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Masashi Misawa

Local Hölder regularity of gradients for evolutional p-Laplacian systems

Dedicated to Professor Norio Kikuchi on his sixtieth birthday

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Abstract. We study a Hölder regularity of gradients for evolutional p-Laplacian systems with Hölder continuous coefficients and exterior force. We use the perturbation argument with the p-Laplacian systems with constant coefficients and only principal terms. The main task is to make the Hölder estimate of gradients for the systems above well-worked in the perturbation estimate. We also need to make a localization of the Hölder estimate in [2].

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1. Introduction

Let Ω be a domain in an Euclidean space R^m for $m \geq 2$ and T be a positive number. Suppose that $\frac{2m}{m+2} < p < \infty$. We consider the evolutional p-Laplacian system

$$\partial_t u^i - D_\alpha \left(|Du|_g^{p-2} g^{\alpha\beta} D_\beta u^i \right) = \operatorname{div} \left(|F|^{p-2} F^i \right), \quad i = 1, \ldots, n, \tag{1.1}$$

where the function $F = \left(F_\alpha^i \right)$ is defined on $Q = (0, T) \times \Omega$ with values into R^{mn}, $\left(g^{\alpha\beta}(z) \right)$ is a symmetric matrix with entries of measurable functions satisfying the uniformly elliptic and bounded condition: For a positive constants λ, Λ,

$$\lambda |\xi|^2 \leq g^{\alpha\beta}(z) \xi_\alpha \cdot \xi_\beta \leq \Lambda |\xi|^2 \tag{1.2}$$

holds for any $\xi = \left(\xi_\alpha^i \right) \in R^{mn}$ and almost every $z \in Q$. The notation $|\xi|^2 = \xi \cdot \xi = \xi_\alpha^i \xi_\alpha^i$ and $|\xi|_g^2 = g^{\alpha\beta} \xi_\alpha^i \xi_\beta^i$ is used. Here and in what follows, the summation notation over repeated indices is adopted.

Such evolution systems as (1.1) describe the gradient flow of the p-energy functional with variable coefficients and lower-order terms.

M. Misawa: Department of Mathematics, Faculty of Science, Kumamoto University, 2-39-1 Kurokami, Kumamoto-shi, Kumamoto 860-8555, Japan;
e-mail: misawa@math.sci.kumamoto-u.ac.jp

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We are interested in how the regularity of the function F is reflected in the solutions under some assumption on the coefficients. Let us consider a Hölder regularity of the gradient of a solution for a given Hölder continuous function F. Such Hölder regularity is known to hold for elliptic and parabolic systems of divergence form (see [11, pp. 87–89], [17] and the references in them). The $C^{1,\alpha}$-regularity for evolutional p-Laplacian systems with only principal term was established in [7–9, 3] and the results have become fundamental to the regularity theory for evolutional p-Laplacian systems. Concerning p-Laplacian systems with differentiable coefficients and lower-order terms, we have the corresponding results in [6, 4, 18, 19, 12]. For stationary p-Laplacian systems with non-differentiable lower-order terms, a Hölder regularity of the gradient is studied in [10] in the degenerate case $p > 2$. In [10, 13, 14], L^q-estimates for the gradient for stationary p-Laplacian systems are also obtained and these are of interest itself (also see [15]). The L^q-estimates for evolutional p-Laplacian systems will be studied elsewhere. The results above concern an interior regularity of a "local" solution. On the other hand, an interior regularity for evolutional p-Laplacian systems with non-differentiable coefficients and lower-order terms seems to have not been successfully investigated. In particular, the estimate in [6, 2] seems not to be a "local" estimate, because of their decomposition argument of local parabolic cylinders (refer to [6, 2]). In this paper, we add a modification to the argument in [6, 2] to establish a local Hölder regularity of the gradient for (1.1). On basis of the result in this paper, we will study the partial Hölder regularity of gradients for evolutional p-Laplacian systems with natural growth in the forthcoming paper (refer to [14]).

Before stating the main result, we recall the definition of a weak solution of (1.1): a function u defined on Q with values into R^n is said to be a weak solution of (1.1), if $u \in L^\infty(0, T; L^2(\Omega, R^n)) \cap L^p\left(0, T; W^{1,p}(\Omega, R^n)\right)$ satisfies, for all $\phi \in C_0^\infty(Q, R^n)$,

$$\int_Q \left\{-u \cdot \partial_t \phi + |Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha \phi + |F|^{p-2} F \cdot D\phi\right\} dz = 0. \quad (1.3)$$

Then, our main result is the following:

Theorem 1. *Suppose that the coefficients $g^{\alpha\beta}$ and F are Hölder continuous functions in Q with an exponent β, $0 < \beta < 1$, on the usual parabolic metric. Let u be a weak solution of* (1.1). *Then there exists an exponent α, $0 < \alpha < 1$, depending only on m, p and β, and a positive constant C, depending only on m, p, λ, Λ, β, $|Du|_{p,Q}$ and $[g, |F|^{p-2}F]_{\beta,Q}$, such that the gradient of the solution is locally Hölder continuous in Q with an exponent α on the usual parabolic metric and the Hölder constant is bounded by the constant C.*

To prove Theorem 1, we use the perturbation argument with the p-Laplacian system with constant coefficients. Our main tasks are to make a device in the argument in [6, 2] and to choose local parabolic cylinders on which the Hölder estimate of gradients for the p-Laplacian systems with constant coefficients and only principal term can be well-worked in the perturbation estimate. We do not know whether the Hölder exponent α can be chosen to be equal to the Hölder exponent β of the function F.

2. Growth of local p-energy

In this section, we study the growth on the radius for the local p-energy. For any $z = (t, x) \in R^{m+1}$, $\rho > 0$ and $\theta > 0$, put $Q_\rho^\theta(z) = (t - \rho^\theta, t) \times B_\rho(x)$ and $Q_\rho^2(z)$ is abbreviated to $Q_\rho(z)$. For a function f defined on a region $B \subset R^{m+1}$, we denote, by $[f]_{\beta,B}$, the Hölder seminorm in B of f with the exponent β and, by $(f)_B$, the integral average of f in B. Let $z_0 = (t_0, x_0) \in Q$ be taken arbitrarily and $R_0 = \frac{1}{2}\mathrm{dist}_p(z_0, \partial Q)$, where $\mathrm{dist}_p(z_0, z_1) = \min\{|t_1 - t_0|^{\frac{1}{p}}, |x_1 - x_0|\}$ for any $z_i = (t_i, x_i) \in Q$, $i = 0, 1$. By translation and a scaling transformation, let $u \in L^\infty(-2^p, 0, L^2(B_2(0), R^n)) \cap L^p(-2^p, 0; W_0^{1,p}(B_2(0), R^n))$ be a weak solution of (1.1) in $Q_2^p = (-2^p, 0) \times B_2(0)$.

The main lemma in this section is the following:

Lemma 2. *For any* α, $0 < \alpha < 1$, *there exist a positive number* $\tilde{r}_0 < 1$, *depending only on* m, p *and* α, *and a positive constant* C, *depending only on* m, p, λ, Λ, $[g]_{\beta,Q_2^p}$, $[|F|^{p-2}F]_{\beta,Q_2^p}$, $|Du|_{p,Q_2^p}$ *and* α, *such that*

$$\int_{Q_\rho(\tilde{z})} (1 + |Du|^p) dz \le C\, \rho^{m+2-\alpha p} \tag{2.1}$$

holds for any $\tilde{z} \in Q_{\tilde{r}_0}$ *and all* ρ, $0 < \rho \le \tilde{r}_0$.

First, we consider the degenerate case $p > 2$. Let $\theta \ge 2$, R, $0 < R \le 1$ and $\tilde{z} = (\tilde{t}, \tilde{x}) \in Q_1$. By translation, we assume that $\tilde{z}$ is the origin. To prove Lemma 2, we use the perturbation argument with the p-Laplacian system with constant coefficients and only the principal term, similarly to [2] and [6, pp. 292–315]. Let $v \in L^\infty(-R^\theta, 0; L^2(B_R, R^n)) \cap L^p(-R^\theta, 0; W^{1,p}(B_R, R^n))$ be a solution to the p-Laplacian system (for the existence of a weak solution, refer to [16, Theorem 6.7, pp. 466–475])

$$\begin{aligned} \partial_t v = D_\alpha \left(|Dv|_{g(0)}^{p-2} g^{\alpha\beta}(0) D_\beta v \right) \quad &\text{in } Q_R^\theta, \\ v = u \quad &\text{on } \partial_p Q_R^\theta. \end{aligned} \tag{2.2}$$

We know that the L^∞-estimate holds for the gradient of solutions (see [6, Theorem 5.1, pp. 238]).

Lemma 3. *There exists a positive constant* C *depending only on* m *and* p *such that*

$$\sup_{Q_{\frac{R}{2}}^\theta} |Dv|^p \le C \left(\frac{R^{\theta-2}}{|Q_R^\theta|} \int_{Q_R^\theta} |Dv|^p dz \right)^{\frac{p}{2}} + C\, R^{\frac{p(2-\theta)}{p-2}}. \tag{2.3}$$

We now estimate the difference of u from v in the local L^p-norm.

Lemma 4. *There exists a positive constant* C *depending only on* m, p, λ, Λ, $[g]_{\beta,Q_2^p}$ *and* $[|F|^{p-2}F]_{\beta,Q_2^p}$ *such that*

$$\int_{Q_R^\theta} |Du - Dv|^p dz \le C\, R^{\frac{\beta p}{p-1}} \int_{Q_R^\theta} (1 + |Du|^p) dz. \tag{2.4}$$

Proof. Subtract (1.1) from (2.2) and use a test function $v - u$, which is shown to be admissible by the usual approximation argument, in the resulting equation. We utilize algebraic inequalities

$$g^{\alpha\beta}(0)\left(|P|_{g(0)}^{p-2}P_\alpha - |Q|_{g(0)}^{p-2}Q_\alpha\right)\cdot\left(P_\beta - Q_\beta\right) \geq C\,|P-Q|^p \qquad (2.5)$$
$$\left|\left(g^{\alpha\beta}|P|_g^{p-2}P_\beta^i - g^{\alpha\beta}(0)|P|_{g(0)}^{p-2}P_\beta^i\right)\right| \leq C\,|g-g(0)|\,|P|^{p-1},$$

which hold for any $P = \left(P_\alpha^i\right)$, $Q = \left(Q_\alpha^i\right) \in R^{mn}$ with a positive constant C depending only on p, λ and Λ. We use Young's inequality to have, for any $\epsilon > 0$,

$$\begin{aligned}
&C\int_{Q_R^\theta}|Dv-Du|^p dz\\
&\leq \int_{Q_R^\theta}|Dv-Du|\left(|g-g(0)||Du|^{p-1} + \left||F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta}\right|\right)dz\\
&\leq \epsilon\int_{Q_R^\theta}|Dv-Du|^p dz\\
&\qquad + C\int_{Q_R^\theta}|g-g(0)|^{\frac{p}{p-1}}|Du|^p + \left||F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta}\right|^{\frac{p}{p-1}}dz\\
&\leq \epsilon\int_{Q_R^\theta}|Dv-Du|^p dz \qquad (2.6)\\
&\qquad + CR^{\frac{p\beta}{p-1}}\left([g]_{\beta,Q_2^p}^{\frac{p}{p-1}}\int_{Q_R^\theta}(1+|Du|^p)dz + |Q_R^\theta|[|F|^{p-2}F]_{\beta,Q_2^p}^{\frac{p}{p-1}}\right),
\end{aligned}$$

where the positive constant C depends only on ϵ^{-1} and p, and we used (1.2) and, in the last inequality, the Hölder continuity of the coefficients g and the function $|F|^{p-2}F$ in Q_2^p.

Combining (2.4) with (2.3), we arrived at the following estimation:

Lemma 5. *Set $\theta = 2 + \alpha(p-2)$ for any positive number α. Then there exists a positive constant C having the same dependence as the one in Lemma 4 such that*

$$\begin{aligned}
&\int_{Q_\rho^\theta}(1+|Du|^p)dz\\
&\leq C\left\{\left(\frac{R^{\alpha p}}{|Q_R^\theta|}\int_{Q_R^\theta}(1+|Du|^p)dz\right)^{\frac{p-2}{2}} + 2^{m+\theta}\right\}\left(\frac{\rho}{R}\right)^{m+\theta}\int_{Q_R^\theta}(1+|Du|^p)dz\\
&\qquad + C\,\rho^{m+\theta}R^{-\alpha p} + C\,R^{m+\theta-\alpha p+\frac{p\beta}{p-1}}\left(\frac{R^{\alpha p}}{|Q_R^\theta|}\int_{Q_R^\theta}(1+|Du|^p)dz\right) \qquad (2.7)
\end{aligned}$$

holds for all ρ, R, $0 < \rho < R \leq 1$.

Proof. Noting that $0 < R \leq 1$, by (2.6), we have

$$\int_{Q_R^\theta} |Dv|^p dz \leq C \int_{Q_R^\theta} |Dv - Du|^p + |Du|^p dz$$
$$\leq C \int_{Q_R^\theta} (1 + |Du|^p) dz. \tag{2.8}$$

We substitute (2.8) into (2.3) and combine (2.4) with the resulting inequality. Then it follows that, for any ρ, $0 < \rho \leq \frac{R}{2}$,

$$\begin{aligned} \int_{Q_\rho^\theta} (1 + |Du|^p) dz &\leq C \left|Q_\rho^\theta\right| \left(1 + \sup_{Q^\theta_{\frac{R}{2}}} |Dv|^p\right) + C \int_{Q_R^\theta} |Du - Dv|^p dz \\ &\leq C \left|Q_\rho^\theta\right| \left\{ \left(\frac{R^{\theta-2}}{\left|Q_R^\theta\right|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{p}{2}} + R^{\frac{p(2-\theta)}{p-2}} \right\} \\ &\quad + C R^{\frac{\beta p}{p-1}} \int_{Q_R^\theta} (1 + |Du|^p) dz. \end{aligned} \tag{2.9}$$

The first term in the right-hand side of (2.9) is bounded by

$$C \left(\frac{R^{\frac{p(\theta-2)}{p-2}}}{\left|Q_R^\theta\right|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{p-2}{2}} \left(\frac{\rho}{R}\right)^{m+\theta} \int_{Q_R^\theta} (1 + |Du|^p) dz. \tag{2.10}$$

For $\rho > \frac{R}{2}$, we trivially have

$$\int_{Q_\rho^\theta} (1 + |Du|^p) dz \leq 2^{m+\theta} \left(\frac{\rho}{R}\right)^{m+\theta} \int_{Q_R^\theta} (1 + |Du|^p) dz. \tag{2.11}$$

We employ the non-linear iteration introduced in [2, Lemma 3.1, pp. 297–299].

Lemma 6. *Let ϕ be a non-negative, non-decreasing function defined on* [0, 1]. *Suppose that*

$$\phi(\rho) \leq \gamma_0 \left(\frac{\rho}{R}\right)^l \phi(R) + \gamma_0 \left(R^{l-\sigma\kappa} + \rho^l R^{-\kappa}\right) \tag{2.12}$$

holds for all ρ, R, $0 < \rho < R \leq 1$, where γ_0, l, κ and σ are given positive constants with $l > \kappa$ and $0 < \sigma < 1$. Then, for any positive number δ satisfying

$$0 \leq \delta < \kappa\left(\frac{\kappa(1-\sigma)}{\kappa(1-\sigma)+l}\right), \tag{2.13}$$

and any R, $0 < R \leq 1$, there exist a positive constant γ_1, depending only on $\gamma_0, l, \kappa, \sigma$ and δ, and a positive integer n_0, depending on the same constants as γ_1 and also on R, such that

$$\phi(\rho) \leq \gamma_1 \rho^{l-\kappa+\delta} \left(R^{-l q^{n_0+2}} \phi(R) + 1\right) \tag{2.14}$$

holds for all ρ, $0 < \rho < R$, where $q = 1 + \frac{\kappa(1-\sigma)}{l}$.

Define the sequences $\{r_k\}$, $\{\alpha_k\}$ and $\{\theta_k\}$ by

$$\begin{aligned} & r_0 = 1, \quad r_k = \left(\tfrac{1}{2}\right)^k; \\ & \alpha_0 = \tfrac{m+2}{2}, \quad \alpha_{k+1} = \alpha_k \tfrac{\alpha_k (p-2)+m+2}{\alpha_k (p-2+\frac{\beta}{\alpha_0})+m+2}; \\ & \theta_k = 2 + \alpha_k (p-2), \quad k = 0, 1, \ldots . \end{aligned} \tag{2.15}$$

Then we find from induction that $\{\alpha_k\}$ and $\{\theta_k\}$ are positive decreasing sequences with $\alpha_k \searrow 0$ and $\theta_k \searrow 2$ as $k \nearrow \infty$. Now we claim that:

Lemma 7. *For each $k = 0, 1, \ldots$, there exist a positive number $r_k < 1$, depending only on m, p and α_k, and a positive constant C_k, depending only on $\alpha_k, \alpha_0, \beta$, $|Du|_{p,Q_2^p}$ and the same quantities as in Lemma 4, such that*

$$\frac{1}{\left|Q_\rho^{\theta_k}\right|} \int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p) dz \le C_k\, \rho^{-\alpha_k p} \tag{2.16}$$

holds for any $z_0 \in Q_{r_k}$ and all ρ, $0 < \rho \le r_k$.

Proof of Lemma 7. We prove the validity of Lemma 7 by induction on $k = 0, 1, \ldots$. From $u \in L^p((-2^p, 0); W^{1,p}(B_2(0), R^n))$, we see that

$$\frac{1}{\left|Q_\rho^{\theta_0}\right|} \int_{Q_\rho^{\theta_0}(z_0)} (1 + |Du|^p) dz \le \rho^{-\alpha_0 p} \left(1 + C \int_{Q_2^p} |Du|^p dz\right) \tag{2.17}$$

holds for any $z_0 \in Q_1$ and all ρ, $0 < \rho \le 1$, where we use that $m + \theta_0 = p\,\alpha_0$ by $\alpha_0 = \frac{m+2}{2}$. Suppose by induction that (2.16) holds for some $k = 1, \ldots$. Let us show that (2.16) holds for $k + 1$. We now proceed to our estimation for each $z_0 \in Q_{r_k}$. Fix $z_0 \in Q_{r_k}$ and put $\alpha = \alpha_k$, $\theta = \theta_k$. Then we obtain from (2.7) that, for all ρ, R, $0 < \rho < R \le r_k$,

$$\begin{aligned} & \textstyle\int_{Q_\rho^\theta(z_0)} (1 + |Du|^p) dz \\ & \le C \left\{ (C_k)^{\frac{p-2}{2}} + 2^{m+\theta_0} \right\} \left(\tfrac{\rho}{R}\right)^{m+\theta} \textstyle\int_{Q_R^\theta(z_0)} (1 + |Du|^p) dz \\ & \quad + C\, \rho^{m+\theta}\, R^{-p\,\alpha} + C\,(C_k)\, R^{m+\theta-p\,\alpha\left(1-\frac{\beta}{\alpha_0\,(p-1)}\right)}. \end{aligned} \tag{2.18}$$

In Lemma 6, set $R = r_k$, $\phi(\rho) = \int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p) dz$ and

$$l = m + \theta_k, \quad \kappa = p\,\alpha_k, \quad \sigma = 1 - \tfrac{\beta}{\alpha_0\,(p-1)} \tag{2.19}$$

and then, apply Lemma 6 for (2.18). Noting that

$$0 < \tilde{\delta} = \frac{\frac{\beta}{\alpha_0}\,\alpha_k}{\frac{\beta}{\alpha_0}\,\alpha_k + m + \theta_k} < \frac{\alpha_k\,\frac{p\,\beta}{\alpha_0\,(p-1)}}{\alpha_k\,\frac{p\,\beta}{\alpha_0\,(p-1)} + m + \theta_k} = \frac{\kappa\,(1-\sigma)}{\kappa\,(1-\sigma)+l}, \tag{2.20}$$

we obtain from Lemma 6 that, for all ρ, $0 < \rho \le r_k$,

$$\int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p) dz \le \gamma_1\, \rho^{l-\kappa+\tilde{\delta}\,\kappa} \left(2^{l\,q^{n_0+1}} \int_{Q_{r_k}^{\theta_k}(z_0)} (1 + |Du|^p) dz + 1\right), \tag{2.21}$$

where we note that the positive constant γ_1 depends only on C_k, α_k, α_0 and β and n_0 depends on r_k and the same constants as γ_1 and that

$$q = 1 + \frac{\alpha_k\, p\, \beta}{\alpha_0\,(p-1)\,(m+\theta_k)}, \tag{2.22}$$
$$l - \kappa + \tilde{\delta}\,\kappa = m + \theta_k - p\,\alpha_k + p\,\alpha_k \frac{\frac{\beta}{\alpha_0}\,\alpha_k}{\frac{\beta}{\alpha_0}\,\alpha_k + m + \theta_k} = m + \theta_k - p\,\alpha_{k+1}.$$

Thus we can choose a positive constant $\tilde{C}_{k+1}$ depending only on r_k, C_k, α_k, α_0, β and $|Du|_{p,Q_2^p}$ such that

$$\int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p)dz \le \tilde{C}_{k+1}\,\rho^{m+\theta_k - p\,\alpha_{k+1}} \tag{2.23}$$

holds for any $z_0 \in Q_{r_k}$ and all ρ, $0 < \rho \le r_k$.

To show that (2.16) holds for $Q_\rho^{\theta_{k+1}}(z_0)$ with $z_0 \in Q_{r_{k+1}}$ and $0 < \rho \le r_{k+1}$, divide $Q_\rho^{\theta_{k+1}}(z_0)$, in the time direction, into s_0 cylinders $Q_\rho^{\theta_k}(t_i, x_0)$, $i = 0, 1, \ldots, s_0-1$, where $s_0 = [\rho^{\theta_{k+1}}/\rho^{\theta_k}]+1$. Adopt (2.23) in each region $Q_\rho^{\theta_k}(t_i, x_0)$, $i = 0, 1, \ldots, s_0 - 1$. Here note that the vertices (t_i, x_0), $i = 0, 1, \ldots, s_0 - 1$, are contained in Q_{r_k}. Add up the inequalities to have

$$\begin{aligned}\int_{Q_\rho^{\theta_{k+1}}(z_0)} (1 + |Du|^p)dz &\le \sum_{i=0}^{s_0-1} \int_{Q_\rho^{\theta_k}(t_i,x_0)} (1 + |Du|^p)dz \\ &\le \tilde{C}_{k+1}\, s_0\, \rho^{m+\theta_k - p\,\alpha_{k+1}} \\ &\le 2\,\tilde{C}_{k+1}\,\rho^{m+\theta_{k+1} - p\,\alpha_{k+1}}.\end{aligned} \tag{2.24}$$

Hence, Lemma 7 follows from (2.17) and (2.24).

Finally, we derive the assertion in Lemma 2 from Lemma 7. For any positive number α, $0 < \alpha < 1$, let k be a positive integer such that $\alpha_k \le \alpha < \alpha_{k-1}$. Adopt Lemma 7 to find that (2.16) holds for any $z_0 \in Q_{r_k}$ and ρ, $0 < \rho \le r_k$. Then choose $\tilde{r}_0 = r_{k+1}$ and use the decomposition argument above to conclude (2.1) in Lemma 2.

Next, we will show that Lemma 2 holds in the singular case $\frac{2m}{m+2} < p < 2$. Let $u \in L^\infty(-2^p, 0, L^2(B_2(0), R^n)) \cap L^p(-2^p, 0; W^{1,p}(B_2(0), R^n))$ be a weak solution of (1.1) in $Q_2^p = (-2^p, 0) \times B_2(0)$. Let θ, $0 < \theta \le 2$, R, $0 < R \le 1$ and $\tilde{z} = (\tilde{t}, \tilde{x}) \in Q_1$. By translation, we assume that $\tilde{z}$ is the origin. We proceed with our argument similarly to the degenerate case. Let $v \in L^\infty(-R^\theta, 0; L^2(B_R, R^n)) \cap L^p(-R^\theta, 0; W^{1,p}(B_R, R^n))$ be a solution to (2.2). The L^∞-estimate for the gradient also holds in the singular case (see [6, Theorem 5.1, p. 238]).

Lemma 8. *Set $\theta_0 = 2 + \frac{m+2}{2}(p-2)$. There exists a positive constant C depending only on m and p such that*

$$\sup_{Q^\theta_{2^{-\frac{2}{\theta_0}}R}} |Dv|^p \le C \left(\frac{R^{\frac{m(2-\theta)}{2}}}{\left|Q_R^\theta\right|} \int_{Q_R^\theta} |Dv|^p dz \right)^{\frac{2p}{p(m+2)-2m}} + C\, R^{\frac{p(2-\theta)}{p-2}}. \tag{2.25}$$

Proof. Use the L^∞-estimate for the gradient in [2, Proposition 3.1', p. 113] (see also [6, Theorem 5.2, pp. 238–239]), where we put $\rho = \frac{R}{2}$ and $\Lambda = R^{\theta-2}$. Note that $Q^\theta_{2^{-\frac{2}{\theta_0}}R} \subset (\frac{R^\theta}{4}, 0) \times B_{\frac{R}{2}}$, since $0 < \theta_0 \le \theta \le 2$.

As in (2.4), the difference of u from v is estimated in the local L^p-norm,

Lemma 9. *There exists a positive constant C depending only on m, p, λ, Λ, $[g]_{\beta,Q_2^p}$ and $[|F|^{p-2}F]_{\beta,Q_2^p}$ such that*

$$\int_{Q_R^\theta} |Dv - Du|^p dz \le C\, R^{\frac{\theta\beta p(p-1)}{2(2p-1)}} \int_{Q_R^\theta} (1 + |Du|^p) dz \qquad (2.26)$$
$$+ C\, R^{\frac{p-1}{2p-1}\left(m+\theta+\frac{\theta\beta p}{2}\right)} \left(\int_{Q_R^\theta} |Du|^p dz\right)^{\frac{p}{2p-1}}.$$

Proof. Subtract (1.1) from (2.2) and apply a test function $v - u$. Use the algebraic inequalities

$$g^{\alpha\beta}(0)\left(|P|^{p-2}_{g(0)} P_\alpha - |Q|^{p-2}_{g(0)} Q_\alpha\right) \cdot \left(P_\beta - Q_\beta\right) \ge C\,|P-Q|^2\,(|P|+|Q|)^{p-2},$$
$$\left|\left(g^{\alpha\beta} P^i_\beta |P|^{p-2}_g - g^{\alpha\beta}(0) P^i_\beta |P|^{p-2}_{g(0)}\right)\right| \le C\,|g - g(0)|\,|P|^{p-1}, \qquad (2.27)$$

which hold for any $P, Q \in R^{mn}$ with a positive constant C depending only on p, λ and Λ. Then we have

$$C \int_{Q_R^\theta} |Dv - Du|^2\,(|Dv| + |Du|)^{p-2}\,dz \qquad (2.28)$$
$$\le C \int_{Q_R^\theta} |Dv - Du| \left(|g - g(0)|^{p-1}|Du|^{p-1} + \left||F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta}\right|\right) dz.$$

Hölder's and Young's inequalities give, for any $\epsilon > 0$,

$$\int_{Q_R^\theta} |Dv - Du|^p dz$$
$$\le \left(\int_{Q_R^\theta} |Dv - Du|^2\,(|Dv| + |Du|)^{p-2}\,dz\right)^{\frac{1}{2}} \left(\int_{Q_R^\theta} (|Dv| + |Du|)^p\,dz\right)^{\frac{1}{2}}$$
$$\le \left(\int_{Q_R^\theta} |Dv - Du| \left(|g - g(0)|^{p-1}|Du|^{p-1} + \left||F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta}\right|\right) dz\right)^{\frac{1}{2}}$$
$$\cdot \left(\int_{Q_R^\theta} (|Dv - Du|^p + |Du|^p)\,dz\right)^{\frac{1}{2}}$$
$$\le \left(\int_{Q_R^\theta} |Dv - Du|^p dz\right)^{\frac{1}{2p}} \left\{\left(\int_{Q_R^\theta} |Dv - Du|^p dz\right)^{\frac{1}{2}} + \left(\int_{Q_R^\theta} |Du|^p dz\right)^{\frac{1}{2}}\right\}$$
$$\cdot \left(\int_{Q_R^\theta} \left(|g - g(0)|^{p-1}|Du|^{p-1} + \left||F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta}\right|\right)^{\frac{p}{p-1}} dz\right)^{\frac{p-1}{2p}}$$

$$
\begin{aligned}
&\leq \epsilon \int_{Q_R^\theta} |Dv - Du|^p dz \\
&\quad + C \int_{Q_R^\theta} |g - g(0)|^p |Du|^p + \left| |F|^{p-2}F - (|F|^{p-2}F)_{Q_R^\theta} \right|^{\frac{p}{p-1}} dz \\
&\quad + C \left(\int_{Q_R^\theta} |Du|^p dz \right)^{\frac{p}{2p-1}} \qquad (2.29) \\
&\quad \cdot \left(\left(\operatorname{osc}_{Q_{R^{\frac{\theta}{2}}}} (g) \right)^p \int_{Q_R^\theta} |Du|^p dz + |Q_R^\theta| \left(\operatorname{osc}_{Q_{R^{\frac{\theta}{2}}}} \left(|F|^{p-2}F \right) \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{2p-1}},
\end{aligned}
$$

where the positive constant C depends only on ϵ^{-1}, p, λ and Λ. Note that $Q_R^\theta \subset Q_{R^{\frac{\theta}{2}}}$, since $0 < \theta < 2$ and $0 < R \leq 1$ and then use the Hölder continuity of the coefficients g and the function $|F|^{p-2}F$ in $Q_{R^{\frac{\theta}{2}}}$ and the algebraic inequality (2.27) to have

$$
\operatorname*{osc}_{Q_{R^{\frac{\theta}{2}}}} (g) \leq C\, R^{\frac{\theta\beta}{2}}, \quad \operatorname*{osc}_{Q_{R^{\frac{\theta}{2}}}} (|F|^{p-2}F) \leq C\, R^{\frac{\theta\beta(p-1)}{2}}. \qquad (2.30)
$$

Substitute (2.30) into (2.29) to have (2.26).

Combining (2.26) with (2.25), we arrive at the following estimation:

Lemma 10. *Set $\theta = 2 + \alpha(p-2)$ for any positive number α and let $\theta_0 = 2 + \frac{m+2}{2}(p-2)$. Then there exists a positive constant C having the same dependence as the one in Lemma* 9 *such that*

$$
\begin{aligned}
\int_{Q_\rho^\theta} (1 + |Du|^p) dz \leq C &\left\{ \left(\frac{R^{\alpha p}}{|Q_R^\theta|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{m(2-p)}{p(m+2)-2m}} + 2^{\frac{2(m+2)}{\theta_0}} \right\} \\
&\cdot \left(\frac{\rho}{R}\right)^{m+\theta} \int_{Q_R^\theta} (1 + |Du|^p) dz + C\, \rho^{m+\theta}\, R^{-p\alpha} \qquad (2.31) \\
&+ C\, R^{m+\theta-\alpha p + \frac{\theta\beta p(p-1)}{2(2p-1)}} \left(\frac{R^{p\alpha}}{|Q_R^\theta|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right) \\
&+ C\, R^{m+\theta-p\alpha\left(\frac{p}{2p-1}\right)} \left(\frac{R^{p\alpha}}{|Q_R^\theta|} \int_{Q_R^\theta} (1 + |Du|^p) dz \right)^{\frac{p}{2p-1}}
\end{aligned}
$$

holds for all ρ, R, $0 < \rho < R \leq 1$.

Proof. Noting that $0 < R \leq 1$, by (2.26), we have

$$
\int_{Q_R^\theta} |Dv|^p dz \leq C \int_{Q_R^\theta} (1 + |Du|^p) dz. \qquad (2.32)
$$

As in the proof of Lemma 2, substitute (2.32) into (2.25) and combine (2.26) with the resulting inequality to have that (2.31) holds for any ρ, $0 < \rho \leq 2^{-\frac{2}{\theta_0}} R$, where we note that $\frac{m(2-\theta)}{2}\left(1 + \frac{p(m+2)-2m}{m(2-p)}\right) - p\alpha = 0$, since $\theta = 2 + \alpha(p-2)$. Noting that $0 < \theta_0 \leq \theta \leq 2$, the estimate for $\rho > 2^{-\frac{2}{\theta_0}} R$ is simple.

Define the sequences $\{r_k\}$, $\{\alpha_k\}$ and $\{\theta_k\}$ by

$$\begin{aligned}
&r_0 = 1, \quad r_k = \left(\tfrac{1}{2\sqrt{m}}\right)^k; \\
&\alpha_0 = \tfrac{m+2}{2}, \quad \theta_0 = 2 + \alpha_0\,(p-2), \quad \beta_0 = \tfrac{\theta_0\,\beta\,p\,(p-1)}{2\,\alpha_0\,p\,(2\,p-1)}; \\
&\theta_k = 2 + \alpha_k\,(p-2), \quad \tilde{\delta}_k = \left(\tfrac{\alpha_k\,p\,\beta_0}{\alpha_k\,p\,\beta_0+m+\theta_k}\right)\left(\tfrac{p-1}{p}\right), \\
&\alpha_{k+1} = \tfrac{\alpha_k}{2}\left(1 + \tfrac{2\,(1-\tilde{\delta}_k)}{\alpha_k\,\tilde{\delta}_k\,(p-2)+2}\right), \quad k = 0, 1, \ldots .
\end{aligned} \tag{2.33}$$

Since $\alpha_0 < \frac{2}{2-p}$ by $\frac{2m}{m+2} < p$, we find from induction that

$$\frac{2\,(1-\tilde{\delta}_k)}{\alpha_k\,\tilde{\delta}_k\,(p-2)+2} < 1, \quad \alpha_k\,\frac{2\,(1-\tilde{\delta}_k)}{\alpha_k\,\tilde{\delta}_k\,(p-2)+2} < \alpha_{k+1} < \alpha_k. \tag{2.34}$$

Thus, $\{\alpha_k\}$ is a positive decreasing sequence and $\{\theta_k\}$ is a positive increasing one such that $\alpha_k \searrow 0$ and $\theta_k \nearrow 2$ as $k \nearrow \infty$. Now we observe that Lemma 7 also holds in the singular case.

We argue by induction on $k = 0, 1, \ldots$. Since $u \in L^p((-2^p, 0); W^{1,p}(B_2(0), R^n))$, (2.17) is satisfied. Suppose that (2.16) holds for some $k = 1, \ldots$. Let us show that (2.16) holds for $k+1$. We will make our estimation for each $z_0 \in Q_{r_k}$. Fix $z_0 \in Q_{r_k}$ and put $\alpha = \alpha_k$, $\theta = \theta_k$. Then we obtain from (2.31) that, for all ρ, R, $0 < \rho < R \le r_k$,

$$\begin{aligned}
&\int_{Q_\rho^\theta} (1 + |Du|^p)dz \\
&\le C\left\{(C_k)^{\frac{m(2-p}{p(m+2)-2m}} + 2^{\frac{2(m+2)}{\theta_0}}\right\}\left(\tfrac{\rho}{R}\right)^{m+\theta} \int_{Q_R^\theta} (1 + |Du|^p)dz \\
&\quad + C\rho^{m+\theta} R^{-\alpha p} + C\left(C_k + (C_k)^{\frac{p}{2\,p-1}}\right) R^{m+\theta-\alpha p\left(1-\frac{\theta_0\,\beta\,p\,(p-1)}{2\,\alpha_0\,p\,(2\,p-1)}\right)}.
\end{aligned} \tag{2.35}$$

In Lemma 6, choose $R = r_k$, $\phi(\rho) = \int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p)dz$ and

$$l = m + \theta_k, \quad \kappa = p\,\alpha_k, \quad \sigma = 1 - \beta_0 = 1 - \left(\tfrac{\theta_0\,\beta\,p\,(p-1)}{2\,\alpha_0\,p\,(2\,p-1)}\right), \tag{2.36}$$

and then apply Lemma 6 for (2.35). Noting that

$$0 < \tilde{\delta}_k = \frac{\alpha_k\,p\,\beta_0}{\alpha_k\,p\,\beta_0+m+\theta_k}\left(\frac{p-1}{p}\right) < \frac{\kappa\,(1-\sigma)}{\kappa\,(1-\sigma)+l}, \tag{2.37}$$

we find from Lemma 6 that (2.21) with $\tilde{\delta} = \tilde{\delta}_k$ holds for all ρ, $0 < \rho \le r_{k+1}$, where we note that the positive constant γ_1 depends only on C_k, α_k, α_0 and β and the positive number n_0 depends only on r_{k+1} and the same ones as γ_1 and that

$$q = 1 + \frac{\alpha_k\,p\,\beta_0}{m+\theta_k}, \quad l - \kappa + \tilde{\delta}\,\kappa = m + \theta_k - p\,\alpha_k\,(1 - \tilde{\delta}_k). \tag{2.38}$$

Thus we can choose a positive constant $\tilde{C}_{k+1}$ depending only on C_k, α_k, α_0, β and $|Du|_{p,Q_2^p}$ such that

$$\int_{Q_\rho^{\theta_k}(z_0)} (1 + |Du|^p)dz \le \tilde{C}_{k+1}\,\rho^{m+\theta_k-p\,\alpha_k\,(1-\tilde{\delta}_k)} \tag{2.39}$$

holds for any $z_0 \in Q_{r_k}$ and all ρ, $0 < \rho \le r_k$.

Let us show that (2.16) holds for $Q_\rho^{\theta_{k+1}}(z_0)$ with $z_0 \in Q_{r_{k+1}}$ and $0 < \rho \le r_{k+1}$. Divide the cube $\mathcal{C}_\rho(x_0) = \Pi_{i=1}^m \left(x_0^i - \rho, x_0^i + \rho\right)$ into $(s_0)^m$ cubes $\mathcal{C}_r(x_i)$ with $r = \rho^{\frac{\theta_{k+1}}{\theta_k}}$, $i = 1, \ldots, (s_0)^m$, where $s_0 = \left[\rho^{\left(1-\frac{\theta_{k+1}}{\theta_k}\right)}\right] + 1$. Note by definition (2.33) of $\{r_k\}$ that $0 < \sqrt{m}\, r \le r_k$ and the vertices (t_0, x_i), $i = 1, \ldots, (s_0)^m$, are contained in Q_{r_k} and adopt (2.39) in each region $Q_{\sqrt{m}\, r}(t_0, x_i)$, $i = 1, \ldots, (s_0)^m$. Sum up the inequalities to have

$$\begin{aligned}\int_{Q_\rho^{\theta_{k+1}}(z_0)} (1 + |Du|^p) dz &\le \sum_{i=1}^{(s_0)^m} \int_{Q_{\sqrt{m}\, r}(t_0, x_i)} (1 + |Du|^p) dz \\ &\le \tilde{C}_{k+1}\, (s_0)^m\, r^{m+\theta_k - p\,\alpha_k\,(1-\tilde{\delta}_k)} \\ &\le C_{k+1} \rho^{m+\theta_{k+1} - p\,\alpha_{k+1}},\end{aligned} \tag{2.40}$$

where $C_{k+1} = \tilde{C}_{k+1} 2^m (\sqrt{m})^{m+\theta_k - p\,\alpha_k\,(1-\tilde{\delta}_k)}$ and, in the last inequality, we use that (2.34) is equivalent to

$$\tfrac{\theta_{k+1}}{\theta_k} \left(\tilde{\delta}_k - 1\right) \alpha_k\, p \ge -\alpha_{k+1}\, p. \tag{2.41}$$

Hence, Lemma 7 in the singular case follows from (2.17) and (2.40).

Finally, use Lemma 7 and the decomposition argument above to conclude (2.1) in Lemma 2, where we note that $0 < \theta_0 = 2 + \frac{m+2}{2} \le \theta_k < 2$ for any $k = 1, \ldots$.

3. Growth of local mean oscillation

In this section, we study the growth on the radius of the local mean oscillation. We use the same settings as in Sect. 2 Let $u \in L^\infty(-2^p, 0, L^2(B_2, R^n)) \cap L^p(-2^p, 0; W_0^{1,p}(B_2, R^n))$ be a weak solution of (1.1) in $Q_2^p = (-2^p, 0) \times B_2(0)$.

Our main lemma in this section is the following:

Lemma 11. *There exist positive numbers $\hat{r}_0, \beta_1 < 1$, depending only on m, p and β, and a positive constant C, depending only on $m, p, \lambda, \Lambda, [g]_{\beta, Q_2^p}$, $[|F|^{p-2}\, F]_{\beta, Q_2^p}$ and $|Du|_{p, Q_2^p}$, such that*

$$\begin{aligned}&\int_{Q_\rho(z_0)} |Du - (Du)_\rho|^p dz \\ &\le C \rho^{m+2+p\,\beta_1} \left(\tfrac{1}{R^{m+2+p\,\beta_1}} \int_{Q_R(z_0)} |Du - (Du)_R|^p dz + 1 \right)\end{aligned} \tag{3.1}$$

holds for any $z_0 \in Q_{\hat{r}_0}$ and all ρ, R, $0 < \rho < R \le \hat{r}_0$.

We apply the isomorphism theorem due to Campanato (see [5, Theorem 3.1], [1] and also refer to [11, Theorem 1.2, p. 70; Theorem 1.3, p. 72]) to see that Du is Hölder continuous in $Q_{\hat{r}_0}$ with an exponent β_1. Hence, we conclude the assertion in Theorem 1.

First, we treat the degenerate case $p > 2$. As in Sect. 2, we use the perturbation argument. However, here we make a device to change the power of the radius of a local parabolic cylinder, on which we solve the evolutional p-Laplacian systems with constant coefficients and only the principal term. By this device, we can appropriately employ the Hölder estimate for the gradient to make an estimation of the mean oscillation of the gradient of the solution in the L^p-norm.

Let α, $0 < \alpha < 1$, be stipulated later and $\tilde{r}_0$, $0 < \tilde{r}_0 < 1$ be chosen in Lemma 2 for α. Take $z_0 \in Q_{\tilde{r}_0}$ arbitrarily and, for any δ, $0 < \delta < 1$, let R be any positive number R, $0 < R \le (\tilde{r}_0)^{\frac{1}{1-\delta}}$, and set $r = R^{1-\delta}$. For brevity, we assume that z_0 is the origin. Let $v \in L^\infty(-r^2, 0 : L^1(B_r, R^n)) \cap L^p(-r^2, 0 : W^{1,p}(B_r, R^n))$ be a weak solution of (2.2), in which θ and R are replaced by 2 and r. As in (2.6) in the proof of Lemma 4, subtract (1.1) from (2.2), replace θ and R by 2 and r, respectively, and use a test function $v - u$ in the resulting equation. Then we have the estimation

$$\int_{Q_{R^{1-\delta}}} |Dv - Du|^p dz \le C\, R^{(1-\delta)\frac{p\beta}{p-1}} \int_{Q_{R^{1-\delta}}} (1 + |Du|^p) dz, \tag{3.2}$$

where a positive constant C depends only on m, p, λ, Λ, $[g]_{\beta, Q_2^p}$ and $[|F|^{p-2}F]_{\beta, Q_2^p}$.

Now we observe that the L^∞-estimate holds in the following form:

Lemma 12. *For any* α, $0 < \alpha < 1$, *there exists a positive constant* $C(\alpha)$ *having the same dependence as the one in Lemma* 2 *such that*

$$|Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}} \le C(\alpha)\, R^{-p\,\alpha\,(1-\delta)}. \tag{3.3}$$

Proof. We choose $\theta = 2$ and $R = r$ in Lemma 3 to have

$$|Dv|_{\infty, Q_{\frac{r}{2}}} \le C \left(\frac{1}{|Q_r|} \int_{Q_r} |Dv|^p dz \right)^{\frac{1}{2}} + C. \tag{3.4}$$

Noting that $0 < r < 1$, we find from (3.2) that

$$\int_{Q_r} |Dv|^p dz \le C \int_{Q_r} (1 + |Du|^p) dz. \tag{3.5}$$

Substitute (3.5) into (3.4) to have

$$|Dv|_{\infty, Q_{\frac{r}{2}}} \le C \left(\frac{1}{|Q_r|} \int_{Q_r} (1 + |Du|^p) dz \right)^{\frac{1}{2}} + C. \tag{3.6}$$

Note that $0 < r = R^{1-\delta} \le \tilde{r}_0$ and adopt (2.1) with $\rho = r = R^{1-\delta}$ in (3.6) to arrive at (3.3).

We need the estimation for a oscillation of the gradient of the solution v.

Lemma 13. *For any positive number* δ, $0 < \delta < 1$, *there exist a positive number* α_1, $0 < \alpha_1 < 1$, *depending only on* m, p *and* δ, *and a positive constant* $C(\delta)$, *depending only on* δ, $|Du|_{p, Q_2^p}$ *and the same quantities as in Lemma* 12, *such that*

$$\operatorname*{osc}_{Q_{\frac{R}{2}}} (Dv) \le C(\delta)\, R^{\alpha_1}. \tag{3.7}$$

Proof. We know that the Hölder estimate holds in the following form (see [6, Theorem 1.1', pp. 256]). There exist positive constants C and α_0 depending only on m and p such that

$$\operatorname{osc}_{Q_{\frac{R}{2}}}(Dv) \leq C\,|Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}} \left(\frac{\frac{R}{2} + \frac{R}{2} \max\left\{1,\, |Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}}^{\frac{p-2}{2}} \right\}}{\operatorname{dist}_2(Q_{\frac{R}{2}},\, \partial_p Q_{\frac{R^{1-\delta}}{2}})} \right)^{\alpha_0}. \tag{3.8}$$

If $R^\delta \geq \frac{1}{4}$, then we have

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(Dv) \leq 8\,R^\delta\,|Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}}. \tag{3.9}$$

Since, if $0 < R^\delta \leq \frac{1}{4}$, then

$$\begin{aligned} \operatorname{dist}_2(Q_{\frac{R}{2}},\, \partial_p Q_{\frac{R^{1-\delta}}{2}}) &= \tfrac{1}{2}\min\{R^{1-\delta} - R,\, \sqrt{R^{1-\delta} - R}\} \\ &\geq \tfrac{3}{8}\,R^{1-\delta}, \end{aligned}$$

we obtain from (3.3) and (3.8)

$$\begin{aligned} \operatorname*{osc}_{Q_{\frac{R}{2}}}(Dv) &\leq C\,|Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}} \max\left\{1,\, |Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}}^{\frac{\alpha_0(p-2)}{2}} \right\} R^{\delta\alpha_0} \\ &\leq C(\alpha)\,R^{\delta\alpha_0 - p\alpha(1-\delta)\left(1+\frac{\alpha_0(p-2)}{2}\right)}. \end{aligned} \tag{3.10}$$

We choose a positive number α to satisfy

$$0 < \alpha < \frac{\delta\alpha_0}{p(1-\delta)\left(1+\frac{\alpha_0(p-2)}{2}\right)} \tag{3.11}$$

and let $\tilde{r}_0 < 1$ be a positive number determined in Lemma 2 for α above. Then, we put

$$\alpha_1 = \delta\alpha_0 - p\alpha(1-\delta)\left(1 + \frac{\alpha_0(p-2)}{2}\right) > 0 \tag{3.12}$$

to give the conclusion of Lemma 13.

Now let us finish the proof of Lemma 11 in the degenerate case. For all R, $0 < R \leq 1$, and all ρ, $\frac{R}{2} \leq \rho < R$, we trivially have

$$\int_{Q_\rho} |Dv - (Dv)_\rho|^p dz \leq 2^{m+4}\left(\tfrac{\rho}{R}\right)^{m+4} \int_{Q_R} |Dv - (Dv)_R|^p dz. \tag{3.13}$$

From Lemma 13, we find that

$$\int_{Q_\rho} |Dv - (Dv)_\rho|^p dz \leq \tilde{C}\,R^{m+2+p\alpha_1} \tag{3.14}$$

holds for all R, $0 < R \leq (\tilde{r}_0)^{\frac{1}{1-\delta}}$, and all ρ, $0 < \rho < \frac{R}{2}$, where a positive number $\tilde{r}_0 < 1$ depends only on m, p and α, positive constants $\tilde{C}$ and α_1, $0 < \alpha_1 < 1$, depend only on the same quantities as in Lemma 13 for a positive number δ, $0 < \delta < 1$. Gathering (3.13) and (3.14) gives that

$$\int_{Q_\rho} |Dv - (Dv)_\rho|^p dz \leq 4^{m+4} \left(\tfrac{\rho}{R}\right)^{m+4} \int_{Q_R} |Dv - (Dv)_R|^p dz + \tilde{C}\, R^{m+2+p\alpha_1} \tag{3.15}$$

holds for all ρ, R $0 < \rho < R \leq (\tilde{r}_0)^{\frac{1}{1-\delta}}$, where positive constants $\tilde{r}_0 < 1$, $\tilde{C}$ and α_1, $0 < \alpha_1 < 1$, depend only on the same quantities as in (3.14). From (2.1), (3.2) and (3.15), it follows that, for all ρ, R, $0 < \rho < R \leq (\tilde{r}_0)^{\frac{1}{1-\delta}}$,

$$\begin{aligned}
&\int_{Q_\rho} |Du - (Du)_\rho|^p dz \\
&\leq C \int_{Q_R} |Dv - (Dv)_\rho|^p dz + C \int_{Q_R} |Dv - Du|^p dz \\
&\leq C \left(\tfrac{\rho}{R}\right)^{m+4} \int_{Q_R} |Du - (Du)_R|^p dz + \tilde{C}\, R^{m+2+p\,\alpha_1} \\
&\qquad + C\, R^{(1-\delta)\left(\frac{p\,\beta}{p-1}\right)} \int_{Q_{R^{1-\delta}}} (1 + |Du|^p) dz \\
&\leq C \left(\tfrac{\rho}{R}\right)^{m+4} \int_{Q_R} |Du - (Du)_R|^p dz + \tilde{C}\, R^{m+2+p\,\alpha_1} \\
&\qquad + C(\alpha)\, R^{(1-\delta)\left(m+2-p\,\alpha+\frac{p\,\beta}{p-1}\right)}.
\end{aligned} \tag{3.16}$$

Fix a positive number δ to satisfy

$$0 < \delta < \frac{\frac{p\,\beta}{p-1}}{m+2+\frac{p\,\beta}{p-1}}. \tag{3.17}$$

Next, choose a positive number α satisfying (3.11) and

$$0 < \alpha < \frac{(1-\delta)\frac{p\,\beta}{p-1} - \delta\,(m+2)}{p\,(1-\delta)}. \tag{3.18}$$

Then we can choose a positive number α_2 to be

$$m + 2 + p\,\alpha_2 = (1-\delta)\left(m + 2 - p\,\alpha + \frac{p\,\beta}{p-1}\right) > m + 2. \tag{3.19}$$

Finally, let $\tilde{r}_0 < 1$ be a positive number determined in Lemma 2 for α above and set $\hat{r}_0 = (\tilde{r}_0)^{\frac{1}{1-\delta}}$ for δ above and $\beta_1 = \min\{\alpha_1, \alpha_2\}$, where α_1 is in (3.12), and apply the iteration argument in [11, Lemma 2.1, p. 86] for (3.16) to arrive at (3.1).

Next, we show that Lemma 11 holds in the singular case. We argue in the same settings as in the degenerate case. As in (2.30) in the proof of Lemma 9, we find that (2.26) with $\theta = 2$ and $R = r$ holds.

The L^∞-estimate for the gradient holds in the following form:

Lemma 14. *For any α, $0 < \alpha < 1$, there exists a positive constant $C(\alpha)$ having the same dependence as the one in Lemma 2 such that*

$$|Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}} \le C(\alpha)\, R^{-\frac{2\,p\,\alpha\,(1-\delta)}{p\,(m+2)-2\,m}}\,. \tag{3.20}$$

Proof. As in the proof of Lemma 12, choose $\theta = 2$ and $R = r$ in Lemma 8. Note that $0 < r = R^{1-\delta} \le \tilde{r}_0$ and substitute (2.32) and (2.1) with $\rho = r = R^{1-\delta}$ into (2.25) with $\theta = 2$ and $R = r$ to arrive at (3.20).

We make an estimation for a oscillation of the gradient of the solution v.

Lemma 15. *For any positive number δ, $0 < \delta < 1$, there exist a positive number α_1, $0 < \alpha_1 < 1$, depending only on m, p and δ, and a positive constant $C(\delta)$, depending only on δ, $|Du|_{p, Q_2^p}$ and the same quantities as in Lemma 14, such that*

$$\operatorname*{osc}_{Q_{\frac{R}{2}}} (Dv) \le C(\delta)\, R^{\alpha_1}. \tag{3.21}$$

Proof. The Hölder estimate holds in the following form (see [6, Theorem 1.1", pp. 258]). There exist positive constants C and α_0 depending only on m and p such that

$$\operatorname{osc}_{Q_{\frac{R}{2}}} (Dv) \le C\, |Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}} \left(\frac{\frac{R}{2} \max\left\{1, |Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}}^{\frac{2-p}{2}}\right\} + \frac{R}{2}}{\operatorname{dist}_2(Q_{\frac{R}{2}}, \partial_p Q_{\frac{R^{1-\delta}}{2}})} \right)^{\alpha_0}. \tag{3.22}$$

Use (3.20) and argue in an exactly similar way as in the proof of Lemma 13 to have

$$\begin{aligned} \operatorname*{osc}_{Q_{\frac{R}{2}}} (Dv) &\le C\, |Dv|_{\infty, Q_{\frac{R^{1-\delta}}{2}}} \max\left\{1,\ |Dv|_{\infty,\, Q_{\frac{R^{1-\delta}}{2}}}^{\frac{\alpha_0\,(2-p)}{2}}\right\} R^{\delta\,\alpha_0} \\ &\le C(\alpha)\, R^{\delta\,\alpha_0 - \frac{2\,p\,\alpha\,(1-\delta)}{p\,(m+2)-2\,m}\left(1+\frac{\alpha_0\,(2-p)}{2}\right)}. \end{aligned} \tag{3.23}$$

We choose a positive number α to satisfy

$$0 < \alpha < \frac{\delta\,\alpha_0}{\frac{2\,p\,(1-\delta)}{p\,(m+2)-2\,m}\left(1+\frac{\alpha_0\,(2-p)}{2}\right)} \tag{3.24}$$

and let $\tilde{r}_0 < 1$ be a positive number determined in Lemma 2 for α above and then we put

$$\alpha_1 = \delta\,\alpha_0 - \frac{2\,p\,\alpha\,(1-\delta)}{p\,(m+2)-2\,m} \left(1 + \frac{\alpha_0\,(2-p)}{2}\right) > 0 \tag{3.25}$$

to get the conclusion of Lemma 15.

The proof of Lemma 11 is performed as in the degenerate case by using Lemma 15 and Lemma 9. As in (3.16), we have, for all ρ, R, $0 < \rho < R \le \tilde{r}_0$,

$$\begin{aligned}
&\int_{Q_\rho} |Du - (Du)_\rho|^p dz \qquad (3.26)\\
&\le C\big(\tfrac{\rho}{R}\big)^{m+4} \int_{Q_R} |Du - (Du)_R|^p dz + \tilde{C}\, R^{m+2+p\alpha_1} + C\, R^{(1-\delta)(m+2+\beta p)}\\
&\quad + C(\alpha)\left(R^{(1-\delta)\left(m+2-p\alpha+\frac{\beta p(p-1)}{2p-1}\right)} + R^{(1-\delta)\left(m+2+\frac{\beta p(p-1)}{2p-1}-\frac{\alpha p^2}{2p-1}\right)}\right)\\
&\le C\big(\tfrac{\rho}{R}\big)^{m+4} \int_{Q_R} |Du - (Du)_R|^p dz + \tilde{C}(\alpha)\, R^{(1-\delta)\left(m+2-p\alpha+\frac{\beta p(p-1)}{2p-1}\right)}.
\end{aligned}$$

Fix a positive number δ to satisfy

$$0 < \delta < \frac{\frac{\beta p(p-1)}{2p-1}}{m+2+\frac{\beta p(p-1)}{2p-1}}. \qquad (3.27)$$

Next, choose a positive number α satisfying (3.24) and

$$0 < \alpha < \frac{(1-\delta)\frac{\beta p(p-1)}{2p-1} - \delta(m+2)}{p(1-\delta)}. \qquad (3.28)$$

Then we can choose a positive number α_2 to be

$$m + 2 + p\alpha_2 = (1-\delta)\left(m + 2 - p\alpha + \frac{\beta p(p-1)}{2p-1}\right) > m + 2. \qquad (3.29)$$

Finally, let $\tilde{r}_0 < 1$ be a positive number determined in Lemma 2 for α above and set $\beta_1 = \min\{\alpha_1, \alpha_2\}$, where α_1 is in (3.25), and apply the iteration argument in [11, Lemma 2.1, p. 86] for (3.26) to arrive at (3.1).

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