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A remark on the genericity of multiplicity results for forced oscillations on manifolds

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Abstract. We discuss the genericity of some multiplicity results for periodically perturbed autonomous first- and second-order ODEs on manifolds.

In particular, the genericity of the following property is investigated: if the differentiable manifold M is compact, then the equation $\ddot{x}_\pi = h(x, \dot{x}) + f(t, x, \dot{x})$ on M has $|\chi(M)|$ geometrically distinct T-periodic solutions for any small enough T-periodic perturbing function f.

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1. Introduction

Let $M \subset \mathbb{R}^k$ be a boundaryless smooth manifold. In our recent work [6] the genericity of the following property has been proved: *if M is compact, the perturbed autonomous equation on M*

$$\ddot{x}_\pi = g(x) + f(t, x, \dot{x}) \tag{1}$$

has $|\chi(M)|$ geometrically distinct T-periodic solutions for any 'small' perturbation f that is T-periodic in t.

In this paper, which can be seen as a continuation of our research in [6], we want to discuss the same property, relatively to the following equation (Theorem 4.4):

$$\ddot{x}_\pi = h(x, \dot{x}) + f(t, x, \dot{x}), \tag{2}$$

where $h : TM \longrightarrow \mathbb{R}^k$ is C^r and tangent to M, and the perturbing function $f : \mathbb{R} \times TM \longrightarrow \mathbb{R}^k$ is T-periodic in t (with $T > 0$ a fixed number), tangent to M and satisfies the usual Carathéodory and admissibility conditions.

In particular, we shall prove that when M is compact, then the set of h such that (2) admits at least $|\chi(M)|$ geometrically distinct T-periodic solutions for f small enough, is open and dense in the set of all the C^r tangent vector fields (Corollary 4.5).

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The genericity result relative to (2) does not seem to be attainable directly with the methods of [6]. In fact, we proceed in two steps: first, we obtain results in the spirit of [6] but for first-order equations in the non-compact case. Secondly, using the fact that every second-order ODE on M is equivalent to a suitable first-order equation on the tangent bundle TM, we get a genericity result for second-order equations on (not necessarily compact) manifolds (Theorem 4.4) that reduces to the quoted result when M is compact.

In the following, we use the same terminology of [6], and refer to [5,8] for the notions of differential topology.

2. Preliminaries and notation

Let $N \subset \mathbb{R}^l$ be a boundaryless, n-dimensional, smooth manifold. The general form of the first-order ODE on N studied here is the following:

$$\dot{x} = \varphi(x) + \gamma(t, x), \tag{3}$$

where $\varphi : N \longrightarrow \mathbb{R}^l$ is C^r, tangent to N and admissible, i.e. such that $\varphi^{-1}(0)$ is compact. The perturbation $\gamma : \mathbb{R} \times N \longrightarrow \mathbb{R}^l$ is assumed to have the following properties:

(P1) (Carathéodory, T-periodicity in t)
- for any $p \in N$, $\gamma(\cdot, p) : \mathbb{R} \longrightarrow \mathbb{R}^l$ is measurable and T-periodic,
- for a.a. $t \in \mathbb{R}$, $\gamma(t, \cdot) : N \longrightarrow \mathbb{R}^l$ is continuous;

(P2) (tangency)
- for any $p \in N$ and for a.a. $t \in \mathbb{R}$, $\gamma(t, p) \in T_pN$;

(P3) (admissibility)
- for any compact $K \subset N$ there exists a function $h_K \in L^1([0, T], \mathbb{R})$ such that for a.a. $t \in [0, T]$, for any $p \in K$,

$$|\gamma(t, p)| < h_K(t).$$

By TM we mean the tangent bundle to the embedded manifold M, that is the subset of $\mathbb{R}^k \times \mathbb{R}^k$ given by

$$TM = \left\{(p, v) \in \mathbb{R}^k \times \mathbb{R}^k : p \in M\ ,\ v \in T_pM\right\}.$$

We will say that a continuous map $\varphi : \mathbb{R} \times TM \to \mathbb{R}^k$ is *tangent* to M provided that $\varphi(t, q, v) \in T_qM$ for all $(t, q, v) \in \mathbb{R} \times TM$.

In what follows, the symbol $C^1_T(M)$ will denote the metric subspace of the Banach space $\left(C^1_T(\mathbb{R}^k), \|\cdot\|_1\right)$ of all the T-periodic, C^1 functions $x : \mathbb{R} \longrightarrow M$ with the usual C^1 norm $\|\cdot\|_1$. Analogously, by $C_T(TM)$ we mean the metric space of T-periodic, continuous functions $x : \mathbb{R} \longrightarrow TM$, with the metric inherited from the Banach space $C_T(\mathbb{R}^k \times \mathbb{R}^k)$.

As in [4], we tacitly assume some natural identifications; for example, we identify a point $p \in M$ with the constant function $t \mapsto p$ in $C^1_T(M)$, or a function $x \in C^1_T(M)$ with $(x, \dot{x}) \in C_T(TM)$. Moreover, M is regarded as the zero section of

TM, so that, given $h : TM \to \mathbb{R}^k$, by $h|_M : M \longrightarrow \mathbb{R}^k$ we understand the function $h|_M(p) = h(p, 0)$.

Recall that x is a solution of (2) if $\dot{x}$ is absolutely continuous, and for a.a. $t \in \mathbb{R}$

$$\ddot{x}_\pi(t) = h(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t)),$$

where $\ddot{x}_\pi(t)$ is the orthogonal projection of $\mathbb{R}^k$ onto $T_{x(t)}M$.

Equation (2) is equivalent to the following ODE on TM:

$$\dot{\xi} = \widehat{h}(\xi) + \bar{f}(t, \xi), \tag{4}$$

where, given $\xi = (p, v)$ with $p \in M$ and $v \in T_pM$, $\widehat{h}(p, v) = \big(v, r(p, v)+h(p, v)\big)$ and $\bar{f}(t, p, v) = \big(0, f(t, p, v)\big)$. The above map $r : TM \longrightarrow \mathbb{R}^k$ assigns to any fixed $(q, v) \in TM$ the unique vector in $\mathbb{R}^k$ which makes $\big(v, r(q, v)\big)$ tangent to TM at (q, v). It is known that $r(q, v) \in T_qM^\perp$. In this way $\widehat{h}$ (as well as $\bar{f}$) is tangent to TM. In the following, given $h : TM \longrightarrow \mathbb{R}^k$ as above, we will often make use of the associated vector field $\widehat{h}$, which will be referred to as the *second-order vector field associated to h*.

Consider Equation (3). We say that a point $p \in \varphi^{-1}(0) \subset N$ is T-resonant for φ if (see e.g. [3]):

- φ is C^1 in a neighbourhood of p,
- the linearized equation on T_pN (note that $\varphi'(p) \in \mathrm{End}(T_pN)$)

$$\dot{x} = \varphi'(p)x$$

  admits nontrivial (i.e. nonzero) T-periodic solutions.

Note that p is non-T-resonant for φ if and only if the spectrum $\mathrm{spec}\big(\varphi'(p)\big)$ of $\varphi'(p)$ contains no eigenvalues of the form $\frac{2\pi n i}{T}$, $n \in \mathbb{Z}$.

Following [6], we say that a point $p \in (h|_M)^{-1}(0) \subset M$ is second-order T-resonant for h, if $(p, 0) \in TM$ is T-resonant for $\widehat{h}$. In particular, if h is C^1 in a neighbourhood of $(p, 0)$ in TM and $D_2h(p, 0) = 0$, the second-order T-resonancy is equivalent to

$$-\left(\frac{2n\pi}{T}\right)^2 \in \mathrm{spec}(h|_M)'(p) \text{ for some } n \in \mathbb{Z}.$$

As in [6], we denote by $\mathcal{F}(N)$ the topological vector space of all the functions $\gamma : \mathbb{R} \times N \longrightarrow \mathbb{R}^l$ having the properties (P1)–(P3), endowed with the topology given by the following fundamental system of neighbourhoods of 0:

$$\big\{U_{K,\varepsilon} : K \text{ is a compact subset of } N,\ \ \varepsilon > 0\big\},$$

with

$$U_{K,\varepsilon} = \big\{\gamma \in \mathcal{F}(N) :\ \text{for a.a. } t \in [0, T],\ \text{for all } p \in K,\ \ |\gamma(t, p)| < \varepsilon\big\}.$$

Furthermore, by $\mathcal{E}(M)$ we mean the topological vector space of all the functions $f : \mathbb{R}\times TM \longrightarrow \mathbb{R}^k$ with the properties as in Sect. 1, and with the topology inherited from $\mathcal{F}(TM) \supset \mathcal{E}(M)$.

3. Genericity of the multiplicity results for first-order equations

Consider the set $\mathfrak{X}^{r,s}(N)$, $r, s \in \mathbb{N}\cup\{0\}$, of the admissible C^r vector fields φ, tangent to N, and such that $|\deg(\varphi, N)| = s$, and let $\mathfrak{X}_T^{r,s}(N)$ be its subset determined by the additional condition that (3) has at least s geometrically distinct T-periodic solutions for any γ in a suitably 'small' neighbourhood of 0 in $\mathcal{F}(N)$. In this section, that is devoted to first-order ODEs on (not necessarily compact) boundaryless manifolds, we show that $\mathfrak{X}_T^{r,s}(N)$ is open and dense (with an appropriate topology) in $\mathfrak{X}^{r,s}(N)$.

Let $\mathfrak{X}^r(N)$, $r \geq 0$, be the vector space of the C^r tangent vector fields to N endowed with the fine (Whitney) topology [5]. For the purpose of future reference, we recall that, given $\varphi \in \mathfrak{X}^r(N)$, the basis of its open neighbourhoods consists of the sets

$$\mathcal{N}^r(\varphi, \Psi, \mathcal{K}, E) = \Big\{\omega \in \mathfrak{X}^r(N) : \big\| D^k\big(\varphi\psi_i^{-1}\big)(p) - D^k\big(\omega\psi_i^{-1}\big)(p)\big\| < \varepsilon_i, \\ \text{for all } p \in \psi_i(K_i),\ |k| = 0, \dots, r,\ i \in \Lambda\Big\},$$

where $\Psi = \{\psi_i, U_i\}_{i\in\Lambda}$ is a locally finite set of charts on N, indexed by a set Λ, $\mathcal{K} = \{K_i\}_{i\in\Lambda}$ is a family of compact subsets $K_i \subset U_i$, and $E = \{\varepsilon_i\}_{i\in\Lambda}$ a family of positive numbers.

Let $\mathfrak{X}_a^r(N)$ be the subset of $\mathfrak{X}^r(N)$ made up of the C^r admissible vector fields. Observe that $\mathfrak{X}_a^r$ is open whereas, in general, it is not a vector space.

We will say that $s \in \mathbb{N} \cup \{0\}$ is admissible if there exists $\varphi \in \mathfrak{X}_a^r(N)$ such that $|\deg(\varphi, N)| = s$. Given an admissible s, we denote by $\mathfrak{X}^{r,s}(N)$ the set of admissible vector fields $\varphi \in \mathfrak{X}_a^r(N)$ such that $|\deg(\varphi, N)| = s$. Obviously $\mathfrak{X}^{r,n}(N)$ is not a vector space unless N is compact. In fact, as a consequence of the Poincaré–Hopf theorem, when N is compact, $s = |\chi(N)|$ is the only possible admissible number.

In the following, unless stated differently, s will always denote an admissible integer.

Proposition 3.1. *The set $\mathfrak{X}^{r,s}(N)$, $r \geq 0$, is open in $\mathfrak{X}^r(N)$.*

Proof. Fix a vector field $\varphi \in \mathfrak{X}^{r,s}(N)$. Let $\Psi = \{\psi_i, U_i\}_{i\in\Lambda}$ be a locally finite atlas on N. Refining Ψ if necessary, we can assume that $\overline{U}_i$ is compact for any $i \in \Lambda$. Let $\{\nu_i\}_{i\in\Lambda}$ be a partition of unity subordinated to the open covering $\{U_i\}_{i\in\Lambda}$ of N. For any $i \in \Lambda$, put $K_i = \operatorname{supp} \nu_i$. The family of compact subsets $\mathcal{K} = \{K_i\}_{i\in\Lambda}$ is a neighbourhood-finite covering of N. Consequently, since φ is admissible, there exists a finite set of indices $\{i_1, \dots, i_\sigma\} \subset \Lambda$ such that $\varphi^{-1}(0) \subset \bigcup_{j=1}^{\sigma} U_{i_j}$.

Let $E = \{\varepsilon_i\}_{i\in\Lambda}$ be a family of positive numbers. One sees that if for $i \notin \{i_1, \dots, i_\sigma\}$ the ε_i's are small enough, then

$$\mathcal{N}^r(\varphi, \Psi, \mathcal{K}, E) \subset \mathfrak{X}_a^r(N).$$

By the homotopy property of the degree, assuming ε_{i_j} small enough for $j = 1, \dots, \sigma$, one gets

$$\mathcal{N}^r(\varphi, \Psi, \mathcal{K}, E) \subset \mathfrak{X}^{r,s}(N),$$

which completes the proof. □

Remark. Note that Proposition 3.1 is false if the Whitney topology is replaced by the compact-open (in C^r) one.

Proposition 3.2. *Any open set of $\mathfrak{X}^{r,s}(N)$, $r \geq 0$, contains a vector field ω whose zeros are all non-degenerate. Consequently, by the additivity of the degree, $\#\omega^{-1}(0) \geq s$.*

Proof. We recall that by the Thom transversality theorem, in case $r \geq 1$, the set of the C^r tangent vector fields on N whose zeros are non-degenerate is dense in $\mathfrak{X}^r(N)$ [5,9]. Since $\mathfrak{X}^{r,s}(N)$ is open in $\mathfrak{X}^r(N)$, U is open in $\mathfrak{X}^r(N)$.

In the case $r = 0$ it is enough to note that $\mathfrak{X}^1(N)$ is dense in $\mathfrak{X}^0(N)$ and use the argument above. □

Lemma 3.3. *Assume that $\varphi \in \mathfrak{X}^{r,s}(N)$, $r \geq 1$, has σ non-degenerate zeros $p_1, \dots, p_\sigma$. Then, given a neighbourhood U of φ in $\mathfrak{X}^{r,s}(N)$, there exists $\omega \in U$ such that $p_1, \dots, p_\sigma$ are non-T-resonant zeros of ω.*

Proof. For $j = 1, \dots, \sigma$ define a smooth function $w_j : N \longrightarrow \mathbb{R}$ by

$$w_j(p) = \frac{1}{2}\, \eta_j(p) \| p - p_j \|^2,$$

where $\eta_j : N \longrightarrow [0, 1]$ is smooth with compact support and is equal to 1 in a neighbourhood of p_j. Note that $\operatorname{supp} w_j \subset \operatorname{supp} \eta_j$ and

$$(\varphi + \rho \, \operatorname{grad} w_j)'(p_j) = \varphi'(p_j) + \rho \, \mathrm{Id}_{T_p N}.$$

Without loss of generality, one can assume that $\operatorname{supp} \eta_k \cap \operatorname{supp} \eta_j = \emptyset$ for $k \neq j$, and $k, j \in \{1, \dots, \sigma\}$. Define

$$w = \sum_{j=1}^{\sigma} w_j.$$

Take $\omega = \varphi + \rho \, \operatorname{grad} w$. For $j = 1, \dots, \sigma$ and $\rho > 0$ small enough, the spectrum of $\omega'(p_j)$ does not contain elements of the form $2\pi n i/2$, $n \in \mathbb{Z}$. From Proposition 3.1 it follows that ω is in $\mathfrak{X}^{r,s}$ when ρ is small. This proves the assertion. □

Denote by $\mathfrak{X}^{r,s}_T(N)$ the set consisting of those vector fields $\varphi \in \mathfrak{X}^{r,s}(N)$ for which there exists an open neighbourhood of U_φ of 0 in $\mathcal{F}(N)$ with the property that Equation (3) admits at least s geometrically distinct T-periodic solutions whenever γ is taken in U_φ. Our main result states that such a set is generic within $\mathfrak{X}^{r,s}(N)$.

Theorem 3.4. *The set $\mathfrak{X}^{r,s}_T(N)$, $r \geq 0$, is open in $\mathfrak{X}^r(N)$ and dense in $\mathfrak{X}^{r,s}(N)$.*

Proof. To prove the first assertion, take $\varphi \in \mathfrak{X}^{r,s}_T(N)$ and let $U_{K,\varepsilon} \subset \mathcal{F}(N)$ be such that (3) admits at least s geometrically distinct T-periodic solutions whenever

$\gamma \in U_{K,\varepsilon}$. By Proposition 3.1, take $\mathcal{N}^r(\varphi, \Psi, \mathcal{K}, E) \subset \mathfrak{X}^{r,s}(N)$. Obviously, if $\varepsilon_i < \varepsilon/2$ for all $i \in \Lambda$ such that $K_i \cap K \neq \emptyset$, then $\mathcal{N}^r(\varphi, \Psi, \mathcal{K}, E) \subset \mathfrak{X}_T^{r,s}(N)$.

We now prove the density. Since $\mathfrak{X}^1(N)$ is dense in $\mathfrak{X}^0(N)$, without loss of generality we can assume that $r \geq 1$.

Fix an open subset U of $\mathfrak{X}^{r,s}(N)$. By Proposition 3.2 and Lemma 3.3, there exists $\omega \in U$ with at least s non-T-resonant zeros $p_1, \ldots, p_s$. Let us prove that $\omega \in \mathfrak{X}_T^{r,s}(N)$. Indeed, the proof of Theorem 4.1 in [6] shows that for every p_i, $i = 1, \ldots, s$, one can find a sufficiently small compact neighbourhood C_i of p_i in N such that (3) with $\gamma \in U_i$ (U_i a small neighbourhood of 0 in $\mathcal{F}(N)$) has a T-periodic solution whose image is contained in C_i. This finishes the proof. □

As we already remarked, in the case when N is compact the only possible admissible integer is $s = |\chi(N)|$. Indeed, in this case, $\mathfrak{X}^r(N) = \mathfrak{X}^{r,|\chi(N)|}(N)$, and the fine topology coincides with the C^r uniform. Hence we have:

Corollary 3.5. *When N is compact, $\mathfrak{X}_T^{r,|\chi(N)|}(N)$, $r \geq 0$, is open and dense in $\mathfrak{X}^r(N)$ (with the uniform C^r topology).*

We stay with the case N compact and, as in [6], restrict our attention to a particular class of first-order systems (3) whose leading term φ is a gradient of some C^r ($r \geq 1$) function $G : N \longrightarrow \mathbb{R}$, i.e.:

$$\dot{x} = \operatorname{grad} G(x) + \gamma(t, x). \tag{5}$$

Denote by $\mathcal{G}_T^r(N)$ the subspace of $C^r(N, \mathbb{R})$ of all the functions G having the property that there exists a neighbourhood U of 0 in $\mathcal{F}(N)$ such that (5) has at least

$$b(N) = \sum_{i=0}^{n} b_i(N)$$

geometrically distinct T-periodic solutions for any $\gamma \in U$. Here $b_i(N)$ denotes the i-th Betti number of N.

In view of the proof of Lemma 3.3 and Theorem 5.5 in [6], one gets:

Theorem 3.6. $\mathcal{G}_T^r(N)$ ($r \geq 1$) *is open and dense in* $C^r(N, \mathbb{R})$.

Since $b(N)$ is greater than or equal to the Euler–Poincaré characteristic

$$\chi(N) = \sum_{i=0}^{n} (-1)^i b_i(N),$$

the above theorem gives a stronger result than Theorem 3.4 applied to Equation (5). For instance, if N is the two-dimensional torus $\mathbf{T}^2$, one has $b(\mathbf{T}^2) = 4$, whereas $\chi(\mathbf{T}^2) = 0$.

4. Applications to second-order equations

In this section we study the genericity of the multiplicity results for second-order ODEs on (not necessarily compact) boundaryless differentiable manifolds. We shall define the second-order analogues, $\mathcal{S}_T^{r,s}(M)$ and $\mathcal{S}^{r,s}(M)$, of the spaces $\mathfrak{X}_T^{r,s}$ and $\mathfrak{X}^{r,s}$ considered in the previous section, and show that the former is open and dense in the latter one. This result will, in particular, yield a generalization of the main result of [6] (Corollary 4.5).

In what follows, we take $N = TM \subset \mathbb{R}^l$ with $l = 2k$, k being the dimension of the ambient space for M. We will say that $s \in \mathbb{N} \cup \{0\}$ is *second-order admissible* if there exists $h : TM \to \mathbb{R}^k$ tangent to M such that $|\deg(h|_M, M)| = s$. As in the previous section, when M is compact, the only possible second-order admissible integer is $|\chi(M)|$.

Define

$$\mathcal{S}^r(M) = \left\{h \in C^r(TM) : h(p, v) \in T_pM, \text{ for any } p \in M, v \in T_pM\right\}$$

and, for a second-order admissible integer s, let

$$\mathcal{S}^{r,s}(M) = \left\{h \in \mathcal{S}^r(M) : h|_M \in \mathfrak{X}^{s,r}(M)\right\}.$$

Recall that, given $h : TM \to \mathbb{R}^k$ tangent to M, $\widehat{h} : TM \longrightarrow \mathbb{R}^{2k}$ denotes the second-order vector field associated to h. Let $\theta : \mathcal{S}^r(M) \longrightarrow \mathfrak{X}^r(TM)$ be the mapping that takes h to $\widehat{h}$. Clearly θ is injective. Put $\mathcal{Y}^r(M) = \theta\,(\mathcal{S}^r(M))$, and define $\mathcal{Y}^{r,s}(M) = \mathcal{Y}^r(M) \cap \mathfrak{X}^{r,s}(TM)$. By Lemma 3.2 in [4], $\mathcal{Y}^{r,s}(M) = \theta\,(\mathcal{S}^{r,s}(M))$.

Since $\mathcal{Y}^r(M)$ and $\mathcal{Y}^{r,s}(M)$ are contained in $\mathfrak{X}^r(TM)$, they naturally inherit the topology from $\mathfrak{X}^r(TM)$. Moreover, considering $\mathcal{S}^r(M)$ as a topological subspace of $C^r(TM, \mathbb{R}^k)$ endowed with the fine (Whitney) topology, one can check that $\mathcal{S}^r(M)$ and $\mathcal{S}^{r,s}(M)$ are, respectively, homeomorphic to $\mathcal{Y}^r(M)$ and $\mathcal{Y}^{r,s}(M)$.

Lemma 4.1. *Any open subset of $\mathcal{Y}^{r,s}(M)$, $r \geq 1$, contains a vector field ω whose zeros are non-degenerate. Moreover, $\#\omega^{-1}(0) \geq s$.*

Proof. It is enough to prove the assertion for the basic neighbourhoods of the topology on $\mathcal{Y}^{r,s}(M)$. Therefore, given any φ in $\mathcal{Y}^{r,s}(M)$, consider any of its basic neighbourhood U. By the definition of the topology on $\mathcal{Y}^{r,s}(M)$, U is given by

$$U = \mathcal{N}^r(\varphi, \Psi, \mathcal{K}, E) \cap \mathcal{Y}^{r,s}(M),$$

for some fixed families $\Psi = \{\psi_i\}_{i\in\Lambda}$, $\mathcal{K} = \{K_i\}_{i\in\Lambda}$ and $E = \{\varepsilon_i\}_{i\in\Lambda}$.

We shall prove that U contains a vector field ω as in the assertion. This, by the arbitrariness of the choices of φ and U will prove the lemma.

By the definition of $\mathcal{Y}^{r,s}(M)$, there exists $h_0 : TM \longrightarrow \mathbb{R}^k$, C^r, tangent to M, such that $h_0|_M \in \mathfrak{X}^{r,s}(M)$ with the property that $\varphi = \widehat{h_0}$. Define

$$\widetilde{\Psi} = \{\widetilde{\psi}_i, \widetilde{U}_i\}_{i\in\Lambda}, \quad \widetilde{\mathcal{K}} = \{\widetilde{K}_i\}_{i\in\Lambda}, \quad \text{and} \quad \widetilde{E} = \{\widetilde{\varepsilon}_i\}_{i\in\Lambda},$$

where $\widetilde{U}_i = U_i \cap M$, $\widetilde{\psi}_i = \psi_{i|\widetilde{U}_i}$, $\widetilde{K}_i = K_i \cap M$, and $\widetilde{\varepsilon}_i = \varepsilon_i/2$. Let

$$\widetilde{U} = \mathcal{N}^r(h_0|_M, \widetilde{\Psi}, \widetilde{\mathcal{K}}, \widetilde{E}) \cap \mathfrak{X}^{r,s}(M).$$

By Proposition 3.2, there exists $h_1 \in \widetilde{U}$ such that all its zeros are non-degenerate and $\# h_1^{-1}(0) \geq s$.

Let $\sigma : TM \longrightarrow [0, 1]$ be a smooth function such that $\sigma|_M = 1$. If the support of σ is a small enough neighbourhood of M, then one has that the function $h : TM \longrightarrow \mathbb{R}^k$,

$$h(p, v) = \sigma(p, v)\, h_1(p) + (1 - \sigma(p, v))\, h_0(p, v),$$

satisfies $\widehat{h} \in \mathcal{N}^r(\varphi, \Psi, \mathcal{K}, E)$. It is easy to check that h is C^r, tangent to M and $h|_M = h_1 \in \mathfrak{X}^{r,s}(M)$.

Let $\omega = \widehat{h}$. Then $\omega \in U$ and $\omega^{-1}(0) = h_1^{-1}(0)$, consequently $\#\omega^{-1}(0) \geq s$. Take $p \in \omega^{-1}(0)$.

Since

$$T_{(p,0)}TM = T_pM \times T_pM,$$

the linear operator $\omega'(p, 0) : T_{(p,0)}TM \longrightarrow T_{(p,0)}TM$ is represented by the block matrix:

$$\begin{pmatrix} 0 & I \\ D_1h(p, 0) & D_2h(p, 0) \end{pmatrix} = \begin{pmatrix} 0 & I \\ D_1h_1(p, 0) & D_2h(p, 0) \end{pmatrix},$$

where I is the identity on T_pM. Therefore

$$\det \omega'(p, 0) = (-1)^m \det h_1'(p),$$

where m is the dimension of M. Consequently, all zeros of ω are non-degenerate. □

We now establish a technical lemma that, in the framework of second-order differential equations, plays the same role as Lemma 3.3 in the previous section.

Lemma 4.2. *Assume that $\varphi \in \mathcal{Y}^{r,s}(M)$, $r \geq 1$, has σ non-degenerate zeros $z_1, \dots, z_\sigma$. Then, given a neighbourhood U of φ in $\mathcal{Y}^{r,s}(M)$, there exists $\omega \in U$ such that $z_1, \dots, z_\sigma$ are second-order non-T-resonant zeros of ω.*

Proof. Since φ is in $\mathcal{Y}^{r,s}(M)$, we have $\varphi = \widehat{h}_0$ for some $h_0 : TM \longrightarrow \mathbb{R}^k$ of C^r class, tangent to M and such that $h_0|_M \in \mathfrak{X}^{r,s}(M)$, and with the property that the points $p_1, \dots, p_\sigma$, defined by $(p_i, 0) = z_i$, $i = 1, \dots, \sigma$, are non-degenerate zeros of $h_0|_M$.

Exactly as in the proof of Lemma 4.1, but using Lemma 3.3 instead of Proposition 3.2, we get a vector field $\omega = \widehat{h} \in U$ with $p_1, \dots, p_\sigma$ being (first-order) non-T-resonant zeros of $h|_M$. Thus $z_1, \dots, z_\sigma$ are second-order non-T-resonant zeros of ω and the result follows. □

Analogously to the space $\mathcal{X}_T^{r,s}(N)$ introduced in Sect. 3, we define the space $\mathcal{S}_T^{r,s}(M) \subset \mathcal{S}^{r,s}(M)$, made out of those $h : TM \longrightarrow \mathbb{R}^k$, tangent to M, for which Equation (2) admits at least s geometrically distinct T-periodic solutions whenever f belongs to an appropriate open neighbourhood of 0 in $\mathcal{E}(M)$. We also put $\mathcal{Y}_T^{r,s}(M) = \theta\big(\mathcal{S}_T^{r,s}(M)\big)$.

We summarize the relations between the spaces introduced above in the following table:

Relations induced by the correspondence $h \overset{\theta}{\longmapsto} \widehat{h}$		
The space:	consists of:	corresponds to:
$\mathcal{S}^r(M)$	any $h \in C^r(TM, \mathbb{R}^k)$ s.t. $h(p, v) \in T_pM$ for any $(p, v) \in TM$	$\mathcal{Y}^r(M) \subset \mathcal{X}^r(TM)$
$\mathcal{S}^{r,s}(M)$	any $h \in \mathcal{S}^r(M)$ s.t. $\lvert \deg(h\vert_M, M)\rvert = s$	$\mathcal{Y}^{r,s}(M) \subset \mathcal{X}^{r,s}(TM)$
$\mathcal{S}_T^{r,s}(M)$	any $h \in \mathcal{S}^{r,s}(M)$ s.t. (2) has s T-periodic solutions whenever f belongs to a 'small' nbd. of 0 in $\mathcal{E}(M)$	$\mathcal{Y}_T^{r,s}(M) \subset \mathcal{X}_T^{r,s}(TM)$

In view of Lemmas 4.1 and 4.2 above and of Theorem 4.1 in [6], arguing as in the proof of Theorem 3.4, we get the following result:

Lemma 4.3. *The set $\mathcal{Y}_T^{r,s}(M)$, $r \geq 0$, is open and dense in $\mathcal{Y}^{r,s}(M)$.*

We are now ready to state the second-order analogue of Theorem 3.4, which, roughly speaking, asserts that for 'almost any' $h \in \mathcal{S}^{r,s}(M)$ Equation (2) admits at least s geometrically distinct T-periodic solutions for any 'small' T-periodic perturbation f.

Theorem 4.4. *The set $\mathcal{S}_T^{r,s}(M)$, $r \geq 0$, is open and dense in $\mathcal{S}^{r,s}(M)$.*

Proof. It follows immediately from Lemma 4.3 and from the homeomorphism of $\mathcal{S}^r(M)$ and of $\mathcal{S}^{r,s}(M)$ with $\mathcal{Y}^r(M)$ and $\mathcal{Y}^{r,s}(M)$, respectively. □

The following corollary is a generalization of Theorem 5.1 in [6], where the function g in (2) was assumed to depend only on the position p (and not on the speed v).

Corollary 4.5. *When M is compact, $\mathcal{S}_T^{r,|\chi(M)|}(M)$, $r \geq 0$, is open and dense in $\mathcal{S}^r(M)$.*

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