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A continuation theorem with applications to periodically forced Liénard equations in the presence of a separatrix*

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Abstract. In this paper we prove a result on the existence of periodic motions for the periodically forced Liénard differential equation $x'' + f(x)x' + g(x) = e(t)$ in a situation where the phase portrait of the associated autonomous equation is similar to that of a centre limited by an unbounded separatrix. The existence result, which is based on a degree theoretic continuation theorem, enables us to treat some interesting cases not previously considered in the literature.

Mathematics Subject Classification (2000). 34C25

Key words. Liénard equation – periodic solutions – continuation methods

1. Introduction

The study of harmonic oscillations for a conservative system with one degree of freedom subject to external perturbations, which are periodic in the time-variable

$$x'' + g(x) = p(t, x, x'), \tag{1.1}$$

has been widely considered in the literature for its significance from the point of view of possible applications to a broad class of models arising in nonlinear mechanics, physics and engineering.

In the last twenty years, much interest has been addressed to the case in which the restoring force g has superlinear growth at infinity, namely $g(s)/s \to +\infty$ as $s \to \pm\infty$. Indeed, in this situation, the unperturbed equation

$$x'' + g(x) = 0 \tag{1.2}$$

presents an unbounded family of solutions satisfying the desired boundary conditions (like the periodic or the Sturm–Liouville ones) and, in many cases, such

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a feature is preserved for the perturbed equation as well. Hence, some more standard approaches based on the search of a priori bounds for the solutions cannot be directly applied and more refined tools need to be developed. In this direction, classical and recent existence or multiplicity results for various boundary value problems associated with Equation (1.1) usually impose either some growth restrictions for $p(\dots)$ in x and x' (see, e.g., [5], [28], [29], [71] and the references therein), in a way that the term $g(x)$ essentially "dominates" the rest of the nonlinearities and the behaviour of the solutions of Equation (1.1) can be controlled by means of the knowledge of the orbits of (1.2) and their speed. Another way of tackling Equation (1.1) consists of requiring suitable conditions on p which guarantee that the trajectories of (1.1) enter those of (1.2) in the phase plane. Then the existence of periodic solutions for (1.1) follow by standard applications of the Brouwer fixed point theorem or the Massera theorem. In this direction, we are led to consider suitable dissipativity conditions which are quite well developed in the case when $p(\dots)$ consists of a principal part of the form like $f(x, x')x'$ or $f(x')$, plus a bounded term (see, e.g. [3], [4], [12], [22], [66], [82]).

In this paper, we are interested in the case in which $p(t, x, y)$ splits as $e(t, x, y) - f(x)y$, with $e(\dots)$ bounded and f of not definite sign in $\mathbb{R}$, (but only on the positive or the negative semi-axes), so that equation (1.1) takes the familiar form of a Liénard equation

$$x'' + f(x)x' + g(x) = e(t, x, x') \tag{1.3}$$

in which, due to the lack of sign-definiteness of f, it seems not always possible to understand the structure of the trajectories (1.3) in terms of those of (1.2), neither it is clear how to enter in a dissipative setting (if any such setting does exist for the given equation) and, in particular, the known constructions of positively invariant regions for (1.3) in the phase plane, using the trajectories of (1.2), is not successful here.

From the beginning of the modern theory of ODEs, Liénard equations have been considered as a constant source of questions, problems and models of relevant interest both from the theoretical and the applied points of view. Looking for periodic solutions of nonautonomous Liénard equations, besides the classical books like [12], [34], [66], covering the main achievements up to the sixties and further enlarged and updated at the beginning of the seventies by the content of [22], [43], [68], there has been a constant production in this area, continuing today. In recent years the interest in such a class of equations has been even increased after some new directions (e.g., toward the study of bounded solutions [1], [47], [50], [63], the consideration of nonlinearities with singularities [23], [46], the study of nonstandard oscillatory phenomena [73]) renewed and enriched the interest in this field.

Throughout this article, we suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $e : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, T-periodic ($T > 0$) in the t-variable and (uniformly) bounded, that is,

$$|e(t, x, y)| \leq E,$$

for all t, x, y, for some $E > 0$. We also define

$$F(s) = \int_0^s f(\xi)\, d\xi, \quad G(s) = \int_0^s g(\xi)\, d\xi.$$

Since Liénard equations have already been studied so widely, our first goal in this introductory section will be that of excluding a large list of cases in which general existence results are already available. To start, let us confine ourselves to the case in which f and g are of constant sign for $x \ll 0$ and $x \gg 0$. In this manner, we can consider the possibility of f and g being polynomials of degrees $k \geq 1$ and $l \geq 1$, respectively, as has already been done in the classical works of Gomory [21] and Mawhin [43].

In particular, to express the fact that we are discussing the polynomial case for f and g, we usually write

$$f(x) = p_k(x) = b_0 x^k + \cdots + b_k\,, \quad g(x) = q_l(x) = a_0 x^l + \cdots + a_l\,, \tag{1.4}$$

also assuming, tacitly, that

$$b_0 \neq 0, \quad \text{and} \;\; a_0 \neq 0.$$

Note that in this case,

$$|g(s)| \to \infty, \quad \text{as} \;\; s \to \pm\infty. \tag{1.5}$$

This condition will also be assumed from now on.

As a first possibility for the validity of (1.5), we discuss the case in which

$$\lim_{s \to \pm\infty} g(s)\mathrm{sgn}(s) = -\infty.$$

In this situation, due to a theorem of Reissig [64] (see also [21], [43], [44] for previous works in this direction), we know that Equation (1.3) has at least one T-periodic solution for any bounded forcing term e. Thus, we can exclude this case from our further considerations and, for the polynomial model, we may put aside the possibility "l odd and $a_0 < 0$".

Next, let us consider the case when

$$\lim_{s \to \pm\infty} g(s) = +\infty.$$

In this situation, since it is well-known that

$$\mathcal{R}(e) \cap \mathcal{R}(g) \neq \emptyset$$

is a necessary condition for the existence of periodic solutions of (1.3), we can see immediately that for some forcing terms there are no T-periodic solutions of (1.3). In particular, if we keep $e(.)$ fixed and consider the parameter depending equation

$$x'' + f(x)x' + g(x) = \nu + e(t) \tag{1.6}$$

with $\nu \in \mathbb{R}$, we find that here is ν_0 such that (1.6) has no T-periodic solutions for each $\nu < \nu_0$. Indeed, a complete description of the solvability of (1.6) was already given in [15], yielding the existence of at least *two solutions* for $\nu > \nu_0$. Similar features also occur for (1.3) in the case when $\lim_{s\to\pm\infty} g(s) = -\infty$. Thus, in view of [15], we can exclude also these two cases from our further considerations and, for the polynomial model, we may put aside the possibility "l even".

By the above discussion, we are henceforth led to consider the case in which Condition (1.5) takes the form

$$\lim_{s\to\pm\infty} g(s)\mathrm{sgn}(s) = +\infty. \tag{1.7}$$

In this situation, and having in mind the polynomial model (1.4), as a possible application, we can further take into account the following two cases.

Suppose, at first, that $g(s)/s$ is upper bounded as $s \to \pm\infty$ by a positive constant, say ω^2. Then, as a consequence of a line of results due to Lazer, Mawhin, Cesari and Kannan, Martelli, Reissig, Mawhin and Ward, and others (see, for instance [44], [49], [65] and the references therein), we know that Equation (1.3) has at least one T-periodic solution, for an *arbitrary* f, provided that the period of the forcing term is not too large: e.g., $T < 2\pi/\omega$ (according to [65]).

A second possibility is that the function f induces some friction effect to the system. This is a typical situation that occurs when $f(s) \geq c > 0$ for some constant c and $|s|$ large. In this case, we may enter into the theory of dissipative systems and prove the uniform ultimate boundedness of the solutions and the existence of a T-periodic solution via the Massera theorem [42]. Thus, a possible condition to ensure the solvability of the periodic problem for (1.3) is that

$$F(s)\mathrm{sgn}(s) \to +\infty, \quad \text{as } s \to \pm\infty$$

(or the "dual" assumption

$$F(s)\mathrm{sgn}(s) \to -\infty, \quad \text{as } s \to \pm\infty,$$

corresponding to the change in time direction) holds. Results in this direction were obtained by many authors, like Levinson, Cartwright and Littlewood, Langenhop, Loud, Mizohata and Yamaguti, Reuter, Burton, Burton and Townsend, (see [33], [82]). For general results for the damped polynomial case we recall again the works of Gomory [21] and Mawhin [43], as well as Graef [22] and refer to [30] and [57] for extensions to differential systems.

Hence, we can also exclude these two examples from our next considerations and, for the polynomial model, we may put aside the possibilities "k even" or "$l = 1$".

Thus, recalling the sign constraints put on f and g at the beginning, we are led to limit ourselves to the cases when

$$\lim_{s\to\pm\infty} F(s) = +\infty \quad (\text{or } -\infty), \tag{1.8}$$

with g satisfying (1.7) (or even a stronger form of it like $g(s)/s \to +\infty$ as $s \to \pm\infty$) as well. In some applications, we'll also assume the stronger form of (1.8) expressed by

$$\lim_{s\to\pm\infty} f(s)\mathrm{sgn}(s) = +\infty \quad (\text{or } -\infty). \tag{1.9}$$

Anyhow, making reference to the polynomial model (1.4) and excluding the situations which can be treated by means of known results, our aim therefore is to focus attention to the cases

$$k = 2m+1, \quad l = 2n+1,$$

and

$$f(x) = p_{2m+1}(x) = b_0 x^{2m+1} + \ldots, \quad g(x) = q_{2n+1}(x) = a_0 x^{2n+1} + \ldots \tag{1.10}$$

with

$$a_0 > 0, \quad n \geq 1$$

and (without loss of generality)

$$b_0 > 0, \quad m \geq 0.$$

At this point, to proceed further, we had better try to understand which are the planar dynamics associated to (1.3) when (1.7) and (1.8) or (1.9) hold. To this aim, it can be useful to consider the associated autonomous system

$$\begin{cases} x' = y \\ y' = -f(x)y - g(x) \end{cases} \tag{1.11}$$

in the phase plane, or the equivalent one in the Liénard plane

$$\begin{cases} x' = y - F(x) \\ y' = -g(x). \end{cases} \tag{1.12}$$

Here we can observe that (except some special cases), the qualitative behaviour of the trajectories drastically changes according to whether the fact that a suitable "balance" between the growth rates of F and g at infinity is respected or not. Namely, let us consider the negative semitrajectory $\Gamma^-(y_0)$ of (1.11) starting at $(0, y_0)$ with $y_0 \ll 0$. According to a classical result by Filippov (further refined in [9], [58], [25], [79], [24], [72]), we have that if

$$\sup_{s\geq 0} \sqrt{8}\sqrt{G(s)} - F(s) < +\infty, \tag{1.13}$$

then there is some $\hat{y}_0 < 0$ such that for any $y_0 \leq \hat{y}_0$, $\Gamma^-(y_0)$ does not cross the x-axis. On the other hand, if

$$\exists\, a \in]0, \sqrt{8}[\, : \quad \lim_{s\to+\infty} a\sqrt{G(s)} - F(s) = +\infty, \tag{1.14}$$

then such an intersection occurs for any y_0 sufficiently negative. Repeating similar arguments on $(-\infty, 0]$, we can easily find sufficient conditions which, respectively, imply the existence of trajectories in the negative phase plane $y < 0$ which are unbounded in the past and future, or (otherwise), imply the existence of solutions winding around the origin.

It can be interesting to rephrase the above conditions in the polynomial example. Using (1.10), we can easily conclude that the former case occurs when $n < 2m+1$, while if we wish to always have the intersections with the zero-isoclines for the system (1.12), we have to assume $n > 2m + 1$. When $n = 2m + 1$ the dynamics are determined by the values of the coefficients. This possibility will be examined in the following.

In the latter case ("intersection"), a possible way to attack the problem for the nonautonomous system

$$\begin{cases} x' = y \\ y' = -f(x)y - g(x) + e(t, x, y) \end{cases} \tag{1.15}$$

can be that of considering it as a perturbation of a centre described by (1.11) or by a certain autonomous equation very close to it. As we don't suppose that $e(.)$ is small, we have to require that the centre is global outside a compact set and then we could take advantage of some dissipativity-like effect forcing the trajectories of the nonautonomous system to enter the regions bounded by the orbits of the centre. This strategy was employed in [78] and permits us to prove the existence of at least one T-periodic solution of (1.3) when g is odd and the even part f_e of f is not null. Applied to the polynomial example, this yields the existence of periodic solutions for the equation

$$x'' + p_{2m+1}(x)x' + q_{2n+1}(x) = e(t), \tag{1.16}$$

for q_{2n+1} *odd,* p_{2m+1} *not odd and* $n \geq 2(m + 1)$. If q_{2n+1} and p_{2m+1} are both odd, then one has to exploit, in a better way, the symmetry of the equation. Indeed, under the supplementary assumption of $e(.)$ odd too, Ding in [14], with an ingenious proof, obtained the existence of at least *two* harmonic solutions for an high frequency forcing term. This result was subsequently extended by Liu [38], [39], Liu and You [40] using the KAM theory for reversible systems and, in a different direction, by Chow and Pei [8], using a variant of the Aubry–Mather theorem. In particular, (cf. [39], [40]) it is known that *if* q_{2n+1}, p_{2m+1} *and* $e(.)$ *are odd and* $n \geq 2(m+1)$, then all the solutions of (1.16) are bounded and there are infinitely many harmonic solutions as well as subharmonic solutions of any order. Roughly speaking, these results are proved via a very careful and delicate analysis (having its roots in the works of Morris [53] and Dieckerhoff and Zehnder [13]) which shows that under the assumption

$$n \geq 2(m + 1), \tag{1.17}$$

equation (1.16) can be treated as a perturbation of the oscillator

$$x'' + x^{2n+1} = 0.$$

The problem of the existence and multiplicity of T-periodic solutions for (1.16) under only Condition (1.17) seems to still be open [1].

Suppose now that

$$n < 2(m+1) \tag{1.18}$$

holds. In this case, the Liénard Equation (1.3) cannot be considered as a perturbation of an associated conservative system anymore and, indeed, we have to examine whether the trajectories of the nonautonomous equation somehow keep the structure of those of the associated autonomous systems (1.11) or (1.12). Trying to explain this point in a better way: consider as a model example the equation

$$x'' + p_{2m+1}(x)x' + \gamma x = e(t), \tag{1.19}$$

with $\gamma > 0$. Liénard equations of the type (1.19) have been investigated, mainly in the autonomous case

$$x'' + p_{2m+1}(x)x' + x = 0, \tag{1.20}$$

for their significance in connection with the Hilbert 16th problem (see [2], [37]). In [77] the study of the dependence of the separatrices of (1.20) with respect to a parameter was given. It is interesting to remark that, thanks to a change of variable considered by Conti in [9], any autonomous Liénard System (1.12), with $g(s)s > 0$ for $s \neq 0$ and $G(\pm\infty) = \infty$, has the same phase portrait as a system where $g(x) = x$. We point out, however, that in the nonautonomous case, Conti's change of variable would drastically modify the character of the forcing term and, moreover, it does not preserve the time of the motion along the trajectories. On the contrary, the time mapping estimates will be a crucial tool for our argument in the proof of the existence of T-periodic solutions of the nonautonomous equation. Thus, from our point of view, we have to consider the case $g(x) = \gamma x$ in (1.19), just as a possible example which, clearly, does not cover the general situation.

Consider Equation (1.19). If p_{2m+1} is odd, then we have that in the phase plane the orbits of the associated autonomous system,

$$\begin{cases} x' = y \\ y' = -p_{2m+1}(x)y - \gamma x, \end{cases}$$

have mirror symmetry with respect to the y-axis. Moreover, by the Filippov theorem, we also know that there is an unbounded separatrix which lies in the semi-plane $y < 0$ and is symmetric. By considerations similar to those developed in [74] we can also see that such a separatrix is bounded in $y = x'$ and, therefore, the solution on it is globally defined in time, i.e., a point on the separatrix takes infinite time to go to infinity.

A simple example of this behaviour is given by the autonomous equation

$$x'' + xx' + x = 0, \tag{1.21}$$

[1] a recent existence result in this direction has been obtained by D. Papini, Periodic solutions for a class of Liénard equations, *Funkc. Ekvac.* (to appear)

which has the solution $x(t) = -t$. In the phase plane, Equation (1.21) produces the phase portrait of a centre (at the origin) which is unbounded in the x-direction and for $y > 0$, while it is bounded below by a separatrix ($y = -1$) lying on $y < 0$. Note that the fact that the separatrix is an algebraic curve is consistent with a theorem of Odani [54]. These kinds of centres have been investigated by Conti in [10], [11] and related works, introducing the notion of "centres of type B" for those centres having the boundary that does not contain singular points.

Coming back to Equation (1.19) we observe that the presence of a separatrix with the above properties for the autonomous equation, can be used to prove the existence of T-periodic solutions for the periodically forced equation. Indeed, looking at the dynamical structure of the equation, we can see that a solution (x, x') of any Cauchy problem with initial point sufficiently far from the origin, will wind around the origin in the phase plane or will follow the direction of the separatrix, but, in any case, will not be able to complete a turn before the time T. This, in connection with a continuation theorem of Krasnosel'skii (on points of T-irreversibility [31]), would allow us to prove the existence of at least one T-periodic solution of (1.19), for any bounded and T-periodic forcing term. Casting this idea in a little more precise shape along a formal proof, and taking into account the cases when previous theorems can be applied, one can easily arrive at the following claim:

The equation,

$$x'' + p_k(x)x' + \gamma x = e(t, x, x'),$$

with $\gamma > 0$ and $e(\dots)$ bounded continuous and T-periodic in t, has at least one T-periodic solution for any polynomial p_k with degree $k \geq 1$.

This result, indeed, is not new as it can be deduced from a theorem obtained by Omari et al. in [56], where it was proved that the Liénard Equation (1.3) with g satisfying the sign Condition (1.7), has at least one T-periodic solution, provided that

$$\lim_{\substack{s\to+\infty \\ (\text{or } s\to-\infty)}} g(s)/F(s) = 0. \tag{1.22}$$

In particular, applying [56] to the polynomial model, we have that (1.16) has at least one harmonic solution if

$$2n < 2m + 1,$$

(or if $2n = 2m+1$ and T is sufficiently small, the smallness of T being computable in terms of the leading coefficients of the polynomials [56]).

Thus, naturally, we arrive at the following problem which, apparently, has not been treated before: "can we say anything about the solvability of the periodic BVP for (1.16) when

$$m < n < 2(m + 1), \tag{1.23}$$

or, even better, for general f and g not satisfying all the above list of conditions which were already considered in preceding existence results ?"

To prepare a partial answer to this question, we choose the model equation

$$x'' + 3xx' + x(x^2 + 1) = e(t), \tag{1.24}$$

which corresponds to the case $m = 0$, $n = 1$ and therefore fits into Case (1.23). If we analyse the associated autonomous equation

$$x'' + 3xx' + x(x^2 + 1) = 0, \tag{1.25}$$

we find that it has the unbounded solution $x(t) = -\tan(t)$. In the phase plane, Equation (1.25) produces the portrait of a centre (at the origin) which is unbounded in the x-direction and for $y > 0$, while it is bounded below by a trajectory ($y = -x^2 - 1$) lying on $y < 0$. A phase-plane inspection shows the existence of a separatrix between the trajectory $y = -(x^2 + 1)$ and the isocline $y = -(x^2 + 1)/3$. Hence, we are again in the case of a Conti's centre of type B, but, this time, and this is the crucial difference with respect to the Example (1.21) (where we had $m = n = 0$ and thus $2n < 2m + 1$), the separatrix is run in finite time. Here we don't have global continuability in time for all the solutions of (1.25). Therefore, the periods of the orbits of the centre, for initial points which are far from the origin or close to the separatrix, are near to a finite number. We notice that actually, being

$$n = 2m + 1,$$

this is the above mentioned limit case for the intersection/non-intersection properties in the line of the Filippov theorem. Indeed, using (1.13) and (1.14), a straightforward computation gives that, for the equation

$$x'' + bxx' + x(x^2 + 1) = 0, \quad b > 0,$$

we have a global centre if $b < \sqrt{8}$, while there is a separatrix, like in equation (1.25), if $b > \sqrt{8}$.

In conclusion, a possible way to summarize all these possible different behaviours for the solutions of

$$x'' + p_{2m+1}(x)x' + q_{2n+1}(x) = 0 \tag{1.26}$$

(with p_{2m+1} and q_{2n+1} odd), drawing a rough analogy from some well-known cases, would lead to the following description:

- $n \geq 2(m + 1)$: like a perturbation of a superlinear problem; indeed the time map tends to zero as the initial points tend to infinity. The analogous model from a dynamical point of view: $x'' + g(x) = 0$, with $g(s)/s \to \infty$ as $|s| \to \infty$.
- $n \leq m$: like a perturbation of a sublinear problem; indeed the time map tends to infinity as the initial points tend to infinity or to the separatrix. The analogous model from a dynamical point of view: $x'' + g(x) = 0$, with $g(s)/s \to 0^+$ as $|s| \to \infty$.

- $m < n < 2m + 1$: like a perturbation of a semilinear problem; indeed the time map tends to a finite positive limit as the initial points tend to infinity or to the separatrix. The analogous model from a dynamical point of view: $x'' + g(x) = 0$, with $0 < a \le g(s)/s \le b < \infty$ as $|s| \to \infty$.

In view of the above analogy, which, we stress, is imprecise, but not too strange if one is interested in the dynamical properties of the solutions expressed by means of their time maps, in order to solve the periodic problem for the third case, it is natural to put conditions on the period. This will be justified a posteriori by some technical steps in the proofs. Thus, if we wish to adapt the argument described above for the study of the harmonic solutions of (1.19), a natural assumption will be that

$$T < \tau,$$

where τ is the time limit of the orbits of the centre approaching the separatrix and this, in turn, will be related to the blow-up time along the separatrix itself. To prove this, we first develop a continuation lemma in Sect. 2 which is based on [6] and makes use of some arguments previously developed in [55]. In the same section, we combine the continuation lemma with a bound set like a condition (cf. [45]) and prove our main results (Theorem 1 and Theorem 2) which relate the period of the forcing term T to the escape time of a comparison equation (in the applications this is a way to estimate the time along the separatrix). We point out that such limitations on the admissible periods for the forcing term are somehow in the spirit of [32], [60], [62] where computable estimates for the period are given. It is clear that, according to general results on degree theory like those in [6], [31], [45], [48], or by directly applying a theorem of Schmitt [69], one could prove that there is $\varepsilon > 0$ such that Equation (1.3) has at least one T-periodic solution for any T-periodic forcing term $e(.)$ with $0 < T < \varepsilon$. The main point here is that the upper limit for the period is explicitly computable and we are able to specify those equations for which we don't need any bound on T as well.

In Sect. 3 we give some applications of our existence result, in particular, we show, for some concrete equations like (1.16), how to verify the assumptions of the main abstract results. We also produce a broad class of examples which generalize the simple models (1.21) and (1.25) considered above.

2. Continuation results

Let us consider the periodically perturbed Liénard equation,

$$x'' + f(x)x' + g(x) = e(t, x, x'), \tag{2.1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ and $e : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions with $e(\dots)$, T-periodic in the t-variable, i.e., $e(t + T, x, y) = e(t, x, y)$, $\forall\, t, x, y$.
We look for the existence of T-periodic solutions of Equation (2.1), by assuming the *boundedness* of $e(\dots)$:

$$\exists\, E > 0 : \quad |e(t, x, y)| \le E, \ \forall\, t, x, y. \tag{2.2}$$

and a standard sign condition on g :

$$\exists\, d > 0 : \quad g(s)\mathrm{sgn}(s) \geq E,\ \forall\, |s| \geq d. \tag{2.3}$$

Note that (2.2) is trivially satisfied when $e = e(t)$ and (2.3) holds (for a suitable choice of $d > 0$) when $g(x)\mathrm{sgn}(x) \to +\infty$ as $|x| \to \infty$.
The following continuation lemma is used to guarantee the existence of a T-periodic solution to Equation (2.1).

Lemma 1. *Assume* (2.2) *and* (2.3). *Then, there is a continuous function* $\eta : [d, +\infty) \to]d, +\infty)$, *with* $\eta(s) > s$, *for all* $s \geq d$, *such that*

$$|x|_\infty + |x'|_\infty < \eta(R) \tag{2.4}$$

holds for each $R \geq d$, *and each* T*-periodic solution* $x(\cdot)$ *of the equation*

$$x'' + f(x)x' + g(x) = \lambda e(t, x, x'), \tag{2.5}$$

with $\lambda \in [0, 1[$, *satisfying one of the following conditions:*

$$\max x(t) - \min x(t) \leq R, \quad \textit{or} \quad \min x(t) \geq -R, \quad \textit{or} \quad \max x(t) \leq R. \tag{2.6}$$

If, moreover, there exists some $R = R_0 \geq d$ *such that* (2.6) *holds for all the possible* T*-periodic solutions of* (2.5), *then Equation* (2.1) *has at least one* T*-periodic solution.*

Proof. Lemma 1 is strongly related to [6, Corollary 5], and [55, Lemma] where, however, different homotopies of the form $x'' + \lambda f(x)x' + g(x) = \lambda e(t, x, x')$ and $x'' + \lambda f(x)x' + \lambda g(x) = \lambda e(t, x, x')$ were considered, respectively. The way of producing the a priori bounds below, follows some arguments already employed (more or less explicitly), in various previous works like [50], [55], [56], [65]. For this reason, here we'll provide only the main steps of the proof.

Let $x(\cdot)$ be a T-periodic solution of the parametrized Equation (2.5) for some $\lambda \in [0, 1[$. Taking the mean value of (2.5) over $[0, T]$, using the periodicity condition and recalling (2.3), we have that

$$T^{-1}\int_0^T g(x(t))\,dt = \lambda T^{-1}\int_0^T e(t, x(t), x'(t))\,dt \in]-E, E[\,.$$

Then, from (2.2), we see that there is $\tilde{t} = \tilde{t}_{(x,\lambda)} \in [0, T]$ with

$$|x(\tilde{t})| < d. \tag{2.7}$$

Suppose now that $x(\cdot)$ satisfies the first assumption (2.6), for some $R \geq d$, so that (2.7) implies that

$$\min_{t\in\mathbb{R}} x(t) > -(R + d) \quad \text{and} \quad \max_{t\in\mathbb{R}} x(t) < R + d, \tag{2.8}$$

and, therefore, $|x|_\infty < R + d$, where $|\cdot|_\infty$ is the "sup" norm on $[0, T]$. Hence, in any case, from (2.6), we have that

$$\min x(t) > -(R + d), \quad \text{or} \quad \max x(t) < R + d.$$

Let us also set $C_1 = C_1(R) := \max\{|g(s)| : |s| \le R+d\}$. Then, working as in [50, Theorem 2] we find that

$$|g(x(\cdot)) - \lambda e(\cdot, x(\cdot), x'(\cdot))|_1 < 2T(E + C_1) := C_2 = C_2(R)\,,$$

where $|\cdot|_1$ is the L^1-norm over $[0, T]$.

Set $F(s) := \int_0^s f(\xi)\, d\xi$, so that $w(t) := x'(t) + F(x(t))$ satisfies $|w'|_1 < C_2$ and, on the other hand, there is some $t^* \in [0, T]$ (with t^* a point of minimum or a point of maximum of $x(t)$ according to the fact that the first or the second inequality is valid in (2.8)) with $x'(t^*) = 0$ and $|x(t^*)| < R + d$. Hence, $|w(t^*)| \le C_3 = C_3(R) := \max\{|F(s)| : |s| \le R + d\}$ and then

$$|w|_\infty < C_2 + C_3 := C_4 = C_4(R)\,.$$

This, in turns, implies (cf. [55, p.150]) that

$$|x'|_2 < T^{1/2} C_4 \quad \text{and } |x|_\infty < d + TC_4 := C_5\,.$$

Finally, for $C_6 = C_6(R) := \max\{|F(s)| : |s| \le C_5\}$, we have that $|x'|_\infty < C_4 + C_6$ and in this manner, we have found a constant

$$\eta(R) := C_4 + C_5 + C_6$$

(independent of $x(\cdot)$ and λ), with $\eta(R) > R$ such that

$$|x|_\infty + |x'|_\infty < \eta(R)\,,$$

for any T-periodic solution $x(\cdot)$ of (2.5), with $\lambda \in [0, 1[$ and hence the first part of the claim is proved.

Assume now that there is $R_0 \ge d$ such that Condition (2.6) holds with $R = R_0$, for all the T-periodic solutions $x(\cdot)$ of (2.5). In this case, by the first part of the lemma, we have that $\eta(R_0)$ is a bound in the C^1-norm on $[0, T]$ for all the T-periodic solutions of (2.5). As $g(s)s > 0$ for $|s| \ge d$ and $R_0 \ge d$, we have that

$$\deg_B(g,]-r, r[, 0) = 1 \ne 0, \quad \forall\, r \ge R_0\,,$$

where $\deg_B$ is the Brouwer degree. Then all the assumptions of [6, Corollary 6] are satisfied and that continuation theorem ensures the existence of at least one T-periodic solution $\tilde{x}$ for Equation (2.1), satisfying

$$|\tilde{x}|_\infty + |\tilde{x}'|_\infty \le \eta(R).$$

□

Remark 1. Adapting to this setting some results in [48] for parametrized equations along the line of [35, Theoréme Fondamental], we can also prove that if Condition (2.6) holds for some $R \ge d$ and for all the possible T-periodic solutions of (2.5), then there is a compact connected set $\Sigma \subset C^1_T \times [0, 1]$ [2] of solution-pairs (x, λ) with $x(\cdot)$ a T-periodic solution of (2.5), such that for each $\lambda \in [0, 1]$ there

[2] here C^1_T denotes the space of the continuously differentiable and T-periodic functions $u : I\!R \to I\!R$ with the norm $|u|_\infty + |u'|_\infty$

is some $(x, \lambda) \in \Sigma$ and (2.6) holds for each x with $(x, \lambda) \in \Sigma$. Moreover, arguing like in [18], [19], it is possible to see that the connected branch Σ starts at $\lambda = 0$, from the set of the zeros of g, which are the equilibria of the autonomous equation

$$x'' + f(x)x' + g(x) = 0,$$

and reaches, at $\lambda = 1$, the set of the T-periodic solutions of Equation (2.1).

Now, we are going to prove our main results. Clearly, one has to work in order to find sufficient conditions guaranteeing that the key assumption (2.6) is satisfied. One way to approach this problem will consist of writing Equation (2.1) as the equivalent system in the phase-plane

$$\begin{cases} x' = y \\ y' = -f(x)y - g(x) + \lambda e(t, x, y), \quad 0 \le \lambda \le 1. \end{cases} \tag{2.9}$$

Integrating the first equation in (2.9), would give an upper bound on $\max x - \min x$ if we are able to find a bound on y from below. A development of this argument leads to the following result which combines the estimates in Lemma 1 with the "bound sets" technique of Gaines and Mawhin [20] and Mawhin [45].

Theorem 1. *Assume* (2.2) *and* (2.3). *Suppose that there is a continuous function* $a : \mathbb{R} \to \mathbb{R}^- := (-\infty, 0[$, *such that, any T-periodic solution $x(\cdot)$ of* (2.5) *with* $\lambda \in [0, 1[$ *satisfies*

$$x'(t) \neq a(x(t)), \quad \forall t \in [0, T]. \tag{2.10}$$

Suppose also that

$$\int_{-\infty}^{+\infty} \frac{1}{|a(s)|}\, ds > T. \tag{2.11}$$

Then, Equation (2.1) *has at least one T-periodic solution.*

Proof. Let us fix $\bar{R} \ge d$ such that, according to Condition (2.11),

$$\int_{-\bar{R}}^{\bar{R}} \frac{1}{|a(s)|}\, ds > T.$$

We first claim that for such an $\bar{R}$, (2.6) holds for all the possible T-periodic solutions of (2.5), with $\lambda \in [0, 1[$, satisfying the further assumption $x'(t) \ge a(x(t))$, for all $t \in \mathbb{R}$.

Indeed, let $x(\cdot)$ be any T-periodic solution of (2.5) with $x'(t) \ge a(x(t))$, for all $t \in \mathbb{R}$. We denote, respectively, by $t^* = t^*{}_{(x,\lambda)}$ a point of maximum of $x(\cdot)$ and $t_* = t_{*(x,\lambda)}$ a point of minimum of $x(\cdot)$ and using the T-periodicity of $x(\cdot)$, we choose t_* and t^* such that $t^* < t_* < t^* + T$. If, by contradiction,

$$\min x = x(t_*) < -\bar{R}, \quad \text{and} \quad \max x = x(t^*) > \bar{R},$$

then,

$$T > t_* - t^* \geq \int_{t^*}^{t_*} \frac{x'(t)}{a(x(t))}\, dt = \int_{t^*}^{t_*} \frac{-x'(t)}{|a(x(t))|}\, dt$$
$$= \int_{\min x}^{\max x} \frac{ds}{|a(s)|}\, dt > \int_{-\bar{R}}^{\bar{R}} \frac{1}{|a(s)|}\, ds > T,$$

which is absurd. We have thus proved that the second or the third conditions in (2.6) are always satisfied for such special T-periodic solutions and for $R = \bar{R}$ and therefore, the first part of Theorem 1 implies that

$$|x|_\infty + |x'|_\infty < \eta(\bar{R}) \tag{2.12}$$

holds with respect to all the T-periodic solutions of (2.5) satisfying $x'(t) \geq a(x(t))$.

Let us consider now the open bounded set

$$\Omega := \{u \in C^1_T \;:\; |u|_\infty + |u'|_\infty < \eta(\bar{R}),\; u'(t) - a(u(t)) > 0, \forall\, t \in I\!R\} \subset C^1_T\,.$$

Here and in what follows, we use the standard convention of identifying real numbers with constant functions, so that $I\!R \subset C^1_T$.
By (2.12) and (2.10), we have that

$$x(\cdot) \notin \partial\Omega,$$

for each possible solution $x(\cdot)$ of (2.5). As $\Omega \cap I\!R =]-\eta(\bar{R}), \eta(\bar{R})[$ and, like in the proof of Lemma 1, $\deg_B(g,]-r, r[, 0) = 1 \neq 0$, for all $r \geq d$, we can conclude with Theorem 2 and Corollary 6 in [6] and have the existence of at least one T-periodic solution $\tilde{x}$ of Equation (2.1) with $\tilde{x} \in \bar{\Omega}$. □

A simple way to check the key assumption (2.10) will be that of verifying its validity for all the solutions (not necessarily the T-periodic ones) of equation (2.5). This can be achieved, by imposing some conditions guaranteeing the positive (or the negative) invariance of the set

$$\mathcal{M}_a := \{(x, y) \in I\!R^2 : y \leq a(x)\}$$

in the phase plane. This yields the following result whose proof is only sketched.

Theorem 2. *Assume* (2.2) *and* (2.3). *Suppose that there is a continuous and piecewise continuously differentiable function* $a : I\!R \to I\!R^- := (-\infty, 0[$, *such that, either*

$$-a(x)a'(x) - f(x)a(x) - g(x) + E \leq 0, \quad \forall\, x \in I\!R, \tag{2.13}$$

or

$$-a(x)a'(x) - f(x)a(x) - g(x) - E \geq 0, \quad \forall\, x \in I\!R. \tag{2.14}$$

Suppose also that (2.11) *holds. Then, Equation* (2.1) *has at least one T-periodic solution.*

As the function $a(x)$ is only piece-wise continuously differentiable in x, when we consider assumptions (2.13) and (2.14), we suppose they are satisfied with respect both to the right and the left derivatives of $a(\cdot)$ at the "corner points".

Proof. Let us write Equation (2.5) in the form of System (2.9) and assume, without loss of generality, that (2.13) holds (the treatment of (2.14) is completely similar and thus omitted).

Suppose that there is some $t_0 \in I\!R$ such that

$$x'(t_0) = y(t_0) = a(x(t_0)) < 0 \tag{2.15}$$

for some solution $(x(\cdot), y(\cdot))$ of (2.9) with $0 \leq \lambda < 1$. Hence, there is $\varepsilon > 0$, such that $x(t) > x(t_0)$ for $t : t_0 - \varepsilon < t < t_0$ and $x(t) < x(t_0)$ for $t : t_0 < t < t_0 + \varepsilon$. Taking both the right and the left derivatives of $h(t) := x'(t) - a(x(t))$ at $t = t_0$, from (2.13) and (2.15), we see that

$$\max\{h'(t_0^+), h'(t_0^-)\} \leq -E + \lambda e(t_0, x(t_0), y(t_0)) < 0, \tag{2.16}$$

thus, $(h(t) - h(t_0))(t - t_0) < 0$ for $t \neq t_0$ and $|t - t_0|$ small. This can be repeated for any t_0 where $h(t_0) = 0$. If we now suppose that $x(\cdot)$ is a periodic function, we have the map $t \mapsto h(x(t))$ periodic too and therefore we immediately see that $h(x(t)) < 0$ or $h(x(t)) > 0$, for all $t \in I\!R$. Therefore, (2.10) is fulfilled for all the T-periodic solutions of (2.5) and, moreover, the set $\mathcal{M}_a$ is positively invariant in a strong sense (respectively, negatively invariant when (2.14) is assumed). Hence we achieve the conclusion, via Theorem 1. □

An interesting application of such a result can be given when $y = a(x)$ is a solution of the associated autonomous equation. More precisely, in order to enter into the setting of Theorem 2, let $q : I\!R \to I\!R$ be continuous and such that $q(s)\mathrm{sgn}(s) \to +\infty$ as $s \to \pm\infty$ and assume that $\tilde{y}(x)$ is a solution of the autonomous equation

$$y' = -f(x) - \frac{q(x)}{y}, \tag{2.17}$$

with $\tilde{y}(x)$ defined for all $x \in I\!R$ and such that $\tilde{y} < 0$. We define

$$\tau := \int_{-\infty}^{\infty} \frac{1}{|\tilde{y}(x)|}\, dx. \tag{2.18}$$

Let also $h = h(t, x, y) : I\!R \times I\!R^2 \to I\!R$ be a continuous function which is T-periodic in the t-variable and satisfies

$$0 \leq h(t, x, y) \leq H, \quad \forall\, t, x, y. \tag{2.19}$$

Then we have:

Corollary 1. *Under the above assumptions, the equation*

$$x'' + f(x)x' + q(x) = h(t, x, x'), \tag{2.20}$$

has at least one T-periodic solution, if

$$T < \tau.$$

Proof. We apply Theorem 1, with $g(x) = q(x) - (H/2)$, $e(t, x, y) = h(t, x, y) - (H/2)$ and $E = H/2$. In this case, it is possible to check that (2.14) is fulfilled with $a(x) = \tilde{y}(x)$. □

It is clear that a similar result holds if $-H \leq h(t, x, x') \leq 0$.

A more difficult situation occurs when the forcing term changes sign. In this case, it is necessary to add a constant at both sides of the equation in order to be in the previous case. This changes the function $g(x)$ and we should know the properties of new autonomous equations

$$x'' + f(x)x' + g(x) \pm E = 0. \tag{2.21}$$

Its phase-portrait is similar to the first one for $|x|$ large, because the leading term of the restoring part is still $g(x)$, and this ensures the persistence of the separatrix. Nevertheless, some symmetry properties may drastically change. This fact can also be viewed by a change of variables and, indeed, the effect of the term $\pm E$ is equivalent to that of changing the f (see, for instance, some control applications in [80]). For this reason, instead of looking for solutions of (2.21), it may be more convenient for the application to follow a comparison argument. The following example shows this situation:

Example 1. Consider the equation

$$x'' + 3xx' + x(x^2 + 1) = \sin(\omega t + \alpha).$$

In this case, according to Theorem 1, we are led to consider the equation

$$x'' + 3xx' + x(x^2 + 1) + 1 = 0,$$

which is a perturbation of (1.25). Now, a comparison of the respective slopes shows that the separatrix of this equation, is actually below the one of (1.25). Thus, we are still able to say that there are periodic solutions of the same period of the forcing term is ω is large enough. The precise estimate of ω is still unknown, while if we consider the equation

$$x'' + 3xx' + x(x^2 + 1) = 1 + \sin(\omega t + \alpha),$$

we can apply directly Corollary 1 and use the fact that $y = -(x^2 + 1)$ is a solution of (1.25) and, therefore, we have the existence of periodic solutions if $\omega > 2$. Finally, let us consider the equation

$$x'' + 3xx' + x(x^2 + 1) = \nu + \sin(\omega t + \alpha),$$

with $\nu > 0$. Writing it in the form (2.1), with $f(x) = 3x$ and $g(x) = x(x^2 + 1) - \nu$, we can take the C^1-function

$$a(x) = \begin{cases} -1, & \text{for } x \leq 0 \\ -x^2 - 1, & \text{for } x \geq 0. \end{cases}$$

A simple computation shows that, for the above choice of $a(x)$,

$$-a(x)a'(x) - f(x)a(x) - g(x) - 1 \geq 0, \quad \forall\, x \in I\!R,$$

provided that

$$\nu \geq 1 + \sqrt{32/27}.$$

Hence, using the fact that $\int_{-\infty}^{0} |a(x)|^{-1}\, dx = +\infty$, and having (2.14) in Theorem 2 satisfied, we find that there are periodic solutions of period $T = (2\pi/\omega)$ for *each* value of ω, if $\nu > 0$ is large enough. In Example 2, below, we give a general treatment of these kind of applications.

3. Applications and examples

In this section we are going to present some consequences of Theorem 1 and Corollary 1. Throughout this section, besides the regularity conditions already stated at the beginning of Sect. 2, we assume a sign type condition on $f(x)$ and on $g(x)$, namely,

$$\lim_{s\to\pm\infty} f(s)\mathrm{sgn}(s) = +\infty, \qquad \lim_{s\to\pm\infty} g(s)\mathrm{sgn}(s) = +\infty.$$

Similar results may be obtained when

$$\lim_{s\to\pm\infty} f(s)\mathrm{sgn}(s) = -\infty, \qquad \lim_{s\to\pm\infty} g(s)\mathrm{sgn}(s) = +\infty.$$

We stress the fact that such assumptions are not restrictive at all, as we wish to consider only the cases that are not yet investigated, in view of the discussion given in the introduction.

The main idea is to produce a situation in which the function $a(x)$ has the property that

$$\int_{-\infty}^{+\infty} \frac{1}{|a(x)|}\, dx = +\infty. \tag{3.1}$$

At first, we consider the equation

$$x'' + xx' + Lx = h(t, x, x') \tag{3.2}$$

with h defined as in (2.19) and $L > 0$. The line $y = -L$ is the separatrix of the autonomous equation $x'' + xx' + Lx = 0$ and hence a trajectory. If we take $\tilde{y} = -L$, we have that (2.18) holds for $\tau = +\infty$ and, therefore, Equation (3.2) always has a T-periodic solution or every T. However, as already mentioned in the introduction, this fact may be obtained using a result in [56].

Starting from this situation, we consider a more general equation of the form

$$x'' + f(x)x' + g(x) = e(t, x, x'), \tag{3.3}$$

with $f(x)$, $g(x)$ and $|e(t, x, x')| \leq E$ as in the previous section, and try to find a function $a(x)$ like that in Theorem 2, which will be equal to $-L$, only for x large enough.

From (2.13), if $a(x) = -L$, for all $x \geq 0$ and for some $L > 0$, we need to require

$$g(x) \geq L f(x) + E, \quad \forall x \geq 0. \tag{3.4}$$

On the other hand, for $x \leq 0$, we take $a(x) = -L - MF(x)$. Then, for the validity of (2.13), we are led to put the constraint

$$g(x) \geq \frac{L}{2} f(x) + \frac{1}{4} F(x) f(x) + E, \quad \forall x \leq 0, \tag{3.5}$$

in the case $M = 1/2$. For the general case, the leading term is of the form $(M - M^2) f(x) F(x)$. If both (3.4) and (3.5) are satisfied, then $a(x)$ is such that (3.1) holds and therefore we get the existence of T-periodic solutions for every T. In this frame, we can construct the following:

Example 2. Let $f(x) = p_{2m+1}(x) = b_0 x^{2m+1} + \ldots$ be an odd polynomial, of degree $2m + 1$, with $b_0 > 0$ and $g(x) = q_{2n+1}(x) + k$ with $q_{2n+1}(x) = a_0 x^{2n+1} + \ldots$ an odd polynomial of degree $2n + 1$, with $a_0 > 0$. If $n \geq m$, we have (3.4) satisfied for $k > 0$ large enough and any $L > 0$ sufficiently small. On the other hand, if $n < 2m + 1$, we have that (3.5) also holds for $k > 0$ large. The value $n = 2m + 1$ can actually be reached if b_0 is not too small compared to a_0. As a conclusion, we obtain that the condition

$$m \leq n < 2m + 1$$

implies the existence of T-periodic solutions for the equation

$$x'' + p_{2m+1}(x) x' + q_{2n+1}(x) = \nu + e(t, x, x'), \tag{3.6}$$

for any $\nu < 0$ with $|\nu|$ large enough. Similarly, one can get a dual result for $\nu > 0$ and large, in virtue of (2.14). In this manner, for the equation (3.6), with $|\nu|$ large, we are able to cover the full range of polynomial degrees mentioned in (1.23) (for the limit case $n = 2m+1$ an additional condition on the leading coefficients should be required).

A different, more geometrical, approach to treating the problem of how to obtain (3.1), is the following.

Consider again Equation (3.3). Let $d > 0$, be a point such that $g(s) > E$ for $s \geq d$. Assume that f has at least a zero for $s > d$, being eventually positive for s large (the dual case of f being eventually negative can be treated in the same way). Suppose that for $x \leq d$, the associated autonomous equation has a trajectory $\tilde{y}(x)$, with $\tilde{y}(x) < 0$ and such that

$$\int_{-\infty}^{0} \frac{1}{|\tilde{y}(x)|}\, dx < +\infty.$$

Call $\tilde{y}(d) = -c$ and let $x_0 > d$ be the last zero of f. For $d \le x \le x_0$ define Γ, the graph of the function $y = -(F(x) + c)$. Inspection of the slopes of the vector field associated to

$$\begin{cases} x' = y \\ y' = -f(x)y - g(x) + e(t, x, y) \end{cases}$$

shows that its trajectories enter the region bounded above by Γ and the graph of $\tilde{y}$. In this manner, we can define the function $a(x)$ on the interval $(-\infty, x_0\,]$. Next, we consider, for $x > x_0$, the graph of the zero-isocline of the autonomous system

$$\begin{cases} x' = y \\ y' = -f(x)y - g(x) + E. \end{cases}$$

This is given by

$$y = \phi(x) = -\frac{g(x) - E}{f(x)}\,.$$

Observe that y is always negative and that $\lim\limits_{x \to x_0^-} \phi(x) = -\infty$. Notice also that, without loss of generality, we can assume that $\lim\limits_{x \to +\infty} \phi(x) = -\infty$ (because, otherwise, at least in the polynomial case which is more interesting, we have (1.22) and the result may be obtained using the previously mentioned theorem from [56]). Hence, ϕ takes its maximum at some $\tilde{x} > x_0$. Now if

$$\phi(\tilde{x}) \le -c - F(x_0), \tag{3.7}$$

we can take the line $y = -(c + F(x_0)) := -L$ for $x \ge x_0$ to complete the definition of our function $a(x)$ and, clearly, $\int_{x_0}^{+\infty} |a(x)|^{-1}\, dx = +\infty$.

We stress that Condition (3.7) can be easily checked once that f, g and E are given. Otherwise, when

$$\phi(\tilde{x}) > -c - F(x_0),$$

we modify, in $x > d$, the function $f(x)$, by multiplying it by a positive parameter μ. Letting $\mu \to 0$, we get that c tends to d, while the maximum of the corresponding function ϕ, which is taken at the same point $\tilde{x}$, tends to $-\infty$. This means that it is possible to estimate a value μ_0 such that (3.7) is satisfied for $0 < \mu < \mu_0$. Observe that such a constant μ_0 can be effectively computed, differently to what usually occurs in analogous perturbation-like results.

Finally, as a bonus, we present a general construction leading to a class of autonomous equations of Liénard type whose solutions possess some properties which seem to be useful in view of the applicability of Theorem 2.

Let $f, \psi : \mathbb{R} \to \mathbb{R}$ be two given continuous functions and define

$$\gamma(s) := K + \int_0^s (f(\xi) - \psi(\xi))\, d\xi,$$

where $K \in I\!R$ is a fixed constant. We also set

$$q(s) := \gamma(s)\psi(s).$$

We claim that any solution of the first-order differential equation

$$w' = -\gamma(w) \tag{3.8}$$

determines a solution of the autonomous Liénard equation

$$x'' + f(x)x' + q(x) = 0. \tag{3.9}$$

Indeed, if $w(\cdot)$ is a solution of (3.8), we obtain

$$w''(t) = -\gamma'(w(t))w'(t) = \gamma'(w(t))\gamma(w(t)) = -f(w(t))w'(t) - q(w(t))$$

and the claim is immediately checked.

Assume now that

$$\gamma(s) > 0, \quad \forall\, s \in I\!R. \tag{3.10}$$

In this case, $y(t) = x'(t) = -\gamma(x(t)) < 0$ for all t and thus, evaluating y as a function of x, we obtain that

$$\frac{dy}{dx} = -\gamma'(x) = -f(x) + \frac{q(x)}{\gamma(x)}\,.$$

This means that $a(x) = -\gamma(x)$ satisfies the equation

$$-a(x)a'(x) - f(x)a(x) - q(x) = 0.$$

Then, in order to discuss Condition (2.11), we have to consider the integral

$$\int_{-\infty}^{+\infty} \frac{ds}{K + F(s) - \Psi(s)},$$

where

$$F(s) := \int_0^s f(\xi)\,d\xi, \quad \Psi(s) := \int_0^s \psi(\xi)\,d\xi.$$

We present now some simple examples for the solvability of the non-autonomous Liénard equation

$$x'' + f(x)x' + q(x) = e(t), \tag{3.11}$$

with $e : I\!R \to I\!R$ a continuous and T-periodic forcing term.

(*i*) Take $f(x) = \rho q(x)$, with f, q odd polynomials with positive leading coefficients and $\rho > 0$. In this case, Equation (3.9) has $x(t) = -\rho t$ as a solution and we can take $a(x) = -\gamma(x) = -\rho$. As a consequence of the previous results we have that the Liénard equation (3.11) has a T-periodic solution for any $e(\cdot)$.

(*ii*) Take $f(x)$ as an odd polynomial with positive leading coefficient, such that $\liminf_{x\to+\infty} \frac{f(x)}{x} > 2\rho$, for some $\rho > 0$ and let

$$q(x) = (f(x) - 2\rho x)(\rho x^2 + L), \quad \text{with } L > 0.$$

In this case, Equation (3.9) has $x(t) = -(L/\rho)^{\frac{1}{2}} \tan((L\rho)^{\frac{1}{2}} t)$ as a solution and we can take $a(x) = -\gamma(x) = -\rho x^2 - L$. Then the Liénard equation (3.11) has a T-periodic solution for any $e(\cdot)$ of constant sign and period less than $\pi/(L\rho)^{\frac{1}{2}}$. The case $\rho = L = 1$, corresponds to Example 1.

(*iii*) Take $f(x)$ as an odd polynomial with positive leading coefficient. Let k be a positive integer and suppose that $\liminf_{x\to+\infty} \frac{f(x)}{x^{2k-1}} > 2k\rho$, for some $\rho > 0$. If we take

$$q(x) = (f(x) - 2k\rho x^{2k-1})(\rho x^{2k} + L), \quad \text{with } L > 0$$

and denote by $\tilde{u}(t)$ the solution of $u' = \rho u^{2k} + L$, with $u(0) = 0$, we have that Equation (3.9) has $x(t) = -\tilde{u}(t)$ as a solution and thus we can take $a(x) = -\gamma(x) = -(\rho x^{2k} + L)$. By simple computations, we find also that $\tilde{u}(\cdot)$ is defined on the maximal interval $]-\tau_k, \tau_k[$, where

$$2\tau_k = \int_{-\infty}^{+\infty} \frac{1}{\rho x^{2k} + L}\, dx = \frac{1}{L^{1-\frac{1}{2k}} \rho^{\frac{1}{2k}}} \frac{\pi}{k \sin\left(\frac{\pi}{2k}\right)}.$$

Then the Liénard equation (3.11) has a T-periodic solution for any $e(\cdot)$ of constant sign and period less than $2\tau_k$.

(*iv*) Take $f(x)$ as an odd polynomial with positive leading coefficient. Let $\ell \geq 2$ be an integer and suppose that $\liminf_{x\to+\infty} \frac{f(x)}{x^{2\ell-1}} > 2\ell\rho^{\ell}$, for some $\rho > 0$. If we take

$$q(x) = (f(x) - 2\ell\rho x(\rho x^2 + L)^{\ell-1})(\rho x^2 + L)^{\ell}, \quad \text{with } L > 0$$

and denote by $\tilde{u}(t)$ the solution of $u' = (\rho u^2 + L)^{\ell}$, with $u(0) = 0$, we have that Equation (3.9) has $x(t) = -\tilde{u}(t)$ as a solution and thus we can take $a(x) = -\gamma(x) = -(\rho x^2 + L)^{\ell}$. By simple computations, we find also that $\tilde{u}(\cdot)$ is defined on the maximal interval $]-\sigma_\ell, \sigma_\ell[$, where

$$2\sigma_\ell = \int_{-\infty}^{+\infty} \frac{1}{(\rho x^2 + L)^{\ell}}\, dx = \frac{1}{L^{\ell-\frac{1}{2}} \rho^{\frac{1}{2}}} \frac{(2\ell-3)!!}{(2\ell-2)!!} \pi.$$

Then the Liénard equation (3.11) has a T-periodic solution for any $e(\cdot)$ of constant sign and period less than $2\sigma_\ell$.

(*v*) Take $f(x)$ as an odd polynomial with positive leading coefficient. Let k and ℓ be positive integers with $\ell \geq 2$ and suppose that $\liminf_{x\to+\infty} \frac{f(x)}{x^{2k\ell-1}} > 2k\ell\rho^{\ell}$, for some $\rho > 0$. If we take

$$q(x) = (f(x) - 2k\ell\rho x^{2k-1}(\rho x^{2k} + L)^{\ell-1})(\rho x^{2k} + L)^{\ell}, \quad \text{with } L > 0$$

and denote by $\tilde{u}(t)$ the solution of $u' = (\rho u^{2k} + L)^{\ell}$, with $u(0) = 0$, we have that equation (3.9) has $x(t) = -\tilde{u}(t)$ as a solution and thus we can take

$a(x) = -\gamma(x) = -(\rho x^{2k} + L)^{\ell}$. By computations involving the use of the Gamma function, we find also that $\tilde{u}(\cdot)$ is defined on the maximal interval $]-\mu_{\ell,k}, \mu_{\ell,k}[$, where

$$2\mu_{\ell,k} = \int_{-\infty}^{+\infty} \frac{1}{(\rho x^{2k} + L)^{\ell}}\, dx$$
$$= \frac{1}{L^{\ell - \frac{1}{2k}} \rho^{\frac{1}{2k}}} \frac{(\ell - 1 - \frac{1}{2k})(\ell - 2 - \frac{1}{2k}) \ldots (1 - \frac{1}{2k})}{(\ell - 1)!} \frac{\pi}{k \sin\left(\frac{\pi}{2k}\right)}.$$

Then the Liénard equation (3.11) has a T-periodic solution for any $e(\cdot)$ of constant sign and period less than $2\mu_{\ell,k}$.

Remark 2. All the above examples cover the range $m \le n \le 2m+1$, being $2m+1$ and $2n+1$ the degrees of f and q, respectively. If we denote by $\bar{e} = \frac{1}{T}\int_0^T e(t)\,dt$, the mean value of $e(\cdot)$ and consider the decomposition $e(t) = \bar{e} + \tilde{e}(t)$, then, for $\tilde{e}$ fixed, we have that in all the cases from (ii) to (v) it is possible to prove the existence of T-periodic solutions, without any restriction on the period, if $|\bar{e}|$ is sufficiently large.

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