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On Hausdorff dimension of invariant measures arising from non-contractive iterated function systems

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Abstract. Under rather general assumptions we give an upper and lower bound estimate of the Hausdorff dimension of the invariant probability measure for an iterated function systems acting on a Polish space.

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0. Introduction

Hausdorff dimension, introduced in 1919, is a notion of size useful for distinguishing between sets of Lebesgue measure zero. This notion was widely investigated and widely used, among others in the theory of dynamical systems, where many interesting invariant sets are null in the sense of Lebesgue measure. Unfortunately, the Hausdorff measure and dimension of even relatively simple sets can be very hard to calculate. Several ways (for example, packing dimension, box dimension, informatic dimension, correlation dimension, entropy) of estimating the dimension of irregular sets has been proposed, but undoubtedly the most popular is the Hausdorff dimension.

In 1975 Mandelbrot [19] observed that some scientific phenomena can be described by the sets having fractional dimension. He called such sets fractals and tentatively defined them as the sets whose Hausdorff dimension is strictly greater from their topological dimension. It appears that such sets occur in diverse branches of pure mathematics such as number theory, measure theory, and nonlinear differential equations. On the other hand, these sets are a good model for describing a wide variety of natural phenomena such as Brownian motion, turbulence flow, and the growth of plants. Because of this, the theory of fractals has been intensively studied by many people.

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The very explosion of interest in fractal sets started after the observation that some classes of fractals can be constructed by using an iteration process (see Hutchinson [15], Barnsley [1]). More precisely, having N contractive maps, say S_1, \dots, S_N , we can construct the fractal set A_* as the limit of the sequence $\{F^n(A)\}$, where $F(A) = \bigcup_{i=1}^N S_i(A)$, A a compact set. Moreover, if for every map S_i there is given the probability p_i determining the frequency with which the map S_i can be chosen, we can associate to such a system – known under the name iterated function system with probabilities – a Markov operator P acting on the space of Borel measures (see Formula (1.1)). The fractal set can be obtained as the support of the invariant measure with respect to P , provided that P is asymptotically stable. It is interesting that the class of sets which can be defined in this way contains not only fractals but also other sets known as semifractals. Roughly speaking, a semifractal is the support of the invariant measure with respect to an asymptotically stable Markov operator generated by an iterated function system with probabilities (see Lasota and Myjak [17, 18]).

The important question is to find effective criteria for asymptotic stability. Classical results in this area were obtained by Hutchinson [15], Barnsley et al. [2] and Lasota and Yorke [16] in the case of the Menger space (i.e. the space such that every closed bounded subset is compact). Recently such criteria were established in the case of Polish spaces (see Szarek [25, 26]). It seems that natural habitats for fractals and semifractals are Polish spaces. Consequently, the results concerning Hausdorff dimension of invariant measures with respect to Markov operators arising from iterated function systems acting on Polish spaces seem to be of particular interest. For related results see [6, 7, 9, 11–14, 20, 22, 24]. These results are formulated for contractive systems and proved using a multifractal formalism.

In this paper we study a more general case when the system is contracting “on the average”. More precisely, we give an upper and lower bound estimate of the Hausdorff dimension of the probability measure invariant with respect to the system contractive on the average. Observe that for such system we are unable to set up a one-to-one correspondence between general symbolic dynamical systems and the system under consideration, consequently we cannot use the thermodynamic formalism (see [24]). The approach used here is based on the technique initiated by Hutchinson [15] (see also [2, 11, 12]).

This paper is organized as follows. Notation and basic definitions are contained in Section 1. In Section 2 we establish some auxiliary results which play a crucial role in the proof of the theorems and also seem to be interesting in themselves. In Sections 3 and 4 we give, respectively, upper and lower bound estimates of the Hausdorff dimension of an invariant probability measure for an iterated function system acting on Polish spaces.

1. Preliminaries

Throughout this paper X denotes a Polish space, ρ a metric in X and this assumption will not be repeated in the statement of theorems. By $B(x, r)$ we denote the closed ball in X with center at x and radius r . For $A \subset X$, $A \neq \emptyset$, we denote by $\text{diam } A$

the diameter of A and by 1_A the characteristic function of A . As usual, \mathbb{R} stands for the set of all reals and \mathbb{N} for the set of all positive integers. Moreover set $\mathbb{R}_+ = [0, +\infty)$.

By $\mathcal{B}(X)$ we denote the σ -algebra of Borel subsets of X and by \mathcal{M} the family of all finite Borel measures on X . Moreover, denote by \mathcal{M}_1 the family of all $\mu \in \mathcal{M}$ such that $\mu(X) = 1$ and by $\mathcal{M}_s = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}\}$ the space of all finite signed measures.

Finally, $B(X)$ stands for the space of all bounded Borel measurable functions $f : X \rightarrow \mathbb{R}$ and $C(X)$ for the subspace of $B(X)$ of all bounded continuous functions. Both spaces are endowed with the norm of uniform convergence $\|\cdot\|_0$.

For $f \in B(X)$ and $\nu \in \mathcal{M}_s$ we write

$$\langle f, \nu \rangle = \int_X f(x)\nu(dx).$$

We admit that \mathcal{M}_s is endowed with the *Fortet–Mourier norm* (see [10]) given by

$$\|\nu\| = \sup\{|\langle f, \nu \rangle| : f \in F\},$$

where F is the set of all functions $f \in C(X)$ such that $\|f\|_0 \leq 1$ and $|f(x) - f(y)| \leq \rho(x, y)$ for $x, y \in X$.

It is known that the convergence

$$\lim_{n \rightarrow \infty} \|\mu_n - \mu\| = 0 \quad \text{for } \mu_n, \mu \in \mathcal{M}_1$$

is equivalent to the weak convergence of the sequence $(\mu_n)_{n \geq 1}$ to μ (see [4]).

An operator $P : \mathcal{M} \rightarrow \mathcal{M}$ is called a *Markov operator* if it satisfies the following two conditions:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1P\mu_1 + \lambda_2P\mu_2 \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}_+ \text{ and } \mu_1, \mu_2 \in \mathcal{M}$$

and

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}.$$

A measure μ is called *stationary* (or *invariant*) with respect to the operator P if $P\mu = \mu$. A Markov operator P is called *asymptotically stable* if there exists an invariant probability measure μ_* such that

$$\lim_{n \rightarrow \infty} \|P^n\mu - \mu_*\| = 0 \quad \text{for every } \mu \in \mathcal{M}_1.$$

Clearly, if the operator P is asymptotically stable, then the corresponding invariant measure is unique.

Let $A \subset X$ and $s, \delta > 0$. Define

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i=1}^\infty (\text{diam } U_i)^s : A \subset \bigcup_{i=1}^\infty U_i \text{ and } \text{diam } U_i \leq \delta \right\}.$$

Given $s > 0$ we define the *Hausdorff s -dimensional outer measure* by the formula

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A).$$

The restriction of \mathcal{H}^s to the σ -algebra of \mathcal{H}^s -measurable sets is called the *Hausdorff s -dimensional measure*. Note that all Borel sets are \mathcal{H}^s -measurable. For $A \subset X$ the value

$$\dim_H A = \inf \{s > 0 : \mathcal{H}^s(A) < \infty\}$$

is called the *Hausdorff dimension* of A .

Finally, we recall that the *Hausdorff dimension* of a Borel probability measure μ is defined by the formula

$$\dim_H \mu = \inf \{ \dim_H A : A \in \mathcal{B}(X), \mu(A) = 1 \}.$$

Assume now we are given a sequence of continuous transformations

$$S_i : X \rightarrow X, \quad \text{for } i = 1, \dots, N,$$

and a probabilistic vector

$$p_i : X \rightarrow [0, 1], \quad \text{for } i = 1, \dots, N,$$

where p_i are continuous functions satisfying

$$p_i(x) > 0 \quad \text{and} \quad \sum_{i=1}^N p_i(x) = 1 \quad \text{for } x \in X.$$

Such a system is denoted by $(S, p)_N$ and it is called the *Iterated Function System (IFS)*, for brevity).

Having an IFS $(S, p)_N$ we define the corresponding Markov operator $P: \mathcal{M} \rightarrow \mathcal{M}$ by

$$(1.1) \quad P\mu(A) = \sum_{i=1}^N \int_{S_i^{-1}(A)} p_i(x) \mu(dx) \quad \text{for } A \in \mathcal{B}(X).$$

We say that an IFS $(S, p)_N$ is asymptotically stable if the corresponding Markov operator P is asymptotically stable. A measure $\mu_\star \in \mathcal{M}$ is called *invariant* for the IFS $(S, p)_N$ if it is invariant with respect to the corresponding Markov operator P .

We assume that $S_i, i = 1, \dots, N$, are Lipschitzian transformations (not necessarily contractions) with the corresponding Lipschitz constants λ_i and that there exists a constant $\gamma_0 \in (0, 1)$ such that

$$(1.2) \quad \prod_{i=1}^N (\rho(S_i(x), S_i(y)))^{p_i(x)} \leq \gamma_0 \rho(x, y) \quad \text{for every } x, y \in X.$$

Condition (1.2) is commonly called the *average contraction* or *Lyapunov contraction* and the constant γ_0 is strictly connected with Lyapunov exponents (see [8, 23]) Moreover, we assume that

$$(1.3) \quad \kappa = \min_{1 \leq i \leq N} \inf_{x \in X} p_i(x) > 0$$

and

$$(1.4) \quad \sum_{i=1}^N |p_i(x) - p_i(y)| \leq \omega(\rho(x, y)),$$

where $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing and concave function satisfying the Dini condition, i.e.

$$\int_0^\sigma \frac{\omega(t)}{t} dt < \infty,$$

for some $\sigma > 0$.

2. Auxiliary results

Proposition 2.1. *If an IFS $(S, p)_N$ satisfies Conditions (1.2)–(1.4), then it is asymptotically stable.*

The proof for the case when X is a locally and σ -compact metric space was given in [2, 16]. The case when X is a Polish space it can be found in [26].

Let

$$\Omega = \{1, \dots, N\}^\infty = \{(i_1, i_2, \dots) : i_k \in \{1, \dots, N\} \text{ for every } k \in \mathbb{N}\}$$

and

$$\Omega_* = \bigcup_{n=1}^\infty \Omega_n, \text{ where } \Omega_n = \{1, \dots, N\}^n.$$

Observe that Ω_* (Ω , respectively) is the space of all finite (infinite, respectively) sequences of elements $i_k \in \{1, \dots, N\}$. For $\mathbf{i} = (i_1, \dots, i_n) \in \Omega_*$ let $|\mathbf{i}| = n$ denote the length of \mathbf{i} . If $\mathbf{i} \in \Omega$ we assume that $|\mathbf{i}| = \infty$. For $\mathbf{i} \in \Omega \cup \Omega_*$ and $m \in \mathbb{N}$, $m \leq |\mathbf{i}|$, we set

$$\mathbf{i}|m = (i_1, \dots, i_m).$$

A subset $\Lambda \subset \Omega$ is called a *cylinder* if there exists $\mathbf{i} \in \Omega_*$, $\mathbf{i} = (i_1, \dots, i_n)$, such that

$$\Lambda = \Lambda(\mathbf{i}) = \{\mathbf{j} \in \Omega : \mathbf{j}|n = \mathbf{i}\}.$$

By \mathcal{A} we denote the σ -algebra in Ω , which is generated by such cylinders.

Given an IFS $(S, p)_N$, for $x \in X$ we denote by \mathbb{P}_x the probability measure on \mathcal{A} defined on the cylinder $\Lambda(\mathbf{i})$, $\mathbf{i} = (i_1, \dots, i_n) \in \Omega_*$, by the formula

$$\mathbb{P}_x(\Lambda(\mathbf{i})) = p_{i_1}(x)p_{i_2}(S_{i_1}(x)) \cdot \dots \cdot p_{i_n}(S_{i_{n-1}} \circ \dots \circ S_{i_1}(x)).$$

For convenience, in the following we will write $\mathbb{P}_x(\mathbf{i})$ in the place of $\mathbb{P}_x(\Lambda(\mathbf{i}))$ and $\mathbb{P}_x(A)$ in the place of $\mathbb{P}_x(\Lambda(A))$, where $A \subset \Omega_n$ and $\Lambda(A) = \bigcup_{\mathbf{i} \in A} \Lambda(\mathbf{i})$. Moreover, for $\mathbf{i} \in \Omega_n$ we write

$$S_{\mathbf{i}} = S_{i_n} \circ \dots \circ S_{i_1}.$$

Let μ_* be the unique invariant probability measure for $(S, p)_N$. For $\alpha \geq 0$ we define

$$X_\alpha = \left\{ x \in X : \liminf_{r \rightarrow 0} \frac{\ln \mu_*(B(x, r))}{\ln r} \leq \alpha \right\}.$$

Lemma 2.2. *If an IFS $(S, p)_N$ satisfies Conditions (1.2)–(1.4), then*

$$S_i(X_\alpha) \subset X_\alpha,$$

for every $\alpha \geq 0$ and $i \in \{1, \dots, N\}$.

Proof. Fix $\alpha \geq 0$ and $i \in \{1, \dots, N\}$. Let $x \in X_\alpha$ be arbitrary. Using the fact that μ_* is stationary, Formula (1.1), Condition (1.3), and the inclusion $B(x, r/\lambda_i) \subset S_i^{-1}(B(S_i(x), r))$ we obtain

$$\mu_*(B(S_i(x), r)) \geq \kappa \mu_*(S_i^{-1}(B(S_i(x), r))) \geq \kappa \mu_*(B(x, r/\lambda_i)),$$

for every $r > 0$. Consequently

$$\liminf_{r \rightarrow 0} \frac{\ln \mu_*(B(S_i(x), r))}{\ln r} \leq \liminf_{r \rightarrow 0} \frac{\ln [\kappa \mu_*(B(x, r/\lambda_i))]}{\ln r}.$$

From this and the inequality

$$\liminf_{r \rightarrow 0} \frac{\ln \mu_*(B(x, r/\lambda_i))}{\ln(r/\lambda_i)} \leq \alpha,$$

we have

$$\liminf_{r \rightarrow 0} \frac{\ln \mu_*(B(S_i(x), r))}{\ln r} \leq \alpha,$$

which completes the proof. □

Proposition 2.3. *Assume that an IFS $(S, p)_N$ satisfies Conditions (1.2)–(1.4) and μ_* is the corresponding invariant probability measure. Let $\alpha \geq 0$. If $\mu_*(X_\alpha) > 0$, then $\mu_*(X_\alpha) = 1$.*

Proof. Let an IFS $(S, p)_N$ satisfying Conditions (1.2)–(1.4) be given and let P and μ_* be the corresponding Markov operator and invariant measure, respectively. Consider the measure $\bar{\mu} : \mathcal{B}(X) \rightarrow \mathbb{R}$ given by

$$\bar{\mu}(A) = \frac{\mu_*(A \cap X_\alpha)}{\mu_*(X_\alpha)}, \quad \text{for } A \in \mathcal{B}(X).$$

We have

$$\begin{aligned} P\bar{\mu}(A) &= \sum_{i=1}^N \int_{S_i^{-1}(A)} p_i(x) \bar{\mu}(dx) \\ &= \frac{1}{\mu_*(X_\alpha)} \sum_{i=1}^N \int_{S_i^{-1}(A) \cap X_\alpha} p_i(x) \mu_*(dx). \end{aligned}$$

From Lemma 2.2 it follows that $X_\alpha \subset S_i^{-1}(X_\alpha)$. Using this inclusion, Relation (1.1), and the fact that μ_* is invariant with respect to P , we obtain

$$\begin{aligned} P\bar{\mu}(A) &\leq \frac{1}{\mu_*(X_\alpha)} \sum_{i=1}^N \int_{S_i^{-1}(A \cap X_\alpha)} p_i(x) \mu_*(dx) \\ &= \frac{1}{\mu_*(X_\alpha)} \cdot P\mu_*(A \cap X_\alpha) = \frac{\mu_*(A \cap X_\alpha)}{\mu_*(X_\alpha)} = \bar{\mu}(A) \end{aligned}$$

for every $A \in \mathcal{B}(X)$. Since $P\bar{\mu} \leq \bar{\mu}$ and $P\bar{\mu}(X) = \bar{\mu}(X) = 1$, $P\bar{\mu} = \bar{\mu}$. From the uniqueness of the invariant probability measure it follows immediately that $\bar{\mu} = \mu_*$. Consequently $\mu_*(X_\alpha) = 1$. \square

Proposition 2.4. *Assume that an IFS $(S, p)_N$ satisfies Conditions (1.2)–(1.4) and μ_* is the corresponding invariant measure. Let $\alpha \geq 0$. If $\mu_*(X_\alpha) > 0$, then $\dim_H \mu_* \leq \alpha$.*

Proof. Let $\alpha \geq 0$ be such that $\mu_*(X_\alpha) > 0$. By Proposition 2.3 we have $\mu_*(X_\alpha) = 1$. According to the regularity of μ_* and the Ulam theorem, for every $n \in \mathbb{N}$ we can choose a compact set $K_n \subset X_\alpha$ such that $\mu_*(K_n) \geq 1 - 1/n$. Since $\mu_*(\bigcup_{n=1}^\infty K_n) = 1$ and $\dim_H(\bigcup_{n=1}^\infty K_n) = \sup\{\dim_H K_n : n \in \mathbb{N}\}$, to complete the proof it is enough to show that

$$\dim_H K_n \leq \alpha, \quad \text{for every } n \in \mathbb{N}.$$

This can be proved by a standard argument. For details see, for example, [8, Proposition 2.3]. \square

Fix $x_0 \in X$. Let γ_0 be given by Condition (1.2). Now, for arbitrary $\gamma > \gamma_0$ and $n \in \mathbb{N}$ we define the function $\varphi_n^\gamma : X \rightarrow [0, 1]$ by the formula

$$(2.1) \quad \varphi_n^\gamma(x) = \mathbb{P}_{x_0}(Q_n(\gamma; x_0, x)),$$

where

$$(2.2) \quad Q_n(\gamma; x_0, x) = \{\mathbf{i} \in \Omega : \rho(S_{\mathbf{i}|k}(x), S_{\mathbf{i}|k}(x_0)) \leq \gamma^k \rho(x, x_0) \text{ for every } k \geq n\}.$$

Lemma 2.5. *For every $\gamma > \gamma_0$ and $n \in \mathbb{N}$ the function φ_n^γ given by (2.1) and (2.2) is Borel measurable. Moreover, for every $x \in X$ the sequence $(\varphi_n^\gamma(x))_{n \geq 1}$ is increasing and*

$$\lim_{n \rightarrow \infty} \varphi_n^\gamma(x) = 1.$$

Proof. Fix $\gamma > \gamma_0$ and $n \in \mathbb{N}$. For arbitrary $x \in X$ we have

$$\begin{aligned} \varphi_n^\gamma(x) &= 1 - \mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega : \rho(S_{\mathbf{i}|m}(x), S_{\mathbf{i}|m}(x_0)) > \gamma^m \rho(x, x_0) \text{ for some } m \geq n\}) \\ &= 1 - \sum_{m=n}^\infty \sum_{\mathbf{i} \in \Omega_m} \mathbb{P}_{x_0}(\mathbf{i}) 1_{X_{\mathbf{i}}}(x), \end{aligned}$$

where $X_{\mathbf{i}}$, $\mathbf{i} \in \Omega_m$, stands for the set of all $y \in X$ such that $\rho(S_{\mathbf{i}}(y), S_{\mathbf{i}}(x_0)) > \gamma^m \rho(y, x_0)$ and $\rho(S_{\mathbf{i}|k}(y), S_{\mathbf{i}|k}(x_0)) \leq \gamma^k \rho(y, x_0)$ for $k = 1, \dots, m - 1$.

Since the sets $X_{\mathbf{i}}$, $\mathbf{i} \in \Omega_*$, are Borel measurable, the function φ_n^γ is Borel measurable. Obviously $\varphi_n^\gamma(x) \leq \varphi_{n+1}^\gamma(x)$ for every $n \in \mathbb{N}$ and $x \in X$. Moreover, from Lemma 2 in [5] it follows that $\lim_{n \rightarrow \infty} \varphi_n^\gamma(x) = 1$ for every $x \in X$. The proof is complete. \square

From Lemma 2.5 and the theorem of Egorov it follows immediately that:

Corollary 2.6. *Let $\gamma > \gamma_0$ and $x_0 \in X$ be given. Then for every $\varepsilon > 0$ there exist a subset X_0 of X and a number \tilde{n} such that*

$$\mu_*(X_0) > 1 - \varepsilon,$$

and

$$\mathbb{P}_{x_0}(Q_n(\gamma; x_0, x)) \geq 1 - \varepsilon, \quad \text{for every } x \in X_0 \text{ and } n \geq \tilde{n}.$$

Now for $\gamma > \gamma_0$, $x_0, x \in X$ and $n_0, n \in \mathbb{N}$, $n \geq n_0$, we define

$$Q_{n_0}^n(\gamma; x_0, x) = \{\mathbf{i} \in \Omega_n : \rho(S_{\mathbf{i}|k}(x), S_{\mathbf{i}|k}(x_0)) \leq \gamma^k \rho(x, x_0), \text{ for } n_0 \leq k \leq n\}.$$

Lemma 2.7. *Assume that a compact subset K of X , a point $x_0 \in K$ and a number $n_0 \in \mathbb{N}$ are given. Then for every $\gamma > \gamma_0$ there exist $\beta > 0$ such that for every $x \in K$ and $n \geq n_0$ we have*

$$\mathbb{P}_x(\mathbf{i}) \geq \beta \mathbb{P}_{x_0}(\mathbf{i}), \quad \text{for every } \mathbf{i} \in Q_{n_0}^n(\gamma; x_0, x).$$

Proof. Fix a compact set $K \subset X$ and set $d = \text{diam } K$. Since the function ω given by (1.4) satisfies the Dini condition, we have (see [5, 16])

$$\omega_0 = \sum_{k=1}^{\infty} \omega(\gamma^k d) < \infty.$$

Let $n \geq n_0$ and $x \in K$ be arbitrary. For $\mathbf{i} \in Q_{n_0}^n(\gamma; x_0, x)$ we have

$$\begin{aligned} \mathbb{P}_{x_0}(\mathbf{i}) &= p_{i_1}(x_0) p_{i_2}(S_{\mathbf{i}|1}(x_0)) \cdots p_{i_n}(S_{\mathbf{i}|n-1}(x_0)) \\ &= \frac{p_{i_1}(x_0) \cdots p_{i_{n_0}}(S_{\mathbf{i}|n_0-1}(x_0))}{p_{i_1}(x) \cdots p_{i_{n_0}}(S_{\mathbf{i}|n_0-1}(x))} \cdot p_{i_1}(x) \cdots p_{i_{n_0}}(S_{\mathbf{i}|n_0-1}(x)) \\ &\quad \cdot \prod_{k=n_0+1}^n \left(1 + \frac{p_{i_k}(S_{\mathbf{i}|k-1}(x_0)) - p_{i_k}(S_{\mathbf{i}|k-1}(x))}{p_{i_k}(S_{\mathbf{i}|k-1}(x))} \right) \cdot p_{i_k}(S_{\mathbf{i}|k-1}(x)) \end{aligned}$$

Using Conditions (1.3) and (1.4), and the definition of $Q_{n_0}^n(\gamma; x_0, x)$ we obtain

$$\begin{aligned} \mathbb{P}_{x_0}(\mathbf{i}) &\leq \frac{(1 - \kappa)^{n_0}}{\kappa^{n_0}} \cdot \prod_{k=n_0+1}^n \left(1 + \frac{\omega(\rho(S_{\mathbf{i}|k-1}(x), S_{\mathbf{i}|k-1}(x_0)))}{\kappa} \right) \cdot \mathbb{P}_x(\mathbf{i}) \\ &\leq \left(\frac{1 - \kappa}{\kappa} \right)^{n_0} \prod_{k=n_0+1}^n \left(1 + \frac{\omega(\gamma^{k-1} d)}{\kappa} \right) \cdot \mathbb{P}_x(\mathbf{i}). \end{aligned}$$

Consequently

$$\mathbb{P}_{x_0}(\mathbf{i}) \leq \left(\frac{1 - \kappa}{\kappa} \right)^{n_0} \prod_{k=n_0+1}^{\infty} e^{\omega(\gamma^{k-1} d)/\kappa} \cdot \mathbb{P}_x(\mathbf{i}) \leq \left(\frac{1 - \kappa}{\kappa} \right)^{n_0} e^{\omega_0/\kappa} \cdot \mathbb{P}_x(\mathbf{i}).$$

Setting $\beta = \kappa^{n_0} (1 - \kappa)^{-n_0} e^{-\omega_0/\kappa}$ we complete the proof.

Using a standard martingale argument one can prove the following version of the Brin–Katok theorem (see [3])

Lemma 2.8. *Let $x_0 \in X$. Assume that an IFS $(S, p)_N$ satisfies Conditions (1.2)–(1.4). Then*

$$(2.3) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}_{x_0}(\mathbf{i}|n) \geq \ln \delta_0 \quad \mathbb{P}_{x_0}\text{-a.s.},$$

where

$$(2.4) \quad \delta_0 = \inf_{x \in X} \prod_{i=1}^N p_i(x)^{p_i(x)}.$$

Proof. Define the function $X_n : \Omega \rightarrow \mathbb{R}$ by the formula

$$X_n(\mathbf{i}) = -\ln(p_{i_n}(S_{i_{n-1}} \circ \dots \circ S_{i_1}(x_0))), \quad \text{for } n \in \mathbb{N}.$$

For $\mathbf{i} \in \Omega_*$, $\mathbf{i} = (i_1, \dots, i_n)$, we denote by $\mathcal{A}(\mathbf{i})$ the σ -algebra generated by the cylinders $\{\Lambda(\mathbf{j}) : \mathbf{j} \in \Omega_*, \mathbf{j} > \mathbf{i}\}$. Moreover, let \mathbb{E}_{x_0} denote the expectation with respect to the probability measure \mathbb{P}_{x_0} on Ω .

Fix $\mathbf{i} \in \Omega$, $\mathbf{i} = (i_1, i_2, \dots)$ and observe that

$$\mathbb{E}_{x_0}(X_n | \mathcal{A}(i_1, \dots, i_{n-1})) = \sum_{i=1}^N p_i(S_{i_{n-1}} \circ \dots \circ S_{i_1}(x_0)) X_n((i_1, \dots, i_{n-1}, i)).$$

By (2.4) we have

$$-\sum_{i=1}^N p_i(S_{i_{n-1}} \circ \dots \circ S_{i_1}(x_0)) \ln(p_i(S_{i_{n-1}} \circ \dots \circ S_{i_1}(x_0))) \leq -\ln \delta_0.$$

Now let $Y_n = X_n - \mathbb{E}_{x_0}(X_n | \mathcal{A}(i_1, \dots, i_{n-1}))$. Then

$$|Y_n(\mathbf{i})| \leq 2 \sup_{\mathbf{i} \in \Omega} |X_n(\mathbf{i})| \quad \mathbb{P}_{x_0}\text{-a.s.}$$

Write

$$M = 2 \sup_{\mathbf{i} \in \Omega} |X_n(\mathbf{i})| < \infty.$$

Define

$$Z_n = \sum_{k=1}^n \frac{Y_k}{k}, \quad \text{for } n \in \mathbb{N}.$$

It is easy to see that $(Z_n)_{n \geq 1}$ is a martingale. Since Y_k and Y_l for $k \neq l$ are mutually orthogonal, we have

$$\mathbb{E}_x(Z_n^2) \leq M^2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Hence $(Z_n)_{n \geq 1}$ is an \mathbb{L}^2 -bounded martingale, and so $(Z_n)_{n \geq 1}$ is convergent a.s. Then by Kronecker's lemma (see [4])

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n Y_k = 0 \quad \mathbb{P}_{x_0}\text{-a.s.}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\infty} X_k + \ln \delta_0 \leq 0 \quad \mathbb{P}_{x_0}\text{-a.s.},$$

whence

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}_{x_0}(\mathbf{i}|n) \geq \ln \delta_0 \quad \mathbb{P}_{x_0}\text{-a.s.}$$

The proof is complete. □

Finally observe that $-\ln \delta_0$ is some form of entropy for the IFS (see [8]).

3. An upper estimation

Theorem 3.1. *Assume that an IFS $(S, p)_N$ satisfies Conditions (1.2)–(1.4) and let μ_* be the corresponding invariant probability measure. Then*

$$\dim_H \mu_* \leq \frac{\ln \delta_0}{\ln \gamma_0},$$

where γ_0 and δ_0 are given by (1.2) and (2.4), respectively.

Proof. Let μ_* be the unique invariant probability measure for $(S, p)_N$ and let K be a compact subset of X such that $\mu_*(K) \geq 3/4$. Set $d = \text{diam } K$. Choose $x_0 \in K$ and let $\delta \in (0, \delta_0)$ and $\gamma \in (\gamma_0, 1)$ be arbitrary. By virtue of Corollary 2.6 there exist a subset K_0 of K and a number $n_0 \in \mathbb{N}$ such that $\mu_*(K_0) \geq 1/2$ and

$$(3.1) \quad \mathbb{P}_{x_0}(\mathcal{Q}_n(\gamma; x_0, x)) \geq \frac{3}{4}, \quad \text{for every } x \in K_0 \text{ and } n \geq n_0.$$

By virtue of Lemma 2.8 and the theorem of Egorov we can also simultaneously assume that

$$(3.2) \quad \mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega : \mathbb{P}_{x_0}(\mathbf{i}|n) \geq \delta^n \text{ for every } n \geq n_0\}) \geq 3/4.$$

From (3.1), (3.2), and the obvious inclusion $\mathcal{Q}_{n_0}(\gamma; x_0, x) \subset \Lambda(\mathcal{Q}_{n_0}^n(\gamma; x_0, x))$, $n \geq n_0$, it follows that for every $x \in K_0$ and every $n \geq n_0$ we have

$$\mathbb{P}_{x_0}(\{\mathbf{i} \in \Omega : \mathbf{i}|n \in \mathcal{Q}_{n_0}^n(\gamma; x_0, x) \text{ and } \mathbb{P}_{x_0}(\mathbf{i}|n) \geq \delta^n\}) \geq \frac{1}{2},$$

or equivalently that

$$\sum_{\mathbf{i} \in \mathcal{Q}_{n_0}^n(\gamma; x_0, x), P_{x_0}(\mathbf{i}) \geq \delta^n} \mathbb{P}_{x_0}(\mathbf{i}) \geq \frac{1}{2} \quad \text{for } x \in K_0 \text{ and } n \geq n_0.$$

Since $Q_{n_0}^n(\gamma; x_0, x) \subset Q_m^n(\gamma; x_0, x)$ for $n \geq m \geq n_0$, by Lemma 2.7 there exists $\beta > 0$ such that

$$(3.3) \quad \sum_{\mathbf{i} \in Q_{n_0}^n(\gamma; x_0, x), P_{x_0}(\mathbf{i}) \geq \delta^n} \mathbb{P}_x(\mathbf{i}) \geq \frac{\beta}{2}, \quad \text{for } x \in K_0 \text{ and } n \geq n_0.$$

Now, for $n \in \mathbb{N}$ we define

$$Y_n = \bigcup_{\mathbf{i} \in \Omega_n, P_{x_0}(\mathbf{i}) \geq \delta^n} B(S_{\mathbf{i}}(x_0), \gamma^n d).$$

Let $n \geq n_0$. For $x \in K_0$ and $\mathbf{i} \in Q_{n_0}^n(\gamma; x_0, x)$ we have $\rho(S_{\mathbf{i}}(x), S_{\mathbf{i}}(x_0)) \leq \gamma^n \rho(x, x_0) \leq \gamma^n d$, which means that $S_{\mathbf{i}}(x) \in Y_n$. Consequently, for arbitrary $n \geq n_0$ we have

$$(3.4) \quad K_0 \subset S_{\mathbf{i}}^{-1}(Y_n), \quad \text{for every } \mathbf{i} \in Q_{n_0}^n(\gamma; x_0, x).$$

Since μ_* is an invariant measure for the IFS $(S, p)_N$, using (1.1), (3.3), (3.4), and the inequality $\mu_*(K_0) \geq 1/2$ we obtain

$$(3.5) \quad \begin{aligned} \mu_*(Y_n) &= \sum_{\mathbf{i} \in \Omega_n} \int_X 1_{Y_n}(S_{\mathbf{i}}(x)) \mathbb{P}_x(\mathbf{i}) \mu_*(dx) \\ &\geq \sum_{\mathbf{i} \in \Omega_n, P_{x_0}(\mathbf{i}) \geq \delta^n} \int_X 1_{Y_n}(S_{\mathbf{i}}(x)) \mathbb{P}_x(\mathbf{i}) \mu_*(dx) \\ &\geq \int_{K_0} \sum_{\mathbf{i} \in Q_{n_0}^n(\gamma; x_0, x), P_{x_0}(\mathbf{i}) \geq \delta^n} \mathbb{P}_x(\mathbf{i}) \mu_*(dx) \geq \frac{\beta}{2} \cdot \frac{1}{2} = \frac{\beta}{4} \end{aligned}$$

for every $n \geq n_0$.

Set $\eta = \beta/4$. For $n \in \mathbb{N}$ define

$$Z_n = \{x \in Y_n : \mu_*(B(x, 2\gamma^n d)) \leq \eta \delta^n / 2\}.$$

If $x \in Z_n$, by the definition of Y_n there exists $\mathbf{i}_0 \in \Omega_n$ with $P_{x_0}(\mathbf{i}_0) \geq \delta^n$, such that $x \in B(S_{\mathbf{i}_0}(x_0), \gamma^n d)$. Since $B(S_{\mathbf{i}_0}(x_0), \gamma^n d) \subset B(x, 2\gamma^n d)$, it follows that $\mu_*(B(S_{\mathbf{i}_0}(x_0), \gamma^n d)) \leq \eta \delta^n / 2$. This means that the set Z_n can be covered by the balls $B(S_{\mathbf{i}}(x_0), \gamma^n d)$ with $\mathbf{i} \in \Omega_n, P_{x_0}(\mathbf{i}) \geq \delta^n$ and $\mu_*(B(S_{\mathbf{i}}(x_0), \gamma^n d)) \leq \eta \delta^n / 2$. From this and the equality $P_{x_0}(\Omega) = 1$ it follows that

$$(3.6) \quad \mu_*(Z_n) \leq \text{card} \{ \mathbf{i} \in \Omega_n : P_{x_0}(\mathbf{i}) \geq \delta^n \} \cdot \frac{\eta \delta^n}{2} \leq \delta^{-n} \cdot \frac{\eta \delta^n}{2} = \frac{\eta}{2},$$

for every $n \geq n_0$. (Here the symbol card stands for the cardinality.) From (3.5) and (3.6) it follows that $\mu_*(Y_n \setminus Z_n) \geq \eta/2$ for every $n \geq n_0$. Set $X_n = Y_n \setminus Z_n$ and define $X^\infty = \bigcap_{m=n_0}^\infty \bigcup_{n=m}^\infty X_n$. Since $\mu_*(X_n) \geq \eta/2$ for $n \geq n_0$, from the continuity of measure it follows immediately that $\mu_*(X^\infty) \geq \eta/2$. Moreover, a simple calculation shows that

$$X^\infty \subset \left\{ x \in X : \liminf_{r \rightarrow 0} \frac{\ln \mu_*(B(x, r))}{\ln r} \leq \frac{\ln \delta}{\ln \gamma} \right\}.$$

Thus, by Proposition 2.4 we have

$$\dim_H \mu_* \leq \frac{\ln \delta}{\ln \gamma}.$$

Letting $\delta \rightarrow \delta_0$ and $\gamma \rightarrow \gamma_0$, we complete the proof. □

4. A lower estimation

In this section we assume that $S_i : X \rightarrow X, i = 1, \dots, N$ are continuous transformations such that

$$(4.1) \quad \rho(S_i(x), S_i(y)) \geq \ell_i \rho(x, y), \quad \text{for } x, y \in X,$$

where $\ell_i \in (0, 1), i = 1, \dots, N$. Without any loss of generalisation we can assume that

$$(4.2) \quad l_1 \leq \ell_i \leq l_N, \quad \text{for } i = 1, \dots, N.$$

Moreover, we assume that there exists a compact set K such that

$$(4.3) \quad K = \bigcup_{i=1}^N S_i(K),$$

and

$$(4.4) \quad \bigcap_{i=1}^N S_i(K) = \emptyset.$$

Now, for $x \in X$, define

$$(4.5) \quad I_x = \{i : i \in \{1, \dots, N\} \text{ and } x \in S_i(K)\}.$$

Theorem 4.1. *Assume that an IFS $(S, p)_N$ satisfies Conditions (4.1)–(4.4). Moreover, assume that it admits an invariant probability measure μ_* such that $\text{supp } \mu_* \subset K$. If*

$$(4.6) \quad \sum_{i \in I_x} p_i (S_i^{-1}(x)) < 1, \quad \text{for every } x \in K,$$

then

$$(4.7) \quad \dim_H \mu_* \geq s_*,$$

where

$$(4.8) \quad s_* = \sup \left\{ s > 0 : \sum_{i \in I_x} p_i (S_i^{-1}(x)) l_i^{-s} \leq 1 \text{ for } x \in K \right\},$$

and I_x is defined by (4.5).

Proof. Consider the function $\sigma : X \rightarrow \mathbb{R}_+$ given by

$$\sigma(x) = \min_{i \notin I_x} \rho(x, S_i(K)).$$

Simple calculation shows that σ is a continuous function. Moreover, from (4.4) it follows that $\sigma(x) > 0$ for $x \in K$. Thus

$$\sigma_0 = \min_{x \in K} \sigma(x) > 0.$$

Now, let s_* be given by (4.8). Fix an arbitrary $s \in (0, s_*)$. Clearly

$$\sum_{i \in I_x} p_i (S_i^{-1}(x)) l_i^{-s} < 1 \quad \text{for every } x \in K.$$

Owing to the continuity of p_i, S_i for $i = 1, \dots, N$ and compactness of K , there is $\hat{r} \in (0, \sigma_0)$ such that

$$(4.9) \quad \sum_{i \in I_x} p_i(z_i) l_i^{-s} < 1,$$

for every $x \in K$ and every $z_i \in B(S_i^{-1}(x), \hat{r})$ with $i \in I_x$.

Set $r_0 = l_1 \hat{r}$. We claim that

$$(4.10) \quad \mu_*(B(x, r)) \leq \left(\frac{r}{r_0}\right)^s,$$

for every $x \in K$ and $r > 0$.

Obviously (4.10) holds for every $x \in X$ and $r \geq r_0$. Define

$$r_* = \inf \left\{ r > 0 : \mu_*(B(x, t)) \leq \left(\frac{t}{r_0}\right)^s \text{ for } t \geq r \text{ and } x \in K \right\}.$$

We will prove that $r_* = 0$. Suppose, for a contradiction, that $r_* > 0$. Let $r \in [l_N r_*, r_0)$. Using the fact that μ_* is an invariant measure, the inclusion $\text{supp } \mu_* \subset K$, Formula (1.1), and the inequality $r < r_0$, we have

$$\mu_*(B(x, r)) = \sum_{i=1}^N \int_{S_i^{-1}(B(x, r))} p_i(z) \mu_*(dz) = \sum_{i=1}^N \int_{S_i^{-1}(B(x, r)) \cap K} p_i(z) \mu_*(dz).$$

There exists $y \in K$ such that $S^{-1}(B(x, r)) \cap K = \emptyset$ for $i \notin I_y$. Hence

$$\begin{aligned} \mu_*(B(x, r)) &\leq \sum_{i \in I_y} \int_{S_i^{-1}(B(x, r)) \cap K} p_i(z) \mu_*(dz) \\ &= \sum_{i \in I_y} p_i(z_i) \mu_*(S_i^{-1}(B(x, r))), \end{aligned}$$

for some $z_i \in B(S_i^{-1}(x), r/l_i)$, $i \in I_x$. Since $S_i^{-1}(B(x, r) \text{ big}) \subset B(S_i^{-1}(x), r/l_i)$ and $r/l_i > r_*$ we have

$$\begin{aligned} \mu_*(B(x, r)) &\leq \sum_{i \in I_x} p_i(z_i) \mu_*(B(S_i^{-1}(x), r/l_i)) \leq \sum_{i \in I_x} p_i(z_i) \left(\frac{r}{l_i r_0}\right)^s \\ &\leq \left(\frac{r}{r_0}\right)^s \sum_{i \in I_x} p_i(z_i) l_i^s. \end{aligned}$$

From the last inequality and (4.9) it follows that

$$\mu_*(B(x, r)) \leq \left(\frac{r}{r_0}\right)^s, \quad \text{for } r \geq l_N r_* \text{ and } x \in K,$$

which contradicts the definition of r_* . Thus $r_* = 0$ and (4.10) hold for every $x \in K$ and $r > 0$. Consequently, by the mass distribution principle (see [24, p. 43]) we have

$$\dim_H \mu_* \geq s.$$

Since $s \in (0, s_*)$ was arbitrary, Inequality (4.7) follows. The proof is completed. \square

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