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Nonlinear elliptic equations with natural growth in general domains

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Abstract. We prove the existence of solutions of nonlinear elliptic equations with first-order terms having "natural growth" with respect to the gradient. The assumptions on the source terms lead to the existence of possibly unbounded solutions (though with exponential integrability). The domain Ω is allowed to have infinite Lebesgue measure.

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1. Introduction

In this paper we are interested in proving the existence of solutions of nonlinear elliptic problems whose model is:

$$\begin{cases} -\Delta_p(u) + \alpha_0 |u|^{p-2}u = d(x)|\nabla u|^p + f(x) - \operatorname{div} g(x) \text{ in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1}$$

where Ω is an open set of $\mathbf{R}^N$, possibly of infinite measure, p is a constant such that $p > 1$, $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, α_0 is a positive constant, $d(x)$ is a function in $L^\infty(\Omega)$. We assume the following hypotheses on the source terms f and g (supposing $p < N$ for simplicity):

$$f \in L^{N/p}\big(\{x \in \Omega : |f(x)| > 1\}\big), \quad f \in L^{p'}\big(\{x \in \Omega : |f(x)| \le 1\}\big),$$

$$g \in L^{N/(p-1)}(\Omega; \mathbf{R}^N) \cap L^{p'}(\Omega; \mathbf{R}^N).$$

It is clear that in the case of a domain having finite measure, these assumptions become

$$f \in L^{N/p}(\Omega), \qquad g \in L^{N/(p-1)}(\Omega; \mathbf{R}^N).$$

When Ω is a bounded open set of $\mathbf{R}^N$, this kind of problem with lower-order terms $H(x, u, \nabla u)$ having natural growth p with respect to the gradient has been studied by Boccardo, Murat, Puel in [2], [3], where the source terms f, g are

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assumed to be in $L^q(\Omega)$, $L^r(\Omega; \mathbf{R}^N)$, respectively, with $q > N/p$, $r > N/(p-1)$ (an hypothesis which implies that $u \in L^\infty(\Omega)$). The case of unbounded solutions is treated in [1], where a sign condition on $H(x, u, \nabla u)$ is assumed.

In Ferone and Murat [11], Ω is a bounded open set, while the data belong to the "limit spaces", i.e., $f \in L^{N/p}(\Omega)$, $g \in L^{N/(p-1)}(\Omega; \mathbf{R}^N)$ with "sufficiently small" norms, and $\alpha_0 = 0$ is allowed. No sign assumption is made on the term H. The authors of that paper prove an existence result of an unbounded solution which satisfies $e^{\lambda|u|} - 1 \in W_0^{1,p}(\Omega)$ for some $\lambda > 0$. With respect to the results proved in [11] we show that the presence of the term $\alpha_0|u|^{p-2}u$ allows us to drop the assumption of smallness of the source terms f, g, as well as the assumption of boundedness of Ω.

In the case of domains having infinite measures, one can proceed by solving the problem on a sequence Ω_n of bounded sets invading Ω. We look for uniform (with respect to n) estimates of $e^{\lambda|u_n|} - 1$ in $W_0^{1,p}(\Omega_n)$ (for every positive λ), u_n being solutions on Ω_n. This is done in Section 3: to this aim, we cannot use any embedding theorem between L^s spaces, since Ω may have infinite measure, nor any argument involving the measure of Ω_n. As in many of the cited papers, we make use of test functions of exponential growth, which are, in some sense, natural tools to get rid of the term $H(x, u, \nabla u)$ in order to obtain any estimate on u_n.

After that, in Section 4 we pass to the limit in the approximating problems: we need to prove a result of local strong convergence of the gradients ∇u_n, and again this is done through exponential-type functions, using a local adaptation of a technique by Ferone and Murat (see [11]).

In Section 5 we prove that any solution of (1) is bounded if $f \in L^q(\Omega) \cap L^{p'}(\Omega)$, with $q > N/p$ (or a weaker assumption, see (F′) of Section 2) and $g \in L^r(\Omega; \mathbf{R}^N)$, with $r > N/(p-1)$. This is in the spirit of the results of Stampacchia [16] and Boccardo, Murat, Puel [3], but it requires particular care since the techniques of Stampacchia rely heavily on the measure of the domain Ω.

It is worth noticing that, as far as we know, the only result concerning the case of unbounded domains is proved in [9], where the principal part is a quasilinear operator with linear growth ($p = 2$), and $f(x) \in L^2(\Omega) \cap L^\infty(\Omega)$ (see also [10] for results of existence of bounded solutions in the case of degenerate operators).

On the other hand, results concerning sets Ω of infinite measure and terms with growth of order $p-1$ with respect to the gradients have been proved in Bottaro, Marina [4], Lions [14], [15], Chicco, Venturino [6] for the linear setting ($p = 2$) and in Dall'Aglio, De Cicco, Giachetti, Puel [7] for the nonlinear one.

2. Main result

Let Ω be an open subset of $\mathbf{R}^N$, possibly of infinite measure. We are interested in establishing an existence result for the following elliptic problem in Ω:

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) + c(x, u) + H(x, u, \nabla u) = f(x) - \operatorname{div} g(x) \text{ in } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{P}$$

where p satisfies $p > 1$. Let us focus our attention, for the moment, on the case $p < N$. We assume the following hypotheses on the terms which appear in (P):

Assumptions on $a(x, s, \xi)$**:**

(A1) $a(x, s, \xi) = (a_1(x, s, \xi), \ldots, a_N(x, s, \xi)) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function, i.e., it is measurable with respect to x for every $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$, and continuous with respect to (s, ξ) for almost every $x \in \Omega$;

(A2) there exists a constant $\Lambda_1 > 0$ such that

$$|a(x, s, \xi)| \le \Lambda_1 \left(k_1(x) + |s|^{p-1} + |\xi|^{p-1}\right)$$

for almost every $x \in \Omega$ and every $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$, where $k_1(x)$ is a positive function in $L^{p'}(\Omega) \cap L^r_{\mathrm{loc}}(\Omega)$, for some $r > p'$ (here p' denotes Hölder's conjugate exponent of p, defined by $\frac{1}{p} + \frac{1}{p'} = 1$);

(A3) there exists a constant $\alpha > 0$ such that

$$a(x, s, \xi) \cdot \xi \ge \alpha |\xi|^p$$

for almost every $x \in \Omega$ and every $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$;

(A4)
$$[a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) > 0$$

for almost every $x \in \Omega$, for every $s \in \mathbf{R}$ and $\xi, \eta \in \mathbf{R}^N$, with $\xi \neq \eta$.

Remark 1. The assumption $k_1(x) \in L^{p'}(\Omega) \cap L^r_{\mathrm{loc}}(\Omega)$, with $r > p'$, which appears in (A2) instead of the more usual hypothesis $k_1(x) \in L^{p'}(\Omega)$, will be used in the proof of the strong convergence of the gradient of the approximate solutions, in Section 4.

Assumptions on $c(x, s)$**:**

(C1) $c(x, s) : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function;

(C2) there exists a constant $\alpha_0 > 0$ such that, for almost every $x \in \Omega$ and every $s \in \mathbf{R}$

$$c(x, s)s \ge \alpha_0 |s|^p;$$

(C3) there exists a constant $\Lambda_2 > 0$ such that for almost every $x \in \Omega$ and every $s \in \mathbf{R}$

$$|c(x, s)| \le \Lambda_2 \left(k_2(x) + |s|^{p-1}\right),$$

where $k_2(x)$ is a positive function in $L^{p'}(\Omega)$.

Assumptions on $H(x, s, \xi)$**:**

(H1) $H(x, s, \xi) : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is a Carathéodory function;

(H2) there exists a constant $d > 0$ such that for almost every $x \in \Omega$ and every $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$

$$|H(x, s, \xi)| \le d |\xi|^p.$$

Remark 2. One can replace the last inequality with

$$|H(x, s, \xi)| \le d |\xi|^p + h(x),$$

where h satisfies the same assumption as the source term f below.

Assumptions on $f(x)$**:**
$f(x) : \Omega \to \mathbf{R}$ is a measurable function satisfying

$$\text{(F)} \qquad f \in L^{N/p}\big(\{x \in \Omega : |f(x)| > 1\}\big), \quad f \in L^{p'}\big(\{x \in \Omega : |f(x)| \le 1\}\big).$$

From now on we use the compact notation $\{|f| > M\}$ instead of $\{x \in \Omega : |f(x)| > M\}$.

Remark 3. It is easy to check that (F) holds if and only if it holds when 1 is replaced by any $M > 0$.

Remark 4. If p is such that $N/p \ge p'$, then (F) is equivalent to

$$f \in L^{N/p}(\Omega) \cap L^{p'}(\Omega). \tag{2}$$

On the other hand, if $N/p < p'$, then (2) implies (F), but not vice versa.

Remark 5. When Ω has finite measure, (F) means $f \in L^{N/p}(\Omega)$.

Assumptions on $g(x)$**:**
$g(x) : \Omega \to \mathbf{R}^N$ is a measurable function satisfying

$$\text{(G)} \qquad g \in L^{N/(p-1)}(\Omega; \mathbf{R}^N) \cap L^{p'}(\Omega; \mathbf{R}^N).$$

Remark 6. Since $p < N$, assumption (G) is equivalent to

$$|g| \in L^{N/(p-1)}\big(\{|g| > 1\}\big), \quad |g| \in L^{p'}\big(\{|g| \le 1\}\big).$$

The main result we are going to prove is the following:

Theorem 1. *Assume that* $1 < p < N$*, and that hypotheses* (A1)–(A4), (C1)–(C3), (H1)–(H2), (F), (G) *hold. Then there exists a solution* u *of problem* (P) *in the sense that*

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla\psi + \int_\Omega c(x, u)\psi + \int_\Omega H(x, u, \nabla u)\psi = \int_\Omega f\psi - \int_\Omega g \cdot \nabla\psi, \tag{3}$$

for every function $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$*. Moreover* u *satisfies*

$$e^{\lambda|u|} - 1 \in W_0^{1,p}(\Omega), \tag{4}$$

for every $\lambda \ge 0$*.*

Remark 7. In the proof of Theorem 1 we will show that (3) holds for every $\psi \in C_0^\infty(\Omega)$. Therefore, by density, it holds for every $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. In particular, if $\varphi(s) : \mathbf{R} \to \mathbf{R}$ is a locally Lipschitz function satisfying $\varphi(0) = 0$ and $|\varphi'(s)| \leq ce^{\lambda|s|}$ for some $c, \lambda > 0$, we can take $\psi = \varphi(T_k(u))$ in (3), where T_k is the truncation defined by

$$T_k(s) = \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } |s| < k, \\ -k & \text{if } s \leq -k, \end{cases} \tag{5}$$

and then pass to the limit for $k \to \infty$, using (4) and Lebesgue's theorem. It follows that $\psi = \varphi(u)$ is admissible in (3). This will be used for the next result, Theorem 2.

Theorem 1 will be proved by approximating problem (P) with the following problems on the bounded domains $\Omega_n = \Omega \cap B_n(0)$:

$$\begin{cases} -\operatorname{div} a(x, u_n, \nabla u_n) + c(x, u_n) + H_n(x, u_n, \nabla u_n) = f_n - \operatorname{div} g_n \text{ in } \Omega_n, \\ u_n \in W_0^{1,p}(\Omega_n) \cap L^\infty(\Omega_n), \end{cases} \tag{P$_n$}$$

where

$$H_n(x, s, \xi) = T_n(H(x, s, \xi)), \quad f_n(x) = T_n\big(f(x)\big), \quad g_n(x) = \frac{g(x)}{1 + \frac{1}{n}|g(x)|}, \tag{6}$$

and T_n is defined by (5). Since H_n, f_n and g_n are bounded, the existence of a bounded solution of (P$_n$) is classical (see, for instance, Leray, Lions [12] or Lions [13] for the existence, and Stampacchia [16] for the boundedness). Let us remark, moreover, that $|H_n| \leq |H|$, $|f_n| \leq |f|$ and $|g_n| \leq |g|$, so that H_n, f_n and g_n satisfy the same assumptions as H, f and g respectively.

The scheme of the proof of Theorem 1 is a classical one: we first find *a priori* estimates in $W_0^{1,p}(\Omega_n)$ for the functions $e^{\lambda|u_n|} - 1$, where u_n is any solution of (P$_n$). This will be done in Section 3. Then we extract a weakly converging subsequence and we try to pass to the limit in the weak formulation of (P$_n$). In order to do this we need a result of local strong convergence of the gradients, which is proved in Section 4, using a local version of the technique used in [11]. Finally, in Section 5, we prove that, if f and g have higher integrability, or if $p > N$, then every solution u of (P) is bounded. More precisely, we will assume that $1 < p < \infty$, and that (F) and (G) are replaced by

(F′) $\quad f \in L^q\big(\{|f| > 1\}\big)$, for some $q > \max\left\{1, \dfrac{N}{p}\right\}$, $f \in L^{p'}\big(\{|f| \leq 1\}\big)$,

(G′) $\quad g \in L^r(\Omega; \mathbf{R}^N) \cap L^{p'}(\Omega; \mathbf{R}^N)$, for some $r > \dfrac{N}{p-1}$,

respectively.

Remark 8. We point out that, in the case $p > N$, one can take $r = p'$ in (G′), and therefore, in this case (G′) gives $g \in L^{p'}(\Omega; \mathbf{R}^N)$.

We now state the boundedness theorem:

Theorem 2. *Assume that* $1 < p < \infty$*, and that hypotheses* (A1)–(A4), (C1)–(C3), (H1)–(H2), (F′), (G′) *hold. Then every solution u of* (P) *in the sense specified in* (3) *and* (4) *is essentially bounded, and*

$$\|u\|_{L^\infty(\Omega)} \le C\,(N, p, \alpha, \alpha_0, d, f, g)\ . \tag{7}$$

The proof of Theorem 2 relies on the combined use of the well-known technique by Stampacchia (see [16]) and suitable exponential test functions, as in [3].

The previous result can be used as an *a priori* estimate for the approximate solutions u_n in the case $p \ge N$. In this case Theorem 2 shows that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$ (this, by the way, simplifies dramatically the proof of the strong convergence of the gradients, see Remark 10 below). Therefore we can state the following existence result:

Theorem 3. *Assume that* $N \le p < \infty$*, and that hypotheses* (A1)–(A4), (C1)–(C3), (H1)–(H2), (F′), (G′) *hold. Then there exists a solution* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of* (P) *such that* (3) *holds for every* $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

3. A priori estimate

In this section we will prove a uniform estimate for the solutions u_n of (P_n).

Proposition 1. *Under the hypotheses of Theorem* 1*, let* u_n *be any solution of* (P_n)*. Then for every* $\lambda > 0$ *there exists a positive constant* $C = C(N, p, \alpha, \alpha_0, d, f, g, \lambda)$ *such that*

$$\|e^{\lambda|u_n|} - 1\|_{W_0^{1,p}(\Omega_n)} \le C\,. \tag{8}$$

Remark 9. The previous estimate yields an estimate for the functions $e^{\lambda|u_n|}$ in $L^r_{\mathrm{loc}}(\Omega)$ for every r. More precisely, for every $\lambda > 0$, every $r \in [1, +\infty)$ and every set $\Omega_0 \subset\subset \Omega$, one has

$$\|e^{\lambda|u_n|}\|_{L^r(\Omega_0)} \le c(r, \lambda, \Omega_0)\,.$$

Proof of Proposition 1. For simplicity of notation we will always omit the index n of the sequence. For positive λ, let us define the function $\varphi(s) = \varphi_\lambda(s) : \mathbf{R} \to \mathbf{R}$ by

$$\varphi(s) = \left(e^{\lambda|s|} - 1\right) \operatorname{sign} s\,. \tag{9}$$

We take $\varphi(G_k(u))$ as test function in (P_n), where

$$G_k(s) = s - T_k(s) = \begin{cases} s - k & \text{if } s > k, \\ 0 & \text{if } |s| \le k, \\ k - s & \text{if } s < -k, \end{cases} \tag{10}$$

and k will be specified later. Using hypotheses (A3), (C2), (H2), we obtain

$$\begin{aligned}
&\alpha \int_\Omega |\nabla G_k(u)|^p \varphi'(G_k(u)) + \alpha_0 \int_\Omega |u|^{p-1} |\varphi(G_k(u))| \\
&\le d \int_\Omega |\nabla G_k(u)|^p |\varphi(G_k(u))| + \int_\Omega |f|\, |\varphi(G_k(u))| \\
&\quad + \int_\Omega |g|\, |\nabla G_k(u)|\, \varphi'(G_k(u)) \\
&= I + J + K\,.
\end{aligned} \tag{11}$$

It is easy to check that, if λ satisfies

$$\lambda \ge \frac{8d}{\alpha}\,, \tag{12}$$

then

$$d|\varphi_\lambda(s)| \le \frac{\alpha}{8} \varphi'_\lambda(s)\,, \qquad \text{for every } s \in \mathbf{R}. \tag{13}$$

Therefore

$$I \le \frac{\alpha}{8} \int_\Omega |\nabla G_k(u)|^p \varphi'(G_k(u))\,.$$

We now estimate the integral J, by splitting it as follows:

$$\begin{aligned}
J = &\int_{\{|f|>H\,,\,|G_k(u)|\ge 1\}} |f||\varphi(G_k(u))| \\
&\quad + \int_{\{|f|>H\,,\,|G_k(u)|<1\}} |f||\varphi(G_k(u))| + \int_{\{|f|\le H\}} |f||\varphi(G_k(u))| \\
= &\; J_1 + J_2 + J_3\,,
\end{aligned}$$

where H is a positive constant to be chosen later. Before estimating J_1, we remark that, by Sobolev's embedding,

$$\begin{aligned}
&\int_\Omega |\nabla G_k(u)|^p \varphi'(G_k(u)) \\
&= \int_\Omega |\nabla \Psi(G_k(u))|^p \ge c_1(N,p) \left[\int_\Omega \Big(\Psi(G_k(u)) \Big)^{p^*} \right]^{p/p^*}\,,
\end{aligned} \tag{14}$$

where $p^* = pN/(N-p)$ is the Sobolev exponent relative to p, and

$$\Psi(s) = \int_0^{|s|} \varphi'(t)^{1/p}\, dt = \frac{p}{\lambda^{1/p'}} \left(e^{\lambda|s|/p} - 1 \right). \tag{15}$$

Moreover, we observe that there exists a positive constant $c_2 = c_2(p,\lambda)$ such that

$$|\varphi(s)| \le c_2 (\Psi(s))^p \qquad \text{for every } s \text{ such that } |s| \ge 1. \tag{16}$$

Therefore, by Hölder's inequality, the term J_1 can be estimated as follows (see Remark 3)

$$
\begin{aligned}
J_1 &\le \|f\|_{L^{N/p}(\{|f|>H\})} \left[\int\limits_{\{|G_k(u)|\ge 1\}} |\varphi(G_k(u))|^{\frac{N}{N-p}} \right]^{\frac{N-p}{N}} \\
&\le c_2 \|f\|_{L^{N/p}(\{|f|>H\})} \left[\int\limits_{\{|G_k(u)|\ge 1\}} \big(\Psi(G_k(u))\big)^{p^*} \right]^{\frac{p}{p^*}} .
\end{aligned}
$$

We choose $H = H(N, p, \alpha, f, \lambda)$ large enough so that

$$
c_2 \|f\|_{L^{N/p}(\{|f|>H\})} \le \frac{c_1 \alpha}{8} , \tag{17}
$$

and, therefore, J_1 satisfies

$$
J_1 \le \frac{c_1 \alpha}{8} \left[\int\limits_{\{|G_k(u)|\ge 1\}} \big(\Psi(G_k(u))\big)^{p^*} \right]^{\frac{p}{p^*}} .
$$

On the other hand

$$
J_2 \le \varphi(1) \int\limits_{\{|f|>H\}} |f| \le \frac{\varphi(1)}{H^{\frac{N-p}{p}}} \int\limits_{\{|f|>H\}} |f|^{N/p} .
$$

Finally

$$
J_3 \le H \int\limits_{\{|f|\le H\}} |\varphi(G_k(u))| ,
$$

therefore, if we choose $k = k(N, p, \alpha, \alpha_0, f, \lambda)$ such that

$$
\alpha_0 k^{p-1} \ge 4H , \tag{18}
$$

we can write

$$
J_3 \le \frac{\alpha_0}{4} \int_\Omega |u|^{p-1} |\varphi(G_k(u))| .
$$

We now estimate the term K. By Young's inequality

$$
K \le \frac{\alpha}{8} \int_\Omega |\nabla G_k(u)|^p \varphi'(G_k(u)) + c_3(p,\alpha) \int_\Omega |g|^{p'} \varphi'(G_k(u)) = K_1 + K_2 .
$$

The integral K_2 can be estimated as follows:

$$
\begin{aligned}
K_2 &\le c_3 \lambda e^\lambda \int_\Omega |g|^{p'} + c_3 \int\limits_{\{|g|>\tilde{H}, |G_k(u)|>1\}} |g|^{p'} \varphi'(G_k(u)) + c_3 \tilde{H}^{p'} \int\limits_{\{|g|\le\tilde{H}, |G_k(u)|>1\}} \varphi'(G_k(u)) \\
&= K_{2,1} + K_{2,2} + K_{2,3} ,
\end{aligned}
$$

where $\tilde{H}$ is a positive number to be chosen hereafter. Since $\varphi'(s) \le c(\lambda, p)(\Psi(s))^p$ for every s such that $|s| > 1$, one has, by Hölder's inequality,

$$K_{2,2} \le c_4(p, \alpha, \lambda) \left[\int_{\{|g|>\tilde{H}\}} |g|^{N/(p-1)} \right]^{\frac{p}{N}} \left[\int_{\{|G_k(u)|\ge 1\}} \big(\Psi(G_k(u))\big)^{p^*} \right]^{\frac{p}{p^*}} .$$

Choosing $\tilde{H} = \tilde{H}(N, p, \alpha, g, \lambda)$ large enough, so that

$$c_4(p, \alpha, \lambda) \left[\int_{\{|g|>\tilde{H}\}} |g|^{N/(p-1)} \right]^{\frac{p}{N}} < \frac{\alpha c_1}{8} , \tag{19}$$

we obtain

$$K_{2,2} \le \frac{\alpha c_1}{8} \left[\int_\Omega |\Psi(G_k(u))|^{p^*} \right]^{\frac{p}{p^*}} .$$

Finally, using inequality

$$\varphi'(s) \le c_5(\lambda)|\varphi(s)| , \qquad \text{for every } s \text{ such that } |s| \ge 1, \tag{20}$$

and choosing $k = k(p, \alpha, \alpha_0, \lambda, \tilde{H})$ such that

$$\frac{\alpha_0 k^{p-1}}{4} \ge c_3 c_5 \tilde{H}^{p'} , \tag{21}$$

we obtain

$$K_{2,3} \le \frac{\alpha_0}{4} \int_\Omega |u|^{p-1} |\varphi(G_k(u))| .$$

In conclusion, putting all the estimates together, we get

$$\begin{aligned} &\frac{\alpha}{2} \int_\Omega |\nabla G_k(u)|^p \varphi'(G_k(u)) + \frac{\alpha_0}{2} \int_\Omega |u|^{p-1} |\varphi(G_k(u))| \\ &\le \frac{c_6(\lambda)}{H^{(N-p)/p}} \int_{\{|f|>H\}} |f|^{N/p} + c_7(\alpha, \lambda) \int_\Omega |g|^{p'} = c_8(N, p, \alpha, \alpha_0, f, g, \lambda), \end{aligned} \tag{22}$$

for every λ, H, k satisfying (12), (17), (18) and (21), where $\tilde{H}$ verifies (19). Note that (22) implies an estimate in $W_0^{1,p}(\Omega)$ for $G_k(u)$, when k is large enough.

We now fix λ, H, k such that (22) holds, and we use $\varphi(T_k(u))$ as a test function in (P_n). Then we obtain

$$\begin{aligned} &\alpha \int_\Omega |\nabla T_k(u)|^p \varphi'(T_k(u)) + \alpha_0 \int_\Omega |u|^{p-1} |\varphi(T_k(u))| \\ &\le d \int_\Omega |\nabla T_k(u)|^p |\varphi(T_k(u))| + d\varphi(k) \int_\Omega |\nabla G_k(u)|^p \\ &\quad + \int_\Omega |f| |\varphi(T_k(u))| + \int_\Omega |g| |\nabla T_k(u)| \varphi'(T_k(u)) \\ &= L_1 + L_2 + L_3 + L_4 . \end{aligned} \tag{23}$$

As before, by (13),

$$L_1 \le \frac{\alpha}{4} \int_\Omega |\nabla T_k(u)|^p \varphi'(T_k(u)) \,, \tag{24}$$

while, using (22), we obtain

$$L_2 \le c_9(N, p, \alpha, \alpha_0, f, g, \lambda) \,. \tag{25}$$

Let us remark that the integral L_3 is very easy to estimate if meas(Ω) is finite, or more generally if $f \in L^1(\Omega)$, since in this case

$$L_3 \le \varphi(k) \, \|f\|_{L^1(\Omega)} \,.$$

In the general case, we write

$$\begin{aligned} L_3 &= \int_{\{|f|>1\}} |f| \, |\varphi(T_k(u))| + \int_{\{|f|\le 1\}} |f| \, |\varphi(T_k(u))| \\ &\le \varphi(k) \int_{\{|f|>1\}} |f| + \varepsilon \int_\Omega |\varphi(T_k(u))|^p + c(\varepsilon) \int_{\{|f|\le 1\}} |f|^{p'} \,, \end{aligned} \tag{26}$$

where ε will be chosen hereafter. Since

$$|\varphi(T_k(u))|^p \le c_{10}(p, \lambda, k)|u|^{p-1}|\varphi(T_k(u))| \,,$$

choosing ε such that $\varepsilon c_{10} < \alpha_0/2$, we obtain

$$L_3 \le \frac{\alpha_0}{2} \int_\Omega |u|^{p-1}|\varphi(T_k(u))| + c_{11}(p, \alpha_0, f, \lambda, k) \,. \tag{27}$$

Finally, one has

$$L_4 \le \frac{\alpha}{4} \int_\Omega |\nabla T_k(u)|^p \varphi'(T_k(u)) + c_{12}(\alpha, \lambda, k) \int_\Omega |g|^{p'} \,. \tag{28}$$

Putting all the inequalities (23)–(28) together, we get

$$\frac{\alpha}{2} \int_\Omega |\nabla T_k(u)|^p \varphi'(T_k(u)) + \frac{\alpha_0}{2} \int_\Omega |u|^{p-1}|\varphi(T_k(u))| \le c_{13}(N, p, \alpha, \alpha_0, f, g, \lambda, k) \,. \tag{29}$$

Let us show that (22) and (29) imply estimate (8). Indeed they yield

$$\int_{\{|u|\le k\}} |\nabla u|^p e^{\lambda|u|} \le c_{14} \,, \qquad \int_{\{|u|>k\}} |\nabla u|^p e^{\lambda(|u|-k)} \le c_{14} \,,$$

for every λ, k large enough (see (12), (17) and (18)), where c_{14} depends on λ, k and the data. Since

$$\int_\Omega |\nabla u|^p e^{\lambda|u|} = \int_{\{|u|\le k\}} |\nabla u|^p e^{\lambda|u|} + e^{k\lambda} \int_{\{|u|>k\}} |\nabla u|^p e^{\lambda(|u|-k)} \le c_{14}(1 + e^{k\lambda}) = c_{15} \,,$$

if we fix the value of k (depending on λ, see (17), (18), (19) and (21)), we obtain an estimate on $\nabla(e^{\lambda|u|/p}-1)$ in $L^p(\Omega)$ (depending on λ). This implies, by Sobolev's embedding, that

$$\int_\Omega \left(e^{\lambda|u|/p}-1\right)^{p^*} \le c_{16}\,, \tag{30}$$

for every $\lambda \ge 8d/\alpha$ (and *a fortiori* for every $\lambda > 0$), where c_{16} depends on λ and on the data of the problem. Note that (30) does not imply an estimate in $L^p(\Omega)$ for $e^{\lambda|u|/p}-1$, since meas(Ω) may be infinite. To obtain such an estimate, we have to combine (29) and (30), since, for every $k > 0$, one has the inequalities

$$\int\limits_{\{|u|\le k\}} \left(e^{\lambda|u|/p}-1\right)^p \le c_{17}(p,\lambda,k)\int_\Omega |u|^{p-1}|\varphi(T_k(u))|\,,$$

$$\int\limits_{\{|u|>k\}} \left(e^{\lambda|u|/p}-1\right)^p \le c_{18}(p,\lambda,k)\int_\Omega \left(e^{\lambda|u|/p}-1\right)^{p^*}\,.$$

Therefore, if $k = k(\lambda)$ is such that (29) holds, we can write

$$\int_\Omega \left(e^{\lambda|u|/p}-1\right)^p = \int\limits_{\{|u|\le k\}} \left(e^{\lambda|u|/p}-1\right)^p + \int\limits_{\{|u|>k\}} \left(e^{\lambda|u|/p}-1\right)^p \le c_{19}\,, \tag{31}$$

where c_{19} depends on λ and on the data of the problem. □

4. Strong convergence and proof of the main theorem

This section is devoted to the proof of Theorem 1. Let $\{u_n\}$ be any sequence of solutions of problems (P_n); we extend them to zero in $\Omega \setminus \Omega_n$. By (8), there exist a subsequence (still denoted by $\{u_n\}$) and a function $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{32}$$

We wish to show that u is a solution of (P) in the sense of (3) and (4). The main difficulty to be overcome consists of proving the strong convergence of the gradients of u_n in $L^p_{\mathrm{loc}}(\Omega;\mathbf{R}^N)$; to this aim we follow the technique used by Ferone and Murat in [11].

Proposition 2. *For every open set $\Omega_0 \subset\subset \Omega$,*

$$\nabla u_n \to \nabla u \qquad \textit{strongly in } L^p\big(\Omega_0;\mathbf{R}^N\big). \tag{33}$$

Proof. We limit ourselves to the case where $g \equiv 0$, since the additional term $-\mathrm{div}\, g$ in (P) can be treated easily.

Step 1. We will show that, for every $k > 0$,

$$\nabla T_k(u_n) \to \nabla T_k(u) \qquad \text{strongly in } L^p\big(\Omega_0; \mathbf{R}^N\big). \tag{34}$$

This will be proved if we show that

$$\lim_{n\to+\infty} \int_{\Omega_0} [a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))] \cdot (\nabla T_k(u_n) - \nabla T_k(u))\, dx = 0 \tag{35}$$

(see [12]). Let $\Psi(x)$ be a cut-off function such that

$$\Psi \in C_0^\infty(\Omega)\,, \quad 0 \le \Psi(x) \le 1\,, \quad \Psi \equiv 1 \text{ in } \Omega_0.$$

We define

$$z_n(x) = T_k(u_n) - T_k(u)\,;$$

we know that $\nabla z_n \rightharpoonup 0$ weakly in $L^p(\Omega; \mathbf{R}^N)$, and $z_n \to 0$ strongly in $L^p_{\text{loc}}(\Omega)$. We take

$$w = \varphi(z_n) e^{\delta|u_n|} \Psi \tag{36}$$

as a test function in (P_n), where $\varphi = \varphi_\lambda$ is defined by (9), and the positive constants λ, δ will be chosen below. We will always omit the explicit dependence on x of $a(x, s, \xi)$ and $c(x, s)$. Using (C2) and (H2), we obtain

$$\begin{aligned}
&\int_\Omega a(u_n, \nabla u_n) \cdot \nabla z_n \varphi'(z_n) e^{\delta|u_n|}\Psi + \int_\Omega c(u_n)\varphi(z_n) e^{\delta|u_n|}\Psi \\
&\le d \int_\Omega |\nabla u_n|^p |\varphi(z_n)| e^{\delta|u_n|}\Psi + \int_\Omega |f||\varphi(z_n)| e^{\delta|u_n|}\Psi \\
&\quad - \delta \int_\Omega a(u_n, \nabla u_n) \cdot \nabla u_n \varphi(z_n) e^{\delta|u_n|} \text{sign}(u_n)\Psi \\
&\quad + \int_\Omega |a(u_n, \nabla u_n)|\, |\nabla\Psi|\, |\varphi(z_n)| e^{\delta|u_n|} \\
&= C_n + D_n + E_n + F_n\,.
\end{aligned} \tag{37}$$

Moreover, we set

$$A_n = \int_\Omega a(u_n, \nabla u_n) \cdot \nabla z_n \varphi'(z_n) e^{\delta|u_n|}\Psi\,, \qquad B_n = \int_\Omega c(u_n)\varphi(z_n) e^{\delta|u_n|}\Psi\,.$$

We have

$$\begin{aligned}
A_n = & \int_{\{|u_n|\le k\}} a(T_k(u_n), \nabla T_k(u_n)) \cdot \nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\Psi \\
& + \int_{\{|u_n|>k\}} a(u_n, \nabla u_n) \cdot \nabla z_n \varphi'(z_n) e^{\delta|u_n|}\Psi \\
= & \int_{\{|u_n|\le k\}} \big[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))\big] \cdot \nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\Psi \\
& + \int_{\{|u_n|\le k\}} a(T_k(u_n), \nabla T_k(u)) \cdot \nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\Psi \\
& + \int_{\{|u_n|>k\}} a(u_n, \nabla u_n) \cdot \nabla z_n \varphi'(z_n) e^{\delta|u_n|}\Psi \\
= & A_n^{(1)} + A_n^{(2)} + A_n^{(3)} .
\end{aligned}$$

Let us show that $A_n^{(2)}$ tends to zero. Indeed $A_n^{(2)} = \int_\Omega \mu_n \cdot \nabla z_n$, where

$$\mu_n = a(T_k(u_n), \nabla T_k(u))\varphi'(z_n) e^{\delta|T_k(u_n)|}\Psi\chi_{\{|u_n|\le k\}} .$$

Since $\nabla z_n \to 0$ weakly in $L^p(\Omega; \mathbf{R}^N)$, it is enough to show that $\mu_n \to \mu$ strongly in $L^{p'}(\Omega; \mathbf{R}^N)$, where $\mu = a(T_k(u), \nabla T_k(u))\varphi'(0)e^{\delta|T_k(u)|}\Psi\chi_{\{|u|\le k\}}$. Indeed, $\mu_n \to \mu$ almost everywhere in Ω: the only difficulty is on the set where $|u(x)| = k$, but for almost every x in this set $a(T_k(u_n), \nabla T_k(u)) = a(T_k(u_n), 0) = 0 = a(T_k(u), \nabla T_k(u))$. Moreover, by (A2),

$$|\mu_n| \le \Lambda_1\big(k_1(x) + k^{p-1} + |\nabla u|^{p-1}\big)\varphi'(2k)e^{\delta k}\Psi ,$$

which is a fixed function in $L^{p'}(\Omega)$. Therefore the strong convergence of μ_n follows from Lebesgue's theorem. Similarly, $A_n^{(3)} \to 0$, since $\nabla z_n \chi_{\{|u_n|>k\}} = -\nabla T_k(u)\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^p(\Omega; \mathbf{R}^N)$, while $a(u_n, \nabla u_n)\varphi'(z_n)e^{\delta|u_n|}\Psi$ is bounded in $L^{p'}(\Omega; R^N)$, by hypothesis (A2), estimate (8) and Remark 9. Therefore, we have proved that

$$A_n = A_n^{(1)} + o(1) . \tag{38}$$

Let us examine the term B_n:

$$\begin{aligned}
B_n = & \int_{\{|u_n|\le k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\Psi + \int_{\{|u_n|>k\}} c(u_n)\varphi(z_n)e^{\delta|u_n|}\Psi \\
\ge & \int_{\{|u_n|\le k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\Psi ,
\end{aligned}$$

since $\varphi(z_n)$ has the same sign as $c(u_n)$ on the set where $|u_n| > k$. On the other hand, the last integral goes to zero, since the integrand converges pointwise and is bounded by $\Lambda_2(k_2(x) + k^{p-1})\varphi(2k)e^{\delta k}\Psi$. Therefore, we have proved that

$$B_n \geq o(1)\,. \tag{39}$$

Moreover,

$$\begin{aligned} C_n + E_n \leq\ & \frac{d}{\alpha}\int_\Omega a(u_n, \nabla u_n)\cdot\nabla u_n|\varphi(z_n)|e^{\delta|u_n|}\Psi \\ & - \delta\int_\Omega a(u_n, \nabla u_n)\cdot\nabla u_n\varphi(z_n)e^{\delta|u_n|}\operatorname{sign} u_n\Psi \\ \leq\ & \left(\frac{d}{\alpha}+\delta\right)\int_{\{|u_n|\leq k\}} a(T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u_n)|\varphi(z_n)|e^{\delta|T_k(u_n)|}\Psi \\ & + \left(\frac{d}{\alpha}-\delta\right)\int_{\{|u_n|>k\}} a(u_n, \nabla u_n)\cdot\nabla u_n|\varphi(z_n)|e^{\delta|u_n|}\Psi\,, \end{aligned} \tag{40}$$

since $\varphi(z_n)\operatorname{sign} u_n = |\varphi(z_n)|$ on the set $\{|u_n| > k\}$. We first fix δ such that

$$\delta > \frac{d}{\alpha}\,,$$

so that the last term of (40) is negative. Therefore,

$$\begin{aligned} C_n + E_n \leq\ & \left(\frac{d}{\alpha}+\delta\right)\int_{\{|u_n|\leq k\}} a(T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u_n)|\varphi(z_n)|e^{\delta|T_k(u_n)|}\Psi \\ =\ & \left(\frac{d}{\alpha}+\delta\right)\int_{\{|u_n|\leq k\}} \big[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))\big] \\ & \qquad\qquad\qquad\qquad\cdot\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\Psi \\ & + \left(\frac{d}{\alpha}+\delta\right)\int_{\{|u_n|\leq k\}} a(T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u)|\varphi(z_n)|e^{\delta|T_k(u_n)|}\Psi \\ & + \left(\frac{d}{\alpha}+\delta\right)\int_{\{|u_n|\leq k\}} a(T_k(u_n), \nabla T_k(u))\cdot\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\Psi\,. \end{aligned}$$

It is easy to see that the last two integrals converge to zero as $n \to \infty$. We now choose λ such that

$$\lambda \geq 2\left(\frac{d}{\alpha}+\delta\right)\,,$$

so that

$$\left(\frac{d}{\alpha}+\delta\right)|\varphi(s)| \leq \frac{\varphi'(s)}{2}\,, \qquad \text{for every } s \in \mathbf{R},$$

and therefore,

$$C_n + E_n \leq \frac{1}{2} A_n^{(1)} + o(1) . \tag{41}$$

As for the two remaining terms D_n and F_n, it is easy to see that

$$D_n \to 0 , \qquad F_n \to 0 ; \tag{42}$$

for the term D_n, use Remark 9, and for the term F_n, observe that $|\nabla\Psi|\,|\varphi(z_n)|$ converges strongly to zero in $L^r(\Omega)$ for every $r \geq 1$, while $|a(u_n, \nabla u_n)|e^{\delta|u_n|}$ is bounded in $L^{p'}_{\mathrm{loc}}(\Omega)$ (see (A2) and (8)).

From (37), (38), (39), (41), (42) we obtain that

$$A_n^{(1)} = \int_{\{|u_n|\leq k\}} \big[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))\big] \cdot \nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\Psi \to 0 . \tag{43}$$

On the other hand it is easy to see that

$$\int_{\{|u_n|>k\}} \big[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))\big]\nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\Psi = \int_{\{|u_n|>k\}} a(k, \nabla T_k(u)) \cdot \nabla T_k(u)\varphi'(-T_k(u))e^{\delta k}\Psi \quad \to \quad 0 . \tag{44}$$

Convergence (35) follows from (43) and (44). This proves convergence (34).

Step 2. Let us prove that

$$\sup_n \int_\Omega |\nabla G_k(u_n)|^p \overset{k\to\infty}{\longrightarrow} 0 , \tag{45}$$

where G_k is defined by (10). From estimate (22), which holds for all u_n, we obtain

$$\int_\Omega |\nabla G_k(u_n)|^p \leq \frac{1}{\lambda} \int_\Omega |\nabla G_k(u_n)|^p \varphi'(G_k(u_n)) \leq \frac{c(\alpha, \lambda)}{H^{\frac{N-p}{p}}} \int_{\{|f|>H\}} |f|^{N/p} . \tag{46}$$

This holds for every λ satisfying (12), for every H sufficiently large to verify (17), and for every k such that

$$\alpha_0 k^{p-1} \geq 2H \tag{47}$$

(see (18)). If η is an arbitrary positive number, let us choose H such that (17) holds and the right-hand side of (46) is smaller than η. It follows that, for every k satisfying (47) and every $n \in \mathbf{N}$,

$$\int_\Omega |\nabla G_k(u_n)|^p < \eta ,$$

which proves (45).

Step 3. Let Ω_0 be an open set compactly contained in Ω, and let $\eta > 0$. Since

$$\begin{aligned}&\|\nabla u_n - \nabla u\|_{L^p(\Omega_0;\mathbf{R}^N)}\\&\le \|\nabla T_k(u_n) - \nabla T_k(u)\|_{L^p(\Omega_0;\mathbf{R}^N)} + \|\nabla G_k(u_n)\|_{L^p(\Omega;\mathbf{R}^N)} + \|\nabla G_k(u)\|_{L^p(\Omega;\mathbf{R}^N)},\end{aligned}$$

using (45), we can choose k such that the last two terms are both smaller than $\eta/3$, for every $n \in \mathbf{N}$. Once k is fixed, by (34), the first term can be made smaller than $\eta/3$ choosing n large enough. This proves (33). □

Using Proposition 2, it is now very easy to pass to the limit in the distributional formulation of problem (P_n), obtaining (3). Finally, statement (4) follows easily from (8) and Proposition 2, using Fatou's lemma.

Remark 10. If the approximate solutions u_n are uniformly bounded in L^∞ (this happens, for instance, under the assumptions of Theorem 2), then the proof of the strong convergence of the gradients can be achieved in a simpler way, indeed it is sufficient to take $w = \varphi(u - u_n)\Psi$ in (36) (thus avoiding the use of truncations and taking $\delta = 0$).

5. Boundedness of solutions

This section is devoted to the proof of Theorem 2. We will use an adaptation of a classical technique due to Stampacchia. We need the following lemma (see [16]):

Lemma 1. *Let ϕ be a non-negative, non-increasing function defined on the half-line $[k_0, \infty)$. Suppose that there exist positive constants A, γ, β, with $\beta > 1$, such that*

$$\phi(h) \le \frac{A}{(h-k)^\gamma}\phi(k)^\beta,$$

for every $h > k \ge k_0$. Then $\phi(k) = 0$ for every $k \ge k_1$, where

$$k_1 = k_0 + A^{1/\gamma}2^{\beta/(\beta-1)}\phi(k_0)^{(\beta-1)/\gamma}\,.$$

Proof of Theorem 2. In what follows we will denote the constants which appear in the formulas as $\tilde{c}_i$, $i = 1, 2, \ldots$ Let us begin with the case $p < N$. Following the method used in the first part of the proof of Proposition 1 (see (22)), one easily obtains an estimate for $\int_\Omega |u|^{p-1}|\varphi_\lambda(G_{k_0}(u))|$, when $k_0 = k_0(\lambda)$ is large enough. This implies that, for some larger value of k_0,

$$\operatorname{meas}(A_{k_0}) \le 1\,, \tag{48}$$

where we have set

$$A_k = \{x \in \Omega : |u(x)| > k\}\,.$$

Moreover, with the same choice of test function $\varphi(G_k(u))$, with $\lambda = 4d/\alpha$, and $k \geq k_0(\lambda)$, using (A3), (C2) and (H2) one has

$$\begin{aligned} &\alpha \int_{A_k} |\nabla G_k(u)|^p \varphi'(G_k(u)) + \alpha_0 k^{p-1} \int_{A_k} |\varphi(G_k(u))| \\ &\leq d \int_{A_k} |\nabla G_k(u)|^p |\varphi(G_k(u))| + \int_{A_k \cap \{|f|>1\}} |f|\,|\varphi(G_k(u))| \\ &\quad + \int_{A_k \cap \{|f|\leq 1\}} |\varphi(G_k(u))| + \int_{A_k} |g|\, \varphi'(G_k(u)) |\nabla G_k(u))| \,. \end{aligned} \tag{49}$$

As in the proof of Proposition 1, since $\lambda = 4d/\alpha$, the first term of the right-hand side of (49) is absorbed by the first term of the left-hand side. Choosing k_0 such that

$$\alpha_0 k_0^{p-1} \geq 2\,, \tag{50}$$

we can get rid of the term

$$\int_{A_k \cap \{|f|\leq 1\}} |\varphi(G_k(u))| \,.$$

Moreover, one has

$$\begin{aligned} &\int_{A_k} |g| \varphi'(G_k(u)) |\nabla G_k(u))| \\ &\leq \frac{\alpha}{4} \int_{A_k} |\nabla G_k(u)|^p \varphi'(G_k(u)) + \tilde{c}_1(\alpha) \int_{A_k} |g|^{p'}\, \varphi'(G_k(u)) \,. \end{aligned}$$

This gives

$$\begin{aligned} &\frac{\alpha}{2} \int_{\Omega} |\nabla G_k(u)|^p \varphi'(G_k(u)) + \frac{\alpha_0 k^{p-1}}{2} \int_{A_k} |\varphi(G_k(u))| \\ &\leq \int_{A_k \cap \{|f|>1\}} |f|\,|\varphi(G_k(u))| + \tilde{c}_1 \int_{A_k} |g|^{p'}\, \varphi'(G_k(u)) \,. \end{aligned} \tag{51}$$

By (51), using (14) and (15) of Section 3, it is easy to check that

$$\begin{aligned} &\tilde{c}_2 \left[\int_{A_k} \big(\Psi(G_k(u)) \big)^{p^*} \right]^{\frac{p}{p^*}} + \frac{\alpha_0 k^{p-1}}{2} \int_{A_k} |\varphi(G_k(u))| \\ &\leq \int_{(A_k \setminus A_{k+1}) \cap \{|f|>1\}} |f|\,|\varphi(G_k(u))| + \int_{A_{k+1} \cap \{|f|>1\}} |f|\,|\varphi(G_k(u))| \\ &\quad + \tilde{c}_1 \varphi'(1) \int_{A_k \setminus A_{k+1}} |g|^{p'} + \tilde{c}_1 \int_{A_{k+1}} |g|^{p'} \varphi'(G_k(u)) \,, \end{aligned} \tag{52}$$

where Ψ is defined by (15), and $\tilde{c}_2 = \alpha c_1/2$, where $c_1(N, p)$ is the constant which appears in (14). Let us estimate the right-hand side of (52): one has

$$\int_{(A_k \setminus A_{k+1})\cap\{|f|>1\}} |f|\,|\varphi(G_k(u))| \leq \varphi(1) \int_{A_k\cap\{|f|>1\}} |f| \leq \varphi(1)\,\|f\|_{L^q(\{|f|>1\})} (\mathrm{meas}(A_k))^{1/q'} .$$

On the other hand, since, by (F′), $1 < q' < p^*/p$, using Hölder's and the interpolation inequalities and (16), we can write

$$\begin{aligned}
&\int_{A_{k+1}\cap\{|f|>1\}} |f|\,|\varphi(G_k(u))| \leq \|f\|_{L^q(\{|f|>1\})} \|\varphi(G_k(u))\|_{L^{q'}(A_{k+1})} \\
&\leq \|f\|_{L^q(\{|f|>1\})} \|\varphi(G_k(u))\|^{\frac{N}{pq}}_{L^{p^*/p}(A_{k+1})} \|\varphi(G_k(u))\|^{1-\frac{N}{pq}}_{L^1(A_{k+1})} \\
&\leq \frac{\tilde{c}_2}{4} \|\Psi(G_k(u))\|^p_{L^{p^*}(A_k)} + \tilde{c}_3 \|f\|^{\frac{pq}{pq-N}}_{L^q(\{|f|>1\})} \|\varphi(G_k(u))\|_{L^1(A_{k+1})} ,
\end{aligned}$$

where $\tilde{c}_3 = \tilde{c}_3(N, p, q, \alpha, \lambda)$. Therefore, choosing k_0 such that

$$\frac{\alpha_0 k_0^{p-1}}{2} \geq \tilde{c}_3 \|f\|^{\frac{pq}{pq-N}}_{L^q(\{|f|>1\})} , \tag{53}$$

the second integral in the right-hand side of (52) can be absorbed by the left-hand side. Finally, as far as the last two terms in (52) are concerned, one has, with similar calculations, using inequality (20),

$$\tilde{c}_1 \varphi'(1) \int_{A_k} |g|^{p'} \leq \tilde{c}_1 \varphi'(1) \|g\|^{p'}_{L^r(\Omega;\mathbf{R}^N)} (\mathrm{meas}(A_k))^{1-p'/r} ,$$

$$\begin{aligned}
&\tilde{c}_1 \int_{A_{k+1}} |g|^{p'} \varphi'(G_k(u)) \\
&\leq \frac{\tilde{c}_2}{4} \|\Psi(G_k(u))\|^p_{L^{p^*}(A_k)} + \tilde{c}_4(N, p, \alpha, \lambda)\, \|g\|^{\frac{rp}{r(p-1)-N}}_{L^r(\Omega;\mathbf{R}^N)} \|\varphi(G_k(u))\|_{L^1(A_k)} .
\end{aligned}$$

Therefore, by taking k_0 satisfying (48), (50), (53) and the further condition

$$\frac{\alpha_0 k_0^{p-1}}{2} \geq \tilde{c}_4(N, p, \alpha, \lambda)\, \|g\|^{\frac{rp}{r(p-1)-N}}_{L^r(\Omega;\mathbf{R}^N)} , \tag{54}$$

one obtains, for every $k \geq k_0$,

$$\begin{aligned}
&\frac{\tilde{c}_2}{2} \left[\int_{A_k} \Big(\Psi(G_k(u))\Big)^{p^*} \right]^{\frac{p}{p^*}} \\
&\leq \varphi(1) \|f\|_{L^q(\{|f|>1\})} (\mathrm{meas}(A_k))^{1/q'} + \tilde{c}_1 \varphi'(1) \|g\|^{p'}_{L^r(\Omega;\mathbf{R}^N)} (\mathrm{meas}(A_k))^{1-p'/r} \\
&\leq \tilde{c}_5(p, \alpha, f, g, \lambda) (\mathrm{meas}(A_k))^m ,
\end{aligned}$$

where $m = \min\{1/q', 1 - p'/r\}$ (here we have used (48)). We now take $h > k$ and recall that there exists $\tilde{c}_6 = \tilde{c}_6(\lambda, p)$ such that $|\Psi(s)| \geq \tilde{c}_6|s|$ for every $s \in \mathbf{R}$, so that

$$\int_{A_k} |\Psi(G_k(u))|^{p^*} \geq \int_{A_h} |\Psi(G_k(u))|^{p^*} \geq [\tilde{c}_6(h-k)]^{p^*} \operatorname{meas}(A_h)\,.$$

Then it follows from (55) that

$$\operatorname{meas}(A_h) \leq \frac{\tilde{c}_7}{(h-k)^{p^*}} \left(\operatorname{meas}(A_k)\right)^{mp^*/p}$$

for every h and k such that $h > k \geq k_0$, where $\tilde{c}_7 = \tilde{c}_7(N, p, \alpha, d, f, g)$. Since, by (F′), (G′),

$$\frac{mp^*}{p} > 1\,,$$

Lemma 1 applied to the function $\phi(h) = \operatorname{meas}(A_h)$, for k_0 satisfying (48), (50), (53) and (54), gives

$$\|u\|_{L^\infty(\Omega)} \leq C_1 = C_1\,(N, p, \alpha, k_0, d, f, g)\,.$$

In the case $p = N$, one can repeat the same choice of test functions in (P) (i.e., $\varphi_\lambda(G_k(u))$), since one can check that the left-hand side of (51) is greater than

$$\frac{\alpha}{2}\int_\Omega |\nabla\Psi(G_k(u))|^N + \frac{\tilde{c}_8(N,\lambda)\alpha_0}{2}\int_\Omega |\Psi(G_k(u))|^N\,,$$

where Ψ is the function defined in (15). Using the embedding of $W^{1,N}(\mathbf{R}^N)$ into $L^s(\mathbf{R}^N)$, which holds for every $s \geq N$, one can easily follow the same arguments used above to obtain the conclusion.

Let us show briefly how the proof can be achieved in the case $p > N$. For sake of brevity, we take $g \equiv 0$. Once again, using $\varphi_\lambda(G_k(u_n))$ as a test function in (P_n), with $\lambda = 2d/\alpha$, and employing the same techniques used in the proof of Proposition 1, one obtains (we again omit the index n):

$$\begin{aligned}\frac{\alpha}{2}\int_\Omega |\nabla G_k(u)|^p\varphi'(G_k(u)) + \frac{\alpha_0 k^{p-1}}{2}\int_\Omega |\varphi(G_k(u))| \\ \leq \left(\varphi(1) + \tilde{c}_2\|\Psi(G_k(u))\|^p_{L^\infty(\Omega)}\right)\int_{\{|f|>H\}} |f|\,,\end{aligned} \tag{55}$$

where $\tilde{c}_2$ is the same constant appearing in (16), $H = H(N, p, \alpha, \alpha_0, f) > 0$ will be chosen hereafter and $k = k(H)$ is such that $k^{p-1} = \max\{2H/\alpha_0, 1\}$. On the other hand, it is easily checked that

$$|\varphi(s)| \geq \tilde{c}_9(p,\lambda)\Big(\Psi(s)\Big)^p, \qquad \text{for every } s \in \mathbf{R},$$

and therefore, if $k \geq 1$, one has

$$\alpha_0 k^{p-1}\int_\Omega |\varphi(G_k(u))| \geq \tilde{c}_9(p,\lambda)\alpha_0\int_\Omega \left(\Psi(G_k(u))\right)^p.$$

Therefore, using (55) and Sobolev's embedding theorem of $W_0^{1,p}(\Omega)$ into $L^\infty(\Omega)$, one obtains

$$\|\Psi(G_k(u))\|^p_{L^\infty(\Omega)} \le \tilde{c}_{10}(N, p, \alpha, \alpha_0, \lambda) \left(\varphi(1) + \|\Psi(G_k(u))\|^p_{L^\infty(\Omega)}\right) \int_{\{|f|>H\}} |f|,$$

so that, choosing $H = H(N, p, \alpha, \alpha_0, f)$ sufficiently large, one gets an estimate for $\Psi(G_k(u))$ in $L^\infty(\Omega)$. □

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