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The mean value property for harmonic functions on graphs and trees

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Abstract. Following the Euclidean example, we introduce the strong and weak mean value property for finite variation measures on graphs. We completely characterize finite variation measures with bounded support on radial trees which have the strong mean value property. We show that for counting measures on bounded subsets of a tree with root o, the strong mean value property is equivalent to the invariance of the subset under the action of the stabilizer of o in the automorphism group. We finally characterize, using the discrete Laplacian, the finite variation measures on a generic graph which have the weak mean value property and we give a non-trivial example.

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1. Introduction

Harmonic functions are widely studied in classical analysis. One of the most typical properties of complex harmonic functions is the so-called "mean value property" (see [1, Definition 11.12]) which is also a characterization ([1, Theorem 11.13]). A simple extension of this property says that the integral of a harmonic function h on a measurable subset B of the complex plane, which is invariant under all the rotations centred at a point x_0, is equal to $h(x_0)m(B)$ (where $m(B)$ is the Lebesgue measure of the set B), that is

$$\int_B h \, \mathrm{d}m = h(x_0)m(B), \tag{1}$$

for every harmonic function h.

Harmonic functions can also be defined on graphs (see Definition 1.1) and they represent a consolidated branch of studies (see for instance [2] and [3] in the case of trees and also [4], [5] for more general cases).

The mean value property is usually referred to functions (see [6] for a short introduction and also [7]); on homogeneous trees $\mathbb{T}_M$ with $M \geq 3$ it characterizes harmonic functions (see [8]). Here, we change the point of view: according to

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Definition 1.1 below, Equation (1) says that the measure $m_B(\cdot) := m(B \cap \cdot)$ has the *strong mean value property* with respect to x_0. The aim of this paper is to characterize all the measures on a certain type of graphs X, which satisfy the strong or the weak mean value property (see Definition 3.1) with respect to some vertex.

We consider random walks (X, P) where X is the vertex set of a graph and $P = (p(x, y))_{x,y \in X}$ is the stochastic transition operator which describes the one-step transitions of a Markov chain $\{Z_n\}_{n \in \mathbb{N}}$ with state space X.

Given a graph $(X, E(X))$, a stationary random walk (X, P) is adapted to the graph if

$$p(x, y) > 0 \Longleftrightarrow [x, y] \in E(X);$$

if $(X, E(X))$ is a non-oriented graph, an adapted random walk is usually called of *nearest neighbour type*.

On the other hand, every stationary random walk (X, P) is adapted to a unique graph (called *associated graph*).

Given a couple of vertices x, y of a non-oriented graph $(X, E(X))$, we say that x and y are *neighbours* (and we write $x \sim y$) if $[x, y] \in E(X)$ (i.e. $[y, x] \in E(X)$); we define the degree of a vertex x as $\deg(x) := \operatorname{card}\{y \in X : [x, y] \in E(X)\}$.

Throughout this paper if we have a random walk on a graph, it will always be an adapted random walk: its vertex set is then uniquely determined by the matrix P of the transition probabilities. For this reason we will often indicate a graph as X or (X, P) instead of $(X, E(X))$. In particular, with the exception of Section 6, if not otherwise explicitly stated, we consider stationary, irreducible random walks of nearest neighbour type on *locally finite* non-oriented (connected) graphs.

Definition 1.1. *Given a random walk (X, P), we call a function $h : X \to \mathbb{R}$* harmonic *(*superharmonic*, respectively) in $x \in X$ if*

$$h(x) = (Ph)(x) := \sum_{y \sim x} p(x, y)h(y), \quad (h(x) \geq (Ph)(x), \text{ respectively}). \tag{2}$$

A function which is harmonic (superharmonic, respectively) in x for every $x \in X$ is called harmonic *(*superharmonic*, respectively).*

In Equation (2) we implicitly require that $(P|h|)(x) < +\infty$, for every $x \in X$.

In the following paragraph we introduce the basic concepts and some results concerning the general theory of random walks on graphs. In particular we discuss some properties of a special class of trees: the radial trees (see Section 4).

In Section 3 we introduce the general idea of mean value property for a finite variation measure on a graph (see Definition 3.1). We discuss some basic properties and we give a non-trivial example of a family of finite variation measures on locally finite trees which have the strong mean value property (see Proposition 3.3). This result generalizes a previous one (see [8, Lemma 1]). Moreover, we give a necessary and sufficient condition for a certain family of finite variation measures to have the mean value property (Proposition 3.5 and Corollary 3.6), this condition involves the representation of harmonic functions given by the Martin kernel (see Equation (6)).

In Section 4 we characterize all the measures with finite support on radial trees which have the strong mean value property, and we prove that, in this case, the weak mean value property is equivalent to the strong one. To this aim we introduce a particular class of measures that we call *well distributed* (see Definition 4.8). We will show that there are measures which have the mean value property with respect to some vertex o in spite of not being Γ_o-invariant (see Definition 2.2). We give an explicit example of such a measure (see Example 4.7) on the homogeneous tree $\mathbb{T}_3$ with the simple random walk.

As a consequence of these results we prove (Section 5) that, if B is a (non-empty) finite subset of an x_0-radial tree T, then

$$\frac{1}{|B|}\sum_{x\in B} h(x) = h(x_0),$$

for every harmonic function h if and only if B is invariant under the action of Γ_{x_0} (that is, if and only if B is a finite union of spheres centred at x_0; see Theorem 5.3).

In Section 6 we deal with generic random walks on oriented graphs, which in general are not locally finite. In particular we characterize all the measures which have the weak mean value property by using the pre-adjoint of the discrete Laplacian (Theorem 6.3). Using this result we are able to give an explicit example of a positive measure with infinite support which is not Γ_o-invariant but has the weak mean value property (Example 6.4 on the homogeneous tree $\mathbb{T}_M$).

2. Preliminaries

Every connected, non-oriented graph X carries a natural distance d. By means of this metric we define, for all $x \in X$, $n \in \mathbb{N}$,

$$B(x,n) := \{y \in X : d(x,y) \le n\}, \qquad S(x,n) := \{y \in X : d(x,y) = n\}.$$

Given a connected graph X and $x, y \in X$ there is always a *path* connecting x and y (that is a subset $\{x_i\}_{i=1}^n$ of X such that $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$). A *cycle* is a closed path of length at least 3, without repeated vertices (with the exception of the first and the last one); a *tree* is a graph without cycles.

Given a random walk (X, P) and $x, y \in X$, we denote by $p^{(n)}(x,y)$ the transition probability from x to y in n steps (by definition $p^{(0)}(x,y) = \delta_x(y)$, where δ_x is the Dirac measure on x). The generating function of the transition probabilities (called the *Green function*) is defined as the following power series:

$$G(x,y|z) = \sum_{n=0}^{+\infty} p^{(n)}(x,y)z^n, \qquad x, y \in X, z \in \mathbb{C}, \tag{3}$$

and its radius of convergence will be denoted by r. Given a general Markov chain Z_n described by the random walk (X, P), we set the hitting probabilities and their generating function:

$$\begin{aligned} f^{(n)}(x, y) &= \mathbf{Pr}[Z_n = y, Z_k \neq y, k = 1, \dots, n-1 | Z_0 = x] \\ f^{(0)}(x, y) &= 0 \\ F(x, y|z) &= \sum_{n=0}^{+\infty} f^{(n)}(x, y) z^n, \quad x, y \in X, z \in \mathbb{C}. \end{aligned} \tag{4}$$

Strictly related to F is the function

$$U(x, y|z) := \begin{cases} F(x, y|z) & \text{if } x \neq y \\ 1 & \text{if } x = y, \end{cases} \tag{5}$$

which is a probability generating function as well. We define $U(x, y) := U(x, y|1)$.

We recall that a set function $\nu : \mathcal{P}(X) \to \mathbb{R}$ is called a finite variation measure if it is a σ–additive measure and if the measure $|\nu|(A) := \sum_{x \in A} |\nu(x)|$ satisfies $|\nu|(X) < +\infty$. We define its support as $\operatorname{supp}(\nu) := \{x \in X : \nu(x) \neq 0\}$.

We want to emphasize that we are dealing with measures which, in general, are not positive. According to Theorem 6.2 of [1], $|\nu|$ is a positive measure. We could look at a finite variation measure on X as a particular function on X which belongs to $l^1(X) := \{\{\alpha_x\}_{x \in X} : \sum_{x \in X} |\alpha_x| < +\infty\}$ (that is $l^1(X) = L^1((X, \mathcal{P}(X), \mu_c); \mathbb{R})$, where μ_c is the counting measure on the set X and $\mathcal{P}(X)$ is the set of all the subsets of X). It is easy to show that for every $f \in L^1(X, |\nu|)$ we have $\int_X f \, d\nu = \sum_{x \in X} f(x)\nu(x)$, where the right-hand side of the previous equation converges absolutely.

We denote by $\mathcal{H}(X, P)$ the linear space of all the real harmonic functions on X and by $\mathcal{H}^+(X, P)$ the positive cone of all the positive harmonic functions on X.

Let us consider a tree (T, P) and let the root be o, then for every $x, y \in T$ there exists a unique shortest path $\Pi[x, y]$ connecting x to y. We define the vertex $x \wedge y \in T$ by $\Pi[o, x] \cap \Pi[o, y] = \Pi[o, x \wedge y]$. The relation $x \geq y \Leftrightarrow y = x \wedge y$ is a partial ordering on T.

Let $(X, E(X))$ be a graph, we say that a bijective function γ from X onto itself is an automorphism if for all $x, y \in X$ then

$$[x, y] \in E(X), \quad \text{if and only if} \quad [\gamma x, \gamma y] \in E(X),$$

and we denote the set of all the automorphisms from X onto itself by $\mathrm{AUT}(X)$.

In the next lemma we list some elementary properties of automorphisms.

Lemma 2.1. *Let $(X, E(X))$ be a non-oriented graph, then*

(i) $\gamma \in \mathrm{AUT}(X)$ *if and only if γ is a bijective isometry;*
(ii) $\gamma \in \mathrm{AUT}(X)$ *if and only if γ is bijective and preserves all the geodesics;*
(iii) if X is a tree and $\gamma \in \mathrm{AUT}(X)$ such that $\gamma(o_X) = o_X$ then, for all $x, y \in X$, $\gamma x \wedge \gamma y = \gamma(x \wedge y)$.

Proof. The proof is straightforward and we omit it. □

Definition 2.2. *Let X be a graph and $x \in X$ be fixed, then an automorphism $\gamma \in \mathrm{AUT}(X)$ is called a* rotation *centred at x if $\gamma(x) = x$; the subgroup of all the rotations centred at x is denoted by Γ_x.*

The graph X is said to be x-radial ($x \in X$) if Γ_x acts transitively on $S(x, k)$, for every $k \in \mathbb{N}$. Given $x \in X$, then we write $[y]_x$ for the Γ_x-orbit of $y \in X$. A function on X is Γ_x-invariant if and only if it is constant on every trajectory $[y]_x$, for all $y \in X$ and $B \subset X$ is Γ_x-invariant if and only if $B = \cup_{y\in B}[y]_x$.

The following assertions are clearly equivalent:

(i) for all $y \in X$, $[y]_x = S(x, d(x, y))$;
(ii) X is x-radial;
(iii) $B \subset X$ is Γ_x-invariant if and only if there exists $I \subseteq \mathbb{N}$ such that $B = \cup_{k\in I} S(x, k)$.

We say that a tree T is of $T_{\{n_k\}}$-type if it is a tree branching from a root o such that $\deg(x) = n_{d(o,x)}$, for all $x \in T$. We know (see e.g. [9]) that a tree T is radial with respect to $x \in T$ if and only if it is of $T_{\{n_k\}}$-type with respect to x.

Proposition 2.3. *Let T be a tree such that $\mathrm{card}(T) \geq 2$, then the following assertions are equivalent:*

(i) there exist $x, y \in T$, $x \neq y$ such that T is x-radial and y-radial;
(ii) T is w-radial for every $w \in T$.

Proof. (i) $\Rightarrow$ (ii). We first observe that a $T_{\{n_k\}}$-type tree is radial with respect to every $x \in T$ if and only if $n_k = n_{k+2}$, for all $k = 0, 1, \ldots$. Moreover, it is not difficult to see that if (i) holds then T must be infinite. Let us suppose that $d(x, y) = r$ and let T be $T_{\{n_k\}}$ with respect to x and $T_{\{m_k\}}$ with respect to y. Then for all $k = 0, 1, \ldots, r$ and for all $p \geq 0$,

$$\begin{cases} n_p = m_{r+p}, \\ n_{r+p} = m_p, \\ n_{k+p} = m_{r-k+p}. \end{cases}$$

We consider two different cases:

a) $r = 1$: then $n_0 = m_1, m_0 = n_1, n_l = m_{l+1} = m_{l-1}$ which implies $m_{l+1} = m_{l-1}$ and $n_l = n_{l-2}$ and then T is x-radial for every $x \in T$;
b) $r > 1$: then $n_l = n_{l-1+1} = m_{r-1+(l-1)} = m_{r+l-2}$, but $n_l = m_{l+r}$ then $n_l = m_{r+l} = m_{r+l-2} = n_{l-2}$, for every $l \geq 2$.

(ii) $\Rightarrow$ (i). It is trivial. □

From the previous proposition we obtain that if T is a tree with $\mathrm{card}(T) \geq 2$, satisfying (i) or (ii) then it is an infinite bi-homogeneous tree.

Given a random walk (X, P) and a point $o \in X$, we define the Martin kernel to be a function $K : X \times X \to \mathbb{R}$ defined as

$$k(x, y) := \frac{U(x, y)}{U(o, y)}, \quad \forall x, y \in X. \tag{6}$$

Let us remember that if T is a tree with root o and if we consider $x, y, w \in T$ such that $w \in \Pi[x, y]$, then

$$U(x, y|\lambda) = U(x, w|\lambda)U(w, y|\lambda),$$

for every λ in the domain of convergence of the function U. Using this property we can derive the following one:

$$k(x, y) = \frac{U(x, y)}{U(o, y)} = \frac{U(x, x \wedge y)U(x \wedge y, y)}{U(o, x \wedge y)U(x \wedge y, y)} = \frac{U(x, x \wedge y)}{U(o, x \wedge y)} = k(x, x \wedge y). \quad (7)$$

If we denote with $\widehat{T}$ the Martin compactification of T (cf. [2]) and $\mathcal{M}(T) := \widehat{T} \setminus T$, then, for all $\xi \in \mathcal{M}(T)$, there exists a unique sequence $\{x_n\} \subset T$ such that $x_0 = o$, $\Pi[o, x_n] = \{x_0, \ldots, x_n\}$, for all $n \in \mathbb{N}$, and $x_n \to \xi$ in the topology of $\widehat{T}$. We represent this sequence as $\Pi[o, \xi]$ and we can show that for every $\xi_1, \xi_2 \in \widehat{T}$ there exists a unique $\xi_1 \wedge \xi_2 \in \widehat{T}$ such that $\Pi[o, \xi_1 \wedge \xi_2] = \Pi[o, \xi_1] \cap \Pi[o, \xi_2]$. Moreover, if $\xi_1 \neq \xi_2$ then $\xi_1 \wedge \xi_2 \in T$; if $y \in T$ and if $\{x_n\} = \Pi[o, \xi]$ then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\xi \wedge y = x_n \wedge y$. Using these properties we can extend the Martin kernel to the Martin compactification as $k(x, \xi) := k(x, \xi \wedge x)$, for every $\xi \in \widehat{T}$. This extension can be made for any random walk (X, P) (see [10] and [11]). We will need the following lemma (see e.g. [12, Lemma 6.3]):

Lemma 2.4. *Let* (T, P) *be an irreducible, transient random walk on a tree; if* $\xi \in \widehat{T}$ *then the function* $x \mapsto k(x, \xi)$ *is superharmonic. Moreover, if* $\xi \in \mathcal{M}(T)$ *and* $\mathrm{card}\{y : y \sim x_0\} < +\infty$ *then* $k(\cdot, \xi)$ *is harmonic in* x_0.

Let T be a tree with root o; for every $x \in T$ we define

$$T_x := \{y \in T : y \wedge x = x\}, \quad \mathcal{M}(T_x) := \{\xi \in \mathcal{M}(T) : x \in \Pi[o, \xi]\}. \quad (8)$$

We observe that T_x is the connected subtree of T branching from its root x, constituted by all the vertices y satisfying $x \in \Pi[o, y]$.

Remark 2.5. If we are dealing with a $T_{\{n_k\}}$-type tree T with root o then given any couple of vertices $x, y \in T$ such that $d(o, x) = d(o, y)$, if $\Pi[x \wedge y, x] = \{x \wedge y, x_1, \ldots, x\}$ and $\Pi[x \wedge y, y] = \{x \wedge y, y_1, \ldots, y\}$ then there exists a map $\gamma \in \Gamma_o$ such that:

(i) $\gamma(T_{x_1}) = T_{y_1}$ and $\gamma(T_x) = T_y$;
(ii) $\gamma^2 = \mathbb{1}_T$;
(iii) $\gamma(z) = z$, for all $z \in T \setminus \{T_{x_1} \cup T_{y_1}\}$;

where $\mathbb{1}_T \in \mathrm{AUT}(T)$ is the identity map.

With the next proposition we want to point out a couple of properties of the Martin kernel of a random walk on a tree, which will be very useful in the next sections. If we have $\gamma \in \Gamma_o$ and $\xi \in \mathcal{M}(T)$ then there exists a unique $\xi' \in \mathcal{M}(T)$ such that $\Pi[o, \xi'] = \gamma(\Pi[o, \xi]) := \{\gamma(x_0), \gamma(x_1), \ldots\}$, hence we extend the map γ to the Martin boundary as $\gamma(\xi) := \xi'$; this extension is bijective.

Lemma 2.6. *Let us consider a transient random walk* (T, P) *on a tree* T *with root* o. *If* $\xi_1, \xi_2 \in \widehat{T}$, $x \in T$ *then:*

a) if $d(o, \xi_1 \wedge x) > d(o, \xi_1 \wedge \xi_2)$ *then* $k(x, \xi_1) > k(x, \xi_2)$*;*
b) if $\max(d(o, \xi_1 \wedge x), d(o, \xi_2 \wedge x)) \leq d(o, \xi_1 \wedge \xi_2)$ *then* $k(x, \xi_1) = k(x, \xi_2)$.

Moreover, if the random walk is Γ_o*-invariant then* $k(x, \xi) = k(\gamma(x), \gamma(\xi))$*, for all* $\gamma \in \Gamma_o$, $x \in T$, $\xi \in \widehat{T}$. *In particular if* T *is a* $T_{\{n_k\}}$*-type tree then* $\xi_1, \xi_2 \in S(0, k)$ *implies* $k(\xi_1, \xi_2) = k(\xi_2, \xi_1)$.

Proof. a) If $d(o, \xi_1 \wedge x) > d(o, \xi_1 \wedge \xi_2)$ then $\xi_2 \wedge x = \xi_1 \wedge \xi_2$ and

$$\frac{k(x, \xi_1)}{k(x, \xi_2)} = \frac{k(x, \xi_1 \wedge x)}{k(x, \xi_2 \wedge x)} = \frac{U(x, \xi_1 \wedge x)U(o, \xi_2 \wedge x)}{U(x, \xi_2 \wedge x)U(o, \xi_1 \wedge x)}$$

$$= \frac{U(x, \xi_1 \wedge x)U(o, \xi_1 \wedge \xi_2)}{U(x, x \wedge \xi_1)U(x \wedge \xi_1, \xi_1 \wedge \xi_2)U(o, \xi_1 \wedge \xi_2)U(\xi_1 \wedge \xi_2, x \wedge \xi_1)}$$

$$= \frac{1}{U(x \wedge \xi_1, \xi_1 \wedge \xi_2)U(\xi_1 \wedge \xi_2, x \wedge \xi_1)} \geq \frac{1}{F(x \wedge \xi_1, x \wedge \xi_1)} > 1,$$

since $x \wedge \xi_1 \neq \xi_1 \wedge \xi_2$ and the random walk is transient.

b) If $d(o, x \wedge \xi_i) \leq d(o, \xi_1 \wedge \xi_2)$ for $i = 1, 2$ then $x \wedge \xi_1 = x \wedge \xi_2 \in \Pi[o, \xi_1 \wedge \xi_2]$, hence

$$k(x, \xi_1) = k(x, x \wedge \xi_1) = k(x, x \wedge \xi_2) = k(x, \xi_2).$$

If $\gamma \in \Gamma_o$ then $\deg(x) = \deg(\gamma(x))$ and $p(x, y) = p(\gamma(x), \gamma(y))$, for all $x, y \in T$, this implies that $U(x, y|\lambda) = U(\gamma(x), \gamma(y)|\lambda)$ and then

$$k(x, \xi) = \frac{U(x, x \wedge \xi)}{U(o, x \wedge \xi)} = \frac{U(\gamma(x), \gamma(x \wedge \xi))}{U(\gamma(o), \gamma(x \wedge \xi))}$$
$$= \frac{U(\gamma(x), \gamma(x) \wedge \gamma(\xi))}{U(\gamma(o), \gamma(x) \wedge \gamma(\xi))} = k(\gamma(x), \gamma(\xi)). \tag{9}$$

Finally if T is a $T_{\{n_k\}}$-type tree and γ is defined as in Remark 2.5 then $k(\xi_2, \xi_1) = k(\gamma(\xi_2), \gamma(\xi_1)) = k(\xi_1, \xi_2)$. □

3. Mean value properties

Let us look at the mean value property of harmonic functions on Euclidean spaces as we described it in Section 1. If we take a measurable subset B of the complex plane, which is invariant under all the rotations centred at a point x_o then the measure m_B (which is the Lebesgue measure on B) satisfies

$$\int h \, \mathrm{d}m_B = h(x_0)m(B),$$

for every harmonic function h. The previous equation suggests the basic idea of mean value property of a measure on a graph. If not otherwise explicitly stated, the σ-algebra on the vertex set of a graph will always be the set $\mathcal{P}(X)$.

Definition 3.1. *Let (X, Σ_X) be a measureable space, $o \in X$ and let ν be a measure on (X, Σ_X) with finite variation. If $\mathcal{F}$ is a family of functions in $L^1(|\nu|)$ then we say that ν has the* mean value property *with respect to $\mathcal{F}$ and o if*

$$L(h, \nu)(o) = 0, \quad \forall h \in \mathcal{F}, \tag{10}$$

where

$$L(h, \nu)(x) := \int_X h \, \mathrm{d}\nu - \nu(X)h(x), \quad \forall x \in X. \tag{11}$$

In particular, if (X, P) is a graph with an adapted random walk, we say that ν has the strong mean value property with respect to *o (resp.* weak mean value property with respect to *o) if Equation* (10) *holds with $\mathcal{F} \equiv \mathcal{H}(X, P) \cap L^1(|\nu|)$ (resp. $\mathcal{F} \equiv \mathcal{H}^\infty(X, P) := \mathcal{H}(X, P) \cap l^\infty(X)$, where $l^\infty(X) := L^\infty((X, \mathcal{P}(X), \mu_c); \mathbb{R})$). If X is a tree with root o then a mean value property will always be with respect to o.*

Obviously if ν has the strong mean value property then it has the weak mean value property.

We note also that if $\nu(X) = 0$ then it has the mean value property with respect to $\mathcal{F}$ and x if and only if $\mathcal{F} \subseteq \{f \in L^1(|\nu|) : \int_X f \, \mathrm{d}\nu = 0\}$. Obviously if $\nu(X) \neq 0$ and ν has the mean value property with respect to $\mathcal{F}$ and x, then it has the mean value property with respect to $\mathcal{F}$ and y if and only if $\mathcal{F}$ does not separate x and y (that is $h(x) = h(y)$ for every $h \in \mathcal{F}$). Then if $\nu(X) \neq 0$ we easily see that ν has the mean value property with respect to $\mathcal{F}$ and x for every $x \in X$ if and only if $\mathcal{F}$ is a set of constant functions.

Moreover, the weak mean value property is relevant only in the transient case. More precisely if $\mathcal{H}^\infty(X, P)$ is exactly the set of all the constant functions on X, as in the recurrent case (but not only in this case, take for instance $\mathbb{Z}^d$ with the simple random walk, see [13, Theorem 7.1]), then every finite variation measure has the weak mean value property with respect to any point $o \in X$. Obviously if every finite variation measure has the weak (strong, respectively) mean value property with respect to any point $o \in X$ then $\mathcal{H}^\infty(X, P)$ ($\mathcal{H}(X, P)$, respectively) is the set of all the constant functions.

This means that, in general, the weak mean value property does not imply the strong one. Let (X, P) be a random walk such that $\mathcal{H}^\infty(X, P)$ is the set of all the constant functions and $\mathcal{H}(X, P) \setminus \mathcal{H}^\infty(X, P) \neq \emptyset$ (take for instance $\mathbb{Z}^d$ the d-dimensional grid with the simple random walk), then every finite variation measure has the weak mean value property with respect to any point but there exists a couple of points x, y such that δ_x does not have the strong mean value property with respect to y.

In this paragraph we will consider only strong and weak mean value properties or the case $\mathcal{F} = \{k(\cdot, \xi)\}_{\xi \in \mathcal{M}(T)}$, where T is a tree.

We observe that the map $(h, \nu) \mapsto L(h, \nu)$ is bilinear where it is defined; in particular if h is harmonic so is $L(h, \nu)$. It is useful to note that if ν is a finite variation measure and $h, k \in L^1(|\nu|)$ such that $h(x_0) = k(x_0)$ (where $x_0 \in X$) then,

$$L(h, \nu)(x_0) - L(k, \nu)(x_0) = \int_X h \, \mathrm{d}\nu - \int_X k \, \mathrm{d}\nu. \tag{12}$$

Proposition 3.2. *Let $\{\nu_n\}$ be a sequence of finite variation measures on a graph X such that $\sum_{n\in\mathbb{N}} |\nu_n|(X) < +\infty$. Then for all $A \subseteq X$, $\nu(A) := \sum_{n\in\mathbb{N}} \nu_n(A)$ converges absolutely to a finite variation measure ν such that $|\nu| \le \sum_{n\in\mathbb{N}} |\nu_n| =: \mu$. Moreover, if $h \in L^1(\mu)$ then, for all $x \in X$,*

$$L(h, \nu)(x) = \sum_{n\in\mathbb{N}} L(h, \nu_n)(x).$$

Proof. If we take $h \in L^1(\mu)$ then $\sum_{y\in X} \sum_{n\in\mathbb{N}} |h(y)||\nu_n(y)| \le \int_X |h| \, \mathrm{d}\mu < +\infty$; using Fubini's Theorem we have

$$\int_X h \, \mathrm{d}\nu = \sum_{y\in X} \sum_{n\in\mathbb{N}} h(y)\nu_n(y) = \sum_{n\in\mathbb{N}} \sum_{y\in X} h(y)\nu_n(y) = \sum_{n\in\mathbb{N}} \int_X h \, \mathrm{d}\nu_n. \tag{13}$$

Since μ is of finite variation by hypothesis, for every $A \subseteq X$, $\chi_A \in L^1(\mu)$ (where χ_A is the characteristic function of A), and Equation (13) becomes

$$\nu(A) = \int_X \chi_A \, \mathrm{d}\nu = \sum_{n\in\mathbb{N}} \int_X \chi_A \, \mathrm{d}\nu_n = \sum_{n\in\mathbb{N}} \nu_n(A).$$

Hence, taking $A = X$,

$$L(h, \nu)(x) = \sum_{y\in X} \sum_{n\in\mathbb{N}} h(y)\nu_n(y) - \sum_{n\in\mathbb{N}} \nu_n(X)h(x)$$

$$= \sum_{n\in\mathbb{N}} \left(\sum_{y\in X} h(y)\nu_n(y) - \nu_n(X)h(x) \right) = \sum_{n\in\mathbb{N}} L(h, \nu_n)(x).$$

The absolute convergence of $\sum_{n\in\mathbb{N}} \nu_n(A)$ holds since $\nu_n(A) \le |\nu_n|(X)$, which is summable. □

Now we want to find a class of finite variation measures on trees which have the strong mean value property.

From now on, let (T, P) be a random walk on a tree T with root o. Let us take $x \in S(o, k)$ and suppose $\Pi[o, x] \equiv \{o = x_0 \sim x_1 \sim \cdots \sim x_k = x\}$. We remember that, since T is a tree, we have

$$p^{(k)}(o, x) = \prod_{n=0}^{k-1} p(x_i, x_{i+1}),$$

moreover, if $y \in S(o, k + 1)$ and $y \sim x$, then $p^{(k+1)}(o, y) = p^{(k)}(o, x)p(x, y)$.

We say that a finite variation measure ν is of *type* 1 if and only if there exists a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ such that for every $x \in T$,

$$\nu(x) := p^{(d(o,x))}(o, x)\alpha_{d(o,x)}.$$

Proposition 3.3. *Let T be a locally finite tree carrying an adapted random walk such that the function*

$$x \longmapsto \sum_{\substack{y\sim x \\ d(o,y)=d(o,x)+1}} p(x,y)p(y,x) \tag{14}$$

is constant on $S(o,k)$, for all $k \in \mathbb{N}$. If ν is a finite variation measure of type 1 on T then for all $h \in \mathcal{H}(T,P) \cap L^1(|\nu|)$ we have $L(h,\nu)(o) = 0$.

Proof. Note that $\nu = \sum_{n\in\mathbb{N}} \nu_n$ and $|\nu| = \sum_{n\in\mathbb{N}} |\nu_n|$, where $\nu_n(x) := \nu(x)\chi_{S(o,n)}(x)$. Hence, by Proposition 3.2, it is enough to prove the statement when $\mathrm{supp}(\nu) \subseteq S(o,n)$ for some $n \in \mathbb{N}$.

We use induction on n. If $n = 0$ there is nothing to prove. If $n = 1$ then for all $h \in \mathcal{H}(T.P) \cap L^1(|\nu|)$ we have

$$L(h,\nu)(o) = \sum_{x\in S(o,1)} h(x)\nu(x) - \sum_{x\in S(o,1)} \nu(x)h(o)$$

$$= \alpha_1 \left(\sum_{x\in S(o,1)} h(x)p(o,x) - \sum_{x\in S(o,1)} p(o,x)h(o) \right) = 0.$$

We suppose now that the proposition holds for every $n' \leq n$. If $\mathrm{supp}(\overline{\nu}) \subseteq S(o,n)$ and $h \in \mathcal{H}(T,P)$ we have

$$\sum_{x\in S(o,n)} h(x)\overline{\nu}(x) = \sum_{\substack{y\in S(o,n+1) \\ x\in S(o,n)}} p(x,y)h(y)\overline{\nu}(x) + \sum_{\substack{y\in S(o,n-1) \\ x\in S(o,n)}} p(x,y)h(y)\overline{\nu}(x).$$

Now if ν is a finite variation measure of type 1 such that $\mathrm{supp}(\nu) \subseteq S(o,n+1)$, using the previous equation we have

$$\sum_{y\in S(o,n+1)} h(y)\nu(y) = \sum_{x\in S(o,n)} h(x)\overline{\nu}(x) - \sum_{y\in S(o,n-1)} h(y) \sum_{x\in S(o,n)} p(x,y)\overline{\nu}(x),$$

where, if $x \in S(o,n)$ and $y \sim x$, $y \in S(o,n+1)$, then $\overline{\nu}(x) = \nu(y)/p(x,y) \equiv \alpha p^{(n)}(o,x)$ and hence $\overline{\nu}$ is of type 1 and $\mathrm{supp}(\overline{\nu}) \subseteq S(o,n)$. If we define now, for every $y \in S(o,n-1)$, $\widehat{\nu}(y) := \overline{\nu}(x)/p(y,x)$, where $x \in S(o,n)$, $x \sim y$ (which is well defined and of type 1 as well) then

$$\sum_{y\in S(o,n+1)} h(y)\nu(y) = \sum_{x\in S(o,n)} h(x)\overline{\nu}(x) - \sum_{y\in S(o,n-1)} h(y) \sum_{x\in S(o,n)} p(x,y)p(y,x)\widehat{\nu}(y)$$

$$= \sum_{x\in S(o,n)} h(x)\overline{\nu}(x) - \sum_{y\in S(o,n-1)} h(y)\beta\widehat{\nu}(y),$$

where $\beta = \sum_{x\in S(o,n)} p(y,x)p(x,y)$, which is independent of y.

If $h \equiv 1$, from the previous equation, we obtain

$$\nu(T) = \overline{\nu}(T) - \beta\widehat{\nu}(T),$$

then by induction hypothesis

$$L(h, \nu)(o) = \sum_{x \in T} h(x)\nu(x) - \nu(T)h(o)$$

$$= \sum_{x \in T} h(x)\overline{\nu}(x) - \beta \sum_{x \in T} h(x)\widehat{\nu}(x) - (\overline{\nu}(T) - \beta\widehat{\nu}(T))h(o)$$

$$= L(h, \overline{\nu})(o) - \beta L(h, \widehat{\nu})(o) = 0.$$

□

The previous result extends, using a slightly different technique, a result obtained by Picardello and Woess [8, Lemma 1]; see also [3] and [5].

Remark 3.4. If T is a tree and P is an isotropic random walk (see [9, Definition 1.1]), then a finite variation measure is of type 1 if and only if it is constant on every sphere. If T is a tree of $T_{\{n_k\}}$ type and P is the simple random walk, then every finite variation measure which is Γ_o-invariant has the strong mean value property (Equation (14) is easily verified).

We now consider a general irreducible, transitive random walk (X, P). Let $\mathcal{M}_{min} \equiv \mathcal{M}_{min}(X) := \{\xi \in \mathcal{M}(X) : k(\cdot, \xi) \in \mathcal{H}(X, P)\}$. For every Borel measure ν on the Martin compactification $\widehat{X}$ if

$$h(x) := \int_{\widehat{X}} k(x, \xi)\, \mathrm{d}\nu(\xi), \qquad \forall x \in X, \tag{15}$$

then

$$(Ph)(x) := \int_{\widehat{X}} (Pk)(x, \xi)\, \mathrm{d}\nu(\xi), \qquad \forall x \in X,$$

where $(Pk)(x, \xi) := \sum_{w \in X} p(x, w)k(w, \xi)$. In particular, if $\mathrm{supp}(\nu) \subseteq \mathcal{M}_{min}$ then Equation (15) defines an harmonic function.

If $(\nu_x)_{x \in X}$ is the family of *harmonic measures* (see [14, Paragraph 20] or [12, Paragraph 6.D]) then $\nu_x(B) = \int_B k(x, \xi)\, \mathrm{d}\nu_o(\xi)$, where $B \subseteq \widehat{X}$ is a Borel set and $o \in X$ is the same point involved in the definition of the Martin kernel k. If $\varphi \in L^1(\mathcal{M}, \nu_o)$ then

$$h(x) := \int_{\mathcal{M}_{min}} \varphi(\xi)k(x, \xi)\, \mathrm{d}\nu_o(\xi) \equiv \int_{\mathcal{M}_{min}} \varphi(\xi)\, \mathrm{d}\nu_x(\xi), \qquad \forall x \in X, \tag{16}$$

is a well-defined harmonic function (for every fixed $x \in X$, $\xi \mapsto k(x, \xi)$ is a continuous function on the compact space $\widehat{X}$). In particular Equation (16) defines a linear, bicontinuous map from $L^\infty(\mathcal{M}_{min}, \nu_o)$ onto $\mathcal{H}^\infty(X, P)$ (it is clearly bounded, the boundedness of the inverse map is guaranteed by the Open Mapping Theorem).

Proposition 3.5. *Let (X, P) be an irreducible, transient random walk and let us fix $o \in X$, Δ a family of finite variation Borel measures on $\mathcal{M}$ and ν a finite variation measure on X such that the Martin kernel $k \in L^1(X \times \mathcal{M}_{min}, |\nu| \times |\mu|)$ for every $\mu \in \Delta$. If we define $\mathcal{F}_\Delta := \{h : h(x) := \int_{\mathcal{M}_{min}} k(x, \xi)\, \mathrm{d}\mu(\xi),\ \mu \in \Delta\}$ then $L(k(\cdot, \xi), \nu)(o) = 0$ $|\mu|$ a.e. on $\mathcal{M}_{min}$ for every $\mu \in \Delta$ implies that ν has* *the mean value property with respect to* $\mathcal{F}_\Delta$ *and* o. *Vice versa, let us suppose that* $\Delta \supseteq \{\delta_\xi : \xi \in \mathcal{M}_{min}\}$ *then if* ν *has the mean value property with respect to* $\mathcal{F}_\Delta$ *and* o *we have that* $L(k(\cdot, \xi), \nu)(o) = 0$ *for every* $\xi \in \mathcal{M}_{\min}$.

Proof. Let us note first, using Fubini's Theorem and the continuity of $\xi \mapsto k(x, \xi)$, that $\mathcal{F}_\Delta \subseteq \mathcal{H}(X, P) \cap L^1(X, |\nu|)$. If $h \in \mathcal{F}_\Delta$ (represented by μ) and $f(x, \xi) := k(x, \xi) - k(o, \xi)$, then $f \in L^1(X \times \mathcal{M}, |\nu| \times |\mu|)$ hence, by Fubini's Theorem (applied to the positive measures ν^+, ν^- and μ^+, μ^-),

$$L(h, \nu)(o) = \int_X \int_{\mathcal{M}_{min}} (k(x, \xi) - k(o, \xi))\, \mathrm{d}\mu(\xi)\, \mathrm{d}\nu(x)$$

$$= \int_{\mathcal{M}_{min}} \int_X (k(x, \xi) - k(o, \xi))\, \mathrm{d}\nu(x)\, \mathrm{d}\mu(\xi) = 0.$$

Vice versa, if we choose $\mu = \delta_\xi$ then $k(\cdot, \xi) \in \mathcal{F}_\Delta$ and hence $L(k(\cdot, \xi), \nu)(o) = 0$ for every $\xi \in \mathcal{M}_{\min}$. □

Corollary 3.6. *Let* (X, P) *be an irreducible, transient random walk and let us fix* $o \in X$ *and* ν *a finite variation measure on* X *such that* $k(\cdot, \xi) \in L^1(X, |\nu|)$ ν_o *a.e. (where* ν_o *is the harmonic measure). If* $L(k(\cdot, \xi), \nu)(o) = 0$ ν_o *a.e. on* $\mathcal{M}_{min}$ *then* ν *has the weak mean value property with respect to* o.

Proof. It is enough to apply the first part of the previous theorem to $\Delta := \{\mu : \mu(B) := \int_B \varphi(\xi)\nu_o(\xi),\ B \subseteq \mathcal{M}_{min}$ Borel set, $\varphi \in L^\infty(\mathcal{M}, \nu_o)\}$ recalling that, in this case, $\mathcal{F}_\Delta \equiv \mathcal{H}^\infty(X, P)$. □

4. The mean value property for measures with finite support

In this paragraph we deal only with signed measures with bounded support (which, of course, are of finite variation) on radial trees. Our goal is to find a necessary and sufficient condition for such measures to have the strong or the weak mean value properties (we will see that in this case there is no difference between the two).

Remark 4.1. By Lemma 2.4 and Equation (7), for all $\xi \in \mathcal{M}(T)$,

$$k(x, \xi) = k(x, \xi_k), \quad \forall x \in T^c_{\xi_k} \cup \{\xi_k\},$$

where $\{\xi_k\} = \Pi[o, \xi] \cap S(o, k)$. Moreover, every positive harmonic function (and hence every bounded harmonic function) on T admits an integral representation. Therefore, using this integral representation, we see that a finite variation measure ν on a tree T with $\mathrm{supp}(\nu) \subseteq B(o, k)$ has the strong mean value property with respect to $x \in T$ if and only if $L(k(\cdot, \xi), \nu)(x) = 0$ for every $\xi \in S(o, k)$.

In order to be able to compare the value of the map $L(\cdot, \nu)$ corresponding to two different Martin kernels $k(\cdot, \xi_1)$ and $k(\cdot, \xi_2)$ we need the following proposition:

Proposition 4.2. *Let (T, P) be a Γ_o-invariant random walk on a $T_{\{n_k\}}$-type tree with root o and ν a finite variation measure such that* $\operatorname{supp}(\nu) \subseteq B(o, k)$. *If we take $\xi_1, \xi_2 \in S(o, k)$, then:*

(i) $L(k(\cdot, \xi_1), \nu)(o) - L(k(\cdot, \xi_2), \nu)(o) = \int_T k(\cdot, \xi_1)\, d\nu - \int_T k(\cdot, \xi_2)\, d\nu$;

(ii) $\int_T k(\cdot, \xi_1)\, d\nu - \int_T k(\cdot, \xi_2)\, d\nu = \sum_{x \in T_{x_1}} [k(x, \xi_1) - k(x, \xi_2)](\nu(x) - \nu(\gamma(x)))$, *where $x_i \in \Pi[\xi_1 \wedge \xi_2, \xi_i]$, $d(x_i, \xi_1 \wedge \xi_2) = 1$ $(i = 1, 2)$ and $\gamma \in \Gamma_o$ is exactly as in Remark 2.5 and $\gamma(\xi_1) = \xi_2$. Moreover, if $\xi_1 \neq \xi_2$ then for all $x \in T_{x_1}$ we have that* $k(x, \xi_1) - k(x, \xi_2) > 0$.

Proof. (i) It is an easy consequence of $k(o, \xi_1) = k(o, \xi_2) = 1$ and of Equation (12).

(ii) If $\xi_1 = \xi_2$ there is nothing to prove, hence we may suppose that $\xi_1 \neq \xi_2$. Using Lemma 2.6 we obtain

$$\int_T (k(\cdot, \xi_1) - k(\cdot, \xi_2))\, d\nu = \int_{T_{x_1} \cup T_{x_2}} (k(\cdot, \xi_1) - k(\cdot, \xi_2))\, d\nu. \tag{17}$$

Then if $\gamma \in \Gamma_o$ satisfies the hypotheses and recalling Equation (9), we have

$$\begin{aligned}
&\int_{T_{x_1} \cup T_{x_2}} (k(\cdot, \xi_1) - k(\cdot, \xi_2))\, d\nu \\
&= \sum_{z \in T_{x_1}} k(z, \xi_1)\nu(z) - \sum_{z \in T_{x_1}} k(z, \xi_2)\nu(z) + \sum_{z \in T_{x_2}} k(z, \xi_1)\nu(z) - \sum_{z \in T_{x_2}} k(z, \xi_2)\nu(z) \\
&= \sum_{z \in T_{x_1}} (k(z, \xi_1) - k(z, \xi_2))\nu(z) + \sum_{z \in T_{x_2}} (k(\gamma(z), \gamma(\xi_1)) - k(\gamma(z), \gamma(\xi_2)))\nu(z).
\end{aligned}$$

Recalling that $z \in T_{x_2}$ if and only if $\gamma(z) \in T_{x_1}$ and that $\gamma^2 = \mathbb{1}$, then the last equation can be rewritten as

$$\begin{aligned}
&\sum_{z \in T_{x_1}} (k(z, \xi_1) - k(z, \xi_2))\nu(z) + \sum_{z \in T_{x_1}} (k(z, \xi_2) - k(z, \xi_1))\nu(\gamma(z)) \\
&= \sum_{z \in T_{x_1}} (k(z, \xi_1) - k(z, \xi_2))(\nu(z) - \nu(\gamma(z))).
\end{aligned}$$

Moreover, $z \in T_{x_1}$ if and only if $d(o, z \wedge \xi_1) \geq d(o, x_1) = d(o, \xi_1 \wedge \xi_2) + 1$, hence Lemma 2.6 implies that $k(z, \xi_1) - k(z, \xi_2) > 0$.

Finally we note that the all these sums are finite since ν is a finite support measure. □

Remark 4.3. It follows from Nash–Williams' recurrence criterion (see [15]) that the simple random walk on a T_{n_k}-type tree T is transient if and only if

$$\sum_{i=1}^{+\infty} \prod_{j=1}^{i} \frac{1}{n_j - 1} < +\infty. \tag{18}$$

For the rest of this section we consider only $T_{\{n_k\}}$-type trees with a transient simple random walk.

Using the next theorem we will derive a necessary and sufficient condition for a certain type of measures to have the strong mean value property.

Theorem 4.4. *Let T be a transient $T_{\{n_k\}}$-type tree and ν a finite variation measure such that* $\mathrm{supp}(\nu) \subseteq S(o,k)$*, $k \geq 1$. If for all $\xi_1, \xi_2 \in S(o,k)$ we have $\int_T k(\cdot,\xi_1)\,\mathrm{d}\nu = \int_T k(\cdot,\xi_2)\,\mathrm{d}\nu$ then ν is Γ_o-invariant.*

Proof. Let us suppose by contradiction that ν is not Γ_o-invariant, then we can choose $\xi, \xi_1 \in S(0,k)$ such that $\nu(\xi_1) \geq \nu(z)$, for all $z \in S(o,k)$ and $\nu(\xi_1) > \nu(\xi)$. We define $r = \min\{d(\xi_1,\xi) : \xi \in S(o,k),\ \nu(\xi) < \nu(\xi_1)\}$ (note that r is even and $r \geq 2$) and we fix $\xi_2 \in S(0,k)$ satisfying $d(\xi_1,\xi_2) = r$ and $\nu(\xi_2) < \nu(\xi_1)$. Let $\Pi[\xi_1 \wedge \xi_2, \xi_i] = \{\xi_1 \wedge \xi_2, x_i, \ldots, \xi_i\}$, $i = 1,2$ (it is possible that $x_i = \xi_i$) and let us define $\widetilde{T}_{x_i} := T_{x_i} \cap S(o,k)$, $i = 1,2$. If $z \in \widetilde{T}_{x_1}$ then $d(z,\xi_1) < d(\xi_1,\xi_2)$, hence, recalling the definition of r, we have $\nu(z) = \nu(\xi_1)$. By Proposition 4.2,

$$\int_T k(\cdot,\xi_1)\,\mathrm{d}\nu - \int_T k(\cdot,\xi_2)\,\mathrm{d}\nu = \sum_{x \in \widetilde{T}_{x_1}} [k(x,\xi_1) - k(x,\xi_2)](\nu(x) - \nu(\gamma(x))) > 0,$$

since for every $x \in \widetilde{T}_{x_1}$, $\nu(x) - \nu(\gamma(x)) = \nu(\xi_1) - \nu(\gamma(x)) \geq 0$ and $k(x,\xi_1) - k(x,\xi_2) > 0$. Besides if $x = \xi_1$ then $\nu(\xi_1) - \nu(\gamma(\xi_1)) = \nu(\xi_1) - \nu(\xi_2) > 0$ and this contradicts the hypothesis. □

This theorem has many consequences.

Corollary 4.5. *Let T be a transient $T_{\{n_k\}}$-type tree and ν a finite variation measure such that* $\mathrm{supp}(\nu) \subseteq S(o,k)$*, $k \geq 1$. If ν is not Γ_o-invariant then there exists $\xi_1, \xi_2 \in S(o,k)$ such that both $\int_T k(\cdot,\xi_1)\,\mathrm{d}\nu \neq 0$ and $L(k(\cdot,\xi_2),\nu)(o) \neq 0$.*

Proof. By Theorem 4.4 we can choose $\xi, \zeta \in S(o,k)$ such that $\int_T k(\cdot,\xi)\,\mathrm{d}\nu \neq \int_T k(\cdot,\zeta)\,\mathrm{d}\nu$, then by Proposition 4.2(i) it is not possible that $L(k(\cdot,\xi),\nu)(o) = 0$ and $L(k(\cdot,\zeta),\nu)(o) = 0$. □

Corollary 4.6. *Let T be a transient $T_{\{n_k\}}$-type tree and ν a finite variation measure such that* $\mathrm{supp}(\nu) \subseteq S(o,k)$*, $k \geq 1$; the following assertions are equivalent:*

(i) ν is Γ_o-invariant;
(ii) ν has the strong mean value property;
(iii) for all $\xi \in \mathcal{M}(T)$, we have that $L(k(\cdot,\xi),\nu)(o) = 0$.

Proof. (i) ⇒ (ii). It is an easy consequence of Remark 3.4.

(ii) ⇒ (iii). It is obvious.

(iii) ⇒ (i). It is by Corollary 4.5, recalling that, according to Lemma 2.4 and Equation (7) it is enough to consider $\xi \in S(o,k)$. □

We may easily note that the previous corollary gives us a necessary and sufficient condition for a measure ν with $\mathrm{supp}(\nu) \subseteq S(o,k)$ to have the strong mean value property and the weak mean value property as well.

What happens if $\mathrm{supp}(\nu) \nsubseteq S(o,k)$ for any $k \in \mathbb{N}$? It is possible to find many examples of finite variation measures having the strong mean value property which are not Γ_o-invariant.

Example 4.7. On the homogeneous tree $\mathbb{T}_3$, we define the measure ν with $\operatorname{supp}(\nu) \subseteq B(o, 2)$ as follows (see Figure 1).

$$\nu(o) := 0, \quad \nu(x_{ij}) := \begin{cases} 1 & \text{if } i = 1, j = 1, 2 \\ 0 & \text{if } i \neq 1, j = 1, 2 \end{cases}, \quad \nu(x_i) := \begin{cases} 3 & \text{if } i \neq 1 \\ 0 & \text{if } i = 1. \end{cases}$$

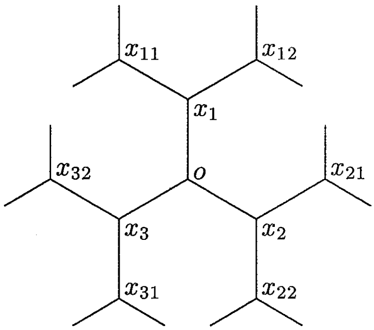

Fig. 1. The homogeneous tree $\mathbb{T}_3$

In fact if $h \in \mathcal{H}(\mathbb{T}_3)$ then, by Equation (2),

$$\begin{aligned} L(h, \nu)(o) = \int_{\mathbb{T}_3} h \, \mathrm{d}\nu - \nu(\mathbb{T}_3) h(o) &= 3h(x_2) + 3h(x_3) + h(x_{11}) + h(x_{12}) - 8h(o) \\ &= 3(h(x_2) + h(x_3) + h(x_1)) - h(o) - 8h(o) = 0. \end{aligned}$$

Definition 4.8. *Let T be a tree with root o and ν a finite variation measure on T such that* $\operatorname{supp}(\nu) \subseteq B(o, k)$ *and there exists $x \in S(o, k)$ with $\nu(x) \neq 0$. We call ν a* well-distributed *measure (with respect to o) if $k = 0$ or if $k \geq 1$ and for all $x, y \in S(o, k)$ such that $x \wedge y \in S(o, k-1)$ we have $\nu(x) = \nu(y)$.*

We observe that in this definition only the values of the measure on $S(o, k)$ are involved. If ν_1 and ν_2 are well distributed and $\alpha, \beta \in \mathbb{R}$, then $\alpha\nu_1 + \beta\nu_2$ is well distributed (note that in this case $\operatorname{supp}(\nu_1) \subseteq B(o, k_1)$ and $\operatorname{supp}(\nu_2) \subseteq B(o, k_2)$ and we do not necessarily suppose that $k_1 = k_2$). Obviously, if $\operatorname{supp}(\nu) \subseteq B(o, 1)$ then ν is a well-distributed measure if and only if it is constant on $S(o, 1)$.

We now introduce an operator which could be regarded as a contraction of the measure. If we are dealing with a finite variation measure ν with $\operatorname{supp}(\nu) \subseteq B(o, k)$, we want to construct a suitable new measure whose support is a subset of $B(o, k-1)$; we denote this new measure by $\mathbb{K}_k(\nu)$.

Definition 4.9. *If ν is a finite variation measure on a $T_{\{n_k\}}$-type tree with* $\operatorname{supp}(\nu) \subseteq B(o, k)$*, $k \geq 1$, then we define a new measure $\mathbb{K}_k(\nu)$ such that* $\operatorname{supp}(\nu) \subseteq B(o, k-1)$ *as follows:*

a) *if* $k = 0$ *then* $\mathbb{K}_0(\nu) := \nu$;
b) *if* $k = 1$, *then,*

$$\mathbb{K}_1(\nu)(x) := \begin{cases} \sum_{y \in B(o,1)} \nu(y) & \text{if } x = o \\ 0 & \text{if } x \neq o; \end{cases}$$

c) *if* $k \geq 2$, *then,*

$$\mathbb{K}_k(\nu)(x) := \begin{cases} 0 & \text{if } x \notin B(o,k-1) \\ \nu(x) + \dfrac{n_{k-1}}{n_{k-1}-1} \sum_{\substack{y \sim x \\ y \in S(o,k)}} \nu(y) & \text{if } x \in S(o,k-1) \\ \nu(x) - \dfrac{1}{n_{k-1}-1} \sum_{y \in T_x \cap S(o,k)} \nu(y) & \text{if } x \in S(o,k-2) \\ \nu(x) & \text{if } k \geq 3,\ x \in B(o,k-3). \end{cases}$$

Moreover, if $n \in \{2, \dots, k\}$, $\mathbb{K}_k^{(n)}(\nu) := \mathbb{K}_{k-n+1} \circ \mathbb{K}_{k-n+2} \circ \cdots \mathbb{K}_k(\nu)$, *besides if* $n = 1$ *(*$n = 0$*, respectively) we define* $\mathbb{K}_k^{(1)}(\nu) := \mathbb{K}_k(\nu)$ *(*$\mathbb{K}_k^{(0)}(\nu) := \nu$*, respectively).*

We note immediately that if $\nu(x) = 0$ for all $x \in S(o,k)$ then $\mathbb{K}_k(\nu) \equiv \nu$. The measure $\mathbb{K}_k(\nu)$ has two important properties: the first one (Proposition 4.10) holds for every finite variation measure ν, while for the second one (Proposition 4.11) we must suppose that ν is well distributed.

Proposition 4.10. *If* ν *is a finite variation measure on a* $T_{\{n_k\}}$*-type tree* T *with* $\mathrm{supp}(\nu) \subseteq B(o,k)$ *then* $\mathbb{K}_k(\nu)(T) = \nu(T)$.

Proof. If $k = 0$ there is nothing to prove and the very easy case $k = 1$ is left to the reader. We can suppose $k \geq 2$.

$$\mathbb{K}_k(\nu)(T) = \sum_{x \in B(o,k-1)} \nu(x) + \sum_{y \in S(o,k-1)} \sum_{\substack{x \sim y \\ x \in S(o,k)}} \frac{n_{k-1}}{n_{k-1}-1} \nu(x)$$
$$- \sum_{x \in S(0,k-2)} \sum_{y \in T_x \cap S(o,k)} \frac{\nu(y)}{n_{k-1}-1}$$
$$= \sum_{x \in B(o,k-1)} \nu(x) + \sum_{x \in S(o,k)} \frac{n_{k-1}}{n_{k-1}-1} \nu(x) - \sum_{z \in S(o,k)} \frac{\nu(z)}{n_{k-1}-1} = \sum_{x \in B(o,k)} \nu(x) = \nu(T).$$

□

The case when ν is a well-distributed measure is very important and it is worth looking to the explicit expression of $\mathbb{K}_k(\nu)$ in this case.

a) if $k = 0$ then $\mathbb{K}_0(\nu) := \nu$;
b) if $k = 1$ then

$$\mathbb{K}_1(\nu)(x) := \begin{cases} \sum_{y \in B(o,1)} \nu(y) & \text{if } x = o \\ 0 & \text{if } x \neq o; \end{cases}$$

c) if $k \geq 2$ then

$$\mathbb{K}_k(\nu)(x) := \begin{cases} 0 & \text{if } x \notin B(o, k-1) \\ \nu(x) + n_{k-1}\nu(y_x) & \text{if } x \in S(o, k-1) \\ \nu(x) - \sum\limits_{\substack{z \sim x \\ z \in S(o,k-1)}} \nu(y_z) & \text{if } x \in S(o, k-2) \\ \nu(x) & \text{if } k \geq 3, \ x \in B(o, k-3), \end{cases}$$

where $y_z \sim z$, $y_z \in S(o, k)$ (the definition does not depend on the choice of y_z since ν is constant on $\{y \sim z : y \in S(o, k)\}$ when $z \in S(o, k-1)$ is fixed).

Proposition 4.11. *Let ν be a well-distributed measure on a $T_{\{n_k\}}$-type tree T with* $\operatorname{supp}(\nu) \subseteq B(o, k)$. *If $\widetilde{\nu} := \mathbb{K}_k(\nu)$ and $h \in \mathcal{H}(T, P)$ then*

$$\int_T h \, \mathrm{d}\nu = \int_T h \, \mathrm{d}\widetilde{\nu}, \tag{19}$$

and

$$L(h, \nu)(x) = L(h, \widetilde{\nu})(x), \ \forall x \in T. \tag{20}$$

Proof. If $k = 0$ there is nothing to prove and if $k = 1$ the proposition follows immediately from Definition 1.1. If $k \geq 2$ and if $\nu_1 := \widetilde{\nu} - \nu$ then

$$\int_T h \, \mathrm{d}\widetilde{\nu} = \sum_{x \in B(o,k-1)} h(x)\widetilde{\nu}(x)$$

$$= \sum_{x \in S(o,k-1)} h(x)\nu(x) + \sum_{x \in S(o,k-1)} \sum_{\substack{y \sim x \\ y \in S(o,k)}} \frac{h(y)}{n_{k-1}} \nu_1(x) + \sum_{x \in S(o,k-1)} \sum_{\substack{y \sim x \\ y \in S(o,k-2)}} \frac{h(y)}{n_{k-1}} \nu_1(x)$$

$$+ \sum_{x \in S(o,k-2)} h(y)\widetilde{\nu}(y) + r(h) = \sum_{x \in S(o,k-1)} h(x)\nu(x)$$

$$+ \sum_{x \in S(o,k)} h(x)\nu(x) + \sum_{y \in S(o,k-2)} h(y) \left[\sum_{\substack{x \sim y \\ x \in S(o,k-1)}} \frac{\nu_1(x)}{n_{k-1}} + \widetilde{\nu}(y) \right] + r(h)$$

$$= \sum_{x \in B(o,k)} h(x)\nu(x) = \int_T h \, \mathrm{d}\nu,$$

where

$$r(h) := \begin{cases} \sum\limits_{x \in B(o,k-3)} h(x)\widetilde{\nu}(x) \equiv \sum\limits_{x \in B(o,k-3)} h(x)\nu(x) & \text{if } k \geq 3 \\ 0 & \text{if } k = 2. \end{cases}$$

This proves Equation (19).

Equation (20) is an immediate consequence of Equation (19) and Proposition 4.10. ◻

We have proved that the contracted measure has the same total mass as the original measure (Proposition 4.10) and, in the case of a well-distributed measure, the integrals of all the harmonic functions are invariant (Proposition 4.11). This last property does not hold for general measures; if we take, for instance, the homogeneous tree $\mathbb{T}_3$, the Dirac measure $\nu(x) := \delta_{x_{11}}(x)$ and $h(x) := k(x, x_{11})$ (see Remark 4.1) then $\widetilde{\nu} := \mathbb{K}_k(\nu) = (3/2)\delta_{x_1} - (1/2)\delta_o$. Therefore $\int_T h \,\mathrm{d}\nu = 4$, while $\int_T h \,\mathrm{d}\widetilde{\nu} = 2$ (see Figure 1).

Now we are able to characterize all the finite variation measures with finite support on a $T_{\{n_k\}}$-type tree which have the strong mean value property.

Lemma 4.12. *Let ν be a finite variation measure with* $\mathrm{supp}(\nu) \subseteq B(o,k)$ *on a $T_{\{n_k\}}$-type tree with transient simple random walk. If $L(k(\cdot,\xi),\nu)(o) = 0$ for every $\xi \in S(o,k)$ then ν is well distributed.*

Proof. If ν is not well distributed, then there exist $\xi_1, \xi_2 \in S(o,k)$ such that $\nu(\xi_1) > \nu(\xi_2)$ and $d(\xi_1, \xi_2) = 2$ and, by Proposition 4.2,

$$\begin{aligned} L(k(\cdot,\xi_1),\nu)(o) - L(k(\cdot,\xi_2),\nu)(o) &= \int_T (k(\cdot,\xi_1) - k(\cdot,\xi_2))\,\mathrm{d}\nu \\ &= [k(\xi_1,\xi_2) - k(\xi_2,\xi_1)](\nu(\xi_1) - \nu(\xi_2)) > 0, \end{aligned}$$

which contradicts the hypothesis. □

Proposition 4.13. *Let ν be a finite variation measure with* $\mathrm{supp}(\nu) \subseteq B(o,k)$ *on a $T_{\{n_k\}}$-type tree with a transient simple random walk. The following assertions are equivalent:*

(i) *ν has the weak mean value property;*
(ii) *ν has the strong mean value property;*
(iii) *$L(k(\cdot,\xi),\nu)(o) = 0$ for every $\xi \in S(o,k)$;*
(iv) *$\mathbb{K}_k^{(n)}(\nu)$ is well distributed for all $n = 0, 1, \ldots k$.*

Proof. (i) ⇒ (ii). It is enough to note that since T is a transient T_{n_k}-type tree, it is possible to extend $h|_{B(o,k)}$ ($h \in \mathcal{H}(T,P)$) to a function $g \in \mathcal{H}^\infty(T,P)$.

If $k = 0$ then we can choose $g \equiv h(o)$. Let $k \geq 1$. Then $T \setminus B(o,k-1) = \bigcup_{x \in S(o,k)} T_x$, where $T_x \cap T_y = \emptyset$ if $x, y \in S(o,k)$, $x \neq y$. It is enough to consider each T_x separately; therefore, we fix $x_0 \in S(o,k)$, $a_k := h(x_0)$ while $a_{k-1} := h(z_0)$ (where $z_0 \in S(o,k-1)$, $z_0 \sim x_0$). We look for a harmonic function g which is constant on $S(o,r) \cap T_{x_0}$, for all $r > k$ (i.e. $g(x) := a_{d(o,x)}$ for every $x \in T_{x_0}$). By Equation (2),

$$a_{i+1} = \frac{a_i n_i - a_{i-1}}{n_i - 1}, \quad \forall i > k, \tag{21}$$

then

$$a_{i+1} - a_i = \frac{a_i - a_{i-1}}{n_i - 1}, \quad \forall i > k,$$

hence

$$a_{i+1} - a_i = \left[\prod_{j=k}^{i} \frac{1}{n_j - 1}\right](h(x_0) - h(z_0)),$$

and finally using Remark 4.3 and Equation (18),

$$|a_{i+1} - h(x_0)| = \left[\sum_{l=k}^{i} \prod_{j=k}^{l} \frac{1}{n_j - 1}\right] |h(x_0) - h(z_0)| < +\infty.$$

The function g is bounded since $S(o, k)$ is finite.

(ii) $\Rightarrow$ (i). It is obvious.

(ii) $\Rightarrow$ (iii). It is an easy consequence of Remark 4.1.

(iii) $\Rightarrow$ (iv). Lemma 4.12 implies that ν is well distributed; besides, using Proposition 4.11, we have that $L(k(\cdot, \xi), \mathbb{K}_k(\nu))(o) = 0$, for all $\xi \in S(o, k)$ and again from Lemma 4.12 we obtain that $\mathbb{K}_k(\nu)$ is well distributed. Now if $\mathbb{K}_k^{(n)}(\nu)$ is such that $L(k(\cdot, \xi), \mathbb{K}_k^{(n)}(\nu))(o) = 0$, for all $\xi \in S(o, k)$ then by Lemma 4.12 it is well distributed, hence by Proposition 4.11, since $\mathbb{K}_k^{(n+1)}(\nu) = \mathbb{K}_{k-n} \circ \mathbb{K}_k^{(n)}(\nu)$ we have that $L(k(\cdot, \xi), \mathbb{K}_k^{(n+1)}(\nu))(o) = L(k(\cdot, \xi), \mathbb{K}_k^{(n)}(\nu))(o) = 0$, for all $\xi \in S(o, k)$. This proves (iv).

(iv) $\Rightarrow$ (ii). If $\mathbb{K}_k^{(n)}$ is well distributed for all n then, using Proposition 4.11, we have that

$$L(h, \nu)(o) = L(h, \mathbb{K}_k(\nu))(o) = L\big(h, \mathbb{K}_k^{(n)}(\nu)\big)(o), \qquad \forall h \in \mathcal{H}(T, P),\ \forall n = 0, 1, \ldots, k.$$

If we take $n = k - 1$ then $\operatorname{supp}(\mathbb{K}_k^{(k-1)}(\nu)) \subseteq B(o, 1)$ and it is well distributed (i.e. constant on $S(o, 1)$), then, by Proposition 3.3, we have that

$$0 = L(h, \nu)(o) = L\big(h, \mathbb{K}_k^{(k-1)}(\nu)\big)(o), \quad \forall h \in \mathcal{H}(T, P).$$

□

The meaning of the previous proposition is that the set of finite variation measures on a $T_{\{n_k\}}$-type tree with finite support which have the strong mean value property is the maximal $\mathbb{K}$-invariant subset of the set of all the finite variation measures with finite support.

5. The case of counting measures with finite support

In this paragraph we are interested in a particular class of measures: the counting measures on finite subsets of a $T_{\{n_k\}}$-type tree T, given a non-empty finite subset $B \subset T$ and a measure ν_B defined by

$$\nu_B(A) := \operatorname{card}(A \cap B), \quad \forall A \in \mathcal{P}(T). \tag{22}$$

In particular, we want to characterize all the counting measures with finite support which have the strong mean value property.

We recall that, since T is a $T_{\{n_k\}}$-type tree, then

$$\mathrm{card}(S(o,k)) = \begin{cases} 1 & \text{if } k=0 \\ n_0 & \text{if } k=1 \\ n_0 \prod_{i=1}^{k-1}(n_i - 1) & \text{if } k \geq 2, \end{cases}$$

$$\mathrm{card}(B(o,k)) = \begin{cases} 1 & \text{if } k=0 \\ n_0+1 & \text{if } k=1 \\ 1+n_0+n_0 \sum_{j=2}^{k} \prod_{i=1}^{j-1}(n_i - 1) & \text{if } k \geq 2. \end{cases} \tag{23}$$

The next lemma compares the cardinality of a ball of radius k with the cardinality of the sphere of radius $k+1$.

Lemma 5.1. *If T is a $T_{\{n_k\}}$-type tree with $n_0 \geq 2$ and $n_k \geq 3$ for all $k \geq 1$ then*

$$\mathrm{card}(S(o,k)) \geq 1 + \mathrm{card}(B(o,k-1)), \quad \forall k \geq 1. \tag{24}$$

Proof. If $k=1$ then $\mathrm{card}(S(o,k)) = n_0 \geq 2 = 1 + \mathrm{card}(B(o,0))$. We will prove the lemma by induction on k: let it be true for $k-1$ then

$$\begin{aligned} \mathrm{card}(B(o,k-1)) &= 1 + \sum_{x \sim o} \mathrm{card}(T_x \cap B(o,k-1)) \\ &= 1 + n_0 \mathrm{card}(T_{x_0} \cap B(o,k-1)) \\ \mathrm{card}(S(o,k)) &= 1 + \sum_{x \sim o} \mathrm{card}(T_x \cap S(o,k)) \\ &= 1 + n_0 \mathrm{card}(T_{x_0} \cap S(o,k)), \end{aligned} \tag{25}$$

where $x_0 \sim o$ is fixed. If one looks at T_{x_0} as a tree of $T_{\{m_k\}}$-type with root x_0, where $m_0 = n_1 - 1 \geq 2$ and $m_i = n_{i+1} \geq 3$ for all $i \geq 1$ then $T_{x_0} \cap S(o,k)$ and $T_{x_0} \cap B(o,k-1)$ are, respectively, the sphere of radius $k-1$ and the ball of radius $k-2$ of T_{x_0}. Applying the induction hypothesis

$$\mathrm{card}(T_{x_0} \cap S(o,k)) \geq 1 + \mathrm{card}(T_{x_0} \cap B(o,k-1)), \tag{26}$$

hence, using Equations (25) and (26), we have

$$\mathrm{card}(S(o,k)) \geq n_0 + \mathrm{card}(B(o,k-1)) > 1 + \mathrm{card}(B(o,k-1)).$$

□

Estimate (24) can be improved but it is sufficient for our purpose. Before proving the main result of this section (Theorem 5.3) we need another lemma.

Lemma 5.2. *Let T be a $T_{\{n_k\}}$-type tree with $n_0 \geq 2$ and $n_k \geq 3$ for all $k \geq 1$ and let us take $\overline{x}, \overline{y} \in T$ such that $\overline{x} \neq \overline{y}$, $d(o, \overline{x}) = d(o, \overline{y}) = r \geq 1$. If $B \subset B(o,k)$ is such that $\nu := \nu_B$ (see Equation (22)) has the mean value property and $\nu(x) = 1$, for all $x \in T_{\overline{x}} \cap S(o,k)$ while $\nu(y) = 0$, for all $y \in T_{\overline{y}} \cap S(o,k)$ then*

$$\mathbb{K}_k^{(k-r)}(\nu)(\overline{x}) > \mathbb{K}_k^{(k-r)}(\nu)(\overline{y}). \tag{27}$$

Proof. If $k = r$ this is obvious. Let us prove the statement by induction on $k - r$. If $k > r$, Definition 4.9 implies that $\mathbb{K}_k^{(k-r-1)}(\nu)(x) = \nu(x)$ for every $x \in B(o, r-1)$, whence

$$\mathbb{K}_k^{(k-r-1)}(\nu)(T_{\overline{x}}) = \nu(T_{\overline{x}}), \qquad \mathbb{K}_k^{(k-r-1)}(\nu)(T_{\overline{y}}) = \nu(T_{\overline{y}}). \tag{28}$$

Moreover, by Proposition 4.13, $\widetilde{\nu} := \mathbb{K}_k^{(n-r-1)}(\nu)$ is well distributed (since ν has the mean value property) and Equation (28) implies that there exists $p, q \in \mathbb{R}$ such that

$$\widetilde{\nu}(x) := \begin{cases} 0 & \text{if } x \in S(o,s),\ \ s > r+1 \\ p & \text{if } x \in S(o, r+1) \cap T_{\overline{x}} \\ q & \text{if } x \in S(o, r+1) \cap T_{\overline{y}} \\ \nu(T_{\overline{x}}) - (n_r - 1)p & \text{if } x = \overline{x} \\ \nu(T_{\overline{y}}) - (n_r - 1)q & \text{if } x = \overline{y}. \end{cases}$$

Let $\widetilde{x} \sim \overline{x}$ such that $\widetilde{x} \in S(o, r+1)$ and $\widetilde{y} \sim \overline{y}$ such that $\widetilde{y} \in S(o, r+1)$. By induction hypothesis (on $T_{\widetilde{x}}$ and $T_{\widetilde{y}}$) we have that

$$p = \widetilde{\nu}(\widetilde{x}) = \mathbb{K}_k^{(n-r-1)}(\nu)(\widetilde{x}) > \mathbb{K}_k^{(n-r-1)}(\nu)(\widetilde{y}) = \widetilde{\nu}(\widetilde{y}) = q. \tag{29}$$

If we apply $\mathbb{K}_{r+1}$ to $\widetilde{\nu}$ we obtain

$$\mathbb{K}_k^{(k-r)}(\nu)(\overline{x}) = \nu(T_{\overline{x}}) + p, \qquad \mathbb{K}_k^{(k-r)}(\nu)(\overline{y}) = \nu(T_{\overline{y}}) + q.$$

Hence, by Lemma 5.1 (note that the trees $T_{\overline{x}}$ and $T_{\overline{y}}$ satisfy all the hypotheses of the lemma),

$$\begin{aligned} \nu(T_{\overline{x}}) \geq \operatorname{card}(T_{\overline{x}} \cap S(o,k)) > \operatorname{card}(T_{\overline{x}} \cap B(o, k-1)) \\ = \operatorname{card}(T_{\overline{y}} \cap B(o, k-1)) \geq \nu(T_{\overline{y}}), \end{aligned} \tag{30}$$

and finally, from Equations (29) and (30), we obtain

$$\mathbb{K}_k^{(k-r)}(\nu)(\overline{x}) = \nu(T_{\overline{x}}) + p > \nu(T_{\overline{y}}) + q = \mathbb{K}_k^{(k-r)}(\nu)(\overline{y}).$$

□

Theorem 5.3. *Let ν_B be the counting measure on a finite non-empty subset B of a $T_{\{n_k\}}$-type tree T with $n_0 \geq 2$ and $n_k \geq 3$ for all $k \geq 1$. If $B \subseteq B(o,k)$ and $B \cap S(o,k) \neq \emptyset$ then ν_B has the strong ($\Longleftrightarrow$ weak) mean value property if and only if B is Γ_o-invariant.*

Proof. Proposition 4.13 guarantees the equivalence between weak and strong mean value property.

If B is Γ_o-invariant then Proposition 3.3 implies that ν_B has the strong mean value property.

Let us suppose that ν_B has the strong mean value property. We use induction on k. We first recall that on a $T_{\{n_k\}}$-type tree, a measure is of type 1 if and only if it is Γ_o-invariant (see Remark 3.4).

If $k = 1$ then Corollary 4.6 implies the result. Let us suppose it holds for $k-1$: we already know (Proposition 4.13) that (i) is equivalent to $\mathbb{K}_k^{(n)}(\nu_B)$ being well distributed for all $n = 0, 1, \dots k$. Let us suppose also, by contradiction, that ν_B is not Γ_o-invariant, then there exist $x, y \in S(o, k')$ ($k' \le k$) such that $\nu_B(x) = 1$ and $\nu_B(y) = 0$.

If $k' < k$ then for all $r > k'$ and for all $z, w \in S(o, r)$, we have that $\nu_B(z) = \nu_B(w)$: let us consider the measure $\nu'(x) := \nu_B(x)\chi_{B(o,k')}$. It is easy to show that $\nu_B - \nu'$ is Γ_o-invariant, hence by Proposition 3.3 it has the strong mean value property. By induction hypothesis, ν' (which is not Γ_o-invariant) has not the strong mean value property, then ν_B cannot have the strong mean value property as well and this is a contradiction.

If $k' = k$ we define $r := \min\{d(x, y) : x, y \in S(o, k), \nu_B(x) = 1, \nu_B(y) = 0\}$ (obviously r is even and $r \ge 2$), we fix $x_1, y_1 \in S(o, k)$ such that $d(x_1, y_1) = r$, $\nu_B(x_1) = 1$ and $\nu_B(y_1) = 0$. Let $\Pi[x_1 \wedge y_1, x_1] = \{x_1 \wedge y_1, \overline{x}, \dots, x_1\}$ and $\Pi[x_1 \wedge y_1, y_1] = \{x_1 \wedge y_1, \overline{y}, \dots, y_1\}$; hence for every $z \in T_{\overline{x}} \cap S(o, k)$ ($w \in T_{\overline{y}} \cap S(o, k)$, respectively) we have $d(z, \overline{x}) \le r - 2$ ($d(w, \overline{y}) \le r - 2$, respectively) and then $\nu_B(z) = 1$ ($\nu_B(w) = 0$, respectively).

Since $d(\overline{x}, o) = d(\overline{y}, o) = d(x_1 \wedge y_1, o) + 1 = k - r/2 + 1 \ge 1$ then by Lemma 5.2 we have

$$\mathbb{K}_k^{(r/2-1)}(\nu_B)(\overline{x}) > \mathbb{K}_k^{(r/2-1)}(\nu_B)(\overline{y}). \tag{31}$$

Again we obtained a contradiction, since $d(\overline{x}, \overline{y}) = 2$, $d(\overline{x}, o) = d(\overline{y}, o)$ and $\mathbb{K}_k^{(n)}(\nu_B)$ is well distributed for all $n = 0, 1, \dots, k$. □

6. A remark on the mean value property for a general Markov chain

Let (X, P) be an irreducible random walk with state space X (please note that the random walk is not required to be of nearest neighbour type). We could think of X as the associated oriented graph ($x \mapsto y$ if and only if $p(x, y) > 0$). We already introduced, in Section 1, the Banach space of the finite variation measures on X as $l^1(X) = L^1((X, \mathcal{P}(X), \mu_c); \mathbb{R})$ (with the usual norm $\|\nu\|_1 := \sum_{x \in X} |\nu(x)|$) whose (topological) dual is $l^\infty(X) := L^\infty((X, \mathcal{P}(X), \mu_c); \mathbb{R})$; the duality relationship is as follows: $l \in l^1(X)^*$ if and only if there is $h \in l^\infty(X)$ such that, for all $f \in l^1(X)$,

$$l(f) = l_h(f) := \sum_{x \in X} h(x) f(x). \tag{32}$$

In this paragraph we want to characterize all the finite variation measures which have the weak mean value property with respect to a fixed point $o \in X$ (see

Definition 3.1). To this aim we need some definitions and some elementary result of the Banach space theory.

We recall that if $\{v_\alpha\}_{\alpha\in A} \subset B$ (where B is a Banach space) then $\overline{span}(\{v_\alpha\}_{\alpha\in A})$ is the smallest closed linear subspace of B which contains $\{v_\alpha\}_{\alpha\in A}$. If $v \in B$ and $\{v_\alpha\}_{\alpha\in A} \subset B$ we say that v is independent of $\{v_\alpha\}_{\alpha\in A}$ if $\sum_{\alpha\in A} a_\alpha v_\alpha + av = 0$ implies $a = 0$. Moreover, we say that $\{v_\alpha\}_{\alpha\in A}$ is a family of independent vectors if and only if each one is independent of the others (that is $\sum_{\alpha\in A} a_\alpha v_\alpha = 0$ if and only if $a_\alpha = 0$, for all $\alpha \in A$). Let us recall that if $Y \subset B$ and B^* is the (topological) dual space of B, then $Y^\perp := \{l \in B^* : l(v) = 0, \forall v \in Y\}$: it is easy to show that $Y^\perp = \overline{span}(Y)^\perp$ and that $Y^\perp$ it is a closed linear subspace of B^*. An important result is the following (see [16, Proposition II.12]):

Proposition 6.1. *If B is a Banach space and $M \subset B$, $N \subset B^*$ are linear subspaces, then*

$$(M^\perp)^\perp = \overline{M}, \qquad (N^\perp)^\perp \supseteq \overline{N}, \tag{33}$$

where $\overline{M}$ and $\overline{N}$ are the topological closures of M and N.

It is well known that if $w = \sum_{\alpha\in A} a_\alpha v_\alpha$ (where the sum is strong convergent in B) and $l \in B^*$ then $l(w) = \sum_{\alpha\in A} a_\alpha l(v_\alpha)$.

Remark 6.2. We note immediately that $\mathcal{H}^\infty(X, P)$ is a linear closed subspace of $l^\infty(X)$: in fact it is easy to show that $\mathcal{H}^\infty(X, P) = \{\nu_x\}^\perp_{x\in X} = \overline{span}(\{\nu_x\}_{x\in X})^\perp$, where

$$\nu_x(y) := \begin{cases} p(x, x) - 1 & \text{if } y = x \\ p(x, y) & \text{if } y \neq x. \end{cases} \tag{34}$$

We note that, for every $x \in X$

$$\|\nu_x\|_1 = \sum_{y\in X} |\nu_x(y)| = 2(1 - p(x, x)), \tag{35}$$

then, if the graph has at least two points, ν_x is not the null measure for any x, since (X, P) is irreducible. Equation (34) could also be written as $\nu_x := \sum_{y\in X} p(x, y)\delta_y - \delta_x$ which converges in $l^1(X)$. Moreover, $\mathcal{H}^\infty(X, P) = \mathrm{Ker}(P - \mathbb{I}_\infty)$, where P is defined by Equation (2) and is a bounded operator from $l^\infty(X)$ into itself, $\mathbb{I}_\infty$ is the identity operator on $l^\infty(X)$ and $P - \mathbb{I}_\infty$ is the discrete Laplacian. If we consider the linear bounded operator Q from $l^1(X)$ into itself

$$(Q\nu)(x) := \sum_{y\in X} \nu(y)p(y, x), \quad \forall x \in X,$$

then $Q^* = P$ (where Q^* is the adjoint operator of Q), that is Q is the pre-adjoint of P.

We are ready to state and prove the main theorem of this section.

Theorem 6.3. *Let (X, P) be a random walk and let $o \in X$ be a fixed point in X. If $\{\nu_x\}_{x\in X}$ is defined by Equation* (34) *and ν is a finite variation measure on X, then the following assertions are equivalent:*

(i) *ν has the weak mean value property with respect to o;*
(ii) *there exist $a \in \mathbb{R}$, $\overline{\nu} \in \overline{span}(\{\nu_x\}_{x\in X})$ such that $\nu = a\delta_o + \overline{\nu}$ (in this case $a = \nu(X)$ and $\overline{\nu}$ is uniquely determined);*
(iii) *$\nu - \nu(X)\delta_o \in \overline{\mathrm{Rg}(Q - \mathbb{I}_1)}$ (where $\mathbb{I}_1$ is the identity operator on $l^1(X)$).*

Proof. (i) $\Leftrightarrow$ (ii) Let us define $\mathbb{1} \in l^\infty(X)$ as the constant function $\mathbb{1}(x) := 1$, for all $x \in X$. Obviously ν has the mean value property with respect to o if and only if

$$\nu \in A := \left\{\nu_1 \in l^1(X) : l_h(\nu_1) = l_{\mathbb{1}}(\nu_1) l_h(\delta_o), \forall h \in \mathcal{H}^\infty(X, P)\right\}; \tag{36}$$

since $l_{\mathbb{1}}(\nu) = \nu(X)$ and $l_h(\delta_o) = h(o)$ (see Equation (32)). Using the linearity of l_h and $l_{\mathbb{1}}$, it is easy to show that

$$l_h(\nu) = l_{\mathbb{1}}(\nu) l_h(\delta_o), \quad \forall h \in \mathcal{H}^\infty(X, P) \quad \Longleftrightarrow \quad l_h(\nu - l_{\mathbb{1}}(\nu)\delta_o) = 0, \quad \forall h \in \mathcal{H}^\infty(X, P), \tag{37}$$

which is equivalent to

$$\nu - l_{\mathbb{1}}(\nu)\delta_o \in \mathcal{H}^\infty(X, P)^\perp = \overline{span}(\{\nu_x\}_{x\in X})$$

by Proposition 6.1 and Remark 6.2. From the last equation we finally derive that ν has the mean value property with respect to o if and only if there exists (a unique) $\overline{\nu} \in \overline{span}(\{\nu_x\}_{x\in X})$ such that

$$\nu = l_{\mathbb{1}}(\nu)\delta_o + \overline{\nu} = \nu(X)\delta_o + \overline{\nu}.$$

(i) $\Leftrightarrow$ (iii) Since $\mathcal{H}^\infty(X, P) = \mathrm{Ker}(P - \mathbb{I}_\infty) = \mathrm{Ker}((Q - \mathbb{I}_1)^*)$, Corollary II.17 of [16] and Equation (37) imply that $\{\nu \in l^1(X) : l_h(\nu - l_{\mathbb{1}}(\nu)\delta_o) = 0, \quad \forall h \in \mathcal{H}^\infty(X, P)\} = \overline{\mathrm{Rg}(Q - \mathbb{I}_1)}$. □

If $\lambda \in l^1(X)$ then $\sum_{y\in Y} \lambda(y)\nu_y$ is well defined and convergent in $l^1(X)$. Obviously if $x \in X$,

$$\left(\sum_{y\in X} \lambda(y)\nu_y\right)(x) = \sum_{y\in X} \lambda(y) p(x, y) - \lambda(x) = (Q\lambda)(x) - \lambda(x);$$

this equation describes the relationship between the set $\{\nu_x\}_{x\in X}$ and the pre-adjoint of the Laplacian operator $Q - \mathbb{I}_1$.

Obviously if (X, P) is recurrent we know (see Section 4) that every finite variation measure has the weak mean value property with respect to any point o, then $\overline{\mathrm{Rg}(Q - \mathbb{I}_1)} = \overline{span}(\{\nu_x\}_{x\in X} \cup \{\delta_o\}) = l^1(X)$.

Using elementary techniques of Banach space theory and some basic properties of the Green function, we can show that:

(i) $\delta_o \notin \overline{span}(\{\nu_x\}_{x\in X})$;
(ii) if (X, P) is transient $\{\nu_x\}_{x\in X} \cup \{\delta_o\}$ are independent.

Moreover one can show (in a similar way as in [12, Theorem 3.9]) that $Q - \mathbb{I}_1$ is not injective if and only if P is a positive-recurrent transition operator.

We note that Theorem 6.3 characterizes all the finite variation measures which have the weak mean value property, while Proposition 4.13 characterizes only those with finite support. ν has the weak mean value property with respect to x if and only if $\nu = \lim_{n\to\infty} \nu_n$ where $\nu_n(y) = a\delta_x(y) + \sum_{w\in X} \mu_n(w)p(w, y)$ for a suitable sequence $\{\mu_n\}_{n\in\mathbb{N}} \subseteq l^1(X)$ (the previous limit is with respect to the l^1-norm).

It is easy to understand the connection between the operator $\mathbb{K}$ (Definition 4.9) and the set $\{\nu_x\}_{x\in X}$. First of all it is clear that ν_x (δ_o, respectively) is well distributed, for every $x \in X$. Let us now take a $T_{\{n_k\}}$-type tree T, if ν is well distributed on T and $\mathrm{supp}(\nu) \subseteq B(o, k)$ (for some $k \geq 1$) then for every $x \in S(o, k-1)$ there exists $\{c_x\}_{x\in T} \subset \mathbb{R}$ such that

$$\nu(y) = c_x, \quad \forall y \in S(o, k), \, y \sim x.$$

We easily verify that

$$\nu - \mathbb{K}_k(\nu) = \sum_{x\in S(o,k-1)} c_x n_{k-1} \nu_x. \tag{38}$$

By means of the last equation, ν could be written as

$$\nu = \nu(X)\delta_o + \sum_{x\in J} a_x \nu_x,$$

where J is a finite subset of X.

Theorem 6.3 also allows us to find explicit examples of finite variation measures with unbounded support which have the weak mean value property with respect to a vertex o without being Γ_o-invariant.

Example 6.4. We construct a positive finite variation measure with unbounded support on $\mathbb{T}_M$ which has the weak mean value property with respect to the root o. Let us fix $x_0 \in S(o, 1)$ and let us define

$$\nu(x) := \begin{cases} (M-1)/M^{n+2} & \text{if } x \in S(o, n) \cap T_{x_0} \\ 1/M & \text{if } x \in S(o, 1) \setminus \{x_0\} \\ 1 & \text{if } x = o \\ 0 & \text{if } x \in B(o, 1)^c \cap T_{x_0}^c, \end{cases}$$

it is easy to show that $\nu = (2 - 1/M^2)\delta_o + \sum_{x\in T_{x_0}\cup\{o\}} \nu_x / M^{d(o,x)}$. One can immediately see that ν is not Γ_o-invariant.

Here are some questions which, as far as we know, are still open:

(i) Is there a more explicit characterization of the set $\overline{span}(\{\nu_x\}_{x\in X})$ (i.e. $\overline{\mathrm{Rg}(Q - \mathbb{I}_1)}$)? To this aim it could be useful to understand when $\{\nu_x\}_{x\in X}$ (or equivalently $\{\nu_x\}_{x\in X} \cup \{\delta_o\}$) is a basis of $\overline{span}(\{\nu_x\}_{x\in X})$ ($\overline{span}(\{\nu_x\}_{x\in X} \cup \{\delta_o\})$, respectively); in this case $\{\nu_x\}_{x\in X}$ is called a *basic sequence* (see [17, Definition 4.5]).

(ii) One could easily show that if a finite variation measure ν has the weak mean value property with respect to o and $f \in L^1(X; |\nu|)$ such that there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}^\infty(X, P)$ satisfying $|h_n(x)| \le |f(x)|$ $|\nu|$-a.e., $\lim_{n\to+\infty} h_n(x) = f(x)$ $|\nu|$-a.e. and $\lim_{n\to+\infty} h_n(o) = f(o)$ then $L(f, \nu)(o) = 0$. It could be interesting to find non-trivial examples of graphs where any harmonic function could be approximated by a sequence of bounded harmonic functions as described above. In these cases the weak and the strong mean value properties are completely equivalent.

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