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A priori and universal estimates for global solutions of superlinear degenerate parabolic equations

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Abstract. We prove an a priori estimate and a universal bound for any global solution of the nonlinear degenerate reaction-diffusion equation $u_t = \Delta u^m + u^p$ in a bounded domain with zero Dirichlet boundary conditions.

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1. Introduction and main results

Consider the following problem:

$$\begin{cases} u_t = \Delta u^m + u^p, & 0 < t < T, \quad x \in \Omega, \\ u(t,x) = 0, & 0 < t < T, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Throughout the paper, Ω is a C^3-smooth bounded domain of $\mathbb{R}^N$, $p > \max(1, m)$ and $m > 0$. We consider solutions which may change sign and define u^k as $|u|^k \operatorname{sign}(u)$ for all real $k > 0$. We assume that the initial datum $u_0 \in L^\infty(\Omega)$ satisfies $u_0^m \in H_0^1(\Omega)$.

It is well known that solutions of (1.1) blow up in finite time if u_0 is suitably large, while they exist globally and decay as $t \to \infty$ if u_0 is small. In view of a classification of all solutions of (1.1), it is then a natural question to ask whether unbounded global solutions may exist or not.

The question of the boundedness of global solutions of (1.1) was initiated in [15] and further investigated in [4]. Denoting $p_S = (N+2)/(N-2)$ (∞ if $N \le 2$), the result of [4] says that if $p/m < p_S$, then any global solution of (1.1) is uniformly bounded for $t \ge 0$, that is,

$$\sup_{t\ge 0} \|u(t)\|_\infty < \infty. \tag{1.2}$$

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On the other hand, it is known from [15] that some unbounded global solutions do exist if $N \geq 3$ and $p/m \geq p_S$ (see also [9]).

In the semilinear case $m = 1$, the conclusion of [4] had been obtained earlier in [3]. A more precise result was given in [10] for $m = 1$ and $u_0 \geq 0$, where an *a priori* estimate of the form

$$\sup_{t\geq 0} \|u(t)\|_\infty \leq C(\|u_0\|_\infty, \Omega, p) \tag{1.3}$$

was established for $p < p_S$. In [16], (1.3) was then proved for $m = 1$ without the restriction $u_0 \geq 0$. It is to be noted that boundedness of global solutions does not imply the a priori estimate (1.3) in general (see [9, 6] for some counter-examples).

Recently, the question of whether the bound in (1.3) might be independent of u_0 for $t \geq \tau > 0$ was raised in [7], and it was proved there that

$$\sup_{t\geq \tau} \|u(t)\|_\infty \leq C(\tau, \Omega, p), \qquad \text{for all } \tau > 0, \tag{1.4}$$

holds for all global solutions of (1.1), provided $m = 1$, $u_0 \geq 0$ and $p < (N + 1)/(N - 1)$ or $N = 1$. We call (1.4) a *universal* bound. In other words, (1.4) says that after any positive time τ, every global non-negative trajectory of (1.1) enters into an absorbing bounded set A_τ. Shortly thereafter, it was shown in [17] that (1.4) still holds for $N \leq 2$ or $N = 3$ and $p < 5 = p_S$ (with $m = 1$ and $u_0 \geq 0$).

For more results on boundedness of global solutions and on a priori estimates for other classes of evolution equations, we refer to the surveys [18, 5], to [13, 20] and to the references in [7]. Let us also mention the paper [19] which contains some results on semilinear parabolic systems.

The aim of the present paper is to establish both a priori estimates and universal bounds for global solutions of (1.1) in the degenerate case $m > 1$. By a solution of (1.1), we mean a weak solution (see Section 2 below for a precise definition). Our main results are the following:

Theorem 1 (a priori estimate). *Assume that*

$$1 < m < p < p_1(m, N) := \begin{cases} \infty, & \text{if } N = 1 \\ m + \frac{10m+2}{3N-4}, & \text{if } N \geq 2. \end{cases} \tag{1.5}$$

Then any global solution of (1.1) *satisfies*

$$\sup_{t\geq 0} \|u(t)\|_\infty \leq C(\|u_0\|_\infty, \Omega, p, m), \tag{1.6}$$

where the constant $C(\|u_0\|_\infty, \Omega, p, m) > 0$ *remains bounded for* $\|u_0\|_\infty$ *bounded.*

Theorem 2 (universal bound). *Assume that*

$$1 < m < p < p_2(m, N) := \begin{cases} \infty, & \text{if } N = 1 \\ \frac{N+2}{N}\, m, & \text{if } N \geq 2. \end{cases} \tag{1.7}$$

Then, for all $\tau > 0$, *there exists a constant* $C(\tau, \Omega, p, m) > 0$ *such that any global non-negative solution of* (1.1) *satisfies*

$$\sup_{t\geq \tau} \|u(t)\|_\infty \leq C(\tau, \Omega, p, m). \tag{1.8}$$

Remark 1.1. It is easy to show that (1.8) implies (1.6). On the other hand, (1.8) cannot be true for $\tau = 0$, since there exist global solutions starting from unbounded initial data in L^q (see, e.g., [1]). Furthermore, Theorem 2 cannot be true for solutions of a mixed sign. Indeed, if $1 < p/m < p_S$, then there exist sign-changing stationary solutions of arbitrary large sup norm.

Remark 1.2. The paper [4] on boundedness of global solutions also gives an a priori estimate similar to (1.6) under the condition $m < p < m + (m+1)\min(1, 2/N)$ (see [4, Remark 1.10 p. 236], where $m+1$ is substituted by 2 due to a misprint). Our condition (1.5) in Theorem 1 is weaker. However, we restrict to the "slow-diffusion" case $m > 1$, while the results of [4] also work for the "fast-diffusion" case $m < 1$, with $m > (N-2)/(N+2)$.

The proof of our a priori estimate relies on suitable modifications of ideas in [3], based on energy and interpolation arguments. For the proof of universal bounds, besides the previously established a priori estimate, we use suitable test functions and certain smoothing properties of solutions of (1.1).

The rest of the paper is organized as follows. Section 2 contains some useful preliminary material, including local existence and smoothing properties of solutions. Sections 3 and 4 are then devoted to the proofs of Theorems 1 and 2, respectively.

2. Preliminaries

For $u_0 \in L^\infty(\Omega)$, $u_0^m \in H_0^1(\Omega)$, by a (weak) solution of (1.1) on $[0, T]$, we mean a function u such that

$$(2.1)\qquad \begin{cases} u \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega)), \quad u^m \in L^\infty\big(0,T; H_0^1(\Omega)\big), \\ \int_0^t \int_\Omega \big(u\varphi_t - \nabla u^m \cdot \nabla\varphi + u^p\varphi\big) = \int_\Omega u(t)\varphi(t) - \int_\Omega u_0\varphi(0) \\ \qquad \text{for all } t \in (0,T] \text{ and } \varphi \in L^2\big(0,T; H_0^1(\Omega)\big) \cap H^1(0,T; L^2(\Omega)). \end{cases}$$

It is known (see, e.g., [12]) that there exists $T^* = T^*(u_0) \in (0, \infty]$ such that for each $T \in (0, T^*)$, (P) admits a unique solution on $[0, T]$. If $T^* < \infty$, then $\|u(t)\|_\infty \to \infty$ as $t \to T^*$. Moreover, u satisfies the energy inequality

$$(2.2)\qquad E(t_1) + m\int_{t_0}^{t_1}\int_\Omega |u|^{m-1}u_t^2 \le E(t_0), \quad 0 \le t_0 \le t_1 < T^*,$$

where the energy is defined by

$$(2.3)\qquad E(t) := \int_\Omega \left(\frac{1}{2}|\nabla u^m(t)|^2 - \frac{m}{p+m}|u(t)|^{p+m}\right).$$

In particular, the energy is non-increasing in time. Also, we have the useful identity

$$(2.4)\qquad \frac{d}{dt}\int_\Omega \frac{|u|^{m+1}}{m+1} = -\int_\Omega |\nabla u^m(t)|^2 + \int_\Omega |u(t)|^{p+m}, \quad \text{a.e. } t \in (0, T^*).$$

(This follows from (2.1) with the test function $\varphi = u^m$ – note that $\varphi_t \in L^2(0, T; L^2(\Omega))$ is due to (2.2) and $u \in L^\infty(0, T; L^\infty(\Omega))$.)

Finally, in what follows, $C(\dots)$ will denote positive constants which may vary from line to line and depend only on the indicated arguments.

In the proof of Theorems 1 and 2, we will need the following two lemmas which give useful regularizing properties for local solutions of (1.1). In particular Lemma 2.2 (i), which is essentially a consequence of results in [1], provides smoothing properties from L^q into L^∞.

Lemma 2.1. *Assume* $p > m > 1$. *For all* $t \in (0, T^*)$, *it holds*

$$\|\nabla u^m(t)\|_2^2 \le C(p, m).\Big(t^{-1}\|u_0\|_{m+1}^{m+1} + \sup_{s\in(0,t)} \|u(s)\|_{m+p}^{m+p}\Big). \tag{2.5}$$

Moreover, we have

$$T^* > T_1 := C(p)\|u_0\|_\infty^{1-p}, \qquad \|u(t)\|_\infty \le 2\|u_0\|_\infty, \quad 0 < t \le T_1 \tag{2.6}$$

and

$$\|\nabla u^m(T_1)\|_2^2 \le C(p, m)|\Omega|\, \|u_0\|_\infty^{p+m}. \tag{2.7}$$

Lemma 2.2. *Assume* $p > m > 1$ *and* $q \ge 1$, $q > \frac{N}{2}(p - m)$.
(i) There exist positive constants $L_1, L_2, \alpha, \beta, \gamma$ *depending only on* N, p, m, q, *such that* $T^* > T_0 := L_2(1 + \|u_0\|_q)^{-\gamma}$ *and*

$$\|u(t)\|_\infty \le L_1\|u_0\|_q^\beta t^{-\alpha}, \quad 0 < t \le T_0. \tag{2.8}$$

(ii) For all $M > 0$, *there exists* $C(M, N, p, m, q) > 0$ *such that if*

$$0 < T \le T^*, \quad \|u_0\|_\infty \le M \quad \textit{and} \quad \sup_{t\in(0,T)} \|u(t)\|_q \le M,$$

then

$$\|u(t)\|_\infty \le C(M, N, p, m, q), \quad 0 \le t < T.$$

Proof of Lemma 2.1. Integrating (2.4) over $(0, t)$, we have

$$\int_0^t\int_\Omega |\nabla u^m(s)|^2 \le \int_0^t\int_\Omega |u(s)|^{m+p} + \int_\Omega \frac{|u_0|^{m+1}}{m+1}.$$

Due to the non-increasing property of the energy, it follows that

$$tE(t) \le \int_0^t E(s)\,ds \le \frac{1}{2}\int_0^t\int_\Omega |\nabla u^m(s)|^2.$$

By combining these two inequalities, we deduce that

$$\begin{aligned}\int_\Omega |\nabla u^m(t)|^2 &= 2E(t) + \frac{2m}{m+p}\int_\Omega |u(t)|^{m+p}\\ &\le \frac{1}{t}\Big(\int_0^t\int_\Omega |u(s)|^{m+p} + \int_\Omega \frac{|u_0|^{m+1}}{m+1}\Big) + \frac{2m}{m+p}\int_\Omega |u(t)|^{m+p},\end{aligned}$$

which implies (2.5).

On the other hand, by comparing with the solution of the ODE $y' = y^p$ for $y(0) = \|u_0\|_\infty$, we get

$$\|u(t)\|_\infty \le y(t) = \left(\|u_0\|_\infty^{1-p} - (p-1)t\right)^{-1/(p-1)}, \\ 0 < t < \min\left(T^*, (p-1)^{-1}\|u_0\|_\infty^{1-p}\right),$$

which immediately yields (2.6).

Finally, (2.7) follows from (2.6) and (2.5) with $t = T_1$. □

Proof of Lemma 2.2. (i) For each $u_0 \in L^q(\Omega)$ with $u_0 \ge 0$, it follows from [1, Theorem 3.1 and Remark 3.3] that there exists (at least) a solution v of (1.1) which exists on $[0, T_0]$ with $T_0 := L_2(1 + \|u_0\|_q)^{-\gamma}$ and satisfies (2.8). However, the notion of solution in [1] is weaker than (2.1), so that we cannot immediately identify v with our solution u.

To do so, we note that the solution v in [1] is constructed as a limit, pointwise on $(0, T_0) \times \Omega$, of solutions of the approximating problems

$$(2.9)_n \qquad \begin{cases} v_{n,t} = \Delta v_n^m + \min(v_n^p, n), & t > 0, \quad x \in \Omega, \\ v_n(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ v_n(0,x) = \min(u_0(x), n), & x \in \partial\Omega. \end{cases}$$

For each $n \ge 1$, $(2.9)_n$ admits a unique (global) solution v_n in the sense of (2.1). For each $0 < T' < T^*(u_0)$, since $u_0 \in L^\infty$ and $u \in L^\infty((0, T') \times \Omega)$, it follows by uniqueness for $(2.9)_n$ that $v_n = u$ on $(0, T') \times \Omega$ for all $n \ge n_0(T')$ large enough. Passing to the limit, we deduce that $v = u$ on $(0, \min(T', T_0)) \times \Omega$, hence on $(0, \min(T^*(u_0), T_0)) \times \Omega$. Since v satisfies (2.8), we conclude that u satisfies (2.8).

Finally, in the case when u_0 changes sign, it suffices to compare u with the solutions $u^\pm$ corresponding to the initial data $\pm|u_0|$.

(ii) Let $T_2 = \min(T_0, T_1)$, where T_0 is defined in Lemma 2.2 (i) and T_1 is defined in (2.6). By Lemma 2.1 and Lemma 2.2 (i), we have

$$\|u(t)\|_\infty \le \begin{cases} 2\|u_0\|_\infty, & \text{if } 0 \le t \le T_2 \\ L_1\|u(t - T_2)\|_q^\beta T_2^{-\alpha}, & \text{if } T_2 \le t < T, \end{cases}$$

and the result follows. □

Remark 2.1. The result of Lemma 2.2 (ii) can also be found in [8] under the assumption $q > \max(1, N/2)(p - m)$ (stronger if $N = 1$). Under this assumption, the result of Lemma 2.2 (i) can be proved alternatively by combining the arguments in [4, Lemma 1.6] and [14, pp. 46–48] with [11, Théorème 1].

3. Proof of Theorem 1

The proof of Theorem 1 relies on a suitable adaptation of the arguments in [3], based on energy estimates and interpolation.

Proof of Theorem 1. Let u be a global solution of (1.1). From (2.4) and (2.3) we have, for a.e. $t \geq 0$,

$$\frac{d}{dt}\int_\Omega \frac{|u|^{m+1}}{m+1} = -2E(t) + \frac{p-m}{p+m}\int_\Omega |u(t)|^{p+m}. \tag{3.1}$$

By Hölder's inequality and the non-increasing property of the energy, it follows that for each $t_0 \geq 0$,

$$\frac{d}{dt}\int_\Omega \frac{|u|^{m+1}}{m+1} \geq C(p,m,\Omega)\Big(\int_\Omega |u(t)|^{m+1}\Big)^\alpha - 2E(t_0), \qquad \text{a.e. } t \geq t_0,$$

where $\alpha = \frac{m+p}{m+1} > 1$. Since u exists globally, this implies

$$E(t_0) \geq 0, \qquad \text{for all } t_0 \geq 0 \tag{3.2}$$

and

$$\int_\Omega |u(t)|^{m+1} \leq C(p,m,\Omega)E^{\frac{m+1}{m+p}}(0), \qquad \text{for all } t \geq 0. \tag{3.3}$$

On the other hand, by (2.2) and (3.2), we have

$$\int_0^\infty \int_\Omega |u|^{m-1}u_t^2 \leq \frac{E(0)}{m}. \tag{3.4}$$

Now using (3.1) and $E(t) \leq E(0)$, we get

$$\frac{p-m}{p+m}\int_\Omega |u(t)|^{p+m} \leq 2E(0) + \int_\Omega |u(t)|^{(m+1)/2}\,|u(t)|^{(m-1)/2}u_t;$$

hence, by Cauchy–Schwarz's inequality and (3.3),

$$\begin{aligned}\Big(\int_\Omega |u(t)|^{p+m}\Big)^2 &\leq 2\left(\frac{p+m}{p-m}\right)^2\Big(4E^2(0) + \int_\Omega |u(t)|^{m+1}\int_\Omega |u(t)|^{m-1}u_t^2\Big)\\ &\leq C(p,m,\Omega)\Big(E^2(0) + E^{\frac{m+1}{m+p}}(0)\int_\Omega |u(t)|^{m-1}u_t^2\Big).\end{aligned} \tag{3.5}$$

By integrating over $(t, t+1)$, setting $a = \frac{p+2m+1}{p+m}$, it follows from (3.4) that

$$\int_t^{t+1}\Big(\int_\Omega |u|^{p+m}\Big)^2 \leq C(p,m,\Omega)\big(E^2(0) + E^a(0)\big), \quad t \geq 0. \tag{3.6}$$

Let $v = u^{(m+1)/2}$ and $r = \frac{2(m+p)}{m+1} > 2$. In the rest of the proof, k will denote various positive constants depending only on p and m. Rewrite the inequalities (3.4) and (3.6) as

$$\int_0^\infty \int_\Omega v_t^2 \leq C(m)E(0)$$

and

$$\int_t^{t+1}\Big(\int_\Omega |v|^r\Big)^2 \le C(p,m,\Omega)\big(1+E^k(0)\big), \quad t\ge 0,$$

hence, in particular,

$$\|v\|_{L^{2r}(t,t+1;L^r(\Omega))}+\|v\|_{H^1(t,t+1;L^2(\Omega))}\le C(p,m,\Omega)(1+E^k(0)), \quad t\ge 0.$$

By interpolation (see [3, Appendice] or also [2] for more general results), it follows that

$$\|v\|_{L^\infty(t,t+1;L^a(\Omega))}\le C(p,m,\Omega,a)(1+E^k(0)), \quad t\ge 0,$$

for all $a\in[1,a_0)$, where $a_0=r-\frac{r-2}{3}=\frac{2(r+1)}{3}>2$, so that

$$\|u\|_{L^\infty(0,\infty;L^q(\Omega))}\le C(p,m,\Omega,q)(1+E^k(0)), \tag{3.7}$$

for all $q\in[1,q_0)$, where $q_0=\frac{(m+1)(r+1)}{3}=\frac{3m+2p+1}{3}>1$.

By combining (2.6) and (2.7) from Lemma 2.2 with (3.7) (shifting the origin of time from 0 to T_1 in (3.7)), we obtain

$$\|u\|_{L^\infty(0,\infty;L^q(\Omega))}\le C(p,m,\Omega,q)\big(1+\|u_0\|_\infty^k\big), \quad \text{for all } q\in[1,q_0).$$

Finally, since the assumption (1.5) implies that $q_0>\frac{N}{2}(p-m)$, the conclusion follows from Lemma 2.2 (ii). □

Remark 3.1. By using the estimates (3.5), (3.4), Lemma 2.2 (i) and arguing similarly as in the proof of [3, Proposition 6], we obtain a new proof of the boundedness of global solutions of (1.1) under the assumption $1<p/m<p_S$, $m>1$. This proof is completely different from that in [4] (note, however, that the proof in [4] works also for $(N-2)/(N+2)<m<1$).

4. Proof of Theorem 2

In this section, we denote, respectively, by $\lambda_1>0$ and $\varphi_1(x)>0$ the first eigenvalue and first eigenfunction of $-\Delta$ in Ω with Dirichlet boundary conditions, satisfying

$$\begin{cases}-\Delta\varphi_1=\lambda_1\varphi_1, & x\in\Omega,\\ \varphi_1\in H_0^1(\Omega),\\ \int_\Omega\varphi_1=1.\end{cases}$$

Let u be a non-negative global solution of (1.1). We first claim that

$$\int_\Omega u(t)\varphi_1\le C(p,m,\lambda_1), \quad t\ge 0 \tag{4.1}$$

and

$$\int_0^\tau\int u^p\varphi_1\le C(p,m,\lambda_1,\tau), \quad \tau>0. \tag{4.2}$$

Indeed, taking $\varphi = \varphi_1$ in (2.1), we get for a.e. $t \geq 0$,

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 = \int_\Omega u^p(t)\varphi_1 - \lambda_1 \int_\Omega u^m(t)\varphi_1;$$

hence, by Young's and Jensen's inequalities, and since $p > m$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(t)\varphi_1 &\geq \frac{1}{2}\int_\Omega u^p(t)\varphi_1 - C(p, m, \lambda_1) \\ &\geq \frac{1}{2}\Big(\int_\Omega u(t)\varphi_1\Big)^p - C(p, m, \lambda_1). \end{aligned} \tag{4.3}$$

This implies (4.1) since otherwise $\int_\Omega u(t)\varphi_1$ would blow up in finite time, contradicting $T^* = \infty$. Integrating (4.3) over $(0, \tau)$ and using (4.1) we deduce

$$\frac{1}{2}\int_0^\tau \int_\Omega u^p(t)\varphi_1 \leq C(p, m, \lambda_1)\tau + \int_\Omega u(\tau)\varphi_1 \leq C(p, m, \lambda_1, \tau),$$

which is (4.2).

We next claim that there exists $t_0 \in (0, \tau/2)$ such that for all $l \in (0, p/2)$,

$$\int_\Omega u^m(t_0) + \int_\Omega u^l(t_0) \leq C(p, m, l, \Omega, \tau). \tag{4.4}$$

To this end, we introduce the solution χ of the problem

$$\begin{cases} -\Delta\chi = 1, & x \in \Omega, \\ \chi \in H_0^1(\Omega). \end{cases}$$

Using (2.1) with test function χ, we get

$$\int_0^\tau \int_\Omega u^m = \int_0^\tau \int_\Omega u^p\chi + \int_\Omega u_0\chi - \int_\Omega u(\tau)\chi.$$

Since

$$c\chi(x) \leq \text{dist}(x, \partial\Omega) \leq c^{-1}\varphi_1(x), \quad x \in \Omega,$$

for some $c = c(\Omega) > 0$, we deduce from (4.1) and (4.2) that

$$\int_0^\tau \Big(\int_\Omega u^m + \int_\Omega u^p\varphi_1\Big) \leq C(p, m, \Omega, \tau).$$

In particular, there exists $t_0 \in (0, \tau/2)$ such that $\int_\Omega u^m(t_0) + \int_\Omega u^p(t_0)\varphi_1 \leq C(p, m, \Omega, \tau)$. Now, by Hölder's inequality,

$$\int_\Omega u^l(t_0) = \int_\Omega u^l(t_0)\varphi_1^{l/p}.\varphi_1^{-l/p} \leq \Big(\int_\Omega u^p(t_0)\varphi_1\Big)^{l/p}\Big(\int_\Omega \varphi_1^{-l/(p-l)}\Big)^{(p-l)/p}.$$

Since $l/(p-l) < 1$ and $\varphi_1(x) \geq c\ \text{dist}(x, \partial\Omega)$, we have $\int_\Omega \varphi_1^{-l/(p-l)} < \infty$ and (4.4) follows.

Now, let us first consider the case $N \geq 2$, for which $p < p_2(N,m)$ implies that $m > \frac{N}{2}(p-m)$. We may thus apply Lemma 2.1 (i) with $q = m$. From $\|u(t_0)\|_m \leq C(p,m,\Omega,\tau)$ (see (4.4)), we then deduce that

$$\|u(t_1)\|_\infty \leq C(p,m,\Omega,\tau) \quad \text{for some } t_1 \in (t_0,\tau). \tag{4.5}$$

In the case $N = 1$, we have $p/2 > \frac{N}{2}(p-m)$ and since $m > 1$, we can choose $\tilde{m} \geq 1$ such that $\frac{N}{2}(p-m) < \tilde{m} < \max(m, p/2)$. We may thus apply Lemma 2.1 (i) with $q = \tilde{m}$. Since $\|u(t_0)\|_{\tilde{m}} \leq C(p,m,\Omega,\tau)$ due to (4.4), we then deduce that (4.5) is still true.

Finally, since $p_1(N,m) > p_2(N,m)$, we conclude from (4.5) and Theorem 1 that

$$\sup_{t\geq\tau} \|u(t)\|_\infty \leq C(p,m,\Omega,\tau).$$

□

Remark 4.1. The control on $\int_\Omega u^l(t_0)$ for all $l < p/2$ (cf. (4.4)) does not enable one to improve the value of $p_2(N,m)$ if $N \geq 2$.

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