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On a quasilinear parabolic–hyperbolic system arising in MEMS modeling

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Abstract

A coupled system consisting of a quasilinear parabolic equation and a semilinear hyperbolic equation is considered. The problem arises as a small aspect ratio limit in the modeling of a MEMS device taking into account the gap width of the device and the gas pressure. The system is regarded as a special case of a more general setting for which local well-posedness of strong solutions is shown. The general result applies to different cases including a coupling of the parabolic equation to a semilinear wave equation of either second or fourth order, the latter featuring either clamped or pinned boundary conditions.

Keywords Quasilinear parabolic equation · Wave equation · Well-posedness

Mathematics Subject Classification 35K59 · 35L05 · 35A01

1 Introduction

Electrostatically actuated micro-electromechanical systems (MEMS) are ubiquitous in today's electronic devices. Idealized MEMS often consist of a fixed ground plate and an elastic membrane (or plate) that are close. Keeping the two components at different potential induces a Coulomb force deflecting the membrane. In the past two decades MEMS devices have been a highly active mathematical research focus, in particular due to their interesting qualitative behaviors with respect to pull-in instabilities (as a result from a possible touching of membrane and ground plate) and the inherent challenges related to local and global well-posedness of the corresponding models. We refer to [7] and the references therein for more details on MEMS models and their mathematical investigation in general.

In this paper, we consider a model introduced in [5, 6] arising as a small aspect ratio limit of equations governing an electrostatically actuated MEMS, where the narrow gap separating the membrane and the ground plate is filled with a rarefied gas. More precisely, we consider

$$\partial_t(wu) = \operatorname{div}\big(w^3 u \nabla u\big), \quad t>0, \quad x\in\Omega\,, \tag{1.1a}$$

$$\partial_t^2 w + \sigma\partial_t w = \Delta w - \frac{a}{w^2} + b(u-1), \quad t>0, \quad x\in\Omega, \tag{1.1b}$$

$$u(t,x)=\theta_1, \quad w(t,x)=\theta_2, \quad t>0, \quad x\in\partial\Omega, \tag{1.1c}$$

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$$u(0,x)=u_0(x), \quad w(0,x)=w_0(x), \quad \partial_t w(0,x)=w_0'(x), \quad x\in\Omega, \tag{1.1d}$$

where $w = w(t,x)$ denotes the varying width of the gap and $u = u(t,x)$ is the local pressure of the gas. The (sufficiently) smooth, bounded open subset Ω of $\mathbb{R}^n$ with $n \in \{1,2\}$ represents the shape of the membrane and the ground plate. The constants $a, b, \theta_1, \theta_2 > 0$ and $\sigma \geq 0$ in (1.1b) and (1.1c) as well as the initial data u_0, w_0, and w_0' in (1.1d) are given. The degeneracy of (1.1a) and the singularity in (1.1b) occurring for a vanishing gap width $w(t,x) = 0$ capture the instabilities related to a touchdown of the membrane on the ground plate. A detailed account of the modeling aspects is given in [5, 6] to which we refer.

In [5] the short-time existence of solutions to this MEMS model is established for the one-dimensional case $n = 1$. The approach chosen therein consists of solving first the hyperbolic equation for w (via a fixed point argument for a given, fixed u) and so reducing the coupled system (1.1) to a single fixed point equation for u which is then solved using parabolic semigroup theory. Instead of decoupling the system, we proceed differently and solve the mild formulation of (1.1) simultaneously for u and w, also relying on semigroup theory for semilinear hyperbolic and quasilinear parabolic equations described in [4, 8] respectively [2, 3]. A key ingredient for this is the observation that mild solutions to the hyperbolic equation (1.1b) enjoy *a priori* Hölder continuity properties with respect to time (and values in spaces of sufficiently high spatial regularity) that guarantee an evolution operator for the quasilinear parabolic equation (1.1a) in the sense of [3] (see also Remark 3.2 and Proposition A.1 below). In this way we provide a considerably shorter proof for local existence including also the case $n = 2$:

Theorem 1.1 *Let $r > 0$ and $u_0 \in H^{2+r}(\Omega)$ with $u_0 > 0$ in Ω and $u_0 = \theta_1$ on $\partial\Omega$. Let $w_0 \in H^2(\Omega) \cap C^1(\bar{\Omega})$, $w_0' \in H^1(\Omega)$ with $w_0 > 0$ in Ω and $w_0 - \theta_2 = w_0' = 0$ on $\partial\Omega$. Then there is a unique solution*

$$u \in C^1\big([0,T], L_2(\Omega)\big) \cap C\big([0,T], H^2(\Omega)\big),$$
$$w \in C^2\big([0,T], L_2(\Omega)\big) \cap C^1\big([0,T], H^1(\Omega)\big) \cap C\big([0,T], H^2(\Omega)\big)$$

to (1.1) on some interval $[0,T]$.

The initial values are compatible with the Dirichlet boundary conditions and the regularity of strong solutions at $t = 0$. In fact, the solution component u has even better regularity properties than stated in Theorem 1.1, see Remark 4.1. Moreover, the solution can be extended to a maximal solution on $[0, T_{max})$ existing as long as $u(t) > 0$ and $w(t) > 0$ in $\bar{\Omega}$ as well as the C^1-norm of (u, w) does not blow up. It is worth pointing out that a common feature in MEMS models [7] is the possible occurrence of finite time quenching $\inf_{x\in\Omega} w(t,x) \to 0$ as $t \to T_{max}$ preventing global existence of solutions.

A similar result as Theorem 1.1 can also be shown for a related fourth-order equation when the Laplacian Δ in (1.1b) is replaced by $-\Delta^2 + \Delta$ subject to pinned or clamped boundary conditions (see Theorem 5.1 below for details). This corresponds to a MEMS device involving an elastic plate instead of a membrane.

In fact, Theorem 1.1 (and Theorem 5.1) can be regarded as a special case of a more general setting including a quasilinear parabolic equation coupled to a semilinear wave equation of the form

$$\partial_t u = \mathcal{A}(u,w)u + g(u,w,\partial_t w), \quad t>0, \quad u(0)=u_0, \tag{1.2a}$$

$$\partial_t^2 w + \sigma\partial_t w = -Aw + f(u,w,\partial_t w), \quad t>0, \quad (w(0),\partial_t w(0)) = (w_0, w_0'), \tag{1.2b}$$

where $\mathcal{A}(u, w)$ are generators of analytic semigroups on a Banach space and $-A$ is a generator of a cosine function on a Hilbert space (see Sect. 3 below for details).

In Sect. 2 we first identify (1.1) as a special case of (1.2), see (2.11) below. The latter is treated in Sect. 3 in an abstract functional analytic framework that is not restricted to the particular setting of (1.1). The main result of this research is Theorem 3.1 on the local well-posedness of (1.2) that is established using semigroup theory and then implies Theorem 1.1 for the particular case (1.1) as shown in Sect. 4. In Sect. 5 we briefly show how to apply Theorem 3.1 for the case of the fourth-order problem (5.1) including a bi-Laplacian.

2 Functional formulation of the problem

We demonstrate how to express the system (1.1) in the abstract form of problem (1.2) and list relevant properties of the functions involved.

Setting

$$(\bar{u}, \bar{w}) := (u - \theta_1, w - \theta_2), \quad (\bar{u}_0, \bar{w}_0) := (u_0 - \theta_1, w_0 - \theta_2)$$

and dropping then again the bars for simplicity, problem (1.1) is equivalent to

$$\partial_t u = \frac{1}{w+\theta_2}\mathrm{div}\big((w+\theta_2)^3(u+\theta_1)\nabla u\big) - \frac{\partial_t w\,(u+\theta_1)}{w+\theta_2}, \quad t > 0, \quad x \in \Omega, \tag{2.1a}$$

$$\partial_t^2 w + \sigma \partial_t w = \Delta w - \frac{a}{(w+\theta_2)^2} + b(u + \theta_1 - 1), \quad t > 0, \quad x \in \Omega, \tag{2.1b}$$

$$u(t,x) = w(t,x) = 0, \quad t > 0, \quad x \in \partial\Omega, \tag{2.1c}$$

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x), \quad \partial_t w(0,x) = w_0'(x), \quad x \in \Omega, \tag{2.1d}$$

as long as $w > -\theta_2$. In the following, we shall focus on (2.1) with initial data $u_0 \in H^{2+r}(\Omega)$, $w_0 \in H^2(\Omega) \cap C^1(\bar{\Omega})$, and $w_0' \in H^1(\Omega)$ satisfying $u_0 + \theta_1 > 0$ and $w_0 + \theta_2 > 0$ in $\bar{\Omega}$ and

$$u_0 = w_0 = w_0' = 0 \quad \text{on } \partial\Omega.$$

For technical reasons we handle the parabolic equation for u in an L_q-setting, denoting by $H_q^s(\Omega)$ the scale of Bessel potential spaces [2] (that coincide in the Hilbert space case $q = 2$ with the Sobolev–Slobodeckii spaces $H^s(\Omega) = H_2^s(\Omega)$).

Since $r > 0$ and $n \in \{1, 2\}$ we may choose $q \in (2, 4)$ and $\alpha, \beta, \mu \in (0, 1)$ such that

$$\begin{aligned} &\min\{r, 1/2\} > n/2 - n/q, \quad \alpha \in \big(n/2 - n/q, 1/2\big), \\ &\mu \in (2 - 2/q - \alpha, 3/2 - \alpha), \quad 2\beta \in (1, 2). \end{aligned} \tag{2.2}$$

Then, since $q > 2 \geq n$ and $\mu + \alpha > 2 - 2/q \geq 1 + n/2 - n/q$, we have

$$H_q^{2\beta}(\Omega) \hookrightarrow H_q^1(\Omega) \hookrightarrow C(\bar{\Omega}), \quad H^{1+\alpha}(\Omega) \hookrightarrow H^{\mu+\alpha}(\Omega) \hookrightarrow H_q^1(\Omega) \hookrightarrow C(\bar{\Omega}) \tag{2.3}$$

and

$$u_0 \in H^{2+r}(\Omega) \hookrightarrow H_q^2(\Omega) \hookrightarrow C^1(\bar{\Omega}), \quad w_0 \in H^2(\Omega) \hookrightarrow H^{1+\alpha}(\Omega)$$

with $u_0 + \theta_1 \geq 2\varsigma > 0$ and $w_0 + \theta_2 \geq 2\varsigma > 0$ in $\bar{\Omega}$ for some $\varsigma > 0$. Hence we may choose $\varepsilon > 0$ such that

$$u + \theta_1 \geq \varsigma \quad \text{in } \bar{\Omega}, \quad u \in \mathbb{B}_{H_q^{2\beta}}(u_0, \varepsilon), \tag{2.4a}$$

$$w+\theta_2 \geq \varsigma \quad \text{in } \bar{\Omega}, \quad w \in \mathbb{B}_{H^{\mu+\alpha}}(w_0,\varepsilon). \tag{2.4b}$$

Set

$$\begin{aligned}\mathcal{A}(u,w)v &:= \frac{1}{w+\theta_2}\mathrm{div}\big((w+\theta_2)^3(u+\theta_1)\nabla v\big)\\ &= \mathrm{div}\big((w+\theta_2)^2(u+\theta_1)\nabla v\big) + (w+\theta_2)(u+\theta_1)\nabla w\cdot\nabla v \end{aligned}\tag{2.5}$$

for $u, w \in C^1(\bar{\Omega})$ and v belonging to

$$H^2_{q,D}(\Omega) := H^2_q(\Omega)\cap \mathring{H}^1_q(\Omega)\,,$$

[i.e. $H^2_{q,D}(\Omega)$ incorporates homogeneous Dirichlet boundary conditions]. Using the embeddings (2.3) along with the fact that $H^1_q(\Omega)$ is an algebra (since $q > n$) we obtain that the pointwise multiplications

$$H^{\mu+\alpha}(\Omega)\bullet H^{\mu+\alpha}(\Omega)\bullet H^{2\beta}_q(\Omega)\bullet H^1_q(\Omega)\longrightarrow H^1_q(\Omega)$$

and, choosing $\epsilon > 0$ small with $\mu+\alpha-1-\epsilon > n/2-n/q$,

$$H^{\mu+\alpha}(\Omega)\bullet H^{2\beta}_q(\Omega)\bullet H^{\mu+\alpha-1}(\Omega)\bullet H^1_q(\Omega)\longrightarrow H^{\mu+\alpha-1-\epsilon}(\Omega)\hookrightarrow L_q(\Omega)$$

are continuous (see [1, Theorem 4.1] for the latter). Consequently, using the first multiplication for the divergence term and the second multiplication for the first-order term of $\mathcal{A}(u,w)v$ in (2.5), we derive that

$$\mathcal{A}\in C^{1-}\big(H^{2\beta}_q(\Omega)\times H^{\mu+\alpha}(\Omega),\mathcal{L}\big(H^2_{q,D}(\Omega),L_q(\Omega)\big)\big), \tag{2.6a}$$

where C^{1-} means (locally) Lipschitz continuous. Moreover, since $u_0, w_0 \in C^1(\bar{\Omega})$ satisfy (2.4), $\mathcal{A}(u_0,w_0)$ is a second-order normally elliptic differential operator in divergence form with C^1-coefficients. Hence, it follows from e.g. [2, Theorem 4.1, Examples 4.3] that

$$\mathcal{A}(u_0,w_0)\in\mathcal{H}\big(H^2_{q,D}(\Omega),L_q(\Omega)\big), \tag{2.6b}$$

where $\mathcal{H}(E_1,E_0)$ denotes the set of generators of analytic semigroups on the Banach space E_0 with domain E_1. Also observe the identities

$$H^{2\theta}_{q,D}(\Omega) := \big[L_q(\Omega),H^2_{q,D}(\Omega)\big]_\theta \doteq \begin{cases}\{v\in H^{2\theta}_q(\Omega);\ v=0 \text{ on } \partial\Omega\}, & 1/q<2\theta\leq 2,\\ H^{2\theta}_q(\Omega), & 0\leq 2\theta<1/q,\end{cases} \tag{2.7}$$

for complex interpolation [2, Theorem 5.2]. Setting formally

$$g(u,w,w') := -\frac{w'(u+\theta_1)}{w+\theta_2}, \tag{2.8}$$

we may now reformulate (2.1a) subject to the initial and boundary conditions as quasilinear parabolic problem

$$\partial_t u = \mathcal{A}(u,w)u + g(u,w,\partial_t w),\quad t>0,\quad u(0)=u_0,$$

in $L_q(\Omega)$.

We focus next on the hyperbolic problem for w. Let $H^{2\theta}_D(\Omega) := H^{2\theta}_{2,D}(\Omega)$ be as in (2.7). Clearly, the Laplacian subject to homogeneous Dirichlet boundary conditions

$$-A := \Delta \in \mathcal{H}\big(H^2_D(\Omega),L_2(\Omega)\big)$$

is the generator of an analytic semigroup on $L_2(\Omega)$. In fact,

$$\begin{aligned} &A : H^2_D(\Omega) \subset L_2(\Omega) \to L_2(\Omega) \text{ is a closed, densely defined,} \\ &\quad \text{self-adjoint, positive operator with compact inverse.} \end{aligned} \tag{2.9}$$

Introducing

$$f(u,w) := -\frac{a}{(w+\theta_2)^2} + b(u+\theta_1-1), \tag{2.10}$$

we can rewrite (2.1) now in the form

$$\partial_t u = \mathcal{A}(u,w)u + g(u,w,\partial_t w), \quad t>0, \quad u(0)=u_0, \tag{2.11a}$$

$$\partial_t^2 w + \sigma\partial_t w = -Aw + f(u,w), \quad t>0, \quad (w(0),\partial_t w(0)) = \big(w_0,w_0'\big). \tag{2.11b}$$

To handle the semilinear terms we define the open subsets

$$O_\beta := \mathbb{B}_{H^{2\beta}_{q,D}}(u_0,\varepsilon), \quad \mathbb{O}_\alpha := \mathbb{B}_{H^{1+\alpha}_D}(w_0,\varepsilon/c_0) \times H^\alpha_D(\Omega)$$

of $H^{2\beta}_{q,D}(\Omega)$ respectively $H^{1+\alpha}_D(\Omega) \times H^\alpha_D(\Omega)$, where $\varepsilon > 0$ stems from (2.4) and $c_0 > 0$ denotes the norm of the embedding $H^{1+\alpha}_D(\Omega) \hookrightarrow H^{\mu+\alpha}_D(\Omega)$. Recalling $\alpha > n/2 - n/q$ we find $2\eta \in (0, 1/q)$ and $\epsilon > 0$ small with $\alpha - \epsilon - n/2 > 2\eta - n/q$, hence the embedding

$$H^{\alpha-\epsilon}(\Omega) \hookrightarrow H^{2\eta}_q(\Omega) = H^{2\eta}_{q,D}(\Omega).$$

Therefore, noticing that

$$(u+\theta_1)(w+\theta_2)^{-1} \in H^1_q(\Omega), \quad u \in O_\beta, \quad (w,w') \in \mathbb{O}_\alpha,$$

due to (2.3) and (2.4b), it follows from the continuity of the multiplication (see [1, Theorem 4.1])

$$H^1_q(\Omega) \bullet H^\alpha(\Omega) \longrightarrow H^{\alpha-\epsilon}(\Omega) \hookrightarrow H^{2\eta}_{q,D}(\Omega)$$

and (2.8) that

$$g \in C^{1-}\big(O_\beta \times \mathbb{O}_\alpha, H^{2\eta}_{q,D}(\Omega)\big), \tag{2.12}$$

while (2.3), (2.4b), (2.10), and

$$H^1_q(\Omega) \hookrightarrow H^\alpha(\Omega) = H^\alpha_D(\Omega)$$

(since $q > 2$ and $\alpha < 1/2$) ensure that (of course, f is independent of the w'-component)

$$f \in C^{1-}\big(O_\beta \times \mathbb{O}_\alpha, H^\alpha_D(\Omega)\big). \tag{2.13}$$

To guarantee later on sufficient regularity of solutions it is worth noting that

$$f(\hat u,\hat w) = -\frac{a}{\big(\hat w+\theta_2\big)^2} + b(\hat u+\theta_1-1) \in C^1\big([0,T], L_2(\Omega)\big) \tag{2.14}$$

whenever $\hat u \in C^1([0,T], L_2(\Omega))$ and $\hat w \in C([0,T], H^{1+\alpha}(\Omega)) \cap C^1([0,T], H^\alpha(\Omega))$ with $\hat w(t) + \theta_2 \geq \varsigma$ in Ω. Also note from (2.2) tha $H^{2+r}(\Omega) \hookrightarrow H^2_q(\Omega)$

$$u_0 \in O_\beta \cap H^2_{q,D}(\Omega), \quad \big(w_0,w_0'\big) \in \mathbb{O}_\alpha \cap \big(H^2_D(\Omega) \times H^1_D(\Omega)\big), \tag{2.15}$$

by the assumptions of Theorem 1.1 and (2.3). In fact, since $u_0, w_0 \in H^{2+r}(\Omega)$ and since $H^{1+r}(\Omega)$ is an algebra, it readily follows from (2.5) that

$$u_0 \in H^2_{q,D}(\Omega), \quad \mathcal{A}(u_0, w_0)u_0 \in H^r(\Omega) \hookrightarrow H^{2\eta}_{q,D}(\Omega) \tag{2.16}$$

since we may make $\eta > 0$ smaller to guarantee $0 < 2\eta < \min\{r - n/2 + n/q, 1/q\}$. That is, the initial value u_0 belongs to the domain of the $H^{2\eta}_{q,D}(\Omega)$-realization of the generator $\mathcal{A}(u_0, w_0) \in \mathcal{H}(H^2_{q,D}(\Omega), L_q(\Omega))$.

The previous considerations ensure that problem (2.11) (and thus problem (1.1)) fits into the more general framework of Theorem 3.1 of the next section. We shall then continue from here in Sect. 4 and finish off the proof of Theorem 1.1 by applying Theorem 3.1.

3 Main theorem

As just pointed out above, Theorem 1.1 is a special case of a more general setting: Consider

$$\partial_t u = \mathcal{A}(u, w)u + g(u, w, \partial_t w), \quad t > 0, \quad u(0) = u_0, \tag{3.1a}$$

$$\partial_t^2 w + \sigma \partial_t w = -Aw + f(u, w, \partial_t w), \quad t > 0, \quad (w(0), \partial_t w(0)) = \left(w_0, w_0'\right), \tag{3.1b}$$

where $\mathcal{A}(u, w) \in \mathcal{L}(E_1, E_0)$ for some continuously and densely injected Banach couple $E_1 \hookrightarrow E_0$ is such that $\mathcal{A}(u_0, w_0) \in \mathcal{H}(E_1, E_0)$, i.e. $\mathcal{A}(u_0, w_0)$ with domain E_1 generates an analytic semigroup on E_0], and

$$\begin{aligned} &A : D(A) \subset H \to H \text{ is a closed, densely defined, self-adjoint,} \\ &\quad \text{positive operator with bounded and compact inverse} \end{aligned} \tag{3.2}$$

on a Hilbert space H with scalar product $(\cdot|\cdot)$. Here, positive operator means that $(Ax|x) \geq 0$ for $x \in D(A)$. Let $\sigma \geq 0$.

We formulate (3.1) as a coupled system of two first order equations relying on results for cosine functions [4, Sections 5.5 and 5.6], see also Appendix A. To this end note that (3.2) ensures that the powers A^z for $z \in \mathbb{C}$ are well-defined closed operators (bounded for $\operatorname{Re} z \leq 0$). Consequently, the matrix operator

$$\mathbb{A} := \begin{pmatrix} 0 & 1 \\ -A & -\sigma \end{pmatrix}, \quad D(\mathbb{A}) := D(A) \times D(A^{1/2}),$$

generates a strongly continuous semigroup $(e^{t\mathbb{A}})_{t\geq 0}$ on the Hilbert space

$$\mathbb{H} := D(A^{1/2}) \times H$$

[in fact, it generates a group $(e^{t\mathbb{A}})_{t\in\mathbb{R}}$]. Using the notion $\mathbf{w} = (w, w')$ and setting

$$\mathbf{F}(u, \mathbf{w}) := \begin{pmatrix} 0 \\ f(u, w, w') \end{pmatrix},$$

we can write (3.1b) as a semilinear hyperbolic Cauchy problem

$$\partial_t \mathbf{w} = \mathbb{A}\mathbf{w} + \mathbf{F}(u, \mathbf{w}), \quad t > 0, \quad \mathbf{w}(0) = \mathbf{w}_0 := \left(w_0, w_0'\right),$$

in $\mathbb{H}$. In fact, for greater flexibility (and to cope with the particular case (2.1)) we shift this problem to the interpolation space (for some $\alpha \in [0, 1)$)

$$\mathbb{H}_\alpha := [\mathbb{H}, D(\mathbb{A})]_\alpha \doteq D(A^{(1+\alpha)/2}) \times D(A^{\alpha/2}),$$

where we recall (due to the Fourier series representation of A^α or [9, Theorem 1.15.3]) that

$$\big[D(A^{\alpha_0}), D(A^{\alpha_1})\big]_\theta \doteq D\big(A^{(1-\theta)\alpha_0+\theta\alpha_1}\big), \quad \theta \in [0,1], \quad 0 \le \alpha_0 < \alpha_1. \tag{3.3}$$

Then, the $\mathbb{H}_\alpha$-realization $\mathbb{A}_\alpha$ of $\mathbb{A}$, given by

$$\mathbb{A}_\alpha \mathbf{w} := \mathbb{A}\mathbf{w}, \quad \mathbf{w} \in D(\mathbb{A}_\alpha) := \{\mathbf{w} \in D(\mathbb{A});\ \mathbb{A}\mathbf{w} \in \mathbb{H}_\alpha\} = D(A^{1+\alpha/2}) \times D(A^{(1+\alpha)/2}),$$

generates a strongly continuous semigroup $(e^{t\mathbb{A}_\alpha})_{t\ge 0}$ on $\mathbb{H}_\alpha$ according to [3, Chapter V]. We shall then consider (3.1) in the equivalent form

$$\partial_t u = \mathcal{A}(u, w)u + g(u, w, \partial_t w), \quad t > 0, \quad u(0) = u_0, \tag{3.4a}$$

$$\partial_t \mathbf{w} = \mathbb{A}_\alpha \mathbf{w} + \mathbf{F}(u, \mathbf{w}), \quad t > 0, \quad \mathbf{w}(0) = \mathbf{w}_0 = \big(w_0, w_0'\big). \tag{3.4b}$$

In the following, let $(\cdot,\cdot)_\theta$ be arbitrary admissible interpolation functors [3, I.Section 2.11] and set

$$E_\theta := (E_0, E_1)_\theta, \quad \theta \in [0,1].$$

Let $O_\beta \subset E_\beta$ and $\mathbb{O}_\alpha \subset \mathbb{H}_\alpha$ be open sets for some $\alpha, \beta \in [0,1)$.

Theorem 3.1 *Let $\alpha, \beta, \mu \in [0,1)$ and $\tau \in (\beta, 1]$. Consider initial values $u_0 \in O_\beta \cap E_\tau$ and $(w_0, w_0') \in \mathbb{O}_\alpha$, let*

$$\mathcal{A} \in C^{1-}\big(E_\beta \times D\big(A^{(\alpha+\mu)/2}\big), \mathcal{L}(E_1, E_0)\big), \quad \mathcal{A}(u_0, w_0) \in \mathcal{H}(E_1, E_0), \tag{3.5}$$

and suppose (3.2). Moreover, assume that

$$g \in C^{1-}(O_\beta \times \mathbb{O}_\alpha, E_0), \quad f \in C^{1-}\big(O_\beta \times \mathbb{O}_\alpha, D(A^{\alpha/2})\big). \tag{3.6}$$

(a) *There is a unique mild solution*

$$u \in C\big([0,T], E_\beta\big), \quad \mathbf{w} = (w, \partial_t w) \in C\big([0,T], D(A^{(1+\alpha)/2}) \times D(A^{\alpha/2})\big) \tag{3.7}$$

to the Cauchy problem (3.4) on some interval $[0,T]$.

(b) *If*

$$g \in C(O_\beta \times \mathbb{O}_\alpha, E_\eta) \quad \text{for some } \eta > 0 \tag{3.8}$$

or if

$$g \in C^{1-}\big(O_\beta \times D(A^{(\alpha+\mu)/2}), E_0\big) \quad \text{is independent of } w', \tag{3.9}$$

then $u \in C^1((0,T], E_0) \cap C((0,T], E_1)$ is a strong solution to (3.1a).

(c) *Let $u_0 \in E_1$. If (3.8) is satisfied and*

$$\mathcal{A}(u_0, w_0)u_0 \in E_\eta \tag{3.10}$$

or if (3.9) is satisfied, then

$$u \in C^1([0,T], E_0) \cap C([0,T], E_1) \tag{3.11}$$

is a strict solution to (3.1a). In this case, if

$$f(u, w, \partial_t w) \in C\big([0,T], D(A^{1/2})\big) \tag{3.12}$$

or if

$$f \text{ is independent of } w' \quad \text{and} \quad f(u,w) \in C^1([0,T], H), \tag{3.13}$$

then

$$w \in C^2\big([0,T],H\big) \cap C^1\big([0,T],D(A^{1/2})\big) \cap C\big([0,T],D(A)\big)$$

is a strong solution to (3.1b) provided that $(w_0, w_0') \in \mathbb{O}_\alpha \cap D(A) \times D(A^{1/2})$.

We emphasize that one may rely on the regularity properties (3.7) and (3.11) when checking (3.12) or (3.13).

Proof **(i)** It follows from (3.5) and [3, I.Theorem 1.3.1] that there are $\varepsilon > 0$, $\kappa \geq 1$, and $\omega > 0$ such that[1]

$$\mathcal{A}(u,w) \in \mathcal{H}(E_1, E_0; \kappa, \omega), \quad (u,w) \in \bar{\mathbb{B}}_{E_\beta \times D(A^{(\alpha+\mu)/2})}((u_0, w_0), \varepsilon), \tag{3.14}$$

and

$$\begin{aligned} &\|\mathcal{A}(u,w) - \mathcal{A}(\hat{u},\hat{w})\|_{\mathcal{L}(E_1,E_0)} \leq c\,\|(u,w)-(\hat{u},\hat{w})\|_{E_\beta \times D(A^{(\alpha+\mu)/2})} \\ &\text{for } (u,w), (\hat{u},\hat{w}) \in \bar{\mathbb{B}}_{E_\beta \times D(A^{(\alpha+\mu)/2})}\big((u_0,w_0),\varepsilon\big), \end{aligned} \tag{3.15}$$

for some constant $c = c(u_0, w_0) > 0$. Let $\rho \in (0, \min\{\tau - \beta, 1-\mu\})$ and let

$$c_* := \max\left\{1, \|i\|_{\mathcal{L}(D(A^{(\alpha+1)/2}), D(A^{(\alpha+\mu)/2}))}\right\},$$

where $i : D(A^{(\alpha+1)/2}) \hookrightarrow D(A^{(\alpha+\mu)/2})$ is the natural inclusion. Writing $z = (u, \mathbf{w})$ with $\mathbf{w} = (w, w')$ in the following and noticing that $z_0 = (u_0, \mathbf{w}_0) \in O_\beta \times \mathbb{O}_\alpha$ with open subsets $O_\beta \subset E_\beta$ and $\mathbb{O}_\alpha \subset \mathbb{H}_\alpha$, it follows from (3.6) that we may assume without loss of generality that

$$\|g(z) - g(\hat{z})\|_{E_0} + \|\mathbf{F}(z) - \mathbf{F}(\hat{z})\|_{\mathbb{H}_\alpha} \leq c_1 \|z - \hat{z}\|_{E_\beta \times \mathbb{H}_\alpha}, \quad z, \hat{z} \in \bar{\mathbb{B}}_{E_\beta \times \mathbb{H}_\alpha}(z_0, \varepsilon/c_*), \tag{3.16}$$

for some constant $c_1 = c_1(z_0) > 0$ and that

$$\|(u,w) - (u_0,w_0)\|_{E_\beta \times D(A^{(\alpha+\mu)/2})} \leq \varepsilon, \quad z \in \bar{\mathbb{B}}_{E_\beta \times \mathbb{H}_\alpha}(z_0, \varepsilon/c_*).$$

Given $T \in (0,1)$ (to be specified later) we then introduce

$$\begin{aligned} \mathcal{V}_T := \Big\{ &z = (u, \mathbf{w}) \in C\big([0,T], E_\beta \times \mathbb{H}_\alpha\big);\ w \in C^1\big([0,T], D(A^{\alpha/2})\big), \\ &\|z(t) - z_0\|_{E_\beta \times \mathbb{H}_\alpha} \leq \varepsilon/c_*, \ \ \|u(t) - u(s)\|_{E_\beta} \leq |t-s|^\rho, \\ &\|\partial_t w(t)\|_{D(A^{\alpha/2})} \leq \|\mathbf{w}_0\|_{\mathbb{H}_\alpha} + \varepsilon/2c_*, \quad t,s \in [0,T] \Big\}, \end{aligned}$$

where $z_0 = (u_0, \mathbf{w}_0)$ and the notation $\mathbf{w} = (w, w')$ is used throughout. Then $\mathcal{V}_T$ is a complete metric space when equipped with the metric

$$d_{\mathcal{V}_T}(z, \hat{z}) := \|z - \hat{z}\|_{C([0,T], E_\beta \times \mathbb{H}_\alpha)} + \|\partial_t w - \partial_t \hat{w}\|_{C([0,T], D(A^{\alpha/2}))}.$$

Then, for $z = (u, \mathbf{w}) \in \mathcal{V}_T$, we have by interpolation [see (3.3)]

$$w \in C\big([0,T], D(A^{(1+\alpha)/2})\big) \cap C^1\big([0,T], D(A^{\alpha/2})\big) \hookrightarrow C^{1-\mu}\big([0,T], D(A^{(\alpha+\mu)/2})\big)$$

[1] The notation $\mathcal{A} \in \mathcal{H}(E_1, E_0; \kappa, \omega)$ means that $\omega - \mathcal{A} \in \mathcal{L}is(E_1, E_0)$ and

$$\kappa^{-1} \leq \frac{\|(\lambda - \mathcal{A})x\|_{E_0}}{|\lambda|\,\|x\|_{E_0} + \|x\|_{E_1}} \leq \kappa, \quad x \in E_1 \setminus \{0\}, \quad \operatorname{Re}\lambda \geq \omega,$$

see [3, I.Section 1.2]. Note that $\mathcal{H}(E_1, E_0; \kappa, \omega) \subset \mathcal{H}(E_1, E_0)$.

(3.17)

and it thus follows from (3.15) and $\rho < 1 - \mu$ that

$$\sup_{0 \le s < t \le T} \frac{\|\mathcal{A}(u(t), w(t)) - \mathcal{A}(u(s), w(s))\|_{\mathcal{L}(E_1, E_0)}}{|t-s|^\rho} \le r(u_0, \mathbf{w}_0) \tag{3.18a}$$

for some constant $r(u_0, \mathbf{w}_0) > 0$ (independent of $z \in \mathcal{V}_T$) and from (3.14) that

$$\mathcal{A}(u(t), w(t)) \in \mathcal{H}(E_1, E_0; \kappa, \omega), \quad t \in [0, T]. \tag{3.18b}$$

Now, [3, II.Corollary 4.4.2] and (3.18) imply that for each $z = (u, \mathbf{w}) \in \mathcal{V}_T$, the operator $\mathcal{A}(u, w)$ generates a parabolic evolution operator

$$U_{\mathcal{A}(u,w)}(t, s), \quad 0 \le s \le t \le T,$$

on E_0 with regularity subspace E_1 and that we may apply the results of [3, II.Section 5]. Introduce now

$$\Gamma_1(z)(t) := U_{\mathcal{A}(u,w)}(t, 0)u_0 + \int_0^t U_{\mathcal{A}(u,w)}(t, s)\, g(z(s))\, \mathrm{d}s, \tag{3.19a}$$

$$\Gamma_2(z)(t) := e^{t\mathbb{A}_\alpha}\mathbf{w}_0 + \int_0^t e^{(t-s)\mathbb{A}_\alpha}\, \mathbf{F}(z(s))\, \mathrm{d}s, \tag{3.19b}$$

for $t \in [0, T]$ and $z = (u, \mathbf{w}) \in \mathcal{V}_T$ recalling $u_0 \in O_\beta \cap E_\tau$ and $\mathbf{w}_0 = (w_0, w_0') \in \mathbb{H}_\alpha$. Then, mild solutions to (3.4) correspond to fixed points of the operator $\Gamma = (\Gamma_1, \Gamma_2)$.

(ii) We claim that $\Gamma : \mathcal{V}_T \to \mathcal{V}_T$ is a contraction for $T \in (0, 1)$ sufficiently small. To see this, let $z = (u, \mathbf{w}) \in \mathcal{V}_T$. It then follows from (3.19a), (3.16), (3.18), and [3, II.Theorem 5.3.1] that (for some $c > 0$ depending only on the parameters in (3.18))

$$\|\Gamma_1(z)(t) - \Gamma_1(z)(s)\|_{E_\beta} \le c\,(t-s)^{\tau-\beta}\left(\|u_0\|_{E_\tau} + \|g(z)\|_{C([0,T],E_0)}\right) \le |t-s|^\rho$$

(we recall that $\rho < \tau - \beta$) and thus, in particular,

$$\|\Gamma_1(z)(t) - u_0\|_{E_\beta} \le T^\rho \le \frac{\varepsilon}{2c_*}$$

for $0 \le s \le t \le T$ with $T \in (0, 1)$ sufficiently small. Moreover, we deduce from (3.16) that $\mathbf{F}(z) \in C([0, T], \mathbb{H}_\alpha)$ so that (3.19b), the assumption $\mathbf{w}_0 \in \mathbb{H}_\alpha$, and Proposition A.1 entail that

$$\Gamma_2(z) = \left(\Gamma_2(z), \Gamma_2(z)'\right) \in C([0, T], \mathbb{H}_\alpha)$$

with

$$\Gamma_2(z) \in C^1\big([0, T], D(A^{\alpha/2})\big), \quad \partial_t \Gamma_2(z) = \Gamma_2(z)', \tag{3.20}$$

and

$$\|\Gamma_2(z)(t) - \mathbf{w}_0\|_{\mathbb{H}_\alpha} \le \|e^{t\mathbb{A}_\alpha}\mathbf{w}_0 - \mathbf{w}_0\|_{\mathbb{H}_\alpha} + \int_0^t \|e^{(t-s)\mathbb{A}_\alpha}\|_{\mathcal{L}(\mathbb{H}_\alpha)}\, \|\mathbf{F}(z(s))\|_{\mathbb{H}_\alpha}\, \mathrm{d}s \le \frac{\varepsilon}{2c_*}$$

for $t \in [0, T]$ with $T \in (0, 1)$ sufficiently small since $(e^{t\mathbb{A}_\alpha})_{t \ge 0}$ is strongly continuous on $\mathbb{H}_\alpha$. In particular, (3.20) implies

$$\|\partial_t \Gamma_2(z)(t)\|_{D(A^{\alpha/2})} \le \|\Gamma_2(z)(t)\|_{\mathbb{H}_\alpha} \le \|\mathbf{w}_0\|_{\mathbb{H}_\alpha} + \frac{\varepsilon}{2c_*}, \quad t \in [0, T].$$

Consequently, $\Gamma : \mathcal{V}_T \to \mathcal{V}_T$ is well-defined.

(iii) To check the Lipschitz property consider $z = (u, \mathbf{w}) \in \mathcal{V}_T$ and $\hat{z} = (\hat{u}, \hat{\mathbf{w}}) \in \mathcal{V}_T$. Then (3.18) and [3, II.Theorem 5.2.1] imply

$$\begin{aligned}
&\|\Gamma_1(z)(t) - \Gamma_1(\hat{z})(t)\|_{E_\beta} \\
&\quad \le c\Big\{t^{\tau-\beta}\|\mathcal{A}(u,w) - \mathcal{A}(\hat{u},\hat{w})\|_{C([0,T],\mathcal{L}(E_1,E_0))}\big[\|u_0\|_{E_\tau} + t^{1-\tau}\|g(z)\|_{C([0,T],E_0)}\big] \\
&\qquad + t^{1-\beta}\|g(z) - g(\hat{z})\|_{C([0,T],E_0)}\Big\} \\
&\quad \le \frac{1}{4} d_{\mathcal{V}_T}(z,\hat{z})
\end{aligned}$$

for $t \in [0, T]$ with $T \in (0, 1)$ sufficiently small, where we used (3.15) and (3.16) for the last estimate. Moreover, due to (3.16) we have

$$\|\Gamma_2(z)(t) - \Gamma_2(\hat{z})(t)\|_{\mathbb{H}_\alpha} \le \int_0^t \|e^{(t-s)\mathbb{A}_\alpha}\|_{\mathcal{L}(\mathbb{H}_\alpha)}\|\mathbf{F}(z(s)) - \mathbf{F}(\hat{z}(s))\|_{\mathbb{H}_\alpha}\,\mathrm{d}s \le \frac{1}{8} d_{\mathcal{V}_T}(z,\hat{z})$$

for $t \in [0, T]$ with $T \in (0, 1)$ sufficiently small. Consequently, taking into account (3.20), we deduce that

$$d_{\mathcal{V}_T}\big(\Gamma(z), \Gamma(\hat{z})\big) \le \frac{1}{2} d_{\mathcal{V}_T}\big(z, \hat{z}\big), \quad z, \hat{z} \in \mathcal{V}_T.$$

Banach's fixed point theorem now ensures that there is a unique $z = (u, \mathbf{w}) \in \mathcal{V}_T$ with $z = \Gamma(z)$; that is, $(u, \mathbf{w})$ is a mild solution to (3.1).

This proves statement **(a)** of Theorem 3.1.

(iv) Setting for $t \in [0, T]$

$$\tilde{\mathcal{A}}(t) := \mathcal{A}(u(t), w(t)), \quad \tilde{g}(t) := g(u(t), w(t), \partial_t w(t)), \tag{3.21a}$$

we see that u is a mild solution to the linear Cauchy problem

$$\partial_t u = \tilde{\mathcal{A}}(t)u + \tilde{g}(t), \quad t \in (0, T], \quad u(0) = u_0. \tag{3.21b}$$

If (3.8) holds, then $\tilde{g} \in C([0, T], E_\eta)$ with $\eta > 0$ and we infer from [3, II.Theorem 1.2.2] that

$$u \in C^1((0, T], E_0) \cap C((0, T], E_1)$$

is a strong solution to (3.1a). If (3.9) holds, then (3.16) and (3.17) imply $\tilde{g} \in C^\rho([0, T], E_0)$ and we obtain again that u is a strong solution to (3.1a) with regularity properties as above in view of [3, II.Theorem 1.2.1].

This proves statement **(b)** of Theorem 3.1.

(v) Let now $u_0 \in E_1$. Then (3.9) or (3.10) both imply (3.11) due to [3, II.Theorem 1.2.1] respectively [3, II.Theorem 1.2.2]; that is,

$$u \in C^1([0, T], E_0) \cap C([0, T], E_1)$$

is a strict solution to (3.1a). Set now $\tilde{\mathbf{F}}(t) := \mathbf{F}(z(t))$ and note that $\mathbf{w}$ is a mild solution to the linear Cauchy problem

$$\partial_t \mathbf{w} = \mathbb{A}\mathbf{w} + \tilde{\mathbf{F}}(t), \quad t \in (0, T], \quad \mathbf{w}(0) = \mathbf{w}_0.$$

Then (3.12) implies $\tilde{\mathbf{F}} \in C([0, T], D(\mathbb{A}))$ while (3.13) yields $\tilde{\mathbf{F}} \in C^1([0, T], \mathbb{H})$. In either case we derive from Proposition A.1 that

$$w \in C^2([0, T], H) \cap C^1\big([0, T], D(A^{1/2})\big) \cap C([0, T], D(A))$$

is a strong solution to (3.1b) provided that $(w_0, w_0') \in D(\mathbb{A})$. This proves statement **(c)** of Theorem 3.1. □

Remark 3.2 It is worth emphasizing that one of the key ingredients of the proof of Theorem 3.1 is the observation that the first component w of a *mild* solution $\mathbf{w} = (w, w')$ to the hyperbolic equation (3.4b) enjoys a Hölder continuity property with respect to time and values in spaces of sufficiently high spatial regularity [see (3.17)] as stated in Proposition A.1. In fact, this ensures the Hölder continuity of the operator $t \mapsto \mathcal{A}(u(t), w(t))$ and thus that the associated evolution operator is well-defined according to [3, II.Corollary 4.4.2].

4 Proof of Theorem 1.1

We can now complete the proof of Theorem 1.1. From Sect. 2 we know that problem (1.1) is equivalent to (2.1) (recalling that (u, w) is identified with $(u - \theta_1, w - \theta_2)$) which, in turn, is a special case of (3.1), see (2.11).

Choose $q \in (2, 4)$ and $\alpha, \beta, \mu \in (0, 1)$ as in (2.2) and $\eta \in (0, 1)$ as in (2.12) and (2.16). Setting

$$E_0 := L_q(\Omega), \quad E_1 := H^2_{q,D}(\Omega),$$

we notice from (2.7) that $E_\theta \doteq H^{2\theta}_{q,D}(\Omega)$ (with complex interpolation functor) for $2\theta \neq 1/q$ while the operator $A := -\Delta$ in $H := L_2(\Omega)$ with domain $H^2_D(\Omega)$ satisfies (3.2), see (2.9). Moreover, (2.7) and (3.3) imply

$$D(A^s) \doteq H^{2s}_D(\Omega), \quad s \in [0, 1], \quad 2s \neq 1/2. \tag{4.1}$$

It now follows from (2.6)–(2.16) that problem (2.11) (and thus problem (1.1)) fits into the framework of Theorem 3.1 with assumptions (3.5), (3.6), (3.8), (3.10), and (3.13) satisfied (and $\tau = 1$). Therefore, Theorem 3.1 implies that (2.1) admits a unique solution

$$\begin{aligned} &u \in C^1\big([0, T], L_q(\Omega)\big) \cap C\big([0, T], H^2_q(\Omega)\big), \\ &w \in C^2\big([0, T], L_2(\Omega)\big) \cap C^1\big([0, T], H^1(\Omega)\big) \cap C\big([0, T], H^2(\Omega)\big). \end{aligned}$$

Since $q > 2$, this proves Theorem 1.1. □

Remark 4.1 As shown above, u belongs in fact to $C^1([0, T], L_q(\Omega)) \cap C([0, T], H^2_q(\Omega))$ for some $q > 2$. Parabolic smoothing effects ensure additional regularity properties. For instance, the regularity of (u, w) stated in Theorem 1.1 implies that u solves a (linear) equation of the form (3.21) with $\tilde{\mathcal{A}} \in C^\rho([0, T], \mathcal{H}(H^2_D(\Omega), L_2(\Omega)))$ [see (3.18)] and $\tilde{g} \in C^\rho([0, T], L_2(\Omega))$ for some $\rho > 0$ [see (2.8)]. The maximal regularity result of [3, I.Theorem 1.2.1] yields $u \in C^\rho((0, T], H^2(\Omega)) \cap C^{1+\rho}((0, T], L_2(\Omega))$. Moreover, since $\tilde{g} \in C([0, T], H^{2\eta}_{q,D}(\Omega))$ by (2.12), a higher spatial regularity of u is derived from [3, I.Theorem 1.2.2] taking into account the regularity (2.16) of the initial value.

5 A fourth-order problem

As pointed out in the introduction, Theorem 3.1 also applies to certain fourth-order wave equations. Indeed, consider

$$\partial_t(wu) = \operatorname{div}\big(w^3 u \nabla u\big), \quad t > 0, \quad x \in \Omega, \tag{5.1a}$$

$$\partial_t^2 w + \sigma \partial_t w = -D_1 \Delta^2 w + D_2 \Delta w - \frac{a}{w^2} + b(u-1), \quad t > 0, \quad x \in \Omega, \tag{5.1b}$$

$$u(t,x) = \theta_1, \quad w(t,x) = \theta_2, \quad \mathcal{B}w(t,x) = 0, \quad t > 0, \quad x \in \partial\Omega, \tag{5.1c}$$

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x), \quad \partial_t w(0,x) = w_0'(x), \quad x \in \Omega, \tag{5.1d}$$

with

$$\mathcal{B}w := (1-\delta)\partial_\nu w + \delta \Delta w \quad \text{on } \partial\Omega, \tag{5.1e}$$

where $\delta \in \{0,1\}$. Equations (5.1) govern the gap width w and the gas pressure u for a MEMS device involving an elastic plate (instead of a membrane) of shape Ω, where $\Omega \subset \mathbb{R}^n$ with $n \in \{1,2\}$ is assumed to be a (sufficiently) smooth, bounded open set. The elastic plate is either clamped at its boundary (corresponding to $\delta = 0$) or is hinged along its boundary so that it is free to rotate (corresponding to $\delta = 1$). We assume that $D_1 > 0$ and $D_2 \geq 0$ and that $a, b, \theta_1, \theta_2 > 0$ and $\sigma \geq 0$.

This MEMS model was introduced in [6] and the short-time existence of solutions was established for the pinned case $\delta = 1$ (for both cases $n = 1, 2$). We derive the result for pinned and clamped boundary conditions simultaneously as a consequence of Theorem 3.1 (the assumptions on the initial values are compatible with the regularity of the solution):

Theorem 5.1 *Let $r > 0$ and $u_0 \in H^{2+r}(\Omega)$ with $u_0 > 0$ in Ω and $u_0 = \theta_1$ on $\partial\Omega$. Let $(w_0, w_0') \in H^4(\Omega) \times H^2(\Omega)$ with $w_0 > 0$ in Ω and $w_0 - \theta_2 = \mathcal{B}w_0 = 0$ on $\partial\Omega$ and $w_0' = (1-\delta)\mathcal{B}w_0' = 0$ on $\partial\Omega$. Then there is a unique solution*

$$u \in C^1\big([0,T], L_2(\Omega)\big) \cap C\big([0,T], H^2(\Omega)\big),$$
$$w \in C^2\big([0,T], L_2(\Omega)\big) \cap C^1\big([0,T], H^2(\Omega)\big) \cap C\big([0,T], H^4(\Omega)\big)$$

to (5.1) on some interval $[0,T]$.

Proof The proof is very much the same as for Theorem 1.1 and we thus only sketch it and point out the new aspects. Arguing as in Sect. 2 by shifting u and w we may focus on

$$\partial_t u = \mathcal{A}(u,w)u + g(u,w,\partial_t w), \quad t > 0, \quad u(0) = u_0, \tag{5.2a}$$

$$\partial_t^2 w + \sigma \partial_t w = -Aw + f(u,w), \quad t > 0, \quad (w(0), \partial_t w(0)) = \big(w_0, w_0'\big), \tag{5.2b}$$

with $\mathcal{A}$ defined in (2.5) and g and f in (2.8) respectively (2.10). The only difference now is that we consider the fourth-order operator

$$-A := -D_1\Delta^2 + D_2\Delta \in \mathcal{L}\big(H^4_{\mathcal{B}}(\Omega), L_2(\Omega)\big),$$

where its domain $H^4_{\mathcal{B}}(\Omega)$ incorporates the homogeneous pinned ($\delta = 0$) or clamped ($\delta = 1$) boundary conditions. More generally, we set for $s \in [0,4]$

$$H^s_{\mathcal{B}}(\Omega) := \begin{cases} \big\{v \in H^s(\Omega);\ v = \mathcal{B}v = 0 \text{ on } \partial\Omega\big\}, & s > 3/2 + \delta, \\ \big\{v \in H^s(\Omega);\ v = 0 \text{ on } \partial\Omega\big\}, & 1/2 < s < 3/2 + \delta, \\ H^s(\Omega), & 0 \leq s < 1/2. \end{cases}$$

Then $H^{4\theta}_{\mathcal{B}}(\Omega)$ coincides with the complex interpolation space

$$H^{4\theta}_{\mathcal{B}}(\Omega) \doteq \big[L_2(\Omega), H^4_{\mathcal{B}}(\Omega)\big]_\theta, \quad 4\theta \in [0,4] \setminus \{1/2, 3/2 + \delta\}, \tag{5.3}$$

up to equivalent norms, see [9, Theorem 4.3.3]. Moreover, $-A \in \mathcal{H}(H^4_{\mathcal{B}}(\Omega), L_2(\Omega))$ is the generator of an analytic semigroup on $L_2(\Omega)$ with exponential decay, e.g., see [2, Remarks 4.2] or [8, Theorem 7.2.7]. In fact, we have again that

$$A : H^4_{\mathcal{B}}(\Omega) \subset L_2(\Omega) \to L_2(\Omega) \text{ is a closed, densely defined, self-adjoint, positive operator with compact inverse.} \tag{5.4}$$

Moreover, (5.3) and (3.3) entail

$$D(A^s) \doteq H^{4s}_{\mathcal{B}}(\Omega), \quad s \in [0,1], \quad 4s \neq 1/2, 3/2+\delta. \tag{5.5}$$

As in Sect. 2 we choose $q > 2$ and $\alpha, \beta, \mu \in (0,1)$ such that

$$r \geq n/2 - n/q, \quad 2\alpha \in (n/2 - n/q, 1/2), \quad \mu \in (1-\alpha, 1), \quad 2\beta \in (1,2),$$

and define the open subsets

$$O_\beta := \mathbb{B}_{H^{2\beta}_{q,D}}(u_0, \varepsilon), \quad \mathbb{O}_\alpha := \mathbb{B}_{H^{2+2\alpha}_{\mathcal{B}}}(w_0, \varepsilon/c_0) \times H^{2\alpha}_{\mathcal{B}}(\Omega)$$

of $H^{2\beta}_{q,D}(\Omega)$ respectively $H^{2+2\alpha}_{\mathcal{B}}(\Omega) \times H^{2\alpha}_{\mathcal{B}}(\Omega)$, where $c_0 > 0$ denotes the norm of the embedding $H^{2+2\alpha}_{\mathcal{B}}(\Omega) \hookrightarrow H^{2\mu+2\alpha}_{\mathcal{B}}(\Omega)$ and $\varepsilon > 0$ is such that, for some $\varsigma > 0$,

$$u + \theta_1 \geq \varsigma \quad \text{in } \bar{\Omega}, \quad u \in O_\beta = \mathbb{B}_{H^{2\beta}_{q,D}}(u_0, \varepsilon),$$
$$w + \theta_2 \geq \varsigma \quad \text{in } \bar{\Omega}, \quad w \in \mathbb{B}_{H^{2\mu+2\alpha}}(w_0, \varepsilon).$$

Exactly as in Sect. 2 we have

$$\mathcal{A} \in C^{1-}\big(H^{2\beta}_q(\Omega) \times H^{2\mu+2\alpha}(\Omega), \mathcal{L}\big(H^2_{q,D}(\Omega), L_q(\Omega)\big)\big) \tag{5.6}$$

and

$$\mathcal{A}(u_0, w_0) \in \mathcal{H}\big(H^2_{q,D}(\Omega), L_q(\Omega)\big), \tag{5.7}$$

while

$$g \in C^{1-}\big(O_\beta \times \mathbb{O}_\alpha, H^{2\eta}_{q,D}(\Omega)\big), \quad f \in C^{1-}\big(O_\beta \times \mathbb{O}_\alpha, H^{2\alpha}_{\mathcal{B}}(\Omega)\big) \tag{5.8}$$

for some $\eta > 0$ small enough. Moreover,

$$f(\hat{u}, \hat{w}) = -\frac{a}{(\hat{w}+\theta_2)^2} + b(\hat{u}+\theta_1-1) \in C^1\big([0,T], L_2(\Omega)\big) \tag{5.9}$$

whenever $\hat{u} \in C^1([0,T], L_2(\Omega))$ and $\hat{w} \in C([0,T], H^{2+2\alpha}(\Omega)) \cap C^1([0,T], H^{2\alpha}(\Omega))$ with $\hat{w}(t) + \theta_2 \geq \varsigma$ in Ω. Finally, by premise of the theorem,

$$u_0 \in O_\beta \cap H^2_{q,D}(\Omega), \quad \big(w_0, w_0'\big) \in \mathbb{O}_\alpha \cap \big(H^4_{\mathcal{B}}(\Omega) \times H^2_{\mathcal{B}}(\Omega)\big) \tag{5.10}$$

and, as before, since $u_0, w_0 \in H^{2+r}(\Omega)$,

$$\mathcal{A}(u_0, w_0)u_0 \in H^{2\eta}_q(\Omega) = H^{2\eta}_{q,D}(\Omega) \tag{5.11}$$

making $\eta > 0$ smaller, if necessary, so that $0 < 2\eta < \min\{r, 1/q\}$. Setting

$$E_0 := L_q(\Omega), \quad E_1 := H^2_{q,D}(\Omega), \quad H := L_2(\Omega),$$

it then follows from (5.4)–(5.11) that problem (5.2) (and thus problem (5.1)) fits into the framework of Theorem 3.1 with assumptions (3.5), (3.6), (3.8), (3.10), and (3.13) satisfied (and $\tau = 1$). Therefore, Theorem 3.1 implies Theorem 5.1. □

Appendix A

Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a closed, densely defined, self-adjoint, positive operator with bounded and compact inverse, where its domain $D(A)$ is equipped with the graph norm. Then the square root $A^{1/2}$ (more generally: $f(A)$ for $f : (0, \infty) \to \mathbb{C}$) is a well-defined closed operator on H by Fourier series representation.

Proposition A.1 *Suppose (3.2) and let $\sigma \in \mathbb{R}$. The matrix operator*

$$\mathbb{A} := \begin{pmatrix} 0 & 1 \\ -A & -\sigma \end{pmatrix}, \quad D(\mathbb{A}) := D(A) \times D(A^{1/2}),$$

generates a strongly continuous group on the Hilbert space $\mathbb{H} := D(A^{1/2}) \times H$ (with an exponential decay if $\sigma > 0$). Consider $\mathbf{w_0} \in \mathbb{H}$ and

$$\mathbf{F} := \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad f \in C([0, T], H).$$

Then

$$\mathbf{w}(t) = e^{t\mathbb{A}}\mathbf{w_0} + \int_0^t e^{(t-s)\mathbb{A}}\,\mathbf{F}(s)\,\mathrm{d}s, \quad t \in [0, T],$$

satisfies $\mathbf{w} = (w, w') \in C([0, T], \mathbb{H})$ and

$$w \in C^1([0, T], H) \quad \text{with } \partial_t w = w'.$$

If $\mathbf{w_0} \in D(\mathbb{A})$ and $\mathbf{F} \in C^1([0, T], \mathbb{H}) + C([0, T], D(\mathbb{A}))$, then

$$\mathbf{w} \in C^1([0, T], \mathbb{H}) \cap C([0, T], D(\mathbb{A}))$$

is a strong solution to

$$\partial_t \mathbf{w} = \mathbb{A}\mathbf{w} + \mathbf{F}(t), \quad t \in [0, T], \quad \mathbf{w}(0) = \mathbf{w}_0.$$

Proof Let $\sigma = 0$. Then

$$e^{t\mathbb{A}} = \begin{pmatrix} \cos\left(tA^{1/2}\right) & \sin\left(tA^{1/2}\right)A^{-1/2} \\ -\sin\left(tA^{1/2}\right)A^{1/2} & \cos\left(tA^{1/2}\right) \end{pmatrix}, \quad t \in \mathbb{R},$$

and the mild formulation for $\mathbf{w} = (w, w')$ yields explicit formulas for both w and w' which readily imply that $\mathbf{w} = (w, w') \in C([0, T], \mathbb{H})$ with

$$w \in C^1([0, T], H) \quad \text{and} \quad \partial_t w = w'.$$

If $\sigma \neq 0$, then replace f by $f - \sigma w' \in C([0, T], H)$ to reduce the problem to the case $\sigma = 0$. The statement about strong solutions is classical. □

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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