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On the density of "wild" initial data for the barotropic Euler system

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Abstract
We show that the set of "wild data", meaning the initial data for which the barotropic Euler system admits infinitely many *admissible entropy* solutions, is dense in the L^p-topology of the phase space.

Keywords Compressible Euler system · Wild data · Convex integration

1 Introduction

The concept of wild data/solution appeared recently in the context of the ill-posedness results for the incompressible Euler equations obtained via the method of convex integration. In their seminal work [9] on the existence of infinitely many solutions to the incompressible Euler equations satisfying various admissibility criteria, De Lellis and Székelyhidi also proved, as an outcome, the existence of bounded initial data allowing for infinitely many admissible weak solutions to the barotropic Euler system. However, the data of [9] for the compressible Euler system were extremely irregular and raised the question whether the ill-posedness was due to the irregularity of the data, rather than to the irregularity of the solution. In [6] and [11], it was shown that initial data with regular density but very irregular velocity still allow for such non-uniqueness phenomena in compressible gas dynamics. Remarkably, in [7], the first author with De Lellis and Kreml proved that even some Lipschitz initial data can generate non-uniqueness among admissible solutions and finally in [8] the same result was achieved for smooth initial data. All these non-unique weak solutions generated by the

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method of convex integration may go under the umbrella name of "wild solutions" (see e.g. Buckmaster et al [3], [4] and the references cited therein for further developments of the convex integration method). While the concept of wild solution may be ambiguous, that of wild data can be distinctively identified with those data that give rise to infinitely many solutions of a given problem on any (short) time interval, see Definition 1.2 below. A natural problem consists then in studying the size of the class of wild data.

In the context of the incompressible Euler equations, Székelyhidi and Wiedemann [13] showed that the set of wild data in the framework of weak solutions satisfying the global energy inequality is dense in the L^p-topology of the phase space. A comparable result for the isentropic Euler system was obtained by Chen et al. [5]. The admissibility criterion used in [5], based on the stipulation that the energy of the system never exceeds the initial energy, may be rough and still compatible with non-physical (increasing) energy profiles. Our goal is to discuss the problem in the class of weak physically admissible entropy solutions of the barotropic Euler system describing the motion of a compressible fluid.

1.1 Barotropic Euler system

We consider the *barotropic Euler system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0 \tag{1.2}$$

describing the time evolution of the mass density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ of a compressible inviscid fluid. The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas under consideration and it is assumed to satisfy $p' > 0$. For simplicity, we impose the space periodic boundary conditions identifying the fluid domain with the flat torus:

$$\mathbb{T}^d = \left([-1,1]|_{\{-1;1\}}\right)^d, \ d = 2,3. \tag{1.3}$$

The same method can be used to obtain similar results for fluids occupying a bounded domain $\Omega \subset R^d$, with impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{n} - \text{the outer normal vector to } \partial\Omega, \tag{1.4}$$

see Sect. 5.

Definition 1.1 *(Admissible entropy solution)* We say that $(\varrho, \mathbf{u})$ is an *admissible entropy solution* to the Euler system (1.1)–(1.3) in $(0,T) \times \mathbb{T}^d$ with initial data

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

if the following holds:

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \,\mathrm{d}x \,\mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_0 \varphi(0,\cdot) \,\mathrm{d}x, \quad \text{for any } \varphi \in C^1_c([0,T) \times \mathbb{T}^d), \tag{1.5}$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \,\mathrm{d}x \,\mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \,\mathrm{d}x \quad \text{for any } \boldsymbol{\varphi} \in C^1_c([0,T) \times \mathbb{T}^d; R^d), \tag{1.6}$$

with the energy inequality:

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) \partial_t \varphi + \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{u} \cdot \nabla_x \varphi \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \geq - \int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0) \right) \varphi(0, \cdot) \, \mathrm{d}x \\
&\text{for any } \varphi \in C^1([0,T) \times \mathbb{T}^d), \ \varphi \geq 0,
\end{aligned} \tag{1.7}
$$

where we have introduced the pressure potential:

$$
P'(\varrho)\varrho - P(\varrho) = p(\varrho).
$$

1.2 Wild data, main result

Definition 1.2 *(Wild data)* We say that the initial data ϱ_0, $\mathbf{u}_0$ are *wild* if there exists $T_{\mathrm{w}} > 0$ such that the Euler system admits infinitely many admissible entropy solutions $(\varrho, \mathbf{u})$ on any interval $[0, T]$, $0 < T < T_{\mathrm{w}}$ such that

$$
\varrho \in L^\infty((0,T) \times \mathbb{T}^d), \ \varrho > 0, \ \mathbf{u} \in L^\infty((0,T) \times \mathbb{T}^d; R^d).
$$

We are ready to state our main result.

Theorem 1.3 (Density of wild data) *Suppose $p \in C^\infty(a,b)$, $p' > 0$ in (a,b), for some $0 \leq a < b \leq \infty$.*

Then for any

$$
\varrho_0 \in W^{k,2}(\mathbb{T}^d), \ a < \inf_{\mathbb{T}^d} \varrho_0 \leq \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d), \ k > \frac{d}{2} + 1,
$$

any $\varepsilon > 0$, and any $1 \leq p < \infty$, there exist wild data $\varrho_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$ such that

$$
\|\varrho_{0,\varepsilon} - \varrho_0\|_{L^p(\mathbb{T}^d)} < \varepsilon, \ \|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\|_{L^p(\mathbb{T}^d; R^d)} < \varepsilon.
$$

Recently, Chen et al. [5] established density of wild data for the isentropic Euler system in the class of weak solutions satisfying the total energy inequality

$$
\int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x \leq \int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0) \right] \, \mathrm{d}x \text{ for any } \tau > 0. \tag{1.8}
$$

These solutions are global in time; however, the associated total energy profile

$$
\int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x
$$

may not be non-increasing in time. In this paper, we revisit the problem in the class of physically admissible entropy solutions satisfying the energy inequality in the differential form on a possibly short time interval, thus ruling out non-physical (increasing) energy profiles.

The proof of Theorem 1.3 is based on a combination of the strong solution ansatz proposed by Chen, Vasseur, and You [5] with the abstract convex integration results concerning weak solutions with a given energy profile established in [10].

2 Convex integration ansatz

Similarly to Chen, Vasseur, and You [5], our convex integration ansatz is based on strong solutions to the Euler system.

2.1 Local in time smooth solutions

Proposition 2.1 (Local existence for smooth data) *Suppose $p \in C^\infty(a,b)$, $p' > 0$ in (a,b), for some $0 \le a < b \le \infty$.*

Then for any initial data

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ a < \inf_{\mathbb{T}^d} \varrho_0 \le \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1 \tag{2.1}$$

there exists $T_{\max} > 0$ such that the compressible Euler system admits a classical solution $(\varrho, \mathbf{u})$ unique in the class

$$\varrho \in C([0,T]; W^{k,2}(\mathbb{T}^d)),\ a < \varrho < b,\ \mathbf{u} \in C([0,T]; W^{k,2}(\mathbb{T}^d; R^d)) \tag{2.2}$$

for any $0 < T < T_{\max}$.

The proof of Proposition 2.1 is nowadays standard and essentially attributed to Kato [12], cf. also Benzoni-Gavage and Serre [2, Chapter 13, Theorem 13.1].

2.2 Basic convex integration ansatz

Consider the initial data $(\varrho_0, \mathbf{u}_0)$ in the regularity class (2.1) together with the associated smooth solution $(\tilde\varrho, \tilde{\mathbf{u}})$ in $[0,T] \times \mathbb{T}^d$, $T < T_{\max}$. In addition, denote $\widetilde{\mathbf{m}} = \tilde\varrho\tilde{\mathbf{u}}$. The Euler system written in the conservative variables $(\tilde\varrho, \widetilde{\mathbf{m}})$ reads

$$\partial_t \tilde\varrho + \mathrm{div}_x \widetilde{\mathbf{m}} = 0, \tag{2.3}$$

$$\partial_t \widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\tilde\varrho} + p(\tilde\varrho)\mathbb{I} \right) = 0. \tag{2.4}$$

We look for solutions in the form

$$\varrho = \tilde\varrho,\ \mathbf{m} = \varrho\mathbf{u} = \widetilde{\mathbf{m}} + \mathbf{v},$$

where

$$\mathrm{div}_x \mathbf{v} = 0, \tag{2.5}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\tilde\varrho} - \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\tilde\varrho} \right) = 0, \tag{2.6}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{2.7}$$

Here, $\mathbf{v}_0 \in L^2(\mathbb{T}^d; R^d)$, $\mathrm{div}_x \mathbf{v}_0 = 0$, is an initial condition for the problem (2.5), (2.6) that will be specified below.

To apply the abstract results of [10], we rewrite problem (2.5)–(2.7) in the form

$$\mathrm{div}_x \mathbf{v} = 0, \tag{2.8}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\tilde{\varrho}} - \frac{1}{d} \frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} \mathbb{I} - \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\tilde{\varrho}} + \frac{1}{d} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} \mathbb{I} \right) = 0, \tag{2.9}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{2.10}$$

together with the prescribed "kinetic energy"

$$\frac{1}{2} \frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} = \frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \Lambda, \tag{2.11}$$

with a suitable spatially homogeneous function $\Lambda = \Lambda(t)$ determined below.

3 Application of convex integration

Setting

$$\mathbb{H} = \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\tilde{\varrho}} - \frac{1}{d} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} \mathbb{I} \in C^1([0, T] \times \mathbb{T}^d; R^{d \times d}_{0,\mathrm{sym}}),$$
$$e = \frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \Lambda \in C([0, T] \times \mathbb{T}^d), \tag{3.1}$$

we may rewrite (2.8)–(2.11) in the form

$$\mathrm{div}_x \mathbf{v} = 0, \tag{3.2}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\tilde{\varrho}} - \frac{1}{d} \frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} \mathbb{I} - \mathbb{H} \right) = 0, \tag{3.3}$$

$$\frac{1}{2} \frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} = e, \tag{3.4}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \tag{3.5}$$

for fixed $\mathbb{H}$ and e given by (3.1).

3.1 Subsolutions

To apply the abstract results obtained in [10], we introduce the set of *subsolutions*

$$\begin{aligned} X_0 = \Big\{ & \mathbf{v} \in C_{\mathrm{weak}}([0, T]; L^2(\mathbb{T}^d; R^d)) \cap L^\infty((0, T) \times \mathbb{T}^d; R^d) \ \Big| \\ & \mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T; \cdot) = \mathbf{v}_T, \mathbf{v} \in C((0, T) \times \mathbb{T}^d; R^d), \\ & \mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0 \text{ in } \mathcal{D}'((0, T) \times \mathbb{T}^d) \\ & \text{for some } \mathbb{F} \in L^\infty((0, T) \times \mathbb{T}^d; R^{d \times d}_{0,\mathrm{sym}}) \cap C((0, T) \times \mathbb{T}^d; R^{d \times d}_{0,\mathrm{sym}}), \\ & \sup_{0 < \tau < t < T; x \in \mathbb{T}^d} \frac{d}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H} \right] - e < 0, \\ & \text{for any } 0 < \tau < T \Big\}. \end{aligned} \tag{3.6}$$

Here, the symbol $\lambda_{\max}[\mathbb{A}]$ denotes the maximal eigenvalue of a symmetric matrix $\mathbb{A}$. The function $\mathbf{v}_T \in L^2(\mathbb{T}^d; R^d)$, $\mathrm{div}_x \mathbf{v}_T = 0$ specifies the value of the solution at the final time T. Prescribing $\mathbf{v}_0$ together with $\mathbf{v}_T$ may seem like over-determining the problem, however, the method of convex integration used below provides adequate solutions even in this case.

3.2 First existence result

The following results is a special case of [10, Theorem 13.2.1].

Proposition 3.1 *Suppose that set of subsolutions X_0 is non-empty and bounded in $L^\infty((0,T) \times \mathbb{T}^d; R^d)$, $d = 2, 3$.*

Then problem (3.2)–(3.5) admits infinitely many weak solutions.

Fix $\mathbf{v}_0 = \mathbf{v}_T = 0$. Next, using the algebraic inequality

$$\frac{1}{2} \frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} \leq \frac{d}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H} \right] \tag{3.7}$$

we can see that the set X_0 is bounded in $L^\infty((0,T) \times \mathbb{T}^d; R^d)$ as long as $\Lambda \in C[0,T]$ (recall also that $a < \tilde{\varrho} < b$). Finally, we observe that $\mathbf{v} \equiv 0$ is a subsolution as soon as

$$\Lambda(t) > 0 \text{ for any } t \in [0, T]. \tag{3.8}$$

Indeed we may consider $\mathbb{F} \equiv 0$ and compute

$$\frac{d}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H} \right] = \frac{d}{2} \lambda_{\max} \left[\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\tilde{\varrho}} - \mathbb{H} \right] = \frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}$$

while (see (3.1))

$$e = \frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \Lambda.$$

Thus, a direct application of Proposition 3.1 yields the following result.

Theorem 3.2 (Existence with a small initial energy jump) *Let $\Lambda \in C[0,T]$, $\inf_{t \in [0,T]} \Lambda(t) > 0$ be given. Let $\mathbf{v}_0 = 0$.*

Then problem (3.1)–(3.4) admits infinitely many weak solutions $\mathbf{v}$ in $(0,T) \times \mathbb{T}^d$.

As $\mathbf{v}_0 = 0$ and $\Lambda > 0$, the solutions $\varrho = \tilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}} + \mathbf{v}$ necessarily experience an initial energy jump therefore they are not physically admissible. This problem will be fixed in the next section.

3.3 Second existence result

The following results is a special case of [10, Theorem 13.6.1].

Proposition 3.3 *Suppose that set of subsolutions X_0 is non-empty and bounded in $L^\infty((0,T) \times \mathbb{T}^d; R^d)$, $d = 2, 3$.*

Then there exists a set of time $\mathfrak{T} \subset (0,T)$ dense in $(0,T)$ with the following properties:

For any $\tau \in \mathfrak{T}$, there exists $\mathbf{v}^\tau \in \overline{X_0}$ satisfying:

- $$\mathbf{v}^\tau \in C_{\rm weak}([0,T]; L^2(\mathbb{T}^d; R^d)) \cap L^\infty((0,T)\times \mathbb{T}^d; R^d) \cap C((\tau,T)\times \mathbb{T}^d; R^d), \\ \mathbf{v}^\tau(0,\cdot) = \mathbf{v}_0,\ \mathbf{v}^\tau(T,\cdot) = \mathbf{v}_T; \tag{3.9}$$

- $$\partial_t \mathbf{v}^\tau + {\rm div}_x \mathbb{F} = 0 \ in \ \mathcal{D}'((\tau,T)\times\mathbb{T}^d) \tag{3.10}$$

  for some $\mathbb{F} \in L^\infty \cap C((\tau;T)\times \mathbb{T}^d; R^{d\times d}_{0,{\rm sym}})$;

- $$\sup_{\tau+s<t<T, x\in\mathbb{T}^d} \frac{d}{2}\lambda_{\max}\left[\frac{(\mathbf{v}^\tau + \widetilde{\mathbf{m}})\otimes(\mathbf{v}^\tau+\widetilde{\mathbf{m}})}{\tilde{\varrho}} - \mathbb{F} - \mathbb{H}\right] - e < 0 \tag{3.11}$$

  for any $0 < s < T - \tau$;

- $$\frac{1}{2}\int_{\mathbb{T}^d} \frac{|\mathbf{v}^\tau + \widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau,\cdot)\ {\rm d}x = \int_{\mathbb{T}^d} e(\tau,\cdot)\ {\rm d}x. \tag{3.12}$$

In accordance with (3.9)–(3.12), the function $\mathbf{v}^\tau$ can be used as a subsolution on the time interval (τ, T). Then, Proposition 3.1 yields the following result.

Theorem 3.4 (Existence without initial energy jump) *Let* $\Lambda \in C[0,T]$, $\inf_{t\in[0,T]} \Lambda(t) > 0$ *be given.*

Then, there exists a sequence $\tau_n \to 0$ *and* $\mathbf{v}_{0,n}$,

$$\mathbf{v}_{0,n} \to 0 \ weakly\text{-}(*) \ in \ L^\infty(\mathbb{T}^d; R^d)$$

such that problem (3.1)–(3.4) *admits infinitely many weak solutions in* $(\tau_n, T)\times\mathbb{T}^d$ *satisfying*

$$\mathbf{v}(\tau_n,\cdot) = \mathbf{v}_{0,n},\ \mathbf{v}(T,\cdot) = 0,\ \frac{1}{2}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau_n,\cdot) = e(\tau_n). \tag{3.13}$$

Note carefully that continuity of the initial energy stated in (3.13) follows from (3.12) and weak continuity of $\mathbf{v}$.

4 Adjusting the energy profile

To complete the proof of Theorem 1.3, it remains to adjust the energy profile Λ so that:

- $$\limsup_{n\to 0} \|\mathbf{v}_{0,n}\|_{L^2(\mathbb{T}^d;R^d)} < \varepsilon; \tag{4.1}$$
- the energy inequality (1.7) holds for $\mathbf{u} = \mathbf{v} + \widetilde{\mathbf{m}}$, $\varrho = \tilde{\varrho}$, at least on a short time interval.

As for (4.1), it is enough to choose $\Lambda(0) > 0$ small enough. Indeed (3.1), (3.13) yield

$$\frac{1}{2}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau_n,\cdot) = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}}(\tau_n) + \Lambda(\tau_n).$$

Seeing that

$$\mathbf{v}(\tau_n,\cdot) = \mathbf{v}_{0,n} \to 0 \text{ weakly in } L^2(\mathbb{T}^d; R^d)$$

we easily conclude.

Finally, the total energy of the system reads

$$\frac{1}{2}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \Lambda + P(\tilde{\varrho}) \text{ a.a. in } (0,T)\times\mathbb{T}^d.$$

In particular, the energy is continuously differentiable as soon as $\Lambda \in C^1[0,T]$. The desired energy inequality reads

$$\partial_t\left(\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho})\right) + \Lambda' + \mathrm{div}_x\left[\left(\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho})\right)\frac{\widetilde{\mathbf{m}}+\mathbf{v}}{\tilde{\varrho}}\right] \le 0. \quad (4.2)$$

Seeing that the smooth solution $(\tilde{\varrho}, \widetilde{\mathbf{m}})$ satisfies the energy equality we may simplify (4.2) to

$$\Lambda' + \Lambda\mathrm{div}_x\tilde{\mathbf{u}} + \mathrm{div}_x\left[\left(\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho})\right)\frac{\mathbf{v}}{\tilde{\varrho}}\right] \le 0. \quad (4.3)$$

Moreover, as $\mathrm{div}_x\mathbf{v} = 0$,

$$\mathrm{div}_x\left[\left(\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho})\right)\frac{\mathbf{v}}{\tilde{\varrho}}\right] = \nabla_x\left[\frac{1}{\tilde{\varrho}}\left(\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho})\right)\right]\cdot\mathbf{v},$$

and (4.3), reduces to

$$\Lambda' + \Lambda\mathrm{div}_x\tilde{\mathbf{u}} + \nabla_x\left[\frac{1}{\tilde{\varrho}}\left(\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) + \Lambda + p(\tilde{\varrho})\right)\right]\cdot\mathbf{v} \le 0. \quad (4.4)$$

As Λ is decreasing in t, we get

$$\Lambda(t) \le \Lambda(0).$$

Similarly, we control $\|\mathbf{v}\|_{L^\infty((0,T)\times\mathbb{T}^d;R^d)}$ by means of $\Lambda(0)$ and certain norms of the strong solution $\tilde{\varrho}$, $\widetilde{\mathbf{m}}$.

Choosing

$$\Lambda(t) = \varepsilon\exp\left(-\frac{t}{\varepsilon^2}\right),$$

with $\varepsilon > 0$ small enough, we obtain the desired energy inequality at least on a short time interval $(0, T_{\mathrm{w}})$, $T_{\mathrm{w}} > 0$. We have proved Theorem 1.3 for $p = 2$. The same statement holds for a general $1 \le p < \infty$ as all solutions in question are uniformly bounded.

5 Concluding remarks

A similar result can be shown for the more realistic complete slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

imposed on a bounded domain $\Omega \subset R^d$. Note that the result is local in time and that the smooth solutions of the Euler system enjoy the finite speed of propagation property (see e.g. Wang [14] for smooth solutions or Wiedemann [15] for generalized solutions of the compressible Euler system). Consequently, the problem of compatibility conditions may be solved by considering the initial data in the form

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ a < \inf_{\mathbb{T}^d} \varrho_0 \le \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1,$$

$$\mathbf{u} = 0, \varrho = \overline{\varrho} - \text{a positive constant in a meighborhood of } \partial\Omega.$$

The relevant local existence result for strong solutions was proved by Beirão da Veiga [1].

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