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A Clifford algebra approach to simple Lie algebras of real rank two, II: The G_2 case

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1. Introduction

In a couple of papers, Cowling, Dooley, Korányi, and Ricci [3,4] proposed an approach to real rank one simple Lie algebras which is based on the observation, due to Korányi [7], that the nilpotent subalgebras obtained in the Iwasawa decomposition of these simple Lie algebras are generalized Heisenberg algebra (see [6]). This simply means that the orthogonal complement of the centre of an Iwasawa algebra is a module of the Clifford algebra built on the centre. Recently the author [2] succeeded in extending this approach to simple Lie algebras with root system A_2 . Here, we study with the same technique the exceptional simple Lie algebras with root system G_2 , showing that the same Clifford algebra approach works also in this exceptional context despite the complications due to the richer structure of these algebras.

Here is the outline of the paper. In Sect. 2 we briefly recall the structure of the root system G_2 .

In Sect. 3 we study the subalgebras $\mathfrak{n}$ and $\mathfrak{m}$ obtained by performing the Iwasawa decomposition $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of the Lie algebra. In particular, in Subsect. 3.1 we decompose $\mathfrak{n}$ into the semidirect product of a generalized Heisenberg algebra $\mathfrak{n}_0$ and the root space corresponding to the shortest of the simple roots, say α . In Subsect. 3.2 we study the action of the rank one subalgebra $\mathfrak{g}_{(\alpha)}$ generated by the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ on $\mathfrak{n}_0$. Finally in Subsect. 3.3 we examine the structure of $\mathfrak{m}$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and its action on $\mathfrak{n}$. We use this analysis to show that all root spaces must have the same multiplicities and that these may only be 1, or 2.

In Sect. 4 we construct a skew-symmetric algebra $\mathfrak{g}$ of dimension 14. We start in Subsect. 4.1 considering two generalized Heisenberg algebras which correspond to $\mathfrak{n}_0$ and $\theta\mathfrak{n}_0$. Then in Subsect. 4.2 we build the algebra of deriva-

tions of n_0 corresponding to $\mathfrak{g}_{(\alpha)}$. Eventually in Subsect. 4.3 we obtain the full algebra $\mathfrak{g}$.

In Sect. 6 we show that the bracket of $\mathfrak{g}$ satisfies the Jacobi identity, proving that $\mathfrak{g}$ is a Lie algebra.

Finally, in Sect. 7 we prove that any simple Lie algebra with root system G_2 and dimension 28 is the complexification of $\mathfrak{g}$.

2. The root system G_2

Denote by θ the Cartan involution and by $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ the Killing form of $\mathfrak{g}$, and set

$$\langle X, Y \rangle = -cB(X, \theta Y), \quad \forall X, Y \in \mathfrak{g},$$

where c is a positive constant to be fixed later. It is clear that $\langle \cdot, \cdot \rangle$ is an inner product. Note the identity

$$(2.1.0) \quad \langle [X, Y], Z \rangle = \langle X, [\theta Y, Z] \rangle,$$

which is equivalent to the invariance of the Killing form.

Denote by Δ the set of restricted roots, that is G_2 , by Δ_+ the set of positive roots, and by α and β the simple roots. Then

$$\Delta_+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\},$$

$$(\alpha | \beta) = -\frac{1}{2}(\beta | \beta),$$

and the highest root [5, p. 475] is

$$\omega = 3\alpha + 2\beta.$$

We choose c in $\langle \cdot, \cdot \rangle$ so that

$$(2.1.1) \quad (\beta | \beta) = (3\alpha + \beta | 3\alpha + \beta) = (\omega | \omega) = 2,$$

it follows that

$$(\alpha | \beta) = -1, \quad (\alpha | \alpha) = \frac{2}{3},$$

and

$$(2.1.2) \quad (\omega | \beta) = (\omega | \alpha + \beta) = (\omega | 2\alpha + \beta) = (\omega | 3\alpha + \beta) = 1.$$

3. The structure of the algebra

3.1. The Clifford structure

Proposition 3.1.1. *Let*

$$\mathfrak{v} = \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha+2\beta} \oplus \mathfrak{g}_{\alpha+3\beta}.$$

For $Z \in \mathfrak{g}_\omega$ define the linear operator $J_Z : \mathfrak{v} \rightarrow \mathfrak{v}$ by

$$J_Z X = [Z, \theta X],$$

with $X \in \mathfrak{v}$. Then for all $X, Y \in \mathfrak{v}$

$$(3.1.1) \quad [X, J_Z X] = \|X\|^2 Z,$$

and

$$(3.1.2) \quad J_Z^2 X = -\|Z\|^2 X,$$

that is, $\mathfrak{v}$ is a Clifford module of the Clifford algebra $\mathcal{C}(d_\omega)$, where d_ω denote the dimension of $\mathfrak{g}_\omega$.

Proof. The statement follows from the Jacobi identity and (2.1.2), using

$$[\theta X, X] = \|X\|^2 H_\gamma,$$

which holds for $X \in \mathfrak{g}_\gamma$ [5, p. 94]. □

Remark. Since J_Z is skew-symmetric, $n_0 \equiv \mathfrak{v} \oplus \mathfrak{g}_\omega$ endowed with $\langle \cdot, \cdot \rangle$ is by (3.1.2) a generalized Heisenberg algebra with centre $\mathfrak{g}_\omega$ ([6] and [7]).

Corollary 3.1.2. (1) $d_\omega \leq d_{\alpha+\beta} = d_{2\alpha+\beta} = d_\alpha$;

(2) $d_\omega = d_\beta = d_{3\alpha+\beta} \in \{1, 2, 4, 8\}$.

Proof. Since by (3.1.4) the restrictions of J_Z to $\mathfrak{g}_\beta$ and $\mathfrak{g}_{\alpha+\beta}$ are, for $Z \neq 0$, invertible maps onto $\mathfrak{g}_{3\alpha+\beta}$ and $\mathfrak{g}_{2\alpha+\beta}$, respectively, $d_\beta = d_{3\alpha+\beta}$ and $d_{\alpha+\beta} = d_{2\alpha+\beta}$.

By (3.1.1) $d_\beta \geq d_\omega$ and $d_{\alpha+\beta} \geq d_\omega$. Let $X \in \mathfrak{g}_\beta$, take a unit $Y \in \mathfrak{g}_{3\alpha+\beta}$, then $[X, Y] \in \mathfrak{g}_\omega$. By Jacobi, since $\omega - 3\alpha - \beta \notin \Delta$, we find

$$[\theta Y, [Y, X]] = -X.$$

Hence, $d_\omega \geq d_\beta$, from which it follows $d_\omega = d_\beta$. Similarly taking a unit $Y \in \mathfrak{g}_\beta$ we prove that $d_\alpha = d_{\alpha+\beta}$, providing (1).

(2) From the proof of Proposition 3.1.1 we see that $\mathfrak{g}_\beta \oplus \mathfrak{g}_{3\alpha+\beta}$ is a module of $\mathcal{C}(d_\omega)$, which by (1) has dimension $2d_\omega$. By the classification of Clifford algebras the algebra $\mathcal{C}(d_\omega)$ has a module of dimension $2d_\omega$ only when $d_\omega \in \{1, 2, 4, 8\}$, and this provides the statement. □

3.2. The action of $\mathfrak{g}_{(\alpha)}$

Lemma 3.2.1. *Let $S \in \mathfrak{g}_\alpha$ and $X \in \mathfrak{g}_\beta$, then*

$$(3.2.1) \quad [\theta S, [S, X]] = -\|S\|^2 X,$$

$$(3.2.2) \quad [\theta S, [S, [S, X]]] = -\frac{4}{3}\|S\|^2 [S, X],$$

and

$$(3.2.3) \quad [\theta S, [S, [S, [S, X]]]] = -\|S\|^2 [S, [S, X]].$$

In particular, if both S and X are non-zero, then

$$[S, X] \neq 0, \quad [S, [S, X]] \neq 0, \quad [S, [S, [S, X]]] \neq 0.$$

Proof. Since $\beta - \alpha \notin \Delta$, the Jacobi identity implies

$$[\theta S, [S, X]] = [X, [S, \theta S]] = \beta ([\theta S, S]) X = (\beta | \alpha) \|S\|^2 X = -\|S\|^2 X \neq 0.$$

Using (3.2.1) and the Jacobi identity we get

$$\begin{aligned} [\theta S, [S, [S, X]]] &= [S, [\theta S, [S, X]]] + [[S, X], [S, \theta S]] \\ &= -\|S\|^2 [S, X] + (\alpha + \beta) ([\theta S, S]) [S, X] \\ &= -\frac{4}{3}\|S\|^2 [S, X] \neq 0. \end{aligned}$$

Similarly, from (3.2.2) we find

$$\begin{aligned} [\theta S, [S, [S, [S, X]]]] &= [S, [\theta S, [S, [S, X]]]] + [[S, [S, X]], [S, \theta S]] \\ &= -\frac{4}{3}\|S\|^2 [S, [S, X]] + (2\alpha + \beta) ([\theta S, S]) [S, [S, X]] \\ &= -\|S\|^2 [S, [S, X]] \neq 0. \end{aligned}$$

□

Proposition 3.2.2. $d_\omega \in \{1, 2\}$.

Proof. Assume $d_\omega = d \in \{4, 8\}$, and let $\{Z_1, Z_2, \dots, Z_{d_\omega}\}$ be an orthonormal basis of $\mathfrak{g}_\omega$. Set

$$\epsilon = J_1 J_2 \dots J_d,$$

then ϵ is symmetric and $\epsilon^2 = 1$. Therefore, ϵ has eigenvalues ± 1 . Let $V \in \mathfrak{v} \setminus \{0\}$ and $\epsilon V = V$, V has a non-zero component in one of the root spaces in $\mathfrak{v}$, say $X \in \mathfrak{g}_\gamma$. It is clear that $\{X, J_1 J_2 X, \dots, J_1 J_d X\}$ and $\{J_1 X, J_2 X, \dots, J_d X\}$ span $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{\omega-\gamma}$. Since $\epsilon J_k = -J_k \epsilon$, the restriction of ϵ to $\mathfrak{g}_\gamma$ is the identity and the restriction to $\mathfrak{g}_{\omega-\gamma}$ is minus the identity. Hence, the component of V in $\mathfrak{g}_{\omega-\gamma}$ is

zero. It is not restrictive to assume that ϵ is the identity on $\mathfrak{g}_{\alpha+\beta}$ and minus the identity on $\mathfrak{g}_{2\alpha+\beta}$.

Take a unit vector $S \in \mathfrak{g}_\alpha$. Since $\text{ad}S$ commutes on $\mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta}$ with $J_i J_k$, it also commutes with ϵ . Hence, for all $Y \in \mathfrak{g}_{\alpha+\beta}$,

$$\epsilon [S, Y] = [S, \epsilon Y] = [S, Y].$$

This provides a contradiction, since $[S, Y] \neq 0$ if $Y = [S, Y']$ with $Y' \in \mathfrak{g}_\beta \setminus \{0\}$ by (3.2.2). $\square$

3.3. The structure of $\mathfrak{m}$

In Subsect. 3.2 we established $d_\omega \in \{1, 2\}$. Now exploiting the structure of $\mathfrak{m}$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, we shall determine the allowed multiplicities of the roots. Let $\gamma \in \Delta$, we denote by $\mathfrak{g}_{(\gamma)}$ the rank one subalgebra generated by $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\gamma}$ and set

$$\mathfrak{m}_\gamma = \mathfrak{m} \cap \mathfrak{g}_\gamma.$$

We collect in the following lemma some results more or less known, whose proof can, however, be found in [2].

Lemma 3.3.0. *Suppose $\gamma, \delta \in \Delta$ and $2\gamma \notin \Delta$, and fix an orthonormal basis of $\mathfrak{g}_\gamma$ $\{X_1, \dots, X_{d_\gamma}\}$.*

(1)

$$\mathfrak{m}_\gamma = \text{span} \{[X_i, \theta X_j] : 1 \leq i < j \leq d_\gamma\};$$

(2)

$$[[X_i, \theta X_j], X_i] = (\gamma \mid \gamma) X_j \quad \text{if } i \neq j;$$

(3)

$$[[X_i, \theta X_j], X_k] = 0 \quad \text{if } i \neq j, i \neq k, \text{ and } j \neq k;$$

(4) $\mathfrak{m}_\gamma \simeq \mathfrak{so}(d_\gamma)$;

(5) $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta$ is an ideal in $\mathfrak{m}_\gamma$ and $\mathfrak{m}_\delta$;

(6) if the action of $\mathfrak{m}_\delta$ on $\mathfrak{g}_\gamma$ is trivial, then $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta = \{0\}$. When $d_\gamma \geq 3$, if $\mathfrak{m}_\gamma \cap \mathfrak{m}_\delta = \{0\}$, the action of $\mathfrak{m}_\delta$ on $\mathfrak{g}_\gamma$ is trivial.

Proof. (1) follows from Proposition 2.3 of [2]. (2) is Formula (8) of [2]. (3) is Formula (9) of [2]. (4) is Corollary 2.5 of [2]. (5) is obvious. (6) is Proposition 2.8 of [2]. $\square$

Proposition 3.3.1. *Suppose $d_\omega = 2$.*

(1) Let $\gamma \in \{\beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta\}$, then $\mathfrak{m}_\omega \cap \mathfrak{m}_\gamma = \{0\}$;

(2) $d_\gamma = 2$ for all $\gamma \in \Delta$;

(3) $\mathfrak{m} = \mathfrak{m}_\omega \oplus \mathfrak{m}_\alpha \simeq \mathfrak{so}(2) \oplus \mathfrak{so}(2)$.

Proof. Since $d_\omega \geq 2$, $\mathfrak{m}_\omega$ is non-trivial. By Corollary 3.1.2 $d_\gamma \geq 2$.

(1) Suppose by contradiction that $\mathfrak{m}_\omega \cap \mathfrak{m}_\gamma \neq \{0\}$. Being $\dim \mathfrak{m}_\omega = 1$, it follows by Lemma 3.3.0 (5) that $\mathfrak{m}_\omega$ is a one-dimensional ideal in $\mathfrak{m}_\gamma$. Since, by Lemma 3.3.0 (4), $\mathfrak{m}_\gamma$ is isomorphic to $\mathfrak{so}(d_\gamma)$, which has a one-dimensional ideal only for $d_\gamma = 2$, we obtain $\mathfrak{m}_\gamma = \mathfrak{m}_\omega$. Therefore, if $\{Z_1, Z_2\}$ is an orthonormal basis of $\mathfrak{g}_\omega$ and $X \in \mathfrak{g}_\gamma$ a unit vector,

$$(3.3.1) \quad [Z_1, \theta Z_2] = c [X, \theta J_1 J_2 X], \quad \text{with } c \in \mathbb{R}.$$

Applying both sides of (3.3.1) to Z_1 , by Jacobi we get, using Lemma 3.3.0 (2), (3.3.1), and (3.1.2),

$$2Z_2 = c [[X, \theta J_1 J_2 X], Z_1] = c [[Z_1, \theta J_1 J_2 X], X] = c [J_1^2 J_2 X, X] = c Z_2,$$

whence $c = 2$. On the other hand, (3.3.1) applied to X gives, by Lemma 3.3.0 (2),

$$J_1 J_2 X = [[\theta Z_1, Z_2], X] = 2 [[\theta X, J_1 J_2 X], X] = 2\gamma ([\theta X, X]) J_1 J_2 X,$$

which implies $(\gamma \mid \gamma) = 1/2$, yielding a contradiction.

(2) By Corollary 3.1.2 (2) we consider $\gamma \in \{\alpha + \beta, 2\alpha + \beta\}$. Since $\mathfrak{m}_\omega \cap \mathfrak{m}_\gamma = \{0\}$, if it were $d_\gamma \geq 3$ the action of $\mathfrak{m}_\omega$ on $\mathfrak{g}_\gamma$ would be trivial by Lemma 3.3.0 (6), providing a contradiction being $(\text{ad} [Z_1, \theta Z_2]) X = J_1 J_2 X$ for all pairs of orthogonal vectors Z_1 and Z_2 in $\mathfrak{g}_\omega$ and all $X \in \mathfrak{g}_\gamma$. Hence, $d_\gamma = 2$. $\square$

Lemma 3.3.2. *If $d_\omega = 1$, then $d_\gamma = 1$ for all $\gamma \in \Delta$.*

Proof. By Corollary 3.1.2 (2) $d_\beta = d_{3\alpha+\beta} = 1$. We prove that $d \equiv d_{\alpha+\beta} = d_\alpha = d_{2\alpha+\beta} = 1$. Fix an orthonormal basis $\{X_1, \dots, X_d\}$ of $\mathfrak{g}_{\alpha+\beta}$ and a unit $Z \in \mathfrak{g}_\omega$. Then set $Y_i = J_Z X_i$, it follows that $\{Y_1, \dots, Y_d\}$ is an orthonormal basis of $\mathfrak{g}_{2\alpha+\beta}$,

$$(3.3.2) \quad [X_i, Y_k] = \delta_{ik} Z \quad \text{and} \quad [X_i, \theta X_k] = [Y_i, \theta Y_k],$$

where δ_{ik} is the Kroeneker delta, the first relation is a consequence of (3.1.3), and the second is obtained using the Jacobi identity.

We prove the proposition deducing from $d_\alpha \geq 2$ that for $i \neq j$

$$[X_i, \theta Y_i], \quad [X_j, \theta Y_j], \quad \text{and} \quad [X_i, \theta Y_j]$$

are linear independent vectors in $\mathfrak{g}_\alpha$, contradicting Corollary 3.1.2 (1). The Jacobi identity and (2.1.0) provide

$$\begin{aligned} \langle [X_j, \theta Y_i], [X_l, \theta Y_k] \rangle &= \langle X_j, [Y_i, [X_l, \theta Y_k]] \rangle \\ &= \langle X_j, [X_l, [Y_i, \theta Y_k]] \rangle + \langle X_j, [\theta Y_k, [X_l, Y_i]] \rangle. \end{aligned}$$

This relation for $i = j = k = l$ yields

$$(3.3.3) \quad \|[X_i, \theta Y_i]\|^2 = \frac{1}{3} \|X_i\|^2 + \langle X_i, [\theta Y_i, Z] \rangle = \frac{4}{3},$$

and also, using the first relation in (3.3.2),

$$\|[X_i, \theta Y_j]\|^2 = -(\alpha + \beta) ([Y_j, \theta Y_j]) \|X_i\|^2 = \frac{1}{3}.$$

Moreover, from the relations (3.3.2) and Lemma 3.3.0 (2) we get for $i \neq j$

$$\langle [X_i, \theta Y_i], [X_j, \theta Y_j] \rangle = \langle X_i, [X_j, [Y_i, \theta Y_j]] \rangle + \langle X_i, [\theta Y_j, [X_j, Y_i]] \rangle = \frac{1}{3},$$

which together with (3.3.3) implies, by the converse of the Schwarz inequality, that $[X_i, \theta Y_i]$ and $[X_j, \theta Y_j]$ are linearly independent. Finally, by (3.3.2) we find for $i \neq l \neq k \neq i$

$$\begin{aligned} \langle [X_i, \theta Y_i], [X_l, \theta Y_k] \rangle &= \langle X_i, [X_l, [Y_i, \theta Y_k]] \rangle + \langle X_i, [\theta Y_k, [X_l, Y_i]] \rangle \\ &= \langle X_i, [X_l, [X_i, \theta X_k]] \rangle = 0. \end{aligned}$$

This completes the proof. $\square$

Summarizing the above results we obtain:

Proposition 3.3.3. (1) *There are two possible cases: all the roots have multiplicity 1, and all the roots have multiplicity 2.*

(2) *If $d_\omega = 1$, then $\mathfrak{m} = \{0\}$. If $d_\omega = 2$, then $\mathfrak{m} = \mathfrak{m}_\omega \oplus \mathfrak{m}_\beta \simeq \mathfrak{so}(2) \oplus \mathfrak{so}(2)$.*

Lemma 3.3.4. *Let $Z \in \mathfrak{g}_\omega$, $X \in \mathfrak{g}_\beta$, and $Y \in \mathfrak{g}_{\alpha+\beta}$ be unit vectors, then*

$$\| [\theta Y, J_Z Y] \|^2 = \frac{4}{3} \| [\theta X, Y] \|^2 = 1.$$

Proof. The first relation is Formula (3.3.3) in the proof of Lemma 3.3.2, for the second we find,

$$\begin{aligned} \| [\theta X, Y] \|^2 &= \langle [[\theta X, Y], X], Y \rangle \\ &= (\alpha + \beta) (\langle \theta X, X \rangle \| Y, Y \|^2) = (\alpha + \beta | \beta) = 1. \end{aligned}$$

$\square$

We denote by M_γ the element of $\mathfrak{m}_\gamma$ normalized by the condition

$$(3.3.4) \quad \| M_\gamma \| = (\gamma | \gamma).$$

Lemma 3.3.5. *Suppose $d_\omega = 2$. Let γ and γ' be orthogonal roots. For $\delta \in \Delta$, M_δ is given by*

$$(3.3.5) \quad M_\delta = \frac{(\delta | \gamma)}{(\gamma | \gamma)} M_\gamma + \frac{(\delta | \gamma')}{(\gamma' | \gamma')} M_{\gamma'}.$$

If $X \in \mathfrak{g}_\gamma$, then

$$(3.3.6) \quad [M_\delta, X] = \frac{(\delta | \gamma)}{(\gamma | \gamma)} [M_\gamma, X].$$

Proof. The statement makes sense since for all $\gamma \in \Delta$ there are two roots orthogonal to γ . Moreover, $\gamma + \gamma' \notin \Delta$. There are two orthogonal vectors X_1 and X_2 in $\mathfrak{g}_\gamma$ such that $M_\gamma = [X_1, \theta X_2]$. Using the Jacobi identity and (2.1.0) we find

$$(3.3.7) \quad \langle M_\gamma, M_{\gamma'} \rangle = \langle [X_1, \theta X_2], M_{\gamma'} \rangle = \langle X_1, [X_2, M_{\gamma'}] \rangle = 0.$$

since, being $\gamma + \gamma' \notin \Delta$, $[X_2, M_{\gamma'}] = 0$. Therefore, by Lemma 3.3.0 (1) we obtain

$$M_\delta = a M_\gamma + b M_{\gamma'}, \quad \text{with } a, b \in \mathbb{R}.$$

Now, (3.3.5) follows from

$$\|M_\delta\|^2 = (\delta | \delta) = \frac{(\delta | \gamma)^2}{(\gamma | \gamma)} + \frac{(\delta | \gamma')^2}{(\gamma' | \gamma')},$$

(3.3.4), and (3.3.7). Since $[M_{\gamma'}, X] = 0$, (3.3.6) is now an immediate consequence of (3.3.5). $\square$

Lemma 3.3.6. *Let $Z \in \mathfrak{g}_\omega$ be a unit vector, and let $X \in \mathfrak{g}_\beta$ and $Y \in \mathfrak{g}_{\alpha+\beta}$,*

$$(3.3.8) \quad [\theta X, Y] = [J_Z X, \theta J_Z Y].$$

When $d = 2$, if $Z_1, Z_2 \in \mathfrak{g}_\omega$ is an orthonormal basis of $\mathfrak{g}_\omega$, then

$$(3.3.9) \quad [J_1 J_2 X, \theta Y] = [\theta J_1 J_2 Y, X].$$

Proof. For (3.3.8), we have

$$[J_Z X, \theta J_Z Y] = [J_Z X, [\theta Z, Y]] = [Y, [\theta Z, J_Z X]] = [\theta X, Y].$$

For (3.3.9), since $\omega \pm \alpha \notin \Delta$,

$$0 = [[Z_1, \theta Z_2], [X, \theta Y]] = [J_1 J_2 X, \theta Y] + [X, \theta J_1 J_2 Y].$$

$\square$

4. The construction of $\mathfrak{g}$

4.1. Construction of $\mathfrak{n}_{0+}$ and $\mathfrak{n}_{0-}$

Let $\mathfrak{z} = \mathbb{R}$ with $\langle Z, Z' \rangle_{\mathfrak{z}} = ZZ'$. Take two irreducible modules $(\mathfrak{v}_1, J^{(1)})$ and $(\mathfrak{v}_2, J^{(2)})$ of $\mathcal{C}(1) \simeq \mathbb{C}$, where $\mathfrak{v}_1 \simeq \mathfrak{v}_2 \simeq \mathbb{R}^2$, and $J^{(i)} : \mathcal{C}(1) \rightarrow \text{End}(\mathfrak{v}_i)$ for $i = 1, 2$. Endow $\mathfrak{v}_1$ and $\mathfrak{v}_2$ with the euclidean inner product of $\mathbb{R}^2$ $\langle \cdot, \cdot \rangle_i$, $i = 1, 2$. The operators $J_a^{(1)}$ and $J_a^{(2)}$ are skew-symmetric with respect to these inner products for all $a \in \mathcal{C}(1)$. Define

$$\mathfrak{n}_0 = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{z} \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_2 \oplus \langle \cdot, \cdot \rangle_{\mathfrak{z}},$$

and equip $(\mathfrak{n}_0, \langle \cdot, \cdot \rangle)$ with the corresponding structure of generalized Heisenberg algebra. Notice that this is just the Heisenberg algebra $\mathbb{H}_2$. Fix a unit vector $Z_0 \in \mathfrak{z}$

and set $J = J_{Z_0}$. Fix moreover a unit vector $X_0 \in \mathfrak{v}_1$ and a unit vector $Y_0 \in \mathfrak{v}_2$, obtaining

$$\mathfrak{v}_1 = \text{span}\{X_0, JX_0\} \mathfrak{v}_2 = \text{span}\{Y_0, JY_0\}.$$

Put

$$(4.1.1) \quad \mathfrak{v}_{11} = \mathbb{R} X_0, \quad \mathfrak{v}_{12} = \mathbb{R} JX_0, \quad \mathfrak{v}_{21} = \mathbb{R} Y_0, \quad \mathfrak{v}_{22} = \mathbb{R} JY_0.$$

To complete the picture one needs another copy of $\mathfrak{n}_0$. Therefore, consider the cartesian product $\mathfrak{n}_0 \times \{+, -\}$ and set

$$\mathfrak{n}_{0+} = \mathfrak{n}_0 \times \{+\}, \quad \mathfrak{n}_{0-} = \mathfrak{n}_0 \times \{-\}, \quad \mathfrak{n}_0 \times \{+, -\} = \mathfrak{n}_{0+} \oplus \mathfrak{n}_{0-}.$$

Correspondingly, denote by V^+ the pair $(V, +)$ and by V^- the pair $(V, -)$. Then introduce the isomorphism of Heisenberg algebras θ defined by

$$\theta V^+ = V^- \theta V^- = V^+, \quad \forall V \in \mathfrak{n}_0.$$

Clearly,

$$(4.1.2) \quad \theta^2 = I \theta J_{Z^+} V^+ = (J_Z V)^- = J_{Z^-} V^-, \quad \forall Z \in \mathfrak{z} \forall V \in \mathfrak{v}.$$

Let $\mathfrak{n}_{0+} = \mathfrak{z}_+ \oplus \mathfrak{v}_+$, and $\mathfrak{n}_{0-} = \mathfrak{z}_- \oplus \mathfrak{v}_-$, where $\mathfrak{z}_+$ and $\mathfrak{z}_-$ are the centres of $\mathfrak{n}_{0+}$ and $\mathfrak{n}_{0-}$, and consider the decompositions

$$\mathfrak{v}_\sigma = \mathfrak{v}_{11\sigma} \oplus \mathfrak{v}_{21\sigma} \oplus \mathfrak{v}_{22\sigma} \oplus \mathfrak{v}_{12\sigma}, \quad \text{where } \sigma \in \{+, -\},$$

corresponding to (4.1.1).

4.2. Construction of $\mathfrak{g}_0$

Let $\mathcal{D} = \mathcal{D}(\mathfrak{n}_0)$ denote the space of derivations of $\mathfrak{n}_0$. The commutator $[\cdot, \cdot]$ provides $\mathcal{D}$ with a structure of Lie algebra. The operator Θ on $\mathcal{D}$, defined by

$$\Theta D = -{}^t D,$$

where t is the transpose of D , is an involution. We denote by $\mathcal{D}_T = \mathcal{D}_T(\mathfrak{n}_0)$ the subspace of $\mathcal{D}$ consisting of derivations acting trivially on the centre.

Proposition 4.2.1. *If the linear map $D : \mathfrak{n}_0 \rightarrow \mathfrak{n}_0$ is a derivation, then*

$$(4.2.1) \quad J_{\Theta D Z} = -J_Z D + (\Theta D) J_Z, \quad \forall Z \in \mathfrak{z}.$$

In particular, if $D \in \mathcal{D}_T$,

$$(4.2.2) \quad \Theta D \in \mathcal{D}_T J_Z (\Theta D) = D J_Z, \quad \forall Z \in \mathfrak{z},$$

and $\mathcal{D}_T$ is a subalgebra of $\mathcal{D}$.

Proof. If $D \in \mathcal{D}$, for all $Z \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$

$$\begin{aligned} \langle J_{tDZ}X, Y \rangle &= \langle {}^tDZ, [X, Y] \rangle = \langle Z, D[X, Y] \rangle \\ &= \langle Z, [DX, Y] + [X, DY] \rangle \\ &= \langle J_ZDX, Y \rangle + \langle {}^tDJ_ZX, Y \rangle, \end{aligned}$$

which implies

$$J_{tDZ} = J_ZD + {}^tDJ_Z,$$

from which the first part of the statement follows. This clearly implies (4.2.2) for any $D \in \mathcal{D}_T$, from which it follows that $\mathcal{D}_T$ is an algebra. $\square$

To any derivation D of $\mathfrak{n}_{0+}$ the rule

$$(4.2.3) \quad U_- \mapsto -\theta({}^tDU_+),$$

associates a derivation of $\mathfrak{n}_{0-}$ which shall still be denoted by D . From (4.2.3) it follows that

$$\theta(DU_-) = (\Theta D)\theta U_-.$$

Accordingly we will first define a derivation on $\mathfrak{n}_{0+}$, and then extend it to $\mathfrak{n}_{0+} \oplus \mathfrak{n}_{0-}$. As a first example, consider the operator H_V defined by

$$H_V|_{\mathfrak{z}+} = 2I, \quad H_V|_{\mathfrak{v}+} = I, \quad H_V|_{\mathfrak{v}-} = -I, \quad H_V|_{\mathfrak{z}-} = -2I,$$

where I is the identity and $H|_{\mathfrak{u}}$ is the restriction of H to the subspace $\mathfrak{u}$. Consider a linear map E on $\mathfrak{n}_0$ defined by

$$EZ = 0, \quad EX_0 = aY_0, \quad EY_0 = bJY_0, \quad EJY_0 = cJX_0, \quad EJX_0 = 0,$$

$a, b, c \in \mathbb{R}$. Notice that E is a derivation if and only if $c = -a$. From the second formula in (4.2.2) we get

$$\Theta EZ = 0, \quad \Theta EX_0 = 0, \quad \Theta EY_0 = -aX_0, \quad \Theta EJY_0 = -bY_0, \quad EJX_0 = aJY_0,$$

from which, setting

$$H_T \equiv [\Theta E, E]$$

(which gives $H_T|_{\mathfrak{v}} = EJEJ - JEJE$), we deduce

$$[H_T, E]X_0 = (2a^2 - b^2)EX_0[H_T, E]Y_0 = 2(b^2 - a^2)EY_0.$$

Therefore, $[H_T, E] = \lambda E$ if and only if $4a^2 = 3b^2$. Take $b = 1$, then

$$\mathfrak{g}_T \equiv \text{span}\{E, \Theta E, H_T\}$$

is a Lie algebra isomorphic to $\mathfrak{sl}(1, 1)$. Clearly, Θ plays the rôle of the Cartan involution of $\mathfrak{g}_T$. We set $a = 1$ and $b = \frac{2}{\sqrt{3}}$, obtaining

$$(4.2.4) \quad EX_0 = Y_0, \quad EY_0 = \frac{2}{\sqrt{3}}JY_0, \quad EY_0 = -JX_0, \quad EY_0 = 0,$$

which yield

$$(4.2.5) \quad \begin{aligned} 2(\Theta E)X_0 &= 0, & (\Theta E)Y_0 &= -X_0, \\ (\Theta E)JY_0 &= -\frac{2}{\sqrt{3}}Y_0, & (\Theta E)JX_0 &= JY_0, \end{aligned}$$

and

$$(4.2.6) \quad [H_T, E] = \frac{2}{3}E, \quad [H_T, \Theta E] = -\frac{2}{3}\Theta E.$$

Furthermore,

$$H_T|_{\mathfrak{g}_{3+}} = 0,$$

and

$$H_T|_{\mathfrak{v}_{11+}} = -I, \quad H_T|_{\mathfrak{v}_{12+}} = I, \quad H_T|_{\mathfrak{v}_{21+}} = -\frac{1}{3}I, \quad H_T|_{\mathfrak{v}_{22+}} = \frac{1}{3}I.$$

On $\mathfrak{g}_T$ we introduce an inner product by setting

$$\langle D_1, D_2 \rangle \equiv \frac{3}{10} \operatorname{tr} (D_1 \Theta D_2).$$

It is easy to see that E , ΘE , and H_T are mutually orthogonal, and that

$$\langle E, E \rangle = \langle \Theta E, \Theta E \rangle = 1 \quad \langle H_T, H_T \rangle = \frac{2}{3}.$$

It is clear that H_V and H_T are symmetric derivations and that

$$[H_V, E] = [H_V, \Theta E] = [H_V, H_T] = 0.$$

Denote by $\mathfrak{a}$ the real span of H_V and H_T , and set

$$\mathfrak{g}_0 \equiv \mathfrak{a} + \mathfrak{g}_T = \mathbb{R}H_V \oplus \mathfrak{g}_T.$$

Observe that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathcal{D}$ and endow $\mathfrak{g}_0 \oplus \mathfrak{n}_{0+}$ and $\mathfrak{g}_0 \oplus \mathfrak{n}_{0-}$ with the structure of semidirect product of Lie algebras. Extend $\langle \cdot, \cdot \rangle$ to $\mathfrak{g}_0$ by setting

$$\langle H_V, H_V \rangle = 2 \quad \langle H_V, D \rangle = 0, \quad \forall D \in \mathfrak{g}_T.$$

4.3. The algebra $\mathfrak{g}$

Let

$$\mathfrak{g} = \mathfrak{n}_{0-} \oplus (\mathfrak{a} + \mathfrak{g}_T) \oplus \mathfrak{n}_{0+},$$

and $\langle \cdot, \cdot \rangle$ be the orthogonal sum of the given inner products on the single components. Extend θ to $\mathfrak{g}$ by putting

$$\theta|_{\mathfrak{g}_0} = \Theta.$$

It is now time to introduce the root spaces:

$$\begin{array}{llll} \mathfrak{g}_\alpha = \mathbb{R}E, & \mathfrak{g}_{-\alpha} = \mathbb{R}\theta E, & \mathfrak{g}_{2\alpha+3\beta} = \mathfrak{z}_+, & \mathfrak{g}_{-2\alpha-3\beta} = \mathfrak{z}_-, \\ \mathfrak{g}_\beta = \mathfrak{v}_{11+}, & \mathfrak{g}_{\alpha+\beta} = \mathfrak{v}_{21+}, & \mathfrak{g}_{\alpha+2\beta} = \mathfrak{v}_{22+}, & \mathfrak{g}_{\alpha+3\beta} = \mathfrak{v}_{11+}, \\ \mathfrak{g}_{-\beta} = \mathfrak{v}_{11-}, & \mathfrak{g}_{-\alpha-\beta} = \mathfrak{v}_{21-}, & \mathfrak{g}_{-\alpha-2\beta} = \mathfrak{v}_{22-}, & \mathfrak{g}_{-\alpha-3\beta} = \mathfrak{v}_{11-}. \end{array}$$

The vectors $H_V, H_T, Z, X_0, Y_0, JX_0, JY_0, \theta Z, \theta X_0, \theta Y_0, \theta JX_0, \theta JY_0, E$, and θE form a basis of $\mathfrak{g}$. The task at hand now is to write down the multiplication table in terms of this basis. Of course, some brackets have been already defined: indeed, we know the brackets in the semidirect product Lie algebras $\mathfrak{g}_0 \oplus \mathfrak{n}_{0+}$ and $\mathfrak{g}_0 \oplus \mathfrak{n}_{0-}$. If $Y \in \mathfrak{g}_\gamma$ and $Y' \in \mathfrak{g}_\delta$, and $\gamma + \delta \notin \Delta$, we set $[Y, Y'] = 0$. We require that the bracket is skewsymmetric and satisfies

$$(4.3.1) \quad \theta[U, V] = [\theta U, \theta V], \quad \forall U, V.$$

Then, if $\gamma \in \Delta_+$ and $Y \in \mathfrak{g}_\gamma$, we set

$$(4.3.2) \quad [Y^-, Y] = [\theta Y, Y] = \frac{1}{2} (\gamma \mid 3\alpha + 2\beta) H_V + \frac{3}{2} (\gamma \mid \alpha) H_T.$$

For $Z \in \mathfrak{z}$ and $V \in \mathfrak{v}$, we define

$$(4.3.3) \quad [Z^+, V^-] = J_{Z^+} V^+,$$

which by (4.3.1) and the second equation in (4.1.2) gives

$$[Z^-, V^+] = J_{Z^-} V^-.$$

Finally, we put

$$(4.3.4) \quad E = [\theta X_0, Y_0] = [JX_0, \theta JY_0] = \frac{\sqrt{3}}{2} [\theta Y_0, JY_0],$$

which as before provides

$$(4.3.5) \quad \theta E = [X_0, \theta Y_0] = [\theta JX_0, JY_0] = \frac{\sqrt{3}}{2} [Y_0, \theta JY_0].$$

5. Some subalgebras of $\mathfrak{g}$

Lemma 5.1.1. *The linear map σ defined by*

$$\sigma|_{\mathfrak{z}} = \sigma|_{\theta\mathfrak{z}} = I, \quad \sigma|_{\mathfrak{v}} = J, \quad \sigma|_{\theta\mathfrak{v}} = \theta \circ J \circ \theta, \quad \sigma|_{\mathfrak{g}_T} = \theta, \sigma|_{\mathbb{R}H_V} = -\theta I,$$

satisfies

$$(5.1.1) \quad \sigma[U, V] = [\sigma U, \sigma V], \quad \forall V, U \in \mathfrak{g},$$

$$(5.1.2) \quad \sigma \circ \theta = \theta \circ \sigma.$$

Proof. It is clear from the definition that σ commutes with θ . The restriction of σ to the Heisenberg algebra $\mathfrak{n}_0$ is an automorphism of Lie algebras since

$$[\sigma U, \sigma JU] = [JU, J^2 U] = -[JU, U] = Z = \sigma Z, \quad \forall Z \in \mathfrak{z}.$$

The same holds for the restriction of σ to $\theta\mathfrak{n}_0$ by (4.3.1) since θ is an invertible linear map from $\mathfrak{n}_0$ onto $\theta\mathfrak{n}_0$. Moreover, it is clear that the restriction of θ , and hence of σ , to $\mathfrak{g}_T$ is an automorphism. Evidently σ commutes with the action of H_V on $\mathfrak{g}$. This, being $\sigma H_V = H_V$, implies that

$$\sigma[H_V, U] = [\sigma H_V, \sigma U], \quad \forall U \in \mathfrak{g}.$$

In particular, σ preserves the decomposition of $\mathfrak{g}$ in eigenspaces of H_V . Hence, (5.1.1) is trivially satisfied for all $U \in \mathfrak{g}_T$ and $V \in \mathfrak{z} \oplus \theta\mathfrak{z}$. What remains is to prove that (5.1.1) follows by direct computations. From (5.1.2) and (4.3.5) we obtain

$$[\sigma\theta X_0, \sigma Y_0] = [\theta JX_0, JY_0] = \theta E = \sigma E,$$

which by (4.3.4) yields

$$[\sigma\theta X_0, \sigma Y_0] = \sigma E = \sigma[\theta X_0, Y_0].$$

Similarly, we get

$$\begin{aligned} [\sigma JX_0, \sigma\theta JY_0] &= [J^2 X_0, \theta J^2 Y_0] = [X_0, \theta Y_0] \\ &= \theta E = \sigma E = \sigma[\theta X_0, Y_0], \end{aligned}$$

and

$$\begin{aligned} \frac{\sqrt{3}}{2} [\sigma\theta Y_0, \sigma JY_0] &= \frac{\sqrt{3}}{2} [\theta JY_0, J^2 Y_0] = \frac{\sqrt{3}}{2} [Y_0, \theta JY_0] \\ &= \theta E = \sigma E = \frac{\sqrt{3}}{2} \sigma[\theta Y_0, JY_0]. \end{aligned}$$

From these identities it follows, by (5.1.2), that

$$\sigma\theta E = [\sigma X_0, \sigma\theta Y_0] = [\sigma\theta JX_0, \sigma JY_0] = \frac{\sqrt{3}}{2} [\sigma Y_0, \sigma\theta JY_0].$$

To conclude the proof it is now enough to prove that $[\sigma U, \sigma V] = \sigma [U, V]$ for $U \in \mathfrak{g}_T$ and $V \in \mathfrak{v}$, since this formula will provide (5.1.1) for $U \in \mathfrak{g}_T$ and $V \in \theta \mathfrak{v}$ by (5.1.2). From (4.2.5) we deduce

$$[\sigma E, \sigma X_0] = [\theta E, JX_0] = JY_0,$$

which being, by (4.2.4),

$$\sigma [E, X_0] = \sigma Y_0 = JY_0,$$

yields

$$\sigma [E, X_0] = [\sigma E, \sigma X_0].$$

We find, by (4.2.5),

$$[\sigma E, \sigma Y_0] = [\theta E, JY_0] = -\frac{2}{\sqrt{3}}Y_0,$$

which gives the desired identity, being, by (4.2.4),

$$\sigma [E, Y_0] = \frac{2}{\sqrt{3}}\sigma JY_0 = -\frac{2}{\sqrt{3}}Y_0.$$

Similalrly, we get

$$\begin{aligned} [\sigma E, \sigma JY_0] &= -[\theta E, Y_0] = X_0 = -J^2 X_0 \\ &= -\sigma JX_0 = \sigma [E, JY_0], \end{aligned}$$

and

$$[\sigma E, \sigma JX_0] = -[\theta E, X_0] = 0 = \sigma [E, JX_0].$$

□

Proposition 5.1.2. *Let $\gamma \in \Delta$, and $U \in \mathfrak{g}_\gamma$ a unit vector. Put $H_\gamma = [\theta U, U]$, then*

$$\mathfrak{g}_{(\gamma)} = \mathfrak{g}_\gamma \oplus (\mathbb{R}H_\gamma) \oplus \mathfrak{g}_{-\gamma}$$

is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(1, 1)$.

Proof. It follows from (4.3.2) that

$$[H_\gamma, U] = \frac{1}{2}[(\omega | \gamma) H_V + (\alpha | \gamma) H_T, U] = (\gamma | \gamma) U, \quad \forall U \in \mathfrak{g}_\gamma.$$

and hence, the statement follows. □

Let

$$\Delta_{l+} = \{\beta, 3\alpha + 2\beta, 3\alpha + \beta\} \Delta_l = \Delta_{l+} \cup (-\Delta_{l+}).$$

Comparing the multiplication table introduced in this section with that defined in Sect. 5 of [2], we obtain the following proposition:

Proposition 5.1.3. *The subspace*

$$\mathfrak{h} = \sum_{\gamma \in \Delta_I} \oplus \mathfrak{g}_\gamma$$

is a subalgebra of $\mathfrak{g}$ isomorphic to the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$.

Let

$$\begin{aligned} \mathfrak{n}_\beta &= \mathfrak{g}_{-3\alpha-\beta} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta} \oplus \mathfrak{g}_\beta, \\ \mathfrak{n}_{3\alpha+\beta} &= \mathfrak{g}_{3\alpha+2\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{3\alpha+\beta}. \end{aligned}$$

Proposition 5.1.4. *The subspaces $\mathfrak{n}_\beta \oplus (\mathfrak{g}_{(2\alpha+\beta)} + \mathfrak{a})$, $\mathfrak{n}_{3\alpha+\beta} \oplus (\mathfrak{g}_{(\alpha+\beta)} + \mathfrak{a})$, $\theta \mathfrak{n}_\beta \oplus (\mathfrak{g}_{(2\alpha+\beta)} + \mathfrak{a})$, $\theta \mathfrak{n}_{3\alpha+\beta} \oplus (\mathfrak{g}_{(\alpha+\beta)} + \mathfrak{a})$, and $\theta \mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$ are Lie algebras isomorphic to $\mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$.*

Proof. $\mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$ and $\theta \mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$ are semidirect products of Lie algebras, and θ is, by (4.3.1), an isomorphism of Lie algebras between them. We show that $\mathfrak{n}_\beta \oplus (\mathfrak{g}_{(2\alpha+\beta)} + \mathfrak{a})$ is isomorphic to $\mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$. From this it follows by (4.3.1), that $\theta \mathfrak{n}_\beta \oplus (\mathfrak{g}_{(2\alpha+\beta)} + \mathfrak{a})$ is isomorphic to $\mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$. Similarly, $\mathfrak{n}_{3\alpha+\beta} \oplus (\mathfrak{g}_{(\alpha+\beta)} + \mathfrak{a})$ is, by (5.1.1), isomorphic to $\mathfrak{n}_\beta \oplus (\mathfrak{g}_{(2\alpha+\beta)} + \mathfrak{a})$, and hence to $\mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$. This finally implies that $\theta \mathfrak{n}_{3\alpha+\beta} \oplus (\mathfrak{g}_{(\alpha+\beta)} + \mathfrak{a})$ is isomorphic to $\mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$.

We prove that $\mathfrak{n}_\beta \oplus (\mathfrak{g}_{(2\alpha+\beta)} + \mathfrak{a})$ is isomorphic to $\mathfrak{n}_0 \oplus (\mathfrak{g}_T + \mathfrak{a})$, showing that these two algebras have the same constants of structure. To make this evident we relabel the vectors in the basis of $\mathfrak{g}$, setting

$$(5.1.3) \quad Z_1 \equiv X_0, \quad X_1 \equiv \theta JX_0, \quad Y_1 \equiv \theta E.$$

Let

$$\mathfrak{v}_{(\beta)} \equiv \mathfrak{g}_{-3\alpha-\beta} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta},$$

and define

$$J^{(\beta)} U \equiv [Z_1, \theta U], \quad \text{for } U \in \mathfrak{v}_{(\beta)}.$$

We have, by (3.1.5) and (4.2.5),

$$(5.1.4) \quad J^{(\beta)} X_1 = [Z_1, \theta X_1] = [X_0, JX_0] = Z,$$

$$(5.1.5) \quad J^{(\beta)} Y_1 = [Z_1, \theta Y_1] = [X_0, E] = -Y_0,$$

from which we get

$$[X_1, J^{(\beta)} X_1] = [\theta JX_0, Z] = X_0 = Z_1$$

by (3.1.1) and (3.1.2),

$$[Y_1, J^{(\beta)} Y_1] = -[\theta E, Y_0] = X_0 = Z_1$$

by (4.2.4),

$$[X_1, J^{(\beta)} Y_1] = -[\theta JX_0, Y_0] = 0, [Y_1, J^{(\beta)} X_1] = [\theta E, Z] = 0.$$

Hence, $\mathfrak{n}_\beta$ is isomorphic to $\mathfrak{n}_0$.

The algebra $\mathfrak{g}_{(\alpha+\beta)}$ is, by Proposition 5.1.2, isomorphic to $\mathfrak{g}_T$. Thus, $\mathfrak{g}_{(\alpha+\beta)} + \mathfrak{a}$ is isomorphic to $\mathfrak{g}_T + \mathfrak{a}$. Set $E_1 \equiv [\theta X_1, Y_1]$, plugging the last two relations in (5.1.3) into this formula and using the fourth in (4.2.4), we get

$$(5.1.6) \quad E_1 = [\theta X_1, Y_1] = [JX_0, \theta E] = -JY_0.$$

We show that

$$(5.1.7) \quad \begin{aligned} [E_1, X_1] &= Y_1, & [E_1, Y_1] &= \frac{2}{\sqrt{3}} J^{(\beta)} Y_1, \\ [E_1, J^{(\beta)} Y_1] &= -J^{(\beta)} X_1, & [E_1, J^{(\beta)} X_1] &= 0. \end{aligned}$$

From (5.1.6) and the first equation in (4.3.5), it follows that

$$[E_1, X_1] = -[JY_0, \theta JX_0] = -\theta E = Y_1.$$

Similarlry, from (5.1.6), we deduce

$$[E_1, Y_1] = -[JY_0, \theta E] = \frac{2}{\sqrt{3}} Y_0 = \frac{2}{\sqrt{3}} J^{(\beta)} Y_1$$

by the third equation in (4.2.5),

$$[E_1, J^{(\beta)} Y_1] = -[JY_0, Y_0] = -Z = -J^{(\beta)} X_1$$

by (5.1.5) and (3.1.5), and finally,

$$[E_1, J^{(\beta)} X_1] = -[JY_0, Z] = 0$$

by (5.1.4), and (4.3.5). The relations analogous to the (5.1.7)s with θE_1 in place of E_1 may be proved similarlry. Now the statement follows. $\square$

6. The jacobi identity for $\mathfrak{g}$

Theorem 6.1. *$\mathfrak{g}$ is a simple Lie algebra.*

Proof. We will prove that $\mathfrak{g}$ is a Lie algebra, showing that the Jacobi identity holds. In Sect. 5 we showed that many subspaces of $\mathfrak{g}$ are indeed Lie algebras, thus, few Jacobi identities remain to be proved. Using the linear maps θ and σ which satisfy (4.3.1) and (5.4.1), the number of identities which must be proved is further reduced to the relations involving triples of vectors lying in (1) $\mathfrak{g}_\alpha$, $\mathfrak{g}_{\alpha+\beta}$, and $\mathfrak{g}_{-2\alpha-\beta}$, (2) $\mathfrak{g}_\omega$, $\mathfrak{g}_{-\alpha-\beta}$, and $\mathfrak{g}_{-2\alpha-\beta}$, and (3) $\mathfrak{g}_\omega$, $\mathfrak{g}_{-\alpha-\beta}$, and $\mathfrak{g}_\alpha$.

For (1), since $[\theta JY_0, Y_0] = -\left(\sqrt{3}/2\right)\theta E$ by the second relation in (4.3.5), (4.3.2) yields

$$[[\theta JY_0, Y_0], E] = -\frac{2}{\sqrt{3}}[\theta E, E] = -\frac{2}{\sqrt{3}}H_T.$$

By the second equation in (4.2.4) and (4.3.2), we get

$$[[Y_0, E], \theta JY_0] = -\frac{2}{\sqrt{3}}[JY_0, \theta JY_0] = \frac{1}{\sqrt{3}}H_V + \frac{1}{\sqrt{3}}H_T,$$

and similarly, by the third equation in (4.2.5) and (4.3.2),

$$[[E, \theta JY_0], Y_0] = -\frac{2}{\sqrt{3}}[\theta Y_0, Y_0] = -\frac{1}{\sqrt{3}}H_V + \frac{1}{\sqrt{3}}H_T.$$

The sum of these three formulas is zero, and the result follows.

For (2), (3.1.5) and (4.3.2) give

$$[[\theta JY_0, \theta Y_0], Z] = -[\theta Z, Z] = -H_V.$$

Since, by (4.3.3),

$$[Z, \theta Y_0] = JY_0[Z, \theta JY_0] = J^2Y_0 = -Y_0,$$

we find, by (4.3.2),

$$[[\theta Y_0, Z], \theta JY_0] = \frac{1}{2}H_V + \frac{1}{2}H_T[[Z, \theta JY_0], \theta Y_0] = -\frac{1}{2}H_T + \frac{1}{2}H_V.$$

Finally for (3), the second equation in (5.2.5) and (4.3.3) provide

$$[[\theta Y_0, E], Z] = [\theta X_0, Z] = -JX_0,$$

(4.3.3) and the third in (4.2.5) give

$$[[Z, \theta Y_0], E] = [JY_0, E] = JX_0,$$

from which, being $[Z, E] = 0$, we obtain the Jacobi identity.

Now we prove that $\mathfrak{g}$ is simple. Suppose $\mathfrak{h}$ is a nontrivial ideal in $\mathfrak{g}$ and let $U \in \mathfrak{h}$ be nonzero. Since $\text{ad}Z$ is nilpotent there is a nonnegative integer k such that

$$(\text{ad}Z)^{k+1}U = 0V = (\text{ad}Z)^kU \neq 0.$$

Being $[Z, V] = 0$, it follows that $V \in (\mathfrak{n}_0 \oplus \mathfrak{g}_T) \cap \mathfrak{h}$. Hence

$$(6.1.1) \quad V = a_1X_0 + a_2Y_0 + a_3JY_0 + a_4JX_0 + a_5Z + a_6E + a_7\theta E + a_8H_T,$$

where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ are real numbers. It is easy to see that

$$\text{span} \{[V, H_\gamma] : \gamma = \alpha, -\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\} \subset \mathfrak{h}$$

has dimension given by the number of nonzero coefficients among $a_1, a_2, a_3, a_4, a_5, a_6, a_7$. It follows, therefore, that there is a nonvanishing $W \in \mathfrak{h}$ lying in some

root space $\mathfrak{g}_\gamma$. Thus, $\mathfrak{g}_\gamma \subset \mathfrak{h}$, and by Corollary 4.2 of [2] all the root spaces are contained in $\mathfrak{h}$, which implies $\mathfrak{h} = \mathfrak{g}$.

Now we prove that there is only one simple Lie algebra with root system G_2 and dimension 14, i.e. $d_\gamma = 1$ for all $\gamma \in \Delta$. Let $\mathfrak{l}$ be such a Lie algebra, and let $Z \in \mathfrak{l}_\omega$, $X \in \mathfrak{l}_\beta$, and $Y \in \mathfrak{l}_{\alpha+\beta}$ be unit vectors. Using the Jacobi identity and the geometry of the root system, we easily deduce from $E = [\theta X, Y]$ the following relations:

$$(6.1.2) \quad \begin{aligned} [E, X] &= Y, & [\theta E, JX] &= JY, \\ [\theta E, Y] &= -X, & [E, JY] &= -JX, \end{aligned}$$

which, since θ is an automorphism, provide

$$(6.1.3) \quad \begin{aligned} [\theta E, \theta X] &= \theta Y, & [E, \theta JX] &= \theta JY, \\ [E, \theta Y] &= -\theta X, & [\theta E, \theta JY] &= -\theta JX. \end{aligned}$$

We also have

$$\begin{aligned} [[\theta E, E], X] &= (\alpha \mid \beta) X = -X, \\ [[\theta E, E], Y] &= (\alpha \mid \alpha + \beta) Y = -\frac{1}{3}Y, \\ [[\theta E, E], JY] &= (\alpha \mid 2\alpha + \beta) JY = \frac{1}{3}JY, \\ [[\theta E, E], JX] &= (\alpha \mid 3\alpha + \beta) JX = JX. \end{aligned}$$

Since $d_\gamma = 1$ for all $\gamma \in \Delta$, we put

$$[E, Y] = [[\theta X, Y], Y] = a JY, \quad \text{with } a \in \mathbb{R},$$

from which, using the Jacobi identity, the third equation in (6.1.3), and the definition of E , we deduce

$$\begin{aligned} -\frac{1}{3}E &= (\alpha + \beta \mid \alpha) E = [[\theta Y, Y], E] \\ &= [[E, Y], \theta Y] + [[\theta Y, E], Y] \\ &= a [JY, \theta Y] + [\theta X, Y] \\ &= a [JY, \theta Y] + E, \end{aligned}$$

that is,

$$(6.1.4) \quad a [\theta Y, JY] = \frac{4}{3}E.$$

From this relation, by Lemma 3.3.4, it follows that

$$\left(\frac{4}{3}\right)^2 = \left(\frac{4}{3}\right)^2 \|E\|^2 = a^2 \|\theta Y, JY\|^2 = \frac{4}{3}a^2.$$

Hence, $a = \pm \frac{2}{\sqrt{3}}$. The arbitrariness in the sign of a is a consequence of the arbitrariness of the versus of Z , which is defined as a vector of unit norm in $\mathfrak{l}_\omega$. To fix the ideas we assume $a = \frac{2}{\sqrt{3}}$, which means, by (6.1.4).

$$(6.1.5) \quad E = \frac{\sqrt{3}}{2} [\theta Y, JY],$$

and implies

$$(6.1.6) \quad [E, Y] = [[\theta X, Y], Y] = \frac{2}{\sqrt{3}} JY.$$

All the other brackets can be now easily deduced from these, ones employing the Jacobi identity and the results of Sect. 3. For instance, from the definition of J we obtain, using the Jacobi identity,

$$[\theta E, JY] = [\theta E, [Z, \theta Y]] = [\theta Y, [Z, \theta E]] + [Z, [\theta E, \theta Y]],$$

which, being $(\omega \mid \alpha) = 0$, by (6.1.6) and (3.1.5), implies

$$[\theta E, JY] = [Z, \theta [E, Y]] = \frac{2}{\sqrt{3}} [Z, \theta JY] = -\frac{2}{\sqrt{3}} Y.$$

Hence, $\mathfrak{l}$ and $\mathfrak{g}$, having the same constants of structure, are isomorphic. $\square$

7. The complexification of $\mathfrak{g}$

Lemma 7.1.1. *Let*

$$\mathfrak{g}_{(\gamma)} = \mathfrak{g}_{-\gamma} \oplus (\mathfrak{m} \oplus \mathfrak{a}) \oplus \mathfrak{g}_\gamma$$

be a real rank one simple Lie algebra with $d_\gamma = 2$, and let $H_\gamma \in \mathfrak{a}$ and $M_\gamma \in \mathfrak{m}$ satisfy

$$\|H_\gamma\|^2 = \|M_\gamma\|^2 = \gamma(H_\gamma) = \langle \gamma, \gamma \rangle.$$

Define a linear map $\mathcal{J} : \mathfrak{g}_{(\gamma)} \rightarrow \mathfrak{g}_{(\gamma)}$ by

$$\mathcal{J} : Z \mapsto \frac{1}{\langle \gamma, \gamma \rangle} (\text{ad} M_\gamma) Z, \quad \mathcal{J} : \theta Z \mapsto -\frac{1}{\langle \gamma, \gamma \rangle} \theta (\text{ad} M_\gamma) Z, \quad \text{for } Z \text{ in } \mathfrak{g}_\gamma,$$

and by

$$\mathcal{J} : H_\gamma \mapsto M_\gamma, \quad \mathcal{J} : M_\gamma \mapsto -H_\gamma.$$

Then $\mathcal{J}$ is a complex structure on $\mathfrak{g}$, i.e.

$$(7.1.1) \quad \mathcal{J}^2 = -I \mathcal{J} [U, V] = [\mathcal{J} U, V] = [U, \mathcal{J} V], \quad \forall U, V \in \mathfrak{g}_{(\gamma)}.$$

We omit the proof which consists of a straightforward computation.

Theorem 7.1.2. *There exists one, and only one, simple Lie algebra with root system G_2 and dimension 28, i.e. the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$.*

Proof. By Theorem 6.1 $\mathfrak{g}$ is a Lie algebra, hence, its complexification is a Lie algebra [5, p. 178–179]. Since $\mathfrak{g}$ is simple, $\mathfrak{g}_{\mathbb{C}}$ is semisimple by Cartan criterion for semisimplicity (see [5, p. 180]) and has root system of type G_2 . In particular, there exists a semisimple Lie algebra with root system G_2 and dimension 28.

Let $\mathfrak{h}$ be a semisimple Lie algebra with root system G_2 and multiplicities two, we shall prove that $\mathfrak{h}$ is the complexification of a Lie algebra isomorphic to $\mathfrak{g}$. This shows that this class of Lie algebras contains just one element. We know that the subalgebras $\mathfrak{n}$ and $\theta\mathfrak{n}$ of $\mathfrak{h}$ defined in Subsect. 3.1.1 are Heisenberg-type Lie algebras. Denote, as before, by $\mathfrak{g}_{\omega}$ the centre of $\mathfrak{n}$.

Take an orthonormal basis $\{Z_1, Z_2\}$ of $\mathfrak{g}_{\omega}$, a unit vector $X \in \mathfrak{g}_{\beta}$, and a unit vector $Y \in \mathfrak{g}_{\alpha+\beta}$, and set

$$E_1 = [\theta X, Y] \quad E_2 = [\theta X, J_1 J_2 Y].$$

Then $E_2 = -[\theta J_1 J_2 X, Y]$, by (3.3.9), and $\{E_1, E_2\}$ is an orthonormal basis of $\mathfrak{g}_{\alpha}$. Also set

$$\begin{aligned} H_{\omega} &= [\theta Z_1, Z_1] = [\theta Z_2, Z_2], & M_{\omega} &= [Z_1, \theta Z_2], \\ H_{\alpha} &= [\theta E_1, E_1] = [\theta E_2, E_2], & M_{\alpha} &= [E_1, \theta E_2]. \end{aligned}$$

Then define a linear map $\mathcal{J} : \mathfrak{h} \rightarrow \mathfrak{h}$ by

$$(7.1.2) \quad \mathcal{J} \circ \theta + \theta \circ \mathcal{J} = 0,$$

$$(7.1.3) \quad \begin{aligned} \mathcal{J} : Z &\mapsto \frac{1}{2} \text{ad} M_{\omega} Z && \text{for } Z \text{ in } \mathfrak{g}_{\omega}, \\ \mathcal{J} : U &\mapsto \text{ad} M_{\omega} U && \text{for } U \text{ in } \mathfrak{v}, \\ \mathcal{J} : E &\mapsto \frac{3}{2} \text{ad} M_{\alpha} E && \text{for } E \text{ in } \mathfrak{g}_{\alpha}, \end{aligned}$$

and

$$(7.1.4) \quad \begin{aligned} \mathcal{J} : H_{\omega} &\mapsto M_{\omega}, & \mathcal{J} : M_{\omega} &\mapsto -H_{\omega}, \\ \mathcal{J} : H_{\alpha} &\mapsto M_{\alpha}, & \mathcal{J} : M_{\alpha} &\mapsto -H_{\alpha}. \end{aligned}$$

Clearly $\mathcal{J}^2 = -I$, we must show $\mathcal{J}[U, V] = [\mathcal{J}U, V] = [U, \mathcal{J}V]$. We start proving that for all $\gamma \in \Delta$ the restriction of $\mathcal{J}$ to the subalgebra $\mathfrak{g}_{(\gamma)}$ generated by $\mathfrak{g}_{\gamma}$ and $\mathfrak{g}_{-\gamma}$ is the complex structure introduced in Lemma 7.1.1, which by (7.1.4), shows

$$(7.1.5) \quad \mathcal{J}V = \frac{1}{\langle \gamma, \gamma \rangle} \text{ad}(M_{\gamma})V, \quad \forall V \in \mathfrak{g}_{\gamma}.$$

This is obvious when $\gamma = \pm\omega$, or $\gamma = \pm\alpha$. For the other root spaces it is a consequence of the following relations which hold for all $Z \in \mathfrak{g}_\omega$, $X \in \mathfrak{g}_\beta$, and $Y \in \mathfrak{g}_{\alpha+\beta}$,

$$\begin{aligned} \operatorname{ad}([Z_1, \theta Z_2]) X &= J_1 J_2 X = \frac{1}{2} \operatorname{ad}([X, \theta J_1 J_2 X]) X, \\ \operatorname{ad}([Z_1, \theta Z_2]) J_Z X &= J_1 J_2 J_Z X = \frac{1}{2} \operatorname{ad}([J_1 X, \theta J_2 X]) J_Z X, \\ \operatorname{ad}([Z_1, \theta Z_2]) Y &= J_1 J_2 Y = \frac{3}{2} \operatorname{ad}([Y, \theta J_1 J_2 Y]) Y, \\ \operatorname{ad}([Z_1, \theta Z_2]) J_Z Y &= J_1 J_2 J_Z Y = \frac{3}{2} \operatorname{ad}([J_1 X, \theta J_2 X]) J_Z Y. \end{aligned}$$

Let $\gamma, \delta \in \Delta$ and $V \in \mathfrak{g}_\gamma$, we show that $\mathcal{J}[H_\delta, V] = [\mathcal{J}H_\delta, V] = [H_\delta, \mathcal{J}V]$. Since $[H_\delta, V] \in \mathfrak{g}_\gamma$, we have, by (7.1.5),

$$\mathcal{J}[H_\delta, V] = \frac{1}{\langle \gamma, \gamma \rangle} [M_\gamma, [H_\delta, V]],$$

which, recalling that the action of $\mathfrak{m}$ preserves the root spaces, yields by Jacobi and (7.1.5),

$$\begin{aligned} \mathcal{J}[H_\delta, V] &= \frac{1}{\langle \gamma, \gamma \rangle} [M_\gamma, [H_\delta, V]] = \frac{1}{\langle \gamma, \gamma \rangle} [H_\delta, [M_\gamma, V]] \\ &= \frac{\langle \delta, \gamma \rangle}{\langle \gamma, \gamma \rangle} [M_\gamma, V] = \langle \delta, \gamma \rangle \mathcal{J}V = [H_\delta, \mathcal{J}V]. \end{aligned}$$

Since, by (3.3.6),

$$[M_\delta, V] = \frac{\langle \delta, \gamma \rangle}{\langle \gamma, \gamma \rangle} [M_\gamma, V],$$

we get, by (7.1.4),

$$[\mathcal{J}H_\delta, V] = [M_\delta, V] = \frac{\langle \delta, \gamma \rangle}{\langle \gamma, \gamma \rangle} [M_\gamma, V] = \langle \delta, \gamma \rangle \mathcal{J}V = [H_\delta, \mathcal{J}V],$$

from which the result follows. We now prove that $\mathcal{J}[M_\delta, V] = [\mathcal{J}M_\delta, V] = [M_\delta, \mathcal{J}V]$. Since the action of $\mathfrak{m}$ preserves the root spaces, we have, by (7.1.5),

$$\mathcal{J}[M_\delta, V] = \frac{1}{\langle \gamma, \gamma \rangle} [M_\gamma, [M_\delta, V]],$$

which, by Jacobi, yields

$$\mathcal{J}[M_\delta, V] = \frac{1}{\langle \gamma, \gamma \rangle} [M_\delta, [M_\gamma, V]] = [M_\delta, \mathcal{J}V].$$

Moreover, being

$$[M_\gamma, [M_\gamma, V]] = -\langle \gamma, \gamma \rangle^2 V,$$

we get, by (3.3.6),

$$\mathcal{J}[M_\delta, V] = \frac{1}{\langle \gamma, \gamma \rangle} [M_\gamma, [M_\delta, V]] = \frac{\langle \delta, \gamma \rangle}{\langle \gamma, \gamma \rangle^2} [M_\gamma, [M_\gamma, V]] = -\langle \delta, \gamma \rangle V,$$

which provides the result, since

$$[\mathcal{J}M_\delta, V] = -[H_\delta, V] = -\langle \delta, \gamma \rangle V.$$

Since $\mathcal{J}$ preserves the root spaces, the second relation in (7.1.1) is clear for vectors in root spaces corresponding to pairs of roots with sum not in Δ . Let $U, V \in \mathfrak{v}$ and $[U, V] = Z$. Write $Z = a Z_1 + b Z_2$, then

$$\begin{aligned} \langle Z_1, [J_1 J_2 U, V] \rangle &= \langle J_1^2 J_2 U, V \rangle = -\langle J_2 U, V \rangle = -\langle Z_2, [U, V] \rangle \\ &= -\langle Z_2, Z \rangle = -b, \end{aligned}$$

and, similarly,

$$\langle Z_2, [J_1 J_2 U, V] \rangle = \langle J_2 J_1 J_2 U, V \rangle = \langle J_1 U, V \rangle = \langle Z_1, [U, V] \rangle = \langle Z_1, Z \rangle = a,$$

which together imply

$$[\mathcal{J}U, V] = [J_1 J_2 U, V] = a Z_2 - b Z_1.$$

Being

$$\begin{aligned} \mathcal{J}[U, V] &= \frac{1}{2} \text{ad}([Z_1, \theta Z_2])[U, V] = \frac{1}{2} [[Z_1, \theta Z_2], Z] \\ &= a Z_2 - b Z_1, \end{aligned}$$

we find

$$\mathcal{J}[U, V] = [\mathcal{J}U, V],$$

which, since U and V are arbitrary, implies $\mathcal{J}[U, V] = [U, \mathcal{J}V]$. Let $U \in \mathfrak{v}$ and $Z = a Z_1 + b Z_2 \in \mathfrak{g}_\omega$. The definition of $\mathcal{J}$ yields

$$\mathcal{J}[Z, \theta U] = J_1 J_2 [Z, \theta U] = J_1 J_2 (a J_1 U + b J_2 U) = a J_2 U - b J_1 U,$$

$$[Z, \mathcal{J}\theta U] = -[Z, \theta J_1 J_2 U] = -a J_1^2 J_2 U - b J_2 J_1 J_2 U = a J_2 U - b J_1 U,$$

and

$$[\mathcal{J}Z, \theta U] = \frac{1}{2} [[[Z_1, \theta Z_2], Z], \theta U] = \frac{1}{2} [a Z_2 - b Z_1, \theta U] = a J_2 U - b J_1 U,$$

from which we deduce the second of the (7.1.1)'s with Z in place of U and θU in

place of V . Let $U \in \mathfrak{v}$ and $E = a E_1 + b E_2 \in \mathfrak{g}_\alpha$. Since on $\mathfrak{v}$

$$J_1 J_2 \circ \text{ad} E = \text{ad} E \circ J_1 J_2 \text{ad} E_2 = J_1 J_2 \circ \text{ad} E_1,$$

we obtain

$$\mathcal{J}[E, U] = J_1 J_2 [E, U] = [E, J_1 J_2 U] = [E, \mathcal{J}U],$$

and

$$\begin{aligned} \mathcal{J}[E, U] &= J_1 J_2 [a E_1 + b E_2, U] = a J_1 J_2 [E_1, U] + b (J_1 J_2)^2 [E_1, U] \\ &= a [E_2, U] - b [E_1, U]. \end{aligned}$$

Since by definition of $\mathcal{J}$

$$[\mathcal{J}E, U] = \frac{3}{2} [[[E_1, \theta E_2], E], U] = [a E_2 - b E_1, U],$$

we have the result. Similarly, we might prove the second in (7.1.1) for θE and U . Now (7.1.2) yields the result.

We have proved that $\mathfrak{h}$ has a complex structure $\mathcal{J}$. Therefore, there is a subalgebra $\mathfrak{g}_0$ such that

$$\mathfrak{h} = \mathfrak{g}_0 \oplus \mathcal{J}\mathfrak{g}_0.$$

Clearly, $\mathfrak{g}_0$ has root system G_2 and $\dim \mathfrak{g}_0 = 14$. Since by Theorem 6.1 there is up to isomorphisms only one Lie algebra belonging to this class, $\mathfrak{g}_0 \simeq \mathfrak{g}$.

Finally, the argument used in the proof of Theorem 6.1 to show that $\mathfrak{g}$ is simple works also here, and $\mathfrak{g}_{\mathbb{C}}$ is simple. $\square$

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