Annali di Matematica Pura ed Applicata (1923 -) (2024) 203:779–804
https://doi.org/10.1007/s10231-023-01381-4

A comparison principle for doubly nonlinear parabolic partial differential equations

Verena Bögelein[1] · Michael Strunk[1]

Received: 28 June 2023 / Accepted: 28 August 2023 / Published online: 22 September 2023

Abstract

In this paper, we derive a comparison principle for non-negative weak sub- and super-solutions to doubly nonlinear parabolic partial differential equations whose prototype is

$$\partial_t u^q - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \qquad \text{in } \Omega_T,$$

with $q > 0$ and $p > 1$ and $\Omega_T := \Omega \times (0, T) \subset \mathbb{R}^{n+1}$. Instead of requiring a lower bound for the sub- or super-solutions in the whole domain Ω_T, we only assume the lateral boundary data to be strictly positive. The main results yield some applications. Firstly, we obtain uniqueness of non-negative weak solutions to the associated Cauchy–Dirichlet problem. Secondly, we prove that any weak solution is also a viscosity solution.

Keywords Doubly nonlinear parabolic PDE · Comparison principle

Mathematics Subject Classification 35K55 · 35K65 · 35K67 · 35A02

1 Introduction and main results

The parabolic partial differential equation

$$\partial_t u^q - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \qquad \text{in } \Omega_T, \tag{1.1}$$

with some arbitrary exponents $q > 0$ and $p > 1$ is a non-trivial generalization of some well-studied problems. Here and in the following $\Omega_T = \Omega \times (0, T)$ denotes a space-time cylinder over a bounded domain $\Omega \subset \mathbb{R}^n$ and $T > 0$. In its general form, (1.1) is called a doubly nonlinear pde. Only for the specific choice $q = 1$ and $p = 2$, it is linear and yields the heat equation. If $q = p - 1$, it is homogeneous with respect to multiplication. The resulting pde is sometimes called Trudinger's equation. In the case $p = 2$, we obtain the porous medium equation, whereas the case $q = 1$ yields the parabolic p-Laplace equation.

✉ Michael Strunk
michael.strunk@plus.ac.at

Verena Bögelein
verena.boegelein@plus.ac.at

[1] Fachbereich Mathematik, Universität Salzburg, Hellbrunner Str. 34, 5020 Salzburg, Austria

Properties of weak solutions to the porous medium equation and the parabolic p-Laplace equation are by now better understood than for the general doubly nonlinear pde (1.1). In this paper, we will investigate comparison principles for weak sub- and super-solutions to (1.1) as well as generalizations of (1.1). Roughly speaking, the comparison principle states that a sub-solution u and a super-solution v which satisfy $u \le v$ on the parabolic boundary $\partial_p \Omega_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0, T))$ of the domain, must have the same property in the whole domain Ω_T. Although it is generally understood to be a rather simple property, the comparison principle for doubly nonlinear equations is still far from being understood, and only special cases could be treated so far. The difficulties occur due to the lack of a weak time derivative and in particular in points where the solution is close to zero. Note that these difficulties do not occur for parabolic p-Laplace type equations, i.e., in the case $q = 1$, in which the comparison principle can be shown by standard methods. Moreover, comparison principles for the prototype porous medium equation are presented in [30]. For more general equation of porous medium type, the situation is less clear.

In [2], Bamberger proved a comparison principle for weak solutions to doubly nonlinear equations under the additional assumption $\partial_t u^q, \partial_t v^q \in L^1(\Omega_T)$. In a similar spirit, Alt and Luckhaus [1] obtained a comparison principle for weak sub- and super-solutions, provided that $(\partial_t u^q - \partial_t v^q) \in L^1(\Omega_T)$. Also, the result of Diaz [12] requires an additional assumption on the time derivative. Unfortunately, these assumptions are quite restrictive, since they are not inherent in the definition of weak solution and in general not easy to verify.

Otto followed a different approach in [27]. He proved a comparison principle for weak sub- and super-solutions whose lateral boundary data are time independent. In particular, he avoided any extra regularity assumption on the sub- and super-solutions. Yet another approach was chosen by Ivanov, Mkrtychan, and Jäger in [20] for the case $q \in (0, 1]$ and $p \in (1, 2)$. Note that the parameter ℓ in [20] corresponds to $\frac{(1-q)(p-1)}{q}$ in (1.1). They allow time-dependent boundary data and prove a comparison principle for bounded and strictly positive sub- and super-solutions, i.e., the infimum of u and v on Ω_T is assumed to be strictly positive. Subsequently, Ivanov [18] extended the result to the range of exponents $q \in (0, 1]$ and $p > 1$. A similar result for Trudinger's equation, i.e., the case $p > 1$ and $q = p - 1$ was established by Lindgren and Lindqvist in [26].

Our aim in this paper is to treat the full range of exponents $q > 0$ and $p > 1$. Moreover, we are able to weaken the infimum assumption. Instead of requiring the infimum of the sub- and super-solution to be strictly positive, we only assume the lateral boundary data of the super-solution to be strictly positive. Postponing a formal definition of weak sub/super-solutions to Sect. 2.2, our first main result is the following.

Theorem 1.1 *Let $q > 0$, $p > 1$ and suppose that u is a non-negative weak sub-solution and v a non-negative weak super-solution of* (1.1) *in Ω_T satisfying*

$$\operatorname*{ess\,inf}_{\partial\Omega\times(0,T)} v > 0 \quad \text{and} \quad \operatorname*{ess\,sup}_{\partial\Omega\times(0,T)} u < \infty \text{ if } q > 1. \tag{1.2}$$

If

$$u \le v \quad \text{on } \partial\Omega \times (0, T), \tag{1.3}$$

then the following inequality holds

$$\int_{\Omega\times\{t_2\}} \left(u^q - v^q\right)_+ \mathrm{d}x \le \int_{\Omega\times\{t_1\}} \left(u^q - v^q\right)_+ \mathrm{d}x \tag{1.4}$$

for every $0 \le t_1 < t_2 \le T$.

As usual, the assumption $u \le v$ on $\partial\Omega \times (0, T)$ has to be understood in the sense that $(u - v)_+ \in L^p(0, T; W_0^{1,p}(\Omega))$. Applying Theorem 1.1 in the special situation where additionally $u(\cdot, 0) \le v(\cdot, 0)$ a.e. in Ω yields a comparison principle on parabolic cylinders.

Theorem 1.2 *Let $q > 0$, $p > 1$ and u be a non-negative weak sub-solution and v a non-negative weak super-solution of (1.1) in Ω_T satisfying (1.2). If*

$$u \le v \qquad \text{on } \partial_p\Omega_T,$$

then we have

$$u \le v \qquad \text{a.e. in } \Omega_T.$$

The approach in this paper is inspired by the proofs given in [18, 20, 26]. As mentioned above, the assumed lower bound of either the weak sub-solution or the weak super-solution in the whole of Ω_T is a strong restriction one would like to relinquish. In this paper, we were able to relax this condition to a lower bound on the lateral boundary. This has been achieved with the two expedient Lemmas 2.4 and 3.1. The first one allows to replace the sub-solution by another sub-solution which is bounded from below by a positive constant, as well as to replace the super-solution by a bounded super-solution. Assumption (1.2) ensures that the condition on the lateral boundary data is not violated. The difficulty in the proof of the comparison principle is firstly to choose a test-function which is regular enough. As we do not impose any assumption on the time derivatives, the choice of test-function is a delicate issue, in particular when $q \ne 1$. Therefore, a suitable mollification is necessary. Secondly, without a lower bound on the weak sub/super-solution in Ω_T, we somehow have to work around this assumption by determining at least suitable boundary conditions. The latter allows us to apply Lemma 2.4 in order to construct an auxiliary sub-solution which is on the one hand strictly positive in Ω_T and on the other hand smaller than the super-solution on the lateral boundary of Ω_T. This is achieved by working with $\max\{u, \kappa\}$, for a suitable constant $\kappa > 0$, instead of u, where u denotes the weak sub-solution. Similarly, in the case $q > 1$ we also make use of Lemma 2.4 in order to replace the weak super-solution v by the auxiliary super-solution $\min\{v, M\}$ for appropriate M large enough. We emphasize that no upper bound of weak sub-solutions on the lateral boundary is necessary, except in the case $q > 1$. This is achieved with the help of Lemma 3.1 that also has been used in [20]. The application of Lemma 3.1 allows to avoid a time mollification such as Steklov average or exponential mollification in the test-function. Note that the case $q = 1$, which yields the parabolic p-Laplace equation, is easier and neither a lower nor an upper bound for the lateral boundary data is needed. Since this is classical, we do not go into further detail.

For particular ranges of exponents q and p, we obtain stronger results in a local setting. If either $0 < q \le p - 1$, or $0 < p - 1 < q < \frac{n(p-1)+p}{(n-p)_+}$, then weak sub-solutions to (1.1) are locally bounded. This property is exploited in Corollary 3.4 below. A further restriction of the exponents to the range $0 < p - 1 \le q < \frac{n(p-1)}{(n-p)_+}$ even allows to prove in a local setting a comparison principle for weak solutions without any additional assumptions like upper or lower bounds.

Theorem 1.3 *Let $0 < p - 1 \le q < \frac{n(p-1)}{(n-p)_+}$ and u, v be non-negative local weak solutions of (1.1) in Ω_T. Further, let $K \Subset \Omega$ and $0 < t_1 < t_2 < T$. If*

$$u \le v \qquad \text{on } \partial_p\big(K \times (t_1, t_2)\big),$$

then we have

$$u \le v \qquad \text{in } K \times (t_1, t_2).$$

Note that we can also allow $t_1 = 0$ if u and v are defined until the initial time $t = 0$. The key ingredient to the proof of Theorem 1.3 is a Harnack inequality which ensures that non-negative local weak solutions of (1.1) are either zero or strictly positive on any time slice.

Naturally, the interest in a comparison principle for (1.1) with a nonzero right-hand side f arises. Thus, instead of (1.1), one could rather consider its inhomogeneous version

$$\partial_t u^q - \Delta_p u = f \qquad \text{in } \Omega_T. \tag{1.5}$$

We obtain similar comparison principles for the preceding equation by slightly adapting the proofs of the main results in Theorems 1.1 and 1.2, provided f belongs to a suitable parabolic Lebesgue space; see Definition 4.1 below. A further generalization concerns the vector field in the diffusion part of (1.1). Instead of the pure p-Laplace operator, our results continue to hold for vector fields of the form

$$A(x, t, u, \xi) \colon \Omega_T \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$$

and the associated doubly nonlinear differential equation

$$\partial_t u^q - \operatorname{div} A(x, t, u, \nabla u) = f \qquad \text{in } \Omega_T. \tag{1.6}$$

Here, we assume A to be a Carathéodory function which satisfies suitable p-growth, Lipschitz and monotonicity conditions; see the set of assumptions (4.5). We obtain similar comparison principles also for (1.6). However, in contrast to the comparison principle derived for the prototype equation, the proof in the general setting requires more care and a careful use of the assumed monotonicity and Lipschitz conditions is required. Since our results for both equations (1.5) and (1.6) are similar to those for the model equation (1.1), we only state the latter here.

Finally, we note that also the comparison principles shown in [18, 20] apply to more general doubly nonlinear partial differential equations than the prototype one (1.1). To obtain the addressed pde in [18, 20], one may substitute $v = u^q$ in (1.1) to derive the equivalent form

$$\partial_t v - \operatorname{div}\left(q^{1-p} v^{\frac{(1-q)(p-1)}{q}} |\nabla v|^{p-2} \nabla v\right) = 0 \qquad \text{in } \Omega_T, \tag{1.7}$$

for $q > 0$ and $p > 1$. The preceding presentation illustrates the correspondence $\ell = \frac{(1-q)(p-1)}{q}$. Therefore, the assumption $\ell \geq 0$ in [18, 20] corresponds to $q \in (0, 1]$ in (1.1).

Plan of the paper. Firstly, in Sect. 2 we will introduce the setting and notations we are working with, including the definition of (non-negative) weak (sub-/super-)solutions to (1.1). We also define the two auxiliary functions H_δ and G_δ, $\delta > 0$, used in the proof of the comparison principle in Theorem 1.1. Additionally, we introduce two different mollifications in time, namely the Steklov-average and the exponential mollification.

Section 3 contains the main part of the paper, where the comparison principles from Theorems 1.1 and 1.2 are proved. Respective results for the local setting are given in Subsection 3.2, where the comparison principle from Theorem 1.3 is shown. We will then, in Sect. 4, discuss possible generalizations of the comparison principle to inhomogeneous doubly nonlinear equations and more general vector fields. In Sect. 5, we provide uniqueness results for Cauchy–Dirichlet problems associated with a doubly nonlinear equation, which are a direct consequence of the comparison principles obtained before.

Finally, in Sect. 6 we will show as application of the comparison principle that every weak solution of (1.1) is also a viscosity solution in the sense of [11]. In particular, this result implies existence of viscosity solutions.

2 Preliminaries

2.1 Notation

Throughout $\Omega_T = \Omega \times (0, T)$ denotes a space-time cylinder, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $(0, T)$ represents a time interval for a certain time $T > 0$. The parabolic boundary of Ω_T will be denoted by

$$\partial_p \Omega_T = \big(\overline{\Omega} \times \{0\}\big) \cup \big(\partial\Omega \times (0, T)\big).$$

For a function $f \in L^1(\Omega_T) \cong L^1(0, T; L^1(\Omega))$, we also write $f(t)$ instead of $f(\cdot, t)$ whenever it is convenient. Moreover, we will abbreviate the p-Laplace operator by

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big). \tag{2.1}$$

Throughout the paper, we will not distinguish between the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^n$ for $n \geq 2$ and the absolute value $|\cdot|$ in $\mathbb{R}$. Both shall be denoted by $|\cdot|$ and the meaning will be clear from the context. For matrices $X \in \mathbb{R}^{n\times n}$, we will always use the spectral norm given by $\|X\| = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ denotes the largest eigenvalue of $X^\top X$. Recall that the spectral norm is consistent with the Euclidean vector norm, that is

$$|Xv| \leq \|X\|\,|v| \qquad \text{for any } v \in \mathbb{R}^n \text{ and } X \in \mathbb{R}^{n\times n}.$$

Furthermore, the trace of a matrix $X \in \mathbb{R}^{n\times n}$ shall be expressed by $\operatorname{Tr}(X)$.

The positive part of some quantity $a \in \mathbb{R}$ is denoted by $a_+ = \max\{a, 0\}$, whereas the negative part by $a_- = \max\{-a, 0\}$. Constants will always be denoted by c or $c(\cdot)$, where only the dependence of the constants is stated. However, constants may change from line to line without further explanation.

2.2 Definition of weak solution

Although it is standard, we briefly state the definition of a (local) weak solution that we use throughout the paper.

Definition 2.1 (Weak solution) A non-negative measurable function $u\colon \Omega_T \to \mathbb{R}_{\geq 0}$ in the class

$$u \in C\big([0, T]; L^{q+1}(\Omega)\big) \cap L^p\big(0, T; W^{1,p}(\Omega)\big)$$

is a non-negative weak sub(super)-solution of (1.1) if

$$\iint_{\Omega_T} \big[-u^q \partial_t \phi + |\nabla u|^{p-2}\nabla u \cdot \nabla\phi\big]\,\mathrm{d}x\mathrm{d}t \leq (\geq)\, 0 \tag{2.2}$$

for any non-negative function

$$\phi \in W_0^{1,q+1}\big(0, T; L^{q+1}(\Omega)\big) \cap L^p\big(0, T; W_0^{1,p}(\Omega)\big).$$

A non-negative function u is a non-negative weak solution of (1.1) if it is both, a weak sub-solution and a weak super-solution.

Definition 2.2 (Local weak solution) A non-negative measurable function $u\colon \Omega_T \to \mathbb{R}_{\geq 0}$ in the class

$$u \in C\big(0, T; L^{q+1}_{\mathrm{loc}}(\Omega)\big) \cap L^p_{\mathrm{loc}}\big(0, T; W^{1,p}_{\mathrm{loc}}(\Omega)\big)$$

is a non-negative local weak sub(super)-solution of (1.1) if for every $K \Subset \Omega$ and every sub-interval $[t_1, t_2] \subset (0, T)$ we have

$$\int_K u^q \phi \,\mathrm{d}x \Big|_{t_1}^{t_2} + \iint_{K\times(t_1,t_2)} \big[- u^q \partial_t \phi + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \big] \mathrm{d}x\mathrm{d}t \le (\ge)\, 0$$

for any non-negative function

$$\phi \in W^{1,q+1}_{\mathrm{loc}}\big(0, T; L^{q+1}(K)\big) \cap L^p_{\mathrm{loc}}\big(0, T; W^{1,p}_0(K)\big).$$

A non-negative function u is a non-negative local weak solution of (1.1) if it is both, a local weak sub-solution and a local weak super-solution.

Existence of weak solutions to the Cauchy–Dirichlet problem associated with (1.1) has been shown in [1]. It is worth noticing that due to Definition 2.1 weak sub/super-solutions belong to the space

$$u \in C\big([0, T]; L^{q+1}(\Omega)\big) \cap L^p\big(0, T; W^{1,p}(\Omega)\big)$$

and thus, are assumed to be continuous functions in time. However, this is not restrictive as shown in [6, Proposition 4.9].

2.3 Mollification in time

In view of their definition, weak solutions are not necessarily weakly differentiable with respect to the time variable. This difficulty is usually overcome by certain regularization procedures. We will work with two different mollifications. The first one is the Steklov-average, cf. [10]. For a function $f \in L^1(\Omega_T)$ and $0 < h < T$, we define its *Steklov-average* $[f]_h$ by

$$[f]_h(x,t) := \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} f(x,\tau)\,\mathrm{d}\tau, & t \in (0, T-h),\\ 0, & t \in [T-h, T). \end{cases} \tag{2.3}$$

Rewriting inequality (2.2) in terms of Steklov-means $[u]_h$ of u, yields

$$\int_{\Omega\times\{t\}} \Big[\partial_t [u^q]_h \phi + \big[|\nabla u|^{p-2}\nabla u\big]_h \cdot \nabla\phi \Big] \mathrm{d}x \le (\ge)\, 0 \tag{2.4}$$

for any non-negative function $\phi \in W^{1,p}_0(\Omega)$ and any $t \in (0, T)$.

In the course of the paper, we will also need another mollification in time. For any $f \in L^1(\Omega_T)$ and $h > 0$, we introduce the exponential mollification

$$[\![f]\!]_h(x,t) := \frac{1}{h}\int_0^t \mathrm{e}^{\frac{\tau-t}{h}} f(x,\tau)\,\mathrm{d}\tau \quad \text{and} \quad [\![f]\!]_{\bar h}(x,t) := \frac{1}{h}\int_t^T \mathrm{e}^{\frac{t-\tau}{h}} f(x,\tau)\,\mathrm{d}\tau, \tag{2.5}$$

as defined in [23].

2.4 Auxiliary material

The following lemma that can be found in [14, Lemma 2.2] will be useful in order to deal with the nonlinearity of the differential equation.

Lemma 2.3 *Let $k \in \mathbb{N}$. For any $\alpha > 1$, there exists a constant $c = c(\alpha)$ such that*

$$\frac{1}{c}\big||a|^{\alpha-1}a - |b|^{\alpha-1}b\big| \le \big(|a|^{\alpha-1} + |b|^{\alpha-1}\big)|a-b| \le c\big||a|^{\alpha-1}a - |b|^{\alpha-1}b\big|$$

for all $a, b \in \mathbb{R}^k$.

Weak sub(super)-solutions preserve this property when taking the maximum, respectively, minimum, with a constant. For the proof of this fact, we proceed similar as in [7, Lemma A.1]. For the sake of completeness, we provide the details.

Lemma 2.4 *Let $q > 0$, $p > 1$ and u be a non-negative weak sub-solution of (1.1) in the sense of Definition 2.1. Then, for any $\kappa > 0$ the function $\max\{u, \kappa\}$ is also a weak sub-solution of (1.1).*

Similarly, if v is a non-negative weak super-solution of (1.1), then for any $M > 0$ also $\min\{v, M\}$ is a weak super-solution.

Proof Only the sub-solution case is treated. The super-solution case may be treated in a similar way. For $h, \mu > 0$ and $\eta \in C_0^1(\Omega_T)$ such that $\eta \ge 0$ in Ω_T, we choose the test function

$$\phi = \eta\phi_h, \quad \text{where } \phi_h := \frac{([\![u]\!]_{\bar h} - \kappa)_+}{([\![u]\!]_{\bar h} - \kappa)_+ + \mu}$$

in the weak form (2.2) of the differential equation, i.e.,

$$\iint_{\Omega_T} \big[-u^q\partial_t\phi + |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\big]\,\mathrm{d}x\mathrm{d}t \le 0. \tag{2.6}$$

We start by considering the term involving the time derivative. We obtain

$$\begin{aligned}
&-\iint_{\Omega_T} u^q\partial_t\phi\,\mathrm{d}x\mathrm{d}t\\
&= \iint_{\Omega_T}\big([\![u]\!]_{\bar h}^q - u^q\big)\partial_t\phi\,\mathrm{d}x\mathrm{d}t - \iint_{\Omega_T}[\![u]\!]_{\bar h}^q\partial_t\phi\,\mathrm{d}x\mathrm{d}t\\
&= \iint_{\Omega_T}\eta\big([\![u]\!]_{\bar h}^q - u^q\big)\partial_t\phi_h\,\mathrm{d}x\mathrm{d}t\\
&\quad + \iint_{\Omega_T}\partial_t\eta\big([\![u]\!]_{\bar h}^q - u^q\big)\phi_h\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_T}\partial_t[\![u]\!]_{\bar h}^q\phi\,\mathrm{d}x\mathrm{d}t\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned} \tag{2.7}$$

with the obvious meaning of I – III. The first term on the right-hand side of (2.7) is non-negative, which can be seen by the following computation

$$\begin{aligned}
\mathrm{I} &= \mu\iint_{\Omega_T}\eta\big([\![u]\!]_{\bar h}^q - u^q\big)\frac{\partial_t([\![u]\!]_{\bar h} - \kappa)_+}{\big[([\![u]\!]_{\bar h} - \kappa)_+ + \mu\big]^2}\,\mathrm{d}x\mathrm{d}t\\
&= \frac{\mu}{h}\iint_{\Omega_T}\eta\big([\![u]\!]_{\bar h}^q - u^q\big)\big([\![u]\!]_{\bar h} - u\big)\frac{\chi_{\{[\![u]\!]_{\bar h} > \kappa\}}}{\big[([\![u]\!]_{\bar h} - \kappa)_+ + \mu\big]^2}\,\mathrm{d}x\mathrm{d}t \ge 0.
\end{aligned}$$

From the second to last line, we used the identity $\partial_t[\![u]\!]_{\bar h} = \frac{1}{h}([\![u]\!]_{\bar h} - u)$. Now, we turn our attention to the third term in (2.7).To this aim, we define

$$f(u, \kappa, \mu) := \kappa^q + q\int_\kappa^u \frac{s^{q-1}(s-\kappa)_+}{(s-\kappa)_+ + \mu}\,\mathrm{d}s.$$

In view of the chain rule, it is easy to see that

$$\partial_t f\big([\![u]\!]_{\bar h},\kappa,\mu\big)=\partial_t [\![u]\!]_{\bar h}^q \frac{([\![u]\!]_{\bar h}-\kappa)_+}{([\![u]\!]_{\bar h}-\kappa)_+ +\mu}=\partial_t [\![u]\!]_{\bar h}^q \phi_h.$$

Using the previous computation and integrating by parts yields

$$\begin{aligned}\mathrm{III}&=\iint_{\Omega_T}\eta\partial_t [\![u]\!]_{\bar h}^q\phi_h\,\mathrm{d}x\mathrm{d}t=\iint_{\Omega_T}\eta\partial_t f\big([\![u]\!]_{\bar h},\kappa,\mu\big)\,\mathrm{d}x\mathrm{d}t\\&=-\iint_{\Omega_T}\partial_t\eta f\big([\![u]\!]_{\bar h},\kappa,\mu\big)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Inserting these informations into (2.7), we obtain

$$-\iint_{\Omega_T}u^q\partial_t\phi\,\mathrm{d}x\mathrm{d}t\ge\iint_{\Omega_T}\partial_t\eta\big([\![u]\!]_{\bar h}^q-u^q\big)\phi_h\,\mathrm{d}x\mathrm{d}t-\iint_{\Omega_T}\partial_t\eta f\big([\![u]\!]_{\bar h},\kappa,\mu\big)\,\mathrm{d}x\mathrm{d}t.$$

The first term on the right-hand side vanishes in the limit $h\downarrow 0$. Therefore, inserting this inequality into (2.6) and then, letting $h\downarrow 0$, we arrive at

$$-\iint_{\Omega_T}\partial_t\eta f(u,\kappa,\mu)\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}|\nabla u|^{p-2}\nabla u\cdot\nabla\bigg[\eta\frac{(u-\kappa)_+}{(u-\kappa)_++\mu}\bigg]\mathrm{d}x\mathrm{d}t\le 0.$$

Next, we will treat the diffusion term, i.e., the second term on the left hand side of the preceding inequality. We have

$$\begin{aligned}&\iint_{\Omega_T}|\nabla u|^{p-2}\nabla u\cdot\nabla\bigg[\eta\frac{(u-\kappa)_+}{(u-\kappa)_++\mu}\bigg]\mathrm{d}x\mathrm{d}t\\&\quad=\iint_{\Omega_T}\bigg[|\nabla u|^{p-2}\nabla u\cdot\nabla\eta\frac{(u-\kappa)_+}{(u-\kappa)_++\mu}+|\nabla u|^{p-2}\nabla u\cdot\frac{\eta\mu\nabla(u-\kappa)_+}{[(u-\kappa)_++\mu]^2}\bigg]\mathrm{d}x\mathrm{d}t\\&\quad\ge\iint_{\Omega_T}|\nabla u|^{p-2}\nabla u\cdot\nabla\eta\frac{(u-\kappa)_+}{(u-\kappa)_++\mu}\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Inserting this above yields

$$-\iint_{\Omega_T}\partial_t\eta f(u,\kappa,\mu)\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}|\nabla u|^{p-2}\nabla u\cdot\nabla\eta\frac{(u-\kappa)^+}{(u-\kappa)^++\mu}\,\mathrm{d}x\mathrm{d}t\le 0.$$

A direct calculation shows that

$$\lim_{\mu\downarrow 0}f(u,\kappa,\mu)=\max\{u,\kappa\}^q$$

and

$$\lim_{\mu\downarrow 0}\frac{(u-\kappa)^+}{(u-\kappa)^++\mu}=\chi_{\{u>\kappa\}}.$$

Furthermore, note that $\chi_{\{u>\kappa\}}|\nabla u|^{p-2}\nabla u=|\nabla\max\{u,\kappa\}|^{p-2}\nabla\max\{u,\kappa\}$. Therefore, letting $\mu\downarrow 0$ and using an approximating argument in order to obtain the desired inequality for an arbitrary test function

$$\phi\in W_0^{q+1}\big(0,T;L^{q+1}(\Omega)\big)\cap L^p\big(0,T;W_0^{1,p}(\Omega)\big)$$

yields

$$\iint_{\Omega_T}\big[-\max\{u,\kappa\}^q\partial_t\phi+|\nabla\max\{u,\kappa\}|^{p-2}\,\nabla\max\{u,\kappa\}\cdot\nabla\phi\big]\,\mathrm{d}x\mathrm{d}t\le 0,$$

proving that $\max\{u, \kappa\}$ is a weak sub-solution of (1.1). □

3 Comparison principles

Our aim in this section is to prove the comparison principles for the doubly nonlinear equation (1.1). We first turn our attention to weak sub- and super-solutions in Ω_T and subsequently consider the local setting.

3.1 Comparison principles in a global setting

In this subsection, we will accomplish the proofs of Theorems 1.1 and 1.2. The main difficulty stems from the nonlinearity appearing in the time derivative part of (1.1). As illustrated for the homogeneous equation, i.e., the case $q = p - 1$ in [26, (3.1)], a comparison principle can be derived quite easily if the weak time derivative of u^q exists. However, such a property is not implemented in the definition of a weak solution. Without existence of a weak time derivative, the test function has to be chosen very carefully and certain approximation arguments are needed.

Throughout the proof, we shall use the following two auxiliary functions. The first one is a piecewise affine approximation of the indicator function

$$H_\delta(x) := \begin{cases} 1, & x \geq \delta, \\ \frac{x}{\delta}, & 0 < x < \delta, \\ 0, & x \leq 0 \end{cases} \tag{3.1}$$

for $\delta > 0$. The second one is its primitive and approximates the positive part

$$G_\delta(x) := \begin{cases} x - \frac{\delta}{2}, & x \geq \delta, \\ \frac{x^2}{2\delta}, & 0 < x < \delta, \\ 0, & x \leq 0. \end{cases}$$

Note that $G'_\delta(x) = H_\delta(x)$ for any $x \in \mathbb{R}$. The inequality stated in the next Lemma was already used in the proof of [20, Proposition 2.1]. It allows to choose a test function without dependency on any mollifiers like Steklov-average or exponential mollification in the proof of the comparison principle.

Lemma 3.1 *Let $\delta > 0$ and $f \in C(0, T; L^1(\Omega))$. Then, for any $0 < h < T$ the following inequality holds*

$$\partial_t [G_\delta(f)]_h \leq \partial_t [f]_h H_\delta(f) \qquad \text{a.e. in } \Omega_T \tag{3.2}$$

Proof For $t \in [T - h, T)$ inequality, (3.2) is trivial. Therefore, it remains to consider $t \in (0, T - h)$. The definition of the Steklov-average in (2.3) yields

$$\partial_t [G_\delta(f)]_h(t) = \tfrac{1}{h}\big[G_\delta\left(f(t+h)\right) - G_\delta(f(t))\big]$$

and

$$\partial_t [f]_h = \tfrac{1}{h}\big[f(t+h) - f(t)\big].$$

Thus, inequality (3.2) simplifies to

$$G_\delta\left(f(t+h)\right) - G_\delta\left(f(t)\right) \leq \big(f(t+h) - f(t)\big) H_\delta\left(f(t)\right).$$

In view of the convexity of the mapping $\mathbb{R} \ni x \mapsto G_\delta(x)$, we have

$$G_\delta(y) - G_\delta(x) \geq G'_\delta(x)(y - x) \qquad \text{for any } x, y \in \mathbb{R}.$$

Thus, setting $y = f(t + h)$ and $x = f(t)$ yields the desired inequality. □

We start with the following preliminary version of the comparison principle, where we additionally require either the sub- or the super-solution to be bounded from above and below by positive constants.

Proposition 3.2 *Let $q > 0$, $p > 1$ and u be a non-negative weak sub-solution and v a non-negative weak super-solution of (1.1) in Ω_T. Suppose that either*

$$u \geq \epsilon \quad \text{or} \quad v \geq \epsilon \quad \text{a.e. in } \Omega_T \tag{3.3}$$

for some $\epsilon > 0$ and in the case $q > 1$ assume furthermore that either u or v is bounded. If

$$u \leq v \quad \text{on } \partial\Omega \times (0, T),$$

then the following inequality holds

$$\int_{\Omega\times\{t_2\}} \big(u^q - v^q\big)_+ \,\mathrm{d}x \leq \int_{\Omega\times\{t_1\}} \big(u^q - v^q\big)_+ \,\mathrm{d}x \tag{3.4}$$

for every $0 \leq t_1 < t_2 \leq T$.

Proof For $h \in (0, T)$, we consider the Steklov formulation (2.4) of (2.2) for u and v. Adding both inequalities yields

$$\int_{\Omega\times\{t\}} \partial_t[u^q - v^q]_h \phi \,\mathrm{d}x \leq \int_{\Omega\times\{t\}} \big[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\big]_h \cdot \nabla\phi \,\mathrm{d}x$$

for any $\phi \in W_0^{1,p}(\Omega)$ and any $t \in (0, T)$. Note that a weak time derivative for the test functions is not needed in this formulation. We now integrate this inequality with respect to $t \in (t_1, t_2) \subset (0, T)$ and choose the test-function $\phi = H_\delta(u^q - v^q)$ with $0 < \delta \leq \min\{1, \frac{\epsilon^q}{2}\}$, which is admissible since $u^q \leq v^q$ on the lateral boundary $\partial\Omega \times (0, T)$. Recall that H_δ is defined in (3.1). In this way, we obtain

$$\begin{aligned} &\iint_{\Omega\times(t_1,t_2)} \partial_t[u^q - v^q]_h H_\delta(u^q - v^q) \,\mathrm{d}x\mathrm{d}t \\ &\qquad \leq \iint_{\Omega\times(t_1,t_2)} \big[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\big]_h \cdot \nabla H_\delta(u^q - v^q) \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.5}$$

Applying Lemma 3.1 with $f = u^q - v^q$ to the integrand on the left-hand side, we find

$$\begin{aligned} &\iint_{\Omega\times(t_1,t_2)} \partial_t[G_\delta(u^q - v^q)]_h \,\mathrm{d}x\mathrm{d}t \\ &\qquad \leq \iint_{\Omega\times(t_1,t_2)} \big[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\big]_h \cdot \nabla H_\delta(u^q - v^q) \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.6}$$

We now focus on the integral on the left-hand side of (3.6). Letting $h \downarrow 0$, we obtain

$$\begin{aligned} \lim_{h\downarrow 0} \iint_{\Omega\times(t_1,t_2)} \partial_t[G_\delta(u^q - v^q)]_h \,\mathrm{d}x\mathrm{d}t &= \lim_{h\downarrow 0} \int_\Omega [G_\delta(u^q - v^q)]_h \,\mathrm{d}x\bigg|_{t_1}^{t_2} \\ &= \int_\Omega G_\delta(u^q - v^q) \,\mathrm{d}x\bigg|_{t_1}^{t_2}. \end{aligned}$$

Next, we justify the passage to the limit $h \downarrow 0$ for the integral on the right-hand side of (3.6). A direct computation yields

$$\iint_{\Omega\times(t_1,t_2)} \left[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\right]_h \cdot \nabla H_\delta\left(u^q - v^q\right) \mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{\delta}\iint_{\Omega_\delta} \left[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\right]_h \cdot \nabla(u^q - v^q)\, \mathrm{d}x\mathrm{d}t,$$

where

$$\Omega_\delta := \left\{(x,t) \in \Omega \times (t_1,t_2) : 0 < u^q(x,t) - v^q(x,t) < \delta\right\}.$$

Since

$$|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u \in L^{\frac{p}{p-1}}(\Omega_T),$$

we have

$$\left[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\right]_h \to |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u \quad \text{in } L^{\frac{p}{p-1}}(\Omega_T)$$

as $h \downarrow 0$. Our next aim is to ensure that $\nabla(u^q - v^q) \in L^p(\Omega_\delta)$. This will be a consequence of assumption (3.3) and the definition of Ω_δ. We first consider the case $0 < q \le 1$. If $v \ge \epsilon$ in Ω_T, then we have $u > \epsilon$ in Ω_δ. Otherwise, if $u \ge \epsilon$ in Ω_T, then we have $v > 2^{-\frac{1}{q}}\epsilon$ in Ω_δ by the choice of δ. In any case, we find that

$$|\nabla u^q| = qu^{q-1}|\nabla u| \le q\epsilon^{q-1}|\nabla u| \quad \text{in } \Omega_\delta$$

and

$$|\nabla v^q| = qv^{q-1}|\nabla v| \le c(q)\epsilon^{q-1}|\nabla v| \quad \text{in } \Omega_\delta.$$

On the other hand, in the case $q > 1$ we assume that either u or v is bounded. Therefore, there exists a constant $M > 0$ such that either $u \le M$ or $v \le M$ in Ω_T. Since $\delta \le 1$, this implies $u^q < 1 + M^q$ and $v^q < 1 + M^q$ in Ω_δ, so that

$$|\nabla v^q| \le c(q,M)|\nabla v| \;\text{ in } \Omega_T \quad \text{and} \quad |\nabla u^q| \le c(q,M)|\nabla u| \;\text{ in } \Omega_\delta.$$

Thus, we have shown in any case that $\nabla(u^q - v^q) \in L^p(\Omega_\delta)$ and therefore, we may pass to the limit $h \downarrow 0$ also on the right-hand side of (3.6) and derive

$$\lim_{h\downarrow 0}\iint_{\Omega\times(t_1,t_2)} \left[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\right]_h \cdot \nabla H_\delta(u^q - v^q)\, \mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{\delta}\iint_{\Omega_\delta} \left(|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\right) \cdot \nabla(u^q - v^q)\, \mathrm{d}x\mathrm{d}t.$$

In conclusion, after passing to the limit $h \downarrow 0$ on both sides of (3.6), we obtain

$$\int_{\Omega\times\{t_2\}} G_\delta\left(u^q - v^q\right) \mathrm{d}x - \int_{\Omega\times\{t_1\}} G_\delta\left(u^q - v^q\right) \mathrm{d}x$$
$$\le \frac{1}{\delta}\iint_{\Omega_\delta} \left(|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\right) \cdot \nabla(u^q - v^q)\, \mathrm{d}x\mathrm{d}t. \tag{3.7}$$

A simple calculation yields the identity

$$\nabla(u^q - v^q) = qu^{q-1}(\nabla u - \nabla v) + q\nabla v(u^{q-1} - v^{q-1}), \tag{3.8}$$

so that

$$\begin{aligned}\int_{\Omega\times\{t_2\}} & G_\delta(u^q - v^q)\,\mathrm{d}x - \int_{\Omega\times\{t_1\}} G_\delta(u^q - v^q)\,\mathrm{d}x \\ &\le -\frac{q}{\delta}\iint_{\Omega_\delta} u^{q-1} \underbrace{\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot(\nabla u - \nabla v)}_{\ge 0}\,\mathrm{d}x\mathrm{d}t \\ &\quad - \frac{q}{\delta}\iint_{\Omega_\delta} \big(u^{q-1} - v^{q-1}\big)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot\nabla v\,\mathrm{d}x\mathrm{d}t \\ &\le -\frac{q}{\delta}\iint_{\Omega_\delta} \big(u^{q-1} - v^{q-1}\big)\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot\nabla v\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

In view of Lemma 2.3 and assumption (3.3), we obtain in the set Ω_δ the following estimate

$$0 < u^{q-1} - v^{q-1} = \left| u^{q\frac{q-1}{q}} - v^{q\frac{q-1}{q}} \right| \le c(q)(u^q + v^q)^{-\frac{1}{q}}|u^q - v^q| \le \frac{c(q)}{\epsilon}\,\delta.$$

This yields

$$\begin{aligned}\int_{\Omega\times\{t_2\}} & G_\delta(u^q - v^q)\,\mathrm{d}x - \int_{\Omega\times\{t_1\}} G_\delta(u^q - v^q)\,\mathrm{d}x \\ &\le \frac{c(q)}{\epsilon}\iint_{\Omega_\delta} \big||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big|\,|\nabla v|\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We now pass to the limit $\delta \downarrow 0$ on both sides. The integral on the right-hand side vanishes, since $|\Omega_\delta| \to 0$ as $\delta \downarrow 0$. Therefore, we obtain

$$\int_{\Omega\times\{t_2\}} (u^q - v^q)_+\,\mathrm{d}x \le \int_{\Omega\times\{t_1\}} (u^q - v^q)_+\,\mathrm{d}x,$$

which finishes the proof of the proposition. □

We are now in the position to prove our first main result.

Proof of Theorem 1.1 The assumptions of the theorem ensure that there exists $\epsilon > 0$ such that $v \ge \epsilon$ on $\partial\Omega \times (0, T)$.

We first consider the case $0 < q \le 1$. We choose $\kappa \in (0, \epsilon]$ and define

$$u_\kappa := \max\{u, \kappa\}.$$

Due to Lemma 2.4, we know that u_κ is a weak sub-solution to (1.1) in Ω_T. Moreover, in view of assumptions (1.2) and (1.3) we have $(u_\kappa - v)_+ \in L^p(0, T; W_0^{1,p}(\Omega))$. Therefore, we may apply Proposition 3.2 to u_κ and v to conclude that

$$\int_{\Omega\times\{t_2\}} \big(u_\kappa^q - v^q\big)_+\,\mathrm{d}x \le \int_{\Omega\times\{t_1\}} \big(u_\kappa^q - v^q\big)_+\,\mathrm{d}x$$

for every $0 \le t_1 < t_2 \le T$. Letting $\kappa \downarrow 0$ finishes the proof for $0 < q \le 1$.

Next, we consider the case $q \ge 1$. By assumption, there exists a constant $M > 0$ such that $u \le M$ on $\partial\Omega \times (0, T)$. For $\kappa \in (0, \epsilon]$, we now define

$$u_\kappa := \max\{u, \kappa\} \quad \text{and} \quad v_M := \min\{v, M\}. \tag{3.9}$$

Thanks to Lemma 2.4, we know that u_κ is a weak sub-solution and v_M is a bounded weak super-solution to (1.1) in Ω_T. Moreover, in view of (1.2) and (1.3) we have $(u_\kappa - v_M)_+ \in L^p(0,T;W_0^{1,p}(\Omega))$. As before, we apply Proposition 3.2 to u_κ and v_M to conclude that

$$\int_{\Omega\times\{t_2\}} \big(u_\kappa^q - v_M^q\big)_+ \,\mathrm{d}x \le \int_{\Omega\times\{t_1\}} \big(u_\kappa^q - v_M^q\big)_+ \,\mathrm{d}x$$

for every $0 \le t_1 < t_2 \le T$. The claim now follows by letting $\kappa \downarrow 0$ and $M \to \infty$. □

Theorem 1.2 is an immediate consequence of Theorem 1.1.

Proof of Theorem 1.2 Applying Theorem 1.1 with the choice $t_1 = 0$, we obtain

$$\int_{\Omega\times\{t\}} (u^q - v^q)_+ \,\mathrm{d}x \le \int_{\Omega\times\{0\}} (u^q - v^q)_+ \,\mathrm{d}x$$

for any $t \in (0,T)$. Since $u(\cdot,0)^q \le v(\cdot,0)^q$ a.e. in Ω, the right-hand side of the preceding inequality vanishes, so that

$$\int_{\Omega\times\{t\}} (u^q - v^q)_+ \,\mathrm{d}x \le 0$$

for any $t \in (0,T)$. This yields $(u^q - v^q)_+ = 0$ a.e. in Ω for any $t \in (0,T)$, which implies the desired inequality. □

3.2 Comparison principles in a local setting

The comparison principles in Theorems 1.1 and 1.2 require an upper bound of the weak sub-solution on the lateral boundary of Ω_T in the case $q > 1$. However, some typical applications of the comparison principle are in a local setting. For instance, two solutions shall be compared on a compactly contained subset of Ω_T. For certain ranges of exponents, it is known that weak sub-solutions are locally bounded. We summarize these results in the following remark.

Remark 3.3 Let $q > 0$ and $p > 1$ satisfy either $0 < q \le p-1$, or $0 < p-1 < q < \frac{n(p-1)+p}{(n-p)_+}$. Then, any non-negative weak sub-solution u of (1.1) in Ω_T is locally bounded.

The results are scattered in the literature for different ranges of exponents. A natural classification is the following one:

- $0 < q < p-1$ (slow diffusion case), cf. [19] or [8, Theorem 4.1];
- $0 < q = p-1$ (homogeneous case), cf. [19] or [22, Lemma 5.1];
- $0 < p-1 < q < \frac{n(p-1)+p}{(n-p)_+}$ (fast diffusion case), cf. [19] or [6, Theorem 1.3].

This information allows to omit the boundedness assumption in the comparison principle in a local setting.

Corollary 3.4 *Let $q > 0$ and $p > 1$ satisfy either $0 < q \le p-1$, or $0 < p-1 < q < \frac{n(p-1)+p}{(n-p)_+}$, and let u be a non-negative local weak sub-solution and v a non-negative local weak super-solution of (1.1) in Ω_T. Further, let $K \Subset \Omega$ and $0 < t_1 < t_2 < T$ and suppose that*

$$\operatorname*{ess\,inf}_{\partial K\times(t_1,t_2)} v > 0$$

holds. If

$$u \le v \qquad \text{on } \partial_p\big(K \times (t_1, t_2)\big),$$

then we have

$$u \le v \qquad \text{a.e. in } K \times (t_1, t_2).$$

Proof In view of Remark 3.3, we know that under the present assumptions u is locally bounded in Ω_T. Hence, u is bounded in $\overline{K} \times [t_1, t_2]$ and u is a non-negative weak sub-solution and v a non-negative weak super-solution of (1.1) in $K \times (t_1, t_2)$. This allows to apply Theorem 1.2 to u and v on the parabolic cylinder $K \times (t_1, t_2)$ with the result that $u \le v$ a.e. in $K \times (t_1, t_2)$. □

Corollary 3.4 still requires the super-solution to be strictly positive on the lateral boundary of the considered subcylinder. We are able to omit this assumption in the smaller range of exponents $p - 1 < q < \frac{n(p-1)}{(n-p)_+}$. In fact, in this case there holds a Harnack inequality without time gap [6, 22]. This allows to prove the comparison principle for non-negative weak solutions stated in Theorem 1.3 without a lower bound on the lateral boundary data.

Proof of Theorem 1.3 From [22] in the case $q = p - 1$, respectively, [6, Theorem 1.11] in the case $p - 1 < q < \frac{n(p-1)}{(n-p)_+}$ we know that for any $t \in [t_1, t_2]$ either $u(\cdot, t) > 0$ or $u(\cdot, t) \equiv 0$ in $\overline{K}$. Moreover, from [6, 7, 24] we know that u and v are Hölder continuous in $\overline{K} \times [t_1, t_2]$. We now let

$$\tau_o := \sup\big\{t \in [t_1, t_2] : u(\cdot, t) \le v(\cdot, t) \text{ in } \overline{K}\big\}.$$

Note that $u(\cdot, \tau_o) \le v(\cdot, \tau_o)$ in $\overline{K}$ by the continuity of u and v if $\tau_o > t_1$, respectively, by the initial condition $u(\cdot, t_1) \le v(\cdot, t_1)$ if $\tau_o = t_1$.

We claim that $\tau_o < t_2$. As explained above, we either have $u(\cdot, \tau_o) > 0$ or $u(\cdot, \tau_o) \equiv 0$ in $\overline{K}$. In the former case, there exist $\epsilon > 0$ and $0 < \delta \le t_2 - \tau_o$ such that $u \ge \epsilon$ in $\overline{K} \times [\tau_o, \tau_o + \delta]$. Moreover, we have $u \le v$ on $\partial_p(K \times (\tau_o, \tau_o + \delta))$. This allows to apply Theorem 1.2 to conclude that $u \le v$ in $K \times (\tau_o, \tau_o + \delta)$, contradicting $\tau_o < t_2$.

In the latter case, where $u(\cdot, \tau_o) \equiv 0$ in $\overline{K}$, there exists $\tau_1 \in (\tau_o, t_2]$ such that $u > 0$ in $\overline{K} \times (\tau_o, \tau_1]$. Moreover, there exist $\epsilon > 0$ and $0 < \delta < \tau_1 - \tau_o$ such that $u \ge \epsilon$ in $\overline{K} \times [\tau_o + \delta, \tau_1]$. Since $v \ge u \ge \epsilon$ on $\partial K \times [\tau_o + \delta, \tau_1]$, Theorem 1.1 implies

$$\int_{K \times \{t\}} \big(u^q - v^q\big)_+ \,\mathrm{d}x \le \int_{K \times \{\tau_o + \delta\}} \big(u^q - v^q\big)_+ \,\mathrm{d}x$$

for any $\tau_o + \delta \le t \le t_2$. Letting $\delta \downarrow 0$ in the inequality above, the integral on the right hand side vanishes, since u and v are continuous and $u(\cdot, \tau_o) = 0$ in $\overline{K}$. This, however, implies $u \le v$ in $\overline{K} \times [\tau_o, \tau_1]$, again contradicting $\tau_o < t_2$.

Hence, we have $\tau_o = t_2$, which implies $u \le v$ in $K \times (t_1, t_2)$ as claimed. □

4 General structures

In this section, we present some generalizations under which the statements of the comparison principles continue to hold.

4.1 Inhomogeneous equations

The first generalization concerns the presence of a right-hand side. Instead of (1.1), we now consider its inhomogeneous variant

$$\partial_t u^q - \Delta_p u = f \qquad \text{in } \Omega_T \tag{4.1}$$

for some

$$f \in L^{\tilde{p}'}(\Omega_T),$$

where

$$\tilde{p} := \max\{p, q+1\}$$

and $\tilde{p}' = \frac{\tilde{p}}{\tilde{p}-1}$ denotes the Hölder conjugate of $\tilde{p}$.

Definition 4.1 A non-negative measurable function $u\colon \Omega_T \to \mathbb{R}_{\geq 0}$ in the class

$$u \in C\big([0,T]; L^{q+1}(\Omega)\big) \cap L^p\big(0,T; W^{1,p}(\Omega)\big)$$

is a weak sub(super)-solution of (4.1) if

$$\iint_{\Omega\times(t_1,t_2)} \big[-u^q \partial_t \phi + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi\big]\,\mathrm{d}x\mathrm{d}t \leq (\geq) \iint_{\Omega\times(t_1,t_2)} f\phi\,\mathrm{d}x\mathrm{d}t \tag{4.2}$$

for any non-negative function

$$\phi \in W_0^{1,q+1}\big(0,T; L^{q+1}(\Omega)\big) \cap L^p\big(0,T; W_0^{1,p}(\Omega)\big).$$

A function u is a non-negative weak solution of (4.1) if it is both, a weak sub-solution and a weak super-solution.

The next lemma is a generalization of Lemma 2.4 for the inhomogeneous case.

Lemma 4.2 *Let $q > 0$, $p > 1$ and u be a non-negative weak sub-solution of (4.1) in the sense of Definition 4.1. Then, for any $\kappa > 0$ the function $\max\{u,\kappa\}$ is a weak sub-solution of*

$$\partial_t u^q - \Delta_p u = f\chi_{\{u>\kappa\}} \quad \mathit{in}\,\Omega_T.$$

Similarly, if v is a non-negative weak super-solution of (4.1), then for any $M > 0$ also $\min\{v, M\}$ is a weak super-solution of

$$\partial_t v^q - \Delta_p v = f\chi_{\{v<M\}} \quad \text{in } \Omega_T.$$

Proof We only treat the first part of the Lemma concerning sub-solutions, since the second one follows with a similar reasoning. We argue exactly as in the proof of Lemma 2.4 with the only exception that we have to treat the additional term

$$\iint_{\Omega_T} f\phi\,\mathrm{d}x\mathrm{d}t$$

that appears on the right-hand side of (2.6). Inserting the test-function

$$\phi = \eta\phi_h, \quad \text{where } \phi_h := \frac{([\![u]\!]_{\bar{h}} - \kappa)_+}{([\![u]\!]_{\bar{h}} - \kappa)_+ + \mu},$$

as defined in the proof of Lemma 2.4, passing first to the limit $h \downarrow 0$ and afterward to the limit $\mu \downarrow 0$, the integral converges to

$$\iint_{\Omega_T \cap \{u>\kappa\}} f\eta \, \mathrm{d}x\mathrm{d}t.$$

As in the proof of Lemma 2.4, we now use an approximation argument in order to replace η by an arbitrary testing function

$$\phi \in W_0^{1,q+1}\big(0,T; L^{q+1}(\Omega)\big) \cap L^p\big(0,T; W_0^{1,p}(\Omega)\big).$$

This proves that $\max\{u,k\}$ is a sub-solution as claimed. □

In the inhomogeneous case, we obtain the following variant of Theorem 1.1.

Corollary 4.3 *Let $p>1$, $q>0$ and*

$$f_1,\ f_2 \in L^{\tilde{p}'}(\Omega_T).$$

Further, let u be a non-negative weak sub-solution of

$$\partial_t u^q - \Delta_p u = f_1 \quad \text{in } \Omega_T$$

and v be a weak non-negative super-solution of

$$\partial_t v^q - \Delta_p v = f_2 \quad \text{in } \Omega_T$$

satisfying

$$\operatorname*{ess\,inf}_{\partial\Omega\times(0,T)} v > 0 \quad \textit{and} \quad \operatorname*{ess\,sup}_{\partial\Omega\times(0,T)} u < \infty \textit{ if } q>1. \tag{4.3}$$

If

$$u \le v \quad \textit{on } \partial\Omega \times (0,T),$$

then the following inequality holds

$$\int_{\Omega\times\{t_2\}} (u^q - v^q)_+ \,\mathrm{d}x \le \int_{\Omega\times\{t_1\}} (u^q - v^q)_+ \,\mathrm{d}x + \iint_{\Omega\times(t_1,t_2)\cap\{v<u\}} (f_1 - f_2)\,\mathrm{d}x\mathrm{d}t$$

for every $0 \le t_1 < t_2 \le T$.

Proof The claimed inequality may be shown in a similar way as Theorem 1.1 taking also into account the additional terms containing f_1 and f_2. In the following, we will explain in the case $q \ge 1$ how these terms are dealt with. We choose $0<\kappa<\epsilon\le M<\infty$ and define $u_\kappa = \max\{u,\kappa\}$ and $v_M = \min\{v,M\}$ as in the proof of Theorem 1.1. Instead of Lemma 2.4, we now apply Lemma 4.2 to infer that u_κ is a weak sub-solution to

$$\partial_t u_\kappa^q - \Delta_p u_\kappa = f_1 \chi_{\{u>\kappa\}} \quad \text{in } \Omega_T$$

and v_M is a weak super-solution to

$$\partial_t^q v_M - \Delta_p v_M = f_2 \chi_{\{v<M\}} \quad \text{in } \Omega_T.$$

Subsequently, we need a variant of Proposition 3.2 for inhomogeneous equations. Performing the same arguments as in the proof of Proposition 3.2, we obtain in inequality (3.5) the additional term

$$\iint_{\Omega\times(t_1,t_2)} \big[f_1\chi_{\{u>\kappa\}} - f_2\chi_{\{v<M\}}\big]_h \nabla H_\delta(u_\kappa^q - v_M^q)\,\mathrm{d}x\mathrm{d}t$$

on the right-hand side. Passing to the limit $h \downarrow 0$ and $\delta \downarrow 0$, we obtain instead of (3.4) the following inequality:

$$\int_{\Omega\times\{t_2\}} \big(u_\kappa^q - v_M^q\big)_+ \,\mathrm{d}x \le \int_{\Omega\times\{t_1\}} \big(u_\kappa^q - v_M^q\big)_+ \,\mathrm{d}x + \iint_{\Omega\times(t_1,t_2)\cap\{u_\kappa > v_M\}} \big(f_1\chi_{\{u>\kappa\}} - f_2\chi_{\{v<M\}}\big)\,\mathrm{d}x\mathrm{d}t.$$

Note that $\{u_\kappa > v_M\} = \{u > v\}$, since $\kappa < M$. Finally, passing to the limits $\kappa \downarrow 0$ and $M \to \infty$, yields the claimed inequality for $q \ge 1$. The modifications in the case $0 < q \le 1$ are similar. □

In the case $f_1 = f_2$, integral term on the right-hand side vanishes and therefore, we obtain the following variant of Theorem 1.2, which immediately follows from Corollary 4.3.

Corollary 4.4 *Let $q > 0$, $p > 1$ and*

$$f \in L^{\tilde{p}'}(\Omega_T)$$

and u be a non-negative weak sub-solution and v a non-negative weak super-solution of (4.1) in Ω_T satisfying (4.3). If

$$u \le v \qquad \text{on } \partial_p\Omega_T,$$

then we have

$$u \le v \qquad \text{a.e. in } \Omega_T.$$

4.2 General coefficients

Instead of the model equation (1.1), respectively, (4.1), one may consider some more general doubly nonlinear equations. More precisely, instead of the p-Laplacian operator we consider vector fields

$$A(x,t,u,\xi)\colon \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$$

and the associated doubly nonlinear equation

$$\partial_t u^q - \operatorname{div} A(x,t,u,\nabla u) = f \qquad \text{in } \Omega_T, \tag{4.4}$$

where $q > 0$. The vector field A is supposed to be a Carathéodory function, which means

$$(x,t) \mapsto A(x,t,u,\xi) \text{ is measurable for every} (u,\xi) \in \mathbb{R}\times\mathbb{R}^n,$$
$$(u,\xi) \mapsto A(x,t,u,\xi) \text{ is continuous for almost every } (x,t) \in \Omega_T,$$

and further to satisfy the following conditions

$$\begin{cases} A(x,t,u,0) = 0 \\ \big(A(x,t,u,\xi) - A(x,t,u,\eta)\big)\cdot(\xi-\eta) \ge 0 \\ |A(x,t,u,\xi)| \le C\big(1+|\xi|^{p-1}\big) \\ |A(x,t,u,\xi) - A(x,t,v,\xi)| \le L\,|u-v|\,\big(1+|\xi|^{p-1}\big) \end{cases} \tag{4.5}$$

for a.e. $(x,t) \in \Omega_T$ and any $u, v \in \mathbb{R}$, and any $\xi, \eta \in \mathbb{R}^n$, where $p > 1$ and C and L denote positive constants.

Definition 4.5 A non-negative measurable function $u\colon \Omega_T \to \mathbb{R}_{\geq 0}$ in the class

$$u \in C\big(0, T; L^{q+1}(\Omega)\big) \cap L^p\big(0, T; W^{1,p}(\Omega)\big)$$

is a non-negative weak sub(super)-solution of (4.4) if

$$\iint_{\Omega\times(t_1,t_2)} \big[-u^q \partial_t\phi + A(x,t,u,\nabla u)\cdot\nabla\phi\big]\,\mathrm{d}x\mathrm{d}t \leq (\geq) \iint_{\Omega\times(t_1,t_2)} f\phi\,\mathrm{d}x\mathrm{d}t \qquad (4.6)$$

for any non-negative function

$$\phi \in W_0^{1,q+1}\big(0, T; L^{q+1}(\Omega)\big) \cap L^p\big(0, T; W_0^{1,p}(\Omega)\big).$$

A function u is a non-negative weak solution of (4.4) if it is both, a weak sub-solution and a weak super-solution.

Due to the structure condition $(4.5)_3$ and the definition of $\tilde{p}'$, both integrals in (4.6) are finite. Moreover, we mention that the assumed continuity in time of weak sub/super-solutions in the sense of Definition 4.5 is not restrictive, see [6, Proposition 4.9]. Note that the doubly nonlinear equation (1.1) is a special case of (4.4), since $A(x,t,u,\nabla u) = A(\nabla u) = |\nabla u|^{p-2}\nabla u$ satisfies hypothesis (4.5).

The subsequent Lemma is a variant of Lemma 2.4 for the more general equations considered above.

Lemma 4.6 *Let $q > 0$, $p > 1$ and u be a non-negative weak sub-solution of (4.4) in the sense of Definition 4.5. Then, for any $\kappa > 0$ the function $\max\{u,\kappa\}$ is a weak sub-solution of*

$$\partial_t u^q - \operatorname{div} A(x,t,u,\nabla u) = f\chi_{\{u>\kappa\}} \quad \text{in } \Omega_T.$$

Similarly, if v is a non-negative local weak super-solution of (4.4), then for any $M > 0$ also $\min\{v, M\}$ is a local weak super-solution of

$$\partial_t u^q - \operatorname{div} A(x,t,u,\nabla u) = f\chi_{\{v<M\}} \quad \text{in } \Omega_T.$$

Proof The proof is similar to the one of Lemma 2.4, respectively, Lemma 4.2 for the model pdes (1.1) and (4.1). The proof for the case $f \equiv 0$ a.e. in Ω_T can be found in [6, Proposition 4.7]. Note that assumption $(4.5)_1$ is needed here in order to avoid a multiplicative factor $\chi_{\{u>\kappa\}}$, respectively, $\chi_{\{v<M\}}$ of the vector field A, see [6, Remark 4.8]. Moreover, the right-hand side f can be treated as in the proof of Lemma 4.2. □

The following Corollary illustrates another version of Theorem 1.1, which is the most general comparison principle in this paper.

Corollary 4.7 *Let $q > 0$, $p > 1$ and*

$$f_1,\ f_2 \in L^{\tilde{p}'}(\Omega_T)$$

Further, let u be a non-negative weak sub-solution of

$$\partial_t u^q - \operatorname{div}\big(A(x,t,u,\nabla u)\big) = f_1 \qquad \text{in } \Omega_T$$

and v be a non-negative weak super-solution of

$$\partial_t v^q - \operatorname{div}\big(A(x,t,u,\nabla v)\big) = f_2 \qquad \text{in } \Omega_T$$

satisfying

$$\operatorname*{ess\,inf}_{\partial\Omega\times(0,T)} v > 0 \quad \textit{and} \quad \operatorname*{ess\,sup}_{\partial\Omega\times(0,T)} u < \infty \textit{ if } q > 1. \tag{4.7}$$

If

$$u \le v \quad \textit{on } \partial\Omega \times (0, T),$$

then the following inequality holds

$$\int_{\Omega\times\{t_2\}} (u^q - v^q)_+ \,\mathrm{d}x \le \int_{\Omega\times\{t_1\}} (u^q - v^q)_+ \,\mathrm{d}x + \iint_{\Omega\times(t_1,t_2)\cap\{v<u\}} (f_1 - f_2)\,\mathrm{d}x\mathrm{d}t$$

for every $0 \le t_1 < t_2 \le T$.

Proof The proof can be achieved by similar arguments as in Theorem 1.1, taking into account the more general vector field A. The right-hand side can be treated exactly as in Corollary 4.3. Therefore, we only explain the arguments needed to treat the vector field A and omit the terms containing the functions f_1 and f_2. In the case $q \ge 1$, a similar approach to the proof of Theorem 1.1 leads us to the following version of (3.7)

$$\begin{aligned}
&\int_{\Omega\times\{t_2\}} G_\delta\left(u_\kappa^q - v_M^q\right)\mathrm{d}x - \int_{\Omega\times\{t_1\}} G_\delta\left(u_\kappa^q - v_M^q\right)\mathrm{d}x \\
&\qquad \le -\frac{1}{\delta}\iint_{\Omega_\delta} \big(A(x,t,u_\kappa,\nabla u_\kappa) - A(x,t,v_M,\nabla v_M)\big)\cdot\nabla(u_\kappa^q - v_M^q)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Here, Ω_δ denotes the set

$$\Omega_\delta := \big\{(x,t) \in \Omega \times (t_1,t_2) : 0 < u_\kappa^q(x,t) - v_M^q(x,t) < \delta\big\}.$$

Due to identity (3.8), the right-hand side of the preceding inequality may be re-written as

$$\begin{aligned}
&-\frac{1}{\delta}\iint_{\Omega_\delta} \big(A(x,t,u_\kappa,\nabla u_\kappa) - A(x,t,v_M,\nabla v_M)\big)\cdot\nabla(u_\kappa^q - v_M^q)\,\mathrm{d}x\mathrm{d}t \\
&= -\frac{1}{\delta}\iint_{\Omega_\delta} u_\kappa^{q-1}\big(A(x,t,u_\kappa,\nabla u_\kappa) - A(x,t,v_M,\nabla v_M)\big)\cdot(\nabla u_\kappa - \nabla v_M)\,\mathrm{d}x\mathrm{d}t \\
&\quad -\frac{1}{\delta}\iint_{\Omega_\delta} q\big(u_\kappa^{q-1} - v_M^{q-1}\big)\big(A(x,t,u_\kappa,\nabla u_\kappa) - A(x,t,v_M,\nabla v_M)\big)\cdot\nabla v_M\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

The second term on the right-hand side of the above identity vanishes in the limit $\delta \downarrow 0$, which follows similarly as in the proof of Theorem 1.1. Therefore, we will concentrate on the first term. Using assumptions $(4.5)_2$ and $(4.5)_4$ together with the fact that $u_\kappa \ge \kappa$ in Ω_T, we obtain

$$
\begin{aligned}
&-\frac{1}{\delta}\iint_{\Omega_\delta} u_\kappa^{q-1}\big(A(x,t,u_\kappa,\nabla u_\kappa)-A(x,t,v_M,\nabla v_M)\big)\cdot(\nabla u_\kappa-\nabla v_M)\,\mathrm{d}x\mathrm{d}t\\
&\quad=-\frac{1}{\delta}\iint_{\Omega_\delta} u_\kappa^{q-1}\underbrace{\big(A(x,t,u_\kappa,\nabla u_\kappa)-A(x,t,u_\kappa,\nabla v_M)\big)\cdot(\nabla u_\kappa-\nabla v_M)}_{\geq 0}\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\frac{1}{\delta}\iint_{\Omega_\delta} u_\kappa^{q-1}\big(A(x,t,u_\kappa,\nabla v_M)-A(x,t,v_M,\nabla v_M)\big)\cdot(\nabla u_\kappa-\nabla v_M)\,\mathrm{d}x\mathrm{d}t\\
&\quad\leq\frac{1}{\delta}\iint_{\Omega_\delta} u_\kappa^{q-1}\,|A(x,t,u_\kappa,\nabla v_M)-A(x,t,v_M,\nabla v_M)|\,|\nabla u_\kappa-\nabla v_M|\,\mathrm{d}x\mathrm{d}t\\
&\quad\leq\frac{L}{\delta}\iint_{\Omega_\delta} u_\kappa^{q-1}\,|u_\kappa-v_M|\,\big(1+|\nabla v_M|^{p-1}\big)\,|\nabla u_\kappa-\nabla v_M|\,\mathrm{d}x\mathrm{d}t\\
&\quad\leq L\,c(q,\kappa)\iint_{\Omega_\delta} u_\kappa^{q-1}\big(1+|\nabla v_M|^{p-1}\big)\,|\nabla u_\kappa-\nabla v_M|\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

The last integral vanishes in the limit $\delta \downarrow 0$. Finally, letting $\kappa \downarrow 0$ and $M \to \infty$ finishes the proof in the case $q \geq 1$. Since the case $0 < q < 1$ is similar, we omit the details. □

The following corollary represents a generalization of Theorem 1.2 for the doubly nonlinear equation (4.4).

Corollary 4.8 *Let $q > 0$, $p > 1$ and*

$$f \in L^{\tilde{p}'}(\Omega_T)$$

and u be a non-negative weak sub-solution and v a non-negative weak super-solution of (4.4) in Ω_T satisfying (4.7). If

$$u \leq v \qquad \text{on } \partial_p\Omega_T,$$

then we have

$$u \leq v \qquad \text{a.e. in } \Omega_T.$$

Remark 4.9 Similar local results as obtained in Sect. 3.2 also hold true for the doubly-nonlinear equation (4.4). Corollary 3.4 still holds true, provided the right-hand side f is integrable enough to ensure local boundedness of the sub-solution. Theorem 1.3 continues to hold for homogeneous equations of the more general structure (4.4), i.e., $f \equiv 0$. Note that the main ingredient in the proof is a time insensitive Harnack inequality, which is available also under these more general assumptions; see [6, Theorem 1.10].

5 Uniqueness

The comparison principles derived so far imply uniqueness of weak solutions to the associated Cauchy–Dirichlet problem. Since only non-negative weak solutions are considered, the boundary and initial data are assumed to be non-negative as well. Note that due to Corollary 4.3 we are able to also consider a nontrivial right-hand side f in the Cauchy–Dirichlet problem.

Theorem 5.1 *Consider the data*

$$\begin{cases} f \in L^{\tilde{p}'}(\Omega_T), \\ g \in L^p\big(0, T; W^{1,p}(\Omega)\big), \\ u_o \in L^2\big(\Omega, \mathbb{R}_{\geq 0}\big). \end{cases}$$

Suppose furthermore that $g \geq \epsilon$ for some $\epsilon > 0$ and in the case $q > 1$ additionally assume that g is bounded. Then, there exists a unique non-negative weak solution of the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u^q - \Delta_p u = f & \text{in } \Omega \times (0, T), \\ u = g & \textit{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_o & \text{in } \Omega. \end{cases} \tag{5.1}$$

Proof The existence of a weak solution can be inferred for instance from [1]. Let u_1 and u_2 be two non-negative weak solutions of (5.1). Then, we have

$$u_1 - u_2 = (u_1 - g) - (u_2 - g) \in L^p\big(0, T; W_0^{1,p}(\Omega)\big)$$

and similarly for the initial datum

$$u_1(\cdot, 0) - u_2(\cdot, 0) = 0 \qquad \text{a.e. in } \Omega.$$

Applying Corollary 4.4 twice, we first derive $u_1 \leq u_2$ and similarly $u_1 \geq u_2$ a.e. in Ω_T. In turn, this yields $u_1 = u_2$ a.e. in Ω_T. □

A similar uniqueness result for non-negative weak solutions holds for the more general doubly nonlinear equation (4.4). In the proof, Corollary 4.4 has to be replaced by 4.8.

Theorem 5.2 *Let f, g, u_o be as in Theorem 5.1 and suppose that the vector field A satisfies the set of assumptions* (4.5). *Then, there exists a unique non-negative weak solution of the Cauchy–Dirichlet problem*

$$\begin{cases} \partial_t u^q - \operatorname{div} A(x, t, u, \nabla u) = f & \text{in } \Omega \times (0, T), \\ u = g & \textit{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_o & \text{in } \Omega. \end{cases}$$

Remark 5.3 A uniqueness result is also available in the case that the lateral boundary datum g vanishes entirely, see [26, 27]. Moreover, in the case $0 < p - 1 \leq q < \frac{n(p-1)}{(n-p)_+}$, Theorem 1.3 ensures local uniqueness of weak solutions without imposing any additional upper or lower bounds.

6 Viscosity solutions

In this final section, we will give an application of the comparison principle and show that every weak solution of (1.1) is also a viscosity solution. The result is interesting in itself as existence of a weak solution thus guarantees existence of a viscosity solution. In a similar fashion, we are also able to give a respective result for the homogeneous version of the generalized pde (4.4). Throughout we refer to [3, 11] for the definition and properties of viscosity solutions.

Definition 6.1 Let $q > 0$, $p \geq 2$ and $u : \Omega_T \to \mathbb{R}_{\geq 0}$ be an upper semi-continuous function. In the case $0 < q < 1$, we additionally require $u > 0$. u is a viscosity sub-solution of (1.1) if for any function $\phi \in C^1((0,T); C^2(\Omega))$ such that $\phi(x_0,t_0) = u(x_0,t_0)$ and $\phi > u$ in a deleted neighborhood of (x_0,t_0), we have

$$\partial_t \phi^q(x_0,t_0) - \Delta_p \phi(x_0,t_0) \leq 0.$$

Similarly, a lower semi-continuous function $u : \Omega_T \to \mathbb{R}_{\geq 0}$ is a viscosity super-solution of (1.1) if for any function $\phi \in C^1((0,T); C^2(\Omega))$ such that $\phi(x_0,t_0) = u(x_0,t_0)$ and $\phi < u$ in a deleted neighborhood of (x_0,t_0), we have

$$\partial_t \phi^q(x_0,t_0) - \Delta_p \phi(x_0,t_0) \geq 0.$$

Finally, a function u is a viscosity solution of (1.1) if it is both, a viscosity sub-solution and a viscosity super-solution.

Remark 6.2 In the case $1 < p < 2$, the definition of viscosity solution is delicate, since $\Delta_p \phi$ is not well defined for test functions ϕ whose gradient vanishes at the touching point; see [21, 28] for more discussion on this topic. For this reason, we restrict ourselves to the case $p \geq 2$.

Remark 6.3 In the literature, often strict inequalities are used, cf. [9, 21, 28]. Note that viscosity sub/super-solutions may equivalently be defined without strict inequalities of the test functions touching u from either below or above. However, it is always possible to obtain strict inequalities by modifying the test-function, which leads to equivalent Definitions.

We will need the following Lemma to prove the result for viscosity solutions afterward. In the theory of viscosity solutions, the stated property usually is referred to as *degenerate ellipticity*, see [11].

Lemma 6.4 *Let $p \geq 2$ and $\phi \in C^2(\Omega)$ such that $D^2\phi \in \mathbb{R}^{n \times n}$ is positive semi-definite. Then, there holds $\Delta_p \phi \geq 0$.*

Proof Let $x_0 \in \Omega$. In order to simplify notation, we abbreviate $v = \nabla\phi(x_0)$ and $X = D^2\phi(x_0)$. We compute

$$\begin{aligned}
\Delta_p \phi(x_0) &= (p-2)|v|^{p-4}(Xv \cdot v) + |v|^{p-2}\,\mathrm{Tr}(X) \\
&\geq |v|^{p-2}\big(-|v|^{-2}(Xv, v) + \mathrm{Tr}(X)\big) \\
&\geq |v|^{p-2}\Big(-\max_{i \in \{1,\dots,n\}}\{\lambda_i\} + \sum_{i=1}^{n} \lambda_i\Big) \geq 0,
\end{aligned}$$

where λ_i for $i \in \{1, \dots, n\}$ denote the eigenvalues of X and the estimate

$$\frac{\langle Xv, v\rangle}{|v|^2} = \frac{\langle Xv, v\rangle}{\langle v, v\rangle} \leq \max\{\lambda_1, \dots, \lambda_n\}$$

was used. □

We now state the result about viscosity solutions as an application of the comparison principle in Theorem 1.2. We only show that any weak solution is a viscosity solution in the sense of Definition 6.1. We did not attempt to prove the reverse implication, which is more involved. In the elliptic case, this property has for example been shown in [21], whereas the parabolic p-Laplace equation with a more general right hand side has been considered in [28]. In both cases, the weak and viscosity solutions coincide.

Theorem 6.5 *Let $q > 0$, $p \geq 2$ and u be a bounded non-negative weak solution of*

$$\partial_t u^q - \Delta_p u = 0 \quad \text{in } \Omega_T.$$

Then, u is a viscosity solution of

$$\partial_t u^q - \Delta_p u = 0 \quad \text{in } \Omega_T \cap \{u > 0\}.$$

If $1 \leq p-1 < q < \frac{n(p-1)}{(n-p)_+}$, then u is a viscosity solution of

$$\partial_t u^q - \Delta_p u = 0 \quad \text{in } \Omega_T.$$

Proof Instead of u, we consider its upper semi-continuous regularization u_*, which is, for locally bounded solutions, uniquely determined and verifies $u = u_*$ a.e. in Ω_T, see [25, Theorem 2.3].

We first show that any upper semi-continuous non-negative weak sub-solution is a viscosity sub-solution in the set $\Omega_T \cap \{u > 0\}$. Let $z_0 = (x_0, t_0) \in \Omega_T$ with $u(z_0) > 0$ and consider a test-function $\phi \in C^1((0,T); C^2(\Omega))$ with $\phi(z_0) = u(z_0)$ and $\phi > u$ in a deleted neighborhood of z_0. Arguing by contradiction, we assume

$$\partial_t \phi^q(z_0) - \Delta_p \phi(z_0) > 0.$$

Since $\phi \in C^1((0,T); C^2(\Omega))$, this inequality continues to hold in a neighborhood of z_0. Hence, we may find $\gamma_0 \in (0,1)$ and $\epsilon, \delta, \lambda \in (0,1)$ such that

$$\partial_t \phi^q - \Delta_p \phi \geq \lambda > 0 \quad \text{and} \quad \phi \geq \epsilon \qquad \text{in } Q_\delta(z_0) \tag{6.1}$$

and

$$u \leq \gamma_0 \phi \qquad \text{on } \partial_p Q_\delta(z_0), \tag{6.2}$$

where $Q_\delta(z_0) := B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$. The latter is a consequence of the upper semi-continuity of u. We abbreviate

$$M := 1 + \left\| \partial_t \phi^q \right\|_{L^\infty(Q_\delta(z_0))} < \infty.$$

Note that this expression is bounded for any $q > 0$, since $\phi \geq \epsilon$ in $Q_\delta(z_0)$. Choosing $\gamma \in [\gamma_0, 1)$ large enough to have

$$|\gamma^{q-p+1} - 1| \leq \frac{\lambda}{M},$$

we obtain

$$\begin{aligned} \partial_t (\gamma\phi)^q - \Delta_p(\gamma\phi) &= \gamma^{p-1}\big[\partial_t \phi^q - \Delta_p \phi + (\gamma^{q-p+1} - 1)\partial_t \phi^q\big] \\ &\geq \gamma^{p-1}\big[\lambda - |\gamma^{q-p+1} - 1| M\big] \geq 0 \end{aligned}$$

in $Q_\delta(z_0)$. Thus, $\gamma\phi \geq \gamma\epsilon > 0$ is a classical super-solution and therefore, also a weak super-solution in $Q_\delta(z_0)$. Now, Theorem 1.2 applied with u as weak sub-solution and $\gamma\phi$ as weak super-solution yields $u \leq \gamma\phi$ in $Q_\delta(z_0)$. Since $0 < \gamma < 1$, this contradicts $u(x_0, t_0) = \phi(x_0, t_0) > 0$. This ensures that u is a viscosity sub-solution.

Next, we prove that any lower semi-continuous non-negative weak super-solution is a viscosity super-solution in the set $\Omega_T \cap \{u > 0\}$. To this aim, we consider $z_0 \in \Omega_T$ with $u(z_0) > 0$ and a function $\phi \in C^1((0,T); C^2(\Omega))$ with $\phi(z_0) < u(z_0)$ and $\phi < u$ in a deleted neighborhood of z_0. Again we argue by contradiction and assume

$$\partial_t \phi^q(z_0) - \Delta_p \phi(z_0) < 0.$$

Similarly to before, we find $\gamma_0 > 1$ and $\epsilon, \delta, \lambda \in (0, 1)$ such that

$$\partial_t \phi^q - \Delta_p \phi \le -\lambda < 0 \quad \text{and} \quad \phi \ge \epsilon \qquad \text{in } Q_\delta(z_0)$$

and

$$u \ge \gamma_0 \phi \qquad \text{on } \partial_p Q_\delta(z_0).$$

With M defined as above, we choose $\gamma \in (1, \gamma_0]$ small enough to have

$$|\gamma^{q-p+1} - 1| \le \frac{\lambda}{M}.$$

In this way, we obtain

$$\begin{aligned}\partial_t(\gamma\phi)^q - \Delta_p(\gamma\phi) &= \gamma^{p-1}\big[\partial_t \phi^q - \Delta_p \phi + (\gamma^{q-p+1} - 1)\partial_t \phi^q\big]\\ &\le \gamma^{p-1}\big[-\lambda + |\gamma^{q-p+1} - 1|M\big] \le 0\end{aligned}$$

in $Q_\delta(z_0)$. Now, applying Theorem 1.2 with $\gamma\phi \ge \gamma\epsilon > 0$ as weak sub-solution and u as weak super-solution we derive a contraction as in the viscosity sub-solution case. This finishes the first part of the Theorem.

To show the second part of the Theorem, we consider $z_0 = (x_0, t_0) \in \Omega_T$. If $u(z_0) > 0$, the first part of the theorem applies. Therefore, it remains to consider the case $u(z_0) = 0$.

In view of the Harnack inequality from [6, Theorem 1.11], we have $u(\cdot, t_0) = 0$ a.e. in Ω.

We first consider some test function $\phi \in C^1((0, T); C^2(\Omega))$ such that $\phi(z_0) = u(z_0)$ and $\phi > u$ in a deleted neighborhood of z_0. Since u and ϕ both vanish in z_0, it follows that ϕ and hence, also ϕ^q attains a minimum there which implies $\partial_t \phi^q(z_0) = 0$ and $\nabla\phi(z_0) = 0$ and $D^2\phi(z_0)$ is positive semi-definite. Now, in view of Lemma 6.4 we obtain the desired inequality

$$\partial_t \phi^q(z_0) - \Delta_p \phi(z_0) = -\Delta_p \phi(z_0) \le 0.$$

Next, we consider a test function $\phi \in C^1((0, T); C^2(\Omega))$ such that $\phi(z_0) = u(z_0)$ and $\phi < u$ in a deleted neighborhood of z_0. Since $u(\cdot, t_0) = 0$ a.e. in Ω, we have that $\nabla\phi(z_0) = 0$ and $D^2\phi(z_0)$ is negative semi-definite. Moreover, since $q > 1$, we have $\partial_t \phi^q(z_0) = (q - 1)$ $\phi^{q-1}(z_0)\partial_t \phi(z_0) = 0$, so that

$$\partial_t \phi^q(z_0) - \Delta_p \phi(z_0) = -\Delta_p \phi(z_0) \ge 0.$$

Overall, this shows that u is a viscosity solution of (1.1) in Ω_T. □

Note that the Theorem also holds in the range of parameters p and q, where weak solutions might fail to be locally continuous. This is achieved through the semi-continuous regularization u_* which is defined in the proof. The second part of Theorem 6.5 holds in the whole of Ω_T due to infinite speed of propagation of weak solutions as shown in [6].

Remark 6.6 Note that in the second part of Theorem 6.5 we are able to show that any non-negative weak sub-solution is a viscosity sub-solution in Ω_T for any $q \ge 1$ and $p \ge 2$. The restriction $1 \le p - 1 < q < \frac{n(p-1)}{(n-p)_+}$ is only necessary for the argument ensuring that u is a viscosity super-solution.

Acknowledgements V. Bögelein and M. Strunk have been supported by the FWF-Project P31956-N32 "Doubly nonlinear evolution equations".

Funding Open access funding provided by Austrian Science Fund (FWF).

References

1. Alt, H.W., Luckhaus, S.: Quasilinear elliptic-parabolic differential equations. Math. z **183**(3), 311–341 (1983)
2. Bamberger, A.: Étude d'une équation doublement non linéaire. J. Funct. Anal. **24**(2), 148–155 (1977)
3. Bhattacharya, T., Marazzi, L.: On the viscosity solutions to Trudinger's equation. Nonlinear Differ. Equ. Appl. **22**(5), 1089–1114 (2015)
4. Bhattacharya, T., Marazzi, L.: On the viscosity solutions to a class of nonlinear degenerate parabolic differential equations. Rev. Matemática Complut. **30**, 621–656 (2017)
5. Bhattacharya, T., Marazzi, L.: A Phragmén-Lindelöf property of viscosity solutions to a class of doubly nonlinear parabolic equations. Bounded case. Rendiconti del Seminario Matematico della Università di Padova. **142** (2019)
6. Bögelein, V., Duzaar, F., Gianazza, U., Liao, N., Scheven, C.: Hölder Continuity of the Gradient of Solutions to Doubly Non-Linear Parabolic Equations (2023). arXiv preprint arXiv:2305.08539
7. Bögelein, V., Duzaar, F., Liao, N.: On the Hölder regularity of signed solutions to a doubly nonlinear equation. J. Funct. Anal. **281**(9), 109173 (2021)
8. Bögelein, V., Heran, A., Schätzler, L., Singer, T.: Harnack's inequality for doubly nonlinear equations of slow diffusion type. Calc. Var. Partial. Differ. Equ. **60**(6), 1–35 (2021)
9. Brändle, C., Vázquez, J. L.: Viscosity solutions for quasilinear degenerate parabolic equations of porous medium type. Indiana Univ. Math. J. 817–860 (2005)
10. Chagas, J. Q., Diehl, N. M. L., Guidolin, P. L.: Some properties for the Steklov averages (2017). arXiv preprint arXiv:1707.06368
11. Crandall, M.G., Ishii, H., Lions, P.L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. **27**(1), 1–67 (1992)
12. Diaz, J.I.: Qualitative study of nonlinear parabolic equations: an introduction. Extr. Math. **16**(3), 303–341 (2001)
13. DiBenedetto, E.: Degenerate Parabolic Equations. Springer Science & Business Media, Berlin (1993)
14. Giaquinta, M., Modica, G.: Partial regularity of minimizers of quasiconvex integrals. In: Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire, vol. 3, pp. 185–208. Elsevier, (1986)
15. Hajłasz, P., Koskela, P.: Sobolev met Poincaré, vol. 688. American Mathematical Society, Providence (2000)
16. Henriques, E., Laleoglu, R.: Local and global boundedness for some nonlinear parabolic equations exhibiting a time singularity. Differ. Integral Equ. **29**(11/12), 1029–1048 (2016)
17. Henriques, E., Laleoglu, R.: Boundedness for some doubly nonlinear parabolic equations in measure spaces. J. Dyn. Diff. Equat. **30**(3), 1029–1051 (2018)
18. Ivanov, A.V.: Existence and uniqueness of a regular solution of the Cauchy–Dirichlet problem for doubly nonlinear parabolic equations. Zeitschrift für Anal. und ihre Anwendungen **14**(4), 751–777 (1995)
19. Ivanov, A.V.: Maximum modulus estimates for generalized solutions of doubly nonlinear parabolic equations. Zapiski Nauchnykh Seminarov POMI **221**, 83–113 (1995)
20. Ivanov, A.V., Mkrtychan, P., Jäger, W.: Existence and uniqueness of a regular solution of the Cauchy–Dirichlet problem for a class of doubly nonlinear parabolic equations. J. Math. Sci. **1**(84), 845–855 (1997)
21. Juutinen, P., Lukkari, T., Parviainen, M.: Equivalence of viscosity and weak solutions for the $p(x)$-Laplacian. In: Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire, **27**(6), 1471–1487. Elsevier, (2010)
22. Kinnunen, J., Kuusi, T.: Local behaviour of solutions to doubly nonlinear parabolic equations. Math. Ann. **337**(3), 705–728 (2007)
23. Kinnunen, J., Lindqvist, P.: Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. Ann. di Matematica **185**(3), 411–435 (2006)

24. Kuusi, T., Laleoglu, R., Siljander, J., Urbano, J.M.: Hölder continuity for Trudinger's equation in measure spaces. Calc. Var. Partial. Differ. Equ. **45**, 193–229 (2012)
25. Liao, N.: Regularity of weak supersolutions to elliptic and parabolic equations: lower semicontinuity and pointwise behavior. J. de Math. Pures et Appl. **147**, 179–204 (2021)
26. Lindgren, E., Lindqvist, P.: On a comparison principle for Trudinger's equation. Adv. Calc. Var. **15**, 401–415 (2020)
27. Otto, F.: L^1-Contraction and Uniqueness for quasilinear elliptic–parabolic equations. J. Differ. Equ. **131**(1), 20–38 (1996)
28. Siltakoski, J.: Equivalence of viscosity and weak solutions for a p-parabolic equation. J. Evol. Equ. **21**, 2047–2080 (2021)
29. Trudinger, N.S.: Pointwise estimates and quasilinear parabolic equations. Commun. Pure Appl. Math. **21**(3), 205–226 (1968)
30. Vázquez, J.L.: The Porous Medium Equation: Mathematical Theory. Oxford University Press on Demand, Oxford (2007)
31. Vespri, V.: On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations. Manuscr. Math. **75**(1), 65–80 (1992)