# Descent of tautological sheaves from Hilbert schemes to Enriques manifolds 

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#### Abstract

Let $X$ be a K3 surface which doubly covers an Enriques surface $S$. If $n \in \mathbb{N}$ is an odd number, then the Hilbert scheme of $n$-points $X^{[n]}$ admits a natural quotient $S_{[n]}$. This quotient is an Enriques manifold in the sense of Oguiso and Schröer. In this paper we construct slope stable sheaves on $S_{[n]}$ and study some of their properties.


Keywords Enriques manifolds • Stable sheaves • Moduli spaces
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In 1896 Federigo Enriques gave examples of smooth projective surfaces with irregularity $q=0$ and geometric genus $p_{g}=0$ which are not rational. Therefore these surfaces were counterexamples to a conjecture by Max Noether, which stated that surfaces with $q=p_{g}=0$ are rational. Nowadays such a surface is called an Enriques surface.

The canonical bundle $\omega_{S}$ of an Enriques surface $S$ has order two in the Picard group of $S$. The induced double cover turns out to be a K3 surface (a two dimensional hyperkähler manifold), hence it is the universal cover of $S$. On the other hand, every K3 surface $X$ which admits a fixed point free involution doubly covers an Enriques surface $S$.

Mimicking this correspondence Oguiso and Schröer defined higher dimensional analogues of Enriques surfaces, the so called Enriques manifolds in [1]. To be precise a connected complex manifold that is not simply connected and whose universal cover is a hyperkäler manifold is called an Enriques manifold.

The following class of examples is of interest to us in this work: take an odd natural number $n \in \mathbb{N}$ and an Enriques surface $S$. We have the induced K3 surface $X$ with a fixed point free involution $\iota$ such that $S=X / \iota$. Since $n$ is odd we get an induced fixed point free involution $\iota^{[n]}$ on the Hilbert scheme of $n$-points $X^{[n]}$. The quotient of $X^{[n]}$ by the involution $\iota^{[n]}$ is an Enriques manifold $S_{[n]}$ of dimension $2 n$. We have an étale Galois cover $\rho: X^{[n]} \rightarrow S_{[n]}$.

In this article we construct and study stable sheaves on Enriques manifolds of type $S_{[n]}$. The main idea is to start with slope stable sheaves on $X^{[n]}$ and check if they descend to

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$S_{[n]}$. Known examples of stable sheaves on $X^{[n]}$ are given by the tautological bundles $E^{[n]}$ associated to slope stable locally free sheaves $E$ on $X$.

For example, we prove that $E^{[n]}$ descends to $S_{[n]}$ if and only if $E$ descends to $S$. If $E^{[n]}$ descends we have $E^{[n]} \cong \rho^{*} F_{[n]}$ for some locally free sheaf $F_{[n]}$ on $S_{[n]}$. We then show that it is possible to find an ample divisor $D \in \operatorname{Amp}\left(S_{[n]}\right)$ such that $F_{[n]}$ is slope stable with respect to $D$. Finally using results from Kim [2] and Yoshioka [3], we are able to prove that, given certain conditions are satisfied, we have in fact a morphism

$$
(-)_{[n]}: \mathrm{M}_{S, d}(v, L) \rightarrow \mathcal{M}_{S_{[n]}, D}\left(v_{[n]}\right), \quad F \mapsto F_{[n]}
$$

between a moduli spaces of stable sheaves on $S$ and moduli space of stable sheaves on $S_{[n]}$. This morphism identifies the former moduli space as a smooth connected component in the latter.

This paper consists of four sections. In Sect. 1 we generalize some results concerning tautological bundles on Hilbert schemes of points. Section 2 contains results about the descent of tautological sheaves from $X^{[n]}$ to $S_{[n]}$. We compute certain Ext-spaces in Sect. 3. In the final Sect. 4 we study the stability of sheaves on Enriques manilfolds of type $S_{[n]}$.

## 1 Stability of tautological sheaves on Hilbert schemes of points

Let $X$ be a smooth projective surface. The Hilbert scheme $X^{[n]}:=\operatorname{Hilb}^{n}(X)$ classifies length $n$ subschemes in $X$, that is

$$
X^{[n]}=\left\{[Z] \mid Z \subset X, \operatorname{dim}(Z)=0 \text { and } \operatorname{dim}\left(\mathrm{H}^{0}\left(Z, \mathcal{O}_{Z}\right)\right)=n\right\} .
$$

In fact $X^{[n]}$ is smooth itself and has dimension $2 n$, see [4, Theorem 2.4]. Moreover $X^{[n]}$ is a fine moduli space for the classification of length $n$ subschemes and comes with the universal length $n$ subscheme

$$
\mathcal{Z}=\left\{(x,[Z]) \in X \times X^{[n]} \mid x \in \operatorname{supp}(Z)\right\} \subset X \times X^{[n]}
$$

The universal subscheme $\mathcal{Z}$ comes with two projections $p: \mathcal{Z} \rightarrow X^{[n]}$ and $q: \mathcal{Z} \rightarrow X$. Note that the morphism $p$ is finite and flat of degree $n$.

To any locally free sheaf $E$ of rank $r$ on $X$ one can associate the so called tautological vector bundle $E^{[n]}$ on $X^{[n]}$ via

$$
E^{[n]}:=p_{*} q^{*} E .
$$

As $p$ is finite and flat of degree $n$ the sheaf $E^{[n]}$ is indeed locally free and has rank $n r$. The fiber at $[Z] \in X^{[n]}$ can be computed to be

$$
E^{[n]} \otimes \mathcal{O}_{[Z]} \cong \mathrm{H}^{0}\left(Z, E_{\mid Z}\right)
$$

Remark 1.1 Note that the definition of $E^{[n]}$ makes sense for $E$ a coherent sheaf on $X$ or even a complex $E \in \mathrm{D}^{\mathrm{b}}(X)$ in the derived category of $X$, see [5, Definition 2.4].

In [6, Theorem 1.4, Theorem 4.9] Stapleton proves that if $h \in \operatorname{Amp}(X)$ is an ample divisor on $X$ and $E \not \not \mathcal{O}_{X}$ is a slope stable (with respect to $h$ ) locally free sheaf, then there is $H \in \operatorname{Amp}\left(X^{[n]}\right)$ such that the associated tautological bundle $E^{[n]}$ is slope stable with respect to $H$ on $X^{[n]}$.

In fact Stapleton's result remains true, if we drop the locally free condition and allow for torsion free sheaves, see for example [7, Proposition 2.4] for a first step toward the following observation:

Lemma 1.2 Assume $E$ is torsion free and slope stable with respect to $h \in \operatorname{Amp}(X)$ such that its double dual satisfies $E^{* *} \neq \mathcal{O}_{X}$, then the associated tautological sheaf $E^{[n]}$ is slope stable with respect to some $H \in \operatorname{Amp}\left(X^{[n]}\right)$.

Proof Since $X$ is a smooth projective surface and $E$ is torsion free we can canonically embed $E$ into its double dual. This gives an exact sequence

$$
\begin{equation*}
0 \longrightarrow E \longrightarrow E^{* *} \longrightarrow Q \longrightarrow 0 . \tag{1}
\end{equation*}
$$

Here $E^{* *}$ is locally free and also slope stable with respect to $h$. Furthermore $Q$ has support of codimension two.

By [8, Corollary 6] the functor $(-)^{[n]}: \operatorname{Coh}(X) \rightarrow \operatorname{Coh}\left(X^{[n]}\right)$ is exact. So we get an exact sequence on $X^{[n]}$

$$
0 \longrightarrow E^{[n]} \longrightarrow\left(E^{* *}\right)^{[n]} \longrightarrow Q^{[n]} \longrightarrow 0 .
$$

By our assumptions $\left(E^{* *}\right)^{[n]}$ is slope stable with respect to some $H \in \operatorname{Amp}\left(X^{[n]}\right)$. But $Q^{[n]}$ has support of codimension two in $X^{[n]}$ so that $E^{[n]}$ is isomorphic to $\left(E^{* *}\right)^{[n]}$ in codimension one and thus must be also be slope stable with respect to $H$.

The previous lemma shows that for every slope stable $E$ with $E^{* *} \not \not \mathcal{O}_{X}$ there is $H \in$ $\operatorname{Amp}\left(X^{[n]}\right)$ such that the tautological sheaf $E^{[n]}$ is slope stable with respect to $H$. Since $E$ belongs to some moduli space $\mathrm{M}_{X, h}\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right)$, one may ask how $H$ varies if $E$ varies in its moduli. We can answer this question in the case that all sheaves classified by $\mathrm{M}_{X, h}\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right)$ are locally free.

Proposition 1.3 If $\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right) \neq(1,0,0)$ is chosen such that for every $[E] \in M_{X, h}\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right)$ the sheaf $E$ is slope stable and locally free, then there is $H \in \operatorname{Amp}\left(X^{[n]}\right)$ such that $E^{[n]}$ is slope stable with respect to $H$ for all $[E] \in \mathrm{M}_{X, h}\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right)$.

Proof By a result of Stapleton, see [6, Theorem 1.4], we know that for $[E] \in \mathrm{M}_{X, h}\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right)$ the locally free sheaf $E^{[n]}$ is slope stable with respect to the induced nef divisor $h_{n} \in$ $\mathrm{NS}\left(X^{[n]}\right)$. It is also well known that the Hilbert - Chow morphism $\mathrm{HC}: X^{[n]} \rightarrow X^{(n)}$ is semi-small and that $q: \mathcal{Z} \rightarrow X$ is flat, see [9, Theorem 2.1].

The proof is now exactly the same as for tautological bundles on the generalized Kummer variety $\operatorname{Kum}_{n}(A)$ associated to an abelian surface $A$, see [10, Proposition 2.9].

Remark 1.4 The condition that all sheaves in $\mathrm{M}_{X, h}\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right)$ are slope stable can be achieved (for example) in the following two different ways: the first is by a special choice of the numerical invariants, see [11, Lemma 1.2.14]. The second way is by choosing a special ample class $h$, see [11, Theorem 4.C.3].

To find a moduli space such that all sheaves are locally free, one can do the following: if the tuple ( $r, \mathrm{c}_{1}$ ) is fixed, then by Bogomolov's inequality the second Chern class is bounded from below, see [11, Theorem 3.4.1]. Choose the minimal $\mathrm{c}_{2}$, then every sheaf in $\mathrm{M}_{X, h}\left(r, \mathrm{c}_{1}, \mathrm{c}_{2}\right)$ is locally free. Indeed, if such an $E$ is not locally free, then $E^{* *}$ is locally free, stable with respect to $h$ and has the same tuple ( $r, \mathrm{c}_{1}$ ), but it has smaller $\mathrm{c}_{2}$ by exact sequence (1) as $\mathrm{c}_{2}(Q)<0$, contradicting minimality. See also [11, Remark 6.1.9] for a similar argument.

Now let $X$ be a K3 surface. Denote the Mukai vectors of $E$ and $E^{[n]}$ by $v$ respectively $v^{[n]} \in \mathrm{H}^{*}\left(X^{[n]}, \mathbb{Q}\right)$. If $E^{[n]}$ is slope stable, then it belongs to the moduli space $\mathcal{M}_{X^{[n]}, H}\left(v^{[n]}\right)$ of semistable sheaves on $X^{[n]}$ with Mukai vector $v^{[n]}$. In fact we can generalize [7, Corollary 4.6] to get the following

Theorem 1.5 If $v \neq v\left(\mathcal{O}_{X}\right)$ is a Mukai vector such that for every $[E] \in \mathrm{M}_{X, h}(v)$ the sheaf $E$ is slope stable, locally free and $h^{i}(X, E)=0$ for $i=1,2$, then the functor $(-)^{[n]}$ induces a morphism

$$
(-)^{[n]}: \mathrm{M}_{X, h}(v) \rightarrow \mathcal{M}_{X^{[n]}, H}\left(v^{[n]}\right), \quad[E] \mapsto\left[E^{[n]}\right]
$$

which identifies $\mathrm{M}_{X, h}(v)$ with a smooth connected component of $\mathcal{M}_{X^{[n]}, H}\left(v^{[n]}\right)$.
Proof First note that the map $[E] \mapsto\left[E^{[n]}\right]$ is indeed a regular morphism, see for example [12, Proposition 2.1]. Furthermore this morphism is injective on closed points, which follows immediately from [13, Theorem 1.1] (see also [9, Theorem 1.2] for a generalization).

By [5, Corollary 4.2 (11)] we find

$$
\operatorname{Ext}_{X^{[n]}}^{1}\left(E^{[n]}, F^{[n]}\right) \cong \operatorname{Ext}_{X}^{1}(E, F)
$$

since $h^{0}\left(X, E^{\vee}\right)=h^{2}(X, E)=0$ as well as $h^{1}\left(X, E^{\vee}\right)=h^{1}(X, E)=0$. For $E=F$ this isomorphisms translates to

$$
\operatorname{dim}\left(T_{\left[E^{[n]}\right]} \mathcal{M}_{X^{[n]}, H}\left(v^{[n]}\right)\right)=\operatorname{dim}\left(T_{[E]} \mathbf{M}_{X, h}(v)\right)
$$

These two facts imply that we can identify $\mathrm{M}_{X, h}(v)$ with a smooth connected component in $\mathcal{M}_{X^{[n]}, H}\left(v^{[n]}\right)$.

## 2 Descent of tautological sheaves to Enriques manifolds

Let $G$ be a finite group. Consider an étale Galois $\operatorname{cover} \varphi: Y \rightarrow Z$ with Galois group $G$, that is there is a free $G$-action on $Y$ such that $Z=Y / G$ and $\varphi$ is the quotient map. In this situation there is an equivalence between the categories $\operatorname{Coh}(Z)$ of coherent sheaves on $Z$ and $\operatorname{Coh}^{G}(Y)$ of $G$-equivariant coherent sheaves on $Y$ given by the functors

$$
\begin{aligned}
& \varphi^{*}: \operatorname{Coh}(Z) \rightarrow \operatorname{Coh}^{G}(Y), \quad E \mapsto \varphi^{*} E \text { and } \\
& \varphi_{*}^{G}: \operatorname{Coh}^{G}(Y) \rightarrow \operatorname{Coh}(Z), \quad F \mapsto\left(\varphi_{*}(F)\right)^{G}
\end{aligned}
$$

We say that a coherent sheaf $E$ on $Y$ descends to $Z$ if $E$ is in the image of $\varphi^{*}$, that is there is a coherent sheaf $F$ on $Z$ together with an isomorphism $E \cong \varphi^{*}(F)$.

A coherent sheaf $E$ on $X$ is said to be $G$-invariant, if there are isomorphisms $E \cong g^{*} E$ for every $g \in G$. A $G$-equivariant coherent sheaf is $G$-invariant, but the converse is not true. For our purposes the following will suffice, see [14, Lemma 1]:

Proposition 2.1 Assume that $G$ is a cyclic group and $E$ is a simple $G$-invariant coherent sheaf on $Y$, then $E$ descends to $Z$.

Remark 2.2 Recall that if $(X, \iota)$ is a pair consisting of a K3 surface and a fixed point free involution, then $G=\langle\iota\rangle \cong \mathbb{Z} / 2 \mathbb{Z}$ acts freely on $X$ and the quotient $S$ is an Enriques surface. The morphism $\pi: X \rightarrow S$ is an étale $\mathbb{Z} / 2 \mathbb{Z}$-Galois cover.

On the other hand if $S$ is an Enriques surface, then its canonical bundle $\omega_{S}$ is 2-torsion. One can consider the induced canonical cover $\phi: \tilde{S}:=\operatorname{Spec}\left(\mathcal{O}_{S} \oplus \omega_{S}\right) \rightarrow S$. The morphism $\phi$ is an étale $\mathbb{Z} / 2 \mathbb{Z}$-Galois cover and $\tilde{S}$ is a K3 surface with fixed point free involution, the covering involution of $\phi$. Furthermore $\phi_{*} \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{S} \oplus \omega_{S}$.

In [1] Oguiso and Schröer generalized the notion of an Enriques surface to that of an Enriques manifold by mimicking the above constructions:

Definition 2.3 A manifold $Y$ is called an Enriques manifold if it is a connected complex manifold that is not simply connected and whose universal cover is a hyperkähler manifold.

Remark 2.4 In [15] the authors also gave a definition of higher dimensional Enriques varieties, which slightly differs from the one of Enriques manifolds in [1].

Remark 2.5 An Enriques manifold is of even dimension, say $\operatorname{dim}(Y)=2 n$. The fundamental group $\pi_{1}(Y)$ is finite of order $d$ with $d \mid n+1$. This number $d$ is called the index of $Y$. In addition $Y$ is projective and the canonical bundle $\omega_{Y}$ has finite order $d$ and generates the torsion group of $\operatorname{Pic}(Y)$, see [1, Sect. 2].

We will work with the following class of Enriques manifolds, see [1, Proposition 4.1]:
Example 2.6 Let ( $X, \iota$ ) be a pair consisting of a K3 surface together with a fixed point free involution $\iota$ on $X$. Then $X$ covers the Enriques surface $S=X / \iota$. If $n \in \mathbb{N}$ is odd, then $(X, \iota)$ induces the pair $\left(X^{[n]}, l^{[n]}\right)$ of the Hilbert scheme of $n$-points on $X$ and the induced fixed point free involution $\iota^{[n]}$ on $X^{[n]}$. Thus $G=\left\langle\iota^{[n]}\right\rangle \cong \mathbb{Z} / 2 \mathbb{Z}$ acts freely on $X^{[n]}$ and the quotient $S_{[n]}$ is an Enriques manifold with index $d=2$ coming with an étale $\mathbb{Z} / 2 \mathbb{Z}$-cover $\rho: X^{[n]} \rightarrow S_{[n]}$.

We want to study the descent of sheaves from $X$ to $S$ respectively from $X^{[n]}$ to $S_{[n]}$. To do this we need the following lemma:

Lemma 2.7 There is an isomorphism of functors from $\operatorname{Coh}(X)$ to $\operatorname{Coh}\left(X^{[n]}\right)$ :

$$
\left(\iota^{[n]}\right)^{*}\left((-)^{[n]}\right) \cong\left(\iota^{*}(-)\right)^{[n]} .
$$

Proof Recall that $(-)^{[n]}=\mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}(-)$ can be written as the Fourier—Mukai transform with kernel the structure sheaf of universal family $\mathcal{Z}$ in $X \times X^{[n]}$, see for example [9, Sect. 2.3]. Define a group isomorphism

$$
\mu:\langle\iota\rangle \rightarrow\left\langle\iota^{[n]}\right\rangle, \quad \iota \mapsto \iota^{[n]}
$$

and note that this is a so-called $c$-isomorphism, see [16, Definitions 3.1 and 3.3]. By the definition of the universal family we see that there is an isomorphism

$$
(\iota \times \mu(\iota))^{*} \mathcal{O}_{\mathcal{Z}}=\left(\iota \times \iota^{[n]}\right)^{*} \mathcal{O}_{\mathcal{Z}} \cong \mathcal{O}_{\mathcal{Z}}
$$

Thus $\mathcal{O}_{\mathcal{Z}}$ is $\mu$-invariant, see [16, Definition 3.4], which implies

$$
\left(\iota^{[n]}\right)^{*}\left(\mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}(-)\right) \cong \mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}\left(\iota^{*}(-)\right)
$$

by [16, Lemma 3.6 (iii)].
We can now prove the main result of this section:
Theorem 2.8 Assume $(X, \iota)$ is a $K 3$ surface together with a fixed point free involution and let $n \in \mathbb{N}$ be an odd number. If a torsion free sheaf $E$ on $X$ is simple, then the associated tautological sheaf $E^{[n]}$ on $X^{[n]}$ descends to $S_{[n]}$ if and only if $E$ descends to $S$.

Proof First we note that if $E$ is simple then $E^{[n]}$ is also simple. Indeed by [5, Corollary 4.2 (11)] there is an isomorphism

$$
\operatorname{End}_{X^{[n]}}\left(E^{[n]}\right) \cong \operatorname{End}_{X}(E) \oplus \mathrm{H}^{0}\left(X, E^{*}\right) \otimes \mathrm{H}^{0}(X, E)
$$

Since $E$ is simple the second summand must vanish, since otherwise $E$ would have an endomorphism, which has image of rank one and thus is no homothety.

Proposition 2.1 shows

$$
E^{[n]} \text { decends to } S_{[n]} \Leftrightarrow\left(l^{[n]}\right)^{*} E^{[n]} \cong E^{[n]}
$$

By Lemma 2.7 we get

$$
\left(\iota^{[n]}\right)^{*} E^{[n]} \cong E^{[n]} \Leftrightarrow\left(\iota^{*} E\right)^{[n]} \cong E^{[n]} .
$$

But [9, Theorem 1.2] shows

$$
\left(\iota^{*} E\right)^{[n]} \cong E^{[n]} \Leftrightarrow \iota^{*} E \cong E .
$$

Thus $E^{[n]}$ descends to $S_{[n]}$ if and only if $E$ descends to $S$.
The theorem shows that given a simple $\iota$-invariant torsion free sheaf $E$ on $X$ then there is $F \in \operatorname{Coh}(S)$ and $G \in \operatorname{Coh}\left(S_{[n]}\right)$ such that

$$
E \cong \pi^{*} F \text { as well as } E^{[n]} \cong \rho^{*} G
$$

In fact, there is a close relationship between the sheaves $F$ and $G$ : as $\mathcal{O}_{\mathcal{Z}}$ is $\mu$-invariant on $X \times X^{[n]}$, the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ is naturally $\mu$-linearizable on $\mathcal{Z}$, hence so is $\mathcal{O}_{\mathcal{Z}}$ as a sheaf on $X \times X^{[n]}$.

Therefore by [16, Proposition 4.2] the functor $(-)^{[n]}$ descends to a functor

$$
(-)_{[n]}: \mathrm{D}^{\mathrm{b}}(S) \rightarrow \mathrm{D}^{\mathrm{b}}\left(S_{[n]}\right)
$$

together with a commutative diagram


Here For is the functor forgetting the linearizations.
That is if we start with a simple sheaf $E$ on $X$, which descends to $S$ i.e. $E \cong \pi^{*} F$, then $E^{[n]}$ descends to $S_{[n]}$ with $E^{[n]} \cong \rho^{*} F_{[n]}$.

Remark 2.9 As $\mathcal{O}_{\mathcal{Z}}$ has two choices of a $\mu$-linearization (differing by the non-trivial character), there are actually two choices of the descent $(-)_{[n]}: \mathrm{D}^{\mathrm{b}}(S) \rightarrow \mathrm{D}^{\mathrm{b}}\left(S_{[n]}\right)$ (differing by tensor product by $\omega_{S_{[n]}}$ ).

We end this section by giving a more explicit description of $(-)_{[n]}$ similar to $(-)^{[n]}$. For this recall that by $[12,2.4]$ we have

$$
(-)^{[n]}=\mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}(-)=p_{X^{[n]} *}\left(p_{X}^{*}(-)\right),
$$

where $p_{X}: \mathcal{Z} \rightarrow X$ and $p_{X^{[n]}}: \mathcal{Z} \rightarrow X^{[n]}$ are the projections.
The group $G=\mathbb{Z} / 2 \mathbb{Z}$ acts freely on $X$ via $\iota$ with quotient $S$, freely on $X^{[n]}$ via $\iota^{[n]}$ with quotient $S_{[n]}$ and thus also freely on $X \times X^{[n]}$ via $\iota \times \iota^{[n]}$. As the universal family $\mathcal{Z} \hookrightarrow$
$X \times X^{[n]}$ is $G$-invariant, we get a closed subvariety $\mathcal{Z} / G \hookrightarrow\left(X \times X^{[n]}\right) / G$. Furthermore the projections $p_{X}$ and $p_{X^{[n]}}$ are $G$-equivariant. By [17, Lemma 2.3.3] we get cartesian squares

$$
\begin{array}{cccc}
X & \stackrel{p_{X}}{\longleftarrow} & \mathcal{Z} \xrightarrow{p_{X}^{[n]}} & X^{[n]}  \tag{3}\\
\pi \downarrow & \downarrow \alpha & \downarrow^{\rho} \\
S & { }^{p_{S}} & \mathcal{Z} / G & \\
p_{[n]} & S_{[n]}
\end{array}
$$

Theorem 2.10 The functor $(-)_{[n]}: \mathrm{D}^{\mathrm{b}}(S) \rightarrow \mathrm{D}^{\mathrm{b}}\left(S_{[n]}\right)$ has the following description:

$$
(-)_{[n]}=p_{S_{[n]} *}\left(p_{S}^{*}(-)\right)
$$

Proof From diagram (2) we see that $\rho^{*}((-)[n])=\left(\pi^{*}(-)\right)^{[n]}$. Since $\rho_{*}\left(\rho^{*}(-)\right)^{G}=$ id we find

$$
\begin{aligned}
(-)_{[n]} & =\rho_{*}\left(\left(\pi^{*}(-)\right)^{[n]}\right)^{G}=\rho_{*}\left(p_{X^{[n]} *}\left(p_{X}^{*}\left(\pi^{*}(-)\right)\right)\right)^{G} \\
& =p_{S_{[n]}}\left(\alpha_{*}\left(\alpha^{*}\left(p_{S}^{*}(-)\right)\right)\right)^{G}=p_{S_{[n]} *}\left(\alpha_{*}\left(\alpha^{*}\left(p_{S}^{*}(-)\right)\right)^{G}\right) \\
& =p_{S_{[n]}}\left(p_{S}^{*}(-)\right) .
\end{aligned}
$$

Here we used the commutativity of diagram (3), the fact that $G$ acts trivially on $\mathcal{Z} / G$ and $S_{[n]}$ hence by [5, Equation (5)] we have (-) ${ }^{G} p_{S_{[n] *}}=p_{S_{[n] *}}(-)^{G}$ and the $G$-equivariant projection formula.

## 3 Computation of certain extension spaces

In [18, Theorem 3.17] Krug gave explicit formulas for homological invariants of tautological objects in $\mathrm{D}^{\mathrm{b}}\left(X^{[n]}\right)$ in terms of those in $\mathrm{D}^{\mathrm{b}}(X)$, for example for $E, F \in \mathrm{D}^{\mathrm{b}}(X)$ there is an isomorphism of graded vector spaces:

$$
\begin{aligned}
\operatorname{Ext}_{X^{[n]}}^{*}\left(E^{[n]}, F^{[n]}\right) \cong & =\operatorname{Ext}_{X}^{*}(E, F) \otimes S^{n-1} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right) \\
& \oplus \operatorname{Ext}_{X}^{*}\left(E, \mathcal{O}_{X}\right) \otimes \operatorname{Ext}_{X}^{*}\left(\mathcal{O}_{X}, F\right) \otimes S^{n-2} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)
\end{aligned}
$$

See also [5, Sect. 4] for a considerably simplified proof of this formula.
In this section we want to find homological invariants of sheaves of the form $G_{[n]}$ on $S_{[n]}$ in terms of the sheaf $G$ on $S$. It is certainly possible to find a general formula similar to Krug's result, but to keep formulas and proofs short and readable and since it is enough for our purposes, we will restrict our attention to Hom- and Ext ${ }^{1}$ - spaces as well as sheaves without higher cohomology. We will use the notations and results from [5].

We start by studying how Krug's result behaves with respect to the group actions by $\mathbb{Z} / 2 \mathbb{Z}$ on $X^{[n]}$ via $\iota^{[n]}$ and on $X$ via $\iota$. We will denote the various versions of the group $G=\mathbb{Z} / 2 \mathbb{Z}$ in the following by their nontrivial element, that is by $\iota$ or $\iota^{[n]}$ etc.

Lemma 3.1 Assume $(X, \iota)$ is a K3 surface together with a fixed point free involution. For $\iota$-equivariant coherent sheaves $E, F \in \operatorname{Coh}_{\iota}(X)$ there is an isomorphism of graded vector spaces:

$$
\begin{aligned}
\left(\operatorname{Ext}_{X^{[n]}}^{*}\left(E^{[n]}, F^{[n]}\right)\right)^{[n]} \cong & \left(\operatorname{Ext}_{X}^{*}(E, F) \otimes S^{n-1} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\iota} \\
& \oplus\left(\operatorname{Ext}_{X}^{*}\left(E, \mathcal{O}_{X}\right) \otimes \operatorname{Ext}_{X}^{*}\left(\mathcal{O}_{X}, F\right) \otimes S^{n-2} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\iota}
\end{aligned}
$$

Proof Note that on the right hand side of the formula we take invariants with respect to the actions induced by the linearizations of $E, F$ and $\mathcal{O}_{X}$. On the left hand side we take invariants with respect to the induced linearizations on $E^{[n]}$ and $F^{[n]}$. The existence of the induced linearizations follows from the right-hand side of diagram (2).

By [5, Theorem 3.6] there is an isomorphism of functors

$$
\begin{equation*}
(-)^{[n]} \cong \Psi \circ \mathrm{C}, \tag{1}
\end{equation*}
$$

where $\mathrm{C}: \operatorname{Coh}(X) \rightarrow \operatorname{Coh}_{\mathfrak{S}_{n}}\left(X^{n}\right)$ is the exact functor with

$$
\mathrm{C}(E):=\operatorname{lnd}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_{n}} \mathrm{pr}_{1}^{*} E \cong \bigoplus_{i=1}^{n} \operatorname{pr}_{i}^{*} E
$$

Furthermore $\Psi: \mathrm{D}_{\mathfrak{S}_{n}}^{\mathrm{b}}\left(X^{n}\right) \rightarrow \mathrm{D}^{\mathrm{b}}\left(X^{[n]}\right)$ is the Fourier - Mukai transform with kernel the structure sheaf of the isospectral Hilbert scheme $I^{n} X$. Here the isosprectral Hilbert scheme is the reduced fiber product $I^{n} X:=\left(X^{[n]} \times S^{n} X X^{n}\right)_{\text {red }}$ of the quotient map $v: X^{n} \rightarrow S^{n} X$ to the symmetric power and the Hilbert - Chow morphism $\mu: X^{[n]} \rightarrow S^{n} X$. This Fourier Mukai transform is an equivalence, see [5, Proposition 2.8] and satisfies

$$
\begin{equation*}
\left(\iota^{[n]}\right)^{*} \circ \Psi=\Psi \circ\left(\iota^{\times n}\right)^{*} \tag{2}
\end{equation*}
$$

see for example [16, Sect. 5.6]. Here $\iota^{\times n}$ is the induced involution on $X^{n}$.
We have the following chain of isomorphisms:

$$
\begin{aligned}
\left(\operatorname{Ext}_{X^{[n]}}^{*}\left(E^{[n]}, F^{[n]}\right)\right)^{[n]} & \cong\left(\operatorname{Ext}_{X^{[n]}}^{*}(\Psi(C(E)), \Psi(C(F)))\right)^{\iota^{[n]}} \\
& \cong\left(\operatorname{Ext}_{X^{n}, \mathfrak{S}_{n}}^{*}(\mathrm{C}(E), \mathrm{C}(F))\right)^{\iota^{\times n}} \\
& \cong\left(\operatorname{Ext}_{X^{n}, \mathfrak{S}_{n-1}}^{*}\left(\mathrm{pr}_{1}^{*} E, \operatorname{pr}_{1}^{*} F\right)\right)^{\iota^{\times n}} \oplus\left(\operatorname{Ext}_{X^{n}, \mathfrak{S}_{n-2}}^{*}\left(\operatorname{pr}_{1}^{*} E, \mathrm{pr}_{2}^{*} F\right)\right)^{\iota^{\times n}}
\end{aligned}
$$

Here the first isomorphism is (1). The second isomorphism uses that $\Psi$ is an equivalence and (2). The last isomorphism can be extracted from [5, Proposition 4.1].

We look at the first summand, the second working similarly. First note that

$$
\operatorname{pr}_{1}^{*} E=E \boxtimes \mathcal{O}_{X} \boxtimes \cdots \boxtimes \mathcal{O}_{X}
$$

Applying the Künneth formula shows

$$
\begin{aligned}
\operatorname{Ext}_{X^{n}, \mathfrak{S}_{n-1}}^{*}\left(\mathrm{pr}_{1}^{*} E, \mathrm{pr}_{1}^{*} F\right) & =\operatorname{Ext}_{X^{n}, \mathfrak{S}_{n-1}}^{*}\left(E \boxtimes \mathcal{O}_{X} \boxtimes \cdots \boxtimes \mathcal{O}_{X}, F \boxtimes \mathcal{O}_{X} \boxtimes \cdots \boxtimes \mathcal{O}_{X}\right) \\
& \cong\left(\operatorname{Ext}_{X}^{*}(E, F) \otimes \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)^{\otimes n-1}\right)^{\mathfrak{S}_{n-1}}
\end{aligned}
$$

But the group $\mathbb{Z} / 2 \mathbb{Z}$ acts on sheaves of the form $\operatorname{pr}_{1}^{*} E$ by definition of $\iota^{\times n}$ as

$$
\left(\iota^{\times n}\right)^{*} \mathrm{pr}_{1}^{*} E=\iota^{*} E \boxtimes \iota^{*} \mathcal{O}_{X} \boxtimes \cdots \boxtimes \iota^{*} \mathcal{O}_{X}
$$

that is simply by the pullback via $\iota$ on each factor in the box product. Since the action of $\mathbb{Z} / 2 \mathbb{Z}$ via $\iota^{\times n}$ and the $\mathfrak{S}_{n}$ action commute we finally see that:

$$
\left(\operatorname{Ext}_{X^{n}, \mathfrak{S}_{n-1}^{*}}\left(\operatorname{pr}_{1}^{*} E, \operatorname{pr}_{1}^{*} F\right)\right)^{\iota \times n} \cong\left(\operatorname{Ext}_{X}^{*}(E, F) \otimes S^{n-1} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\iota}
$$

Theorem 3.2 Let $(X, \iota)$ be a K3 surface together with a fixed point free involution and let $n \in \mathbb{N}$ be an odd number. If $G, H \in \operatorname{Coh}(S)$ are such that $\pi^{*} G$ and $\pi^{*} H$ have no higher cohomology (here $S=X / \iota$ is the associated Enriques surface), then

$$
\operatorname{Hom}_{S_{[n]}}\left(G_{[n]}, H_{[n]}\right) \cong \operatorname{Hom}_{S}(G, H) \text { and } \operatorname{Ext}_{S_{[n]}}^{1}\left(G_{[n]}, H_{[n]}\right) \cong \operatorname{Ext}_{S}^{1}(G, H)
$$

Proof Define $E:=\pi^{*} G$ and $F:=\pi^{*} H$. It follows from diagram (2) that $E^{[n]} \cong \rho^{*} G_{[n]}$ and $F^{[n]} \cong \rho^{*} H_{[n]}$. We therefore have an isomorphism

$$
\operatorname{Ext}_{S_{[n]}^{*}}^{*}\left(G_{[n]}, H_{[n]}\right) \cong\left(\operatorname{Ext}_{X^{[n]}}^{*}\left(\rho^{*} G_{[n]}, \rho^{*} H_{[n]}\right)\right)^{[n]} \cong\left(\operatorname{Ext}_{X^{[n]}}^{*}\left(E^{[n]}, F^{[n]}\right)\right)^{[n]}
$$

By Lemma 3.1 the last space is isomorphic to

$$
\left(\operatorname{Ext}_{X}^{*}(E, F) \otimes S^{n-1} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\iota} \oplus\left(\operatorname{Ext}_{X}^{*}\left(E, \mathcal{O}_{X}\right) \otimes \mathrm{H}^{*}(X, F) \otimes S^{n-2} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\iota}(3)
$$

We begin investigating the first summand. The natural $\mathbb{Z} / 2$-linearization of $\mathcal{O}_{X}$ induces an $\mathbb{Z} / 2$-linearization on $\pi_{*} \mathcal{O}_{X} \cong \mathcal{O}_{S} \oplus \omega_{S}$ given by the generator of $\mathbb{Z} / 2$ acting by +1 on $\mathcal{O}_{S}$ and by -1 on $\omega_{S}$, see for example [19, Remarks on p.72]. Hence $\iota$ acts as +1 on $\mathrm{H}^{0}\left(X, \mathcal{O}_{X}\right) \cong \mathrm{H}^{0}\left(S, \mathcal{O}_{S}\right)$ and by -1 on $\mathrm{H}^{2}\left(X, \mathcal{O}_{X}\right) \cong \mathrm{H}^{2}\left(S, \omega_{S}\right)$. Furthermore, by the adjunction between $\pi^{*}$ and $\pi_{*}$ together with the projection formula, we get a splitting

$$
\operatorname{Ext}_{X}^{*}(E, F) \cong \operatorname{Ext}_{S}^{*}(G, H) \oplus \operatorname{Ext}_{S}^{*}\left(G, H \otimes \omega_{S}\right)
$$

where $\iota$ acts as +1 on the first summand and by -1 on the second summand.
Thus writing $\mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)=\mathbb{C}[t] /\left(t^{2}\right)$ with $\operatorname{deg}(t)=2$ we get

$$
S^{n-1} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)=\mathbb{C}[t] /\left(t^{n}\right), \quad \operatorname{deg}(t)=2
$$

and $\iota$ acts as +1 on the constants and as -1 on $t$.
We can now compute the invariants and find

$$
\left(\operatorname{Ext}_{S}^{*}(G, H) \otimes \mathbb{C}[t] /\left(t^{n}\right)\right)^{\iota}=\operatorname{Ext}_{S}^{*}(G, H) \otimes \mathbb{C}\left[t^{2}\right] /\left(t^{n}\right), \quad \operatorname{deg}(t)=2
$$

as well as

$$
\left(\operatorname{Ext}_{S}^{*}\left(G, H \otimes \omega_{S}\right) \otimes \mathbb{C}[t] /\left(t^{n}\right)\right)^{\iota}=\operatorname{Ext}_{S}^{*}\left(G, H \otimes \omega_{S}\right) \otimes t \mathbb{C}\left[t^{2}\right] /\left(t^{n}\right), \quad \operatorname{deg}(t)=2
$$

Looking at the components in degree zero and one sees

$$
\begin{aligned}
& \left(\left(\operatorname{Ext}_{X}^{*}(E, F) \otimes S^{n-1} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\imath}\right)_{0} \cong \operatorname{Hom}_{S}(G, H) \text { as well as } \\
& \left(\left(\operatorname{Ext}_{X}^{*}(E, F) \otimes S^{n-1} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\imath}\right)_{1} \cong \operatorname{Ext}_{S}^{1}(G, H)
\end{aligned}
$$

Next we study the second summand in (3): since $E$ and $F$ have no higher cohomology we have

$$
\operatorname{Ext}_{X}^{*}\left(E, \mathcal{O}_{X}\right) \otimes \mathrm{H}^{*}(X, F) \cong \operatorname{Ext}_{X}^{2}\left(E, \mathcal{O}_{X}\right) \otimes \mathrm{H}^{0}(X, F)
$$

which already lives in degree two. As we also have

$$
S^{n-2} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)=\mathbb{C}[t] /\left(t^{n-1}\right), \quad \operatorname{deg}(t)=2,
$$

we see that the second summand in (3) can possibly have nontrivial components starting in degrees at least two. Especially for $k \in\{0,1\}$ we find

$$
\left(\left(\operatorname{Ext}_{X}^{*}\left(E, \mathcal{O}_{X}\right) \otimes \mathrm{H}^{*}(X, E) \otimes S^{n-2} \mathrm{H}^{*}\left(X, \mathcal{O}_{X}\right)\right)^{\imath}\right)_{k}=0
$$

Therefore we must have the desired isomorphisms

$$
\operatorname{Hom}_{S_{[n]}}\left(G_{[n]}, H_{[n]}\right) \cong \operatorname{Hom}_{S}(G, H) \text { and } \operatorname{Ext}_{S_{[n]}}^{1}\left(G_{[n]}, H_{[n]}\right) \cong \operatorname{Ext}_{S}^{1}(G, H)
$$

## 4 Stable sheaves on Enriques manifolds

In this section we want to study the slope stability of sheaves of the form $F_{[n]}$ on $S_{[n]}$. For this we first recall the following fact: let $\varphi: Y \rightarrow Z$ be an étale Galois cover with finite Galois group $G$ then there is the following relationship between slopes with respect to $h \in \operatorname{Amp}(Z)$ :

$$
\begin{equation*}
\mu_{\varphi^{*} h}\left(\varphi^{*} F\right)=|G| \mu_{h}(F) . \tag{4}
\end{equation*}
$$

Using this fact we can prove the following lemma:
Lemma 4.1 Let $E$ be a torsion free coherent sheaf on $Y$, slope stable with respect to $\varphi^{*} h$ for some $h \in \operatorname{Amp}(Z)$. If $E$ descends to $Z$, that is $E \cong \varphi^{*} F$, then $F$ is slope stable with respect to $h$.

Proof Let $H \subset F$ be a subsheaf of $F$. Then $\varphi^{*} H$ is a subsheaf of $\varphi^{*} F \cong E$. Since $E$ is slope stable with respect to $\varphi^{*} h$ we have

$$
\mu_{\varphi^{*} h}\left(\varphi^{*} H\right)<\mu_{\varphi^{*} h}(E)=\mu_{\varphi^{*} h}\left(\varphi^{*} F\right)
$$

which by (4) implies

$$
\mu_{h}(H)<\mu_{h}(F) .
$$

Hence $F$ is slope stable with respect to $h$.
For the rest of this section we let $(X, \iota)$ be a K3 surface together with a fixed point free involution $\iota$. We denote the associated Enriques surface by $S$.

To prove the main theorem in this section we need the following isomorphism:

$$
\operatorname{NS}\left(X^{[n]}\right) \cong \mathrm{NS}(X)_{n} \oplus \mathbb{Z} \delta
$$

Remark 4.2 The summand $\mathrm{NS}(X)_{n}$ is constructed as follows: take $d \in \mathrm{NS}(X)$ and consider the element

$$
D^{n}:=\sum_{i=1}^{n} \operatorname{pr}_{i}^{*} d \in \mathrm{NS}\left(X^{n}\right)
$$

This element is $\mathfrak{S}_{n}$-invariant and thus descends to the symmetric product $S^{n} X$ by [20, Lemma 6.1]. More exactly, there is an element $D_{n} \in \operatorname{NS}\left(S^{n} X\right)$ such that $v^{*} D_{n}=D^{n}$ for the quotient map $v: X^{n} \rightarrow S^{n} X$. Then we define $d_{n}:=\mu^{*} D_{n}$, where $\mu: X^{[n]} \rightarrow S^{n} X$ is the Hilbert Chow morphism.

By [21, Sect. 3] the involution $\iota^{[n]}$ acts on $\operatorname{NS}(X)_{n}$ via:

$$
\begin{equation*}
\left(\iota^{[n]}\right)^{*}\left(d_{n}\right)=\left(\iota^{*} d\right)_{n} . \tag{5}
\end{equation*}
$$

We are now ready to prove the main result of this section:

Theorem 4.3 Assume $E \in \operatorname{Coh}(X)$ satisfies $E^{* *} \not \not \mathcal{O}_{X}$, is torsion free and slope stable with respect to $h=\pi^{*} d$ for some $d \in \operatorname{Amp}(S)$. If $E$ descends to $S$, that is $E \cong \pi^{*} F$ for some $F \in \operatorname{Coh}(S)$, then the induced torsion free sheaf $F_{[n]}$ is slope stable with respect to some ample divisor $D$ on $S_{[n]}$.

Proof By the results of Stapleton in [6] and in Sect. 1 we know that for a given slope stable torsion free sheaf $E$ on $X$ with $E^{* *} \neq \mathcal{O}_{X}$, the associated tautological sheaf $E^{[n]}$ is slope stable on $X^{[n]}$.

By Theorem 2.8 the sheaf $E^{[n]}$ descends to $S_{[n]}$ if and only if $E$ descends to $S$. In this case $E^{[n]} \cong \rho^{*} F_{[n]}$. Now by Theorem 4.1 the sheaf $F_{[n]}$ is slope stable with respect to some $D \in \operatorname{Amp}\left(S_{[n]}\right)$ if $E^{[n]}$ is slope stable with respect to $H \in \operatorname{Amp}\left(X^{[n]}\right)$ of the form $H=\rho^{*} D$ for some $D \in \operatorname{Amp}\left(S_{[n]}\right)$.

To see that we find such a $D \in \operatorname{Amp}\left(S_{[n]}\right)$, we note that the divisor $H$ is described quite explicitly in [6, Proposition 4.8]: it is of the form

$$
H=h_{n}+\epsilon A
$$

for an arbitrary ample divisor $A$ on $X^{[n]}$ and $\epsilon$ sufficiently small. We choose $A$ of the form $A=\rho^{*} C$ for some $C \in \operatorname{Amp}\left(S_{[n]}\right)$. By (5) we also have

$$
\left(\iota^{[n]}\right)^{*}\left(h_{n}\right)=\left(\iota^{*} h\right)_{n}=\left(\iota^{*} \pi^{*} d\right)_{n}=\left(\pi^{*} d\right)_{n}=h_{n}
$$

which implies that we must have that $h_{n}=\rho^{*} B$ for some divisor $B$ on $S_{[n]}$. Putting both facts together shows

$$
H=\rho^{*} D \text { for } D=B+\epsilon C .
$$

It remains to see that $D$ is ample. But since $\rho$ is finite and surjective $D$ is ample if and only if $\rho^{*} D=H$ is ample, see [22, Proposition I.4.4].

In the rest of this section we want to study the moduli spaces containing the slope stable sheaves $F$ on $S$ and $F_{[n]}$ on $S_{[n]}$. For this we let $v \in \mathrm{H}_{\text {alg }}^{*}(S, \mathbb{Z})$ be a Mukai vector on $S$, that is $v=\operatorname{ch}(F) \sqrt{\operatorname{td}(S)}$ for some $F \in \operatorname{Coh}(S)$. Here

$$
\mathrm{H}_{\mathrm{alg}}^{*}(S, \mathbb{Z})=\mathrm{H}^{0}(S, \mathbb{Z}) \oplus \operatorname{Num}(S) \oplus \frac{1}{2} \mathbb{Z} \xi_{S} .
$$

where $\xi_{S}$ denotes the fundamental class of $S$.
We begin with the following result:
Theorem 4.4 Let $F$ be a torsion free coherent sheaf with $F \not \approx F \otimes \omega_{S}$. If $F$ is slope stable with respect to $d \in \operatorname{Amp}(S), F^{* *} \not \not \mathcal{O}_{S}$ and $F^{* *} \not \not \omega_{S}$, then $F_{[n]}$ is a slope stable torsion free coherent sheaf on $S_{[n]}$.

Proof The assumptions imply that $F$ is simple and that $\operatorname{Hom}_{S}\left(F, F \otimes \omega_{S}\right)=0$. Hence $E:=\pi^{*} F$ is is simple due to the formula

$$
\operatorname{Hom}_{X}(E, E) \cong \operatorname{Hom}_{S}(F, F) \oplus \operatorname{Hom}_{S}\left(F, F \otimes \omega_{S}\right)
$$

By [11, Lemma 3.2.3], the sheaf $E$ is polystable with respect to $h=\pi^{*} d$. Being simple and polystable, E is stable.

Since $E^{* *} \not \not \mathcal{O}_{X}$ the sheaf $E^{[n]}$ is slope stable with respect to some $H \in \operatorname{Amp}\left(X^{[n]}\right)$ and descends to $S_{[n]}$ via $E^{[n]} \cong \rho^{*} F_{[n]}$. Now Theorem 4.3 implies that $F_{[n]}$ is slope stable with respect to some $D \in \operatorname{Amp}\left(S_{[n]}\right)$ satisfying $\rho^{*} D=H$.

Remark 4.5 Every torsion free coherent sheaf $F$ of odd rank satisfies the condition $F \not \neq$ $F \otimes \omega_{S}$.

Assume from now on, that $S$ is an unnodal Enriques surface, that is $S$ contains no smooth rational curves (that is no ( -2 )-curves). Note that in the moduli space of Enriques surfaces, a very general element will be unnodal by [23, Corollary 5.7].

Denote the moduli space of slope semistable sheaves (with respect to $d \in \operatorname{Amp}(S)$ ) with Mukai vector $v$ on $S$ by $\mathrm{M}_{S, d}(v)$. Assume that $v$ is primitive and chosen such that every slope semistable sheaf is slope stable and the rank of $v$ is odd. Then for a generic choice of $d \in \operatorname{Amp}(S)$ the moduli space $\mathrm{M}_{S, d}(v)$ is smooth of dimension $v^{2}+1$ and $\mathrm{M}_{S, d}(v) \neq \emptyset$ if and only if $v^{2} \geq-1$, see [3, Proposition 4.2, Theorem 4.6 (i)].

Furthermore in this situation there is a decomposition

$$
\begin{equation*}
\mathrm{M}_{S, d}(v)=\mathrm{M}_{S, d}\left(v, L_{1}\right) \coprod \mathrm{M}_{S, d}\left(v, L_{2}\right) \tag{6}
\end{equation*}
$$

where $\mathrm{M}_{S, d}\left(v, L_{i}\right)$ contains those $[E] \in \mathrm{M}_{S, d}(v)$ with $\operatorname{det}(E)=L_{i}$ where $L_{2}=L_{1} \otimes \omega_{S}$, that is $\mathrm{c}_{1}=\mathrm{c}_{1}\left(L_{1}\right)=\mathrm{c}_{1}\left(L_{2}\right) \in \operatorname{Num}(S)$. By [3, Theorem 4.6.(ii)] for a general choice of $d \in \operatorname{Amp}(S)$ the moduli space $\mathrm{M}_{S, d}(v, L)$ is irreducible, that is a smooth projective variety.

We also assume that the Mukai vector is chosen such that for all $[F] \in \mathrm{M}_{S, d}(v, L)$ the sheaf $F$ is locally free on $S$ and does not have higher cohomology. Denote the Mukai vector of the associated sheaf $F_{[n]}$ on $S_{[n]}$ by $v_{[n]}$. If $F_{[n]}$ is slope stable with respect to some $D \in \operatorname{Amp}\left(S_{[n]}\right)$, denote its moduli space by $\mathcal{M}_{S_{[n]}, D}\left(v_{[n]}\right)$.

Proposition 4.6 If $v \neq v\left(\mathcal{O}_{S}\right)=v\left(\omega_{S}\right)$ then there is a class $D \in \operatorname{Amp}\left(S_{[n]}\right)$ such that $F_{[n]}$ is slope stable with respect to $D$ for all $[F] \in \mathrm{M}_{S, d}(v, L)$.

Proof Since all sheaves classified by $\mathrm{M}_{S, d}(v, L)$ are locally free on $S$, so are all the $E=\pi^{*} F$ on $X$. Proposition 1.3 shows that there is one $H \in \operatorname{Amp}\left(X^{[n]}\right)$ such that all $E^{[n]}$ are slope stable with respect to $H$ since $E \not \approx \mathcal{O}_{X}$. But then by the construction of $D \in \operatorname{Amp}\left(S_{[n]}\right)$ with $H=\rho^{*} D$ in Theorem 4.3, it follows that there is one such desired $D$.

We have the following corollary:
Corollary 4.7 If $v \neq v\left(\mathcal{O}_{S}\right)=v\left(\omega_{S}\right)$, then functor $(-)_{[n]}$ induces a morphism

$$
(-)_{[n]}: \mathrm{M}_{S, d}(v, L) \rightarrow \mathcal{M}_{S_{[n]}, D}\left(v_{[n]}\right),[F] \mapsto\left[F_{[n]}\right]
$$

which identifies $\mathrm{M}_{S, d}(v, L)$ with a smooth connected component of $\mathcal{M}_{S_{[n]}, D}\left(v_{[n]}\right)$.
Proof We use the explicit description $(-)_{[n]}=p_{[n] *}\left(p_{S}^{*}(-)\right)$ given by Theorem 2.10. Since $p_{X}$ and $p_{X^{[n]}}$ are flat we know by faithfully flat descent for $\pi$ resp. $\rho$ that the induced projections $p_{S}$ and $p_{S_{[n]}}$ are flat. Similarly since $p_{X^{[n]}}$ is a finite morphism so is $p_{S_{[n]}}$.

Using these facts together with Theorem 4.4 and Proposition 4.6 shows that Krug's argument in the proof of [12, Proposition 2.1] also works in this case. Hence $[F] \mapsto\left[F_{[n]}\right]$ is a regular morphism.

Similar to Theorem 1.5 it follows from Theorem 3.2 that $(-)_{[n]}$ is injective on closed points as $\operatorname{Hom}_{S_{[n]}}\left(F_{[n]}, G_{[n]}\right) \cong \operatorname{Hom}_{S}(F, G)$. By Theorem 3.2 we also have

$$
\operatorname{dim}\left(\operatorname{Ext}_{S_{[n]}}^{1}\left(F_{[n]}, F_{[n]}\right)\right)=\operatorname{dim}\left(\operatorname{Ext}_{S}^{1}(F, F)\right)
$$

Both facts together imply that $(-)_{[n]}$ identifies $\mathrm{M}_{S, d}(v, L)$ with a smooth connected component of $\mathcal{M}_{S_{[n]}, D}\left(v_{[n]}\right)$.

Remark 4.8 There is a decomposition

$$
\mathcal{M}_{S_{[n]}, D}\left(v_{[n]}\right)=\mathcal{M}_{S_{[n]}, D}\left(v_{[n]}, \mathcal{L}_{1}\right) \coprod \mathcal{M}_{S_{[n]}, D}\left(v_{[n]}, \mathcal{L}_{2}\right)
$$

analogous to (6) and, depending on the choice of ( -$)_{[n]}$ (see Remark 2.9), $\mathrm{M}_{S, d}(v, L)$ is mapped to a component of $\mathcal{M}_{S_{[n]}, D}\left(v_{[n]}, \mathcal{L}_{1}\right)$ or a component of $\mathcal{M}_{S_{[n]}, D}\left(v_{[n]}, \mathcal{L}_{2}\right)$.

Denote the Mukai vector of $E=\pi^{*} F$ on $X$ by $w$, that is $w=\pi^{*} v$. In the rest of this section we want to study the fixed loci of $\iota^{*}$ in $\mathrm{M}_{X, h}(w)$ and $\left(\iota^{[n]}\right)^{*}$ in $\mathcal{M}_{X^{[n]}, H}\left(w^{[n]}\right)$. In our situation we have a well defined morphism

$$
\pi^{*}: \mathrm{M}_{S, d}(v) \rightarrow \mathrm{M}_{X, h}(w), \quad F \mapsto \pi^{*} F
$$

which has image in $\operatorname{Fix}\left(\iota^{*}\right)$. More exactly the image of $\pi^{*}$ is the fixed locus of $\iota^{*}$ and the morphism restricts to an étale 2:1-morphism

$$
\pi^{*}: \mathrm{M}_{S, d}(v) \rightarrow \operatorname{Fix}\left(\iota^{*}\right) .
$$

Furthermore Fix $\left(\iota^{*}\right)$ is a Lagrangian subscheme in $\mathrm{M}_{X, h}(w)$, see for example [2, Theorem (1)] or [24, Theorem 2.3 (c)].

As the morphism $\pi^{*}: \mathrm{M}_{S, d}(v) \rightarrow \operatorname{Fix}\left(\iota^{*}\right)$ is an étale 2:1-morphism, the decomposition (6) shows that $\pi^{*}$ induces an isomorphism $\mathrm{M}_{S, d}(v, L) \cong \operatorname{Fix}\left(\iota^{*}\right)$. As $\mathrm{M}_{S, d}(v, L)$ is irreducible, so is $\operatorname{Fix}\left(\iota^{*}\right)$.

Theorem 4.9 The fixed locus Fix $\left(\iota^{*}\right)$ is a smooth projective variety. The morphism $(-)^{[n]}$ in Theorem 1.5 restricts to a morphism

$$
(-)^{[n]}: \operatorname{Fix}\left(\iota^{*}\right) \rightarrow \operatorname{Fix}\left(\left(\iota^{[n]}\right)^{*}\right)
$$

which identifies $\operatorname{Fix}\left(l^{*}\right)$ with a smooth connected component of $\operatorname{Fix}\left(\left(l^{[n]}\right)^{*}\right)$.
Proof The fixed locus Fix $\left(\iota^{*}\right)$ is smooth and projective since $\mathrm{M}_{X, h}(w)$ is smooth and projective. Since it is also irreducible, it is a smooth projective variety.

By Lemma 2.7 the morphism $(-)^{[n]}$ restricts to a morphism between the fixed loci. Since $(-)^{[n]}$ is injective on closed points, so is its restriction to $\operatorname{Fix}\left(\iota^{*}\right)$.

To identify $\operatorname{Fix}\left(\iota^{*}\right)$ as a smooth connected component it is therefore enough to prove

$$
\operatorname{dim}\left(T_{[E]} \operatorname{Fix}\left(\iota^{*}\right)\right)=\operatorname{dim}\left(T_{\left[E^{[n]}\right]} \operatorname{Fix}\left(\left(\iota^{[n]}\right)^{*}\right)\right)
$$

But a general fact says that the tangent space of the fixed locus satisfies

$$
T_{y}\left(Y^{G}\right) \cong\left(T_{y} Y\right)^{G}
$$

see for example [25, Proposition 3.2]. As we have $E \cong \pi^{*} F$ for some sheaf $F$ on $S$, this shows

$$
T_{[E]} \operatorname{Fix}\left(\iota^{*}\right) \cong\left(T_{[E]} \mathrm{M}_{X, h}(w)\right)^{\iota} \cong\left(\operatorname{Ext}_{X}^{1}(E, E)\right)^{\iota} \cong \operatorname{Ext}_{S}^{1}(F, F)
$$

A similar computation shows

$$
T_{\left[E^{[n]}\right]} \operatorname{Fix}\left(\left(\iota^{[n]}\right)^{*}\right) \cong\left(\operatorname{Ext}_{X^{[n]}}^{1}\left(E^{[n]}, E^{[n]}\right)\right)^{[n]} \cong \operatorname{Ext}_{S_{[n]}^{1}}^{1}\left(F_{[n]}, F_{[n]}\right) \cong \operatorname{Ext}_{S}^{1}(F, F)
$$

by Theorem 3.2 since $E^{[n]} \cong \rho^{*} F_{[n]}$.

Corollary 4.10 The diagram (2) induces the commutative diagram:


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