquad I(\varphi,\lambda)=\int_{[H]}\Lambda^T_r E(h,\varphi,\lambda)\,dh$$

is convergent and does not depend on T.

Remark **5.1.4.2.** — The expression $I(\varphi,\lambda)$ is nothing else but the regularized Rankin-Selberg period of $E(\varphi,\lambda)$ as defined by Ichino-Yamana in [IY15].

Proof. — The convergence follows from Proposition 3.3.2.1 and the fact that Eisenstein series are of moderate growth. It remains to prove that the integral does not depend on T. Recall that ι induces an isomorphism from G_n onto H. In the proof, it will be more convenient to work with G_n instead of H. However, by abuse of notations, for any $g\in G_n(\mathbf{A})$ and any function φ on $G(\mathbf{A})$ we shall write $\varphi(g)$ instead of $\varphi(\iota(g))$.

Let $T'\in\mathfrak{a}^+_{n+1}$. By lemma 2.2 of [IY15], we have

$$\Lambda^{T+T'}_r E(g,\varphi,\lambda)=\sum_{P\in\mathcal{F}_{RS}}\sum_{\delta\in(P\cap H)(F)\backslash H(F)}\Lambda^{T,P}_r E_{G_n\times P_{n+1}}(\delta g,\varphi,\lambda)$$
$$\times\,\Gamma'_{P_{n+1}}(H_{P_{n+1}}(\delta g)-T_{P_{n+1}},T')$$

where the notations are those of Sections 3.2.3 and 3.3.2. The other notations are borrowed from [Zyd18, eq. (4.4)]; the operator $\Lambda_r^{\mathrm{T,P}}$ is the obvious variant of Λ_r^{T} and Γ'_{P} is an Arthur function whose precise definition is irrelevant here. We denote by $\mathrm{E}_{\mathrm{G}_n\times \mathrm{P}_{n+1}}$ the constant term of E along $\mathrm{G}_n \times \mathrm{P}_{n+1}$. Thus, we have

$$\int_{[\mathrm{H}]} \Lambda_r^{\mathrm{T+T'}}\mathrm{E}(g,\varphi,\lambda)\,dg = \sum_{\mathrm{P}\in\mathcal{F}_{\mathrm{RS}}} \int_{(\mathrm{P}\cap \mathrm{H})(\mathrm{F})\backslash \mathrm{H}(\mathbf{A})} (\Lambda_r^{\mathrm{T,P}}\mathrm{E}_{\mathrm{G}_n\times\mathrm{P}_{n+1}}\mathrm{P})(g,\varphi,\lambda)\Gamma'_{\mathrm{P}_{n+1}}(\mathrm{H}_{\mathrm{P}_{n+1}}(g)-\mathrm{T}_{\mathrm{P}_{n+1}},\mathrm{T}')\,dg.$$

Let $\mathrm{P}\in\mathcal{F}_{\mathrm{RS}}$ be such that $\mathrm{P}\subsetneq \mathrm{G}$. It suffices to show that the terms corresponding to P vanish. We identify H with G_n. Then $\mathrm{P}\cap\mathrm{H}$ is identified with P_n. Let $\mathrm{M}_n = \mathrm{M}_{\mathrm{P}_n}$. For an appropriate choice of a Haar measure on K_n, such a term can be written as

$$\int_{[\mathrm{M}_n]}\int_{\mathrm{K}_n} \exp(-\langle 2\rho_{\mathrm{P}_n}, \mathrm{H}_{\mathrm{P}_n}(m)\rangle)(\Lambda_r^{\mathrm{T,P}}\mathrm{E}_{\mathrm{P}})(mk,\varphi,\lambda) \times \Gamma'_{\mathrm{P}_{n+1}}(\mathrm{H}_{\mathrm{P}_{n+1}}(m)-\mathrm{T}_{\mathrm{P}_{n+1}},\mathrm{T}')\,dkdm,$$

where E_{P} denotes the constant term of E along $\mathrm{P}=\mathrm{P}_n\times\mathrm{P}_{n+1}$. At this point, we may and shall assume that P is standard (if not, we may change B_n by a conjugate for the arguments). We have the usual formula for the constant term

$$\mathrm{E}_{\mathrm{P}}(m,\varphi,\lambda) = \sum_{w\in \mathrm{W}(\mathrm{Q};\mathrm{P})} \mathrm{E}^{\mathrm{P}}(m,\mathrm{M}(w,\lambda)\varphi,w\lambda),$$

where $\mathrm{W}(\mathrm{Q};\mathrm{P})$ is the set of elements $w\in\mathrm{W}$ that are of minimal length in double cosets $\mathrm{W}^{\mathrm{P}}w\mathrm{W}^{\mathrm{Q}}$. Let $w\in\mathrm{W}(\mathrm{Q};\mathrm{P})$. Notice that the representation $w\pi$ is also $(w\mathrm{M}_{\mathrm{Q}'}w^{-1},\eta_{\mathrm{G}'})$-distinguished. For the argument, we may and shall assume $w=1$ (that is we assume that $\mathrm{Q}\subset\mathrm{P}$). Thus it suffices to show for all $k\in\mathrm{K}$ the integral

$$\int_{[\mathrm{M}_n]^1} \Lambda_r^{\mathrm{T,P}}\mathrm{E}^{\mathrm{P}}(mk,\varphi,\lambda)\,dm \tag{5.1.4.3}$$

vanishes.

The group $\mathrm{M}_{n+1}=\mathrm{M}_{\mathrm{P}_{n+1}}$ has a decomposition $\mathrm{G}_{d_1}\times\cdots\times\mathrm{G}_{d_r}$ with $d_1+\cdots+d_r = n+1$. Each factor corresponds to a subset of the canonical basis $(e_1,\ldots,e_{n+1})$. We may assume that the factor G_{d_1} corresponds to a subset which does not contain e_{n+1}. As a consequence G_{d_1} is also a factor of M_n. We view $\mathrm{G}_{d_1}\times\mathrm{G}_{d_1}$ as a subgroup of $\mathrm{M}_n\times\mathrm{M}_{n+1}$. Let $\mathrm{Q}_1\times\mathrm{Q}_2 = (\mathrm{G}_{d_1}\times\mathrm{G}_{d_1})\cap\mathrm{Q}\subset\mathrm{G}_n\times\mathrm{G}_{n+1}$. The representation π restricts to $\mathrm{M}_{\mathrm{Q}_1}(\mathbf{A})$ and $\mathrm{M}_{\mathrm{Q}_2}(\mathbf{A})$: this gives representations respectively denoted by π_1 and π_2. As a factor of (5.1.4.3), we get

$$\int_{[\mathrm{G}_{d_1}]^1} \mathrm{E}(g,\varphi_1,\lambda_1)\Lambda^{\mathrm{T}}\mathrm{E}(g,\varphi_2,\lambda_2)\,dg \tag{5.1.4.4}$$

where $\varphi_i \in \mathcal{A}_{Q_i,\pi_i}(G_{d_1})$. Here the truncation is the usual Arthur's truncation operator on the group G_{d_1}. It is clear from Langlands' formula for the integral (5.1.4.4) (see [Art82]) that (5.1.4.4) vanishes unless there exists $w \in W^{G_{d_1}}(Q_1, Q_2)$ such that the contragredient of π_2 is isomorphic to $w\pi_1$. But then π_2 would be both $(M_{Q_2'}, (\eta_{d_1})^n)$-distinguished and $(M_{Q_2'}, (\eta_{d_1})^{n+1})$-distinguished with $\eta_{d_1} = \eta \circ \det_{d_1}$ and $M_{Q_2'} = M_{Q_2} \cap G'_{d_1}$. This is not possible. □

5.1.5. *Relative characters.* — Let (P, π) be a pair for which P be a standard parabolic subgroup of G and π be a cuspidal automorphic representation of its standard Levi factor M_P. Building upon the truncation operator Λ_r^T and the linear form J_η, we define the relative character $I_{P,\pi}^T$ for any $f \in \mathcal{S}(G(\mathbf{A}))$ by

$$I_{P,\pi}^T(f) = \sum_{\varphi \in \mathcal{B}_{P,\pi}} \int_{[H]} \Lambda_r^T E(h, I_P(0, f)\varphi, 0)\, dh \cdot \overline{J_\eta(\varphi)}$$

where the K-basis $\mathcal{B}_{P,\pi}$ is defined in Section 2.8.3. Using Proposition 5.1.4.1, we have

$$I_{P,\pi}^T(f) = I_{P,\pi}(f)$$

where we define:

$$I_{P,\pi}(f) = \begin{cases} \sum_{\varphi \in \mathcal{B}_{P,\pi}} I(I_P(0, f)\varphi, 0) \cdot \overline{J_\eta(\varphi)} \text{ if } \pi \text{ is } (M_{P'}, \eta_{G'})\text{-distinguished;} \\ 0 \text{ otherwise.} \end{cases}$$

Proposition **5.1.5.1.** — *Let $\chi \in \mathfrak{X}^*(G)$. Let (P, π) be a representative. The map $f \mapsto I_{P,\pi}^T(f)$ (and thus $f \mapsto I_{P,\pi}(f)$) is well-defined and gives a continuous linear form on $\mathcal{S}(G(\mathbf{A}))$. It depends only on χ and not on the choice of (P, π).*

Proof. — First we claim that $\varphi \mapsto \int_{[H]} \Lambda_r^T E(h, \varphi, 0)\, dh$ is a continuous map: this is an easy consequence of properties of Eisenstein series and the truncation operator Λ_r^T (see Proposition 3.3.2.1). On the other hand $\varphi \mapsto J_\eta(\varphi)$ is also continuous (see Section 4.1). Thus the first assertion results from an application of Proposition 2.8.4.1. The arguments of the proof of Lemma 4.3.7.2 give the independence on the choice of (P, π). □

5.2. *The ∗-regular contribution*

5.2.1. Let $\chi \in \mathfrak{X}(G)$. Recall that we defined in Theorem 3.2.4.1 a distribution I_χ on $\mathcal{S}(G(\mathbf{A}))$. Let (M, π) be a representative of χ where M is the standard Levi factor of the standard parabolic subgroup P of G. The following theorem is the main result of this section.

Theorem **5.2.1.1.** — *Assume moreover $\chi \in \mathfrak{X}^*(G)$. We have*

$$I_\chi = 2^{-\dim(\mathfrak{a}_P)} I_{P,\pi}.$$

In particular, we have $I_\chi = 0$ *unless* χ *is relevant.*

The theorem is a direct consequence of the following proposition.

Proposition **5.2.1.2.** — *Assume moreover* $\chi \in \mathfrak{X}^*(G)$ *We have for* $T \in \mathfrak{a}_{n+1}^+$

$$\int_{[H]}\int_{[G']} \Lambda_r^T K_\chi(x,y)\,\eta_{G'}(y)dxdy = 2^{-\dim(\mathfrak{a}_P)} I_{P,\pi}(f), \tag{5.2.1.1}$$

where the left-hand side is absolutely convergent (see Proposition 3.3.3.1). In particular, the left-hand side does not depend on T.

Indeed, by Theorem 3.3.9.1, I_χ is the constant term in the asymptotic expansion in T of the left-hand side of (5.2.1.1) hence $I_\chi = 2^{-\dim(\mathfrak{a}_P)} I_{P,\pi}$.

The rest of the section is devoted to the proof of Proposition 5.2.1.2.

5.2.2. *Proof of Proposition 5.2.1.2.* — We assume that $\chi \in \mathfrak{X}^*(G)$. The proof is a straightforward consequence of Theorem 4.3.3.1 and some permutations between integrals, summations and the truncation. These permutations are provided by Lemmas 5.2.2.1 and 5.2.2.3 below.

Lemma **5.2.2.1.** — *For all* $x \in [H]$, *we have*

$$\int_{[G']} (\Lambda_r^T K_\chi)(x,y)\eta_{G'}(y)\,dy = \Lambda_r^T\left(\int_{[G']} K_\chi(\cdot,y)\eta_{G'}(y)\,dy\right)(x).$$

Remark **5.2.2.2.** — On the left-hand side we apply the truncation operator Λ_r^T to the function $K_\chi(\cdot,y)$ (where y is fixed) and then we evaluate at x whereas on the right-hand side we apply the same operator to the function we get by integration of $K_\chi(\cdot,y)\eta_{G'}(y)$ over $y \in [G']$ and then we evaluate at x.

Proof. — Since x is fixed, the operator Λ_r^T is a finite sum of constant terms (see [Art78] lemma 5.1 for the finiteness). Then the lemma follows from Fubini's theorem which holds because we have

$$\int_{[N_Q]}\int_{[G']} |K_\chi(nx,y)|\,dndy < \infty$$

for all parabolic subgroups Q of G_{n+1} containing B_n. Here we identify N_Q with the subgroup $\{1\}\times N_Q$ of $G = G_n \times G_{n+1}$. The convergence of the integral results from the bound (3.3.2.3) above. □

Lemma **5.2.2.3.** — *We have*

$$\int_{[H]} \Lambda_r^T\left(\int_{[G']} K_\chi(\cdot,y)\eta_{G'}(y)\,dy\right)(h)\,dh = 2^{-\dim(\mathfrak{a}_P)} I_{P,\pi}(f).$$

Proof. — First, by Theorem 4.3.3.1, we have for any $x \in [\mathrm{G}]$:

$$\int_{[\mathrm{G}']} \mathrm{K}_\chi(x,y)\eta_{\mathrm{G}'}(y)\,dy = 2^{-\dim(\mathfrak{a}_\mathrm{P})} \sum_{\varphi\in\mathcal{B}_{\mathrm{P},\pi}} \mathrm{E}(x, \mathrm{I}_\mathrm{P}(0,f)\varphi, 0)\,dh \cdot \overline{\mathrm{J}_\eta(\varphi)}$$

where the notations are borrowed from Section 5.1.5. Then we want to apply the truncation operator Λ_r^{T} and evaluate at $h \in [\mathrm{H}]$. We want to show that this operation commutes with the summation over the orthonormal basis. As in the proof of Lemma 5.2.2.1, it suffices to prove

$$\sum_{\varphi\in\mathcal{B}_{\mathrm{P},\pi}} \int_{[\mathrm{N}_\mathrm{Q}]} |\mathrm{E}(ng, \mathrm{I}_\mathrm{P}(0,f)\varphi, 0)|\,dn \cdot |\overline{\mathrm{J}_\eta(\varphi)}| < \infty$$

for any parabolic subgroups Q of G_{n+1} containing B_n, which is an easy consequence of continuity properties of Eisenstein series.

In this way, we get for $h \in [\mathrm{H}]$:

$$\Lambda_r^{\mathrm{T}}\left(\int_{[\mathrm{G}']} \mathrm{K}_\chi(\cdot,y)\eta_{\mathrm{G}'}(y)\,dy\right)(h)$$
$$= 2^{-\dim(\mathfrak{a}_\mathrm{P})} \sum_{\varphi\in\mathcal{B}_{\mathrm{P},\pi}} (\Lambda_r^{\mathrm{T}}\mathrm{E})(h, \mathrm{I}_\mathrm{P}(0,f)\varphi, 0) \cdot \overline{\mathrm{J}_\eta(\varphi)}.$$

By integration over $h \in [\mathrm{H}]$, we have:

$$\int_{[\mathrm{H}]} \Lambda_r^{\mathrm{T}}\left(\int_{[\mathrm{G}']} \mathrm{K}_\chi(\cdot,y)\eta_{\mathrm{G}'}(y)\,dy\right)(h)\,dh$$
$$= 2^{-\dim(\mathfrak{a}_\mathrm{P})} \sum_{\varphi\in\mathcal{B}_{\mathrm{P},\pi}} \int_{[\mathrm{H}]} (\Lambda_r^{\mathrm{T}}\mathrm{E})(h, \mathrm{I}_\mathrm{P}(0,f)\varphi, 0)\,dh \cdot \overline{\mathrm{J}_\eta(\varphi)}.$$

The right-hand side is nothing else but $2^{-\dim(\mathfrak{a}_\mathrm{P})}\mathrm{I}_{\mathrm{P},\pi}(f)$. Still we have to justify the change of order of the integration and the summation. But it is easy to show that

$$\sum_{\varphi\in\mathcal{B}_{\mathrm{P},\pi}} \int_{[\mathrm{H}]} |\Lambda_r^{\mathrm{T}}\mathrm{E}(h, \mathrm{I}_\mathrm{P}(0,f)\varphi, 0)|\,dh \cdot |\overline{\mathrm{J}_\eta(\varphi)}| < \infty. \qquad \square$$

6. Spectral decomposition of the Flicker-Rallis period for ∗-regular cuspidal data

The goal of this section is to give another proof of the spectral decomposition of the Flicker-Rallis period for the same cuspidal data as in Section 4.3.2. The main result of this section (obtained as a combination of Theorem 6.2.5.1 and Theorem 6.2.6.1) can be used to get another version of Theorem 4.3.3.1 with a seemingly different relative character than $J_{P,\pi}$ (this will actually be done in Section 8.2.4). Of course, these two relative characters are the same. A direct proof of this fact will be given in Section 9.

6.1. *Notation*

6.1.1. In this section we adopt the set of notation introduced in Section 3.1: E/F is a quadratic extension of number fields, $G'_n = GL_{n,F}$, $G_n = \mathrm{Res}_{E/F}\, GL_{n,E}$, (B'_n, T'_n), (B_n, T_n) are the standard Borel pairs of G'_n, G_n and K'_n, K_n the standard maximal compact subgroups of $G'_n(\mathbf{A})$, $G_n(\mathbf{A})$ respectively. Besides, we denote by N'_n, N_n the unipotent radicals of B'_n, B_n and we set

$$w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in G'_n(F).$$

We write $e_n = (0, \dots, 0, 1)$ for the last element in the standard basis of F^n and we let $\mathcal{P}_n = \begin{pmatrix} \star & \star \\ 0\dots 0 & 1 \end{pmatrix}$, $\mathcal{P}'_n = \mathcal{P}_n \cap G'_n$ be the mirabolic subgroups of G_n, G'_n respectively (that is the stabilizers of e_n for the natural right actions). The unipotent radicals of $\mathcal{P}_n$, $\mathcal{P}'_n$ will be denoted by U_n and U'_n respectively. For nonnegative integers $m \leqslant n$, we embed G_m in G_n (resp. G'_m in G'_n) in the "upper left corner" by $g \mapsto \begin{pmatrix} g & \\ & I_{n-m} \end{pmatrix}$. Thus, in particular, we have $\mathcal{P}_n = G_{n-1}U_n$ and $\mathcal{P}'_n = G'_{n-1}U'_n$.

The entries of a matrix $g \in G_n(\mathbf{A})$ are written as $g_{i,j}$, $1 \leqslant i, j \leqslant n$, and the diagonal entries of an element $t \in T_n(\mathbf{A})$ as t_i, $1 \leqslant i \leqslant n$.

6.1.2. We fix a nontrivial additive character $\psi' : \mathbf{A}/F \to \mathbf{C}^\times$. For $\phi \in \mathcal{S}(\mathbf{A}^n)$, we define its Fourier transform $\widehat{\phi} \in \mathcal{S}(\mathbf{A}^n)$ by

$$\widehat{\phi}(x_1, \dots, x_n) = \int_{\mathbf{A}^n} \phi(y_1, \dots, y_n)\psi'(x_1y_1 + \cdots + x_ny_n)dy_1 \dots dy_n$$

the Haar measure on $\mathbf{A}^n$ being chosen such that $\widehat{\widehat{\phi}}(x) = \phi(-x)$.

We denote by c the nontrivial Galois involution of E over F. Then, c acts naturally on $G_n(\mathbf{A})$ and thus on cuspidal automorphic representations of the latter. We denote this action by $\pi \mapsto \pi^c$. We fix $\tau \in E^\times$ such that $\tau^c = -\tau$ and we define $\psi : \mathbf{A}_E/E \to \mathbf{C}^\times$ by $\psi(z) = \psi'(\mathrm{Tr}_{E/F}(\tau z))$, $z \in \mathbf{A}_E$, where $\mathbf{A}_E$ denotes the adèle ring of E and $\mathrm{Tr}_{E/F} : \mathbf{A}_E \to \mathbf{A}$ the trace map. We also define a generic character $\psi_n : [N_n] \to \mathbf{C}^\times$ by

$$\psi_n(u) = \psi\left((-1)^n \sum_{i=1}^{n-1} u_{i,i+1}\right), \quad u \in [N_n].$$

(The appearance of the sign $(-1)^n$ is only a convention that will be justified a posteriori in Section 7.) Note that ψ is trivial on $\mathbf{A}$ and therefore ψ_n is trivial on $N'_n(\mathbf{A})$. To any $f \in \mathcal{T}([G_n])$, we associate its *Whittaker function* W_f defined by

$$W_f(g) = \int_{[N_n]} f(ug)\psi_n(u)^{-1}du, \quad g \in G_n(\mathbf{A}).$$

6.2. *Statements of the main results*

6.2.1. Let $n \geqslant 1$ be a nonnegative integer. For $f \in \mathcal{T}([G_n])$, $\phi \in \mathcal{S}(\mathbf{A}^n)$ and $s \in \mathbf{C}$ we set

$$Z_\psi^{FR}(s,f,\phi) = \int_{N'_n(\mathbf{A})\backslash G'_n(\mathbf{A})} W_f(h)\phi(e_n h)|\det h|^s dh$$

provided this expression converges absolutely.

6.2.2. Let $\chi \in \mathfrak{X}^*(G_n)$ be a $*$-regular cuspidal datum (see Section 4.3.2 for the definition of $*$-regular) represented by a pair (M_P, π) and set $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G_n(\mathbf{A})}(\pi)$. We can write

$$M_P = G_{n_1} \times \cdots \times G_{n_k}$$

where $n_1, \ldots, n_k$ are positive integers such that $n_1 + \cdots + n_k = n$. Then, π decomposes accordingly as a tensor product

$$\pi = \pi_1 \boxtimes \ldots \boxtimes \pi_k$$

where for each $1 \leqslant i \leqslant k$, π_i is a cuspidal automorphic representation of $G_{n_i}(\mathbf{A})$.

6.2.3. Let $L(s, \Pi, \mathrm{As})$ be the Shahidi's completed Asai L-function of Π [Sha90], [Gol94]. We have the decomposition

$$L(s, \Pi, \mathrm{As}) = \prod_{i=1}^{k} L(s, \pi_i, \mathrm{As}) \times \prod_{1\leqslant i<j\leqslant k} L(s, \pi_i \times \pi_j^c).$$

(We emphasize that since $L(s, \pi_i \times \pi_j^c) = L(s, \pi_j \times \pi_i^c)$, this decomposition does not depend on the order of the π_i's.) As χ is $*$-regular, the Rankin-Selberg L-functions $L(s, \pi_i \times \pi_j^c)$ are entire and non-vanishing at $s = 1$ [JS81b], [JS81a], [Sha81] whereas by [Fli88], $L(s, \pi_i, \mathrm{As})$ has at most a simple pole at $s = 1$. Therefore, $L(s, \Pi, \mathrm{As})$ has a pole of order at most k at $s = 1$ and this happens if and only if $L(s, \pi_i, \mathrm{As})$ has a pole at $s = 1$ for every $1 \leqslant i \leqslant k$.

We say that the cuspidal datum χ is *distinguished* if $L(s, \Pi, \mathrm{As})$ has a pole of order k at $s = 1$. By [Fli88], it is equivalent to ask π to be $M_{P'} = M_P \cap G'_n$-distinguished.

6.2.4. For $f \in \mathcal{C}([G_n])$, we set $W_{f,\Pi} = W_{f_\Pi}$ where f_Π is defined as in Section 2.9.8. Then, $W_{f,\Pi}$ belongs to the Whittaker model $\mathcal{W}(\Pi, \psi_n)$ of Π with respect to ψ_n.

We define a continuous linear form β_n on $\mathcal{W}(\Pi, \psi_n)$ as follows. For S a finite set of places of F and $W \in \mathcal{W}(\Pi, \psi_n)$, we set

$$\beta_{n,S}(W) = \int_{N'_n(F_S)\backslash \mathcal{P}'_n(F_S)} W(p_S)dp_S$$

the integral being convergent by (the same proof as) [BP21b, Proposition 2.6.1, Lemma 3.3.1] and the Jacquet-Shalika bound [JS81b]. By [Fli88, Proposition 3] and (2.3.2.3), for a given $W \in \mathcal{W}(\Pi, \psi_n)$, the quantity

$$\beta_n(W) = (\Delta_{G'_n}^{S,*})^{-1} L^{S,*}(1, \Pi, As)\beta_{n,S}(W)$$

is independent of S as long as it is sufficiently large (i.e. it contains all the Archimedean places as well as the non-Archimedean places where the situation is "ramified") where we recall that following our general convention of Section 2.1, $L^{S,*}(1, \Pi, As)$ stands for the leading coefficient of the Laurent expansion of the partial L-function $L^S(s, \Pi, As)$ at $s = 1$ and we refer the reader to Section 2.3.3 for the definition of $\Delta_{G'_n}^{S,*}$. This defines the linear form β_n.

6.2.5. For every $f \in \mathcal{C}([G_n])$, we set

$$^0f(g) = \int_{A_{G_n}^\infty} f(ag)da, \quad g \in [G_n].$$

Theorem **6.2.5.1.**

1. *Let* $N \geqslant 0$. *There exists* $c_N > 0$ *such that for every* $f \in \mathcal{T}_N([G_n])$ *and* $\phi \in \mathcal{S}(\mathbf{A}^n)$, *the expression defining* $Z_\psi^{FR}(s, f, \phi)$ *is absolutely convergent for* $s \in \mathcal{H}_{>c_N}$ *and the function* $s \in \mathcal{H}_{>c_N} \mapsto Z_\psi^{FR}(s, f, \phi)$ *is holomorphic and bounded in vertical strips. Moreover, for every* $s \in \mathcal{H}_{>c_N}$, $(f, \phi) \mapsto Z_\psi^{FR}(s, f, \phi)$ *is a (separately) continuous bilinear form on* $\mathcal{T}_N([G_n]) \times \mathcal{S}(\mathbf{A}^n)$.
2. *Let* $\chi \in \mathfrak{X}^*(G_n)$. *For every* $f \in \mathcal{C}_\chi([G_n])$ *and* $\phi \in \mathcal{S}(\mathbf{A}^n)$, *the function* $s \mapsto (s-1)Z_\psi^{FR}(s, {}^0f, \phi)$ *admits an analytic continuation to* $\mathcal{H}_{>1}$ *with a limit at* $s = 1$. *Moreover, we have*

$$Z_\psi^{FR,*}(1, {}^0f, \phi) := \lim_{s \to 1^+} (s-1)Z_\psi^{FR}(s, {}^0f, \phi) = \begin{cases} 2^{1-k}\widehat{\phi}(0)\beta_n(W_{f,\Pi}) & \text{if } \chi \text{ is distinguished}, \\ 0 & \text{otherwise.} \end{cases}$$

6.2.6.

Theorem **6.2.6.1.** — *Let* $\chi \in \mathfrak{X}^*(G_n)$. *The linear form*

$$P_{G'_n} : f \in \mathcal{C}([G_n]) \mapsto \int_{[G'_n]} f(h)dh$$

is well-defined (i.e. the integral converges) and continuous. Moreover, for every $f \in \mathcal{S}_\chi([G_n])$ *and* $\phi \in \mathcal{S}(\mathbf{A}^n)$ *we have*

$$\widehat{\phi}(0)P_{G'_n}(f) = \frac{1}{2}Z_\psi^{FR,*}(1, {}^0f, \phi). \tag{6.2.6.1}$$

6.2.7. A direct consequence of Theorem 6.2.5.1 and Theorem 6.2.6.1 is the following corollary.

Corollary **6.2.7.1.** — *Let* $\chi \in \mathfrak{X}^*(G_n)$ *be represented by a pair* (M_P, π) *and set* $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G_n(\mathbf{A})}(\pi)$. *Then, for every* $f \in \mathcal{S}_\chi([G_n])$ *we have*

$$P_{G'_n}(f) = \begin{cases} 2^{-\dim(A_P)}\beta_n(W_{f,\Pi}) \text{ if } \chi \text{ is distinguished,} \\ \\ 0 \text{ otherwise.} \end{cases}$$

6.3. *Proof of Theorem 6.2.5.1.2*

Part 1. of Theorem 6.2.5.1 will be established in Section 6.5. Here, we give the proof of part 2. of this theorem. Let $f \in \mathcal{C}_\chi([G_n])$, $\phi \in \mathcal{S}(\mathbf{A}^n)$ and (M_P, π) be a pair representing the cuspidal datum χ as in Section 6.2. We make the identification

$$i\mathfrak{a}_M^* \simeq (i\mathbf{R})^k$$

such that for every $\underline{x} = (x_1, \ldots, x_k) \in (i\mathbf{R})^k$ we have

$$\pi_{\underline{x}} := \pi_1|\det|_E^{x_1} \boxtimes \ldots \boxtimes \pi_k|\det|_E^{x_k}. \tag{6.3.0.1}$$

Let $i\mathfrak{a}_{M,0}^*$ be the subspace of $\underline{x} \in i\mathfrak{a}_M^*$ such that $n_1x_1 + \cdots + n_kx_k = 0$. We equip $i\mathfrak{a}_{M,0}^*$ with the unique Haar measure such that the quotient measure on

$$i\mathfrak{a}_M^*/i\mathfrak{a}_{M,0}^* \simeq i\mathbf{R}, \; \underline{x} \mapsto \frac{n_1x_1 + \cdots + n_kx_k}{n}$$

is $(2\pi)^{-1}$ times the Lebesgue measure. For every $\underline{x} \in i\mathfrak{a}_M^*$, we set $\Pi_{\underline{x}} = \mathrm{Ind}_{P(\mathbf{A})}^{G_n(\mathbf{A})}(\pi_{\underline{x}})$ and $f_{\underline{x}} = f_{\Pi_{\underline{x}}}$ following the definition of Section 2.9.8 (so that in particular $\Pi_0 = \Pi$ and $f_0 = f_\Pi$ with notation from the previous section).

We have an isomorphism $A^{\infty}_{G_n} \simeq \mathbf{R}^*_+$, $a \mapsto |\det a|_E$, sending the Haar measure on $A^{\infty}_{G_n}$ to $\frac{dt}{|t|}$ where dt is the Lebesgue measure. Therefore, by Theorem 2.9.8.1 and Fourier inversion, we have

(**6.3.0.2**) $$^0f = \int_{A^{\infty}_{G_n}} \int_{i\mathfrak{a}^*_M} a \cdot f_{\underline{x}} d\underline{x} da = \int_{A^{\infty}_{G_n}} \int_{i\mathfrak{a}^*_M} |\det a|_E^{\frac{n_1x_1+\dots+n_kx_k}{n}} f_{\underline{x}} d\underline{x} da = \int_{i\mathfrak{a}^*_{M,0}} f_{\underline{x}} d\underline{x}$$

where the right-hand side is an absolutely convergent integral in $\mathcal{T}_N([G_n])$ for some $N > 0$. Therefore, by the first part of Theorem 6.2.5.1, there exists $c > 0$ such that for every $s \in \mathcal{H}_{>c}$ we have

(**6.3.0.3**) $$Z^{FR}_{\psi}(s, {}^0f, \phi) = \int_{i\mathfrak{a}^*_{M,0}} Z^{FR}_{\psi}(s, f_{\underline{x}}, \phi) d\underline{x}.$$

Let S_0 be a finite set of places of F including the Archimedean ones and outside of which π is unramified and let $S_{0,f} \subset S_0$ be the subset of finite places. Let $I \subseteq \{1, \dots, k\}$ be the subset of $1 \leqslant i \leqslant k$ such that $L(s, \pi_i, As)$ has a pole at $s = 1$. We choose, for each $1 \leqslant i \leqslant k$ and $v \in S_{0,f}$, polynomials $Q_i(T), Q_{i,v}(T) \in \mathbf{C}[T]$ with roots in $\mathcal{H}_{]0,1[}$ and $\mathcal{H}_{]q_v^{-1},1[}$ respectively such that $s \mapsto Q_i(s)L_{\infty}(s, \pi_i, As)$ and $s \mapsto Q_{i,v}(q_v^{-s})L_v(s, \pi_i, As)$ have no pole in $\mathcal{H}_{]0,1[}$. Finally, we set

$$P(s, \underline{x}) = \prod_{i \in I}(s + 2x_i)(s - 1 + 2x_i) \prod_{1 \leqslant i \leqslant k} Q_i(s + 2x_i) \prod_{\substack{1 \leqslant i \leqslant k \\ v \in S_{0,f}}} Q_{i,v}(q_v^{-s-2x_i}) \text{ and}$$

$$\widetilde{f_{\underline{x}}}(g) = f_{\underline{x}}({}^tg^{-1})$$

for every $\underline{x} \in \mathcal{A}_0$, $s \in \mathbf{C}$ and $g \in G_n(\mathbf{A})$. We will now check that the functions

(**6.3.0.4**) $$(s, \underline{x}) \in \mathbf{C} \times \mathcal{A}_0 \mapsto P(s + \frac{1}{2}, \underline{x}) Z^{FR}_{\psi}(s + \frac{1}{2}, f_{\underline{x}}, \phi)$$

and

(**6.3.0.5**) $$(s, \underline{x}) \in \mathbf{C} \times \mathcal{A}_0 \mapsto P(\frac{1}{2} - s, \underline{x}) Z^{FR}_{\psi^{-1}}(\frac{1}{2} - s, \widetilde{f_{\underline{x}}}, \widehat{\phi})$$

satisfy the conditions of Corollary A.0.11.1.

From the first part of Theorem 6.2.5.1, Theorem 2.9.8.1 and Lemma A.0.9.1, we deduce that these functions satisfy the first condition of Corollary A.0.11.1. To check that they also satisfy the second condition of Corollary A.0.11.1, we need to analyze more carefully the function $s \mapsto Z^{FR}_{\psi}(s, f_{\underline{x}}, \phi)$ for a fixed $\underline{x} \in \mathcal{A}_0$.

For S a sufficiently large finite set of places of F, that we assume to contain Archimedean places as well as the places where π, ψ' or ψ are ramified (thus $S_0 \subset S$), we have decompositions

$$\phi = \phi_S \phi^S \text{ and } W_{f_{\underline{x}}} = W_{S,\underline{x}} W^S_{\underline{x}}$$

for every $\underline{x} \in \mathcal{A}_0$, where $\phi_S \in \mathcal{S}(F_S^n)$, ϕ^S is the characteristic function of $(\widehat{\mathcal{O}_F^S})^n$, $W_{S,\underline{x}} \in \mathcal{W}(\Pi_{\underline{x},S}, \psi_{n,S})$ (that is the Whittaker model of the representation $\Pi_{\underline{x},S}$ with respect to the character $\psi_{n,S} = \psi_{n|N(F_S)}$) and $W_{\underline{x}}^S \in \mathcal{W}(\Pi_{\underline{x}}^S, \psi_n^S)^{K_n^S}$ is such that $W_{\underline{x}}^S(1) = 1$. By [Fli88, Proposition 3] and (2.3.3.4), we then have

$$(\mathbf{6.3.0.6}) \qquad Z_\psi^{FR}(s, f_{\underline{x}}, \phi) = (\Delta_{G'_n}^{S,*})^{-1} L(s, \Pi_{\underline{x}}, As) \frac{Z_\psi^{FR}(s, W_{S,\underline{x}}, \phi_S)}{L_S(s, \Pi_{\underline{x}}, As)}$$

for $s \in \mathcal{H}_{>c}$ where we have set

$$Z_\psi^{FR}(s, W_{S,\underline{x}}, \phi_S) = \int_{N'_n(F_S)\backslash G'_n(F_S)} W_{S,\underline{x}}(h_S)\phi_S(e_n h_S)|\det h_S|^s dh_S.$$

Moreover, by [BP21b, Theorem 3.5.1] the function $Z_\psi^{FR}(s, W_{S,\underline{x}}, \phi_S)$ extends meromorphically to the complex plane and satisfies the functional equation

$$(\mathbf{6.3.0.7}) \qquad \frac{Z_{\psi^{-1}}^{FR}(1-s, \widetilde{W}_{S,\underline{x}}, \widehat{\phi_S})}{L_S(1-s, (\Pi_{\underline{x}})^\vee, As)} = \epsilon(s, \Pi_{\underline{x}}, As) \frac{Z_\psi^{FR}(s, W_{S,\underline{x}}, \phi_S)}{L_S(s, \Pi_{\underline{x}}, As)}$$

where $\widetilde{W_{S,\underline{x}}}(g) = W_{S,\underline{x}}(w_n {}^t g^{-1})$, $\widehat{\phi_S}$ is the (normalized) Fourier transform of ϕ_S with respect to the bicharacter $(\underline{u}, \underline{v}) \mapsto \psi'(u_1 v_1 + \cdots + u_n v_n)$ and $\epsilon(s, \Pi_{\underline{x}}, As)$ denotes the global epsilon factor of the Asai L-function $L(s, \Pi_{\underline{x}}, As)$.

By (6.3.0.6), (6.3.0.7) as well as the meromorphic continuation and functional equation of $L(s, \Pi_{\underline{x}}, As)$ [Sha90, Theorem 3.5(4)], we conclude that $Z_\psi^{FR}(s, f_{\underline{x}}, \phi)$ has a meromorphic continuation to $\mathbf{C}$ satisfying the functional equation

$$(\mathbf{6.3.0.8}) \qquad Z_{\psi^{-1}}^{FR}(1-s, \widetilde{f_{\underline{x}}}, \widehat{\phi}) = Z_\psi^{FR}(s, f_{\underline{x}}, \phi).$$

On the other hand, we have the decomposition

$$L(s, \Pi_{\underline{x}}, As) = \prod_{i=1}^k L(s + 2x_i, \pi_i, As) \times \prod_{1 \leqslant i < j \leqslant k} L(s + x_i + x_j, \pi_i \times \pi_j^c),$$

and, as $\chi \in \mathfrak{X}^*(G_n)$, the Rankin-Selberg L-functions $L(s, \pi_i \times \pi_j^c)$ are entire and bounded in vertical strips [Cog08, Theorem 4.1]. By the Jacquet-Shalika bound [JS81b] and the fact that the gamma function is of exponential decay in vertical strips, $Q_i(s)L_\infty(s, \pi_i, As)$ and $Q_{i,v}(q_v^{-s})L_v(s, \pi_i, As)$ are holomorphic and bounded in vertical strips of $\mathcal{H}_{>0}$ for each $1 \leqslant i \leqslant k$ and $v \in S_{0,f}$. By [FL17, Lemma 5.2], $s \mapsto (s-1)L^{S_0}(s, \pi_i, As)$, for $i \in I$, and $s \mapsto L^{S_0}(s, \pi_i, As)$, for $i \notin I$, are also holomorphic and of finite order in vertical strips of $\mathcal{H}_{>0}$. Therefore, by the definition of P and the functional equation, $P(s, \underline{x})L(s, \Pi_{\underline{x}}, As)$ is entire and of finite order in vertical strips. By (6.3.0.6), (6.3.0.8) and [BP21b, Theorem 3.5.2], it follows that the functions (6.3.0.4), (6.3.0.5) are entire and of finite order in vertical strips in the first variable i.e. they also satisfy the second condition of Corollary A.0.11.1.

Thus, the conclusion of this corollary is valid and in particular the map

$$s \mapsto \left(\underline{x} \mapsto \prod_{i \in I} (s - 1 + 2x_i) Z_\psi^{FR}(s, f_{\underline{x}}, \phi) \right)$$

induces a holomorphic function $\mathcal{H}_{>1-\epsilon} \to \mathcal{S}(\mathcal{A}_0)$ for some $\epsilon > 0$. By (6.3.0.3) and [BP21c, Lemma 3.1.1, Proposition 3.1.2],[9] it follows that $s \mapsto Z_\psi^{FR}(s, {}^0 f, \phi)$ extends analytically to $\mathcal{H}_{>1}$ and that

$$\textbf{(6.3.0.9)} \qquad \lim_{s \to 1^+} (s-1) Z_\psi^{FR}(s, {}^0 f, \phi) = \begin{cases} 2^{1-k} \lim_{s \to 1} (s-1)^k Z_\psi^{FR}(s, f_0, \phi) \text{ if } I = \{1, \ldots, k\}, \\ 0 \text{ otherwise} \end{cases}$$

Recall that $I = \{1, \ldots, k\}$ if and only if $L(s, \Pi, As)$ has a pole of order $k = rk(A_P)$ at $s = 1$. Moreover, by [BP21b, Lemma 3.3.1] and the Jacquet-Shalika bound [JS81b], the integral defining $Z_\psi^{FR}(s, W_{S,0}, \phi_S)$ is absolutely convergent in $\mathcal{H}_{>1-\varepsilon}$ for some $\varepsilon > 0$. Combining this with [BP21c, Lemma 2.16.3] and (6.3.0.6), in the case $I = \{1, \ldots, k\}$ identity (6.3.0.9) can be rewritten as

$$\begin{aligned} \lim_{s \to 1^+} (s-1) Z_\psi^{FR}(s, {}^0 f, \phi) &= 2^{1-k} (\Delta_{G'_n}^{S,*})^{-1} L^{S,*}(1, \Pi, As) Z_\psi^{FR}(1, W_{S,0}, \phi_S) \\ &= 2^{1-k} (\Delta_{G'_n}^{S,*})^{-1} L^{S,*}(1, \Pi, As) \beta_{n,S}(W_{S,0}) \widehat{\phi_S}(0) \\ &= 2^{1-k} \widehat{\phi}(0) \beta_n(W_{f,\Pi}) \end{aligned}$$

and this ends the proof of Theorem 6.2.5.1.2.

6.4. *Proof of Theorem 6.2.6.1*

By (2.4.5.23), since any Siegel domain of $[G'_n]$ is contained in a Siegel domain of $[G_n]$ and $\rho_{B_n} = 2\rho_{B'_n}$, we have $\Xi^{G_n}(h) \ll \Xi^{G'_n}(h)^2$ for $h \in [G'_n]$. Hence, by (2.4.5.24), the linear form $P_{G'_n}$ is well-defined and continuous on $\mathcal{C}([G_n])$. This shows the first part of Theorem 6.2.6.1.

Let $f \in \mathcal{S}_\chi([G_n])$. Recall that $A_{G_n}^\infty = A_{G'_n}^\infty$ but the Haar measure on $A_{G_n}^\infty$ is twice the Haar measure on $A_{G'_n}^\infty$ (see Remark 3.1.3.1). Therefore, we have

$$P_{G'_n}(f) = \frac{1}{2} \int_{[G'_n]_0} {}^0 f(h) dh.$$

[9] More precisely, with the notation of Proposition 3.12 of *loc. cit.* we have an isomorphism $i\mathfrak{a}_{M,0}^* \simeq (i\mathbf{R})_0^k$, $\underline{x} \mapsto \underline{x}'$ given by $x'_i = n_i x_i$ and which sends the measure on $i\mathfrak{a}_{M,0}^*$ to $\frac{n}{n_1 \ldots n_k}(2\pi)^{1-k}$ times the measure on $(i\mathbf{R})_0^k$ used in *loc. cit.*

Let $\phi \in \mathcal{S}(\mathbf{A}^n)$. We form the Epstein-Eisenstein series

$$\mathrm{E}(h, \phi, s) = \int_{\mathrm{A}^\infty_{\mathrm{G}'_n}} \sum_{\gamma \in \mathcal{P}'_n(\mathrm{F}) \backslash \mathrm{G}'_n(\mathrm{F})} \phi(e_n \gamma a h) |\det(ah)|^s da, \quad h \in [\mathrm{G}'_n], s \in \mathbf{C}.$$

This expression converges absolutely for $\Re(s) > 1$ and the map $s \mapsto \mathrm{E}(\phi, s)$ extends to a meromorphic function valued in $\mathcal{T}([\mathrm{G}'_n])$ with simple poles at $s = 0$, 1 of respective residues $\phi(0)$ and $\widehat{\phi}(0)$ (cf. [JS81b, Lemma 4.2]).

Consequently, the function

$$s \mapsto \mathrm{Z}^{\mathrm{FR}}_n(s, {}^0 f, \phi) := \int_{[\mathrm{G}'_n]_0} {}^0 f(h) \mathrm{E}(h, \phi, s) dh$$

is well-defined for $s \in \mathbf{C} \setminus \{0, 1\}$, meromorphic on $\mathbf{C}$ with a simple pole at $s = 1$ whose residue is

$$(\mathbf{6.4.0.1}) \qquad \mathrm{Res}_{s=1} \mathrm{Z}^{\mathrm{FR}}_n(s, {}^0 f, \phi) = 2\widehat{\phi}(0) \mathrm{P}_{\mathrm{G}'_n}(f).$$

Unfolding the definition, we arrive at the identity

$$(\mathbf{6.4.0.2}) \qquad \mathrm{Z}^{\mathrm{FR}}_n(s, {}^0 f, \phi) = \int_{\mathcal{P}'_n(\mathrm{F}) \backslash \mathrm{G}'_n(\mathbf{A})} {}^0 f(h) \phi(e_n h) |\det h|^s dh$$

valid for $\Re(s) > 1$.

More generally, for every $1 \leqslant r \leqslant n$, let $\mathrm{N}_{r,n}$ be the unipotent radical of the standard parabolic subgroup of G_n with Levi component $\mathrm{G}_r \times (\mathrm{G}_1)^{n-r}$, $\mathrm{N}'_{r,n}$ be its intersection with G'_n and set

$${}^0 f_{\mathrm{N}_{r,n}, \psi}(g) = \int_{[\mathrm{N}_{r,n}]} {}^0 f(ug) \psi_n(u)^{-1} du, \quad g \in \mathrm{G}(\mathbf{A}),$$

$$\mathrm{Z}^{\mathrm{FR}}_r(s, {}^0 f, \phi) = \int_{\mathcal{P}'_r(\mathrm{F}) \mathrm{N}'_{r,n}(\mathbf{A}) \backslash \mathrm{G}'_n(\mathbf{A})} {}^0 f_{\mathrm{N}_{r,n}, \psi}(h) \phi(e_n h) |\det h|^s dh, \quad s \in \mathbf{C},$$

provided the last expression above is convergent. The proof of the next lemma will be given in Section 6.5.

Lemma **6.4.0.1.** — *For every $1 \leqslant r \leqslant n$, there exists $c_r > 0$ such that the expression defining $\mathrm{Z}^{\mathrm{FR}}_r(s, {}^0 f, \phi)$ converges absolutely for $\Re(s) > c_r$.*

When $r = 1$, we have $\mathrm{N}_{1,n} = \mathrm{N}_n$ and ${}^0 f_{\mathrm{N}_{1,n}, \psi} = \mathrm{W}_{{}^0 f}$ so that $\mathrm{Z}^{\mathrm{FR}}_1(s, {}^0 f, \phi) = \mathrm{Z}^{\mathrm{FR}}_\psi(s, {}^0 f, \phi)$. Therefore, by (6.4.0.1) and (6.4.0.2), the second part of Theorem 6.2.6.1 is a consequence of the following proposition.

Proposition **6.4.0.2.** — *For every* $1 \leqslant r \leqslant n$, *the function* $s \mapsto (s-1)Z_r^{FR}(s, {}^0 f, \phi)$ *extends to a holomorphic function on* $\{s \in \mathbf{C} \mid \Re(s) > 1\}$ *admitting a limit at* $s = 1$. *Moreover, we have*

$$\lim_{s \to 1^+} (s-1)Z_n^{FR}(s, {}^0 f, \phi) = \lim_{s \to 1^+} (s-1)Z_r^{FR}(s, {}^0 f, \phi).$$

Proof. — By descending induction on r, it suffices to establish the following:

(**6.4.0.3**) Let $1 \leqslant r \leqslant n-1$. There exists a function F_r holomorphic on $\mathcal{H}_{>1-\epsilon}$ for some $\epsilon > 0$ such that

$$Z_{r+1}^{FR}(s, {}^0 f, \phi) = Z_r^{FR}(s, {}^0 f, \phi) + F_r(s)$$

for all $s \in \mathbf{C}$ satisfying $\Re(s) > \max(c_r, c_{r+1})$.

Indeed, as $\mathcal{P}'_{r+1} = G'_r U'_r$, we have

$$Z_{r+1}^{FR}(s, {}^0 f, \phi) = \int_{G'_r(F)N'_{r,n}(\mathbf{A})\backslash G'_n(\mathbf{A})} \int_{[U'_{r+1}]} {}^0 f_{N_{r+1,n},\psi}(uh)\, du\, \phi(e_n h) |\det h|^s dh. \tag{6.4.0.4}$$

By Fourier inversion on the locally compact abelian group $U_{r+1}(F)U'_{r+1}(\mathbf{A})\backslash U_{r+1}(\mathbf{A})$, we have

$$\int_{[U'_{r+1}]} {}^0 f_{N_{r+1,n},\psi}(uh)\, du = \sum_{\gamma \in \mathcal{P}'_r(F)\backslash G'_r(F)} ({}^0 f_{N_{r+1,n},\psi})_{U_{r+1},\psi}(\gamma h) + ({}^0 f_{N_{r+1,n},\psi})_{U_{r+1}}(h) \tag{6.4.0.5}$$

for all $h \in G'_n(\mathbf{A})$ where we have set

$$({}^0 f_{N_{r+1,n},\psi})_{U_{r+1},\psi}(h) = \int_{[U_{r+1}]} {}^0 f_{N_{r+1,n},\psi}(uh)\psi_n(u)^{-1} du = {}^0 f_{N_{r,n},\psi}(h),$$

$$({}^0 f_{N_{r+1,n},\psi})_{U_{r+1}}(h) = \int_{[U_{r+1}]} {}^0 f_{N_{r+1,n},\psi}(uh)\, du.$$

By (6.4.0.4) and (6.4.0.5), we obtain

$$Z_{r+1}^{FR}(s, {}^0 f, \phi) = Z_r^{FR}(s, {}^0 f, \phi) + F_r(s)$$

for all $s \in \mathbf{C}$ such that $\Re(s) > \max(c_r, c_{r+1})$ and where we have set

$$F_r(s) = \int_{G'_r(F)N'_{r,n}(\mathbf{A})\backslash G'_n(\mathbf{A})} ({}^0 f_{N_{r+1,n},\psi})_{U_{r+1}}(h)\phi(e_n h)|\det h|^s dh.$$

It only remains to check that $F_r(s)$ extends to a holomorphic function on $\mathcal{H}_{>1-\epsilon}$ for some $\epsilon > 0$.

Let P_r be the standard parabolic subgroup of G_n with Levi component $M_r = G_r \times G_{n-r}$ and set $P'_r = P_r \cap G'_n$. We readily check that

$$({}^0 f_{N_{r+1,n},\psi})_{U_{r+1}}(h) = \int_{[N_{n-r}]} {}^0 f_{P_r}(\begin{pmatrix} I_r & \\ & u \end{pmatrix} h)\psi_n(u)^{-1} du$$
$$= \int_{[N_{n-r}]} \int_{A^\infty_{G_n}} f_{P_r}(\begin{pmatrix} I_r & \\ & u \end{pmatrix} ah) da \psi_n(u)^{-1} du.$$

Therefore, by the Iwasawa decomposition $G'_n(\mathbf{A}) = P'_r(\mathbf{A})K'_n$ and since

$$\delta_{P'_r}\begin{pmatrix} h_r & \\ & h_{n-r} \end{pmatrix} = \delta_{P_r}\begin{pmatrix} h_r & \\ & h_{n-r} \end{pmatrix}^{1/2} = |\det h_r|^{n-r} |\det h_{n-r}|^{-r}$$

for all $h_r \in G'_r(\mathbf{A})$, $h_{n-r} \in G'_{n-r}(\mathbf{A})$, we have (for $\Re(s) > \max(c_r, c_{r+1})$ and a suitable choice of Haar measure on K'_n)

$$\textbf{(6.4.0.6)} \qquad F_r(s) = \int_{K'_n \times [G'_r] \times N'_{n-r}(\mathbf{A}) \backslash G'_{n-r}(\mathbf{A})} \int_{[N_{n-r}]} \int_{A^\infty_{G_n}} f_{P_r,k,s}\left(a \begin{pmatrix} h_r & \\ & uh_{n-r} \end{pmatrix}\right) da$$
$$\psi_n(u)^{-1} du |\det h_{n-r}|^{ns/(n-r)}$$
$$\times \phi_{k,n-r}(e_{n-r} h_{n-r}) dh_{n-r} dh_r dk$$

where $f_{P_r,k,s} = \delta_{P_r}^{-1/2+s/2(n-r)} (R(k)f)_{P_r} |_{M_r(\mathbf{A})}$ and $\phi_{k,n-r}$ stands for the composition of $R(k)\phi$ with the inclusion $\mathbf{A}^{n-r} \to \mathbf{A}^n$, $x \mapsto (0, x)$. Let χ^M be the inverse image of χ in $\mathfrak{X}(M_r)$. By Corollary 2.9.7.2, we have $f_{P_r,k,s} \in \mathcal{C}_{\chi^M}([M_r])$ for every $(k, s) \in K_n \times \mathcal{H}_{>0}$ and the map

$$(k, s) \in K_n \times \mathcal{H}_{>0} \mapsto f_{P_r,k,s} \in \mathcal{C}_{\chi^M}([M_r])$$

is continuous, holomorphic in the second variable. In particular, for $\Re(s) > 0$ the integral

$$\int_{A^\infty_{G'_r}} \int_{[N_{n-r}]} \int_{A^\infty_{G_n}} f_{P_r,k,s}\left(a \begin{pmatrix} a'h_r & \\ & uh_{n-r} \end{pmatrix}\right) da da'$$

is absolutely convergent and equals, by the obvious change of variable, to

$$\frac{n}{n-r} \int_{A^\infty_{G'_r}} \int_{[N_{n-r}]} \int_{A^\infty_{G_{n-r}}} f_{P_r,k,s}\begin{pmatrix} a'h_r & \\ & uah_{n-r} \end{pmatrix} da da'.$$

It follows that (6.4.0.6) can be rewritten, for $\Re(s) \gg 1$, as

$$\textbf{(6.4.0.7)} \qquad F_r(s) = \frac{n}{n-r} \int_{K'_n} \int_{[G'_r] \times N'_{n-r}(\mathbf{A}) \backslash G'_{n-r}(\mathbf{A})} \int_{[N_{n-r}]} \int_{A^\infty_{G'_r}} f_{P_r,k,s}\begin{pmatrix} h_r & \\ & uah_{n-r} \end{pmatrix} da$$
$$\times \psi_n(u)^{-1} du |\det h_{n-r}|^{ns/(n-r)} \phi_{k,n-r}(e_{n-r} h_{n-r}) dh_{n-r} dh_r dk$$

$$= \frac{n}{n-r} \int_{K'_n} (P_{G'_r} \widehat{\otimes} \mathcal{Z}^{FR}_{n-r}(\frac{ns}{n-r}))(f_{P_r,k,s} \otimes \phi_{k,n-r}) dk$$

where $\mathcal{Z}^{FR}_{n-r}(s)$ stands for the bilinear form

$$(f', \phi') \in \mathcal{C}([G_{n-r}]) \times \mathcal{S}(\mathbf{A}^{n-r}) \mapsto Z^{FR}_{\psi^{(-1)^r}}(s, {}^0f', \phi').$$

On the other hand, by (2.9.6.10) we have

$$\mathcal{C}_{\chi^M}([M_r]) = \bigoplus_{(\chi_1,\chi_2)\in\mathfrak{X}(M_r)=\mathfrak{X}(G_r)\times\mathfrak{X}(G_{n-r})\mapsto\chi} \mathcal{C}_{\chi_1}([G_r]) \widehat{\otimes} \mathcal{C}_{\chi_2}([G_{n-r}])$$

and, as $\chi \in \mathfrak{X}^*(G_n)$, for every $(\chi_1, \chi_2) \in \mathfrak{X}(G_r) \times \mathfrak{X}(G_{n-r})$ mapping to $\chi \in \mathfrak{X}(G_n)$, we also have $\chi_2 \in \mathfrak{X}^*(G_{n-r})$. Therefore, by the first part of Theorem 6.2.6.1, Theorem 6.2.5.1 and (A.0.5.5), $s \mapsto P_{G'_r} \widehat{\otimes} \mathcal{Z}^{FR}_{n-r}(s)$ extends to an analytic family of (separately) continuous bilinear forms on $\mathcal{C}_\chi([M_r]) \times \mathcal{S}(\mathbf{A}^{n-r})$ for $s \in \mathcal{H}_{>1}$. Thus, by the first part of Theorem 6.2.6.1, (A.0.5.4) and the equality (6.4.0.7), $F_r(s)$ has an analytic continuation to $\{\Re(s) > 1 - r/n\}$. This ends the proof of the proposition and hence of Theorem 6.2.6.1. □

6.5. *Convergence of Zeta integrals*

6.5.1.

Proof of Lemma 6.4.0.1. — We only treat the case $1 \leqslant r \leqslant n-1$. The case $r = n$ can be dealt with in a similar manner, and is in fact easier.

Let Q_r be the standard parabolic subgroup of G_n with Levi component $G_r \times G_1^{n-r}$ and set $Q'_r = Q_r \cap G'_n$. Recall that $N_{r,n}$ is the unipotent radical of Q_r. Identifying $A^\infty_{G_n} \simeq \mathbf{R}_{>0}$, by the Iwasawa decomposition $G'_n(\mathbf{A}) = Q'_r(\mathbf{A})K'_n$, we need to show the convergence of

$$(\mathbf{6.5.1.1}) \quad \int_{K'_n \times \mathcal{P}'_r(F)\backslash G'_r(\mathbf{A}) \times T'_{n-r}(\mathbf{A}) \times \mathbf{R}_{>0}} \left| (R(k)f)_{N_{r,n},\psi} \begin{pmatrix} ah & \\ & at \end{pmatrix} \right| |R(k)\phi(t_{n-r}e_n)| \\ |\det h|^s |\det t|^s \delta_{Q'_r} \begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} dadtdhdk$$

for $\Re(s) \gg 1$. We now apply Lemma 2.6.1.1. For this we note that $\psi_n |_{[N_{r,n}]} = \psi' \circ \ell$ where $\ell : N_{r,n} \to \mathbf{G}_a$ sends $u \in N_{r,n}$ to $\mathrm{Tr}_{E/F}(\tau \sum_{i=r}^{n-1} u_{i,i+1})$ and $\tau \in E^\times$ is the unique trace-zero element such that $\psi(z) = \psi'(\mathrm{Tr}_{E/F}(\tau z))$. We readily check that

$$\|\mathrm{Ad}^*(m)\ell\|_{V_{Q_r}} \sim \|t_1^{-1}e_r h\|_{\mathbf{A}^r} \prod_{i=1}^{n-r-1} \|t_i t_{i+1}^{-1}\|_{\mathbf{A}},$$

$$\text{for } m = \begin{pmatrix} h & \\ & t \end{pmatrix} \in \mathrm{G}'_r(\mathbf{A}) \times \mathrm{T}'_{n-r}(\mathbf{A}).$$

Therefore, by Lemma 2.6.1.1.1, we can find $c > 0$ such that for every $\mathrm{N}_1, \mathrm{N}_2 > 0$ we have

(**6.5.1.2**)
$$\left| (\mathrm{R}(k)f)_{\mathrm{N}_{r,n},\psi} \begin{pmatrix} ah & \\ & at \end{pmatrix} \right| \\ \ll \|ah\|_{\mathrm{G}'_r}^{-\mathrm{N}_2} \|t_1^{-1} e_r h\|_{\mathbf{A}^r}^{-\mathrm{N}_1} \prod_{i=1}^{n-r-1} \|t_i t_{i+1}^{-1}\|_{\mathbf{A}}^{-\mathrm{N}_1} \delta_{\mathrm{Q}_r} \begin{pmatrix} h & \\ & t \end{pmatrix}^{-c\mathrm{N}_2}$$

for $(k, h, t, a) \in \mathrm{K}'_n \times \mathrm{G}'_r(\mathbf{A}) \times \mathrm{T}'_{n-r}(\mathbf{A}) \times \mathbf{R}_{>0}$. On the other hand, for every $\mathrm{N}_1 > 0$, we have

$$|\mathrm{R}(k)\phi(te_n)| \ll \|t\|_{\mathbf{A}}^{-\mathrm{N}_1}, \quad (k, t) \in \mathrm{K}'_n \times \mathbf{A}$$

and it is easy to check that for some $\mathrm{N}_2 > 0$ we have

$$\|e_r h\|_{\mathbf{A}^r} \prod_{i=1}^{n-r} \|t_i\|_{\mathbf{A}} \ll \|t_{n-r}\|_{\mathbf{A}}^{\mathrm{N}_2} \|t_1^{-1} e_r h\|_{\mathbf{A}^r}^{\mathrm{N}_2} \prod_{i=1}^{n-r-1} \|t_i t_{i+1}^{-1}\|_{\mathbf{A}}^{\mathrm{N}_2}, \\ (h, t) \in \mathrm{G}'_r(\mathbf{A}) \times \mathrm{T}'_{n-r}(\mathbf{A}).$$

As $\delta_{\mathrm{Q}'_r} \begin{pmatrix} h & \\ & t \end{pmatrix} = |\det h|^{n-r} \prod_{i=1}^{n-r} |t_i|^{n+1-2(r+i)}$ for every $(h, t) \in \mathrm{G}'_r(\mathbf{A}) \times \mathrm{T}'_{n-r}(\mathbf{A})$, combining this with (6.5.1.2), we deduce the existence of $c > 0$ such that for every $\mathrm{N}_1, \mathrm{N}_2 > 0$, (6.5.1.1) is essentially bounded by the product of

(**6.5.1.3**)
$$\int_{\mathcal{P}'_r(\mathrm{F})\backslash \mathrm{G}'_r(\mathbf{A}) \times \mathbf{R}_{>0}} \|ah\|_{\mathrm{G}'_r}^{-\mathrm{N}_2} \|e_r h\|_{\mathbf{A}^r}^{-\mathrm{N}_1} |\det h|^{s-(2c\mathrm{N}_2+1)(n-r)} da dh$$

and

(**6.5.1.4**)
$$\int_{\mathbf{A}^\times} \|t\|_{\mathbf{A}}^{-\mathrm{N}_1} |t|^{s-(2c\mathrm{N}_2+1)(n+1-2(r+i))} dt$$

for $1 \leqslant i \leqslant n-r$.

Let $\mathrm{C}_1, \mathrm{C}_2 > 0$. By Lemma 2.6.2.1, for N_1 sufficiently large the integral (6.5.1.4) converges absolutely in the range

$$1 + (2c\mathrm{N}_2 + 1)(n+1-2(r+i)) < \Re(s) < \mathrm{C}_1 + (2c\mathrm{N}_2+1)(n+1-2(r+i))$$

and for N_1, N_2 sufficiently large the integral (6.5.1.3) converges absolutely in the range

$$1 + (2c\mathrm{N}_2 + 1)(n-r) < \Re(s) < \mathrm{C}_2 + (2c\mathrm{N}_2 + 1)(n-r).$$

Since $n+1-2(r+i) < n-r$ for every $1 \leqslant i \leqslant n-r$, by taking $C_2 = 2$ and $C_1 \geqslant 2+(2cN_2+1)(r+2i-1)$ for every $1 \leqslant i \leqslant n-r$, it follows that if $N_2 \gg 1$ and $N_1 \gg_{N_2} 1$ the integrals (6.5.1.3) and (6.5.1.4) are convergent in the range

$$1+(2cN_2+1)(n-r) < \Re(s) < 2+(2cN_2+1)(n-r).$$

The union of these open intervals for N_2 sufficiently large as above is of the form $]c_r, +\infty[$ which shows that $Z_r^{FR}(s, {}^0 f, \phi)$ converges absolutely in the range $\Re(s) > c_r$ for a suitable $c_r > 0$. □

6.5.2.

Proof of Theorem 6.2.5.1.1. — Applying Lemma 2.6.1.1.2, the same manipulations as in the proof of Lemma 6.4.0.1 reduce us to showing the existence of $c_N > 0$ such that for every $C > c_N$ there exists $N' > 0$ satisfying that the integral

$$(\mathbf{6.5.2.5}) \qquad \int_{T'_n(\mathbf{A})} \prod_{i=1}^{n} \|t_i\|_{\mathbf{A}}^{-N'} \|t\|_{T'_n}^{N} \delta_{B'_n}(t)^{-1} |\det t|^s dt$$

converges in the range $s \in \mathcal{H}_{]c_n,C[}$ uniformly on compact subsets. But this follows again from Lemma 2.6.2.1 as there exists $M > 0$ such that

$$\|t\|_{T'_n}^{N} \delta_{B'_n}(t)^{-1} \ll \prod_{1 \leqslant i \leqslant n} \max(|t_i|, |t_i|^{-1})^M, \quad t \in [T'_n]. \qquad \square$$

7. Canonical extension of the Rankin-Selberg period for H-regular cuspidal data

This section is a continuation of Section 6 and we shall use the notation introduced there. The main goal is to show the existence of a canonical extension of corank one Rankin-Selberg periods to the space of uniform moderate growth functions for certain cuspidal data (see Theorem 7.1.3.1). Combining this with the results of Section 6, this will enable us to give an alternative proof of the spectral expansion of the Jacquet-Rallis trace formula for certain cuspidal data in Section 8.

7.1. *Statements of the main results*

7.1.1. Let $n \geqslant 1$ be a positive integer. We set $G = G_n \times G_{n+1}$ and $H = G_n$ that we consider as an algebraic subgroup of G via the diagonal inclusion $H \hookrightarrow G$. We also set $w = (w_n, w_{n+1}) \in G(F)$, $K = K_n \times K_{n+1}$, $N = N_n \times N_{n+1}$ and $N_H = N_n$. Put $\psi_N = \psi_n \boxtimes \psi_{n+1}$ (a generic character of [N]). We note that ψ_N is trivial on $[N_H]$ (see the convention in the definition of ψ_n in Section 6.1.2). To any function $f \in \mathcal{T}([G])$, we associate its *Whittaker function*

$$W_f(g) = \int_{[N]} f(ug)\psi_N(u)^{-1}du, \ g \in G(\mathbf{A}).$$

For $f \in \mathcal{T}([G])$, we set

$$Z_\psi^{RS}(s,f) = \int_{N_H(\mathbf{A})\backslash H(\mathbf{A})} W_f(h)|\det h|_E^s dh$$

for every $s \in \mathbf{C}$ for which the above expression converges absolutely. We postpone the proof of the following lemma to Section 7.3.

Lemma **7.1.1.1.** — *Let* $N \geqslant 0$. *There exists* $c_N > 0$ *such that:*

- *For every* $f \in \mathcal{T}_N([G])$ *and* $s \in \mathcal{H}_{>c_N}$, *the expression defining* $Z_\psi^{RS}(s,f)$ *converges absolutely;*
- *For every* $s \in \mathcal{H}_{>c_N}$, *the functional* $f \in \mathcal{T}_N([G]) \mapsto Z_\psi^{RS}(s,f)$ *is continuous;*
- *For every* $f \in \mathcal{T}_N([G])$, *the function* $s \in \mathcal{H}_{>c_N} \mapsto Z_\psi^{RS}(s,f)$ *is holomorphic and bounded in vertical strips.*

7.1.2. H-*regular cuspidal datum.* — Let $\chi \in \mathfrak{X}(G)$ be a cuspidal datum represented by a pair (M_P, π) where $P = P_n \times P_{n+1}$ is a standard parabolic subgroup of G and $\pi = \pi_n \boxtimes \pi_{n+1}$ a cuspidal automorphic representation of $M_P(\mathbf{A})$ (with central character trivial on A_P^∞). We have decompositions

$$M_{P_n} = G_{n_1} \times \cdots \times G_{n_k}, \ \ M_{P_{n+1}} = G_{m_1} \times \cdots \times G_{m_r}$$

and π_n, π_{n+1} decompose accordingly as tensor products

$$\pi_n = \pi_{n,1} \boxtimes \ldots \boxtimes \pi_{n,k}, \ \ \pi_{n+1} = \pi_{n+1,1} \boxtimes \ldots \boxtimes \pi_{n+1,r}.$$

We say that χ is H-*regular* if it satisfies the following condition:

(**7.1.2.1**) For every $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant r$, we have $\pi_{n,i} \neq \pi_{n+1,j}^\vee$.

7.1.3.

Theorem **7.1.3.1.** — *Let* $\chi \in \mathfrak{X}(G)$ *be a* H-*regular cuspidal datum. Then,*

1. *For every* $f \in \mathcal{T}_\chi([G])$, *the function* $s \mapsto Z_\psi^{RS}(s,f)$, *a priori defined on some right half-plane, extends to an entire function on* $\mathbf{C}$*;*
2. *The restriction of the linear form*

$$P_H : f \in \mathcal{S}([G]) \mapsto \int_{[H]} f(h)dh$$

to $\mathcal{S}_\chi([G])$ *extends by continuity to* $\mathcal{T}_\chi([G])$ *and moreover for every* $f \in \mathcal{T}_\chi([G])$, *we have*

$$P_H(f) = Z_\psi^{RS}(0,f). \tag{7.1.3.2}$$

7.2. *Proof of Theorem 7.1.3.1*

Let $f \in \mathcal{S}_\chi([G])$. For $1 \leqslant r \leqslant n$, let $N_{r,n}$ and $N_{r,n+1}$ be the unipotent radicals of the standard parabolic subgroups of G_n and G_{n+1} with Levi components $G_r \times (G_1)^{n-r}$ and $G_r \times (G_1)^{n+1-r}$ respectively and set

$$N_r^G = N_{r,n} \times N_{r,n+1},\ N_r^H = N_r^G \cap H = N_{r,n},$$

$$f_{N_r^G,\psi}(g) = \int_{[N_r^G]} f(ug)\psi_N(u)^{-1}du, \quad \text{for } g \in G(\mathbf{A}).$$

For $1 \leqslant r \leqslant n$ and $s \in \mathbf{C}$, we also set, whenever this expression converges,

$$Z_r^{RS}(s,f) = \int_{\mathcal{P}_r(F)N_r^H(\mathbf{A})\backslash H(\mathbf{A})} f_{N_r^G,\psi}(h)|\det h|_E^s dh.$$

Note that we have $Z_1^{RS}(s,f) = Z_\psi^{RS}(s,f)$. The proof of the next lemma will be given in Section 7.3.

Lemma **7.2.0.1.** — *For every* $1 \leqslant r \leqslant n$, *there exists* $c_r > 0$ *such that the expression defining* $Z_r^{RS}(s,f)$ *converges absolutely for* $s \in \mathcal{H}_{>c_r}$.

For $r = n+1$ and every $s \in \mathbf{C}$, we also set

$$Z_{n+1}^{RS}(s,f) = \int_{[H]} f(h)|\det h|_E^s dh.$$

Note that the above expression is absolutely convergent and defines an entire function of $s \in \mathbf{C}$ which is bounded in vertical strips, satisfying $P_H(f) = Z_{n+1}^{RS}(0,f)$. The following proposition is the crux to the proof of Theorem 7.1.3.1.

Proposition **7.2.0.2.** — *For every* $1 \leqslant r \leqslant n$, *we have*

$$Z_{r+1}^{RS}(s,f) = Z_r^{RS}(s,f) \tag{7.2.0.1}$$

for $\Re(s)$ *sufficiently large.*

Proof. — Let $1 \leqslant r \leqslant n-1$. As $\mathcal{P}_{r+1} = G_r U_r$ and $N_{r,n} = U_r N_{r+1,n}$, for $s \in \mathcal{H}_{>c_{r+1}}$ we have

$$Z_{r+1}^{RS}(s,f) = \int_{G_r(F)N_r^H(\mathbf{A})\backslash H(\mathbf{A})} \int_{[U_{r+1}^H]} f_{N_{r+1}^G,\psi}(uh)du|\det h|_E^s dh \tag{7.2.0.2}$$

where we have set $\mathrm{U}^{\mathrm{H}}_{r+1}=\mathrm{U}_{r+1}$ viewed as a subgroup of $\mathrm{H}=\mathrm{G}_n$ (as always via the embedding in "the upper-left corner"). Similarly, we set $\mathrm{U}^{\mathrm{G}}_{r+1}=\mathrm{U}_{r+1}\times\mathrm{U}_{r+1}$ viewed as a subgroup of G. By Fourier inversion on the compact abelian group $\mathrm{U}^{\mathrm{G}}_{r+1}(\mathrm{F})\mathrm{U}^{\mathrm{H}}_{r+1}(\mathbf{A})\backslash\mathrm{U}^{\mathrm{G}}_{r+1}(\mathbf{A})$, we have

$$(\mathbf{7.2.0.3})\qquad \int_{[\mathrm{U}^{\mathrm{H}}_{r+1}]} f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi}(uh)du=\sum_{\gamma\in\mathcal{P}_r(\mathrm{F})\backslash\mathrm{G}_r(\mathrm{F})}(f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi})_{\mathrm{U}^{\mathrm{G}}_{r+1},\psi}(\gamma h)+(f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi})_{\mathrm{U}^{\mathrm{G}}_{r+1}}(h)$$

for every $h\in\mathrm{H}(\mathbf{A})$, where we have set

$$(f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi})_{\mathrm{U}^{\mathrm{G}}_{r+1},\psi}(h)=\int_{[\mathrm{U}^{\mathrm{G}}_{r+1}]} f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi}(uh)\psi_{\mathrm{N}}(u)^{-1}du=f_{\mathrm{N}^{\mathrm{G}}_{r},\psi}(h),$$

$$(f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi})_{\mathrm{U}^{\mathrm{G}}_{r+1}}(h)=\int_{[\mathrm{U}^{\mathrm{G}}_{r+1}]} f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi}(uh)du.$$

By (7.2.0.2) and (7.2.0.3), we obtain

$$(\mathbf{7.2.0.4})\qquad \mathrm{Z}^{\mathrm{RS}}_{r+1}(s,f)=\mathrm{Z}^{\mathrm{RS}}_{r}(s,f)+\mathrm{F}_r(s)$$

for every $s\in\mathbf{C}$ such that $\Re(s)>\max(c_r,c_{r+1})$ and where we have set

$$\mathrm{F}_r(s)=\int_{\mathrm{G}_r(\mathrm{F})\mathrm{N}^{\mathrm{H}}_r(\mathbf{A})\backslash\mathrm{H}(\mathbf{A})}(f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi})_{\mathrm{U}^{\mathrm{G}}_{r+1}}(h)|\det h|^s_{\mathrm{E}}dh.$$

By a similar argument, (7.2.0.4) still holds when $n=r$ if we set

$$f_{\mathrm{U}^{\mathrm{G}}_{n+1}}(h)=\int_{[\mathrm{U}^{\mathrm{G}}_{n+1}]} f(uh)du \text{ and } \mathrm{F}_n(s)=\int_{[\mathrm{H}]} f_{\mathrm{U}^{\mathrm{G}}_{n+1}}(h)|\det h|^s_{\mathrm{E}}dh$$

where $\mathrm{U}^{\mathrm{G}}_{n+1}=1\times\mathrm{U}_{n+1}$.

From (7.2.0.4), we are reduced to showing that $\mathrm{F}_r(s)=0$ identically for every $1\leqslant r\leqslant n$ and $\Re(s)$ sufficiently large. To uniformize notation, we set $f_{\mathrm{N}^{\mathrm{G}}_{n+1},\psi}=f$. Let P_r be the standard parabolic subgroup of G with Levi component $\mathrm{L}_r=(\mathrm{G}_r\times\mathrm{G}_{n-r})\times(\mathrm{G}_r\times\mathrm{G}_{n+1-r})$ and set $\mathrm{P}^{\mathrm{H}}_r=\mathrm{P}_r\cap\mathrm{H}$. Then, we readily check that

$$(f_{\mathrm{N}^{\mathrm{G}}_{r+1},\psi})_{\mathrm{U}^{\mathrm{G}}_{r+1}}(h)=\int_{[\mathrm{N}_{n-r}]\times[\mathrm{N}_{n+1-r}]} f_{\mathrm{P}_r}\left(\left(\begin{pmatrix}\mathrm{I}_r & \\ & u\end{pmatrix},\begin{pmatrix}\mathrm{I}_r & \\ & u'\end{pmatrix}\right)h\right)$$
$$\times\,\psi_n(u)^{-1}\psi_{n+1}(u')^{-1}du'du$$

for every $h \in H(\mathbf{A})$ and $1 \leqslant r \leqslant n$. Therefore, by the Iwasawa decomposition $H(\mathbf{A}) = P_r^H(\mathbf{A})K_n$, we have

$$F_r(s) = \int_{K_n} \int_{[G_r]\times N_{n-r}(\mathbf{A})\backslash G_{n-r}(\mathbf{A})} \int_{[N_{n-r}]\times[N_{n+1-r}]} f_{P_r}\left(\begin{pmatrix} h_r & \\ & uh_{n-r}\end{pmatrix} k, \right.$$
$$\left.\begin{pmatrix} h_r & \\ & u'h_{n-r}\end{pmatrix} k\right) \psi_n(u)^{-1}\psi_{n+1}(u')^{-1} du' du \delta_{P_r^H}\begin{pmatrix} h_r & \\ & h_{n-r}\end{pmatrix}^{-1}$$
$$\times |\det h_r|_E^s |\det h_{n-r}|_E^s dh_{n-r} dh_r dk.$$

By a painless calculation, left to the reader, we have

$$\delta_{P_r^H}\begin{pmatrix} h_r & \\ & h_{n-r}\end{pmatrix}^{-1} |\det h_r|_E^s |\det h_{n-r}|_E^s$$
$$= \delta_{P_r}\left(\begin{pmatrix} h_r & \\ & h_{n-r}\end{pmatrix}, \begin{pmatrix} h_r & \\ & h_{n-r}\end{pmatrix}\right)^{-\frac{1}{2}+\alpha_r(s)} |\det h_{n-r}|_E^{s+2r\alpha_r(s)}$$

where $\alpha_r(s) = \frac{2s+1}{4n-4r+2}$. Let χ^L be the inverse image of χ in $\mathfrak{X}(L_r)$. By Corollary 2.9.7.2, for $\Re(\alpha_r(s)) > 0$ and every $k \in K_n$ the function $f_{P_r,k,s} := \delta_{P_r}^{-\frac{1}{2}+\alpha_r(s)} R(k) f_{P_r}|_{[L_r]}$ belongs to $\mathcal{C}_{\chi^L}([L_r])$. On the other hand, as $L_r = G_r \times G_{n-r} \times G_r \times G_{n+1-r}$, by (2.9.6.10) we have the decomposition

$$\mathcal{C}_{\chi^L}([L_r]) = \bigoplus_{(\chi_1,\chi_2)\in\mathfrak{X}(G_r^2)\times\mathfrak{X}(G_{n-r}\times G_{n+1-r})\mapsto\chi} \mathcal{C}_{\chi_1}([G_r\times G_r])$$
$$\widehat{\otimes}\mathcal{C}_{\chi_2}([G_{n-r}\times G_{n+1-r}])$$

and the above equality can be rewritten as

$$\textbf{(7.2.0.5)} \qquad F_r(s) = \int_{K_n} \left(P_{G_r^\Delta}\widehat{\otimes}\mathcal{Z}_{n-r}^{RS}(s+2r\alpha_r(s))\right)(f_{P_r,k,s})dk$$

where $P_{G_r^\Delta}$ denotes the period integral over the diagonal subgroup of $G_r \times G_r$ and $\mathcal{Z}_{n-r}^{RS}(s)$ stands for the continuous linear form

$$f' \in \mathcal{C}([G_{n-r}\times G_{n+1-r}]) \mapsto Z^{RS}_{\psi^{(-1)^r}}(s, f').$$

Since χ is H-regular, by (7.1.2.1) for every preimage $(\chi_1, \chi_2) \in \mathfrak{X}(G_r^2)\times\mathfrak{X}(G_{n-r}\times G_{n+1-r})$ of χ with $\chi_1 = (\chi_1', \chi_1'') \in \mathfrak{X}(G_r)^2$ we have $\chi_1'' \neq (\chi_1')^\vee$. Hence, by definition of $\mathcal{C}_{\chi_1}([G_r\times G_r])$, $P_{G_r^\Delta}$ vanishes identically on $\mathcal{C}_{\chi_1}([G_r\times G_r])$. This implies that $F_r(s) = 0$ whenever $\Re(s) \gg 1$ and this ends the proof of the proposition. □

We can now finish the proof of Theorem 7.1.3.1. From the proposition, we deduce that

$$Z_\psi^{RS}(s,f) = \int_{[H]} f(h)|\det h|^s dh \tag{7.2.0.6}$$

for $\Re(s) \gg 1$ and every $f \in \mathcal{S}_\chi([G])$. In particular, it follows from this equality that $s \mapsto Z_\psi^{RS}(s,f)$ extends to an entire function on $\mathbf{C}$ that is bounded in vertical strips and satisfies the functional equation

$$Z_\psi^{RS}(-s,\widetilde{f}) = Z_\psi^{RS}(s,f)$$

where $\widetilde{f}(g) = f({}^t g^{-1})$. By Corollary A.0.11.2, we can now deduce the first part of the theorem from the above functional equation, (2.5.10.8), (2.9.5.9), Lemma 7.1.1.1. This corollary also entails that the linear map $f \in \mathcal{T}_\chi([G]) \mapsto Z_\psi(0,f)$ is continuous. As, by (7.2.0.6), this functional coincides with P_H on $\mathcal{S}_\chi([G])$, this proves the second part of the theorem.

7.3. *Convergence of Zeta integrals*

Proof of Lemma 7.2.0.1. — The argument is very similar to the proof of Lemma 6.4.0.1 so we only sketch it. Let $1 \leqslant r \leqslant n$ and Q_r^G be the standard parabolic subgroup of G with Levi component $(G_r \times (G_1)^{n-r}) \times (G_r \times (G_1)^{n+1-r})$ so that N_r^G is the unipotent radical of Q_r^G. Set $Q_r^H = Q_r^G \cap H$. By the Iwasawa decomposition $H(\mathbf{A}) = Q_r^H(\mathbf{A})K_n$, we need to show the convergence of

$$\int_{K_n \times \mathcal{P}_r(F)\backslash G_r(\mathbf{A}) \times T_{n-r}(\mathbf{A})} \left| (R(k)f)_{N_r^G,\psi} \begin{pmatrix} h & \\ & t \end{pmatrix} \right| \times |\det h|_E^s |\det t|_E^s \delta_{Q_r^H} \begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} dt dh dk \tag{7.3.0.1}$$

for $\Re(s) \gg 1$. We apply Lemma 2.6.1.1.1 to $\psi_F = \psi'$ and

$$\ell : N_r^G \to \mathbf{G}_a,$$

$$(u,u') \mapsto \mathrm{Tr}_{E/F}\left((-1)^n \tau \sum_{i=r}^{n-1} u_{i,i+1} + (-1)^{n+1} \tau \sum_{i=r}^{n} u'_{i,i+1} \right).$$

We readily check that there exists $N_0 > 0$ such that

$$\|e_r h\|_{\mathbf{A}_E^r} \prod_{i=1}^{n-r} \|t_i\|_{\mathbf{A}_E} \ll \left\|\mathrm{Ad}^* \begin{pmatrix} h & \\ & t \end{pmatrix} \ell \right\|_{V_{Q_r^G}}^{N_0}, \text{ for } (h,t) \in G_r(\mathbf{A}) \times T_{n-r}(\mathbf{A}). \tag{7.3.0.2}$$

Therefore, from (7.3.0.2) and Lemma 2.6.1.1, there exists $c > 0$ such that for every $N_1, N_2 > 0$, (7.3.0.1) is essentially bounded by

$$\int_{\mathcal{P}_r(F)\backslash G_r(\mathbf{A})\times T_{n-r}(\mathbf{A})} \|h\|_{G_r}^{-N_2} \|e_r h\|_{\mathbf{A}_E^r}^{-N_1} \prod_{i=1}^{n-r} \|t_i\|_{\mathbf{A}_E}^{-N_1} \delta_{Q_r^G}\begin{pmatrix} h & \\ & t \end{pmatrix}^{-cN_2}$$
$$\times \delta_{Q^H}\begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} |\det h|_E^s |\det t|_E^s\, dt dh.$$

Now, the convergence of the above expression for $\Re(s) \gg 1$, $N_2 \gg_s 1$ and $N_1 \gg_{s,N_2} 1$ can be shown as in the end of the proof of Lemma 6.4.0.1 using Lemma 2.6.2.1. □

Proof of Lemma 7.1.1.1. — Applying Lemma 2.6.1.1.2 in a similar way, we are reduced to showing the existence of $c_N > 0$ such that for every $C > c_N$ there exists $N' > 0$ satisfying that the integral

$$\int_{T_n(\mathbf{A})} \prod_{i=1}^{n} \|t_i\|_{\mathbf{A}_E}^{-N'} \|t\|_{T_n}^{N} \delta_{B_n}(t)^{-1} |\det t|_E^s\, dt$$

converges in the range $s \in \mathcal{H}_{]c_n, C[}$ uniformly on any compact subsets. This is exactly what was established in the proof of Theorem 6.2.5.1.1 (up to replacing the base field F by E). □

8. Contributions of ∗-regular cuspidal data to the Jacquet-Rallis trace formula: second proof

In this section, we adopt the set of notation introduced in Section 5. In particular, $n \geqslant 1$ is a positive integer, $G = G_n \times G_{n+1}$, $G' = G'_n \times G'_{n+1}$, $H = G_n$ with its diagonal embedding in G, $K = K_n \times K_{n+1}$ and $K' = K'_n \times K'_{n+1}$ are the standard maximal compact subgroups of $G(\mathbf{A})$ and $G'(\mathbf{A})$ respectively and $\eta_{G'} : [G'] \to \{\pm 1\}$ is the automorphic character defined in Section 3.1.6. We will also use notation from Sections 6 and 7: $N = N_n \times N_{n+1}$ and $N_H = N_n$ are the standard maximal unipotent subgroups of G and H, $\psi_N = \psi_n \boxtimes \psi_{n+1}$ is a generic character of [N] (where ψ_n and ψ_{n+1} are defined as in Section 6.1.2). We also set $\mathcal{P} = \mathcal{P}_n \times \mathcal{P}_{n+1}$ (resp. $\mathcal{P}' = \mathcal{P}'_n \times \mathcal{P}'_{n+1}$) where $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ (resp. $\mathcal{P}'_n$ and $\mathcal{P}'_{n+1}$) stand for the mirabolic subgroups of G_n and G_{n+1} (resp. of G'_n and G'_{n+1}), $T = T_n \times T_{n+1}$ for the standard maximal torus of G and $N' = N'_n \times N'_{n+1}$ for the standard maximal unipotent subgroup of G'. Finally, as in Section 7.1.1, for every $f \in \mathcal{T}([G])$ we set

$$W_f(g) = \int_{[N]} f(ug)\psi_N(u)^{-1} du, \quad g \in G(\mathbf{A}).$$

8.1. *Main result*

8.1.1. Let $\chi \in \mathfrak{X}^*(G)$ be a $*$-regular cuspidal datum (see Section 5.1.3) represented by a pair (M_P, π). We set $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi)$. We have decompositions $P = P_n \times P_{n+1}$, $\pi = \pi_n \boxtimes \pi_{n+1}$ and $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ where: P_n, P_{n+1} are standard parabolic subgroups of G_n, G_{n+1} respectively with standard Levi components of the form

$$M_{P_n} = G_{n_1} \times \cdots \times G_{n_k}, \;\; M_{P_{n+1}} = G_{m_1} \times \cdots \times G_{m_r},$$

π_n and π_{n+1} are cuspidal automorphic representations of $M_{P_n}(\mathbf{A})$, $M_{P_{n+1}}(\mathbf{A})$ decomposing into tensor products

$$\pi_n = \pi_{n,1} \boxtimes \ldots \boxtimes \pi_{n,k}, \;\; \pi_{n+1} = \pi_{n+1,1} \boxtimes \ldots \boxtimes \pi_{n+1,r}$$

respectively and we have set $\Pi_n = \mathrm{Ind}_{P_n(\mathbf{A})}^{G_n(\mathbf{A})}(\pi_n)$, $\Pi_{n+1} = \mathrm{Ind}_{P_{n+1}(\mathbf{A})}^{G_{n+1}(\mathbf{A})}(\pi_{n+1})$. We write $\chi_n \in \mathfrak{X}^*(G_n)$ and $\chi_{n+1} \in \mathfrak{X}^*(G_{n+1})$ for the cuspidal data determined by the pairs (M_{P_n}, π_n) and $(M_{P_{n+1}}, \pi_{n+1})$ respectively.

The representation Π is generic and we denote by $\mathcal{W}(\Pi, \psi_N)$ its Whittaker model with respect to the character ψ_N. Also, for every $\phi \in \Pi$ we define

$$W_\phi(g) := W_{E(\phi)}(g) = \int_{[N]} E(ug, \phi)\psi_N(u)^{-1} du, \;\; g \in G(\mathbf{A}).$$

Note that $W_\phi \in \mathcal{W}(\Pi, \psi_N)$.

8.1.2. We now define two continuous linear forms λ and β_η as well as a continuous invariant scalar product $\langle ., . \rangle_{\mathrm{Whitt}}$ on $\mathcal{W}(\Pi, \psi_N)$. Let $W \in \mathcal{W}(\Pi, \psi_N)$.

- By [JPSS83] [Jac09], the Zeta integral (already encountered in Section 7)

  $$Z^{RS}(s, W) = \int_{N_H(\mathbf{A})\backslash H(\mathbf{A})} W(h) |\det h|_{\mathbf{A}_E}^s \, dh,$$

  converges for $\Re(s) \gg 0$ and extends to a meromorphic function on $\mathbf{C}$ with no pole at $s = 0$. We set

  $$\lambda(W) = Z^{RS}(0, W).$$

- For S a sufficiently large finite set of places of F, we put

  $$\beta_\eta(W) = (\Delta_{G'}^{S,*})^{-1} L^{S,*}(1, \Pi, \mathrm{As}_G) \int_{N'(F_S)\backslash \mathcal{P}'(F_S)} W(p_S)\eta_{G'}(p_S) dp_S$$

  where we have set $L(s, \Pi, \mathrm{As}_G) = L(s, \Pi_n, \mathrm{As}^{(-1)^{n+1}}) L(s, \Pi_{n+1}, \mathrm{As}^{(-1)^n})$.

- Similarly, for S a sufficiently large finite set of places of F, we put

$$\langle W, W\rangle_{\mathrm{Whitt}} = (\Delta_G^{S,*})^{-1} L^{S,*}(1, \Pi, \mathrm{Ad}) \int_{N(F_S)\backslash \mathcal{P}(F_S)} |W(p_S)|^2 dp_S$$

  where we have set $L(s, \Pi, \mathrm{Ad}) = L(s, \Pi_n \times \Pi_n^\vee) L(s, \Pi_{n+1} \times \Pi_{n+1}^\vee)$.

That the above expressions converge and are independent of S as soon as it is chosen sufficiently large (depending on the level of W) follow from [Fli88] and [JS81b]. Moreover, the inner form $\langle ., .\rangle_{\mathrm{Whitt}}$ is $G(\mathbf{A})$-invariant by [Ber84] and [Bar03].

The next result follows from works of Jacquet-Shalika [JS81b], Shahidi [Sha81] and Lapid-Offen [FLO12, Appendix A]. For completeness, we explain the deduction (see Section 2.7.2 for our normalization of the Petersson inner product).

Theorem **8.1.2.1** *(Jacquet-Shalika, Shahidi, Lapid-Offen). — We have*

$$\langle \phi, \phi\rangle_{\mathrm{Pet}} = \langle W_\phi, W_\phi\rangle_{\mathrm{Whitt}}$$

for every $\phi \in \Pi$.

Proof. — Let $\phi \in \Pi$. By the Iwasawa decomposition, for a suitable Haar measure on K we have

$$\langle \phi, \phi\rangle_{\mathrm{Pet}} = \int_K \int_{[M_P]_0} |\phi(mk)|^2 \delta_P(m)^{-1} dm dk.$$

Set $N^P = N \cap M_P$ and

$$\phi_{N^P,\psi}(g) = \int_{[N^P]} \phi(ug)\psi_N(u)^{-1} du, \quad g \in G(\mathbf{A}).$$

Let $\mathcal{P}^P$ be the product of mirabolic groups $\prod_{i=1}^k \mathcal{P}_{n_i} \times \prod_{j=1}^r \mathcal{P}_{m_j}$. It is a subgroup of M_P. According to Jacquet-Shalika [JS81b, §4] (see also [FLO12, p. 265] or [Zha14a, Proposition 3.1][10]), for S a sufficiently large finite set of places of F we have

$$\begin{aligned}(\mathbf{8.1.2.1})\qquad &\int_{[M_P]_0} |\phi(mk)|^2 \delta_P(m)^{-1} dm \\ &= (\Delta_{M_P}^{S,*})^{-1} \prod_{i=1}^{k} \mathrm{Res}_{s=1} L^S(s, \pi_{n,i} \times \pi_{n,i}^\vee) \prod_{j=1}^{r} \mathrm{Res}_{s=1} L^S(s, \pi_{n+1,j} \times \pi_{n+1,j}^\vee) \\ &\quad \times \int_{N^P(F_S)\backslash \mathcal{P}^P(F_S)} |\phi_{N^P,\psi}(p_S k)|^2 \delta_P(p_S)^{-1} dp_S\end{aligned}$$

[10] Note that our normalization of the Petterson inner product if different from *loc. cit.*

for every $k \in \mathrm{K}$. On the other hand, by [FLO12, Proposition A.2] we have

$$(\mathbf{8.1.2.2}) \qquad \int_{\mathrm{K}} \int_{\mathrm{N^P(F_S)} \backslash \mathcal{P}^{\mathrm{P}}(\mathrm{F_S})} |\phi_{\mathrm{N^P},\psi}(p_{\mathrm{S}} k)|^2 \delta_{\mathrm{P}}(p_{\mathrm{S}})^{-1} dp_{\mathrm{S}} dk$$
$$= \frac{\mathrm{vol}_{\mathrm{G}(\mathbf{A}^{\mathrm{S}})}(\mathrm{K^S})}{\mathrm{vol}_{\mathrm{M_P}(\mathbf{A}^{\mathrm{S}})}(\mathrm{K^S} \cap \mathrm{M_P}(\mathbf{A}^{\mathrm{S}}))}$$
$$\times \int_{\mathrm{P(F_S)} \backslash \mathrm{G(F_S)}} \int_{\mathrm{N^P(F_S)} \backslash \mathcal{P}^{\mathrm{P}}(\mathrm{F_S})} |\phi_{\mathrm{N^P},\psi}(p_{\mathrm{S}} g_{\mathrm{S}})|^2 \delta_{\mathrm{P}}(p_{\mathrm{S}})^{-1} dp_{\mathrm{S}} dg_{\mathrm{S}}$$
$$= (\Delta_{\mathrm{G}}^{\mathrm{S},*})^{-1} \Delta_{\mathrm{M_P}}^{\mathrm{S},*} \int_{\mathrm{N(F_S)} \backslash \mathcal{P}(\mathrm{F_S})} |\mathbf{W}_{\mathrm{S}}(p_{\mathrm{S}}, \phi_{\mathrm{N^P},\psi})|^2 dp_{\mathrm{S}},$$

where $\mathbf{W}_{\mathrm{S}} : \mathrm{Ind}_{\mathrm{P(F_S)}}^{\mathrm{G(F_S)}}(\mathcal{W}(\pi_{\mathrm{S}}, \psi_{\mathrm{N,S}})) \to \mathcal{W}(\Pi_{\mathrm{S}}, \psi_{\mathrm{N,S}})$ stands for the *Jacquet functional*, defined as the value at $s = 0$ of the holomorphic continuation of

$$\mathbf{W}_{\mathrm{S},s}(g_{\mathrm{S}}, \phi') = \int_{(w_{\mathrm{P}}^{\mathrm{G}})^{-1} \mathrm{N^P(F_S)} w_{\mathrm{P}}^{\mathrm{G}} \backslash \mathrm{N(F_S)}} \phi'(w_{\mathrm{P}}^{\mathrm{G}} u_{\mathrm{S}} g_{\mathrm{S}}) \delta_{\mathrm{P}}(w_{\mathrm{P}}^{\mathrm{G}} u_{\mathrm{S}} g_{\mathrm{S}})^s \psi_{\mathrm{N}}(u_{\mathrm{S}})^{-1} du_{\mathrm{S}},$$
$$\Re(s) \gg 1$$

for $g_{\mathrm{S}} \in \mathrm{G(F_S)}$ and $\phi' \in \mathrm{Ind}_{\mathrm{P(F_S)}}^{\mathrm{G(F_S)}}(\mathcal{W}(\pi_{\mathrm{S}}, \psi_{\mathrm{N,S}}))$ where $w_{\mathrm{P}}^{\mathrm{G}} = w^{\mathrm{P}} w^{\mathrm{G}}$ with w^{P} (resp. w^{G}) the permutation matrix representing the longest element in the Weyl group of T in $\mathrm{M_P}$ (resp. in G). Finally, by [Sha81, Sect. 4], we have

$$(\mathbf{8.1.2.3}) \qquad \mathbf{W}_{\mathrm{S}}(\phi_{\mathrm{N^P},\psi}) = \prod_{1 \leqslant i < j \leqslant k} \mathrm{L^S}(1, \pi_{n,i} \times \pi_{n,j}^{\vee}) \prod_{1 \leqslant i < j \leqslant r} \mathrm{L^S}(1, \pi_{n+1,i} \times \pi_{n+1,j}^{\vee}) \mathrm{W}_{\phi} |_{\mathrm{G(F_S)}}$$

where $\mathrm{W}_{\phi} |_{\mathrm{G(F_S)}}$ stands for the restriction of the Whittaker function W_{ϕ} to the subgroup $\mathrm{G(F_S)} \subset \mathrm{G}(\mathbf{A})$. (Note that, as χ is regular, the Rankin-Selberg L-functions $\mathrm{L}(s, \pi_{n,i} \times \pi_{n,j}^{\vee})$ and $\mathrm{L}(s, \pi_{n+1,i} \times \pi_{n+1,j}^{\vee})$ are all regular at $s = 1$.) As, for every $s \in \mathbf{R}$,

$$\mathrm{L^S}(s, \Pi \times \Pi^{\vee}) = \prod_{i=1}^{k} \mathrm{L^S}(s, \pi_{n,i} \times \pi_{n,i}^{\vee}) \times \prod_{j=1}^{r} \mathrm{L^S}(s, \pi_{n+1,j} \times \pi_{n+1,j}^{\vee})$$
$$\times \left| \prod_{1 \leqslant i < j \leqslant k} \mathrm{L^S}(s, \pi_{n,i} \times \pi_{n,j}^{\vee}) \right|^2$$
$$\times \left| \prod_{1 \leqslant i < j \leqslant r} \mathrm{L^S}(s, \pi_{n+1,i} \times \pi_{n+1,j}^{\vee}) \right|^2,$$

we deduce from (8.1.2.1), (8.1.2.2) and (8.1.2.3) the identity of the Theorem. □

8.1.3. *Relative characters.* — Let $\mathcal{B}_{P,\pi}$ be a K-basis of Π as in Section 2.8.3. We define the *relative character* I_Π of Π as the following functional on $\mathcal{S}(G(\mathbf{A}))$:

$$I_\Pi(f) = \sum_{\phi \in \mathcal{B}_{P,\pi}} \frac{\lambda(R(f)W_\phi)\overline{\beta_\eta(W_\phi)}}{\langle W_\phi, W_\phi \rangle_{\mathrm{Whitt}}}, \; f \in \mathcal{S}(G(\mathbf{A})),$$

where the series converges, and does not depend on the choice of $\mathcal{B}_{P,\pi}$, by Proposition 2.8.4.1.

8.1.4. For every $f \in \mathcal{S}(G(\mathbf{A}))$, we set

$$K^1_{f,\chi}(g) = \int_{[H]} K_{f,\chi}(h,g)dh \text{ and}$$
$$K^2_{f,\chi}(g) = \int_{[G']} K_{f,\chi}(g,g')\eta_{G'}(g')dg', \quad g \in [G],$$

where the above expressions are absolutely convergent by Lemma 2.10.1.1.3.

Recall that the notion of relevant $*$-regular cuspidal datum has been defined in Section 5.1.3 and that we have defined for any $\chi \in \mathfrak{X}$ a distribution I_χ (see Theorem 3.2.4.1).

Theorem **8.1.4.1.** — *Let $f \in \mathcal{S}(G(\mathbf{A}))$ and $\chi \in \mathfrak{X}^*(G)$. Then,*

1. *If χ is not relevant, we have $K^2_{f,\chi}(g) = 0$ for every $g \in [G]$ and moreover*

$$I_\chi(f) = 0.$$

2. *If χ is relevant, we have*

$$I_\chi(f) = \int_{[G']} K^1_{f,\chi}(g')\eta_{G'}(g')dg'$$

where the right-hand side converges absolutely and moreover

$$I_\chi(f) = 2^{-\dim(A_P)} I_\Pi(f).$$

The rest of this section is devoted to the proof of Theorem 8.1.4.1. Until the end, we fix a function $f \in \mathcal{S}_\chi(G(\mathbf{A}))$.

8.2. *Proof of Theorem 8.1.4.1*

8.2.1. We fix a character η_G of $[G]$ whose restriction to $[G']$ is equal to $\eta_{G'}$ (such a character exists as the idèle class group of F is a closed subgroup of the idèle class group of E) and we set $\widetilde{\chi} = \eta_G \otimes \chi^\vee \in \mathfrak{X}^*(G)$. We can write $\widetilde{\chi}$ as $(\widetilde{\chi}_n, \widetilde{\chi}_{n+1})$ where $\widetilde{\chi}_k \in \mathfrak{X}^*(G_k)$

for $k = n, n+1$. For every $g \in [\mathrm{G}]$, we denote by $\widetilde{\mathrm{K}}_{f,\chi}(g,.)$ the function $\eta_{\mathrm{G}}\mathrm{K}_{f,\chi}(g,.)$. By Lemma 2.10.1.1.2 and (2.9.6.10), we have

$$(\mathbf{8.2.1.1}) \qquad \widetilde{\mathrm{K}}_{f,\chi}(g,.) \in \mathcal{S}_{\widetilde{\chi}}([\mathrm{G}]) = \mathcal{S}_{\widetilde{\chi}_n}([\mathrm{G}_n])\widehat{\otimes}\mathcal{S}_{\widetilde{\chi}_{n+1}}([\mathrm{G}_{n+1}])$$

for all $g \in [\mathrm{G}]$. Moreover, with the notation of Theorem 6.2.6.1, we have

$$(\mathbf{8.2.1.2}) \qquad \mathrm{K}^2_{f,\chi}(g) = \mathrm{P}_{\mathrm{G}'_n}\widehat{\otimes}\mathrm{P}_{\mathrm{G}'_{n+1}}(\widetilde{\mathrm{K}}_{f,\chi}(g,.)).$$

8.2.2. *The non-relevant case.* — Assume that χ is not relevant. By definition of a relevant cuspidal data (see Section 5.1.3), at least one of $\widetilde{\chi}_n$, $\widetilde{\chi}_{n+1}$ is not distinguished (see Section 6.2.3 for the definition of distinguished). Hence, by Theorem 6.2.5.1 and Theorem 6.2.6.1, $\mathrm{P}_{\mathrm{G}'_k}$ vanishes identically on $\mathcal{S}_{\widetilde{\chi}_k}([\mathrm{G}_k])$ for $k = n$ or $k = n+1$. Thus, by (8.2.1.1) and (8.2.1.2), the function $\mathrm{K}^2_{f,\chi}$ vanishes identically. By Theorem 3.3.9.1 applied to the expression (3.3.5.5), this implies $\mathrm{I}_\chi(f) = 0$. This proves part 1. of Theorem 8.1.4.1.

8.2.3. *Regularized Rankin-Selberg period and convergence.* — From now on, we assume that χ is relevant. By Lemma 2.10.1.1.2, for every $g \in [\mathrm{G}]$ the function $\mathrm{K}_{f,\chi}(.,g)$ belongs to $\mathcal{S}_\chi([\mathrm{G}])$. Since χ is relevant, it is H-regular in the sense of Section 7.1.2 (this follows from the dichotomy of Section 4.1.2). Therefore, by Theorem 7.1.3.1, P_{H} extends to a continuous linear form on $\mathcal{T}_\chi([\mathrm{G}])$ that we shall denote by $\mathrm{P}^*_{\mathrm{H}}$. By definition of this extension and of the linear form λ (see Section 8.1.2), for every $\phi \in \Pi$ we have

$$(\mathbf{8.2.3.3}) \qquad \mathrm{P}^*_{\mathrm{H}}(\mathrm{E}(\phi)) = \lambda(\mathrm{W}_\phi).$$

By Lemma 2.10.1.1.3 there exists $\mathrm{N} \geqslant 0$ such that the function

$$g' \in [\mathrm{G}'] \mapsto \mathrm{K}_{f,\chi}(.,g') \in \mathcal{T}_{\mathrm{N}}([\mathrm{G}])$$

is absolutely integrable. As

$$\mathrm{K}^1_{f,\chi}(g) = \mathrm{P}_{\mathrm{H}}(\mathrm{K}_{f,\chi}(.,g)) = \mathrm{P}^*_{\mathrm{H}}(\mathrm{K}_{f,\chi}(.,g)),$$

combined with Theorem 3.3.9.1 applied to the expression (3.3.5.6), this shows at once that the expression

$$(\mathbf{8.2.3.4}) \qquad \int_{[\mathrm{G}']} \mathrm{K}^1_{f,\chi}(g')\eta_{\mathrm{G}'}(g')dg'$$

converges absolutely, is equal to $\mathrm{I}_\chi(f)$ and that

$$(\mathbf{8.2.3.5}) \qquad \mathrm{I}_\chi(f) = \mathrm{P}^*_{\mathrm{H}}\left(\int_{[\mathrm{G}']} \mathrm{K}_{f,\chi}(.,g')\eta_{[\mathrm{G}']}(g')dg'\right) = \mathrm{P}^*_{\mathrm{H}}(\mathrm{K}^2_{f,\chi}).$$

Let us point out here that the absolute convergence of (8.2.3.4), together with Theorem 3.3.7.1.1, implies that the exponential-polynomial $\mathrm{T} \mapsto \mathrm{I}^{\mathrm{T}}_\chi(f)$ of Theorem 3.2.4.1.2 is actually constant (but we caution the reader that this is not necessarily true for cuspidal data that aren't $*$-regular and relevant).

8.2.4. *Spectral expression of* $K^2_{f,\chi}$. — Set $\widetilde{\Pi} = \Pi^\vee \otimes \eta_G$. We may write $\widetilde{\Pi}$ as a tensor product $\widetilde{\Pi}_n \boxtimes \widetilde{\Pi}_{n+1}$ and we let

$$\beta = \beta_n \widehat{\otimes} \beta_{n+1} : \mathcal{W}(\widetilde{\Pi}, \psi_N) = \mathcal{W}(\widetilde{\Pi}_n, \psi_n) \widehat{\otimes} \mathcal{W}(\widetilde{\Pi}_{n+1}, \psi_{n+1}) \to \mathbf{C}$$

be the (completed) tensor product of the linear forms β_n, β_{n+1} defined in Section 6.2.4. Fix $g \in [G]$ and set $\mathbf{f}_g = \widetilde{K}_{f,\chi}(g, .)$. Since χ is relevant, $\widetilde{\chi}_n$ and $\widetilde{\chi}_{n+1}$ are both distinguished. Note that the linear map

$$\mathbf{f} \in \mathcal{S}([G]) \mapsto W_{\mathbf{f},\widetilde{\Pi}} := W_{\mathbf{f}_{\widetilde{\Pi}}} \in \mathcal{W}(\widetilde{\Pi}, \psi_N)$$

is the (completed) tensor product of the continuous linear maps $\mathbf{f} \in \mathcal{S}([G_k]) \mapsto W_{\mathbf{f},\widetilde{\Pi}_k} \in \mathcal{W}(\widetilde{\Pi}_k, \psi_k)$ for $k = n, n+1$ (as can be checked directly on pure tensors). Therefore, by (8.2.1.1), (8.2.1.2), Theorem 6.2.5.1 and Theorem 6.2.6.1 we have

$$K^2_{f,\chi}(g) = 2^{-\dim(A_P)} \beta(W_{\mathbf{f}_g,\widetilde{\Pi}}). \tag{8.2.4.6}$$

Let $\mathcal{B}_{P,\pi}$ be a K-basis Π as in Section 2.8.3. Then, we have $\mathbf{f}_{g,\widetilde{\Pi}} = \sum_{\phi \in \mathcal{B}_{P,\pi}} \langle \mathbf{f}_g, \eta_G E(\overline{\phi}) \rangle_G \eta_G E(\overline{\phi})$ where the sum converges absolutely in $\mathcal{T}_N([G])$ for some $N \geqslant 0$. Hence,

$$W_{\mathbf{f}_g,\widetilde{\Pi}} = \sum_{\phi \in \mathcal{B}_{P,\pi}} \langle \mathbf{f}_g, \eta_G E(\overline{\phi}) \rangle_G \eta_G \overline{W_\phi}$$

in $\mathcal{W}(\widetilde{\Pi}, \psi_N)$. On the other hand, we easily check that $\beta(\eta_G \overline{W_\phi}) = \overline{\beta_\eta(W_\phi)}$ and

$$\langle \mathbf{f}_g, \eta_G E(\overline{\phi}) \rangle_G = \langle K_{f,\chi}(g, .), E(\overline{\phi}) \rangle_G = E(R(f)\phi)(g)$$

for every $\phi \in \mathcal{B}_{P,\pi}$. Therefore, by (8.2.4.6), we obtain

$$K^2_{f,\chi}(g) = 2^{-\dim(A_P)} \sum_{\phi \in \mathcal{B}_{P,\pi}} E(R(f)\phi)(g) \overline{\beta_\eta(W_\phi)}. \tag{8.2.4.7}$$

Note that by Proposition 2.8.4.1, the series above is actually absolutely convergent in $\mathcal{T}_N([G])$ for some $N \geqslant 0$ (and not just pointwise).

8.2.5. *End of the proof.* — By (8.2.3.5), (8.2.3.3) and (8.2.4.7), we obtain

$$I_\chi(f) = 2^{-\dim(A_P)} \sum_{\phi \in \mathcal{B}_{P,\pi}} \lambda(R(f)W_\phi) \overline{\beta_\eta(W_\phi)}.$$

Using Theorem 8.1.2.1 and since $\mathcal{B}_{P,\pi}$ is an orthonormal basis of Π, this can be rewritten as

$$I_\chi(f) = 2^{-\dim(A_P)} \sum_{\phi \in \mathcal{B}_{P,\pi}} \frac{\lambda(R(f)W_\phi)\overline{\beta_\eta(W_\phi)}}{\langle W_\phi, W_\phi \rangle_{\mathrm{Whitt}}} = 2^{-\dim(A_P)} I_\Pi(f)$$

and this ends the proof of Theorem 8.1.4.1 in the relevant case.

9. Flicker-Rallis functional computation

The goal of this section is to prove Theorem 1.3.2.3 of the introduction that states that two natural functionals are equal. This is established in Theorem 9.2.5.1. The bulk of the work is in proving its local avatar. The case of split algebra E/F amounts to comparing scalar products which was done in Appendix A of [FLO12], which is an inspiration for this section.

9.1. *Local comparison*

9.1.1. Let E/F be an etale quadratic algebra over a local field F. Let $\mathrm{Tr}_{E/F}$: $E \to F$ be the trace map. As in Section 6.1.2, let $\psi' : F \to \mathbf{C}^\times$ be a non-trivial additive character, $\tau \in E^\times$ an element of trace 0 and we set $\psi : E \to \mathbf{C}^\times$ to be $\psi(x) = \psi'(\mathrm{Tr}(\tau x))$. We use ψ' and ψ to define autodual Haar measures on F and E respectively. The duality $F \times E/F \to \mathbf{C}^\times$ given by $(x, y) \mapsto \psi(xy)$ defines a unique Haar measure on E/F dual to the one on F. This measure on E/F coincides with the quotient measure.

9.1.2. Let $k = E$ or F. Let S be a closed subgroup of $GL_n(k)$ equipped with a right-invariant Haar measure denoted by ds. We denote by δ_S the modular character such that $\delta_S(s)^{-1}ds$ is a left-invariant Haar measure on S. Let $R \subset S$ be a closed subgroup equipped with a right-invariant Haar measure dr. We denote by $\int_{R\backslash S}$ the right S-invariant linear form on the space of left $(R, \delta_R\delta_S^{-1})$-equivariant functions on S such that

$$\int_{R\backslash S} \int_R f(rt)\delta_S(r)\delta_R(r)^{-1}\, drdt = \int_S f(s)ds$$

for all continuous and compactly supported functions f on S.

We normalize the measures on $GL_n(k)$ and its subgroups as follows:

- On $GL_n(k)$ we set

$$dx = \frac{dx_{ij}}{|\det x|_k^n}$$

where $x = (x_{ij})$.

- On standard Levi subgroups of $\mathrm{GL}_n(k)$ we set the product measure using the measure defined above.
- On (semi) standard unipotent subgroups $\mathrm{N}(k) \subset \mathrm{GL}_n(k)$ we set the additive measure dn_{ij} where n_{ij} run through coordinates of N.
- Let P is a standard parabolic subgroup of $\mathrm{GL}_n(k)$ with the standard Levi decomposition NM. We have the right-invariant measure $dp := dndm$ on $\mathrm{P}(k)$ and the left invariant measure $\delta_{\mathrm{P}_k}^{-1} dp$ where $\delta_{\mathrm{P}_k} : \mathrm{P}(k) \to \mathbf{R}_{>0}^{\times}$ is the Jacobian homomorphism for the adjoint action of $\mathrm{P}(k)$ on $\mathrm{N}(k)$.

With this normalization, we have for all $f \in \mathrm{C}_c^{\infty}(\mathrm{GL}_n(k))$

$$\int_{\mathrm{GL}_n(k)} f(g)\, dg = \int_{\mathrm{P}(k)} \int_{\bar{\mathrm{N}}(k)} f(pn)\delta_{\mathrm{P}_k}(p)^{-1}\, dpdn,$$

where $\overline{\mathrm{N}}$ is the unipotent radical of the opposite parabolic to P. The linear form $\int_{\mathrm{P}(k)\backslash \mathrm{GL}_n(k)}$ is given in this case by the integration over either $\bar{\mathrm{N}}(k)$ or the standard maximal compact subgroup for a suitable Haar measure. Assume P is moreover maximal of type $(n-1, 1)$. Let $\mathcal{P} \subset \mathrm{P}$ be the mirabolic subgroup (see Section 9.1.5 below). We have $\mathrm{P}(k) = \mathcal{P}(k) \times \mathrm{GL}_1(k)$ and this gives the normalization of the measure on $\mathcal{P}(k)$. Moreover the modular character $\delta_{\mathcal{P}(k)}$ coincides with $|\det|_k$ on $\mathcal{P}(k)$.

9.1.3. We will use the notation introduced in Section 4 with some changes. All groups considered in this section are subgroups of $\mathrm{G}_n = \mathrm{Res}_{\mathrm{E}/\mathrm{F}}\,\mathrm{GL}_n$. We write simply G for G_n, P_0 for the fixed minimal parabolic subgroup of G and N_0 for its unipotent radical. In order to be as compatible with Appendix A of [FLO12] as possible, instead of $\mathrm{G}' = \mathrm{GL}_n$ (defined over F) we write $\mathrm{G}_\mathrm{F} = \mathrm{GL}_n$ and for any subgroup H of G we write H_F for $\mathrm{H} \cap \mathrm{G}_\mathrm{F}$. We will often identify a group with its F points in this section.

9.1.4. We define the character $\psi : \mathrm{N}_0 \to \mathbf{C}^{\times}$ as follows. Write $n \in \mathrm{N}_0$ as

$$n = \begin{pmatrix} 1 & n_{12} & n_{13} & \dots & n_{1n} \\ 0 & 1 & n_{23} & \dots & n_{2n} \\ 0 & \ddots & \ddots & \ddots & n_{2n} \\ 0 & \ddots & \ddots & 1 & n_{n-1n} \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \quad n_{ij} \in \mathrm{E}$$

and set $\psi(n) = \psi((-1)^n(n_{12} + n_{23} + \cdots + n_{n-1n}))$. This is the same character as the one from 6.1.2. By restriction, ψ defines a character of $\mathrm{N}_0 \cap \mathrm{M}$ for all standard Levi subgroups M.

9.1.5. We denote by $\mathcal{P} = \mathcal{P}_n$ the mirabolic subgroup of G defined as the stabilizer of the row vector $\begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$.

For φ an element of $C^\infty(N_0\backslash\mathcal{P},\psi)=\{f\in C^\infty(\mathcal{P})\mid f(nx)=\psi(n)f(x),\ n\in N_0,\ x\in\mathcal{P}\}$ we define

$$\beta(\varphi)=\beta_G(\varphi)=\int_{N_{0,F}\backslash\mathcal{P}_F}\varphi(p)\,dp$$

when it is absolutely convergent. Note that ψ is trivial on $N_{0,F}$. In the same way, we define β_M for all standard Levi subgroups M of G.

9.1.6. Let $\Pi_{gen}(G)$ be the set of irreducible generic complex representations of $G=GL_n(E)$. Let $\mathcal{W}(\pi)=\mathcal{W}^\psi(\pi)$ be the space of the Whittaker model of $\pi\in\Pi_{gen}(G)$ with respect to the character ψ. Let $\delta_g^\pi=\delta_g:\mathcal{W}(\pi)\to\mathbf{C}$ be the evaluation at $g\in G$. The group G acts on $\mathcal{W}(\pi)$ by right multiplication.

Fix $P=MN\in\mathcal{F}^G(P_0)$. Let w_M be the element in the Weyl group of G such that $w_M M w_M^{-1}$ is a standard Levi subgroup and the longest for this property. Let $P^w=N^wM^w\in\mathcal{F}^G(P_0)$ be the parabolic subgroup whose Levi component is $M^w=w_M M w_M^{-1}$.

For $\sigma\in\Pi_{gen}(M)$ let $\mathrm{Ind}_P^G(\mathcal{W}(\sigma))$ be the normalized (smooth) induction to G, from $\mathcal{W}(\sigma)$, seen as a representation of P via the natural map $P\to M$. As in Section 8.1.2 we define for $\varphi\in\mathrm{Ind}_P^G(\mathcal{W}(\sigma))$ the Jacquet's integral.

$$\mathbf{W}(g,\varphi)=\int_{N^w}\delta_e^\sigma(\varphi(w_M^{-1}u'g))\psi^{-1}(u')\,du'.$$

We have then that $\mathbf{W}_e(\varphi):=\mathbf{W}(e,\varphi)$ is a Whittaker functional on $\mathrm{Ind}_P^G(\mathcal{W}(\sigma))$.

9.1.7. Let $\sigma\in\Pi_{gen}(M)$ and $\varphi\in\mathrm{Ind}_P^G(\mathcal{W}(\sigma))$. We assume that σ is unitary. Then $\beta_G(\mathbf{W}(\varphi))$ and $\beta_M(\varphi(g))$ are given by absolutely convergent integrals (cf. [Fli88, Lemma 4] and [BP21b, Proposition 2.6.1, proof of Lemma 3.3.1]). We shall say that σ is distinguished if σ admits a non-zero continuous linear form invariant under G_F. From now on we assume that σ is distinguished.

Theorem **9.1.7.1.** — *Under the assumptions above, the linear map β_M is M_F-invariant and we have*

$$\beta_G(\mathbf{W}(\varphi))=\beta'(\varphi) \qquad (\mathbf{9.1.7.1})$$

where

$$\beta'(\varphi)=\int_{P_F\backslash G_F}\beta_M(\varphi(g))\,dg. \qquad (\mathbf{9.1.7.2})$$

Remark **9.1.7.2.** — The integral (9.1.7.2) makes sense since β_M is invariant and since δ_P restricted to P_F equals $\delta_{P_F}^2$. This latter observation pertains also to similar integrals in the proof below.

Proof. — If E is split then the theorem follows from [Bar03, corollary 10.4] and [FLO12, proposition A.2]. We assume from now on that E/F is a field extension. Let's prove the first assertion. Working on factors of M we are reduced to the case $M = G$. If F is p-adic, the result follows from [GJR01, proposition 2]. More precisely, according to this proposition, any non zero G_F-invariant continuous linear form on (the Whittaker model of) a distinguished unitary generic representation of G should be (up to a non zero constant) β. We claim that the result also holds for $F = \mathbf{R}$. Indeed, following the proof of [GJR01, proposition 2] and using multiplicity one property [AG09, theorem 8.2.5], we see that it suffices to show that any $\mathcal{Z}(\mathfrak{g})$-finite distribution on the symmetric space G/G_F left-invariant by $\mathcal{P}_F$ is also left-invariant by G_F. But this is a variant of [Bar03, theorem 10.4] where the adjoint action of G_F on itself is replaced by the action of G_F on G/G_F. It can be proved as in [Bar03, section 9]: indeed the bulk of the argument there is a descent to the centralizer of semi-simple elements where the use of the exponential reduces everything to an analogous theorem for the adjoint action of G_F on $\mathfrak{g}_F$. In fact such a descent can be performed for G/G_F. The main observation is that the tangent space of G/G_F at the origin can be identified as a G_F-representation to the adjoint action of G_F on $\mathfrak{g}_F$.

Once we have the first assertion, we observe that β' is a non-zero G_F-invariant linear form on $\mathrm{Ind}_P^G(\mathcal{W}(\sigma))$ (since β_M is non-zero as a consequence of [GK72], [Jac10, Proposition 5] and [Kem15]). Still by the same argument as before we deduce that the equality (9.1.7.1) holds up to a non zero constant. We have to show that this constant is 1. Thus it suffices to prove (9.1.7.1) for one specific φ such that $\beta'(\varphi) \neq 0$.

There is an easy reduction to the case where P is maximal. Let $Q = LV \supset P$ be maximal and suppose the assertion holds for $M = M_P$. Then

$$\beta'(\varphi) = \int_{P_F\backslash G_F} \beta_M(\varphi(g))\, dg = \int_{Q_F\backslash G_F} \int_{P_F\backslash Q_F} \delta_{Q_F}^{-1}(q)\beta_M(\varphi(qg))\, dqdg.$$

The inner integral on the RHS by induction hypothesis equals

$$\beta_L(g.\mathbf{W}^L(\varphi)).$$

If we let $\varphi'(g) = g.\mathbf{W}^L(\varphi) \in \mathcal{W}(\mathrm{Ind}_{L\cap P}^L(\mathcal{W}(\sigma)))$ then $\varphi' \in \mathrm{Ind}_Q^G(\mathcal{W}(\mathrm{Ind}_{L\cap P}^L(\mathcal{W}(\sigma))))$ and so by assumption and transitivity of Jacquet's integral we obtain

$$\int_{Q_F\backslash G_F} \beta_L(g.\mathbf{W}^L(\varphi))\, dg = \beta_G(\mathbf{W}(\varphi)).$$

From now on we assume that $P = MN$ is maximal of type (n_1, n_2). In [FLO12], the authors use U instead of N. We will consequently use N in place of U here. Write $M = M_1 \times M_2$ with $M_i \cong \mathrm{Res}_{E/F}\, GL_{n_i}$, M_1 being in the upper and M_2 in the lower diagonal.

Let

$$w = w_M^{-1} = \begin{pmatrix} 0 & I_{n_1} \\ I_{n_2} & 0 \end{pmatrix}.$$

Let $P' = M'N'$ be of type (n_2, n_1) so that $M' = M'_2 \times M'_1$ with $M'_i \cong \mathrm{Res}_{E/F}\, GL_{n_i}$, M'_2 being in the upper and M'_1 in the lower diagonal. Let $\mathcal{P}'_i$ be the mirabolic subgroup of M'_i. Let N'_i be the maximal upper triangular unipotent of M'_i, similarly without $'$. Note that $\mathcal{P}_i = w\mathcal{P}'_i w^{-1}$ etc. We identify N' with the group of $n_2 \times n_1$ matrices. We write $\sigma = \sigma_1 \otimes \sigma_2$ where σ_1 and σ_2 are respectively viewed as representations of M_1 and M_2. We view σ_2 as a representation of M'_2 as well.
Let

$$C_i = \{I_n + \xi \mid \xi \text{ column vector of size } n_2 \text{ in the i-th column}\} \subset N',$$
$$i = n_2 + 1, \ldots, n.$$

Let

$$R_i = \{I_n + \xi \mid \xi \text{ row vector of size } n_2 \text{ in the i-th row}\} \subset \overline{N}',$$
$$i = n_2 + 1, \ldots, n.$$

We can identify C_i and R_j with E^{n_2} which induces a pairing between C_i and R_{i-1} that we will denote $\langle \cdot, \cdot \rangle_i$.

We note some obvious facts

- The groups R_i (resp. C_i) commute with each other and are normalized by M'_2.
- The commutator set $[C_i, R_j]$ is contained in N'_1 for $j < i$.

We define the following groups

1. $X_i = C_{i+1} \cdots C_n$. It is normalized by N'_1.
2. $Y_i = R_{n_2+1} \cdots R_{i-1}$. It is normalized by N'_1.
3. $V_i = N'_1 X_i Y_i$. This is a unipotent group.
4. $V'_i = N'_1 X_{i-1} Y_i \supset V_i$. This is a unipotent group.
5. $$S_i = \begin{cases} M'_2 V_i, & i > n_2, \\ \mathcal{P}'_2 N'_1 N', & i = n_2 \end{cases}$$
6. $S'_i = M'_2 V'_i$ for $i > n_2$.

Note that

- $S'_i = C_i S_i$ for $i > n_2$ as well as $S'_i = R_{i-1} S_{i-1}$ for $i > n_2 + 1$.
- Let δ_i and δ'_i be modular characters of S_i and S'_i respectively. It follows that $(\delta'_i)_{|S_i} = |\det|_E \delta_i$ and $(\delta'_i)_{|S_{i-1}} = |\det|_E^{-1} \delta_{i-1}$ in the above range.
- Let $\delta_{i,F}$ and $\delta'_{i,F}$ be modular characters of $S_{i,F}$ and $S'_{i,F}$ respectively.
- We have $\delta_i = |\det|_E^{n+n_2-2i+1}$ for $i \geq n_2$.

Let us define

$$\begin{cases} \mathcal{A}_i = \mathrm{Ind}_{\mathrm{S}_i}^{\mathcal{P}_n}(\mathcal{W}(\sigma_2)\otimes\psi_i), & i = n_2+1,\dots,n. \\ \mathcal{A}_i = \mathrm{Ind}_{\mathrm{N}_0}^{\mathcal{P}_n}\psi, & i = n_2. \end{cases}$$

Here, ψ_i is the character of V_i - the unipotent radical of S_i - whose restriction to $\mathrm{X}_i\mathrm{Y}_i$ is trivial and that coincides with ψ on N_1'.

Explicitly, for $i > n_2$ we have

$$\mathcal{A}_i = \{\varphi : \mathcal{P} \to \mathcal{W}(\sigma_2) \mid \varphi(mvg) = \left(\frac{\delta_i(m)}{|\det m|_{\mathrm{E}}}\right)^{1/2} \psi_i(v)\sigma_2(m)\varphi(g),$$
$$g \in \mathcal{P},\ m \in \mathrm{M}_2',\ v \in \mathrm{V}_i\}.$$

We also denote $\mathcal{A}_i^2$ the L^2-induction version of the above as in [FLO12]. Note that

- $\mathcal{A}_i = \mathrm{Ind}_{\mathrm{S}_i'}^{\mathcal{P}}(\mathrm{Ind}_{\mathrm{S}_i}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_i)))$ for $i > n_2$.
- $\mathcal{A}_{i-1} = \mathrm{Ind}_{\mathrm{S}_i'}^{\mathcal{P}}(\mathrm{Ind}_{\mathrm{S}_{i-1}}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_{i-1})))$ for $i > n_2+1$.
- $\mathcal{A}_{n_2} = \mathrm{Ind}_{\mathrm{S}_{n_2+1}'}^{\mathcal{P}}(\mathrm{Ind}_{\mathrm{N}_0}^{\mathrm{S}_{n_2+1}'}\psi)$.

For any $i > n_2$ the restriction map to C_i identifies $\mathrm{Ind}_{\mathrm{S}_i}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_i))$ with $\mathrm{C}^\infty(\mathrm{C}_i,\mathcal{W}(\sigma_2))$ because $\mathrm{S}_i'/\mathrm{S}_i = \mathrm{C}_i$. Let us denote $\varphi \mapsto \varphi|_{\mathrm{C}_i}$ the restriction map and ι_{C_i} the map in the reverse order. Similarly, restriction to R_{i-1} identifies $\mathrm{Ind}_{\mathrm{S}_{i-1}}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_{i-1}))$ with $\mathrm{C}^\infty(\mathrm{R}_{i-1},\mathcal{W}(\sigma_2))$. Let us denote $\varphi \mapsto \varphi_{\mathrm{R}_{i-1}}$ the restriction map and $\iota_{\mathrm{R}_{i-1}}$ the map in the reverse order.

Given that C_i and R_{i-1} are in duality we have a Fourier transform

$$\mathcal{F}_i' : \mathrm{L}^2(\mathrm{C}_i, \overline{\mathcal{W}(\sigma_2)}) \to \mathrm{L}^2(\mathrm{R}_{i-1}, \overline{\mathcal{W}(\sigma_2)})$$

where $\overline{\mathcal{W}(\sigma_2)}$ is the L^2 completion of $\mathcal{W}(\sigma_2)$.

Lemma **9.1.7.3.** — *For $i = n,\dots,n_2+2$, the above Fourier transform induces a map*

$$\mathcal{B}_i : \mathcal{A}_i^2 = \mathrm{Ind}_{\mathrm{S}_i'}^{\mathcal{P}}(\mathrm{Ind}_{\mathrm{S}_i}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_i))) \to \mathcal{A}_{i-1}^2$$
$$= \mathrm{Ind}_{\mathrm{S}_i'}^{\mathcal{P}}(\mathrm{Ind}_{\mathrm{S}_{i-1}}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_i)))$$

induced from the equivalence $\mathrm{Ind}_{\mathrm{S}_i}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_i)) \to \mathrm{Ind}_{\mathrm{S}_{i-1}}^{\mathrm{S}_i'}((\mathcal{W}(\sigma_2)\otimes\psi_{i-1}))$ *given by* $\varphi \mapsto \iota_{\mathrm{R}_{i-1}}(\mathcal{F}_i'(\varphi|_{\mathrm{C}_i}))$. *It is an equivalence of unitary representations.*

Similarly, we have the map

$$\mathcal{F}_{n_2+1} : \mathrm{Ind}_{\mathrm{S}_{n_2+1}}^{\mathrm{S}_{n_2+1}'}((\mathcal{W}(\sigma_2)\otimes\psi_{n_2+1})) \to \mathrm{Ind}_{\mathrm{N}_0}^{\mathrm{S}_{n_2+1}'}\psi$$

given by

$$\mathcal{F}_{n_2+1}\varphi(vm) = \psi(v)|\det m|_{\mathrm{E}}^{1/2}\hat{\varphi}(\chi_m)(m), \quad m \in \mathrm{M}'_2,\ v \in \mathrm{V}'_{n_2+1} = \mathrm{N}'_1\mathrm{N}'$$

where

$$\chi_m : \mathrm{C}_{n_2+1} \to \mathbf{C}^\times, \quad \chi_m(c) = \psi(mcm^{-1}), \quad \hat{\varphi}(\chi) = \int_{\mathrm{C}_{n_2+1}} \varphi(c)\chi(c)\,dc.$$

Lemma **9.1.7.4.** — *The above Fourier transform induces the equivalence of unitary representations*

$$\mathcal{B}_{n_2+1} : \mathcal{A}^2_{n_2+1} = \mathrm{Ind}^{\mathcal{P}}_{\mathrm{S}'_{n_2+1}}(\mathrm{Ind}^{\mathrm{S}'_{n_2+1}}_{\mathrm{S}_{n_2+1}}((\mathcal{W}(\sigma_2)\otimes\psi_{n_2+1})))$$
$$\to \mathcal{A}^2_{n_2} = \mathrm{Ind}^{\mathcal{P}}_{\mathrm{S}'_{n_2+1}}(\mathrm{Ind}^{\mathrm{S}'_{n_2+1}}_{\mathrm{N}_0}\psi).$$

From now on we take $\varphi \in \mathrm{Ind}^{\mathrm{G}}_{\mathrm{P}}(\mathcal{W}(\sigma_1\otimes\sigma_2))$ supported on the big cell $\mathrm{P}w\mathrm{P}'$. We shall show (9.1.7.1) for such functions which suffices to conclude.

For $m \in \mathrm{M}_1$, let $\delta^1_m : \mathcal{W}(\sigma_1\otimes\sigma_2) \to \mathcal{W}(\sigma_2)$ be the evaluation map in the first variable. Define for $p \in \mathcal{P}$

$$\varphi_n(p) = \delta^1_e\varphi(wp) \in \mathcal{A}_n.$$

We have then

$$\int_{\mathrm{P}\backslash\mathrm{G}} \|\varphi(g)\|^2_{\mathrm{L}^2(\mathcal{W}(\sigma_1\otimes\sigma_2))}\,dg = \|\varphi_n\|^2_{\mathcal{A}_n}.$$

Let $i \in \{n,\dots,n_2+1\}$. The restriction of $\left(\dfrac{\delta_i}{|\det|_{\mathrm{E}}}\right)^{1/2}$ to $\mathrm{S}_{i,\mathrm{F}}$ is equal to $\delta_{i,\mathrm{F}}\delta^{-1}_{\mathcal{P}_\mathrm{F}}$. Thus for $\phi \in \mathcal{A}_i$, the map $p \in \mathcal{P}_\mathrm{F} \mapsto \beta_{\mathrm{M}'_2}(\phi(p))$ is $(\mathrm{S}_{i,\mathrm{F}}, \delta_{i,\mathrm{F}}\delta^{-1}_{\mathcal{P}_\mathrm{F}})$-equivariant. We may introduce (at least formally)

$$\beta_i(\phi) = \int_{\mathrm{S}_{i,\mathrm{F}}\backslash\mathcal{P}_\mathrm{F}} \beta_{\mathrm{M}'_2}(\phi(p))\,dp.$$

Lemma **9.1.7.5.** — *We have*

$$\beta'(\varphi) = \beta_n(\varphi_n).$$

Proof. — Indeed by various changes of variables we get the equality of absolutely convergent integrals:

$$\beta'(\varphi) = \int_{\mathrm{P}_\mathrm{F}\backslash\mathrm{G}_\mathrm{F}} \beta_\mathrm{M}(\varphi(g))\,dg = \int_{\mathrm{P}_\mathrm{F}\backslash\mathrm{G}_\mathrm{F}} \beta_\mathrm{M}(\varphi(gw))\,dg$$

$$
\begin{aligned}
&= \int_{\overline{N}_F} \beta_M(\varphi(uw))\, du = \int_{N'_F} \beta_M(\varphi(wu'))\, du' \\
&= \int_{N'_F} \int_{N_{1,F}\backslash \mathcal{P}_{1,F}} \beta_{M_2}(\delta^1_{m_1}(\varphi(wu')))\, dm_1\, du' \\
&= \int_{N'_F} \int_{N_{1,F}\backslash \mathcal{P}_{1,F}} \beta_{M_2}(\delta^1_e(\varphi(m_1 wu')))\delta_P^{-1/2}(m_1)\, dm_1\, du' \\
&= \int_{N'_F} \int_{N'_{1,F}\backslash \mathcal{P}'_{1,F}} \beta_{M'_2}(\delta^1_e(\varphi(wm'_1 u')))\delta_{P'}^{1/2}(m'_1)\, dm'_1\, du' \\
&= \int_{S_{n,F}\backslash \mathcal{P}_F} \beta_{M'_2}(\delta^1_e(\varphi(wp)))\, dp. \qquad \square
\end{aligned}
$$

Define recursively $\varphi_{i-1} = \mathcal{B}_i(\varphi_i)$ for $i = n, \dots, n_2 + 1$. As shown at the end of the Appendix A.3 of [FLO12], we have

(**9.1.7.3**) $$\varphi_{n_2} = \mathbf{W}(\varphi).$$

Lemma **9.1.7.6.** — *For $i = n, \dots, n_2 + 2$ we have*

$$\beta_i(\varphi_i) = \beta_{i-1}(\varphi_{i-1}).$$

Proof. — Observe that the map

(**9.1.7.4**) $$g \in \mathcal{P}_F \mapsto \int_{R_{i-1,F}} \beta_{M'_2}(\varphi_{i-1}(rg))\, dr$$

is $(S'_i, \delta'_{i,F}\delta^{-1}_{\mathcal{P}_F})$-equivariant. We have

$$\beta_{i-1}(\varphi_{i-1}) = \int_{S'_{i,F}\backslash \mathcal{P}_F} \int_{R_{i-1,F}} \beta_{M'_2}(\varphi_{i-1}(rg))\, dr dg.$$

Let $g \in \mathcal{P}_F$. By construction we have:

$$\varphi_{i-1}(rg) = \int_{C_i} \varphi_i(cg)\psi(\langle c, r\rangle_i) dc.$$

Since the duality $\langle\cdot,\cdot\rangle_i$ between C_i and R_{i-1} restricts to a duality between $C_i/C_{i,F}$ and $R_{i-1,F}$ we get that (9.1.7.4) is equal to

$$\int_{C_{i,F}} \beta_{M'_2}(\varphi_i(cg))\, dc.$$

But we have

$$\beta_i(\varphi_i) = \int_{S'_{i,F}\backslash \mathcal{P}_F} \int_{C_{i,F}} \beta_{M'_2}(\varphi_i(cg))\, dcdg.$$

Note that the inner function in c is compactly supported and the convergence of the integrals we have considered follows. □

Lemma **9.1.7.7.** — *We have*

$$\beta_{n_2+1}(\varphi_{n_2+1}) = \beta_G(\mathbf{W}(\varphi)).$$

Proof. — Recall that $\mathbf{W}(\varphi) = \varphi_{n_2} = \mathcal{B}_{n_2+1}(\varphi_{n_2+1})$, see (9.1.7.3). Thus the right-hand side is

$$\int_{N_{0,F}\backslash \mathcal{P}_F} \mathcal{B}_{n_2+1}(\varphi_{n_2+1})(p)\, dp = \int_{S'_{n_2+1,F}\backslash \mathcal{P}_F} \int_{N'_{2,F}\backslash M'_{2,F}} \mathcal{B}_{n_2+1}(\varphi_{n_2+1})(mp)\, dmdp.$$

Let us fix $p \in \mathcal{P}_F$ and let us work on the inner integral. By definition of $\mathcal{B}_{n_2+1}$, it is equal to

$$\int_{N'_{2,F}\backslash M'_{2,F}} \int_{C_{n_2+1}} \varphi_{n_2+1}(cp)(m)\psi(mcm^{-1})dc|\det m|_F\, dm$$

We denote by $\mathcal{P}'_{n_2}$ the standard mirabolic subgroup of M'_2. Let $\mathcal{P}'_{n_2} \subset Q \subset M'_2$ be the maximal parabolic subgroup of type $(n_2 - 1, 1)$. Let $\bar{N}_Q$ the unipotent radical of the opposite parabolic subgroup. The integral above is also equal to

$$\int_{GL_{1,F}} \int_{\bar{N}_{Q,F}} \int_{C_{n_2+1}} \int_{N'_{2,F}\backslash \mathcal{P}'_{n_2,F}} \varphi_{n_2+1}(cp)(q\lambda\bar{n})\psi(q\lambda\bar{n}c(q\lambda\bar{n})^{-1})dqdc|\lambda|_F^{n_2}\, d\bar{n}d\lambda$$

where we identify λ with the diagonal matrix $diag(1, \dots, 1, \lambda) \in M'_2$. A first observation is that we have

$$\psi(q\lambda\bar{n}c(q\lambda\bar{n})^{-1}) = \psi(\lambda\bar{n}c(\lambda\bar{n})^{-1}).$$

Hence the inner integral over q is:

$$\begin{aligned}\int_{N'_{2,F}\backslash \mathcal{P}'_{n_2,F}} \varphi_{n_2+1}(cp)(q\lambda\bar{n})dq &= \beta_{M'_2}(\varphi_{n_2+1}(cp)(\cdot\lambda\bar{n}))\\ &= \beta_{M'_2}(\varphi_{n_2+1}(cp)).\end{aligned}$$

For the latter equality, we use the fact that σ_2 is distinguished and thus $\beta_{M'_2}$ is left $M'_{2,F}$-invariant. By a change of variables we get

$$\int_{GL_{1,F}} \int_{\bar{N}_{Q,F}} \int_{C_{n_2+1}} \beta_{M'_2}(\varphi_{n_2+1}(cp))\psi(\bar{n}\lambda c(\bar{n}\lambda)^{-1})dqdc|\lambda|_F\, d\bar{n}d\lambda.$$

Now $|\lambda|_F\, d\lambda$ is the additive measure. We can identify $c \in C_{n_2+1}$ with an element $X \in E^n$ and $\bar{n}\lambda$ with an element $Y \in F^{n-1} \times F^\times \subset E$ in such a way that we have

$$\psi(\bar{n}\lambda c(\bar{n}\lambda)^{-1}) = \psi(\langle X, Y\rangle)$$

where $\langle X, Y\rangle$ is the obvious pairing. By Fourier inversion, we deduce that the previous integral is

$$\int_{C_{n_2+1,F}} \beta_{M_2'}(\varphi_{n_2+1}(cp))dc.$$

Hence

$$\begin{aligned}\beta_G(\mathbf{W}(\varphi)) &= \int_{S'_{n_2+1,F}\backslash\mathcal{P}_F} \int_{C_{n_2+1,F}} \beta_{M_2'}(\varphi_{n_2+1}(cp))dcdp \\ &= \beta_{n_2+1}(\varphi_{n_2+1}).\end{aligned}$$

Note that the last integrals are absolutely convergent and our computations are justified. □

Theorem 9.1.7.1 then follows from Lemmas 9.1.7.5, 9.1.7.7, 9.1.7.6 for our specific functions φ and thus holds in general. □

9.2. *Global comparison*

9.2.1. We go back to the global setting and notation introduced in Section 3.1.

9.2.2. We normalize all local and global measures as in Section 2.3, with respect to a fixed character $\psi' : F\backslash\mathbf{A} \to \mathbf{C}^\times$. We have the quadratic character $\eta : F^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$ associated to E/F and the associated character $\eta_{G'}$ of $G'(\mathbf{A})$ as defined in Paragraph 3.1.6.

9.2.3. As in Section 6.1.2, we also fix a non-trivial additive character $\psi : E\backslash\mathbf{A}_E \to \mathbf{C}^\times$, trivial on $\mathbf{A}$ which is then used to define a non-degenerate character ψ_N of the maximal unipotent subgroup of $G(\mathbf{A})$ as in the beginning of Section 8.

9.2.4. Let $\chi \in \mathfrak{X}^*(G)$ be a relevant $*$-regular cuspidal datum (cf. Section 5.1.3) and let (M, π) represent χ. Set $\Pi = \mathrm{Ind}_{P(\mathbf{A})}^{G(\mathbf{A})}(\pi)$.

9.2.5. *The comparison.*

Theorem **9.2.5.1.** — *For all $\phi \in \Pi$ we have*

$$J_\eta(\phi) = \beta_\eta(W_\phi)$$

where

- J_η *is defined in 5.1.2.1;*
- β_η *is defined in 8.1.2.*
- $W_\phi \in \mathcal{W}(\Pi, \psi_N)$ *is defined in 8.1.1.*

Proof. — The proof is essentially the same as of Theorem 8.1.2.1. The only difference is that the natural analogue of (8.1.2.1) is provided by Proposition 3.2 of [Zha14a] and the analogue of (8.1.2.2) is established invoking Theorem 9.1.7.1. □

Corollary **9.2.5.2.** — *We have the equality of distributions on $\mathcal{S}(G(\mathbf{A}))$*

$$I_{P,\pi} = I_\Pi$$

where

1. $I_{P,\pi}$ *is defined in Section 5.1.5.*
2. I_Π *is defined in Section 8.1.3.*

Proof. — Looking at definitions of $I_{P,\pi}$ and I_Π, taking into consideration Theorem 8.1.2.1 and Theorem 9.2.5.1 above, we see that we need to establish for all $\phi \in \Pi$

$$\lambda(W_\phi) = I(\phi, 0)$$

where $\lambda = Z^{RS}(0, \cdot)$ is defined in Section 8.1.2 and $I(\phi, 0)$ is given by Proposition 5.1.4.1. This equality is precisely Theorem 1.1 of [IY15]. □

10. Proofs of the Gan-Gross-Prasad and Ichino-Ikeda conjectures

10.1. *Identities among some global relative characters*

10.1.1. Besides notation of Sections 2 and 3, we shall use notation of Section 1. We fix an integer $n \geqslant 1$ and we will omit the subscript n: we will write $\mathcal{H}$ for $\mathcal{H}_n$.

10.1.2. *Relative characters for unitary groups.* — Let $h \in \mathcal{H}$ be a Hermitian form. Let σ be an irreducible cuspidal automorphic subrepresentation of the group U_h. We define the relative character J_σ^h by

$$J_\sigma^h(f) = \sum_\varphi \mathcal{P}_h(\pi(f)\varphi)\overline{\mathcal{P}_h(\varphi)}, \ \forall f \in \mathcal{S}(U_h(\mathbf{A}))$$

where φ runs over a K_h-basis (see 2.8.3) for some maximal compact subgroup $K_h \subset U_h(\mathbf{A})$. The periods $\mathcal{P}_h$ are those defined in 1.1.5. For any subset $\mathfrak{X}_0 \subset \mathfrak{X}(U_h)$ of cuspidal data which do not come from proper Levi subgroups (that is they are represented by pairs (U_h, τ) where τ is a cuspidal automorphic representation) we define more generally

$$(\mathbf{10.1.2.1}) \qquad J^h_{\mathfrak{X}_0}(f) = \sum_{\chi \in \mathfrak{X}_0} \sum_{\sigma} J^h_\sigma(f)$$

where the inner sum is over the set of the constituents σ of some decomposition of $L^2_\chi([U_h])$ (see Section 2.9.2.1) into irreducible subrepresentations. One can show that the double sum is absolutely convergent (see e.g. [BP21a, Proposition A.1.2]).

10.1.3. Let $V_{F,\infty} \subset S_0 \subset V_F$ be a finite set of places containing all the places that are ramified in E. For every $v \in V_F$, we set $E_v = E \otimes_F F_v$ and when $v \notin V_{F,\infty}$ we denote by $\mathcal{O}_{E_v} \subset E_v$ its ring of integers. Let $\mathcal{H}^\circ \subset \mathcal{H}$ be the (finite) subset of Hermitian spaces of rank n over E that admits a selfdual $\mathcal{O}_{E_v}$-lattice for every $v \notin S_0$.

For each $h \in \mathcal{H}^\circ$, the group U_h is naturally defined over $\mathcal{O}_F^{S_0}$ and we fix a choice of such a model. Since we are going to consider invariant distribution, this choice is irrelevant. We define the open compact subgroups $K_h^\circ = \prod_{v \notin S_0} U_h(\mathcal{O}_v)$ and $K^\circ = \prod_{v \notin S_0} G(\mathcal{O}_v)$ respectively of $U_h(\mathbf{A}^{S_0})$ and $G(\mathbf{A}^{S_0})$.

Let $v \notin S_0$. We denote by $\mathcal{S}^\circ(U_h(F_v))$, resp. $\mathcal{S}^\circ(G(F_v))$, the spherical Hecke algebra[11] of complex functions on $U_h(F_v)$ (resp. $G(F_v)$) that are $U_h(\mathcal{O}_v)$-bi-invariant (resp. $G(\mathcal{O}_v)$-bi-invariant) and compactly supported.

We have the base change homomorphism

$$BC_{h,v} : \mathcal{S}^\circ(G(F_v)) \to \mathcal{S}^\circ(U_h(F_v)).$$

We denote by $\mathcal{S}^\circ(U_h(\mathbf{A}^{S_0}))$, resp. $\mathcal{S}^\circ(G(\mathbf{A}^{S_0}))$, the restricted tensor product of $\mathcal{S}^\circ(U_h(F_v))$, resp. $\mathcal{S}^\circ(G(F_v))$, for $v \notin S_0$. We have also a global base change homomorphism given by $BC_h^{S_0} = \otimes_{v \notin S_0} BC_{h,v}$.

We also denote by $\mathcal{S}^\circ(G(\mathbf{A})) \subset \mathcal{S}(G(\mathbf{A}))$ and $\mathcal{S}^\circ(U_h(\mathbf{A})) \subset \mathcal{S}(U_h(\mathbf{A}))$, for $h \in \mathcal{H}^\circ$, the subspaces of functions that are respectively bi-K°-invariant and bi-K_h°-invariant.

10.1.4. *Transfer.* — Let $h \in \mathcal{H}^\circ$. We shall say that $f_{S_0} \in \mathcal{S}(G(F_{S_0}))$ and $f^h_{S_0} \in \mathcal{S}(U_h(F_{S_0}))$ are transfers if the functions f_{S_0} and $f^h_{S_0}$ have matching regular orbital integrals in the sense of Definition 4.4 of [BPLZZ21]. The Haar measures on the F_{S_0}-points of the involved groups are those defined in Section 2.3.3.

[11] The product structure is given by the convolution where the Haar measure is normalized so that the characteristic functions of $U_h(\mathcal{O}_v)$ and $G(\mathcal{O}_v)$ are units.

10.1.5. Let P be a standard parabolic subgroup of G and π be a cuspidal automorphic representation of M_P. Let $\chi \in \mathfrak{X}(G)$ be the class of the pair (M_P, π). We assume henceforth that χ is a regular relevant cuspidal datum in the sense of Section 5.1.3.

Set $\Pi = \mathrm{Ind}_P^G(\pi)$ for the corresponding parabolically induced representation. The assumption that χ is regular and relevant means exactly that Π is a Hermitian Arthur parameter (see Section 1.1.3). Moreover, we assume, as we may, that S_0 has been chosen such that Π admits K°-fixed vectors.

Attached to these data, we have three distributions denoted by I_χ, $I_{P,\pi}$ and I_Π. The first is constructed as a contribution of the Jacquet-Rallis trace formula and it is defined in Theorem 3.2.4.1. The second and third are relative characters built respectively in Section 5.1.5 and Section 8.1.3. The bulk of the paper was devoted to the proof of the following identities (see Theorem 5.2.1.1, Theorem 8.1.4.1 and Corollary 9.2.5.2)

$$(\mathbf{10.1.5.2}) \qquad I_\chi = 2^{-\dim(\mathfrak{a}_P)} I_{P,\pi} = 2^{-\dim(\mathfrak{a}_P)} I_\Pi.$$

10.1.6. Let S_0' be the union of $S_0 \setminus V_{F,\infty}$ and the set of all finite places of F that are inert in E.

We define $\mathfrak{X}_0^h \subset \mathfrak{X}(U_h)$ as the set of equivalence classes of pairs (U_h, σ) where σ is a cuspidal automorphic representation of $U_h(\mathbf{A})$ that satisfies the following conditions:

- σ is K_h°-unramified;
- for all $v \notin S_0' \cup V_{F,\infty}$ the (split) base change of σ_v is Π_v.

Proposition **10.1.6.1.** — *Let $f \in \mathcal{S}^\circ(G(\mathbf{A}))$ and $f^h \in \mathcal{S}^\circ(U_h(\mathbf{A}))$ for every $h \in \mathcal{H}^\circ$. Assume that the following properties are satisfied for every $h \in \mathcal{H}^\circ$:*

1. $f = (\Delta_H^{S_0,*}\Delta_{G'}^{S_0,*}) f_{S_0} \otimes f^{S_0}$ *with* $f_{S_0} \in \mathcal{S}(G(F_{S_0}))$ *and* $f^{S_0} \in \mathcal{S}^\circ(G(\mathbf{A}^{S_0}))$.
2. $f^h = (\Delta_{U_h'}^{S_0})^2 f_{S_0}^h \otimes f^{h,S_0}$ *with* $f_{S_0}^h \in \mathcal{S}(U_h(F_{S_0}))$ *and* $f^{h,S_0} \in \mathcal{S}^\circ(U_h(\mathbf{A}^{S_0}))$.
3. *The functions* f_{S_0} *and* $f_{S_0}^h$ *are transfers.*
4. $f^{h,S_0} = \mathrm{BC}_h^{S_0}(f^{S_0})$.
5. *The function* f^{S_0} *is a product of a smooth compactly supported function on the restricted product* $\prod'_{v \notin S_0'} G(F_v)$ *by the characteristic function of* $\prod_{v \in S_0' \setminus S_0} G(\mathcal{O}_v)$.

Then we have:

$$(\mathbf{10.1.6.3}) \qquad \sum_{h \in \mathcal{H}^\circ} J^h_{\mathfrak{X}_0^h}(f^h) = 2^{-\dim(\mathfrak{a}_P)} I_\Pi(f) = 2^{-\dim(\mathfrak{a}_P)} I_{P,\pi}(f).$$

Remark **10.1.6.2.** — If the assumptions hold for the set S_0, they also hold for any large enough finite set containing S_0: this follows from the Jacquet-Rallis fundamental lemma (see [Yun11] and [BP]) and the simple expression of the transfer at split places (see [Zha14b, proposition 2.5]). We leave it to the reader to keep track of the different choices of Haar measures in these references.

Proof. — The proof follows the same lines as the proof of [BPLZZ21, Theorem 1.7]. For the convenience of the reader, we recall the main steps.

In Theorem 3.2.4.1 we defined a distribution I on $\mathcal{S}(\mathrm{G}(\mathbf{A}))$: this is the "Jacquet-Rallis trace formula" for G. We have an analogous distribution J^h on $\mathcal{S}(\mathrm{U}_h(\mathbf{A}))$ for each $h \in \mathcal{H}$: it is defined in [Zyd20, théorème 0.3] for compactly supported functions and extended to the Schwartz space in [CZ21, §1.1.3 and théorème 15.2.3.1]. Note that, by the Jacquet-Rallis fundamental lemma [Yun11], [BP], for every $h \in \mathcal{H} \setminus \mathcal{H}^\circ$ there exists a place $v \in \mathrm{S}_0' \setminus \mathrm{S}_0$ such that the characteristic function $\mathbf{1}_{\mathrm{G}(\mathcal{O}_v)}$ admits the zero function on $\mathrm{U}_h(\mathrm{F}_v)$ as a transfer. Therefore, by [CZ21, théorème 1.6.1.1], the hypotheses of the proposition imply:

$$(\mathbf{10.1.6.4}) \qquad \mathrm{I}(f) = \sum_{h \in \mathcal{H}^\circ} \mathrm{J}^h(f^h).$$

We will denote by $\mathcal{M}^{\mathrm{S}_0'}(\mathrm{G}(\mathbf{A}))$, resp. $\mathcal{M}^{\mathrm{S}_0'}(\mathrm{U}_h(\mathbf{A}))$, the algebra of S_0'-multipliers defined in [BPLZZ21, definition 3.5] relatively to the subgroup $\prod_{v \notin \mathrm{S}_0'} \mathrm{G}(\mathcal{O}_v)$, resp. $\prod_{v \notin \mathrm{S}_0'} \mathrm{U}_h(\mathcal{O}_v)$. Any multiplier $\mu \in \mathcal{M}^{\mathrm{S}_0'}(\mathrm{G}(\mathbf{A}))$, resp. $\mu \in \mathcal{M}^{\mathrm{S}_0'}(\mathrm{U}_h(\mathbf{A}))$, gives rise to a linear operator $\mu*$ of the algebra $\mathcal{S}^\circ(\mathrm{G}(\mathbf{A}))$, resp. $\mathcal{S}^\circ(\mathrm{U}_h(\mathbf{A}))$ and for every admissible irreducible representation π of $\mathrm{G}(\mathbf{A})$, resp. of $\mathrm{U}_h(\mathbf{A})$, there exists a constant $\mu(\pi) \in \mathbf{C}$ such that $\pi(\mu * f) = \mu(\pi)\pi(f)$ for all $f \in \mathcal{S}^\circ(\mathrm{G}(\mathbf{A}))$, resp. $f \in \mathcal{S}^\circ(\mathrm{U}_h(\mathbf{A}))$.

Let ξ_Π be the infinitesimal character of Π. By [BPLZZ21, Theorem 4.12 (4)], for every $h \in \mathcal{H}^\circ$ and $(\mathrm{U}_h, \sigma) \in \mathfrak{X}_0^h$, the base-change of the infinitesimal character of σ is ξ_Π. However, the universal enveloping algebras of the complexified Lie algebras of U_h are all canonically identified for $h \in \mathcal{H}$ (since these are inner forms of each other) and base-change is injective at the level of infinitesimal characters. As, by [GRS11], there exists at least one $h \in \mathcal{H}^\circ$ such that the set $\mathfrak{X}_\pi^h$ is nonempty (we may even take for h any quasi-split Hermitian form unramified outside S_0), there exists a common infinitesimal character ξ of all $(\mathrm{U}_h, \sigma) \in \mathfrak{X}_0^h$, for $h \in \mathcal{H}^\circ$, whose base-change is ξ_Π.

By the strong multiplicity one theorem of Ramakrishnan (see [Ram18]) and Theorem 3.17 of [BPLZZ21], one can find a multiplier $\mu \in \mathcal{M}^{\mathrm{S}_0'}(\mathrm{G}(\mathbf{A}))$ such that

i. $\mu(\Pi) = 1$;
ii. For all $\chi' \in \mathfrak{X}(\mathrm{G})$ represented by a pair (M_1, π_1) such that the central character of π_1 is trivial on A_G and $\chi' \neq \chi$ we have

$$\mathrm{K}^0_{\mu * f, \chi'} = 0$$

where the kernel $\mathrm{K}^0_{\mu * f, \chi'}$ is defined as in Section 2.10.1.

By Theorem 3.6 and Theorem 4.12 (3) of [BPLZZ21], for every $h \in \mathcal{H}^\circ$ there exists a multiplier $\mu^h \in \mathcal{M}^{\mathrm{S}_0'}(\mathrm{U}_h(\mathbf{A}))$ such that

iii. $\mu^h(\sigma) = 1$ for all $(\mathrm{U}_h, \sigma) \in \mathfrak{X}_0^h$;

iv. For all $\chi' \in \mathfrak{X}(\mathrm{U}_h)$ such that $\chi' \notin \mathfrak{X}_0^h$ and for all parabolic subgroups P of U_h, we have

$$\mathrm{K}^{\mathrm{U}_h}_{\mathrm{P},\mu^h * f^h,\chi'} = 0$$

where the left-hand side is the kernel of the operator given by the right convolution of $\mu^h * f^h$ on $\mathrm{L}^2_\chi([\mathrm{U}_h]_\mathrm{P})$ (see (2.9.2.1)).

Moreover, by [BPLZZ21, Proposition 4.8, Lemma 4.10], we may choose μ and μ^h such that the functions $\mu * f$ and $\mu^h * f^h$, for $h \in \mathcal{H}^\circ$, still satisfy the assumptions of the proposition. So, in particular, from (10.1.6.4) applied to the functions $\mu * f$ and $(\mu^h * f^h)_{h\in\mathcal{H}^\circ}$ instead of f and $(f^h)_{h\in\mathcal{H}^\circ}$, we get

$$\text{(10.1.6.5)} \qquad \mathrm{I}(\mu * f) = \sum_{h\in\mathcal{H}^\circ} \mathrm{J}^h(\mu^h * f^h).$$

Note that by conditions i. and iii. we have:

$$\text{(10.1.6.6)} \qquad \begin{aligned} &\mathrm{I}_\Pi(\mu * f) = \mathrm{I}_\Pi(f), \ \mathrm{I}_{\mathrm{P},\pi}(\mu * f) = \mathrm{I}_{\mathrm{P},\pi}(f) \text{ and} \\ &\mathrm{J}^h_{\mathfrak{X}_0^h}(\mu^h * f^h) = \mathrm{J}^h_{\mathfrak{X}_0^h}(f^h), \ \text{for every } h \in \mathcal{H}^\circ. \end{aligned}$$

Let $\chi' \in \mathfrak{X}(\mathrm{G})$ be such that $\chi \neq \chi'$. It's easy to see that the integral (3.3.3.4) attached to χ' vanishes for any $f \in \mathcal{S}(\mathrm{G}(\mathbf{A}))$ if χ' does not satisfy the hypothesis in condition ii. If it does, the integral (3.3.3.4) attached to χ' and $\mu * f$ vanishes by condition ii. Thus we can conclude by Theorem 3.3.9.1 that $\mathrm{I}_{\chi'}(\mu * f) = 0$ in any case. By Theorem 3.2.4.1 assertion 4 and by the equality (10.1.5.2), we see that the left-hand side of (10.1.6.5) reduces to

$$\mathrm{I}_\chi(\mu * f) = 2^{-\dim(\mathfrak{a}_\mathrm{P})}\mathrm{I}_\Pi(\mu * f) = 2^{-\dim(\mathfrak{a}_\mathrm{P})}\mathrm{I}_{\mathrm{P},\pi}(\mu * f).$$

On the other hand, by iv. and the very definition of J^h given in [Zyd20], the right-hand side of (10.1.6.5) reduces to

$$\sum_{h\in\mathcal{H}^\circ} \mathrm{J}^h_{\mathfrak{X}_0^h}(\mu^h * f^h).$$

Therefore, (10.1.6.5) and (10.1.6.6) give the identity of the proposition. □

10.2. *Proof of Theorem 1.1.5.1*

10.2.1. Let $\Pi = \mathrm{Ind}_\mathrm{P}^\mathrm{G}(\pi)$ be a Hermitian Arthur parameter of G. Note that by properties 1 and 2 of Section 1.1.3, the cuspidal datum χ associated to the pair $(\mathrm{M}_\mathrm{P}, \pi)$ is regular and relevant in the sense of Section 4.3.2. For $h \in \mathcal{H}$ and σ a cuspidal automorphic representation of $\mathrm{U}_h(\mathbf{A})$, it is readily seen that the linear form $\mathcal{P}_h$ is nonzero on

σ if and only if J^h_σ is not identically zero. On the other hand, the linear form J_η or β_η is always nonzero (this follows either from the fact that χ is relevant or is an easy consequence of [GK72], [Jac10, Proposition 5] and [Kem15]) whereas the linear form I, from Proposition 5.1.4.1, or λ, from Section 8.1.2, is nonzero if and only if $L(\frac{1}{2},\Pi)\neq 0$ (as follows either from the work of Ichino and Yamana, see [IY15, corollary 5.7], or of Jacquet, Piatetski-Shapiro and Shalika [JPSS83], [Jac04]). Therefore, we similarly deduce that the distribution $I_{P,\pi}$ or I_Π is non-zero if and only if $L(\frac{1}{2},\Pi)\neq 0$.

As a consequence, Theorem 1.1.5.1 amounts to the equivalence between the two assertions:

(A) The distribution $I_{P,\pi}$ or I_Π is non-zero.
(B) There exist $h\in\mathcal{H}$, $f\in\mathcal{S}(U_h(\mathbf{A}))$ and a cuspidal subrepresentation σ of U_h such that $BC(\sigma)=\Pi$ and $J^h_\sigma(f)\neq 0$.

10.2.2. *Proof of* (A) $\Rightarrow$ (B). — We choose the S_0 of Section 10.1.3 such that I_Π is not identically zero on $f_1\in\mathcal{S}^\circ(G(\mathbf{A}))$. Then Assertion (B) above is a consequence of Proposition 10.1.6.1: it suffices to take functions f and f^h for $h\in\mathcal{H}^\circ$ satisfying the hypotheses of that theorem and such that $I_\Pi(f)\neq 0$. That it is possible is implied by a combination of a result of [Xue19] and the existence of p-adic transfer [Zha14b].

10.2.3. *Proof of* (B) $\Rightarrow$ (A). — We may choose the set S_0 so that there exist $h_0\in\mathcal{H}^\circ$, $f_0^{h_0}\in\mathcal{S}^\circ(U_{h_0}(\mathbf{A}))$ and a cuspidal representation σ_0 of U_h such that for $v\notin S_0'$ (see Section 10.1.6) $BC(\sigma_{0,v})=\Pi_v$ and $J^{h_0}_{\sigma_0}(f_0^{h_0})\neq 0$. For any other $h\in\mathcal{H}^\circ$ we set $f_0^h=0$. Up to enlarging S_0, we may assume that the family $(f_0^h)_{h\in\mathcal{H}^\circ}$ satisfies conditions 2. and 5. of Proposition 10.1.6.1. Moreover, we have (see [Zha14b, §2.5]) $J^{h_0}_\sigma(f_0^{h_0}*f_0^{h_0})\geqslant 0$ for every $\sigma\in\mathfrak{X}_0^{h_0}$ and $J^{h_0}_{\sigma_0}(f_0^{h_0}*f_0^{h_0})>0$. In particular, the left hand side of (10.1.6.3) for the family $(f_0^h*f_0^h)_{h\in\mathcal{H}^\circ}$ is nonzero. Once again by [Xue19] and the existence of p-adic transfer [Zha14b], this implies that we can find test functions $f\in\mathcal{S}^\circ(G(\mathbf{A}))$ and $f^h\in\mathcal{S}^\circ(U_h(\mathbf{A}))$, for $h\in\mathcal{H}^\circ$, satisfying all the conditions of Proposition 10.1.6.1 and such that the left hand side of (10.1.6.3) is still nonzero. The conclusion of this proposition immediately gives Assertion (A).

10.3. *Proof of Theorem 1.1.6.1*

10.3.1. Let $h\in\mathcal{H}$ and σ be a cuspidal automorphic representation of $U_h(\mathbf{A})$ which is tempered everywhere. By [Mok15], [KMSW], σ admits a weak base-change Π to G. Moreover, by these references Π is also a strong base-change of σ: for every place v of F, the local base-change of σ_v (defined in [Mok15] and [KMSW]) coincides with Π_v. In particular, it follows that Π is also tempered everywhere.

We choose a finite set of places S_0 as in Section 10.1.3 such that $h\in\mathcal{H}^\circ$ and σ as well as the additive character ψ' used to normalize local Haar measures in Section 2.3 are unramified outside of S_0.

For each place v of F, we define a distribution J_{σ_v} on $\mathcal{S}(U_h(F_v))$ by

$$J_{\sigma_v}(f_v^h) = \int_{U_h'(F_v)} \mathrm{Trace}(\sigma_v(h_v)\sigma_v(f_v^h))dh_v, \;\; f_v^h \in \mathcal{S}(U_h(F_v)),$$

where

$$\sigma_v(f_v^h) = \int_{U_h(F_v)} f_v^h(g_v)\sigma_v(g_v)dg_v$$

and the Haar measures are the one defined in Section 2.3.3. Moreover by [Har14], and since the representations σ_v are all tempered, the expression defining J_{σ_v} is absolutely convergent and for every $v \notin S_0$ we have

$$J_{\sigma_v}(\mathbf{1}_{U_h(\mathcal{O}_v)}) = \Delta_{U_h',v}^{-2}\frac{L(\frac{1}{2},\Pi_v)}{L(1,\sigma_v,\mathrm{Ad})}.$$

10.3.2. By [Zha14a, Lemma 1.7] and our choice of local Haar measures, Theorem 1.1.6.1 is equivalent to the following assertion: for all factorizable test function $f^h \in \mathcal{S}(U_{h_0}(\mathbf{A}))$ of the form $f^h = (\Delta_{U_h'}^{S_0})^2 \prod_{v\in S_0} f_v^h \times \prod_{v\notin S_0} \mathbf{1}_{U_h(\mathcal{O}_v)}$, we have

$$J_\sigma^h(f^h) = |S_\Pi|^{-1}\frac{L^{S_0}(\frac{1}{2},\Pi)}{L^{S_0}(1,\sigma,\mathrm{Ad})}\prod_{v\in S_0} J_{\sigma_v}(f_v^h). \tag{10.3.2.1}$$

10.3.3. For every place v of F, we define a local relative character I_{Π_v} on $G(F_v)$ by

$$I_{\Pi_v}(f_v) = \sum_{W_v \in \mathcal{W}(\Pi_v,\psi_{N,v})} \frac{\lambda_v(\Pi_v(f_v)W_v)\overline{\beta_{\eta,v}(W_v)}}{\langle W_v, W_v\rangle_{\mathrm{Whitt},v}}, \;\; f_v \in \mathcal{S}(G(F_v)),$$

where the sum runs over a K_v-basis of the Whittaker model $\mathcal{W}(\Pi_v,\psi_{N,v})$ (in the sense of Section 2.8.3) and λ_v, $\beta_{\eta,v}$, $\langle .,.\rangle_{\mathrm{Whitt},v}$ are local analogs of the forms introduced in Section 8.1.2 given by

$$\lambda_v(W_v) = \int_{N_H(F_v)\backslash H(F_v)} W_v(h_v)dh_v,$$

$$\beta_{\eta,v}(W_v) = \int_{N'(F_v)\backslash \mathcal{P}'(F_v)} W_v(p_v)\eta_{G',v}(p_v)dp_v,$$

$$\text{and } \langle W_v, W_v\rangle_{\mathrm{Whitt},v} = \int_{N(F_v)\backslash\mathcal{P}(F_v)} |W_v(p_v)|^2 dp_v.$$

Note that the above expressions, and in particular $\lambda_v(W_v)$, are all absolutely convergent due to the fact that Π_v is tempered (see [JPSS83, Proposition 8.4]). The above definition also implicitly depends on the choice of an additive character ψ of $\mathbf{A}_E/E$ trivial on $\mathbf{A}$ (through which the generic character ψ_N is defined, see beginning of Section 8 and Section 6.1.2) and up to enlarging S_0, we may assume that ψ is unramified outside of S_0. Then, it follows from the definition of I_Π that for every factorizable test function $f \in \mathcal{S}(G(\mathbf{A}))$ of the form $f = \Delta_H^{S_0,*}\Delta_{G'}^{S_0,*}\prod_{v\in S_0} f_v \times \prod_{v\notin S_0} \mathbf{1}_{G(\mathcal{O}_v)}$, we have

$$I_\Pi(f) = \frac{L^{S_0}(\frac{1}{2},\Pi)}{L^{S_0}(1,\Pi,\mathrm{As}_G^-)}\prod_{v\in S_0} I_{\Pi_v}(f_v) \tag{10.3.3.2}$$

where we have set $L^{S_0}(s,\Pi,\mathrm{As}_G^-) = L^{S_0}(s,\Pi_n,\mathrm{As}^{(-1)^n})L^{S_0}(s,\Pi_{n+1},\mathrm{As}^{(-1)^{n+1}})$.

10.3.4. Let f^h be a test function as in Section 10.3.2. Then, as both sides of (10.3.2.1) are continuous functionals in f_v^h for $v \in V_{F,\infty}$, by the main result of [Xue19] we may assume that for every $v \in V_{F,\infty}$ the function f_v^h admits a transfer $f_v \in \mathcal{S}(G(F_v))$. On the other hand, by [Zha14b], for every $v \in S_0 \setminus V_{F,\infty}$, the function f_v^h admits a transfer $f_v \in \mathcal{S}(G(F_v))$. Moreover, by the results of those references we may also choose the transfers such that for every $h' \in \mathcal{H}^\circ$ with $h' \neq h$, the zero function on $U_{h'}(F_{S_0})$ is a transfer of $f_{S_0} = \prod_{v\in S_0} f_v$. We set $f = \Delta_H^{S_0,*}\Delta_{G'}^{S_0,*} f_{S_0} \times \prod_{v\notin S_0} \mathbf{1}_{G(\mathcal{O}_v)}$. Then, setting $f^{h'} = 0$ for every $h' \in \mathcal{H}^\circ \setminus \{h\}$, the functions f and $(f^{h'})_{h'\in\mathcal{H}^\circ}$ satisfy the assumptions of Proposition 10.1.6.1. Therefore, we have

$$\sum_{\sigma'\in\mathfrak{X}_0^h} J_{\sigma'}^h(f^h) = 2^{-\dim(\mathfrak{a}_P)} I_\Pi(f). \tag{10.3.4.3}$$

10.3.5. If there exists a place $v \in S_0$ such that σ_v does not support any nonzero continuous $U_h'(F_v)$-invariant functional, both sides of (10.3.2.1) are automatically zero.

Assume now that for every $v \in S_0$, the local representation σ_v supports a nonzero continuous $U_h'(F_v)$-invariant functional. By the local Gan-Gross-Prasad conjecture [BP20], and the classification of cuspidal automorphic representations of U_h in terms of local L-packets [Mok15], [KMSW], it follows that all the terms except possibly $J_\sigma^h(f^h)$ in the left hand side of (10.3.4.3) are zero. Moreover, by [BP21c, Theorem 5.4.1] and since Π_v is the local base-change of σ_v, there are explicit constants $\kappa_v \in \mathbf{C}^\times$ for $v \in S_0$ satisfying $\prod_{v\in S_0}\kappa_v = 1$ and such that

$$I_{\Pi_v}(f_v) = \kappa_v J_{\sigma_v}(f_v^h) \tag{10.3.5.4}$$

for every $v \in S_0$. Combining this with (10.3.3.2), we get

$$\begin{aligned} J_\sigma^h(f^h) &= 2^{-\dim(\mathfrak{a}_P)} I_\Pi(f) = 2^{-\dim(\mathfrak{a}_P)} I_{P,\pi}(f) \\ &= 2^{-\dim(\mathfrak{a}_P)} \frac{L^{S_0}(\frac{1}{2},\Pi)}{L^{S_0}(1,\Pi,\mathrm{As}_G^-)}\prod_{v\in S_0} I_{\Pi_v}(f_v) \end{aligned}$$

$$= 2^{-\dim(\mathfrak{a}_P)} \frac{L^{S_0}(\frac{1}{2}, \Pi)}{L^{S_0}(1, \Pi, As_G^-)} \prod_{v \in S_0} J_{\sigma_v}(f_v^h).$$

As $L^{S_0}(s, \Pi, As_G^-) = L^{S_0}(s, \sigma, Ad)$ and $|S_\Pi| = 2^{-\dim(\mathfrak{a}_P)}$, this exactly gives (10.3.2.1) and ends the proof of Theorem 1.1.6.1.

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Appendix A: Topological vector spaces

A.0.1 In this paper, by a *locally convex topological vector space* (LCTVS) we mean a Hausdorff locally convex vector space over $\mathbf{C}$. Most LCTVS encountered in this paper will be Fréchet or LF (that is a countable inductive limit of Fréchet spaces) or even strict LF (that is countable inductive limit $\varinjlim_n V_n$ of Fréchet spaces with closed embeddings

$V_n \to V_{n+1}$ as connecting maps) spaces. Let V and W be LCTVS. We denote by V' the topological dual of V and by Hom(V, W) the space of continuous linear mappings $V \to W$ both equipped with their weak topologies (i.e. topologies of pointwise convergence). Recall that a *total subspace* $H \subset V'$ is a subspace such that $\bigcap_{\lambda \in H} \mathrm{Ker}(\lambda) = 0$. A *bounded* subset $B \subseteq V$ is one that is absorbed by any neighborhood of 0. If $B \subseteq V$ is bounded and absolutely convex,[12] we denote by V_B the subspace generated by B equipped with the norm $\|v\|_B = \inf\{\lambda \geqslant 0 \mid v \in \lambda B\}$. Then, the natural inclusion $V_B \to V$ is continuous. The space V is said to be *quasi-complete* if every closed bounded subset of it is complete. Fréchet spaces and strict LF spaces are quasi-complete.

A.0.2 We recall the notion of integral valued in a LCTVS in the form we use it in the core of the paper. Let X be a σ-compact locally compact topological space, dx be a Radon measure on X and V be a LCTVS. Let $f : X \to V$ be a continuous function. We say that f is *absolutely integrable* if for every continuous semi-norm p on V the integral $\int_X p(f(x))\mu(x)$ converges. If f is absolutely integrable and V is quasi-complete, there exists an unique element $\int_X f(x)\mu(x)$ in V such that

$$\langle \lambda, \int_X f(x)\mu(x)\rangle = \int_X \langle \lambda, f(x)\rangle \mu(x)$$

for every $\lambda \in V'$. This notion applies in particular to series $\sum_{i\in I} v_i$ valued in a quasi-complete LCTVS V in which case we will rather use the terminology *absolutely summable*: a family $(v_i)_{i\in I}$ of vectors in V is *absolutely summable* if for every continuous semi-norm p on V, the series $\sum_{i\in I} p(v_i)$ converges.

We will also use the following weaker notion: a family $(v_i)_{i\in I}$ of vectors in a LCTVS V is said to be *summable* if for every continuous semi-norm p on V and every $\epsilon > 0$, there exists a finite subset $J \subseteq I$ such that

$$p(\sum_{i\in K} v_i) < \epsilon$$

for every finite subset $K \subseteq I \setminus J$. If $(v_i)_{i\in I}$ is a summable family in V and V is quasi-complete then the partial sums $\sum_{i\in J} v_i$ converge to some limit $v \in V$ along the filter associated to the inclusion order on finite subsets of I. In this case, we call v the *sum* of the family $(v_i)_{i\in I}$.

Note that absolutely summable families are automatically summable but the converse is not true e.g. if V is a Hilbert space with a Hilbert decomposition $V = \widehat{\bigoplus}_{i\in I} V_i$ and $v \in V$ then the family of orthogonal projections $(v_i)_{i\in I}$ of v to the subspaces V_i is always summable but not always absolutely summable.

[12] Recall that a subset $S \subseteq V$ is said to be *absolutely convex* is it is convex and circled i.e. $\lambda S \subseteq S$ for every complex number λ with $|\lambda| \leqslant 1$.

A.0.3 We will also freely use the notions of smooth or holomorphic functions valued in a LCTVS. For basic references on these subjects, we refer the reader to [Bou67, §2, §3], [Gro53, §2], [Gro73, Chap. 3, §8]. There are actually two ways to define smooth and holomorphic maps valued in V: either scalarly (that is after composition with any element of V′) or by directly requiring the functions to be infinitely (complex) differentiable. These two definitions coincide when the space V is quasi-complete and, fortunately for us, we will only consider smooth/holomorphic functions valued in such spaces so that we don't have to distinguish.

Let M be a connected complex analytic manifold. A function $f : \mathrm{M} \to \mathrm{V}$ is holomorphic if and only if for every relatively compact open subset $\Omega \subseteq \mathrm{M}$, there exists a bounded absolutely convex subset $\mathrm{B} \subseteq \mathrm{V}$ such that $f\,|_{\Omega}$ factorizes through a holomorphic map $\Omega \to \mathrm{V}_{\mathrm{B}}$ see [Gro53, §2, Remarque 2]. We also record the following convenient criterion of holomorphicity [Bou67, §3.3.1]:

(**A.0.3.1**) Assume that V is quasi-complete. A function $\varphi : \mathrm{M} \to \mathrm{V}$ is holomorphic if and only if it is continuous and for some total subspace $\mathrm{H} \subseteq \mathrm{V}'$, the functions $s \in \mathrm{M} \mapsto \langle \varphi(s), \lambda \rangle$ are holomorphic for every $\lambda \in \mathrm{H}$.

A.0.4 Assume that V is a LF space. As LF spaces are barreled [Trè67, Corollary 33.3] they satisfy the Banach-Steinhaus theorem [Trè67, Theorem 33.1] hence any bounded subset of $\mathrm{Hom}(\mathrm{V}, \mathrm{W})$ is equicontinuous (as $\mathrm{Hom}(\mathrm{V}, \mathrm{W})$ is equipped with the weak topology, a subset $\mathrm{B} \subseteq \mathrm{Hom}(\mathrm{V}, \mathrm{W})$ is bounded if and only if for every $v \in \mathrm{V}$, $\{\mathrm{T}(v) \mid \mathrm{T} \in \mathrm{B}\}$ is a bounded subset of W). This implies in particular that for every bounded subset $\mathrm{B} \subseteq \mathrm{Hom}(\mathrm{V}, \mathrm{W})$, the restriction of the canonical bilinear map $\mathrm{Hom}(\mathrm{V}, \mathrm{W}) \times \mathrm{V} \to \mathrm{W}$ to $\mathrm{B} \times \mathrm{V}$ is continuous. Also, by [Trè67, §34.3 Corollary 2], if W is quasi-complete then so is $\mathrm{Hom}(\mathrm{V}, \mathrm{W})$. In particular, we get:

(**A.0.4.2**) Assume that V is LF, W is quasi-complete and let K be a topological space. Let $s \in \mathrm{M} \mapsto \mathrm{T}_s \in \mathrm{Hom}(\mathrm{V}, \mathrm{W})$ be holomorphic and $(s, k) \in \mathrm{M} \times \mathrm{K} \mapsto v_{s,k} \in \mathrm{V}$ be a continuous map which is holomorphic in the first variable. Then, the map $(s, k) \in \mathrm{M} \times \mathrm{K} \mapsto \mathrm{T}_s(v_{s,k}) \in \mathrm{W}$ is continuous and holomorphic in the first variable.

Indeed, T has locally its image in a bounded set. Hence, by the above discussion, the map $(s, s', k) \in \mathrm{M} \times \mathrm{M} \times \mathrm{K} \mapsto \mathrm{T}_s(v_{s',k}) \in \mathrm{W}$ is continuous. Moreover, this map is separately holomorphic in the variables s, s'. Thus, by Hartog's theorem, this map is holomorphic in the variables (s, s') which immediately implies the claim by "restriction to the diagonal".

(**A.0.4.3**) Assume that V is LF and W is quasi-complete. Let $\mathrm{U} \subseteq \mathrm{M}$ be a nonempty open subset and $s \in \mathrm{U} \mapsto \mathrm{T}_s \in \mathrm{Hom}(\mathrm{V}, \mathrm{W})$ be a holomorphic map. If, for every $v \in \mathrm{V}$ the map $s \mapsto \mathrm{T}_s(v) \in \mathrm{W}$ extends analytically to M then $\mathrm{T}_s \in \mathrm{Hom}(\mathrm{V}, \mathrm{W})$ for every $s \in \mathrm{M}$ and moreover $s \in \mathrm{M} \mapsto \mathrm{T}_s \in \mathrm{Hom}(\mathrm{V}, \mathrm{W})$ is holomorphic.

Indeed, the hypothesis implies that $s \mapsto T_s$ induces a holomorphic map $M \to \mathcal{H}om(V, W)$ where $\mathcal{H}om(V, W)$ stands for the space of *all* linear maps $V \to W$ (not necessarily continuous) equipped with the topology of pointwise convergence. Hence, for every relatively compact connected open subset $\Omega \subseteq M$ such that $\Omega \cap U \neq \emptyset$ there exists a bounded subset $B \subseteq \mathcal{H}om(V, W)$ such that $s \mapsto T_s$ factorizes through a holomorphic map $\Omega \to \mathcal{H}om(V, W)_B$. By the Banach-Steinhaus theorem, $\mathrm{Hom}(V, W) \cap \mathcal{H}om(V, W)_B$ is closed in $\mathcal{H}om(V, W)_B$ and it follows that $s \in \Omega \mapsto T_s$ factors through a holomorphic map $\Omega \to \mathrm{Hom}(V, W) \cap \mathcal{H}om(V, W)_B$. Indeed, by the Hahn-Banach theorem it suffices to show that for every continuous linear form $\lambda : \mathcal{H}om(V, W)_B \to \mathbf{C}$ vanishing on $\mathrm{Hom}(V, W) \cap \mathcal{H}om(V, W)_B$ we have $\lambda(T_s) = 0$ for every $s \in \Omega$. Since $T_s \in \mathrm{Hom}(V, W)$ for $s \in U$ the equality $\lambda(T_s) = 0$ holds at least for $s \in \Omega \cap U$. As $s \in \Omega \mapsto \lambda(T_s)$ is holomorphic Ω is connected and $U \cap \Omega$ non-empty, the claim follows.

A.0.5 Let $\mathrm{Bil}_s(V, W) = \mathrm{Hom}(V, \mathrm{Hom}(W, \mathbf{C}))$ be the space of separately continuous bilinear mappings $V \times W \to \mathbf{C}$ equipped with the topology of pointwise convergence. Applying (A.0.4.2) and (A.0.4.3) twice, we get:

(**A.0.5.4**) Assume that V and W are LF. Let $s \in M \mapsto B_s \in \mathrm{Bil}_s(V, W)$ be holomorphic and $(s, k) \in M \times K \mapsto v_{s,k} \in V$, $(s, k) \in M \times K \mapsto w_{s,k} \in W$ be continuous maps which are holomorphic in the first variable. Then, the function $(s, k) \in M \times K \mapsto B_s(v_{s,k}, w_{s,k})$ is continuous and holomorphic in the first variable.

(**A.0.5.5**) Assume that both V and W are LF. Let $U \subseteq M$ be a nonempty open subset and $s \in U \mapsto B_s \in \mathrm{Bil}_s(V, W)$ be a holomorphic map. If for every $(v, w) \in V \times W$ the function $s \mapsto B_s(v, w)$ extends analytically to M then $B_s \in \mathrm{Bil}_s(V, W)$ for every $s \in M$ and moreover $s \in M \mapsto B_s \in \mathrm{Bil}_s(V, W)$ is holomorphic.

A.0.6 We refer the reader to [Trè67, Definitions 47.2, 47.3] for the definition of nuclear mappings between LCTVS. The following lemma will be useful in proving that certain families are absolutely summable.

Lemma **A.0.6.1.** — *Let* $T : V \to W$ *be a nuclear mapping between LCTVS. Then, for every summable family* $(v_i)_{i \in I}$ *in* V, *the family* $(T(v_i))_{i \in I}$ *is absolutely summable in* W.

Proof. — First, by the very definition, T is a nuclear mapping if and only if it factorizes through a nuclear mapping between Banach spaces. Thus, we may assume without loss in generality that V and W are Banach spaces. Recall [Trè67, Definition 47.2] that this means that T belongs to the image of the natural map $V'_s \widehat{\otimes} W \to \mathrm{Hom}_s(V, W)$ where V'_s (resp. $\mathrm{Hom}_s(V, W)$) is the topological dual of V (resp. the space of continuous linear maps $V \to W$) equipped with the strong topology (aka norm topology) and $\widehat{\otimes}$ stands for the completed projective tensor product. Let $(v_i)_{i \in I}$ be a summable family in V and set

$$M = \sup_J \| \sum_{i \in J} v_i \|_V < \infty$$

where the sup runs over all finite subsets $J \subseteq I$. It follows from the next elementary lemma that for $(\ell, w) \in V' \times W$, the family $(\ell(v_i)w)_{i\in I}$ is absolutely summable in W and moreover

$$\sum_{i\in I} \|\ell(v_i)w\|_W = \|w\|_W \sum_{i\in I} |\ell(v_i)| \leqslant 4M\|\ell\|_{V'}\|w\|_W \tag{A.0.6.6}$$

where $\|.\|_W$ (resp. $\|.\|_{V'}$) denotes the norm on W (resp. dual norm on V′).

Lemma **A.0.6.2.** — *Let $(z_i)_{i\in I}$ be a summable family in* $\mathbf{C}$. *Then, $(z_i)_{i\in I}$ is absolutely summable and moreover*

$$\sum_{i\in I} |z_i| \leqslant 4 \sup_J |\sum_{i\in J} z_i| \tag{A.0.6.7}$$

where the sup runs over all finite subsets of I.

Let $\ell^1(I, W)$ be the vector space of absolutely summable families indexed by I in W. We equip $\ell^1(I, W)$ with the norm

$$\|(w_i)_{i\in I}\|_{\ell^1} = \sum_{i\in I} \|w_i\|_W.$$

It then becomes a Banach space. By (A.0.6.6), the bilinear map $V'_s \times W \to \ell^1(I, W)$, $(\ell, w) \mapsto (\ell(v_i)w)_{i\in I}$, is continuous hence induces a continuous linear map $V'_s \widehat{\otimes} W \to \ell^1(I, W)$. Obviously, this map is the composite of the natural morphism $V'_s \widehat{\otimes} W \to \mathrm{Hom}_s(V, W)$ and of $\mathrm{Hom}_s(V, W) \to W^I$, $T \mapsto (T(v_i))_{i\in I}$. The lemma follows. □

A.0.7 We denote by $V\widehat{\otimes}W$ the completed projective tensor product [Trè67, Chap. 43]. It admits a canonical linear map $V \otimes W \to V\widehat{\otimes}W$ satisfying the following universal property: for every complete LCTVS U, precomposition yields an isomorphism

$$\mathrm{Hom}(V\widehat{\otimes}W, U) \simeq \mathrm{Bil}(V, W; U)$$

where Bil(V, W; U) denotes the space of all *continuous* bilinear mappings $V \times W \to U$. In particular, if U_1, U_2 are two other LCTVS and $T : V \to U_1$, $S : W \to U_2$ are continuous linear mapping, there is an unique continuous linear map $T\widehat{\otimes}S : V\widehat{\otimes}W \to U_1\widehat{\otimes}U_2$ which on $V \otimes W$ is given by $v \otimes w \mapsto T(v) \otimes S(w)$.

Assume now that V and W are spaces of (complex valued) functions on two sets X, Y and that their topologies are finer than the topology of pointwise convergence. When V is moreover a complete nuclear LF space, the following result of Grothendieck [Gro55, Théorème 13, Chap. II, §3 n. 3] generally allows to describe $V\widehat{\otimes}W$ explicitly as a space of functions on $X \times Y$.

(**A.0.7.8**) Let $\mathcal{F}(X \times Y)$ be the space of all complex valued functions on $X \times Y$ equipped with the topology of pointwise convergence. Then the linear map $V \otimes W \to \mathcal{F}(X \times Y)$, $v \otimes w \mapsto ((x, y) \mapsto v(x)w(y))$, extends continuously to a linear embedding $V\widehat{\otimes}W \hookrightarrow \mathcal{F}(X \times Y)$ with image the space of functions $f : X \times Y \to \mathbf{C}$ satisfying the two conditions:

- For every $x \in X$, the function $y \in Y \mapsto f(x, y)$ belongs to the completion of W;
- For every $\lambda \in W'$, the function $x \in X \mapsto \langle f(x, .), \lambda\rangle$ belongs to V.

A.0.8 Let $C \in \mathbf{R} \cup \{-\infty\}$ and $f : \mathcal{H}_{>C} \to V$ be a holomorphic function. We say that f is *of order at most d in vertical strips* if for every $d' > d$ the function $z \mapsto e^{-|z|^{d'}} f(z)$ is *bounded in vertical strips* of $\mathcal{H}_{>C}$. We say that f is of *finite order in vertical strips* if it is of order at most d in vertical strips for some $d > 0$. Finally, we say that f is *rapidly decreasing in vertical strips* if for every $d > 0$ the function $z \mapsto |z|^d F(z)$ is bounded in vertical strips.

A.0.9 Let $\mathcal{A}$ be a real vector space. Denote by $\mathrm{Diff}(\mathcal{A})$ the space of complex polynomial differential operators on $\mathcal{A}$ (which can be identified with $\mathrm{Sym}(\mathcal{A}^*_{\mathbf{C}}) \otimes_{\mathbf{C}} \mathrm{Sym}(\mathcal{A}_{\mathbf{C}})$). When V is quasi-complete, we define the space of *Schwartz functions on* $\mathcal{A}$ *valued in* V, denoted by $\mathcal{S}(\mathcal{A}, V)$, as the space of smooth functions $f : \mathcal{A} \to V$ such that for every $D \in \mathrm{Diff}(\mathcal{A})$, the function Df has bounded image. Note that if W is also quasi-complete and $T : V \to W$ is a continuous linear map then for every $f \in \mathcal{S}(\mathcal{A}, V)$, we have $T \circ f \in \mathcal{S}(\mathcal{A}, W)$. When $V = \mathbf{C}$, we simply set $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{A}, \mathbf{C})$ that we equip with its standard Fréchet topology.

Lemma **A.0.9.1.** — *Assume that* V *is quasi-complete and barreled (e.g. a strict LF space). Let* $C > 0$, $d > 0$ *and* $s \in \mathcal{H}_{>C} \mapsto Z_s \in V'$ *be a map such that such that for every* $v \in V$, $s \in \mathcal{H}_{>C} \mapsto Z_s(v)$ *is a holomorphic function of order at most d in vertical strips. Then, for every* $f \in \mathcal{S}(\mathcal{A}, V)$, *the map*

$$s \in \mathcal{H}_{>C} \mapsto \big(\lambda \in \mathcal{A} \mapsto Z_s(f_\lambda)\big) \in \mathcal{S}(\mathcal{A}) \tag{A.0.9.9}$$

is holomorphic and of finite order in vertical strips.

Proof. — Indeed, by the Banach-Steinhaus theorem, for every $d' > d$, every vertical strip $S \subseteq \mathcal{H}_{>C}$ and every bounded subset $B \subseteq V$ the set

$$\left\{e^{-|s|^{d'}} Z_s(v) \mid s \in S,\, v \in B\right\} \subseteq \mathbf{C}$$

is bounded and, by [Trè67, Corollary 33.1], for every $s_0 \in \mathcal{H}_{>C}$, Z_s converges uniformly on compact subsets to Z_{s_0} as $s \to s_0$. Let $f \in \mathcal{S}(\mathcal{A}, V)$. Moreover, for every $D \in \mathrm{Diff}(\mathcal{A})$, as the function $\lambda \in \mathcal{A} \mapsto Df_\lambda$ is continuous and converges to 0 as $\lambda \to \infty$, the subset

$$\left\{Df_\lambda \mid \lambda \in \mathcal{A}\right\} \cup \{0\}$$

of V is compact. Therefore, for every $s_0 \in \mathcal{H}_{>C}$, $Z_s(Df_\lambda)$ converges to $Z_{s_0}(Df_\lambda)$ as $s \to s_0$ uniformly in $\lambda \in \mathcal{A}$ and $\left\{e^{-|s|^{d'}} Z_s(Df_\lambda) \mid s \in S, \lambda \in \mathcal{A}\right\}$ is bounded for every $d' > d$ and every vertical strip $S \subseteq \mathcal{H}_{>C}$. This shows that the map (A.0.9.9) is continuous and of finite order in vertical strips. To conclude we apply the holomorphicity criterion (A.0.3.1) to the subspace $H \subseteq \mathcal{S}(\mathcal{A})'$ generated by "evaluations at a point of $\mathcal{A}$". □

A.0.10

Lemma **A.0.10.1.** — *Assume that* V *is quasi-complete. Let* $Z_+, Z_- : \mathcal{H}_{>C} \to V$ *be holomorphic functions of finite order in vertical strips for some* $C > 0$. *Assume that there exists a total subspace* $H \subset V'$ *such that for every* $\lambda \in H$, $Z_{+,\lambda} := \lambda \circ Z_+$ *and* $Z_{-,\lambda} := \lambda \circ Z_-$ *extend to holomorphic functions on* $\mathbf{C}$ *of finite order in vertical strips satisfying* $Z_{+,\lambda}(s) = Z_{-,\lambda}(-s)$ *for every* $s \in \mathbf{C}$. *Then,* Z_+ *and* Z_- *extend to holomorphic functions* $\mathbf{C} \to V$ *of finite order in vertical strips satisfying* $Z_+(s) = Z_-(-s)$ *for every* $s \in \mathbf{C}$.

Proof. — Let $d > 0$ be such that Z_+ and Z_- are of order at most d in vertical strips of $\mathcal{H}_{>C}$. Then, by their functional equation and the classical Phragmen-Lindelöf principle, for every $\lambda \in H$, the holomorphic continuations of $Z_{+,\lambda}$ and $Z_{-,\lambda}$ are also of order at most d in vertical strips. Therefore, up to multiplying Z_+ and Z_- by $z \mapsto e^{s^{4n+2}}$ for some $n \geqslant 0$, we may assume that all these functions are rapidly decreasing in vertical strips. Let $D > C$. Then, for every $s \in \mathcal{H}_{]-D,D[}$ and $\epsilon \in \{\pm\}$, we set

$$\Phi_\epsilon(s) = \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} \frac{Z_\epsilon(D+it)}{D+it-s}dt - \int_{-\infty}^{+\infty} \frac{Z_{-\epsilon}(D+it)}{D+it+s}dt\right).$$

Note that, since Z_+ and Z_- are rapidly decreasing in vertical strips and V is quasi-complete, the above integrals converge absolutely and define elements of V. By the usual holomorphicity criterion for parameter integrals, we readily check that the functions Φ_+, Φ_- are holomorphic. Moreover, by the uniform boundedness principle, Φ_+ and Φ_- are bounded in vertical strips. Finally, by Cauchy's integration formula and the fact that the functions $Z_{+,\lambda}$, $Z_{-,\lambda}$ are rapidly decreasing in vertical strips, for every $\epsilon \in \{\pm\}$ and $\lambda \in H$ the functions $\lambda \circ \Phi_\epsilon$ and $Z_{\epsilon,\lambda}$ coincide on $\mathcal{H}_{]-D,D[}$. Therefore, as H is total, Φ_ϵ and Z_ϵ coincide on $\mathcal{H}_{]C,D[}$. This shows that Z_+ and Z_- admit holomorphic extensions bounded in vertical strips to $\mathcal{H}_{>-D}$ for every $D > C$ hence to $\mathbf{C}$. That the functional equation $Z_+(s) = Z_-(-s)$ holds for these extensions easily follows from the assumption. □

A.0.11 Let $\mathcal{A}$ be a real vector space. Specializing the previous lemma to $V = \mathcal{S}(\mathcal{A})$ and H the total subspace of V′ given by "evaluations at a point of $\mathcal{A}$" yields the following corollary.

Corollary **A.0.11.1.** — *Let* $Z_+, Z_- : \mathcal{A} \times \mathbf{C} \to \mathbf{C}$ *be two functions such that:*

1. *There exists* $C > 0$ *such that for every* $s \in \mathcal{H}_{>C}$, *the function* $Z_+(.,s)$, $Z_-(.,s)$ *belong to* $\mathcal{S}(\mathcal{A})$ *and the maps*

$$s \in \mathcal{H}_{>C} \mapsto Z_\epsilon(.,s) \in \mathcal{S}(\mathcal{A}), \quad \epsilon \in \{\pm\},$$

are holomorphic functions of finite order in vertical strips;

2. *For every* $\lambda \in \mathcal{A}$, $s \in \mathbf{C} \mapsto Z_+(\lambda, s)$ *and* $s \in \mathbf{C} \mapsto Z_-(\lambda, s)$ *are holomorphic functions of finite order in vertical strips satisfying the functional equation*

$$Z_+(\lambda, s) = Z_-(\lambda, -s)$$

Then, for every $s \in \mathbf{C}$ *the functions* $Z_+(.,s)$, $Z_-(.,s)$ *belong to* $\mathcal{S}(\mathcal{A})$ *the maps* $s \in \mathbf{C} \mapsto Z_\epsilon(.,s) \in \mathcal{S}(\mathcal{A})$, $\epsilon \in \{\pm\}$, *are holomorphic.*

Assume now that W is a LF space. As W is barreled, W′ is quasi-complete [Trè67, §34.3 Corollary 2]. Specializing Lemma A.0.10.1 to $V = W'$ and H a dense subset of $V' = W$, we obtain the following.

Corollary **A.0.11.2.** — *Let* W *be a LF space,* $C > 0$ *and* $Z_+, Z_- : \mathcal{H}_{>C} \times W \to \mathbf{C}$ *be two functions. Assume that:*

1. *For every* $s \in \mathcal{H}_{>C}$, $Z_+(s,.)$ *and* $Z_-(s,.)$ *are continuous functionals on* W;
2. *There exists* $d > 0$ *such that for every* $w \in W$ *and* $\epsilon \in \{\pm\}$, $s \in \mathcal{H}_{>C} \mapsto Z_\epsilon(s, w)$ *is a holomorphic function of order at most* d *in vertical strips;*
3. *For every* $f \in H$ *and* $\epsilon \in \{\pm\}$, $s \mapsto Z_\epsilon(s,f)$ *extends to a holomorphic function on* $\mathbf{C}$ *of finite order in vertical strips satisfying*

$$Z_+(s,f) = Z_-(-s,f).$$

Then, Z_+ *and* Z_- *extend to holomorphic functions* $\mathbf{C} \to W'$ *of finite order in vertical strips satisfying* $Z_+(s,w) = Z_-(-s,w)$ *for every* $s \in \mathbf{C}$ *and every* $w \in W$.

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