Annali di Matematica 181, 349–363 (2002)

DOI (Digital Object Identifier) 10.1007/s102310100035

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On the cohomology with compact supports for q-complete mixed manifolds

Received: May 3, 2000
Published online: October 30, 2002 –

1. Introduction

The notion of smooth families of complex manifolds appeared in the study of deformation of complex structures by Kodaira and Spencer [20]. Then Andreotti and Grauert [1] used regular families of domains of holomorphy for which they proved vanishing theorems for the sheaf of germs of smooth functions which are holomorphic with respect to the complex variable. This idea of a "mixed variable" (see also [5] and [24]) motivated the introduction of "mixed manifolds" by Jurchescu [14]; the topic was further developed in a series of his papers ([16], [17], [18]) as well as in ([4], [6], [8], [9]). Roughly speaking, a mixed manifold is a smooth manifold which is foliated by locally closed complex manifolds of the same dimension, called "complex leaves."

Within the category of mixed manifolds one has the full subcategory of "Cartan manifolds" (they are analogies to Stein manifolds in the complex set-up) for which several characterizations and properties have been established ([16], [17]); e.g., vanishing theorems in the form of theorem B of Cartan (which suggested the above appellation), embedding as mixed submanifolds into $\mathbb{R}^M \times \mathbb{C}^N$ for suitable integers M and N which depend on the type of the source mixed manifold, etc. In particular, every Cartan manifold admits a smooth exhaustion function which is strongly plurisubharmonic along the complex leaves; hence is 1-complete (see Definition 4). However, in contrast with the pure complex case, there are 1-complete mixed manifolds which are **not** Cartan (*viz.*, Proposition 1 and Remark 7).

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* Supported by the Project MURST "Geometric properties of real and complex manifolds" and European contract ERBFMRXCT 980163 "Complex analysis and analytic geometry".

* Partially supported by ANSTI grant No. 5232/1999.

On the other hand, the complex analytic concept of coherence has to be adapted accordingly due to the presence of real parameters. This, see Definition 3, goes back to Grothendieck's notion of pseudocoherence [13].

Jurchescu ([16], Section 5, Théorème 1), in an attempt to extend a well-known result due, essentially, to Serre ([26], [27]), showed that if X is Cartan and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H_c^i(X, \mathcal{F}) = 0$ for integers $i < \mathrm{codh}_{\mathcal{O}_X}(\mathcal{F}) - k$, provided that X is a smooth family of complex manifolds over a smooth manifold S with $k = \dim S$. (Notice that in this case if X is of type (m, n) then $k \geq m$.)

In this paper, following the bumping method of Andreotti and Grauert [1] we drop the additional assumption on X (cf., Corollary 2) and prove a more general result (Theorem 1). The same idea is applied for the cohomology with constant coefficients. Our results are:

Theorem 1. *Let X be a q-complete mixed manifold of type (m, n) and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then the cohomology group $H_c^i(X, \mathcal{F})$ vanishes for every integer $i \leq \mathit{codh}_{\mathcal{O}_X}(\mathcal{F}) - q - m$.*

Corollary 1. *Let X be a q-complete mixed manifold of type (m, n) and E over X a morphic complex vector bundle. Let $\mathcal{E}$ be the sheaf of germs of morphic sections of E. Then $H_c^i(X, \mathcal{E}) = 0$ for integers $i < n$.*

Corollary 2. *Let X be a Cartan manifold of type (m, n). If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H_c^i(X, \mathcal{F}) = 0$ for integers $i < \mathit{codh}_{\mathcal{O}_X}(\mathcal{F}) - m$.*

Theorem 2. *If X is a q-complete mixed manifold of type (m, n), then the cohomology group $H_c^i(X, \mathbb{Z})$ vanishes for every integer $i \leq n - q$, and is free for $i = n - q + 1$.*

We note that the "dual" result for homology is a simple consequence of Morse theory (see Corollary 4 in Section 2). However, since X may not be orientable, we cannot deduce Theorem 2 from it. As an application, we obtain a Lefschetz-type result.

Corollary 3. *Let $X \subset \mathbb{P}^N$ be a Levi flat CR-submanifold of type (k, n). Also let $\Sigma \subset \mathbb{P}^N$ be a linear subspace of codimension q such that $A := \Sigma \cap X$ is smooth. Then $H_i(X, A; \mathbb{Z}) = 0$ for $i \leq n - q$.*

2. Preliminaries

2.1. Mixed manifolds

Let us briefly recall the elementary notions of mixed manifolds. For a detailed discussion we refer the interested reader to [17].

(•) The category of *mixed manifolds of type* (m, n) is introduced starting with *local models of type* (m, n), i.e., open subsets of $\mathbb{R}^m \times \mathbb{C}^n$. We only say that a function $\Phi \in C^\infty(U, V)$, where U and V are mixed models (not necessarily of the same type), is *morphic* if the real components of Φ are locally independent of the complex variable and the complex components of Φ are holomorphic with respect to the complex variables.

Remark 1. It can be shown that the category with mixed manifolds as objects and morphic maps as morphisms coincides with that of Levi flat CR-manifolds as objects and CR-mappings as morphisms. Also mixed manifolds are called “smooth foliations with complex leaves” [8].

Remark 2. The category of mixed manifolds contains as full subcategories the category of differentiable manifolds and the category of complex manifolds. Differentiable manifolds of dimension m correspond to mixed manifolds of type $(m, 0)$ and complex manifolds of dimension n to mixed manifolds of type $(0, n)$.

Example 1. Given two natural numbers p and q, $p + q \geq 1$, let $\mathrm{Gl}(p, q; \mathbb{R}, \mathbb{C})$ be the subgroup of $\mathrm{Gl}(p + q; \mathbb{C})$ consisting of matrices A of the form

$$A = \begin{pmatrix} A_2 & 0 \\ A_{12} & A_1 \end{pmatrix},$$

where $A_1 \in \mathrm{Gl}(q; \mathbb{C})$, $A_2 \in \mathrm{Gl}(p; \mathbb{R})$, and A_{12} is a $q \times p$ matrix with complex entries. It is easily seen that the above set identifies canonically with an open subset of $\mathbb{R}^{p^2} \times \mathbb{C}^{q(p+q)}$, hence it is a model of type $(p^2, q(p + q))$. In particular, $\mathrm{Gl}(p; \mathbb{R}) = \mathrm{Gl}(p, 0; \mathbb{R}, \mathbb{C})$ is purely real and $\mathrm{Gl}(q; \mathbb{C}) = \mathrm{Gl}(0, q; \mathbb{R}, \mathbb{C})$ is purely complex.

Note also that $\mathrm{Gl}(p, q; \mathbb{R}, \mathbb{C})$ identifies with the set of all $\mathbb{R}$-linear maps of $\mathrm{Aut}(\mathbb{R}^p \times \mathbb{C}^q)$ which are morphic, i.e., given by $(s, z) \mapsto (A_2 s, A_1 z + A_{12} s)$.

This example allows us to introduce the notion of the *morphic vector bundle E* with typical fiber $\mathbb{R}^p \times \mathbb{C}^q$ over a mixed manifold X. If X has type (m, n), then E has type $(m + p, n + q)$. See [17] for more details. *morphic complex vector bundle.*

Example 2. For every integer $k \geq 1$ and every pair of integers $m, n \geq 0$, such that $m + 2n = k + 1$ there is a mixed structure of type (m, n) on $\mathbb{S}^1 \times \mathbb{S}^k$, where $\mathbb{S}^p$ denotes the unit p-dimensional sphere in $\mathbb{R}^{p+1}$; see [18].

Definition 1. *A mixed manifold X of type (m, n) is said to be a* smooth family of complex manifolds *(of dimension n) if there is a smooth manifold S and a morphic map $\Phi : X \longrightarrow S$ of rank m at every point of X.*

Remark 3. In general we have $\dim S \geq m$. Usually it is assumed that $\dim S = m$, e.g., in [20], [25], [19].

A more practical criterion is introduced in [17]:

Definition 2. *A mixed manifold X of type (m, n) has* enough real morphic functions *if for every point $x \in X$ there is a morphic map $\Phi : X \longrightarrow \mathbb{R}^p$, $p = p(x)$, which is of rank m at x.*

Remark 4. In the above definition, if there is a Φ for x, then the choice $p(x) = m$ is possible; moreover, if X is as in Definition 2, then there exists a morphic map $\Phi : X \longrightarrow \mathbb{R}^{2m}$ of rank m at every point of X (see [17]).

Example 3. Let X be the manifold which arises by factoring with respect to a discrete group Γ of parallel translations in $\mathbb{R}^m \times \mathbb{C}^n$. Then X is a mixed manifold of type (m, n); it has sufficiently many real morphic functions if and only if the projection Γ_o of the group Γ in the group of parallel translations of $\mathbb{R}^m$ is discrete. (See [19, p. 68].)

Example 4. Let X be a mixed manifold and $E \longrightarrow Z$ a morphic vector bundle of type (p, q). If X has sufficiently many real morphic functions and $p = 0$, then E has sufficiently many real morphic functions. However, this does not hold for $p \neq 0$. For instance, over $X = \mathbb{C}^\star$ as a mixed manifold of type $(0, 1)$ we define a morphic vector bundle E of type $(1, 0)$ which does not have sufficiently many real morphic functions. In order to show this, consider the covering of X given by $U_1 := \mathbb{C} \setminus (-\infty, 0]$ and $U_2 := \mathbb{C} \setminus [0, \infty)$. Then $U_{12} := U_1 \cap U_2$ has two connected components U_{12}^- and U_{12}^+. Let $\lambda \in \mathbb{R}^\star$ and consider the morphic map $g_{12} : U_{12} \longrightarrow \mathbb{R}^\star$ given by 1 in U_{12}^- and λ in U_{12}^+. Then the corresponding morphic bundle E_λ thus obtained has sufficiently many real morphic functions if and only if $\lambda = 1$, that is if it is trivial. To see this we notice that a morphic function $h : E_\lambda \longrightarrow \mathbb{R}$ is given by two smooth functions $h_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, 2$, such that $h_1(g_{12}(x)t) = h_2(t)$ for every $x \in U_{12}$ and $t \in \mathbb{R}$. Thus $h_1(\lambda t) = h_1(t)$ for every t; hence for $\lambda \neq 1$, the derivative of h_1 vanishes at 0, and whence the desired assertion.

We remark that here $\pi_1(X) = \mathbb{Z}$. There are also similar examples of morphic vector bundles E of type $(1, 0)$ over open subsets X of $\mathbb{C}$ with $\pi_1(X) = \mathbb{Z}^2$ such that every real-valued morphic function on E is constant. For instance, for $X = \mathbb{C}\setminus\{-1, 1\}$ we have a covering by not-empty simply connected open sets $U_1, \ldots, U_5$ such that every intersection of two is simply connected (if not empty), the only not-empty intersections are $U_1 \cap U_2$, $U_1 \cap U_3$, $U_2 \cap U_3$, $U_3 \cap U_4$, $U_3 \cap U_5$, $U_4 \cap U_5$, and the intersection of any three of the U_i's is empty. Consider the transition maps $g_{12} = g_{23} = g_{34} = g_{45} = 1$, $g_{31} = \lambda$, $g_{53} = \mu$, where $\lambda, \mu \in \mathbb{R}^\star$. These define a real morphic bundle $E \longrightarrow X$ of fiber $\mathbb{R}$. Now, if $h : E \longrightarrow \mathbb{R}$ is morphic, then the induced smooth map $h_3 : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $h_3(t) = h_3(\lambda t) = h_3(\mu t)$, $\forall\, t \in \mathbb{R}$. A proper choice of λ and μ implies that h_3 is constant, hence so does h. (For instance we choose λ and μ positive such that $\log \lambda$, $\log \mu$ are independent over $\mathbb{Z}$.)

To see the complex leaves we regard E as the quotient of the disjoint union of $U_i \times \mathbb{R}$ by the obvious equivalence relation. Then the complex leaf of E, which contains $U_3 \times \{t\}$ for some $t \in \mathbb{R}^\star$, has as its trace on $U_3 \times \mathbb{R}$ the union of $U_3 \times \{t\lambda^m \mu^n\}$ over integers m, n.

In order to stress another property of this example we note the following fact. Let $\alpha \in \mathbb{R}$ and $C > \max(1, \alpha)$. Let $A := \{(m, n) \in \mathbb{Z}^2\, ;\ |m + n\alpha| < C\}$. Then for every (m, n), $(m', n') \in A$ there are $(a_i, b_i) \in A$, $i = 1, \ldots, k$, with the folllowing properties:

1) $(a_1, b_1) = (m, n)$ and $(a_k, b_k) = (m', n')$;
2) $|a_{i+1} - a_i| + |b_{i+1} - b_i| = 1$ for $i = 1, \ldots, k - 1$.

(The proof is easy and is left to the reader!) Now, coming back to the example, we observe the following interesting property. We identify X with its zero section in E. Then one has:

For every open neighborhood Ω of X there exists a non-empty open neighborhood Ω' of X, $\Omega' \subset \Omega$, such that every real morphic function on Ω is constant on Ω'.

This will follow by standard arguments from the next discussion. Consider simply connected non-empty open sets $U_i' \Subset U_i$ such that $U_1', \ldots, U_5'$ have similar properties as U_i's. Given a morphic function $h : \Omega \longrightarrow \mathbb{R}$ this induces morphic functions $h_i : \Omega_i \longrightarrow \mathbb{R}$, where $\Omega_i \subset U_i \times \mathbb{R}$ corresponds to $\Omega \cap \pi^{-1}(U_i)$ via the bimorphism Φ_i between $\pi^{-1}(U_i)$ and $U_i \times \mathbb{R}$. Choose now $\epsilon > 0$ sufficiently small such that $U_i' \times (-\epsilon, \epsilon) \Subset \Omega_i$. We get smooth functions $h_i' : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ and the compatibility conditions imply that h_3' is constant. Now take $\omega := \cup \Phi_i^{-1}(U_i' \times (-\epsilon, \epsilon))$, hence h is constant on ω.

The coherence on mixed manifolds is based on Grothendieck's notion of pseudocoherence [13]. Following Jurchescu [14] we give:

Definition 3. *An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be* coherent *if for every point x of X and for every integer $d \geq 0$ there exist an open neighborhood U of x and an exact sequence of $\mathcal{O}_U$-modules*

$$\mathcal{O}_U^{p_d} \longrightarrow \cdots \longrightarrow \mathcal{O}_U^{p_0} \longrightarrow \mathcal{F}|_U \longrightarrow 0,$$

where $\mathcal{O}_U = \mathcal{O}_X|_U$ and p_i are integers ≥ 0.

Let X be a mixed manifold of type (m, n) and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. We define the *homological dimension* of $\mathcal{F}$ by setting

$$\mathrm{dh}_{\mathcal{O}_X}(\mathcal{F}) := \sup_{x \in X} \mathrm{dh}_{\mathcal{O}_{X,x}}(\mathcal{F}_x).$$

Then $0 \leq \mathrm{dh}_{\mathcal{O}_X}(\mathcal{F}) \leq m + n$. (By convention, here the zero sheaf is considered locally free of rank 0.) We refer to [16] and [17] for these facts. Define the *homological codimension* of $\mathcal{F}$ by $\mathrm{codh}_{\mathcal{O}_X}(\mathcal{F}) = m + n - \mathrm{dh}_{\mathcal{O}_X}(\mathcal{F})$. Therefore, one has $0 \leq \mathrm{codh}_{\mathcal{O}_X}(\mathcal{F}) \leq m + n$; moreover, if X is connected, the equality $\mathrm{codh}_{\mathcal{O}_X}(\mathcal{F}) = m + n$ holds if and only $\mathcal{F}$ is locally free.

2.2. Cartan and q-complete mixed manifolds

Let X be a mixed manifold of type (m, n). From [17] we quote:

Definition 4. *X is said to be* Cartan *if the following three conditions hold:*

1) *X is* $\mathcal{O}(X)$-convex, *i.e., for every compact subset $K \subset X$ its* morphic hull $\widehat{K}$ *is compact, where*

$$\widehat{K} := \{x \in X;\ |f(x)| \leq \max_K |f|,\ \forall f \in \mathcal{O}(X)\};$$

2) $\mathcal{O}(X)$-separates the points *of X;*
3) *X has* enough real morphic functions.

Clearly, if X is of pure complex type, we recover the definition of Stein manifolds. For a topological characterization of Cartan open subsets of $\mathbb{R} \times \mathbb{C}$ see [23]. The third condition from above is not easy to verify as we have just seen in Example 4. For some cohomological conditions, see [18].

The canonical sheaf $\mathcal{O}_X$ is a Fréchet sheaf. To see this, we firstly consider $X = U \subset \mathbb{R}^m \times \mathbb{C}^n$ be a local model. For every compact set $K \subset U$ and $j \in \mathbb{N}$ we define the semi-norm $p_{K,j}$ on $\mathcal{O}(U)$ by setting:

$$p_{K,j}(f) := \sum_{|\alpha| \le j} \|\mathcal{X}_s^\alpha f\|_K, \; f \in \mathcal{O}(U),$$

where $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, $|\alpha| := \alpha_1 + \cdots + \alpha_m$, and $\mathcal{X}_s^\alpha$ denotes the derivative $\partial^{|\alpha|}/\partial s_1^{\alpha_1} \cdots \partial s_m^{\alpha_m}$ with respect to the real variables. These semi-norms $\{p_{K,j}\}$ give a Fréchet topology on $\mathcal{O}(U)$. It is then a standard fact to make out of $\mathcal{O}_X$ a Fréchet sheaf.

An open subset D of a Cartan manifold X is said to be *Runge* if D itself is Cartan and the restriction map $\mathcal{O}(X) \longrightarrow \mathcal{O}(D)$ has dense image.

Definition 5. ([8]) *Let X be a mixed manifold of type (m, n).*

1) *A function $\varphi \in C^\infty(W, \mathbb{R})$, where $W \subset X$ is an open subset, is said to be q-*convex *if the restriction of φ along the complex leaves is q-convex.*
2) *We say that X is q-*complete *if there exists an exhaustion function $\varphi \in C^\infty(X, \mathbb{R})$ which is q-convex on X.*

(Recall that if Z is a complex manifold of pure dimension n, then $\psi \in C^\infty(Z, \mathbb{R})$ is called *q-convex* if its Levi form has at least $n - q + 1$ positive (> 0) eigenvalues at every point of Z, see [1].)

It is straightforward to see that Cartan manifolds are 1-complete. However, the converse does not hold as shown by the subsequent proposition.

Proposition 1. *There is a* 1*-complete mixed manifold X of type* $(1, 1)$ *such that every real morphic function on X is constant,* a fortiori *X is* **not** *Cartan.*

Proof. Let G be the group of translations of $\mathbb{R} \times \mathbb{C}$ generated by the vectors $(\sqrt{2}, 1)$ and $(1, 0)$. Set $X := (\mathbb{R} \times \mathbb{C})/G$. Clearly X is a mixed manifold of type $(1, 1)$. Let $\widetilde{\varphi} : \mathbb{R} \times \mathbb{C} \longrightarrow [0, \infty)$ be defined by $\widetilde{\varphi}(s, z) = y^2$, where $z = x + iy$ with $x, y \in \mathbb{R}$. Obviously $\widetilde{\varphi}$ is smooth and 1-convex, and since it is invariant under the action of G it descends to a smooth function $\varphi : X \longrightarrow [0, \infty)$ which is proper and 1-convex on the complex leaves. Therefore X is 1-complete. However, every real-valued morphic function on X is constant (exercise!). □

Remark 5. There exists a Cartan mixed submanifold X of a mixed manifold Y which does not admit a Cartan open neighborhood. (Take, in Example 4 (see the last part of it), $X :=$ the zero section in $E =: Y$; see also Remark 6.)

From [4] we quote:

Theorem 3. *A mixed manifold is Cartan if and only if it is* 1*-complete and has enough real morphic functions.*

Proposition 2. *Let X be a Cartan manifold and D an open subset of X such that $H^i(D, \mathcal{O}_X) = 0$ for every $i > 0$. Then D is a Cartan manifold.*

Proof. It sufficient to check that D is morphically convex. For this, we first remark that for every point $x_o \in \partial D$ there exist $f_1, \dots, f_k \in \mathcal{O}(X)$ such that $\{f_1 = \cdots = f_k = 0\} = \{x_o\}$. Then by [22], there are morphic complex-valued functions $g_1, \dots, g_k \in \mathcal{O}(D)$ such that $f_1 g_1 + \cdots + f_k g_k = 1$ on D. From this we infer that for every compact $K \subset D$, $\widehat{K}$ is contained in D. □

(•) As a way to produce new q-complete mixed manifolds from given ones, we show:

Proposition 3. *Let $\pi : E \longrightarrow X$ be a morphic vector bundle over a mixed manifold X. If X is q-complete, then E is q-complete, too.*

Proof. Since the proof is "standard", we give only a sketch. Let $\varphi : X \longrightarrow \mathbb{R}$ be an exhaustion function which is q-convex. Fix an arbitrary hermitean metric h on E. Then one shows that there $\chi \in C^\infty(\mathbb{R}, \mathbb{R}_+^\star)$ is rapidly increasing and convex such that the function $\psi : E \longrightarrow [0, \infty)$ given by $\psi(\xi) := \log(1 + \|\xi\|^2) + \chi(\varphi(x))$, where $x = \pi(\xi)$, is q-convex. (Note that ψ is always proper.) This is done by straightforward computations using the homogenity of $\|\xi\|$ and the very useful inequality $\epsilon|u|^2 + |v|^2/\epsilon \geq 2|uv|$ valid for $\epsilon > 0$ and any complex numbers u, v. □

Remark 6. This proposition, together with Example 4, furnishes new examples of 1-complete mixed manifolds which are not Cartan.

Proposition 4. *Let $\pi : \widetilde{X} \longrightarrow X$ be a (topological) covering of a mixed manifold X. Then the following statements hold:*

1) *If X is Cartan, then $\widetilde{X}$ is Cartan.*
2) *If X is q-complete, then $\widetilde{X}$ is q-complete.*

Proof. We remark that if Y is an arbitrary mixed manifold which has enough real morphic functions, then the same conclusion holds for any mixed manifold M which is a *spread over* Y (by this we mean that there is a morphic map $p : M \longrightarrow Y$ which is locally homeomorphic). Therefore, according to Theorem 3, it remains to verify 2). This is done using an idea from [21] (see also [28, p. 494]) and it suffices to produce a smooth function $\theta : \widetilde{X} \longrightarrow \mathbb{R}_+$ such that:

a) θ is a *vertical exhaustion function*, this means that, for every compact set $K \subset X$, the restriction of θ to $\pi^{-1}(K)$ is exhaustive.
b) The eigenvalues of the Levi form of θ with respect to the coordinates coming from X are uniformly bounded from below.

Then we construct a q-convex exhaustion function defined on $\widetilde{X}$ in the form $\chi(\varphi \circ \pi) + \theta$, where $\varphi \in C^\infty(X, \mathbb{R})$ is q-convex and exhaustive, and the function $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ is rapidly increasing and convex.

So it remains to construct θ. Let $\{U_i\}_i$ be a locally finite open covering of X such that $U_i \Subset X$ and each U_i is bimorphic to a simply connected model (so U_i is evenly covered). We get a decomposition $\pi^{-1}(U_i) = \cup_k W_{ik}$ into disjoint open sets W_{ik} bimorphic to U_i via the projection map. Let $\{\rho_i\}_i$ be a partition of unity corresponding to $\{U_i\}$ and define $\theta : \widetilde{X} \longrightarrow \mathbb{R}$ as follows: fix some $W_{i_o k_o}$ and define l_{ik} as the length of the shortest chain $W_{i_o k_o}, W_{i_1 k_1}, \ldots, W_{i_s k_s}$ such that each $W_{i_r k_r}$ intersects $W_{i_{r+1} k_{r+1}}$ and $(i_s, k_s) = (i, k)$. We then set

$$\theta := \sum_{i,k} l_{ik} \cdot (\rho_i \circ \pi).$$

Property a) is straightforward. Now, since $\theta = l_{i'k'} + \sum_{i,k} (l_{ik} - l_{i'k'})(\rho_i \circ \pi)$ and $\theta = \lambda \circ \pi$ locally, we easily conclude, by computing $\partial^2 \lambda / \partial z_j \partial \overline{z}_s$, that b) holds. □

Remark 7. Gilligan [11] suggested the question that a kind of Oka theorem for spreads over Cartan manifolds might hold. More precisely, assume that (D, π) is a spread over a mixed manifold X. Suppose that (D, π) is *locally Cartan*, *i.e.*, every point of X has an open neighborhood U such that $\pi^{-1}(U)$ is Cartan. Then we ask:

Does it follow that D is Cartan if X itself is Cartan?

We show that, pointing out, again, the discrepancy between the pure complex case and the mixed one, even in the most trivial set-up of the inclusion, one cannot have such a result. Take $X := \mathbb{R} \times \mathbb{C}$, $D := \mathbb{R} \times \mathbb{C} \setminus (\{0\} \times \mathbb{S}^1)$, and $\pi : D \longrightarrow X$ the inclusion map. Then (D, π) is as desired.

In order to verify this, we let $A \subset \mathbb{C}$ be a closed subset and define $D_A = \mathbb{R} \times \mathbb{C} \setminus (\{0\} \times A)$.

Lemma 1. *D_A is Cartan if and only if A has no compact connected component.*

Proof. In a more general set-up this can be found in [23]. However, because of its simplicity, we supply one ad hoc proof. First we recall a well-known topological result.

Lemma 2. *Let T be a locally compact, Hausdorff topological space with countable topology and $A \subset T$ be a closed set. Then every compact connected component of A has a neighborhood system of open sets V in T such that $A \cap \partial V = \emptyset$.*

Coming back to the proof of Lemma 1, for the "if" part we note that D is a regular family of domains of holomorphy (see Section 2.3; we take $\Omega = \mathbb{C}$ and $V = \mathbb{R}$). For the "only if", we argue by contradiction, using Lemma 2 and the Cauchy–Pompeiu formula. □

Now the example concludes as follows. Note that for $z_o \in \mathbb{S}^1$ and for $r \in (0, 2)$, if $B := \{|z - z_o| < r\}$, then there is an embedding $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ with $B \cap \Gamma = B \cap \mathbb{S}^1$, where $\Gamma = \gamma(\mathbb{R})$. Therefore, $D \cap (\mathbb{R} \times B) = D_\Gamma \cap (\mathbb{R} \times B)$ is Cartan since D_Γ is Cartan. □

The next lemma is well known for complex manifolds (see [30]).

Lemma 3. *Let X be a mixed manifold of type (m, n) and $\varphi \in C^\infty(X, \mathbb{R})$ a q-convex function. If x_o is a non-degenerate critical point of φ, then the index of φ at x_o is $\leq m + n + q - 1$.*

On the other hand, since q-convexity is stable under perturbations with smooth functions whose second-order derivatives are sufficeintly small, and taking into account the density of Morse functions in the C^2-topology we obtain in a standard way:

Proposition 5. *Let X be a mixed manifold and $\varphi \in C^\infty(X, \mathbb{R})$ a q-convex proper function. Then, given an arbitrary $\eta \in C^0(X, \mathbb{R})$, $\eta > 0$, there is $\widetilde{\varphi} \in C^\infty(X, \mathbb{R})$ such that:*

1) *$\widetilde{\varphi}$ is a Morse function;*
2) *$\widetilde{\varphi}$ is q-convex; and,*
3) *$|\widetilde{\varphi} - \varphi| < \epsilon$.*

In particular, if φ is exhaustive, then $\widetilde{\varphi}$ can be chosen exhaustive, too.

From [2, Lemma 2, p. 503] we quote:

Lemma 4. *Let X be a topological space with a countable base and K a compact subset of X such that $X \setminus K$ is a differentiable manifold. Let $\varphi \in C^\infty(X \setminus K, \mathbb{R})$. Assume that the sets*

$$B_c := K \cup \{x \in X \setminus K\,;\ \varphi(x) \leq c\},$$

for $c \in \mathbb{R}$ are compact and that φ has only non-degenerate critical points of index $\leq k$ on $X \setminus K$. Then, if c is such that $K \subset \mathrm{int}(B_c)$, we have

$$H_r(X, B_c; \mathbb{Z}) = 0 \textit{ if } r > k \textit{ and } H_k(X, B_c; \mathbb{Z}) \textit{ is free.}$$

Corollary 4. *If X is a q-complete mixed manifold of type (m, n), then $H_i(X, \mathbb{Z})$ vanishes for $i \geq m + n + q$ and is free for $i = m + n + q - 1$.*

Proof. This is a simple consequence of Lemmas 3 and 4, and Proposition 5. □

Definition 6. *We say that an open subset Ω of a mixed manifold X is q-*Runge *if for every compact set $K \subset \Omega$ there exists a q-convex exhaustion function on X (of course, φ may depend on K) such that*

$$K \subset \{\varphi < 0\} \Subset \Omega.$$

For instance, X is q-complete if and only if the empty set is q-Runge in X. Also, if X is a Cartan manifold, then an open set $\Omega \subset X$ is Runge if and only if Ω is 1-Runge.

Lemma 5. *Let Ω be a q-Runge domain in a mixed manifold X of type (m, n). If G is an arbitrary abelian group, then $H_i(X, \Omega; G) = 0$ for all integers $i \geq m+n+q$.*

Proof. By definition, Proposition 5, and Corollary 4, there exists a sequence of pairs of open subsets of X, (X_k, Ω_k), with the following properties: a) $\{\Omega_k\}_k$ increases to Ω, b) $\{X_k\}_k$ increases to X, and c) $H_i(X_k, \Omega_k; G) = 0$ for $i \geq m + n + q$. As a matter of fact, we take $\Omega_k := \{\varphi_k < 0\} \subset \Omega$ and $X_k := \{\varphi_k < C_k\}$ for suitable Morse exhaustion functions φ_k on X which are q-convex. Then the proof concludes easily by passing to the limit. □

Corollary 5. *Let X be a Cartan manifold of type (m, n) and $\Omega \subset X$ a Runge open set. Then the relative homology group $H_i(X, \Omega; \mathbb{Z})$ vanishes for $i > m + n$ and is torsion free for $i = m + n$.*

We do not know if, in the above set-up, $H_{m+n}(X, \Omega; \mathbb{Z})$ is free.

2.3. Regular families of domains of holomorphy

Let $\pi : \mathbb{R}^m \times \mathbb{C}^n \longrightarrow \mathbb{R}^m$ be the canonical projection and $D \subset \mathbb{R}^m \times \mathbb{C}^n$ a non-empty open subset. Put $D_s := \{z \in \mathbb{C}^n ; (s, z) \in D\}$, $s \in \pi(D)$. We may regard D as a family $\{D_s\}_{s\in\pi(D)}$ of open subsets of $\mathbb{C}^n$.

Following [1] we say that $\{D_s\}_{s\in\pi(D)}$ is a *regular family of domains of holomorphy* if for every point $s_o \in \pi(D)$ there exist an open set $\Omega \subset \mathbb{C}^n$ and an open neighborhood V of s_o in $\pi(D)$ such that:

1) $D_{s_o} \subset \Omega$ and the pair (Ω, D_{s_o}) is Runge.
2) $\pi^{-1}(V) \subset V \times \Omega$.

From [1] we easily get that in this case D is Cartan. However, there are Cartan open subsets of $\mathbb{R} \times \mathbb{C}$ which are not regular families of domains of holomorphy. For instance, if we consider $D := \mathbb{R} \times \mathbb{C} \setminus \Gamma$, where Γ is the graph of the function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$, $\gamma(s) = s + is$, $s \in \mathbb{R}$, then by [10] D is Cartan and, however, D is **not** a regular family in the above sense.

Proposition 6. *Let $D' \subset D$ be two open sets of $\mathbb{R}^m \times \mathbb{C}^n$, each of which being a regular families of domains of holomorphy such that, for any $s \in \mathbb{R}^m$, the pair (D_s, D'_s) is Runge. Then:*

1) *$H^i_c(D, \mathcal{O}) = H^i_c(D', \mathcal{O}) = 0$ for $i < n$ and the canonical extension map $H^n_c(D', \mathcal{O}) \longrightarrow H^n_c(D, \mathcal{O})$ is injective.*
2) *For an arbitrary abelian group G one has $H^i_c(D, G) = H^i_c(D', G) = 0$ for $i < n$ and the extension map $H^n_c(D', G) \longrightarrow H^n_c(D, G)$ is injective.*

Proof. The first statement is proved in [1]. For 2) note that the pair (D, D') is 1-Runge. Then we conclude by using the Poincaré duality and Lemma 5. □

Subsequently we show how one applies Proposition 6 locally. First we extend the notion of q-convexity to continuous functions as follows (see also [3]). Let X be a mixed manifold of type (m, n).

Definition 7. *We say that finitely many functions $\varphi_1, \ldots, \varphi_k \in C^0(X, \mathbb{R})$ are q-*convex with the same positivity directions*, and we write this by $\{\varphi_1, \ldots, \varphi_k\} \in \mathcal{P}^0(X; q)$, if for every point $x_o \in X$ there are: a local model $D \ni x_o$ (we view D as an open subset of $\mathbb{R}^m \times \mathbb{C}^n$), a complex vector space $E \subset \mathbb{C}^n$, and $\psi_{ij} \in C^\infty(W, \mathbb{R})$, $i = 1, \ldots, k$; $j = 1, \ldots, p$, such that, for every i, one has:*

1) *$\varphi_i|_W = \max(\psi_{i1}, \ldots, \psi_{1p})$;*
2) *for every $s \in \mathbb{R}^m$ with $W_s \neq \emptyset$ and $z \in W_s$ the restriction of every $\psi_{ij}(s, \cdot)$ to $W_s \cap (\{z\} + E)$ is strictly plurisubharmonic.*

*If $k = 1$ we say that φ_1 is q-*convex.

Now consider D be a model of type (m, n) and $\{\varphi, \psi\} \in \mathcal{P}^0(D; q)$ be such that $\varphi \leq \psi$. Then every point $x_o \in D$ has a Cartan neighborhood $U^\star$ in D (sufficiently small) such that for every Runge open set $U \subset U^\star$, the pair $(\{x \in U; \varphi(x) < 0\}$, $\{x \in U; \psi(x) < 0\})$ is a Runge family of domains of holomorphy in $\mathbb{C}^{n-q+1}$ over $\mathbb{R}^{m+2q-2}$ (after some morphic coordinates changes).

Proposition 7. *Let X be a mixed manifold and $\varphi \in \mathcal{P}^0(X; q)$. Then, for every $\eta \in C^0(X, \mathbb{R})$, $\eta > 0$, there exists a (smooth) q-convex function $\widetilde{\varphi}$ on X such that*

$$|\widetilde{\varphi} - \varphi| \leq \eta.$$

Proof. This is done by a perturbation argument and a "max regularization" procedure as in [33]. □

3. The proofs

3.1. The bumping method

Let X be a mixed manifold of type (m, n) together with a continuous q-convex exhaustion function φ and $\mathcal{G}$ is a sheaf of abelian groups over X. Suppose we want $H^i_c(X, \mathcal{G}) = 0$ for indices $i \leq p$ (in our case we will have $p = n - q$).

The first natural step is to show that $H^i_c(X_\lambda, \mathcal{G}) = 0$ for $i \leq p$ and every $\lambda \in \mathbb{R}$. As usual, $X_\lambda := \{x \in X\,;\ \varphi(x) < \lambda\}$. This is proved by showing a stronger assertion, namely,

($\star$) for every $\lambda, \mu \in \mathbb{R}$, $\lambda < \mu$, the canonical extension map,

$$H^i_c(X_\lambda, \mathcal{G}) \longrightarrow H^i_c(X_\mu, \mathcal{G}),$$

is bijective for $i \leq p$ and injective for $i = p + 1$.

Then ($\star$) is further implied by:

($\sharp$) for every $\lambda \in \mathbb{R}$ there is $\epsilon > 0$ such that ($\star$) holds for $\mu \in [\lambda, \lambda + \epsilon]$.

We consider this last reduction more closely. Choose a finite open covering $\{U_i\}_{i=1,\dots,k}$ (sufficiently fine, this will be clear from the context) of $L := \overline{X_{\lambda+1}}$ which is "adapted to $\mathcal{G}$" (in a sense that can be precisely specified according to the case we are dealing with, that is, either $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module or $\mathcal{G} = \mathbb{Z}$); then select smooth functions $\rho_i \in C^\infty_o(X, \mathbb{R})$ such that $\rho_i \geq 0$, $\mathrm{Supp}(\rho_i) \subset U_i$, and $\rho_1 + \cdots + \rho_k > 0$ on L. In practice we choose these ρ_i's with sufficiently small derivatives up to order 2.

Then $\epsilon > 0$ is chosen by $\epsilon < \min_L(\rho_1 + \cdots + \rho_k)$ and $\epsilon < 1$. We further reduce ($\sharp$) to a local question using the Mayer–Vietoris sequence as follows. Let $\mu \in (0, \epsilon]$. Define $W_r := \{x \in X\,;\ \varphi(x) < \lambda + \rho_1(x) + \cdots + \rho_r(x)\} \cap X_\mu$ for $r = 1, \dots, k$ and $W_o = X_0$. Then we have $X_o = W_0 \subset W_1 \subset \cdots \subset W_r = X_\mu$, $W_r \setminus W_{r-1} \Subset U_r$ and $W_r = W_{r-1} \cup (W_r \cap U_r)$, for $r = 1, \dots, k$.

Now, taking into account Section 2.3, in order to get ($\sharp$) one should prove that, for every $r = 1, \dots, k$, the following statements hold:

a) $H^i_c(W_{r-1} \cap U_r, \mathcal{G}) = H^i_c(W_r \cap U_r, \mathcal{G}) = 0$ for $i \leq p$;
b) the map $H^{p+1}_c(W_{r-1} \cap U_r, \mathcal{G}) \longrightarrow H^{p+1}_c(W_r \cap U_r, \mathcal{G})$ is injective.

From Section 2.3 we obtain immediately the proof of Theorem 1. □

Remark 8. We note that the injectivity in (♯) was used for the first time in [32].

Remark 9. Let X be a mixed manifold of type (m, n) and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for every family of supports Φ on X, one has the vanishing of $H^i_\Phi(X, \mathcal{F})$ for integers $i > n$. (This follows from a general result due to Reiffen [25]; see also [6].)

3.2. Proofs of Theorem 2 and Corollary 3

First we note that Sections 2.3 and 3.1 give the vanishing of $H^i_c(X, G)$ for $i \leq n-q$ and every abelian group G. To conclude, choose a Morse exhaustion function φ on X which is q-convex, then select a sequence $\{c_\nu\}$ which increases to $+\infty$ and each c_ν is not a critical value for φ. Set $X_\nu := \{\varphi < c_\nu\}$. The Universal Coefficient Theorem (UCT) gives that $H^{n-q+1}_c(X_\nu, \mathbb{Z})$ is torsion free for every ν. On the other hand, the homology groups $H^{n-q+1}_c(X_\nu, \mathbb{Z})$ are finitely generated, hence free. Once more using UCT we get the injectivity of

$$H^{n-q+1}_c(X_\nu, \mathbb{Z}) \otimes G \longrightarrow H^{n-q+1}_c(X_{\nu+1}, \mathbb{Z}) \otimes G.$$

Now, from [32] we quote:

Lemma 6. *Let B be a subgroup of a free abelian group A of finite rank such that the natural map $B \otimes G \longrightarrow A \otimes G$ is injective for every abelian group G. Then B is a direct summand of A.*

Coming back to the proof of Theorem 2, we deduce that $H^{n-q+1}_c(X_\nu, \mathbb{Z})$ embeds into $H^{n-q+1}_c(X_{\nu+1}, \mathbb{Z})$ as a direct summand; hence $H^{n-q+1}_c(X, \mathbb{Z})$ is free, and hence the theorem is completed. □

(•) Here we prove Corollary 3. Let $S := X \cap \Sigma$. Since $H_k(X, S; \mathbb{Z})$ is finitely generated for every k, Corollary 3 is equivalent to $H^i(X, S; \mathbb{Z}) = 0$ for $i \leq n-q$ and $H^{n-q+1}(X, S; \mathbb{Z})$ is free (see [12, p. 136]). On the other hand, since $H^k(X, S; \mathbb{Z}) \simeq H^k_c(X \setminus S; \mathbb{Z})$ for every k, the proof concludes easily from Theorem 2. □

4. Further remarks and open questions

In the circle of ideas presented up to now, we would like to mention some interesting questions.

1) *Let X be a q-complete mixed manifold and $E \longrightarrow X$ a complex morphic vector bundle. Denote by $\mathcal{E}$ the sheaf of germs of morphic sections in E. Then $H^i(X, \mathcal{E}) = 0$ for every integer $i \geq q$.*

Notice that a similar statement for the sheaf of germs of sections in an arbitrary morphic vector bundle E over X does not hold in general. For instance, if X is of type $(0, n)$ (that is, X is a complex manifold) and $E := X \times \mathbb{R}$ is the trivial real morphic vector bundle of rank $(1, 0)$ over X, then the vanishing of some $H^i(X, \mathcal{E})$ is equivalent to $H^i(X, \mathbb{R}) = 0$, as it can be easily checked.

On the other hand, if $E = X \times \mathbb{C}$ and X is a real-analytic Levi flat hypersurface of $\mathbb{C}^{n+1}$ (hence X is mixed of type $(1, n)$) the above vanishing theorem is established in [8].

2) *Let X and Y be mixed manifolds such that X is a mixed submanifold of Y. If X is q-complete, does X always admit a basis of q-complete open subsets in Y?*

Obviously, the condition "X is a mixed submanifold of Y" is necessary since there are simple examples when, without it, the conclusion fails. For instance, take $X := \mathbb{R} \times \mathbb{C}$ as a closed submanifold of $Y := \mathbb{C}^2$ in an obvious way. It is straightforward to see that X is 1-complete and, however, it does not admit a Stein neighborhood basis. Here X is mixed of type $(1, 1)$ and Y of type $(0, 2)$. As a matter of fact, note the following property. Let $A \subset \mathbb{C}^n$ be a closed set. Then $A \times \mathbb{C}$ has a fundamental system of Stein neighborhoods in $\mathbb{C}^{n+1}$ if and only if A is completely pluripolar in $\mathbb{C}^n$ (that is, there exists a plurisubharmonic function $u : \mathbb{C}^n \longrightarrow \mathbb{R} \cup \{-\infty\}$ such that $A = \{u = -\infty\}$.)

3) *Let D be given as the union of an increasing sequence $\{D_k\}_k$ of Cartan open subsets of a Cartan manifold X of type (m, n). Does it follow that D is Cartan?*

If each consecutive pair (D_{k+1}, D_k) is Runge, *i.e.* the restriction maps $\mathcal{O}(D_{k+1}) \longrightarrow \mathcal{O}(D_k)$ have dense range, then D follows Cartan.

On the other hand, if $X = \mathbb{R}^m \times \mathbb{C}$, then the answer is "Yes" as it follows from [10]. For the covenience of the reader, we give a sketch. First, we let $\Omega \subset \mathbb{R}^m \times \mathbb{C}$ be an open set. Consider $\delta : \Omega \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ the distance from the boundary. We say that δ satisfies the *minimum principle in the complex direction* if for every $s \in \mathbb{R}^m$ and every compact set $K \subset \Omega_s := \{z \in \mathbb{C}; (s, z) \in \Omega\}$,

$$\min_{z \in K} \delta(s, z) = \min_{z \in \partial K} \delta(s, z).$$

Then by [10], Ω is Cartan if, and only if, δ satisfies the minimum principle in the complex direction.

We also remark that one cannot characterize Cartan open submanifolds Ω of $\mathbb{R}^m \times \mathbb{C}^n$ for $m > 0$ via the distance function as in the pure complex case. For instance, let $\Omega := \mathbb{R} \times \mathbb{C} \setminus \Gamma$, where Γ is the graph of the function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ given by $\gamma(s) = (s, s + is)$, $s \in \mathbb{R}$. By [10], Ω is Cartan. On the other hand, if $\delta : \Omega \longrightarrow \mathbb{R}$ is the usual boundary distance function computed with respect to the euclidean metric on $\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$, then

$$\delta^2(s, z) = 2(x^2 + y^2 + s^2 - xy - xs - ys)/3, (s, z) \in \Omega, z = x + iy.$$

The condition that $-\log \delta$ be subharmonic in the complex fibres at the point (s, z) is $x^2 + y^2 - 4xy + 2xs + 2ys - 2s^2 \geq 0$, which is not always fulfilled. See also [8] for some other considerations concerning the relation between distance functions and "Stein foliations", or Cartan set according to the terminology used here.

4) *Find a topological characterization for a mixed manifold X of type (m, n) to be n-complete.*

For the case $n = 1$ and X an open subset of $\mathbb{R} \times \mathbb{C}$ we refer to [23].

Acknowledgements. This work was done while the second-named author was a visitor at the Scuola Normale Superiore in Pisa. He would like to express his thanks for the hospitality and the support given to him.

Note added in proof: In the meantime we solved Question 2 from above.

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