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On rational and ruled double planes

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Abstract. Following the ideas of Castelnuovo and Enriques, we classify the birational equivalence classes of double planes which are rational or ruled surfaces. In order to do this, we prove that the vanishing of the m-adjoint linear system to the branch curve of the canonical resolution of a double plane, for $m \geq 2$, is a necessary and sufficient condition for the ruledness of the double plane.

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1. Introduction

A *double plane* is a double covering of the projective plane, i.e., a finite morphism $\pi : X \to \mathbb{P}^2$ of degree 2. We say that $\rho : Y \to \mathbb{P}^2$ is *birationally equivalent* to π if there exist two birational maps $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and $\phi : Y \dashrightarrow X$ such that the following diagram commutes:

$$(1.1) \qquad \begin{array}{ccc} Y & \overset{\phi}{\dashrightarrow} & X \\ \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{P}^2 & \overset{\gamma}{\dashrightarrow} & \mathbb{P}^2. \end{array}$$

The goal of this paper is to classify the birational equivalence classes of double planes which are rational or ruled surfaces. Indeed we will prove the following (see Theorem 9.18 for a more precise statement):

Theorem 1.2. *A double plane is a rational surface if and only if it is birationally equivalent to a normal double plane branched along one of the following:*

- *a smooth quartic;*
- *a sextic with two infinitely near triple points; or*
- *a curve of degree* $2d$ *with a point of multiplicity* $2d-2$.

Moreover a double plane is ruled of genus $q > 0$ *if and only if it is birationally equivalent to a double plane branched along* $2q+2$ *distinct lines through a point.*

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The first part of the previous theorem, concerning rational double planes, was stated by Noether [18] in 1878. Castelnuovo and Enriques [7] claimed to have proved Theorem 1.2 in 1900. Their proof, also improved by Conforto [10] in 1938, has been commonly accepted, but it actually contains some small mistakes and a serious gap, as we will see.

Let $\pi : X \to \mathbb{P}^2$ be a normal double plane and $\tilde{\pi} : \tilde{X} \to S$ its canonical resolution (see Sect. 3), which is a double covering branched along a smooth curve B. If X has Kodaira dimension $-\infty$, then we will see (Equation (3.4)) that:

$$|B + mK_S| = \emptyset, \qquad \text{for } m \geq 2. \tag{1.3}$$

Castelnuovo and Enriques' idea is that Condition (1.3) suffices to find a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ fitting in Diagram (1.1) where $\rho : Y \to \mathbb{P}^2$ is one of the double planes listed in Theorem 1.2, which are known to be rational.

The proof is done by induction on the so-called *simplicity* of the branch curve of the canonical resolution of the double plane (see Sect. 6). In fact, if the branch curve of the double plane is not already in the list of Theorem 1.2, then Condition (1.3) enables us to find a series of quadratic transformations based at some of its singular points in order to make it *simpler*, in a well-defined sense.

In order to handle irrational ruled double planes, it is necessary to apply a theorem of De Franchis [11] (for a modern proof see Catanese and Ciliberto [8] or Khashin [16]) and the birational classification of linear systems of rational plane curves (see Appendix A).

The results contained in this work are part of the author's *Tesi di dottorato* [4] (Ph.D. dissertation), which has been defended successfully on March 1999. In a forthcoming paper, we will study the rationality and ruledness of cyclic triple planes by extending these techniques.

We remark that the classification of birational equivalence classes of rational double planes is equivalent to Bertini's classification of plane involutions, namely Cremona transformations $\iota : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\iota \circ \iota = 1_{\mathbb{P}^2}$ (and $\iota \neq 1_{\mathbb{P}^2}$) modulo coniugacy: $\iota_1 \sim \iota_2$ if and only if there is a Cremona transformation γ with $\gamma \circ \iota_1 = \iota_2 \circ \gamma$ (see [9, l. 2, Chap. IV] for the classical approach or the very recent paper [3] by Bayle and Beauville for a modern proof via Mori theory).

We consider projective surfaces defined over $\mathbb{C}$, but all statements are equally true over any algebraically closed field of characteristic zero. We refer to [15] for the standard algebraic geometry dictionary.

Historical note. In 1900, Castelnuovo and Enriques observed that the adjoint linear systems to the branch curve of a rational double plane vanish. They considered, however, not the branch curve itself, but one with *virtual* multiplicities instead of the real ones. What they really had in mind is indeed the branch curve B of the canonical resolution of the double plane. In fact, such virtual multiplicities are, in general, equal to those of B; they differ only in the case of infinitely near points with the same odd multiplicity (see Sect. 4). This is the reason why we call B the *virtual* branch curve of the double plane.

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2. Notation

In this paper, $\sigma : S \to \mathbb{P}^2$ will always be the composition of finitely many monoidal transformations, i.e., $\sigma = \sigma_0 \circ \cdots \circ \sigma_r$ for some r, where $\sigma_i : S_i \to S_{i-1}$ is the monoidal transformation with its center at a point $x_i \in S_{i-1}$, with $S_{-1} = \mathbb{P}^2$ and $S = S_r$. Regarding the position of the x_i's, the classical notions of infinitely near, proximate and satellite points are useful (see [12, v. 2, pp. 336–386], [15, p. 392], [6, pp. 430–431], [13, 1.2.2] or [4, Sect. 2]). For the readers' convenience, we recall the very basic definitions.

The point x_j is *infinitely near* to x_j, and we write $x_j > x_i$, if

$$x_j \in (\sigma_{j-1} \circ \sigma_{j-2} \circ \cdots \circ \sigma_i)^{-1}(x_i) \subset S_{j-1},$$

while x_j is *proximate* to x_i, and we write $x_j \to x_i$, if x_j lies on the strict transform in S_{j-1} of $\sigma_i^{-1}(x_i)$. A *proper* (or infinitely near of order 0) point is a point $x_j \in S$ which lies on no exceptional curve for σ and we write $x_j \in \mathbb{P}^2$.

The *infinitesimal order* can be defined by induction. Suppose that $x_j > x_i$. If $x_j \in \sigma_{j-1}^{-1}(x_{j-1})$ and x_{j-1} is infinitely near of order $s-1$ to x_i, then x_j is infinitely near of order s to x_i and we write $x_j >^s x_i$. Otherwise, $\sigma_{j-1}(x_j) >^t x_i$ for some t and we set $x_j >^t x_i$.

The point x_j is *satellite* to x_i, and we write $x_j \odot x_i$, if $x_j \to x_i$ and $x_j >^s x_i$ with $s > 1$. A point is called *satellite* if it is satellite to some point, otherwise it is said to be *free*.

Let E_i (E_i^*, respectively) be the strict (total, respectively) transform in S of the exceptional curve $\sigma_i^{-1}(x_i) \subset S_i$. Recall that $\{E_i\}_{0 \le i \le n}$, as well as $\{E_j^*\}_{0 \le j \le n}$, is a set of generators of Pic $S/\sigma^*(\text{Pic}\,\mathbb{P}^2)$ and the base change is given by

$$E_i = E_i^* - \sum_{j : x_j \to x_i} E_j^*. \tag{2.1}$$

If we write a divisor C in Pic S as a linear combination of the E_i^*'s

$$C = dL - \sum_{i=0}^{n} c_i E_i^*, \tag{2.2}$$

where L is the total transform in S of a line in $\mathbb{P}^2$, we say that d is the *degree* of C and c_i is the *multiplicity* of C at x_i. For strict transforms in S of plane curves, these definitions coincide with the usual ones. We say that C verifies the *proximity inequality* at x_i if $(C \cdot E_i) \ge 0$, or equivalently, by (2.1), if

$$c_i \ge \sum_{j : x_j \to x_i} c_j. \tag{2.3}$$

Clearly the strict transform in S of a plane curve satisfies the proximity inequalities, namely it verifies the proximity inequality at x_i, for every $i = 0, \dots, r$ (see [12, v. 2, p. 381], [6, p. 431] or [4, Sect. 9]).

3. Preliminaries on double planes

A double plane $\pi : X \to \mathbb{P}^2$, as any double covering over an algebraic surface whose Picard group has no torsion, is uniquely determined by its branch curve C (of even degree). We recall that X is normal if and only if C is reduced and a point $x \in X$ is singular if and only if C is singular at $\pi(x)$. Moreover X is reducible if and only if X splits into two copies of $\mathbb{P}^2$ which meet transversally, and this happens if and only if there exists a divisor D in $\mathbb{P}^2$ such that $C = 2D$. Later on, we will deal only with irreducible double planes.

We say that two double coverings $\rho : Y \to T$ and $\pi : X \to T$ over a smooth surface T are *strictly birationally equivalent* if there exists a birational map $\phi : Y \dashrightarrow X$ such that $\rho = \pi \circ \phi$. It is easy to show (see [19, p. 19] or [4, Sect. 30]) that there is a one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{strict birational equivalence} \\ \text{classes of double planes} \end{array} \right\} \longleftrightarrow \frac{K^*}{K^{*2}},$$

where K is the field of rational functions on T and $K^* = K \setminus \{0\}$. Furthermore in each strict birational equivalence class there is one and only one normal double covering over T. The normalization process consists of eliminating the even multiplicity components of the branch curve and then taking the reduced part of the rest. Therefore, up to (strict) birational equivalence, it is convenient to assume a double covering over T to be normal, or equivalently its branch curve to be reduced.

Let $\pi : X \to \mathbb{P}^2$ be a double plane. A Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ canonically induces a normal double plane $\pi_\gamma : X_\gamma \to \mathbb{P}^2$ birationally equivalent to π, namely X_γ is the normalization of the double plane branched along the total transform $\gamma^*(C)$, where C is the branch curve of π. Moreover, a birational morphism $\sigma : S \to \mathbb{P}^2$ canonically induces a normal double covering $\pi^\sigma : X^\sigma \to S$, which is the normalization of the fibered product $X \times_{\mathbb{P}^2} S$. We say that π^σ is a *resolution* of π if X^σ is smooth.

If X is not smooth, a resolution of π is obtained by applying a sequence of monoidal transformations centered at singular points of the branch curve and normalizing. In such a way, one gets *the canonical resolution* $\tilde{\pi} : \tilde{X} \to S$, which is uniquely determined up to isomorphism (see [2, Theorem III.7.2] or [5]).

Let B be the (smooth) branch curve of $\tilde{\pi}$. In the next section we will see that $A = B/2$ is well defined in Pic S. Since $\tilde{\pi}_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_S \oplus \mathcal{O}_S(A)$, the projection formula [15, Ex. II.5.1] implies that

$$P_m(X) = P_m(\tilde{X}) = h^0(mA + mK) + h^0((m-1)A + mK), \tag{3.1}$$

where $K = K_S$ and the cohomology groups are computed on S. For $m = 1$, (3.1) becomes $p_g(X) = h^0(A + K)$. Moreover, the irregularity of X is

$$q(X) = q(\tilde{X}) = h^1(-A) = p_g(X) - p_a(A). \tag{3.2}$$

If n is even and $n > 2$, then $h^0(nA + mK) \geq h^0(B + mK)$, because $2A = B$ is an effective divisor, so (3.1) implies that

$$P_m(X) \geq h^0(B + mK), \qquad \text{for } m \geq 2. \tag{3.3}$$

If X has Kodaira dimension $-\infty$, then (3.3) forces

$$|B + mK| = \emptyset, \qquad \text{for } m \geq 2, \tag{3.4}$$

and (3.2) becomes $q(X) = -p_a(A)$. If X is rational, it follows that $p_a(A) = 0$.

4. The virtual branch curve

Let $\tilde{\pi} : \tilde{X} \to S$ be the canonical resolution of a normal double plane $\pi : X \to \mathbb{P}^2$, where $\tilde{\pi} = \pi^\sigma$ and $\sigma : S \to \mathbb{P}^2$ is as in Sect. 2. If the branch curve C of π has odd multiplicity at a point $x_i \in \mathbb{P}^2$, then E_i is a component of the branch curve of $\tilde{\pi}$. More precisely, if one writes the strict trasform of C in Pic S as

$$\tilde{C} = \sigma^* C - \sum_{i=0}^{r} c_i' E_i = \sigma^* C - \sum_{i=0}^{r} c_i E_i^*,$$

where c_i is the multiplicity of C at x_i, then the branch curve of $\tilde{X}$ is

$$B = \tilde{C} + \sum_{i=0}^{r} p_i E_i = \sigma^* C - \sum_{i=0}^{r} b_i' E_i^* = \sigma^* C - \sum_{i=0}^{r} b_i E_i^*, \tag{4.1}$$

where $p_i = c_i' \bmod 2 \in \{0, 1\}$; hence $b_i' = c_i' - p_i$ is even and so are the b_i's (see also [5]). Therefore, $B/2$ is well defined in Pic S. We define b_i the *virtual* multiplicity at x_i of the *virtual* branch curve B of π. To avoid confusion, sometimes we will say that C is the *effective* branch curve of π and c_i is the *effective* multiplicity at x_i. Note that the b_i's equal the c_i's if and only if all the c_i's are even.

We may compute the virtual multiplicities of the branch curve from the effective ones by plugging Formula (2.1) in (4.1). Namely,

$$b_i = \begin{cases} c_i - p_i & \text{if } x_i \in \mathbb{P}^2, \\ c_i + p_j - p_i, & \text{if } x_i >^1 x_j \text{ and } x_i \text{ is free,} \\ c_i + p_j + p_k - p_i & \text{if } x_i >^1 x_j > x_k \text{ and } x_i \odot x_k. \end{cases} \tag{4.2}$$

Let us say that a point x_j is *defective* if there exists $x_i >^1 x_j$ with $b_i > b_j$. Such x_i is called *excessive* and associated to x_j. By (4.2), this may happen only if $p_j = 1$ and $c_i = c_j$, so $b_i = b_j + 2$ and $p_i = 0$. If x_i is free, then $c_i = c_j$ must be odd, while if x_i is satellite, say $x_i \odot x_k$, then $c_i = c_j \equiv 1 + p_k \pmod 2$. Since C verifies the proximity inequalities, x_i is the unique proximate point to x_j.

Lemma 4.3. *Suppose that C contains a line L' through a point x_i and there exists $x_j >^1 x_i$ with $c_i = c_j$ (e.g., x_i is defective). Then $x_j \in L'$.*

Proof. By the proximity inequality for $C - L'$ at x_i. □

Lemma 4.4. *Let x_j be a defective point. For every $m \geq 0$, if $|B + mK|$ (the m-adjoint linear system to B) is not empty, then E_j is one of its fixed components.*

Proof. Let x_i be the excessive proximate point to x_j. Then (2.1) becomes $E_j = E_j^* - E_i^*$, therefore for all $m \geq 0$:

$$(E_j, B + mK) = (E_j^* - E_i^*, -(b_j - m)E_j^* - (b_i - m)E_j^*) = b_j - b_i = -2,$$

i.e., E_j negatively intersects $B + mK$. □

Thus, in general, B does not satisfy the proximity inequalities. However:

Lemma 4.5. *Suppose that x_i is not defective. Then, for $m \geq 2$,*

$$b_i - m \geq \sum_{j:x_j \to x_i} (b_j - m), \tag{4.6}$$

namely $B + mK$ verifies the proximity inequality at x_i, for $m \geq 2$.

Proof. We may assume that $i = 2$ and $x_3, \dots, x_{n+2}$ are the proximate points to x_2. It suffices to show (4.6) for $m = 2$, i.e., since the b_i's are even,

$$b_2 > \sum_{j=3}^{n+2} (b_j - 2). \tag{4.7}$$

If $n = 1$, then (4.7) follows from the hypothesis that x_2 is not defective. Now assume $n \geq 2$. There are many cases depending on the position of x_2.

(a) If x_2 is satellite, we suppose that $x_2 >^1 x_1 > x_0$, $x_2 \odot x_0$ and

$$\begin{gathered} x_{i_h} >^1 x_{i_h - 1} >^1 \cdots >^1 x_{i_{h-1}+1} >^1 x_2, \quad \text{for every } h = 1, \dots, l, \\ \text{where } n + 2 = i_l > i_{l-1} > \cdots > i_1 > i_0 = 2, \end{gathered} \tag{4.8}$$

and, moreover, $x_3 \to x_1$, $x_{i_1+1} \to x_0$. By (4.2), Formula (4.7) is equivalent to

$$c_2 + p_0 + p_1 - p_2 > \sum_{j=3}^{n+2} c_j + p_0 + p_1 + np_2 - \sum_{h=1}^{l} p_{i_h} - 2n. \tag{4.9}$$

By the proximity inequality at x_2, i.e., $c_2 \geq \sum_{j=3}^{n+2} c_j$, Formula (4.9) follows from

$$(n + 1)p_2 - 2n - \sum_{h=1}^{l} p_{i_h} < 0, \tag{4.10}$$

which is clearly true for $n \geq 2$, because $0 \leq p_2 \leq 1$.

(b) If x_2 is satellite as above, but $x_3 \not\to x_1$ and (or, respectively) $x_{i_1+1} \not\to x_0$, then (4.7) is equivalent to (4.9), where p_1 and (or, respectively) p_0 are missing in the right-hand side, and (4.9) is true again.

(c) If x_2 is free, say $x_2 >^1 x_1$ and (4.8), with $x_3 \to x_1$ ($x_3 \not\to x_1$, respectively), then p_0 is missing in both sides (and p_1 is missing in the right-hand side, respectively) of (4.9), so (4.9) holds once more.

(d) If x_2 is proper and (4.8), then (4.7) is equivalent to (4.9), where p_0 and p_1 are missing in both sides and we conclude as before. □

Corollary 4.11. *Let $\bar{B} = B - \sum_j E_j$, where the sum runs over the j's such that x_j is defective. Then $\bar{B} + mK$ verifies the proximity inequalities for $m \geq 2$.*

Proof. It suffices to prove the thesis for $m = 2$. Let $\bar{b}_i$ be the multiplicity of $\bar{B}$ at x_i, for every $i = 0, \dots, r$. Note that $\bar{b}_i \neq b_i$ if and only if x_i is either defective or excessive. Clearly $\bar{B}$, hence $\bar{B} + 2K$, verifies the proximity inequality at defective points: if x_i is defective, then x_i has only one proximate point x_j, which is excessive, and $\bar{b}_i = \bar{b}_j = b_i + 1 = b_j - 1$. Let x_i not be defective. As in Lemma 4.5, we assume $i = 2$, $x_j \to x_2$ for $j = 3, \dots, n+2$ and (4.8). We must show that

$$\bar{b}_2 - 2 \geq \sum_{j=3}^{n+2} \bar{b}_j - 2n. \tag{4.12}$$

If x_j is defective and $x_j \to x_2$, then either $j = i_h$ for some h, or x_{j+1} is excessive and $x_{j+1} \to x_2$. In the former case, $\bar{b}_{i_h} = b_{i_h} + 1$ and $p_{i_h} = 1$, while in the latter case $\bar{b}_j + \bar{b}_{j+1} = b_j + b_{j+1}$. Since $\bar{b}_2 = b_2 - 1 = c_2 + p_0$ ($\bar{b}_2 = b_2$, respectively) if x_2 is (is not, respectively) excessive, the same proof of Lemma 4.5 shows that $\bar{B} + 2K$ verifies the proximity inequality at x_2. Note that if x_2 is excessive, say $x_2 >^1 x_1$, then $p_2 = 0$ and $x_3 \not\to x_1$ and we may conclude more easily. □

The multiplicity of $\bar{B}$ at a point x_i is the *virtual* multiplicity of the branch curve of a double plane π according to Castelnuovo and Enriques [7]. The divisor $\bar{B}+2K$ may be useful to compute the so-called adjoint conditions (see [5]).

5. Double planes and quadratic transformations

A *quadratic* transformation is a birational application $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the net of lines in the domain is carried in a net of generically irreducible conics in the target, whose base point, say x_0, x_1 and x_2, are called the *fundamental* points of α and we write $\alpha = c(x_0, x_1, x_2)$. We also assume that the net of conics through x_0, x_1 and x_2 in the domain is transformed in the net of lines in the target. Let us say that the lines $L_0 = \overline{x_1 x_2}$, $L_1 = \overline{x_0 x_2}$ and $L_2 = \overline{x_0 x_1}$ (whenever defined) are the *exceptional* lines for α. Recall that a quadratic transformation $c(x_0, x_1, x_2)$ is well defined if x_0, x_1, x_2 are not aligned and (after re-ordering the points):

- $x_0 \in \mathbb{P}^2$, $x_1 \in \mathbb{P}^2$ and $x_2 \in \mathbb{P}^2$; or
- $x_0 \in \mathbb{P}^2$, $x_1 \in \mathbb{P}^2$ and $x_2 >^1 x_0$; or
- $x_0 \in \mathbb{P}^2$, $x_2 >^1 x_1 >^1 x_0$ and $x_2 \not\odot x_0$.

We remark that, given a birational morphism $\sigma : S \to \mathbb{P}^2$, a quadratic transformation $\alpha = c(x_0, x_1, x_2)$ canonically induces a birational morphism $\sigma_\alpha : S_\alpha \to \mathbb{P}^2$ and a birational application $\tilde{\alpha} : S_\alpha \dashrightarrow S$ such that $\alpha \circ \sigma_\alpha = \sigma \circ \tilde{\alpha}$, thus $\tilde{\alpha}$ coincides with α over the proper points of S_α. In particular, if σ decomposes in the monoidal transformations with centers at $x_0, x_1, x_2, x_3, \dots, x_r$, then σ_α is the composition of the monoidal transformations centered at $y_0 = x_0$, $y_1 = x_1$, $y_2 = x_2$, $y_3, \dots, y_r$, where $y_i = \alpha^{-1}(x_i)$ if x_i lies on no exceptional line for α.

Noether-Castelnuovo's Theorem (see [17, Theorem 6]) says that a Cremona transformation decomposes in the product of an automorphism and finitely many quadratic transformations. Thus, for our purposes, it suffices to know, given a normal double plane $\pi : X \to \mathbb{P}^2$ and a quadratic transformation $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, what is the branch curve of the induced normal double plane $\pi_\alpha : X_\alpha \to \mathbb{P}^2$.

Let $\pi^\sigma : X^\sigma \to S$ be a resolution of a double plane $\pi : X \to \mathbb{P}^2$, where $\sigma : S \to \mathbb{P}^2$ is a birational morphism. We say that the branch curve B^σ of π^σ is in *good position* with respect to a quadratic transformation $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ if B^σ transversally meets (the strict transfom in S of) any exceptional line for α which is not a component of the branch curve of π.

Lemma 5.1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane and $\alpha = c(x_0, x_1, x_2)$ a quadratic transformation. Then there exist a birational morphism $\sigma : S \to \mathbb{P}^2$ such that the branch curve B^σ of $\pi^\sigma : X^\sigma \to \mathbb{P}^2$ is in good position with respect to α. In* Pic *S, one writes B^σ as*

$$B^\sigma = 2dL - b_0E_0^* - b_1E_1^* - b_2E_2^* - \sum_{j=3}^{r} b_jE_j^*, \tag{5.2}$$

where L is the total transform in S of a line in $\mathbb{P}^2$. Moreover, $\sigma_\alpha : S_\alpha \to \mathbb{P}^2$ induces a resolution $\pi_\alpha^{\sigma_\alpha} : X_\alpha^{\sigma_\alpha} \to S_\alpha$ of $\pi_\alpha : X_\alpha \to \mathbb{P}^2$ and its branch curve in Pic *S_α is*

$$B_\alpha^{\sigma_\alpha} = (2d+e)\bar{L} - \sum_{i=0}^{2}(b_i+e)\bar{E}_i^* - \sum_{j=3}^{r} b_j\bar{E}_j^*, \tag{5.3}$$

where $e = 2d - b_0 - b_1 - b_2$ and $\bar{L}$ ($\bar{E}_i^*$, respectively) is the total transform in S_α of a line in $\mathbb{P}^2$ (of the point y_i, respectively).

Proof. In order to get π^σ, we may start from the canonical resolution and apply monoidal transformations centered at the points where the branch curve does not transversally meet the strict transforms of the exceptional lines for α. After finitely many such transformations, the total inverse image of the branch curve has only normal crossings and contains the branch locus of the induced normal double covering. Then σ and α canonically determine σ_α. Furthermore, (5.3) follows from (5.2) by computing the total transform $\tilde{\alpha}^*(B^\sigma)$ of B^σ in S_α, removing its even multiplicity components and taking the reduced part (cf. [4, 36.2]). □

6. The simplicity of the branch curve

We keep the same notation as the previous sections, namely $\pi : X \to \mathbb{P}^2$ is a normal double plane and B its virtual branch curve. We re-order the points x_0, $\dots$, x_r according to the following rules:

$$b_j > b_i \implies i < j, \qquad \text{and} \qquad b_i = b_j,\ x_j > x_i \implies i < j. \tag{6.1}$$

Thus x_0 is a point where B has the maximal multiplicity. We choose $x_0 \in \mathbb{P}^2$, or, if all the maximal multiplicity points are excessive, we may choose $x_0 >^1 x_i \in \mathbb{P}^2$.

Setting $a_j = b_j/2$ for every $j = 1, \dots, r$, i.e., a_j is the multiplicity of $A = B/2$ at x_j, let us define the *simplicity* of B as the triplet (k, h, s), where

$$(6.2) \qquad 2k = 2d - b_0 = 2(d - a_0), \qquad b_h > k \geq b_{h+1},$$

and s is the number of satellite points among $x_1, \dots, x_h$ (by convention, set $b_{-1} = \infty$ and $b_{r+1} = 0$). Lexicographically ordering the triplets defining the simplicity, we say that π is *simpler* than another double plane $\rho : Y \to \mathbb{P}^2$ if the simplicity of B is less than the simplicity of the virtual branch curve of ρ.

Lemma 6.3. *Let $\alpha = c(x_0, x_i, x_j)$ be a quadratic trasformation. Suppose that either $b_i \geq b_j > k$ or $b_i > b_j = k$. Then π_α is simpler than π.*

Proof. By Lemma 5.1, the virtual branch curve of π_α is

$$B_\alpha = (2d + e)L - (b_0 + e)E_0^* - (b_i + e)E_i^* - (b_j + e)E_j^* - \cdots,$$

where $e = 2k - b_i - b_j < 0$ by hypothesis. The point x_0 is not of maximal multiplicity for B_α if and only if there exists a point x_l with $b_l > b_0 + e$, i.e., if and only if

$$(6.4) \qquad k_\alpha = \frac{2d + e - b_l}{2} < \frac{2d + e - b_0 - e}{2} = d - a_0 = k,$$

where $(k_\alpha, h_\alpha, s_\alpha)$ is the simplicity of B_α. If B_α has the maximal multiplicity at x_0, then

$$\operatorname{mult}_{x_i}(B_\alpha) = b_i + e = 2d - b_0 - b_j = 2k - b_j \leq k = k_\alpha,$$

therefore $k_\alpha = k$ and $h_\alpha < h$. □

Lemma 6.5. *Assume $x_j >^1 x_i >^1 x_0 \in \mathbb{P}^2$, $x_j \odot x_0$, $b_i \geq k > 0$ and $b_j > k$. Apply $\beta = c(x_0, x_i, x_{r+1})$, where x_{r+1} is a general point in $\mathbb{P}^2$. Then π_β is simpler than π.*

Proof. Set $e = 2d - b_0 - b_i = 2k - b_i \leq k$, the virtual branch curve of π_β is

$$B_\beta = (2d + e)L - (b_0 + e)E_0^* - (b_i + e)E_i^* - \cdots - eE_{r+1}^*.$$

Let $(k_\beta, h_\beta, s_\beta)$ be the simplicity of B_β. Like in the proof of Lemma 6.3, either $k_\beta < k$ as (6.4) or B_β has the maximal multiplicity at x_0. In the latter case, it follows that $k_\beta = k$, $h_\beta = h$ and $s_\beta = s - 1$, for $\operatorname{mult}_{x_i}(B_\beta) = 2k_\beta > k_\beta$, $\operatorname{mult}_{x_{r+1}}(B_\beta) = e \leq k_\beta$ and $\beta(x_j)$ is not satellite (cf. [4, Remark 15.3]). Roughly speaking, we 'eliminated' a satellite point. In both cases, π_β is simpler than π. □

Lemma 6.6. *Suppose that x_0, x_i and x_j are aligned, with $0 < i < j \leq h$, $x_0 \in \mathbb{P}^2$, $x_i \in \mathbb{P}^2$ (or $x_i >^1 x_0$) and $x_j \in \mathbb{P}^2$ (or $x_j >^1 x_i$). Then $b_0 = c_0$, $b_i = b_j = k + 1$ and the line $L' = \overline{x_0 x_i x_j}$ is a component of the branch curve C of π.*

Proof. The line L' cannot be double for C, so $c_0 + c_i + c_j \leq 2d + 2$. By the hypothesis, $b_0 + b_i + b_j > b_0 + 2k = 2d$. By (4.2), $c_0 + c_i + c_j \geq b_0 + b_i + b_j$. Hence,

$$2d + 2 \geq c_0 + c_i + c_j \geq b_0 + b_i + b_j \geq 2d + 2,$$

therefore, $b_0 = c_0$ and $b_i + b_j = 2k + 2$. □

Similarly, one can prove the following:

Lemma 6.7. *Suppose that x_0, x_i and x_j are aligned, with $x_0 >^1 x_i \in \mathbb{P}^2$, $b_0 - 2 = b_i \geq k$ and $b_j > k$. Then $b_0 = b_j = c_j = k + 2$ and $x_j \in \mathbb{P}^2$.* □

7. Irrational ruled double planes

The analysis of double planes that are irrational ruled surfaces is based on a theorem by De Franchis, which is remarkable on its own.

Theorem 7.1 (De Franchis). *A double plane $\pi : X \to \mathbb{P}^2$ is an* irregular *surface (i.e., $q(X) > 0$) if and only if, after adding possibly a double curve, its branch curve consists of $2q(X) + 1$ or $2q(X) + 2$ curves belonging to a pencil.*

Proof. See [8, Remark 3.5, Theorem 3.8 and after Corollary 4.9] or [16]. □

The key fact in the proof of De Franchis' Theorem is the following diagram:

$$\begin{array}{ccccc} X & \longleftarrow & \tilde{X} & \xrightarrow{a} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \phi} \\ \mathbb{P}^2 & \longleftarrow & S & \xrightarrow{\tau} & \mathbb{P}^1, \end{array} \tag{7.2}$$

where $\tilde{\pi} : \tilde{X} \to S$ is the canonical resolution of π, $a : \tilde{X} \to E$ is the Albanese map, E is a smooth hyperelliptic curve of genus $q(X)$ and ϕ is the double covering. The fibres of τ are connected, as those of a are, so the virtual branch curve of π is contained in the union of fibres of τ.

Theorem 7.3. *A double plane is ruled of genus $q > 0$ if and only if it is birationally equivalent to a normal double plane branched along a curve C of degree $2q + 2 = 2d$ with a point x_0 of multiplicity $2d$, i.e., C splits into $2d$ distinct lines through x_0.*

Proof. In Diagram (7.2), the fibres of a are rational curves, because $\tilde{X}$ is a ruled surface. So the branch curve of π, as that of $\tilde{\pi}$, is made of rational curves in a (rational) pencil Γ. Then there exists a Cremona transformation γ such that the strict transform of Γ via γ is a pencil of lines through a point, as follows from the classification of pencils of rational plane curves (see Theorem A.10). Conversely, suppose that x_0 is the point at ∞ of the y-axis, where x, y are affine coordinates of $\mathbb{P}^2$. Then C is defined by an equation $\prod_{i=1}^{2q+2}(x - a_i) = 0$, where $a_i \neq a_j$ for $i \neq j$, and π is birationally equivalent to the following surface in $\mathbb{P}^3$:

$$z^2 = (x - a_1)(x - a_2) \cdots (x - a_{2q+2}), \tag{7.4}$$

that is clearly the cone with vertex in x_0 over the hyperelliptic curve of genus q defined in the plane x, z by the same Equation (7.4). □

8. Rational double planes

Now we study the three types of double planes listed in Theorem 1.2.

Let W be a smooth cubic surface in $\mathbb{P}^3$. Recall that W is the plane $\mathbb{P}^2$ blown-up in six general points and W contains exactly 27 lines. Choose a point $q \in W$ which belongs to no line of W. Let $\tau : X \to W$ be the monoidal transformation with center at q. The projection from q over a plane $\mathbb{P}^2$ (not passing through q) defines a rational map $W \dashrightarrow \mathbb{P}^2$ and a double covering $\pi : X \to \mathbb{P}^2$. In affine coordinates, assuming that q is the point at ∞ of the z-axis and $z = 0$ is the plane on which we project (with coordinates x, y), then W is defined by

$$a_1(x, y)z^2 + 2a_2(x, y)z + a_3(x, y) = 0, \tag{8.1}$$

where a_i is a polynomial of degree i. Hence the branch curve of π is the *discriminant* of (8.1) with respect to z

$$a_2^2 - a_1a_3 = 0, \tag{8.2}$$

that is a quartic, which is smooth as X is.

Lemma 8.3. *Any smooth plane quartic is the branch curve of a double plane $\pi : X \to \mathbb{P}^2$, where X is (isomorphic to) the blow-up of $\mathbb{P}^2$ at 7 general points.*

Proof. (Noether) It suffices to show that any smooth quartic C can be written as (8.2), so C will be the branch curve of a double plane $X \to \mathbb{P}^2$, where X is the blow-up at a point of the cubic surface given in $\mathbb{P}^3$ given by (8.1). Let $L_1 = \overline{q_1q_2} : a_1 = 0$ be a bitangent line to C (there are 28 of them [14, p. 282]). Choose an irreducible conic $D : a_2 = 0$ through q_1 and q_2. The pencil $\Gamma = C + 2D$ is made of quartics which are bitangent to C in q_1 and q_2. Imposing on a curve $G \in \Gamma$ the condition to pass through a general point of L_1, then G splits in L_1 and a residual cubic, say $a_3 = 0$. Therefore $C : a_1a_3 - \lambda a_2^2$, for a certain $\lambda \in \mathbb{C}$. □

Proposition 8.4. *A double plane $\pi : X \to \mathbb{P}^2$ branched along a sextic C, whose singularities are only two infinitely near triple points x_0, x_1, is a rational surface.*

Proof. Let $x_0 >^1 x_1$. The virtual branch curve of π is

$$B = C + E_1 = 6L - 4E_0^* - 2E_1^* = 2(3L - 2E_0^* - E_1^*),$$

where L is the total transform in S of a line. Since $|B + 2K| = |0L - 2E_0^*| = \emptyset$ and $|B/2 + 2K| = |-3L + E_1^*| = \emptyset$, it follows that $P_2(X) = 0$. Moreover, $q(X) = -p_a(B/2) = 0$, so X is rational by Castelnuovo's criterion. For a direct proof of the rationality of X, see [9, pp. 419–439] or [19, n. 14]. □

Lemma 8.5. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane branched along a sextic C with two infinitely near triple points $x_0 >^1 x_1$. Then π is birationally equivalent to a double plane branched along either a quartic curve or an irreducible sextic with two infinitely near triple points and no other singularities.*

Proof. We may suppose that the line $\overline{x_1x_0}$ is not a component of C. Indeed, if $\overline{x_1x_0} \subset C$, choose a general point $x_2 \in \mathbb{P}^2$ and apply $\alpha = c(x_0, x_1, x_2)$: the branch curve of π_α is again a sextic with two infinitely near triple points and it does not contain $\overline{x_1x_0}$. If C has no singularity, other than x_1 and x_0, then C is surely irreducible. If C has another singularity, say at x_3, then x_3 cannot lie on $\overline{x_1x_0}$ (otherwise $\overline{x_1x_0}$ would be a component of C) and, if $x_3 >^1 x_0$, then $x_3 \not\succ x_1$, by the proximity inequality at x_1. Therefore, $\beta = c(x_1, x_0, x_3)$ is well defined and π_β is branched along a quartic. □

Lemma 8.6. *A normal double plane branched along a curve C of degree $2d$ with a point x_0 of multiplicity $c_0 = 2d-1$ or $2d-2$ is a rational surface.*

Proof. (Noether) Let $\pi : X \to \mathbb{P}^2$ be the double plane and $\tilde{\pi} : \tilde{X} \to S$ its canonical resolution, branched along the smooth curve B. The strict transform $\tilde{L}$ in S of a general line $L \subset \mathbb{P}^2$ through x_0 meets B in two distinct points, namely the intersections of L with C outside x_0 (if $c_0 = 2d-1$, then one intersection lies on $E_0 \subset B$ and it is the direction of L at x_0). Then $\tilde{\pi}^{-1}(\tilde{L})$ is a double covering over a smooth rational curve branched along two points, hence it is a rational curve in $\tilde{X}$ by the Hurwitz formula. So the pencil of lines through x_0 corresponds to a (rational) pencil of rational curves in $\tilde{X}$. Therefore $\tilde{X}$, and X too, is a rational surface by a well-known theorem of Noether [14, p. 513]. □

Proposition 8.7. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane branched along a curve C of degree $2f$ with a point x_0 of multiplicity $2f-1$ or $2f-2$. Then π is birationally equivalent to a double plane branched along either a smooth conic or an irreducible curve of degree $2d$ with an ordinary singularity of multiplicity $2d-2$, possibly one node and no other singularities.*

Proof. (**a**) First we will show that if $\mathrm{mult}_{x_0}(C) = 2f-1$, then π is birationally equivalent to a double plane branched along a smooth conic. If $f = 1$ there is nothing to prove. Assume $f > 1$, the branch curve C is

$$C = \bar{C} + L_1 + L_2 + \cdots + L_{2f-e}, \tag{8.8}$$

where $\bar{C}$ is irreducible, $\deg(\bar{C}) = e \le 2f$, $\mathrm{mult}_{x_0}(\bar{C}) = e-1$, and the L_i's are distinct lines through x_0. Let $x_i >^1 x_0$ be a point on C and choose a general point $x_{r+1} \in \bar{C}$, so $\alpha = c(x_0, x_i, x_{r+1})$ is well defined. Then π_α is branched along

$$C_\alpha = \bar{C}_\alpha + L'_1 + \cdots + L'_{2f-e} + L'_{2f-e+1}, \tag{8.9}$$

where $\bar{C}_\alpha$ (L'_i, respectively) is the strict transform of $\bar{C}$ (of L_i, for $i = 1, \dots, 2f-e$, respectively) and $\deg \bar{C}_\alpha = e-1$. If $\overline{x_0x_i}$ is not a component of C, then $L'_{2f-e+1} = \overline{x_0x_i}$, otherwise E_i is a component of the virtual branch curve of π and $L'_{2f-e+1} = \overline{x_0x_{r+1}}$. Hence, repeating this argument, after $e-1$ such quadratic transformations, we may assume the branch curve to be as (8.8), where $\bar{C} = L_0$ is a line not passing through x_0. Now let $x_j = L_0 \cap L_{2f-1}$ and $x_l >^1 x_0$ such that $L_{2f-2} = \overline{x_0x_l}$. Applying $c(x_0, x_j, x_l)$, one gets a double plane branched along $2f-2$ distinct lines, all of them but one (the strict transform of L_0) passing through x_0. Therefore, by induction on f, we may suppose that C splits in two lines, say $C = L_1 \cup L_2$. Choose a general point $x_1 \in L_1$ and let $x_2 >^1 x_1$ such that $L_1 = \overline{x_1x_2}$. Choose a general point $x_3 \in \mathbb{P}^2$, then $\alpha = c(x_1, x_2, x_3)$ is well defined and π_α is branched along a smooth conic, that is the strict transform of L_2.

(**b**) Suppose that C has multiplicity $2f-2$ at x_0 and C is irreducible. The other singularities of C, if they exist, must be double points, say at $x_1, \dots, x_h$.

(b^1) If $h = 0$ and the singularity at x_0 is not ordinary, choose a general point $x_{r+1} \in C$ and let $x_{r+2} >^1 x_{r+1}$ be such that the line $\overline{x_{r+1}x_{r+2}}$ is tangent to C.

The double plane π_α, where $\alpha = c(x_0, x_{r+1}x_{r+2})$, is branched along an irreducible curve of degree $2f$ with only an ordinary singularity of multiplicity $2f-2$.

(b^2) If $h = 1$, $x_1 \in \mathbb{P}^2$ and the multiplicity at x_0 or at x_1 is not ordinary, then choose a general point $x_{r+1} \in \mathbb{P}^2$. Applying $c(x_0, x_1, x_{r+1})$, one gets a double plane branched along an irreducible curve of degree $2f$ with an ordinary singularity of multiplicity $2f-2$, a node and no other singularities.

(b^3) If $h = 1$ and $x_1 >^1 x_0$, choose a general point $x_{r+1} \in C$ and let $x_{r+2} >^1 x_{r+1}$ be such that $\overline{x_{r+1}x_{r+2}}$ is tangent to C. Applying $c(x_0, x_1, x_{r+1})$, the branch curve has degree $2f$, an ordinary singularity of multiplicity $2f-2$ and a double point. If the double point is a node, we conclude, otherwise we proceed with (b^2).

(b^4) If $h \geq 2$, we claim that there exists a Cremona transformation γ such that π_γ is branched along an irreducible curve of degree $2f-2$ with a point of multiplicity $2f-4$ and $h-2$ double points, namely π_γ is simpler than π. Hence, by induction on h, we reduce to the case (b^1), (b^2) or (b^3). Now we prove our claim.

(b^4_1) Suppose that there is a proper point x_i among $x_1, \ldots, x_h$. Since $h \geq 2$, there is also x_j such that $x_j \in \mathbb{P}^2$, or $x_j >^1 x_0$ or $x_j >^1 x_i$. In all the cases, x_j does not lie on the line $\overline{x_0x_i}$, otherwise it would be a component of C contradicting the assumption that C is irreducible. Setting $\gamma = c(x_0, x_i, x_j)$, π_γ is simpler than π.

(b^4_2) If there is no proper point among $x_1, \ldots, x_h$, it means that they are all infinitely near to x_0. Choose a general point $x_{r+1} \in C$ and let $x_{r+2} >^1 x_{r+1}$be such that the line $\overline{x_{r+1}x_{r+2}}$ is tangent to C. Applying $c(x_0, x_{r+1}, x_{r+2})$, we reduce to the case (b^4_1).

(**c**) Suppose that C has multiplicity $2f-2$ at x_0 and C is reducible. Let $\bar{C}$ be an irreducible component of C with the highest degree. Then $\bar{C}$ has degree e, with $0 < e < 2f$, and multiplicity either $e-2$ or $e-1$ at x_0.

(c^1) If $\mathrm{mult}_{x_0}(\bar{C}) = e-1$, then $C-\bar{C}$ consists of an irreducible curve C' of degree g and $2f-e-g$ distinct lines through x_0, where $0 < g \leq e$ and $\mathrm{mult}_{x_0}(C') = g-1$. Choose a general point $x_{r+1} \in C'$ and let $x_{r+2} >^1 x_{r+1}$ be such that the line $\overline{x_{r+1}x_{r+2}}$ is tangent to C'. Applying $\alpha = c(x_0, x_{r+1}, x_{r+2})$, the branch curve of π_α splits in $2f-e-g$ distinct lines through x_0 and two irreducible curves $\bar{C}_\alpha$ and $\bar{C}'$ (the strict transforms of $\bar{C}$ and C'), with $\deg \bar{C}_\alpha = e+1 = \mathrm{mult}_{x_0}(\bar{C}_\alpha)+1$ and $\deg C'_\alpha = g-1 = \mathrm{mult}_{x_0}(C'_\alpha)+1$. Repeating this argument, after g such quadratic transformations, one gets a double plane branched along a curve which splits in $2f-e-g$ distinct lines through x_0 and an irreducible curve (the strict transform of $\bar{C}$) of degree $e+g$ and multiplicity $e+g-1$ at x_0. Then we conclude as in part (a).

(c^2) If $\mathrm{mult}_{x_0}(\bar{C}) = e-2$, then C is like (8.8), where the L_i's are distinct lines through x_0. Proceedings as in part (b), namely applying quadratic transformations with x_0 and double points of $\bar{C}$ as fundamental points, we may assume that x_0 and possibly a node are the only singular points of $\bar{C}$. Choose x_i in $\bar{C} \cap L_{2f-e}$ (maybe $x_i >^1 x_0$). Then either $\mathrm{mult}_{x_i}(\bar{C}) = 1$ or $\mathrm{mult}_{x_i}(\bar{C}) = 2$. In the former case, choose a general point $x_{r+1} \in \mathbb{P}^2$ and apply $\alpha = c(x_0, x_i, x_{r+1})$. In the latter case, x_i is proper and we may choose x_j in $\bar{C}$ such that $x_j >^1 x_i$. In particular $\mathrm{mult}_{x_j}(C) = 1$ and x_j does not lie on $L_{2f-e} = \overline{x_0x_i}$, otherwise L_{2f-e} should be a component of C, hence $\alpha = c(x_0, x_i, x_j)$ is well defined. In both cases, the branch curve of π_α is

$$C_\alpha = \bar{C}_\alpha + L'_1 + \cdots + L'_{2f-e-1}, \tag{8.10}$$

where $\bar{C}_\alpha$ (L'_i, respectively) is the strict transform of $\bar{C}$ (of L_i, respectively) and $\deg \bar{C}_\alpha = e + 1 = \mathrm{mult}_{x_0}(\bar{C}_\alpha) + 2$. Therefore, by induction on $2f - e$, the branch curve is irreducible, has degree $2f$ and multiplicity $2f - 2$ at x_0, that is part (b). □

The statement of Proposition 8.7 is the same as [19, Lemma p. 61], but we proved it here because the proof in [19] is incorrect.

9. The classification theorem

The following theorem is the key step towards the classification of double planes which are surfaces with Kodaira dimension $-\infty$.

Theorem 9.1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane. Suppose that its virtual branch curve B is such that* (3.4) *holds. Then π is birationally equivalent to a double plane whose virtual branch curve is a conic, a quartic or a curve of degree $2d'$ with a point of multiplicity at least $2d' - 2$.*

Proof. By induction on the simplicity, it suffices to show that if

$$d > 2 \qquad \text{and} \qquad k = d - b_0/2 > 1, \tag{9.2}$$

then there exists a Cremona transformation $\gamma : \mathbb{P}^2 \to \mathbb{P}^2$ such that the double plane $\pi_\gamma : X_\gamma \to \mathbb{P}^2$ is simpler than π. This will be done in Propositions 9.4 and 9.12: the former amends the "proof" of Castelnuovo and Enriques (see [7], [9, l. 2, n. 27]), while Proposition 9.12 fills the remaining gap.

Lemma 9.3. *Suppose that* (3.4) *and* (9.2) *hold. If $x_0 \in \mathbb{P}^2$, then $b_0 > k$.*

Proof. Let $m = [2d/3]$, i.e., m is the largest integer smaller or equal to $2d/3$. Then $m \geq 2$ by (9.2), so (3.4) forces

$$\emptyset = |B + mK| = |\varepsilon L - (b_0 - m)E_0^* - \cdots|,$$

where $\varepsilon = 0$, 1 o 2. Thus, $b_0 > m$, so $3b_0 > 2d = b_0 + 2k$, that is the thesis. □

Proposition 9.4. *Suppose that* (3.4) *and* (9.2) *hold. Then either there exists a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that π_γ is simpler than π or we may order the points x_j in such a way that*

(9.5) *x_0 is proper and, for every $i = 1, \ldots, h$, x_i is excessive, $x_i >^1 x_{h+i}$, $b_i = k + 2$ and x_0, x_i, x_{h+i} are aligned.*

Proof. It splits in two parts, depending whether x_0 is proper or excessive.

(**a**) Let $x_0 \in \mathbb{P}^2$. Then $b_0 > k$ by Lemma 9.3. Since $k \geq 2$ by (9.2), then (3.4) forces

$$(9.6) \qquad \emptyset = |B + kK| = \left|(b_0 - k)(L - E_0^*) - \sum_{i=1}^{h}(b_i - k)E_i^*\right| + \cdots,$$

hence $h \geq 2$, otherwise (9.6) would contradict (6.1). We want to find x_i and x_j among $x_1, \ldots, x_h$ such that we can apply either Lemma 6.3 or Lemma 6.5, thus there will exist a quadratic transformation α such that π_α is simpler than π. The proof splits again in two sub-parts, depending on the parity of k.

(a^1) Suppose that k is odd. Excessive points of multiplicity $k+1$ do not contribute to make $|B + kK|$ empty. More precisely, let p be the number of such points and $\bar{h} = h - p$; we may order $x_1, \ldots, x_r$ in such a way that $b_{\bar{h}+1} = k + 1$, for every $i = 1, \ldots, p$, $x_{\bar{h}+i}$ is excessive and $x_{\bar{h}+i} >^1 x_{h+i}$. Thus (9.6) becomes

$$(9.7) \qquad \emptyset = \left|(b_0 - k)(L - E_0^*) - \sum_{i=1}^{\bar{h}}(b_i - k)E_i^*\right| + \sum_{j=h+1}^{2h-\bar{h}+1} E_j + \cdots,$$

therefore, $\bar{h} \geq 2$ as above. Moreover, the proximity inequalities imply that

$$(9.8) \qquad x_i >^1 x_l \text{ and } i \leq \bar{h} \Longrightarrow l \leq \bar{h}.$$

The proof splits once more depending on the position of $x_1, \ldots, x_{\bar{h}}$.

(a_1^1) Suppose that there is $j \leq h$ with $x_j \in \mathbb{P}^2$. If a point x_g lies on $L' = \overline{x_0 x_j}$, with $g \leq \bar{h}$, then $b_g = b_j = k + 1$ by Lemma 6.6. Hence (9.7) is

$$\emptyset = \left|(b_0 - b_j)L - \sum_{x_i \notin L'}(b_i - k)E_i^*\right| + (b_j - k)L' + \cdots$$

and there exists $l \leq \bar{h}$ such that x_l does not lie on L'. We claim that we may choose x_l such that $c(x_0, x_i, x_l)$ is well defined, and thus we conclude by using Lemma 6.3. Suppose that $x_l >^1 x_f$, for $f \neq 0, j$. By (9.8), $f \leq \bar{h}$. If $x_f \in L'$, then $b_f = k + 1$ by Lemma 6.6 and $x_l \in L'$ too by Lemma 4.3. Thus L' would be a double component of the effective branch curve C, contradicting the assumption that C is reduced. Therefore, $x_f \notin L'$ and we may choose x_f instead of x_l.

(a_2^1) Suppose that $x_i > x_0$ for every $i = 1, \ldots, \bar{h}$. If $x_i >^1 x_0$ for every $i = 1, \ldots, \bar{h}$, then (9.7) holds only if $b_0 - k < \sum_{i=1}^{\bar{h}}(b_i - k)$, which contradicts Lemma 4.6 for $m = k$. Thus there exist $j, l \leq \bar{h}$ such that $x_j >^1 x_l >^1 x_0$, by (9.8). We claim that we may choose such j, l with $x_j \notin \overline{x_0 x_l}$. Thus we will conclude by Lemma 6.5 (by Lemma 6.3, respectively), if x_j is (is not, respectively) satellite to x_0. It remains to prove our claim. By Lemma 6.6, $x_j \in \overline{x_0 x_l}$ implies that $b_j = b_l = k+1$ and x_j is the unique point infinitely near to x_l. Let q be the number of points like x_j (possibly $q = 0$). Setting $\tilde{h} = \bar{h} - 2q$, we may order $x_1, \ldots, x_{\bar{h}}$ in the following way: $b_{\tilde{h}+1} = k+1$, for every $i = 1, \ldots, q$, $x_{\tilde{h}+i} >^1 x_{\tilde{h}+q+i} >^1 x_0$

and $x_{\tilde{h}+i} \in \overline{x_0 x_{\tilde{h}+q+i}}$. Hence, we cannot find such j, l only if either $\tilde{h} = 0$ or $x_i >^1 x_0$ for $i = 1, \dots, \tilde{h}$. In both cases, (9.7) becomes

$$\emptyset = \left| (b_0 - k - q)(L - E_0^*) - \sum_{i=1}^{\tilde{h}} (b_i - k) E_i^* \right| + \sum_{j=1}^{q} \overline{x_0 x_{\tilde{h}+q+j} x_{\tilde{h}+j}} + \cdots,$$

which holds only if $b_0 - k - q < \sum_{i=1}^{\tilde{h}} (b_i - k)$, contradicting (4.6), that is

$$b_0 - k \geq \sum_{i=1}^{\tilde{h}+q} (b_i - k) = q + \sum_{i=1}^{\tilde{h}} (b_i - k).$$

Thus our claim is proved and sub-parts (a_2^1) and (a^1) are done.

(a^2) Suppose that k is even. Now the proximity inequalities imply that

(9.9) $\quad x_i >^1 x_l$ and $i \leq h \implies$ either $l \leq h$ or x_i is excessive and $b_i = k + 2$.

The proof splits in three cases depending on the position of $x_1, \dots, x_h$.

(a_1^2) Suppose that there is $j \leq h$ with $x_j \in \mathbb{P}^2$. A point x_l, with $l \leq \bar{h}$ and $l \neq 0, j$, cannot lie on $L' = \overline{x_0 x_j}$ by Lemma 6.6. Now (9.6) is

$$\emptyset = \left| (b_0 - b_j) L - \sum_{x_i \notin L'} (b_i - k) E_i^* \right| + (b_j - k) L' + \cdots,$$

which holds only if there exists $l \leq \bar{h}$ such that $x_l \notin L'$. We claim that we may choose x_l in such a way that either $\alpha = c(x_0, x_j, x_l)$ is well defined or $x_l >^1 x_f$, with $b_f = k$ and $\alpha = c(x_0, x_j, x_f)$, is well defined. In both cases, π_α is simpler than π by Lemma 6.3. Suppose that $x_l >^1 x_f$, for $f \neq 0, j$. If $f \leq h$, then we may choose x_f instead of x_l. If $f > h$, then x_l is excessive and $b_l = k + 2$ by (9.9). Moreover, $x_l >^1 x_f$ with $b_f = k$. If $x_f \in L'$, then $x_l \in L'$ by Lemma 4.3 and L' would be a double component of the effective branch curve C, contradicting the assumption that C is reduced. Therefore $x_f \notin L'$ and our claim is proved.

(a_2^2) Suppose that there exists $j \leq h$ such that $x_j \not> x_0$. Clearly x_j is excessive, $x_j >^1 x_l$ and $b_l = k$. We may choose $x_l \in \mathbb{P}^2$, because if $x_l >^1 x_f$ then x_f is excessive and $f \leq h$. Suppose that we cannot find a quadratic transformation γ such that π_γ is simpler than π. We claim that either (9.5) holds or there exists

$(\star)$ $\; x_g >^1 x_f >^1 x_0$ such that $b_g = k + 2$, x_g is excessive and satellite to x_0.

If $(\star)$ occurs, choose a general point $x_{r+1} \in \mathbb{P}^2$ and apply $\alpha = c(x_0, x_f, x_{r+1})$. The simplicity of π_α is $(k, h + 1, s - 1)$ and the point x'_g, corresponding to x_g, is such that $x'_g >^1 x_0$ (cf. Lemma 6.5). Choose another general point $x_{r+2} \in \mathbb{P}^2$ and apply $\beta = c(x_0, x'_g, x_{r+2})$. Then $(\pi_\alpha)_\beta$ is simpler than π, because its simplicity is $(k, h, s - 1)$ by Lemma 6.3. It remains to prove our claim. We suppose that (9.5) does not hold and we cannot find γ as above. Thus, if there is x_j as at the beginning of (a_2^2), then $x_j \in L' = \overline{x_0 x_l}$, otherwise $\gamma = c(x_0, x_j, x_l)$ is well defined.

Moreover $b_0 = c_0$, L' is a component of the effective branch curve C and x_j is the unique point infinitely near to x_l (cf. Lemma 6.6). Similarly for any $x_i \not> x_0$ with $i \leq h$. Hence there exists $x_g > x_0$ with $g \leq h$, otherwise (9.5) holds. Moreover $x_g \not>^1 x_0$, otherwise $\gamma = c(x_0, x_g, x_l)$ is well defined. By (9.9), x_g is excessive with $b_g = k + 2$ and we may choose $x_g >^1 x_f >^1 x_0$. Then $\gamma = c(x_0, x_f, x_g)$ is not defined only if either $x_g \in \overline{x_0 x_f}$ or x_g is like ($\star$). Therefore our claim is proved.

(a_3^2) Suppose that $x_i > x_0$ for every $i = 1, \dots, h$. If $x_i >^1 x_0$ for every $i = 1, \dots, h$, we get a contradiction as in part (a_2^1). Suppose that

($\cdot$) $x_j >^1 x_l >^1 x_0$, x_j is defective, satellite to x_0 and $b_l > b_j = k$.

Choose a general point x_{r+1} in the plane and apply $\alpha = c(x_0, x_l, x_{r+1})$. Then π_α is as simple as π and (the points corresponding to) $x_1, \dots, x_h$ are still all $> x_0$, but (the point corresponding to) x_j is $>^1 x_0$ (cf. Lemma 6.5). Hence, repeating this argument, by applying at most $h/2$ quadratic transformations we may assume that there is no x_j like ($\cdot$). Suppose that we cannot find a quadratic transformation γ such that π_γ is simpler than π. We claim that if (9.5) does not hold, then there is x_g like ($\star$) and we conclude as in part (a_2^2). Let $j \leq h$ be such that $x_j > x_l >^1 x_0$. If $l \leq h$, then $x_j \notin \overline{x_0 x_l}$ by Lemma 6.6 and we find γ either by Lemma 6.3 or by Lemma 6.5. Hence x_j is excessive and $b_j = k + 2$ by (9.9). Either x_j is like x_g in ($\star$) or $x_j \in L' = \overline{x_0 x_l}$, because otherwise $\gamma = c(x_0, x_l, x_j)$ is well defined. If $x_j \in L'$, then $b_0 = c_0$, L' is a component of the effective branch curve C and x_j is the unique infinitely near point to x_l (cf. Lemma 6.6). Similarly for all $j \leq h$ such that $x_j >^2 x_0$. Therefore we may order $x_1, \dots, x_r$ in such a way that: $x_i >^1 x_0$ for every $i = 1, \dots, \bar{h} \leq h$, $b_{\bar{h}+1} = k+2$, for every $i = \bar{h}+1, \dots, h$, x_i is excessive, $x_i >^1 x_{h-\bar{h}+i} >^1 x_0$ and $x_i \in \overline{x_0 x_{h-\bar{h}+i}}$. Set $p = h - \bar{h}$, then $p < h$, otherwise (9.5) holds, and (9.7) becomes:

$$(9.10) \qquad \emptyset = \left| (b_0 - k - p)(L - E_0^*) - \sum_{i=1}^{\bar{h}} (b_i - k) E_i^* \right| + \sum_{i=h+1}^{h+p} \overline{x_0 x_i} + \cdots ,$$

that holds only if

$$b_0 - k - p < \sum_{i=1}^{\bar{h}} (b_i - k),$$

which contradicts the proximity inequality for C at x_0; that is,

$$b_0 - k = c_0 - k \geq \sum_{i: x_i \to x_0} c_i - k \geq \sum_{i=1}^{\bar{h}} (c_i - k) + p.$$

This concludes part (a_3^2), (a^2) and (a).

(**b**) Let $x_0 >^1 x_j \in \mathbb{P}^2$, i.e., all the maximal multiplicity point are excessive. Recall that x_0 is the unique proximate point to x_j. Setting $m = [\frac{2d}{3}]$, Formulae (9.2) and (3.4) imply $m \geq 2$ and

$$\emptyset = |B + mK| = E_j + |\varepsilon L - (b_0 - m - 1)(E_0^* - E_j^*) - \cdots |,$$

where $\varepsilon = 0$, 1 or 2. Thus $b_0 > m+1$, or equivalently $b_0 > k+1$. Since $k \geq 2$ by (9.2), then (3.4) forces:

$$\emptyset = |B + kK| = E_j + (b_0 - k - 2)L' + |2L - E_0^* - E_j^*|, \tag{9.11}$$

where $L' = \overline{x_j x_0}$. We remark that $x_i \notin L'$ for $0 < i \leq h$, $i \neq j$. In fact, if $x_i \in L'$, then $b_0 = b_i = k+2$ and $x_i \in \mathbb{P}^2$ by Lemma 6.7, contradicting the assumption (b). If $j = 1$, set $l = 2$, otherwise set $l = 1$. Then $l \leq h$, otherwise (9.11) does not hold. Moreover $x_l \in \mathbb{P}^2$, or $x_l >^1 x_0$, or x_l is excessive. In the first two cases, $c(x_0, x_j, x_l)$ is well defined and we conclude by Lemma 6.3. In the last case, $x_l >^1 x_f$, with either $x_f \in \mathbb{P}^2$ or $x_f >^1 x_0$. By Lemma 4.3, $x_f \notin L'$, thus $\alpha = c(x_0, x_j, x_f)$ is well defined. If $b_f > k$, then we conclude by Lemma 6.3. Otherwise, if $b_f = k$, then π_α is as simple as π and B_α has the maximal multiplicity at the point corresponding to x_l, which now is proper, therefore we continue with part (a). □

Proposition 9.12. *Suppose that* (3.4), (9.2) *and* (9.5) *hold. Then either X is irrational ruled or $b_0 = b_1 = k+2$ and $h = 3$. In both cases there exists a Cremona transformation γ such that π_γ is simpler than π.*

Proof. Formula (3.4) implies that $p_g(X) = 0$. Then it is well known (see [14, p. 558]) that either X is irrational ruled or $q(X) \leq 1$. In the former case, De Franchis' Theorem 7.1 implies that there exists a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that π_γ is branched along $2q(X)+2$ distinct lines through a point, so π_γ is simpler than π, because its simplicity is (0, 0, 0). In the latter case, we claim that $b_0 = k+2$ and $h = 3$. We know that $k = 2l$ is even, because so is b_0. By (9.2) and (3.4),

$$\emptyset = |B + kK| = \sum_{i=1}^{h} (E_{h+i} + \overline{x_0 x_{h+i} x_i}) + |(b_0 - k - h)(L - E_0^*)| + \cdots$$

that may hold only if $h > b_0 - k$, or equivalently if

$$h \geq 2a_0 - 2l + 1. \tag{9.13}$$

On the other hand, $p_a(A) = -q(X) \geq -1$ implies that

$$d^2 - 3d + 4 - a_0(a_0 - 1) \geq \sum_{i=1}^{r} a_i(a_i - 1) \geq \sum_{i=1}^{2h} a_i(a_i - 1) = 2hl^2, \tag{9.14}$$

where the latter equality follows from (9.5). Since $d = a_0 + 2l$, the left-hand side of (9.14) is $4l^2 - 6l + 4 - 2a_0 + 4la_0$. Therefore, dividing by 2 and applying (9.13), Formula (9.14) forces

$$2l^2 - 3l + 2 - a_0 + 2la_0 \geq hl^2 \geq l^2(2a_0 - 2l + 1), \tag{9.15}$$

which (forgetting the middle term) may be rewritten as:

$$2l^3 + l^2 - 3l + 2 \geq a_0(2l^2 - 2l + 1). \tag{9.16}$$

Recall that $a_0 > k/2 = l$ by Lemma 9.3. If $a_0 \geq l + 2$, then (9.16) implies that

$$2l^3 + l^2 - 3l + 2 \geq (l + 2)(2l^2 - 2l + 1) = 2l^3 + 2l^2 - 3l + 2,$$

that is absurd, for $l > 0$. Therefore $a_0 = l + 1$ and (9.15) become

$$4l^2 - 2l + 1 \geq hl^2 \geq 3l^2,$$

which forces $h = 3$. Thus our claim is proved. The effective branch curve splits in three lines $\overline{x_0 x_i x_{i+3}}$, for $i = 1, 2, 3$, and a curve of degree $6l - 1$ with multiplicity $2l - 1$ at x_0 ($2l$ at $x_1, \dots, x_6$, respectively). Hence x_4, x_5 and x_6 are proper and not aligned, otherwise this line would be a double component of C, contradicting the assumption that C is reduced. Similarly x_0 does not lie on the lines $\overline{x_4 x_5}$, $\overline{x_4 x_6}$ and $\overline{x_5 x_6}$. Apply $\alpha = c(x_1, x_4, x_5)$ and let y_j be the point corresponding to x_j, for $j = 0, 2, 3, 6$. Then π_α is as simple as π. Moreover $y_2 \in \mathbb{P}^2 \setminus \overline{y_6 y_3}$, thus we may apply $\beta = c(y_2, y_3, y_6)$ and $(\pi_\alpha)_\beta$ is simpler than π_α by Lemma 6.3, and therefore is simpler than π too. □

The branch curve as (9.5) in the statement of Proposition 9.12 may appear quite unusual. Nevertheless there are interesting double planes with such a branch curve.

Example 9.17 (Bagnera–De Franchis). Let L_1, L_2, L_3 and L_4 be lines through a point x_0 and L_5 be a line not passing through x_0. Choose two smooth cubics C_1 and C_2 in the pencil spanned by $L_1 + L_2 + L_3$ and $L_4 + 2L_5$ and let $\pi : X \to \mathbb{P}^2$ be the double plane branched along $C = C_1 + C_2 + L_1 + L_2 + L_3 + L_4$. Then X is a *hyperelliptic* (sometimes called *bielliptic*) surface (see [1, Sect. 7, n. 14]).

In particular C is as that in (9.5), with $2d = 10$, $b_0 = 6$, $k = 2$ and $h = 4$. Moreover, it can be shown that C has the smallest degree among the branch curves of the double planes in the birational equivalence class of π. This example also shows that Theorem 2 in [21], which states that a hyperelliptic surface cannot be birationally equivalent to a double plane, is incorrect.

Now we can prove the classification theorem:

Theorem 9.18. *A double plane is a rational surface if and only if it is birationally equivalent to a double plane branched along one of the following:*

1. *a smooth conic;*
2. *a smooth quartic;*
3. *a curve C of degree 10 with four ordinary singularities: one of multiplicity 6 and three aligned of multiplicity 4; moreover, C splits in the line through the three quadruple points and an irreducible curve of degree 9;*
4. *an irreducible curve of degree $2d > 2$ with an ordinary singularity in a point x_0 of multiplicity $2d - 2$ and:*
   (a) *no other singularities;*
   (b) *a node and no other singularities.*

Moreover, Types 1–4 of double planes are birationally distinct.

Proof. Type 3 is birationally equivalent to a double plane branched along an irreducible sextic with two infinitely near triple points, by applying $c(x_0, x_1, x_2)$, where x_0 is the point of multiplicity 6 and x_1, x_2 of multiplicity 4. We showed in Sect. 8 that Types 1–4 of double planes are rational surfaces. Conversely, let $\pi : X \to \mathbb{P}^2$ be a double plane such that X is a rational surface. Up to strict birational equivalence, we may assume that X is normal. By Theorem 9.1, π is birationally equivalent to a double plane whose virtual branch curve B is a conic, a smooth quartic, or a curve of degree $2f$ with a point x_0 of multiplicity at least $2f - 2$. If B is a conic, then π belongs to Type 1, even if B is reducible, by part (a) of the proof of Proposition 8.7. If $\deg B = \mathrm{mult}_{x_0}(B) = 2f > 2$, then X would be irrational ruled by Theorem 7.3. Hence, if $f > 2$, the (virtual) multiplicity of B at x_0 is $2f - 2$. Either x_0 is proper or x_0 is excessive. In the former case, Proposition 8.7 implies the thesis. In the latter one, $x_0 >^1 x_1 \in \mathbb{P}^2$ with $b_i = 2f - 4$ and $c_0 = c_i = 2f - 3$. If $f > 3$, then $2(2f - 3) > 2f + 1$, thus $\overline{x_0 x_1}$ would be a double component of the branch curve, contradicting the assumption that X is normal. Therefore $f = 3$ and the branch curve belongs to Type 3. It remains to show that Types 1–4 are birationally distinct. Recall that the genus of an irreducible plane curve is preserved by Cremona transformations. A curve of Type 1, 2, 3, 4.a, 4.b has genus, respectively, 0, 1, 4, $2d - 2$, $2d - 3$. Finally, a curve of Type 4 is hyperelliptic, while Types 2 and 3 are not hyperelliptic. □

A. Pencils of rational plane curves

This appendix has been included in this paper because we have not found any modern reference about the birational classification of linear systems, in particular pencils, of rational plane curves (see [9, pp. 286–298] for the classical approach).

A *complete linear system* $\Gamma = |C|$ of plane curves is the set of effective divisors on S which are linearly equivalent to a fixed divisor C on S, where S is as in Sect. 2. Writing C in Pic S as (2.2), we say that the complete linear system

$$\Gamma = |dL - c_0 E_0^* - \cdots - c_r E_r^*| \tag{A.1}$$

has *degree* d and *multiplicity* c_i at x_i, for every $i = 0, \ldots, r$. Suppose that Γ is an irreducible pencil of rational curves, namely $\dim \Gamma = 1$ and its generic element is irreducible and rational, so:

$$p_a(C) = (d-1)(d-2) - \sum_{i=0}^{r} c_i(c_i - 1) = 0. \tag{A.2}$$

Hence, by the Riemann-Roch Theorem, Γ is *regular* (i.e., $h^1(S, \mathcal{O}_S(C)) = 0$), therefore $\Gamma^2 = 0$, or equivalently,

$$d^2 = \sum_{i=0}^{r} c_i^2. \tag{A.3}$$

Order the points x_i in such a way that $c_i > c_j$ implies $i < j$. Since C verifies the proximity inequalities, we may choose $x_0 \in \mathbb{P}^2$. Let us define the *simplicity* of Γ

as the triplet (k, h, s), where $k = d - c_0$, $c_h > \frac{k}{2} > c_{h+1}$, and s is the number of satellite points among $c_1, \dots, c_h$ (by convention, we assume $c_{-1} = \infty$ and $c_{r+1} = 0$, so $-1 \leq h \leq r$).

Let $\alpha = c(x_0, x_1, x_2) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a quadratic transformation. The strict transform of Γ via α is again an irreducible pencil of rational plane curves. Moreover, setting $e = d - c_0 - c_1 - c_2$, its general element has degree $d + e$ and multiplicity $c_i + e$ at x_i, for $i = 0, 1, 2$, while the multiplicity at x_j, for $j > 2$, does not change.

Lemma A.4. *Let Γ as above. Suppose that $k > 0$. Then $h \geq 2$ and*

$$\sum_{i=1}^{h} c_i > c_0. \tag{A.5}$$

Proof. For any $m > 0$, add m times (A.2) to $(1 - m)$ times (A.3):

$$d(d - 3m) < d(d - 3m) + 2m = \sum_{i=0}^{r} c_i(c_i - m). \tag{A.6}$$

Replacing m with c_0, the right-hand side of (A.6) must be ≤ 0, therefore $d - 3c_0$ must be negative, or equivalently,

$$c_0 > k/2, \tag{A.7}$$

which implies $h \geq 0$. Plugging $m = k/2$ in (A.6) and moving the term of the sum for $i = 0$ to the left-hand side, it follows that

$$k\left(c_0 - \frac{k}{2}\right) = (d - c_0)\left(d - \frac{3k}{2}\right) < \sum_{i=1}^{r} c_i\left(c_i - \frac{k}{2}\right). \tag{A.8}$$

We remark that $k \geq c_1$, otherwise the line $\overline{x_0 x_1}$ would be a fixed component of Γ. Thus, forgetting the term of the sum for $i > h$, (A.8) forces

$$k\left(c_0 - \frac{k}{2}\right) < \sum_{i=1}^{h} c_i\left(c_i - \frac{k}{2}\right) \leq \sum_{i=1}^{h} k\left(c_i - \frac{k}{2}\right).$$

Dividing by k, the previous formula implies that

$$\left(c_0 - \frac{k}{2}\right) < \sum_{i=1}^{h}\left(c_i - \frac{k}{2}\right). \tag{A.9}$$

If $h = 0$, then (A.9) denies (A.7). Hence $h > 0$ and (A.5) follows from (A.9). If $h = 1$, (A.5) becomes $c_0 < c_1$, contradicting the way we ordered the x_i's. □

Theorem A.10. *Let Γ be an irreducible pencil of rational plane curves. Then there exists a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the strict transform of Γ via γ is a pencil of lines through a fixed point.*

Proof. Setting Γ as in (A.1), let (k, h, s) be the simplicity of Γ. If $k = 0$, then $d = c_0$ and the irreducibility assumption implies $d = 1$, that is the thesis. If $k > 0$, then $h \geq 2$ by Lemma A.4. We claim that we may choose x_j and x_l among $x_1, \dots, x_h$ such that either (**a**) the quadratic transformation $\alpha = c(x_0, x_j, x_l)$ is well defined or (**b**) x_l is satellite to x_0 and $x_l >^1 x_j >^1 x_0$. In case (a), the strict transform of Γ via α is simpler than Γ (the proof is *mutatis mutandis* the same of Lemma 6.3). In case (b), having chosen a general point $x_{r+1} \in \mathbb{P}^2$, the strict transform of Γ via the quadratic transformation $c(x_0, x_j, x_{r+1})$ is simpler than Γ (cf. Lemma 6.5). In both cases, we conclude by induction on the simplicity. It remains to show that there exist x_j and x_l as we claimed. If there exists $j \leq h$ with $x_j \in \mathbb{P}^2$, then set $l = 1$ ($l = 2$, respectively) if $j > 1$ ($j = 1$, respectively). Then $x_l \in \mathbb{P}^2$, or $x_l >^1 x_0$, or $x_l >^1 x_j$. Anyway $x_l \notin L' = \overline{x_0 x_j}$, otherwise L' should be a fixed component of Γ, and case (a) occurs. If $x_i \notin \mathbb{P}^2$ for every $0 < i \leq h$, then $x_i > x_0$ for every $0 < i \leq h$. If $x_i >^1 x_0$ for every $0 < i \leq h$, then (A.5) would contradict the proximity inequality at x_0. Hence there exists j and l such that $0 < j < l \leq h$ and $x_l >^1 x_j >^1 x_0$. If x_l is satellite to x_0, then case (b) occurs. If x_l is not satellite, then α is well defined, that is case (a), because $x_l \notin \overline{x_0 x_j}$, as before. □

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