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A counterexample to L^∞-gradient type estimates for Ornstein–Uhlenbeck operators

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Abstract

Let (λ_k) be a strictly increasing sequence of positive numbers such that $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$. Let f be a bounded smooth function and denote by $u = u^f$ the bounded classical solution to

$$u(x) - \frac{1}{2} \sum_{k=1}^{m} D_{kk}^2 u(x) + \sum_{k=1}^{m} \lambda_k x_k D_k u(x) = f(x), \quad x \in \mathbb{R}^m.$$

It is known that the following dimension-free estimate holds:

$$\int_{\mathbb{R}^m} \left[\sum_{k=1}^{m} \lambda_k \left(D_k u(y) \right)^2 \right]^{p/2} \mu_m(\mathrm{d}y) \le (c_p)^p \int_{\mathbb{R}^m} |f(y)|^p \mu_m(\mathrm{d}y), \quad 1 < p < \infty$$

where μ_m is the "diagonal" Gaussian measure determined by $\lambda_1, \ldots, \lambda_m$ and $c_p > 0$ is independent of f and m. This is a consequence of generalized Meyer's inequalities [4]. We show that, if $\lambda_k \sim k^2$, then such estimate does not hold when $p = \infty$. Indeed we prove

$$\sup_{f \in C_b^2(\mathbb{R}^m), \quad \|f\|_\infty \le 1} \left\{ \sum_{k=1}^{m} \lambda_k \left(D_k u^f(0) \right)^2 \right\} \to \infty \quad \text{as } m \to \infty.$$

This is in contrast to the case of $\lambda_k = \lambda > 0$, $k \ge 1$, where a dimension-free bound holds for $p = \infty$.

Keywords Ornstein–Uhlenbeck operators · Gradient estimates · Dimension-free constant

Mathematics Subject Classification 47D07 · 60H15 · 42B37 · 35R15

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1 Introduction

Let us recall dimension-free L^p-gradient estimates involving Ornstein–Uhlenbeck operators (cf. [4–6, 21, 25]). Let (λ_k) be a strictly increasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty. \tag{1}$$

For any $m \geq 1$ we denote by A_m the $m \times m$ diagonal matrix with negative eigenvalues $-\lambda_k$, $k = 1, \ldots, m$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a bounded C^2-function with all first and second bounded derivatives, i.e., $f \in C_b^2(\mathbb{R}^m)$, and denote by $u \in C_b^2(\mathbb{R}^m)$ the unique bounded classical solution to

$$u(x) - \Big(\frac{1}{2}\triangle_m u(x) + \langle A_m x, Du(x)\rangle\Big) = u(x) - \frac{1}{2}\sum_{k=1}^{m} D_{kk}^2 u(x) + \sum_{k=1}^{m} \lambda_k x_k D_k u(x) = f(x), \tag{2}$$

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^m$. Here, D_k and D_{kk}^2 are first and second partial derivatives with respect to the canonical basis (e_k) in $\mathbb{R}^m$. The operator we consider is an m-dimensional Ornstein–Uhlenbeck operator, namely $L_m = \frac{1}{2}\triangle_m + \langle A_m x, D\rangle$.

Then, introduce the Gaussian measure $\mu_m = \mathcal{N}(0, (-2A_m)^{-1})$ with mean 0 and covariance matrix $(-2A_m)^{-1}$, with density

$$\varphi_m(x) := \left(\frac{1}{2\pi}\right)^{m/2} \left(\prod_{i=1}^{m} \frac{1}{2\lambda_i}\right)^{-1/2} \exp\left\{-\sum_{i=1}^{m} \lambda_i x_i^2\right\}, \qquad x = (x_1, \ldots, x_m) \in \mathbb{R}^m .$$

Note that L_m is a self-adjoint operator on $L^2(\mathbb{R}^m, \mu_m)$, the usual L^2-space with respect to μ_m. See, for instance, [4, 6, 7, 13]. It is known that, if $1 < p < \infty$, there exists a constant c_p, independent of f and the dimension m, such that the following sharp gradient estimate holds:

$$\int_{\mathbb{R}^m} \Big(\sum_{k=1}^{m} \lambda_k \,(D_k u(y))^2\Big)^{p/2} \mu_m(\mathrm{d}y) \leq (c_p)^p \int_{\mathbb{R}^m} |f(y)|^p \mu_m(\mathrm{d}y). \tag{3}$$

The result follows from the general estimates (11) given in Theorem 5.3 of [4], which extends Proposition 3.5 in [25] (see also the references therein). Note that (3) can be rewritten as

$$\|(-A_m)^{1/2} Du\|_{L^p(\mathbb{R}^m, \mu_m)} \leq c_p \|f\|_{L^p(\mathbb{R}^m, \mu_m)}, \tag{4}$$

where

$$(-A_m)^{1/2} Du(x) = \sum_{k=1}^{m} \sqrt{\lambda_k}\, D_k u(x) e_k.$$

Our main result (cf. Theorem 6 below) shows that, when $p = \infty$, the dimension-free estimate (4) in general fails to hold. Indeed, we prove the following stronger assertion. Writing $u = u^f$ to stress the dependence of the solution u on f, we show that if $\lambda_k \sim k^2$ as $k \to \infty$, then, choosing $x = 0$, we have

$$\sup_{\substack{f \in C_b^2(\mathbb{R}^m) \\ \|f\|_\infty \leq 1}} |(-A_m)^{1/2} Du^f(0)|^2_{\mathbb{R}^m} = \sup_{\substack{f \in C_b^2(\mathbb{R}^m) \\ \|f\|_\infty \leq 1}} \Big\{\sum_{k=1}^{m} \lambda_k \,(D_k u^f(0))^2\Big\} \to \infty \ \text{ as } m \to \infty. \tag{5}$$

In contrast to (5), we point out that, when $A_m = -\lambda I_m$ with $\lambda > 0$ and I_m the $m \times m$ identity matrix, then the following dimension-free L^∞-gradient estimates

$$\|(\lambda)^{1/2} Du^f\|_\infty = \sup_{x\in\mathbb{R}^m} |(\lambda)^{1/2} Du^f(x)|_{\mathbb{R}^m} \le \frac{\pi}{\sqrt{2}} \sup_{x\in\mathbb{R}^m} |f(x)|, \quad f \in C_b^2(\mathbb{R}^m) \tag{6}$$

hold true; see Proposition 5.

1.1 Infinite dimensional Ornstein–Uhlenbeck semigroups

Let us comment on the previous dimension-free L^p_μ-estimate (4). Such kind of inequalities can be deduced from known results for infinite dimensional Ornstein–Uhlenbeck operators. This point of view is of interest in probability because of its connection with SPDEs (see also [8] and the references therein).

To introduce this setting, we replace $\mathbb{R}^m$ by a real separable Hilbert space H with orthonormal basis $(e_k)_{k\ge1}$ and inner product $\langle\cdot,\cdot\rangle$. Then, we consider the unbounded self-adjoint operator $A : D(A) \subset H \to H$ such that

$$D(A) = \Big\{x \in H : \sum_{k\ge1} (\langle x, e_k\rangle)^2 \lambda_k^2 < \infty\Big\}, \quad Ae_k = -\lambda_k e_k, \quad k \ge 1 \tag{7}$$

(cf. [1, 8, 9, 22]). Our condition (1) is equivalent to require that the inverse operator A^{-1} : $H \to H$ is a trace class operator. The operator A generates a strongly continuous semigroup (e^{tA}) on H, given by $e^{tA}e_k = e^{-t\lambda_k}e_k$ for any $t \ge 0$ and $k \ge 1$. We can define the corresponding Ornstein–Uhlenbeck semigroup (P_t) by

$$P_t f(x) = \int_H f(e^{tA}x + \sqrt{I - e^{2tA}}\, y)\, \mathcal{N}\big(0, -(2A)^{-1}\big)(\mathrm{d}y), \quad f \in B_b(H), \ x \in H, \ t \ge 0 \tag{8}$$

where $f : H \to \mathbb{R}$ is a Borel, bounded function (i.e., $f \in B_b(H)$); $\mathcal{N}\big(0, -(2A)^{-1}\big)$ stands for the centered Gaussian measure defined on the Borel σ-algebra of H (see Chapter 1 in [7], [9] and Section 2.2); $I : H \to H$ is the identity.

Formula (8) is an extension of a well-known formula used in finite dimension. From the probabilistic point of view (P_t) is the transition Markov semigroup of the OU stochastic process (X_t^x) which solves $dX_t = AX_t\mathrm{d}t + dW_t$, $X_0 = x$ where W is a cylindrical Wiener process on H; cf. [7, 8, 14]. When $f \in C_b^2(H)$, i.e. f is bounded, twice Fréchet-differentiable with first and second bounded and continuous derivatives, we consider $u : H \to \mathbb{R}$ given by

$$u(x) = R(1, L) f(x) = \int_0^\infty e^{-t}(P_t f)(x)\mathrm{d}t, \quad x \in H. \tag{9}$$

Following Chapter 6 in [7], u is the generalized bounded solution to $u - Lu = f$, where L is formally given by $\frac{1}{2}\mathrm{Tr}(D^2) + \langle x, AD\rangle$. Here, we only note that if f is also cylindrical, i.e., there exist $m \ge 1$ and $\tilde{f} \in C_b^2(\mathbb{R}^m)$ such that

$$f(x) = \tilde{f}(\langle x, e_1\rangle, \ldots, \langle x, e_m\rangle), \quad x \in H, \tag{10}$$

then u given in (9) depends only on a finite number of variables, i.e.,

$$u(x) = \tilde{u}(\langle x, e_1\rangle, \ldots, \langle x, e_m\rangle), \ x \in H$$

(cf. Sect. 2.2). Moreover, $\tilde{u}$ solves (2) with f replaced by $\tilde{f}$. In addition, if $f \in C_b^2(H)$, we have that $u = R(1, L)f \in C_b^2(H)$, and $Du(x) \in D((-A)^{1/2})$, $x \in H$ (cf. [9] for stronger results). By Theorem 5.3 of [4] (see also Corollary 5.4 in [4] and Remark 1), there exists a constant c_p (independent of f) such that

$$\|(-A)^{1/2}Du\|_{L^p(H,\mu)} \le c_p \|f\|_{L^p(H,\mu)}, \quad 1 < p < \infty, \tag{11}$$

where $\mu = \mathcal{N}\big(0, -(2A)^{-1}\big)$. Moreover, we have $\|D^2u\|_{L^p(H,\mu)} \le c_p \|f\|_{L^p(H,\mu)}$, i.e.,

$$\int_H \Big(\sum_{k=1}^{\infty} (D_{kk}u(y))^2 \Big)^{p/2} \mu(\mathrm{d}y) \le (c_p)^p \int_H |f(y)|^p \mu(\mathrm{d}y). \tag{12}$$

It is not difficult to show that (11) implies (4) using cylindrical functions f as in (10); see Remark 8. Estimates (11) and (12) are part of the generalized Meyer's inequalities proved in [4] using the elliptic Littlewood-Paley-Stein inequalities associated with the OU semigroup (P_t). For applications of the classical Meyer's inequalities to the Malliavin Calculus we refer to [15, 20, 21, 26] (see also Remark 2). The results given in [4] give a characterization of the domain of the generator of (P_t) in $L^p(H,\mu)$; see also [5] (the case $p = 2$ was obtained earlier in [6]). We also mention the characterization of the domain of non self-adjoint Ornstein–Uhlenbeck generators given in [17, 19]. Estimates (12) have been used to prove strong uniqueness for a class of SPDEs in [10]. For related results on Ornstein–Uhlenbeck operators in Gaussian harmonic analysis we refer to [2, 3, 13, 18] and the references therein.

Our main result implies that (11) fails to hold for $p = \infty$, i.e. it is not true that there exists $C > 0$, independent of f, such that

$$\sup_{x\in H} |(-A)^{1/2}DR(1,L)f(x)|_H \le C \sup_{x\in H} |f(x)|, \quad f \in C_b^2(H), \tag{13}$$

where we have used $u = R(1,L)f$ as in (9) (see in particular Corollary 7). This estimate is stated in [22, Theorem 7] which is based on [22, Lemma 6]. However, there is a mistake in the proof of that lemma. In particular, we show that [22, Theorem 7] cannot hold.

Remark 1 Let us recall the notation used in [4] to study general symmetric Ornstein–Uhlenbeck semigroups in Hilbert spaces. For the sake of notational clarity, the operator C used in [4] corresponds to our $-(2A)^{-1}$, while our semigroup (e^{tA}) corresponds to (e^{-tA}) in [4]. They use the Malliavin gradient $D_I = C^{1/2}D$ (where D is the Fréchet derivative) and $D_A = \frac{1}{\sqrt{2}}D$. Moreover, the symbol $D_{A^2} = AD_I$, which is used in the definition of the Sobolev space $W^{1,p}_{A^2}$ (see Corollary 5.4 in [4]) corresponds to our operator $\frac{1}{\sqrt{2}}(-A)^{1/2}D$.

Remark 2 Let us recall the classical Ornstein–Uhlenbeck semigroup (S_t)

$$S_t f(x) = \int_H f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\, \nu(\mathrm{d}y), \quad f \in B_b(H),\ x \in H, \tag{14}$$

where ν is a centered Gaussian measure on H (see Sect. 2.2). The classical Meyer's inequalities give a complete characterization of the domains of $(I - N_p)^{m/2}$ in $L_p(H,\nu)$ for all $p \in (1,\infty)$ and $m = 1, 2, \dots$ in terms of Gaussian Sobolev spaces related to ν. Here, N_p denotes the generator of (S_t) in $L_p(H,\nu)$ (see [15, 20, 21]). For a discussion of Meyer's inequalities in the Malliavin Calculus we refer to [15] and [26, Chapter 4].

Remark 3 Estimates like (11) and (12) hold true also in Hölder spaces (see [9, 23] for more details). In particular, for any $\theta \in (0,1)$, there exists an absolute constant c_θ only depending on θ such that

$$\|(-A)^{1/2}DR(1,L)f\|_{C_b^\theta(H,H)} \le c_\theta \|f\|_{C_b^\theta(H)}. \tag{15}$$

Remark 4 We do not know if for $p \neq 2$ the constant c_p appearing in (3) is an absolute constant (independent of the positive eigenvalues (λ_k)). Indeed, as we have mentioned before, the dimension-free estimate (3) follows from infinite dimensional estimates like (11) and (12) which are proved in Theorem 5.3 of [4] (extending Proposition 3.5 in [25]). However, Theorem 5.3 uses Lemma 5.1 in [4], whose proof invokes results on sums of operators with bounded imaginary powers (see Theorem 4 and Corollary 2 in [24]). The approach of [11] and [24], which has been also used in [19], does not provide sharp constants in the estimates and so we do not know if c_p also depends on A. We point out that the estimates given in Proposition 3.5 of [25] provide absolute constants.

2 Notations and main results

Let Q be a symmetric and positive definite $m \times m$ matrix. We denote by $\mathcal{N}(0, Q)$ the Gaussian measure with mean 0 and covariance matrix Q, which has density

$$\left(\frac{1}{2\pi}\right)^{m/2} \frac{1}{\sqrt{\det Q}} \exp\left\{-\frac{1}{2}\,|Q^{-1/2}x|^2\right\} \tag{16}$$

with respect to the m-dimensional Lebesgue measure. We first consider for $\lambda > 0$ the equation

$$v(x) - \left(\frac{1}{2}\triangle_m v(x) - \lambda\langle x, Dv(x)\rangle\right) = v(x) - M_m v(x) = f(x), \quad x \in \mathbb{R}^m, \tag{17}$$

with $M_m = \frac{1}{2}\triangle_m - \lambda\langle x, D\rangle$. We assume that $f \in C_b^2(\mathbb{R}^m)$. Equation (17) is similar to (2) with A_m replaced by $-\lambda I_m$. Using the following Ornstein–Uhlenbeck semigroup (S_t^m)

$$S_t^m f(x) = \int_{\mathbb{R}^m} f(e^{-\lambda t}x + \sqrt{1 - e^{-2\lambda t}}\, y)\, \mathcal{N}\Big(0, \frac{1}{2\lambda} I_m\Big)\, (\mathrm{d}y), \quad x \in \mathbb{R}^m,\ t \geq 0, \tag{18}$$

we find (cf. (9), and [7])

$$v(x) = R(1, M_m)f(x) = \int_0^\infty e^{-t}(S_t^m f)(x)\mathrm{d}t, \quad x \in \mathbb{R}^m.$$

Then, we have the following

Proposition 5 *For any $\lambda > 0$ it holds:*

$$\sup_{x\in\mathbb{R}^m} \sqrt{\lambda}|DR(1, M_m)f(x)|_{\mathbb{R}^m} \leq \frac{\pi}{\sqrt{2}} \sup_{x\in\mathbb{R}^m} |f(x)|, \quad f \in C_b^2(\mathbb{R}^m). \tag{19}$$

Proof Let $v(x) = R(1, M_m)f \in C_b^2(\mathbb{R}^m)$. We set $v(x) = u(\sqrt{\lambda}\, x)$ and so, for $y \in \mathbb{R}^m$, we get

$$u(y) - \frac{\lambda}{2}\triangle u(y) + \lambda\langle y, Du(y)\rangle = f(y/\sqrt{\lambda})$$

and

$$\frac{1}{\lambda}u(y) - \frac{1}{2}\triangle u(y) + \langle y, Du(y)\rangle = \frac{1}{\lambda} f(y/\sqrt{\lambda}) = \tilde{f}(y).$$

We have

$$u(x) = \int_0^\infty e^{-\frac{1}{\lambda}t}\mathrm{d}t \int_{\mathbb{R}^m} \tilde{f}(e^{-t}x + y)\mathcal{N}\Big(0, \frac{1 - e^{-2t}}{2}\, I_m\Big)(\mathrm{d}y)$$

and, considering the directional derivative $\langle Du(x), h\rangle = D_h u(x)$, $h \in \mathbb{R}^m$, $|h| = 1$, we get, differentiating under the integral sign,

$$D_h u(x) = 2\int_0^\infty e^{-\frac{1}{\lambda}t} \int_{\mathbb{R}^m} \tilde{f}(e^{-t}x + y)\frac{e^{-t}}{1-e^{-2t}}\,\langle h, y\rangle\, \mathcal{N}\Big(0, \frac{1-e^{-2t}}{2}\, I_m\Big)(\mathrm{d}y)$$

(cf. Theorem 6.2.2 in [7], [9]). Then, changing variable in the integral over $\mathbb{R}^m$, we obtain

$$\begin{aligned}\|D_h u\|_\infty &\le 2\|\tilde{f}\|_\infty \int_0^\infty \frac{e^{-\frac{1}{\lambda}t}\, e^{-t}}{1-e^{-2t}}\mathrm{d}t \int_{\mathbb{R}^m} \Big|\langle h, \big(\frac{1-e^{-2t}}{2}\big)^{1/2} y\rangle\Big| \mathcal{N}\big(0,\ I_m\big)(\mathrm{d}y)\\ &\le \frac{\sqrt{2}}{\lambda}\|f\|_\infty \int_0^\infty \frac{e^{-t}}{(1-e^{-2t})^{1/2}}\mathrm{d}t \int_{\mathbb{R}^m} \big|\langle h, y\rangle\big| \mathcal{N}\big(0,\ I_m\big)\mathrm{d}y \le \frac{\pi}{\lambda\sqrt{2}}\|f\|_\infty.\end{aligned}$$

Since $D_h u(y) = \frac{1}{\sqrt{\lambda}} D_h v(\frac{y}{\sqrt{\lambda}})$, we have $\|D_h u\|_\infty = \frac{1}{\sqrt{\lambda}}\|D_h v\|_\infty$ and (19) follows. □

Let us start the proof of the main estimate (5) concerning equation (2) involving the Ornstein–Uhlenbeck operator L_m. Similarly to the proof of Proposition 5 the solution $u \in C_b^2(\mathbb{R}^m)$ to (2) is given by

$$u(x) = R(1, L_m) f(x) = \int_0^\infty e^{-t}(P_t^m f)(x)\mathrm{d}t \tag{20}$$

with

$$\begin{aligned}P_t^m f(x) &= \int_{\mathbb{R}^m} f(e^{tA_m}x + \sqrt{I_m - e^{2tA_m}}\, y)\, \mathcal{N}\Big(0, -\frac{1}{2}A_m^{-1}\Big)\,(\mathrm{d}y)\\ &= \int_{\mathbb{R}^m} f(e^{tA_m}x + y)\, \mathcal{N}\big(0, Q_t^m\big)\,(\mathrm{d}y), \quad f \in C_b^2(\mathbb{R}^m),\ x \in \mathbb{R}^m,\end{aligned}$$

where

$$Q_t^m = \int_0^t e^{2sA_m} ds = (-2A_m)^{-1}(I_m - e^{2tA_m}), \quad t \ge 0$$

(Q_t^m is a diagonal matrix with positive eigenvalues). Let $\mu_t^m = \mathcal{N}\big(0, Q_t^m\big)$. By differentiating under the integral sign, the following formula holds for the directional derivative of $P_t^m f$ along $h \in \mathbb{R}^m$:

$$D_h P_t^m f(x) = \langle DP_t^m f(x), h\rangle = \int_{\mathbb{R}^m} \langle \Lambda_t^m h, (Q_t^m)^{-\frac{1}{2}} y\rangle\, f(e^{tA_m}x + y)\mu_t^m(\mathrm{d}y),\ x \in \mathbb{R}^m,\ t > 0, \tag{21}$$

where $\Lambda_t^m = (Q_t^m)^{-1/2} e^{tA_m}$; cf. Theorem 6.2.2 in [7]. Hence, the term

$$(-A_m)^{1/2} Du^f(0) = (-A_m)^{1/2} DR(1, L_m) f(0) \in \mathbb{R}^m$$

that appears in (5) has components

$$\begin{aligned}\langle (-A_m)^{1/2} Du^f(0), e_k\rangle &= \int_0^\infty e^{-t}\mathrm{d}t \int_{\mathbb{R}^m} \langle (-A_m)^{1/2}\Lambda_t^m e_k, (Q_t^m)^{-\frac{1}{2}} y\rangle\, f(y)\mu_t^m(\mathrm{d}y)\\ &= \int_0^\infty e^{-t}\mathrm{d}t \int_{\mathbb{R}^m} \langle (-A_m)^{1/2}\Lambda_t^m e_k, y\rangle\, f((Q_t^m)^{\frac{1}{2}} y)\,\mathcal{N}(0, I_m)(\mathrm{d}y)\end{aligned}$$

for $k = 1, \dots, m$. An easy calculation shows that

$$
\begin{aligned}
&|(-A_m)^{1/2} DR(1, L_m) f(0)|^2 \\
&\quad = \sum_{k=1}^{m} \Big(\int_0^\infty \frac{\lambda_k e^{-t} e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2}} \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} f(c_1(t)x_1, \dots, c_m(t)x_m)\, x_k\, e^{-\frac{|x|^2}{2}} \mathrm{d}x \mathrm{d}t \Big)^2,
\end{aligned}
\tag{22}
$$

where, for $t \geq 0$, $c_k(t) = \left(\frac{1 - e^{-2\lambda_k t}}{2\lambda_k} \right)^{1/2}$ and $(Q_t^m)^{1/2} = \mathrm{diag}[c_1(t), \dots, c_m(t)]$.

We will prove the following result.

Theorem 6 *Let (λ_k) be a strictly increasing sequence of positive numbers, such that $\lambda_k \sim k^2$ as $k \to +\infty$. Then, assertion (5) is in force, i.e., taking into account (22), there holds*

$$
\begin{aligned}
\sup_{m \in \mathbb{N}} \sup_{\substack{f \in C_b^2(\mathbb{R}^m) \\ \|f\|_\infty \leq 1}} \sum_{k=1}^{m} \Big(\int_0^\infty \frac{\lambda_k e^{-t} e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2}} \times \\
\times \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} f(c_1(t)x_1, \dots, c_m(t)x_m)\, x_k\, e^{-\frac{|x|^2}{2}} \mathrm{d}x \mathrm{d}t \Big)^2 = \infty.
\end{aligned}
$$

The proof of this theorem is given in Section 3.

Finally, we show that Theorem 6 implies that the infinite dimensional estimate (13) cannot hold.

Corollary 7 *Under the same assumptions of Theorem 6, there holds*

$$
\sup_{f \in C_b^2(H),\ \|f\|_\infty \leq 1} |(-A)^{1/2} D(R(1, L) f)(0)|_H = \infty.
$$

Proof of Corollary 7 Recall that we are considering a real separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. According to Chapter 1 in [7], we can rewrite the OU semigroup (P_t) in (8) as follows

$$
P_t f(x) = \int_H f(e^{tA}x + y)\, \mathcal{N}\big(0, Q_t\big)(\mathrm{d}y), \quad f \in C_b^2(H),\ x \in H,
\tag{23}
$$

where $Q_t = \int_0^t e^{2sA} ds := (-2A)^{-1}(I - e^{2tA}),\ t \geq 0$, and A is given in (7). Suppose that $f \in C_b^2(H)$ is also *cylindrical*, i.e. that (10) holds for some $m \geq 1$ and $\tilde{f} \in C_b^2(\mathbb{R}^m)$. Identifying H with l^2, we have that $f(e^{tA}x + y) = \tilde{f}(e^{tA_m}x^{(m)} + y^{(m)})$, where A_m is the same matrix given in (2) and (20) and $h^{(m)} := (\langle h, e_1 \rangle, \dots, \langle h, e_m \rangle) \in \mathbb{R}^m$, for any $h \in H$. Moreover, we put $\mu := \mathcal{N}\big(0, -(2A)^{-1}\big) = \mathcal{N}\big(0, -(2A_m)^{-1}\big) \times \nu_m$, where $\nu_m = \prod_{k=m+1}^{\infty} N(0, (2\lambda_k)^{-1})$; see Theorem 1.2.1 in [7]. It follows that, for any $x \in H$,

$$
\begin{aligned}
P_t f(x) &= P_t^m(\tilde{f})(x^{(m)}) = \int_{\mathbb{R}^m} \tilde{f}(e^{tA_m}x^{(m)} + \sqrt{I_m - e^{2tA_m}}\, y)\, \mathcal{N}\big(0, -(2A_m)^{-1}\big)(\mathrm{d}y), \\
u(x) &= R(1, L) f(x) = \int_0^\infty e^{-t}(P_t f)(x) \mathrm{d}t = \tilde{u}(\langle x, e_1 \rangle, \dots, \langle x, e_m \rangle)
\end{aligned}
$$

where $\tilde{u}$ is given in (20) with $\tilde{f}$ in place of f therein. Setting $\mu_m := \mathcal{N}\big(0, -(2A_m)^{-1}\big)$ and using that $C_b^2(H)$ contains the cylindrical functions displayed in (10), we get that

$$\sup_{\substack{\tilde{f}\in C_b^2(\mathbb{R}^m)\\ \|\tilde{f}\|_{\infty)}\le 1}} |(-A_m)^{1/2}D\tilde{u}(0)| \le \sup_{\substack{f\in C_b^2(H)\\ \|f\|_\infty\le 1}} |(-A)^{1/2}Du(0)|_H, \tag{24}$$

holds for any $m \ge 1$. Notice that on the left-hand side of (24) we have $0 \in \mathbb{R}^m$ while on the right-hand side we have $0 \in H$. Thus, as a consequence of Theorem 6, we deduce the assertion. □

Remark 8 By the same argument as in the previous proof we get easily

$$\sup_{\substack{\tilde{f}\in C_b^2(\mathbb{R}^m)\\ \|\tilde{f}\|_{L^p(\mathbb{R}^m,\mu_m)}\le 1}} \|(-A_m)^{1/2}D\tilde{u}\|_{L^p(\mathbb{R}^m,\mu_m)} \le \sup_{\substack{f\in C_b^2(H)\\ \|f\|_{L^p(H,\mu)}\le 1}} \|(-A)^{1/2}Du\|_{L^p(H,\mu)}, \tag{25}$$

for any $m \ge 1$ and $p \in (1, +\infty)$. This can be used to deduce that (3) or (4) follow from (11) and (12).

3 Proof of Theorem 6

Let $\delta \in (0, +\infty)$. Then, put $S_m = S_m(\delta)$ with

$$S_m(\delta) = \sup_{\substack{f\in C_b^2(\mathbb{R}^m)\\ \|f\|_\infty\le 1}} \sum_{k=1}^m \Big(\int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1-e^{-2\lambda_k t})^{1/2}} \int_{\mathbb{R}^m} f(c_1(t)x_1, \dots, c_m(t)x_m)\, x_k \, \frac{e^{-\frac{|x|^2}{2}}}{\sqrt{(2\pi)^m}} \mathrm{d}x\mathrm{d}t \Big)^2 .$$

If we show that

$$\sup_{m\ge 2} S_m = \infty \tag{26}$$

holds under the assumption that $\lambda_k \sim k^2$ as $k \to +\infty$, then the validity of Theorem 6 will follow.

3.1 Two useful lemmas

The following identity will be important. Recall that $x_k = \langle x, e_k\rangle$, $k = 1, \dots, m$ where (e_j) denotes the canonical basis in $\mathbb{R}^m$.

Lemma 9 *For any $m \ge 2$, $k \in \{1, \dots, m\}$, $c = (c_1, \dots, c_m) \in \mathbb{R}^m \setminus \{0\}$ and $F \in B_b(\mathbb{R})$, it holds*

$$\begin{aligned} I_{m,k}(F) &= \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} F(\langle c, x\rangle) x_k \, e^{-\frac{|x|^2}{2}} dx \\ &= \frac{2\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)} \frac{c_k}{|c|} \int_0^{+\infty}\int_0^\pi e^{-\frac{1}{2}\rho^2}\rho^m \cos\vartheta (\sin\vartheta)^{m-2} F(|c|\rho\cos\vartheta)\mathrm{d}\rho\mathrm{d}\vartheta. \end{aligned} \tag{27}$$

Proof We provide additional details for the sake of completeness. Let us first consider $m = 2$. We introduce the unitary vectors $\gamma_1 = c/|c|$ and $\gamma_2 \in \mathbb{R}^2$ such that (γ_1, γ_2) is an orthonormal basis in $\mathbb{R}^2$. Using the polar coordinates with respect to such basis we can write

$$x = \rho\cos\theta\,\gamma_1 + \rho\sin\theta\,\gamma_2,$$

which entails that

$$\begin{aligned} I_{2,k}(F) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} \rho^2 F(|c|\rho\cos\theta)\,(\cos\theta\,\langle\gamma_1, e_k\rangle + \sin\theta\,\langle\gamma_2, e_k\rangle)\,e^{-\frac{\rho^2}{2}}\,\mathrm{d}\rho\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\int_0^{+\infty} \rho^2 F(|c|\rho\cos\theta)\,\cos\theta\,\langle\gamma_1, e_k\rangle\,e^{-\frac{\rho^2}{2}}\,\mathrm{d}\rho\mathrm{d}\theta, \quad k = 1, 2 \end{aligned}$$

since $\int_0^{2\pi} F(|c|\rho\cos\theta)\,\sin\theta\,\mathrm{d}\theta = 0$. Indeed, to prove this last identity under the sole assumption that $F \in B_b(\mathbb{R})$, we just notice that

$$\begin{aligned} \int_0^{2\pi} F(|c|\rho\cos\theta)\,\sin\theta\,\mathrm{d}\theta &= \int_0^{\pi} F(|c|\rho\cos\theta)\,\sin\theta\,\mathrm{d}\theta + \int_\pi^{2\pi} F(|c|\rho\cos\theta)\,\sin\theta\,\mathrm{d}\theta \\ &= \int_{-1}^{1} F(|c|\rho t)\,\mathrm{d}t - \int_{-1}^{1} F(|c|\rho t)\,dt = 0 \end{aligned}$$

holds as a consequence of the change of variable $\cos\theta = t$. Finally, we get easily (27) for $m = 2$ upon recalling that $\Gamma(1/2) = \sqrt{\pi}$.

In the general case of $m \geq 3$, we consider an orthonormal basis (γ_k) of $\mathbb{R}^m$ where $\gamma_1 = c/|c|$. Then, we introduce polar coordinates with respect to (γ_k). Let $\rho = |x|$. Proceeding similarly to [12, Sect. 5.9], we have, for $x \neq 0$,

$$x = \rho\cos\theta_1\gamma_1 + \rho\sin\theta_1\cos\theta_2\gamma_2 + \ldots + \rho\sin\theta_1\cdots\sin\theta_{m-2}\sin\theta_{m-1}\gamma_m,$$

where $\rho > 0$ (radial distance), $\theta_1, \ldots, \theta_{m-2} \in [0, \pi]$ (latitudes; θ_1 is the angle between x and γ_1) and $\theta_{m-1} \in [0, 2\pi]$ (longitude). Let $\theta = (\theta_1, \ldots, \theta_{m-1})$. Denote by

$$J(\rho, \theta) = \rho^{m-1}(\sin\theta_1)^{m-2}(\sin\theta_2)^{m-3}\cdots(\sin\theta_{m-2})$$

the Jacobian determinant. Moreover, set $\gamma_i^{(k)} = \langle\gamma_i, e_k\rangle$, for $i, k = 1, \ldots, m$. Let

$$\begin{aligned} &\xi_1(\theta) = \cos\theta_1, \quad \xi_2(\theta) = \sin\theta_1\cos\theta_2, \ \ldots, \\ &\xi_{m-1}(\theta) = \sin\theta_1\cdots\sin\theta_{m-2}\cos\theta_{m-1}, \quad \xi_m(\theta) = \sin\theta_1\cdots\sin\theta_{m-2}\sin\theta_{m-1}. \end{aligned}$$

We infer that

$$\begin{aligned} I_{m,k}(F) &= \frac{1}{\sqrt{(2\pi)^m}}\int_0^\infty\int_{[0,\pi]^{m-2}\times[0,2\pi]} \rho e^{-\frac{\rho^2}{2}} F(|c|\,\rho\cos\theta_1)\left(\sum_{i=1}^m \xi_i(\theta)\gamma_i^{(k)}\right) J(\rho,\theta)\mathrm{d}\rho\mathrm{d}\theta \\ &= \frac{1}{\sqrt{(2\pi)^m}}\int_0^\infty\int_{[0,\pi]^{m-2}\times[0,2\pi]} \rho e^{-\frac{\rho^2}{2}} F(|c|\,\rho\cos\theta_1)\,\xi_1(\theta)\gamma_1^{(k)}\,J(\rho,\theta)\mathrm{d}\rho\mathrm{d}\theta \quad (28) \end{aligned}$$

using that

$$\frac{1}{\sqrt{(2\pi)^m}}\int_0^\infty\int_{[0,\pi]^{m-2}\times[0,2\pi]} \rho e^{-\frac{\rho^2}{2}} F(|c|\,\rho\cos\theta_1)\left(\sum_{i=2}^m \xi_i(\theta)\gamma_i^{(k)}\right) J(\rho,\theta)\mathrm{d}\rho\mathrm{d}\theta = 0. \tag{29}$$

In order to prove (29) we check that if $\rho > 0$ then

$$\int_{[0,\pi]^{m-2}\times[0,2\pi]} F(|c|\,\rho\cos\theta_1)\,\xi_i(\theta)\,J(\rho,\theta)\mathrm{d}\theta = 0, \quad 2 \le i \le m. \tag{30}$$

If $i = m$, we find that

$$\begin{aligned}
&\int_{[0,\pi]^{m-2}\times[0,2\pi]} F(|c|\,\rho\cos\theta_1)\,\xi_m(\theta)\,(\sin\theta_1)^{m-2}(\sin\theta_2)^{m-3}\cdots(\sin\theta_{m-2})\mathrm{d}\theta \\
&= \int_0^\pi F(|c|\,\rho\cos\theta_1)(\sin\theta_1)^{m-2}\sin\theta_1\mathrm{d}\theta_1\times \\
&\times\int_{[0,\pi]^{m-3}\times[0,2\pi]} \sin\theta_2\cdots\sin\theta_{m-1}\,(\sin\theta_2)^{m-3}\cdots(\sin\theta_{m-2})\mathrm{d}\theta_2\cdots\mathrm{d}\theta_{m-1} = 0
\end{aligned}$$

by the Fubini theorem, since $\int_0^{2\pi}\sin\theta_{m-1}\mathrm{d}\theta_{m-1} = 0$. Similarly we obtain that (30) holds with $i = m-1$. Note that up to now we have already proved (30) when $m = 3$. Let $m \ge 4$. We check (30) when $2 < i \le m-2$. We have

$$\begin{aligned}
&\int_{[0,\pi]^{m-2}\times[0,2\pi]} F(|c|\,\rho\cos\theta_1)\,\xi_i(\theta)\,(\sin\theta_1)^{m-2}(\sin\theta_2)^{m-3}\cdots(\sin\theta_{m-2})\mathrm{d}\theta \\
&= \int_0^\pi F(|c|\,\rho\cos\theta_1)(\sin\theta_1)^{m-2}\sin\theta_1\mathrm{d}\theta_1\times \\
&\times\int_{[0,\pi]^{m-3}\times[0,2\pi]} \sin\theta_2\cdots\cos\theta_i\,(\sin\theta_2)^{m-3}\cdots(\sin\theta_{m-2})\mathrm{d}\theta_2\cdots\mathrm{d}\theta_{m-1} = 0,
\end{aligned}$$

because $\int_0^\pi\cos\theta_i\,(\sin\theta_i)^{m-1-i}\mathrm{d}\theta_i = 0$. Similarly, for $i = 2$, we get

$$\begin{aligned}
&\int_0^\pi F(|c|\,\rho\cos\theta_1)(\sin\theta_1)^{m-2}\sin\theta_1\mathrm{d}\theta_1\times \\
&\quad\times\int_{[0,\pi]^{m-3}\times[0,2\pi]} \cos\theta_2\,(\sin\theta_2)^{m-3}\cdots(\sin\theta_{m-2})d\theta_2\cdots\mathrm{d}\theta_{m-1} = 0.
\end{aligned}$$

We have verified (30) and so (28) holds. We rewrite (28) as follow

$$\begin{aligned}
I_{m,k}(F) &= R_m\,\frac{\gamma_1^{(k)}}{\sqrt{(2\pi)^m}}\int_0^\infty\int_0^\pi \rho^m e^{-\frac{\rho^2}{2}}F(|c|\,\rho\cos\theta_1)\,\cos\theta_1\,(\sin\theta_1)^{m-2}\mathrm{d}\rho\mathrm{d}\theta_1 \\
\gamma_1^{(k)} &= \frac{c_k}{|c|},
\end{aligned} \tag{31}$$

where $R_m = 2\pi$ if $m = 3$ and if $m > 3$

$$\begin{aligned}
R_m &= \int_{[0,\pi]^{m-3}\times[0,2\pi]} (\sin\theta_2)^{m-3}(\sin\theta_3)^{m-4}\cdots\sin\theta_{m-2}d\theta_2\cdots\mathrm{d}\theta_{m-1} \\
&= 2\pi\prod_{j=1}^{m-3}\int_0^\pi(\sin\phi)^j\mathrm{d}\phi = 2\pi\prod_{j=1}^{m-3}B\left(\frac{j+1}{2},\frac{1}{2}\right) = 2\pi\prod_{j=1}^{m-3}\frac{\Gamma\left(\frac{j+1}{2}\right)\Gamma(\frac{1}{2})}{\Gamma\left(\frac{j+2}{2}\right)}.
\end{aligned}$$

We have used the Beta function $B(\cdot,\cdot)$ (cf. page 103 of [28]). Hence since $\Gamma(1/2) = \sqrt{\pi}$, we get $R_m = 2\pi(\sqrt{\pi})^{m-3}\Big(\Gamma\big(\frac{m-1}{2}\big)\Big)^{-1}$. Inserting R_m in (31) we obtain (27), i.e.,

$$I_{m,k}(F) = \frac{2\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)}\frac{c_k}{|c|}\int_0^{+\infty}\int_0^{\pi} e^{-\frac{1}{2}\rho^2}\rho^m\cos\vartheta(\sin\vartheta)^{m-2}F(|c|\rho\cos\vartheta)\mathrm{d}\rho\mathrm{d}\vartheta.$$

□

Lemma 10 *If $F \in B_b(\mathbb{R})$ verifies $F(x) = -F(-x)$ for any $x \in \mathbb{R}$, then we have, for any $m \geq 2$, $k \in \{1, \dots, m\}$, $c = (c_1, \dots, c_m) \in \mathbb{R}^m \setminus \{0\}$,*

$$I_{m,k}(F) = \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)}\frac{c_k}{|c|}\int_0^{+\infty} e^{-\frac{1}{2}\rho^2}\rho^m\mathrm{d}\rho\int_0^1 x(1-x^2)^{\frac{m-3}{2}}F(|c|\rho x)\mathrm{d}x \tag{32}$$

(cf. (27)). In the special case of $F = F_0 := \mathbb{1}_{(0,\infty)} - \mathbb{1}_{(-\infty,0)}$, we obtain

$$I_{m,k}(F_0) = \frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbb{R}^m} F_0(\langle c, x\rangle)x_k\, e^{-\frac{|x|^2}{2}}\mathrm{d}x = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{c_k}{|c|}. \tag{33}$$

Proof By changing variable $x = \cos\theta$ and using that $F(x) = -F(-x)$, $x \neq 0$, we have

$$\int_0^{\pi}\cos\vartheta(\sin\vartheta)^{m-2}F(|c|\rho\cos\vartheta)\mathrm{d}\vartheta = 2\int_0^1 x(1-x^2)^{\frac{m-3}{2}}F(|c|\rho x)\mathrm{d}x\ .$$

Whence,

$$I_{m,k}(F) = \frac{2\cdot 2\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)}\frac{c_k}{|c|}\int_0^{+\infty} e^{-\frac{1}{2}\rho^2}\rho^m d\rho\int_0^1 x(1-x^2)^{\frac{m-3}{2}}F(|c|\rho x)\mathrm{d}x.$$

Let us assume that $F = F_0 = \mathbb{1}_{(0,\infty)} - \mathbb{1}_{(-\infty,0)}$. We find

$$I_{m,k}(F_0) = \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)}\frac{c_k}{|c|}\int_0^{+\infty} e^{-\frac{1}{2}\rho^2}\rho^m d\rho\int_0^1 x(1-x^2)^{\frac{m-3}{2}}\mathrm{d}x.$$

Using that $\int_0^1 x(1-x^2)^{\frac{m-3}{2}}\mathrm{d}x = \frac{1}{m-1}$ and $\int_0^{+\infty} e^{-\frac{1}{2}\rho^2}\rho^m\mathrm{d}\rho = \Gamma\left(\frac{m+1}{2}\right)2^{\frac{m-1}{2}}$ we find

$$I_{m,k}(F_0) = \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)}\Gamma\left(\frac{m+1}{2}\right)2^{\frac{m-1}{2}}\frac{1}{m-1}\frac{c_k}{|c|} = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{c_k}{|c|},$$

since $x\Gamma(x) = \Gamma(x+1)$, $x > 0$ and this finishes the proof. □

3.2 Proof of assertion (26)

Recall that $c_k(t) := \left(\frac{1-e^{-2\lambda_k t}}{2\lambda_k}\right)^{1/2}$ for $k \in \{1, \dots, m\}$ and $t \geq 0$. Set $c(t) = (c_1(t), \dots, c_m(t)) \in \mathbb{R}^m$. Fix $m \geq 2$ and $\delta > 0$, and put $S_m := S_m(\delta)$. Then, for $m \geq 2$, define

$$A_m := \frac{2}{\pi}\sum_{k=1}^m\left(\int_0^{\delta}\frac{\lambda_k e^{-\lambda_k t}}{(1-e^{-2\lambda_k t})^{1/2}}\frac{c_k(t)}{|c(t)|}\mathrm{d}t\right)^2. \tag{34}$$

We prove that $\lim_{m\to\infty} S_m = \infty$ in two steps.
I step. We prove

$$S_m \geq A_m, \quad \forall\, m \geq 2. \tag{35}$$

We start by constructing an approximating sequence of smooth functions for $F_0 := \mathbb{1}_{(0,\infty)} - \mathbb{1}_{(-\infty,0)}$. For any $n \geq 1$, consider a non-decreasing $F_n \in C_b^2(\mathbb{R}_+)$ such that $F_n(y) = 0$ if

$0 \le y \le 1/(n+1)$ and $F_n(y) = 1$ if $y \ge 1/n$. Then, extend each F_n to an odd function on $\mathbb{R}$ by the rule $F_n(x) = -F_n(-x)$ if $x < 0$, and define

$$f_n(x_1, \ldots, x_m) = F_n(x_1 + \ldots + x_m), \quad x_1, \ldots, x_m \in \mathbb{R}.$$

It is clear that each $f_n \in C_b^2(\mathbb{R}^m)$ and $\|f_n\|_\infty \le 1$. Whence,

$$\begin{aligned} S_m &\ge \sup_{n\ge1} \sum_{k=1}^m \Big(\int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1-e^{-2\lambda_k t})^{1/2}} \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} f_n(c_1(t)x_1, \ldots, c_m(t)x_m)\, x_k\, e^{-\frac{|x|^2}{2}} \mathrm{d}x \mathrm{d}t \Big)^2 \\ &= \sup_{n\ge1} \sum_{k=1}^m \Big(\int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1-e^{-2\lambda_k t})^{1/2}} \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} F_n(\langle c(t), x\rangle) x_k\, e^{-\frac{|x|^2}{2}} \mathrm{d}x \mathrm{d}t \Big)^2. \end{aligned}$$

Moreover, combining the fact that each F_n is an odd functions with (32), with c replaced by $c(t)$, yields

$$\begin{aligned} &\sup_{n\ge1} \sum_{k=1}^m \Big(\int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1-e^{-2\lambda_k t})^{1/2}} \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} F_n(\langle c(t), x\rangle) x_k\, e^{-\frac{|x|^2}{2}} \mathrm{d}x \mathrm{d}t \Big)^2 \\ &= \sup_{n\ge1} \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)} \sum_{k=1}^m \Big(\frac{c_k(t)}{|c(t)|} \int_0^{+\infty} e^{-\frac{1}{2}\rho^2} \rho^m d\rho \int_0^1 x(1-x^2)^{\frac{m-3}{2}} F_n(|c(t)|\rho x) dx \Big)^2. \end{aligned}$$

Then, using that both $F_n(x) \le F_{n+1}(x)$ and $F_n(x) \to F_0(x)$ hold for any $x \ge 0$, apply the monotone convergence theorem to get

$$\begin{aligned} S_m &\ge \sup_{n\ge1} \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)} \sum_{k=1}^m \Big(\frac{c_k(t)}{|c(t)|} \int_0^{+\infty} e^{-\frac{1}{2}\rho^2} \rho^m d\rho \int_0^1 x(1-x^2)^{\frac{m-3}{2}} F_n(|c(t)|\rho x) \mathrm{d}x \Big)^2 \\ &= \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2}\Gamma\left(\frac{m-1}{2}\right)} \sum_{k=1}^m \Big(\frac{c_k(t)}{|c(t)|} \int_0^{+\infty} e^{-\frac{1}{2}\rho^2} \rho^m d\rho \int_0^1 x(1-x^2)^{\frac{m-3}{2}} F_0(|c(t)|\rho x) \mathrm{d}x \Big)^2 \\ &= \sum_{k=1}^m \Big(\int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1-e^{-2\lambda_k t})^{1/2}} \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} F_0(\langle c(t), x\rangle) x_k\, e^{-\frac{|x|^2}{2}} \mathrm{d}x \mathrm{d}t \Big)^2 = A_m, \end{aligned}$$

for $m \ge 2$. In the last line we have used both (32) and (33) with c replaced by $c(t)$. This proves (35).

II step. We prove that

$$\lim_{m\to\infty} A_m = \infty\,, \tag{36}$$

thus completing the proof of (26). Recalling the definition of $c_k(t)$, we have

$$A_m = \frac{2}{\pi} \sum_{k=1}^m \Big(\int_0^\delta \frac{\sqrt{\lambda_k} e^{-\lambda_k t}}{\sqrt{2}} \frac{1}{|c(t)|} \mathrm{d}t \Big)^2 \ge \frac{1}{\pi} \sum_{k=1}^m \lambda_k \left(\int_0^\delta \frac{e^{-\lambda_k t}}{|c(t)|} \mathrm{d}t \right)^2, \quad m \ge 2. \tag{37}$$

To bound (37) from below, note that

$$|c(t)| = \left(\sum_{k=1}^m \frac{1-e^{-2\lambda_k t}}{2\lambda_k} \right)^{1/2} \le \left(\sum_{k=1}^{+\infty} \frac{1-e^{-2\lambda_k t}}{2\lambda_k} \right)^{1/2} = \left(\int_0^t \left[\sum_{k=1}^{+\infty} e^{-2\lambda_k s} \right] \mathrm{d}s \right)^{1/2}$$

holds for any $t \geq 0$. Now, if there is a positive constant c_0 such that $\lambda_k \geq c_0 k^2$ for any $k \geq 1$, then

$$\sum_{k=1}^{+\infty} e^{-2\lambda_k s} \leq \sum_{k=1}^{+\infty} e^{-2c_0 k^2 s} \leq \int_0^{+\infty} e^{-2c_0 z^2 s} \mathrm{d}z = \sqrt{\frac{\pi}{2c_0 s}}, \quad s > 0,$$

yielding

$$|c(t)| \leq \left(\int_0^t \sqrt{\frac{\pi}{2c_0 s}} \mathrm{d}s \right)^{1/2} = \left(\frac{2\pi t}{c_0} \right)^{1/4}.$$

Up to now we have found that

$$A_m \geq \frac{1}{\pi} \sum_{k=1}^{m} \lambda_k \left(\int_0^{\delta} e^{-\lambda_k t} \left(\frac{c_0}{2\pi t} \right)^{1/4} \mathrm{d}t \right)^2, \quad m \geq 2.$$

Now, exploit that

$$\int_0^{\delta} t^{-\frac{1}{4}} e^{-\lambda t} \mathrm{d}t = \left(\frac{1}{\lambda} \right)^{\frac{3}{4}} \int_0^{\lambda\delta} s^{-\frac{1}{4}} e^{-s} ds \geq \left(\frac{1}{\lambda} \right)^{\frac{3}{4}} \int_0^{c_0\delta} s^{-\frac{1}{4}} e^{-s} \mathrm{d}s$$

holds for every $\lambda \geq c_0$, to get (after recalling that, in particular, $\lambda_k \geq c_0$, for any $k \geq 1$)

$$\begin{aligned} A_m &\geq \frac{1}{\pi} \sqrt{\frac{c_0}{2\pi}} \sum_{k=1}^{m} \lambda_k \left(\int_0^{\delta} t^{-\frac{1}{4}} e^{-\lambda_k t} \mathrm{d}t \right)^2 \geq \frac{1}{\pi} \sqrt{\frac{c_0}{2\pi}} \sum_{k=1}^{m} \lambda_k \left(\frac{1}{\lambda_k} \right)^{\frac{3}{2}} \left(\int_0^{c_0\delta} s^{-\frac{1}{4}} e^{-s} \mathrm{d}s \right)^2 \\ &= \frac{1}{\pi} \sqrt{\frac{c_0}{2\pi}} \left(\int_0^{c_0\delta} s^{-\frac{1}{4}} e^{-s} \mathrm{d}s \right)^2 \sum_{k=1}^{m} \frac{1}{\sqrt{\lambda_k}}. \end{aligned}$$

Thus, if $\lambda_k \sim k^2$ as $k \to +\infty$, then $\sum_{k=1}^{m} \frac{1}{\sqrt{\lambda_k}} \sim \log m$ as $m \to +\infty$, and (36) holds. This finishes the proof.

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