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Ein–Lazarsfeld–Mustopa conjecture for the blow-up of a projective space

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Abstract
We solve the Ein–Lazarsfeld–Mustopa conjecture for the blow up of a projective space along a linear subspace. More precisely, let X be the blow up of $\mathbb{P}^n$ at a linear subspace and let L be any ample line bundle on X. We show that the syzygy bundle M_L defined as the kernel of the evalution map $H^0(X, L) \otimes \mathcal{O}_X \to L$ is L-stable. In the last part of this note we focus on the rigidness of M_L to study the local shape of the moduli space around the point $[M_L]$.

Keywords Stability · Syzygy bundles · Ein–Lazarsfeld–Mustopa conjecture

Mathematics Subject Classification 14J60 · 14M25 · 13D02

1 Introduction

Let (X, L) be a polarized smooth projective variety with L a globally generated line bundle. The syzygy bundle M_L is the kernel of the evaluation map $ev : \mathrm{H}^0(X, L) \otimes \mathcal{O}_X \twoheadrightarrow L$. Thus, M_L is a vector bundle of rank $\mathrm{h}^0(X, L) - 1$ sitting in the following exact sequence:

$$0 \to M_L \to \mathrm{H}^0(X, L) \otimes \mathcal{O}_X \to L \to 0.$$

Arising in a variety of geometric and algebraic problems, the syzygy bundles M_L have been extensively studied from different points of view. In particular, many efforts have been invested on knowing whether M_L is a stable vector bundle with respect to some polarization,

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and in [10] L. Ein, R. Lazarsfeld and Y. Mustopa conjecture that M_L is L-stable for a smooth polarized variety (X, L), when L is very positive (see Conjecture 2.2).

As far as we know, the stability of M_L has been proved in the following cases:

(1) $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ with $n > 1$ (see [1, 15] if $char(k) = 0$ and [2, 5] if $char(k) > 0$),
(2) (X, L) where X is a smooth projective curve of genus $g \geq 1$ and $\deg(L) \geq 2g + 1$ (see [9, Proposition 1.5]),
(3) (X, L) where X is a simple abelian variety and L an ample globally generated line bundle (see [4, Corollary 2.1]),
(4) when (X, L) is a sufficiently positive polarization of an algebraic surface X (see [10, Theorem A]),
(5) (X, L) where X is an Enriques (resp. bielliptic) surface and L an ample globally generated line bundle if $char(k) \neq 2$ (resp. if $char(k) \neq 2, 3$) (see [20, Theorem 3.5]), and more in general
(6) (X, L) where X is a smooth minimal projective surface of Kodaira dimension zero, and L an ample globally generated line bundle if $char(k) \neq 2, 3$ (see [20, Corollary 3.6], [3, Theorem 1] and [21, Theorem p.2]).

More in general, in [17, Question 7.8], M. Hering, M. Mustaţă and S. Payne asks for which choices of polarizations of a projective toric variety (X, L), the syzygy the syzygy bundle M_L is semistable (see Question 2.3).

In this paper we contribute to this conjecture for the blow-up $\mathrm{Bl}_Z(\mathbb{P}^n)$ of a projective space $\mathbb{P}^n$ along a linear subspace $Z \subset \mathbb{P}^n$. We assume that the base field $\mathbb{K}$ has characteristic 0, and we prove that for any very ample line bundle L on $\mathrm{Bl}_Z(\mathbb{P}^n)$, the syzygy bundle M_L is L-stable (Theorem 3.3). To prove this result we identify $\mathrm{Bl}_Z(\mathbb{P}^n)$ with a suitable projective bundle, which endows a toric variety structure. The theory of toric varieties deploys a correspondence between multigraded commutative algebra and geometry. We use this dictionary to give information about the geometric structure of the syzygy bundle M_L by using the syzygies of monomial ideals over non-standard bigraded polynomial rings. This allows us to strengthen Coandă's argument (Lemma 2.4) proving that M_L is L-stable.

One remarkable consequences of M_L being L-stable is that it can be seen as a point $[M_L]$ in its corresponding moduli space $\mathcal{M} = \mathcal{M}_X(N-1; c_1, \ldots, c_{\min\{N-1,n\}})$ where $N = \mathrm{h}^0(X, L)$ and $c_i = c_i(M_L)$ for $1 \leq i \leq \min\{N-1, n\}$. Many few properties, either local or global, are known for the moduli spaces, however in the last part of this note we are able to find that for $n > 2$ the syzygy bundle M_L is infinitessimally rigid and hence $[M_L]$ is an isolated point in $\mathcal{M}$. Furthermore, for $n = 2$ M_L is always unobstructed and we can compute the dimension of the tangent space of $\mathcal{M}$ at $[M_L]$.

This short note is organized as follows. In Sect. 2, we collect all the preliminary results needed to prove our main result. We have subdivided this section into two subsections. First, in Sect. 2.1, we state the basic notions on stability of syzygy bundles. In Sect. 2.2, we recall the relation between blow-ups of projective spaces along linear subspaces and projective bundles, and its toric variety structure. In order to give a more self-contained exposition, we also gather in this subsection the basic definitions and results on toric varieties to establish the algebraic-geometric dictionary used in the sequel. Finally, the core of this note is presented in Sect. 3. A first algebraic result on the syzygies of a monomial ideal yields information on the minimal locally free resolution of the syzygy bundle M_L (Corollary 3.2). Together with Coandă's Lemma (Lemma 2.4), it allows us to prove our main result, namely that M_L is L-stable for any ample line bundle L on $\mathrm{Bl}_Z(\mathbb{P}^n)$ (Theorem 3.3). Finally, we end this note by studying in Sect. 4 the rigidness of the syzygy bundles M_L and the local properties of the moduli space $\mathcal{M}$ around the point $[M_L]$ (Theorem 4.1).

2 Preliminaries

Let (X, L) be a polarized smooth variety defined over an algebraically closed field k of characteristic zero and let L be a globally generated line bundle. The *syzygy bundle* M_L associated to L is defined as the kernel of the evaluation map $ev : \mathrm{H}^0(X, L) \otimes \mathcal{O}_X \longrightarrow L$. Thus, M_L is a vector bundle of rank $\mathrm{h}^0(X, L) - 1$ fitting in the following short exact sequence

$$0 \longrightarrow M_L \longrightarrow \mathrm{H}^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0. \tag{1}$$

In particular we have:

- $c_1(M_L) = -c_1(L)$,
- $\mathrm{rk}(M_L) = \mathrm{h}^0(X, L) - 1$,
- $\mu_L(M_L) = \frac{-L^n}{\mathrm{h}^0(X,L)-1}$.

where for any vector bundle $\mathcal{E}$ on the polarized variety (X, L), we recall that $\mu_L(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot L^{n-1}}{\mathrm{rk}(\mathcal{E})}$ is the so-called *slope* of $\mathcal{E}$.

2.1 Stability of syzygy bundles

In this paper we are interested in determining the stability of syzygy bundles. Let us recall the basic definitions and the key results.

Definition 2.1 Let (X, L) be a polarized smooth variety of dimension n. A vector bundle E on X is L-stable (resp. L-semistable) if for any subsheaf $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$, we have

$$\frac{c_1(F)L^{n-1}}{\mathrm{rk}(F)} < \frac{c_1(E)L^{n-1}}{\mathrm{rk}(E)} \qquad \left(\text{resp.} \quad \frac{c_1(F)L^{n-1}}{\mathrm{rk}(F)} \leq \frac{c_1(E)L^{n-1}}{\mathrm{rk}(E)} \right).$$

In [10, Corollary 2.6], L. Ein, R. Lazarsfeld and Y. Mustopa posed the following conjecture:

Conjecture 2.2 *Let A and P be two line bundles on a smooth projective variety X. Assume that A is ample and set $L_d := dA + P$ for any positive integer d. Then, the syzygy bundle M_{L_d} is A-stable for $d \gg 0$.*

Related to this conjecture, in [17, Question 7.8], M. Hering, M. Mustaţă and S. Payne consider the following question:

Question 2.3 Let L be an ample line bundle on a projective toric variety X. Is the syzygy bundle M_L semistable, with respect to some choice of polarization?

Note that the (semi)stability of M_L with respect to L is equivalent to the (semi)stability of the pull back $\varphi_L^* T_{\mathbb{P}}$ of the tangent bundle of $\mathbb{P} := \mathbb{P}(\mathrm{H}^0(X, L)^\vee)$, where $\varphi_{|L|} : X \longrightarrow \mathbb{P}(\mathrm{H}^0(X, L)^\vee)$ is the morphism associated to L. The goal of this paper is to answer positively this last question for the blow up of a projective space along a linear subspace (see Theorem 3.3).

We end this subsection by stating a preliminary result on which our proof rests, it is a well known cohomological characterization of the stability.

Lemma 2.4 [5, Lemma 2.1] *Let (X, L) be a polarized smooth variety of dimension n. Let E be a vector bundle on X. Suppose that for any integer q and any line bundle G on X such that*

$$0 < q < \mathrm{rk}(E) \quad \text{and} \quad (G \cdot L^{n-1}) \geq q\mu_L(E)$$

one has $\mathrm{H}^0(X, \bigwedge^q E \otimes G^\vee) = 0$. Then, E is L-stable.

2.2 Blow-ups of projective spaces and toric varieties

In this work we focus on proving the stability of the syzygy bundle M_L for any ample line bundle L on a blow-up $\mathrm{Bl}_Z(\mathbb{P}^n)$ of a projective space $\mathbb{P}^n$ along a linear subspace $Z \subset \mathbb{P}^n$. In this subsection, we recall how the blow up $\mathrm{Bl}_Z(\mathbb{P}^n)$ endows a toric structure. We start with the following classical result

Proposition 2.5 *Let $Z \subset \mathbb{P}^n$ be a linear subspace of dimension $r - 1$. Then, the blow-up of $\mathbb{P}^n$ along Z is isomorphic to the projective bundle $\mathbb{P}(\mathcal{O}^r_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1))$, where $s = n - r$.*

Proof See [11, Proposition 9.11]. □

Remark 2.6 There is a conflicting notation in the literature on projective bundles. We use the definition of $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E})$ as found in [7]. It is worthwhile noticing, to avoid any confusion, that in some texts like [11], some authors define a projective bundle: "$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}^\vee)$" that would be, in our notation, $\mathbb{P}(\mathcal{E}^\vee)$.

Moreover, the projectivization of a decomposable vector bundle on a projective space $\mathbb{P}^s$ can be seen as a toric variety. In the following section, we make use of this structure to study the syzygy bundles M_L on $\mathrm{Bl}_Z(\mathbb{P}^n)$. For the sake of completeness, the remaining part of this subsection gathers the basic notions of toric varieties needed in the sequel. For further details on the geometry of toric we refer to [7].

Let X be a *toric variety* of dimension n associated to the *fan* $\Sigma \subset N \otimes \mathbb{R} \cong \mathbb{R}^n$, where $N \cong \mathbb{Z}^n$ is a lattice. If $\mathbb{T} \cong (\mathbb{K}^*)^n$ is the *algebraic torus* acting on X, let $M = \mathrm{Hom}(\mathbb{T}, \mathbb{K}^*) \cong \mathbb{Z}^n$ be the *lattice of characters* and, so, we have $N = \mathrm{Hom}(M, \mathbb{Z})$. For any cone $\sigma \in \Sigma$, we denote $\sigma^\vee \subset M \otimes \mathbb{R}$ its dual cone. We set $S_\sigma := \sigma^\vee \cap M$ and $\mathbb{K}[S_\sigma]$ the corresponding semigroup and semigroup algebra, and $U_\sigma := \mathrm{Spec}(\mathbb{K}[S_\sigma])$ is the corresponding affine toric variety. If τ is a face of σ, we write $\tau \prec \sigma$. There is a character $m \in M$ such that $S_\tau = S_\sigma + \mathbb{Z}\langle m\rangle$. Thus, localizing by χ^m we have $\mathbb{K}[S_\tau] \cong \mathbb{K}[S_\sigma]_{\chi^m}$. In particular, there is an inclusion of affine toric varieties $U_\tau \hookrightarrow U_\sigma$, and X is recovered from the fan Σ by glueing all affine toric varieties U_σ for $\sigma \in \Sigma$ along their intersections. $\Sigma(s)$ denotes the set of all s-dimensional cones in Σ. There is a one to one correspondence between $\mathbb{T}$-invariant orbits of codimension s and cones $\sigma \in \Sigma(s)$. In particular, $\mathbb{T}$-invariant Weil divisors correspond to rays $\rho \in \Sigma(1)$. Moreover, if X has no torus factors then the *class group* of X, $\mathrm{Cl}(X)$, is presented as

$$0 \to M \xrightarrow{\phi} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \to \mathrm{Cl}(X) \to 0, \tag{2}$$

where for any character $m \in M$, $\phi(m) = \sum_\rho \langle m, \rho\rangle$.

The *Cox ring* of X is the polynomial ring $S = \mathbb{K}[x_\rho \mid \rho \in \Sigma(1)]$ endowed with a grading by $\mathrm{Cl}(X)$:

$$\deg(x_\rho) = [D_\rho] \in \mathrm{Cl}(X), \text{ for any ray } \rho \in \Sigma(1).$$

For any cone $\sigma \in \Sigma$, we consider the squarefree monomial $x^{\hat{\sigma}} := \prod_{\rho \notin \sigma(1)} x_\rho$, and we define the *irrelevant ideal* $B = (x^{\hat{\sigma}} \mid \sigma \in \Sigma)$. The localization of S at $x^{\hat{\sigma}}$ is also a $\mathrm{Cl}(X)$-graded algebra $S_{x^{\hat{\sigma}}}$, and there is an isomorphism $\mathbb{K}[S_\sigma] \cong (S_{x^{\hat{\sigma}}})_0$, sending χ^m to the monomial $x_1^{\langle m,\rho_1\rangle} \cdots x_r^{\langle m,\rho_r\rangle}$ for any $m \in S_\sigma$.

Keeping this definitions in mind, we have a correspondence between $\mathrm{Cl}(X)$-graded S-modules and quasi-coherent sheaves on X (see for instance [7, Chapter 5]). By [7, Proposition 5.3.7], for any $\alpha \in \mathrm{Cl}(X)$ and for any Weil divisor $D = \sum_\rho a_\rho D_\rho$ such that $\alpha = [D]$, there is a natural isomorphism $S_\alpha \cong \Gamma(X, \mathcal{O}_X(D))$. Namely, we have the following result:

Proposition 2.7 (i) *If E is a $\mathrm{Cl}(X)-$graded S-module, there is a quasi-coherent sheaf $\widetilde{E}$ on X such that $\Gamma(U_\sigma, \widetilde{E}) = (E_{x^{\hat{\sigma}}})_0$, for any $\sigma \in \Sigma$.*

(ii) *Conversely, if $\mathcal{E}$ is a quasi-coherent sheaf on X, there is a $\mathrm{Cl}(X)-$graded $S-$module E such that $\widetilde{E} = \mathcal{E}$. In particular, $\widetilde{E}$ is coherent if and only if E is finitely generated.*

(iii) $\widetilde{E} = 0$ *if and only if $B^l E = 0$ for all $l \gg 0$.*

(iv) *There is an exact sequence of* $\mathrm{Cl}(X)-$*graded modules*

$$0 \to \mathrm{H}^0_B(E) \to E \to \mathrm{H}^0_*(X, \widetilde{E}) \to \mathrm{H}^1_B(E) \to 0.$$

Where $H^i_B(E)$ is the ith local cohomology module of E with respect to the irrelevant ideal B, and $H^i_*(X, \mathcal{E}) := \bigoplus_{\alpha \in \mathrm{Cl}(X)} H^i(X, \mathcal{E}(\alpha))$.

In particular, for any quasi-coherent sheaf $\mathcal{E}$ on X and any $\alpha \in \mathrm{Cl}(X)$ we identify $\mathrm{H}^0(X, \mathcal{E}(\alpha))$ with the degree $\alpha-$piece of the $S-$module $\mathrm{H}^0_*(X, \mathcal{E})$.

Proof See, for instance, [7, Proposition 5.3.3, Proposition 5.3.6 and Proposition 5.3.10] for (i)–(iii); and [12, Proposition 2.3] for (iv). □

We end this preliminary subsection by presenting the projectivization of a decomposable bundle over $\mathbb{P}^s$ as a toric variety and computing all intersection numbers needed later.

Recall that the only smooth projective toric variety with Picard group $\mathbb{Z}$ is the projective space $\mathbb{P}^n$ and, when $n > 1$, the stability of the syzygy bundle associated to ample line bundles $\mathcal{O}_{\mathbb{P}^n}(d)$ was established in [5, 8, 19]. Now we turn to smooth projective toric varieties with Picard group $\mathbb{Z}^2$. They were classified by Kleinschmidt in [18] who proved that if X is a smooth projective toric variety with $\mathrm{Pic}(X) \cong \mathbb{Z}^2$, then there are integers $r, s \geq 1$ with $s+r = \dim X$ and integers $0 \leq a_1 \leq \cdots \leq a_r$ such that $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \mathcal{O}_{\mathbb{P}^s}(a_r))$ (where $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E})$, see Remark 2.6). To describe the fan of X in the lattice $N = \mathbb{Z}^s \times \mathbb{Z}^r$ we fix the standard basis $\{e_1, \ldots, e_s\}$ and $\{f_1, \ldots, f_r\}$ of $\mathbb{Z}^s$ and $\mathbb{Z}^r$, respectively (see [7, Proposition 7.3.7]). We set

$$\begin{array}{ll} \rho_0 := \mathrm{cone}(-e_1 - \cdots - e_s + a_1 f_1 + \cdots + a_r f_r) & \eta_0 := \mathrm{cone}(-f_1 - \cdots - f_r) \\ \rho_i := \mathrm{cone}(e_i) \quad 1 \leq i \leq s & \eta_j := \mathrm{cone}(f_j) \quad 1 \leq j \leq r, \end{array}$$

and for $1 \leq i \leq s$ and $1 \leq j \leq r$ we define the $r+s-$dimensional cones

$$\sigma_{ij} := \mathrm{cone}(\rho_0, \ldots, \widehat{\rho_i}, \ldots, \rho_s, \eta_0, \ldots, \widehat{\eta_j}, \ldots, \eta_r).$$

Then $\Sigma(1) = \{\rho_0, \ldots, \rho_s, \eta_0, \ldots, \eta_r\}$ and $\Sigma_{max} = \{\sigma_{ij} \mid 1 \leq i \leq s,\ 1 \leq j \leq r\}$. From the exact sequence (2) we obtain the class group of X:

$$\mathrm{Cl}(X) = \mathrm{coker}\,\phi \cong \mathbb{Z}^2 \cong \mathbb{Z}\langle [D_{\rho_0}], [D_{\eta_0}] \rangle.$$

Moreover, we have:

$$[D_{\rho_i}] = [D_{\rho_0}] \text{ for } 1 \leq i \leq s \quad \text{and} \quad [D_{\eta_j}] = -a_j [D_{\rho_0}] + [D_{\eta_0}] \text{ for } 1 \leq j \leq r.$$

In particular, the Cox ring of X is the polynomial ring $\mathbb{K}[x_0, \dots, x_s, y_0, \dots, y_r]$ with $\deg(x_i) = (1, 0)$ for $0 \le i \le s$, $\deg(y_0) = (0, 1)$ and $\deg(y_j) = (-a_j, 1)$ for $1 \le j \le r$.

Finally, a line bundle $\mathcal{O}_X(a, b) := \mathcal{O}_X(a[D_{\rho_0}] + b[D_{\eta_0}])$ is ample (respectively, nef) if and only if $a, b > 0$ (respectively, $a, b \ge 0$). The anticanonical divisor is given by

$$-K_X = D_{\rho_0} + \cdots + D_{\rho_s} + D_{\eta_0} + \cdots + D_{\eta_r} = (s + 1 - a_1 - \cdots - a_r)D_{\rho_0} + (r + 1)D_{\eta_0}.$$

In particular, X is Fano (i.e. $-K_X$ is ample) if and only if $a_1 + \cdots + a_r < s + 1$.

In this setting, $[D_{\rho_0}]$ represents the class of the projective fiber $\pi^*\mathcal{O}_{\mathbb{P}^s}(1)$ and $[D_{\eta_0}]$ represents the class of the tautological line bundle $\mathcal{O}_X(1)$. On the other hand, using the intersection theory of toric varieties, we know that $[D_{\rho_0}] \cdots [D_{\rho_s}] = [D_{\eta_0}] \cdots [D_{\eta_r}] = 0$. In particular, we have $[D_{\rho_0}]^k = 0$ for $k \ge s + 1$, and $[D_{\eta_0}]([D_{\eta_0}] - a_1[D_{\rho_0}]) \cdots ([D_{\eta_0}] - a_r[D_{\rho_0}]) = 0$. From this we deduce that

$$[D_{\rho_0}]^{s-j}[D_{\eta_0}]^{r+j} = \begin{cases} 0 & j < 0 \\ s_0 = 1 & j = 0 \\ s_j = \sigma_1 s_{j-1} - \sigma_2 s_{j-1} + \cdots + (-1)^{j+1}\sigma_j s_0 & 1 \le j \le \min\{r, s\} \\ s_j = \sigma_1 s_{r-1} - \sigma_2 s_{j-1} + \cdots + (-1)^{r+1}\sigma_r s_0 & r < j \le s \end{cases}$$

where $\sigma_k = \sigma_k(a_1, \dots, a_r) = \sum_{1 \le i_1 < \cdots < i_k \le r} a_{i_1} \cdots a_{i_k}$ are the elementary symmetric polynomials.

3 Stability of syzygy bundles on blow-ups of the projective space

We devote this section to prove our main result (Theorem 3.3). From now on, we restrict our attention to blow up $\mathrm{Bl}_Z(\mathbb{P}^n)$ of $\mathbb{P}^n$ along a linear subspace $Z \subset \mathbb{P}^n$ of dimension $r - 1$. We set $s := n - r$, and by Proposition 2.5, $\mathrm{Bl}_Z(\mathbb{P}^n)$ is identified with $X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$. We fix an arbitrary ample line bundle $L = \mathcal{O}_X(a, b) := \mathcal{O}_X(a[D_{\rho_0}] + b[D_{\eta_0}])$ on X, with $a, b > 0$. Our goal is to prove that the syzygy bundle M_L fitting into the exact sequence

$$0 \longrightarrow M_L \longrightarrow \mathrm{H}^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

is L-stable. We start with an algebraic result which plays an important role in the structure of M_L.

Proposition 3.1 *Let $S = \mathbb{K}[x_0, \dots, x_s, y_0, \dots, y_r]$ be the $\mathbb{Z}^2-$graded polynomial ring with $\deg(x_i) = (1, 0)$ for $0 \le i \le s$, $\deg(y_i) = (0, 1)$ for $0 \le i \le r - 1$ and $\deg(y_r) = (-1, 1)$. For any integers $a, b > 0$ we consider the syzygy module K_L of the monomial ideal*

$$I_{a,b} = (x_0^{a_0} \cdots x_s^{a_s} y_0^{b_0} \cdots y_r^{b_r} \mid a_0 + \cdots + a_s = a + b_r,\ b_0 + \cdots + b_r = b).$$

Then, K_L is minimally generated by elements of degree $(a + 1, b)$ and $(a, b + 1)$.

Proof Since $I_{a,b}$ is a monomial ideal generated by forms of degree (a, b), then K_L is generated by syzygies of degree $(a + p, b + q)$ of the form $f w_1 - g w_2 = 0$ with f, g monomials of degree (p, q) (with $q \ge 0$ and $p \ge -q$), and $w_1, w_2 \in I_{a,b}$ monomials of degree (a, b). First of all notice that if $p = -q$, then f and g would be monomials of degree $(-q, q)$, so $f = g = y_r^q$. In particular, $f w_1 - g w_2 = y_r^q(w_1 - w_2)$ which cannot be a syzygy, being w_1 and w_2 different monomials. Therefore we can assume from now on that $q \ge 0$ and $p \ge -q + 1$. Let us write

$$\begin{aligned} f &= x_0^{l_0} \cdots x_s^{l_s} y_0^{m_0} \cdots y_r^{m_r} & w_1 &= x_0^{a_0} \cdots x_s^{a_s} y_0^{b_0} \cdots y_r^{b_r} \\ g &= x_0^{\lambda_0} \cdots x_s^{\lambda_s} y_0^{\mu_0} \cdots y_r^{\mu_r} & w_2 &= x_0^{\alpha_0} \cdots x_s^{\alpha_s} y_0^{\beta_0} \cdots y_r^{\beta_r}. \end{aligned}$$

Let us consider any syzygy $fw_1 - gw_2 = 0$ with $\deg(f) = \deg(g) = (-q+1+z, q)$ with either $z \geq 1$ or $q \geq 1$. We will see that $fw_1 - gw_2 = w(f'w_1' - g'w_2')$ for some monomial w of degree either $(1, 0)$ or $(0, 1)$, and monomials f', g' of degree either $(-q+1+(z-1), q)$ (assuming $z \geq 1$) or degree $(-q+1+z, q-1)$ (assuming $q \geq 1$), and $w_1', w_2' \in I_{a,b}$. We distinguish two main cases:

Case 1 there is $0 \leq i \leq s$ such that $l_i \geq 1$.

Case 2 $l_0 = \cdots = l_s = 0$.

We start analyzing Case 1 and we distinguish two subcases (A) and (B) as follows:

(A) There is an index $0 \leq i_0 \leq s$ such that both $l_{i_0} \geq 1$ and $\lambda_{i_0} \geq 1$. Then, we have

$$fw_1 - gw_2 = x_{i_0}\left(\frac{f}{x_{i_0}}w_1 - \frac{g}{x_{i_0}}w_2\right) = 0,$$

and $f' = \frac{f}{x_{i_0}}$, $g' = \frac{g}{x_{i_0}}$ are monomials of degree $(-q+1+(z-1), q)$.

(B) Otherwise, we may assume without loss of generality (permuting $\{x_0, \ldots, x_s\}$ if necessary), that $l_0 \geq 1$, $\lambda_0 = 0$ and for any $1 \leq i \leq s$, $l_i\lambda_i = 0$. In particular, we have $\alpha_0 = a_0 + l_0 \geq 1$. In this case, we have two options:

(B.1) there is an index j with $1 \leq j \leq s$ such that $\lambda_j \geq 1$, or else

(B.2) $\lambda_0 = \cdots = \lambda_s = 0$.

If (B.1) holds, then we may write

$$fw_1 - gw_2 = x_0\left(\frac{f}{x_0}w_1 - \frac{g}{x_j}\frac{w_2x_j}{x_0}\right).$$

If $w_1 \neq \frac{w_2x_j}{x_0}$, then we may set $f' = \frac{f}{x_0}$, $w_1' = w_1$, $g' = \frac{g}{x_j}$ and $w_2' = \frac{w_2x_j}{x_0}$. Otherwise, we would have $f = \frac{gx_0}{x_j}$, so $fw_1 - gw_2 = \frac{g}{x_j}(x_0w_1 - x_jw_2)$ would be a multiple of a syzygy of degree $(a+1, b)$.

On the other hand, if (B.2) holds it implies that $q \geq 1$. Indeed, if $q = 0$ then $\mu_0 = \cdots = \mu_r = 0$ and we would have $0 = 1 + z$, which cannot occur since in this case $z \geq 1$. We distinguish four more subcases:

(B.2.1) There is $0 \leq j \leq r-1$ such that $m_j \geq 1$ and $\mu_j \geq 1$

(B.2.2) There is $0 \leq j_0 \leq r-1$ such that $m_{j_0} \geq 1$ and $m_j\mu_j = 0$ for all $0 \leq j \leq r-1$.

(B.2.3) There is $0 \leq j_0 \leq r-1$ such that $\mu_{j_0} \geq 1$ and $m_j\mu_j = 0$ for all $0 \leq j \leq r-1$.

(B.2.4) $m_0 = \cdots m_{r-1} = \mu_0 = \cdots = \mu_{r-1} = 0$.

If case (B.2.1) holds, then we may write

$$fw_1 - gw_2 = y_j\left(\frac{f}{y_j}w_1 - \frac{g}{y_j}w_2\right),$$

and we have $\deg(\frac{f}{y_j}) = \deg(\frac{g}{y_j}) = (-q+1+z, q-1)$.

On the other hand, in both cases (B.2.2) and (B.2.3) we may assume without loss of generality (permuting $\{y_0, \ldots, y_{r-1}\}$ if necessary) that $j_0 = 0$. If (B.2.2) holds, then $m_0 \geq 1$ and $\mu_0 = 0$. In particular, $\beta_0 = b_0 + m_0 \geq 1$ and we distinguish two situations:

- There is $1 \leq k \leq r-1$ such that $\mu_k \geq 1$. We have

$$fw_1 - gw_2 = y_0\left(\frac{f}{y_0}w_1 - \frac{g}{y_k}\frac{w_2y_k}{y_0}\right).$$

If $w_1 \neq \frac{w_2 y_k}{y_0}$, then we take $f' = \frac{f}{y_0}$, $w_1' = w_1$, $g' = \frac{g}{y_k}$ and $w_2' = \frac{w_2 y_k}{y_0}$. If not, we have $f = \frac{g y_0}{y_k}$, so $f w_1 - g w_2 = \frac{g}{y_k}(y_0 w_1 - y_k w_2)$ is a multiple of a syzygy of degree $(a, b+1)$.

- Otherwise we may suppose $\mu_0 = \cdots = \mu_{r-1} = 0$, but then $\deg(g) = (-q, q)$ which is a contradiction.

In the case (B.2.3), we have $m_0 = 0$ and $\mu_0 \geq 1$, which in particular gives $b_0 = \beta_0 + \mu_0 \geq 1$. We have two subcases:

- There is $1 \leq k \leq r-1$ such that $m_k \geq 1$, then we have
$$f w_1 - g w_2 = y_0 \left(\frac{f}{y_k} \frac{w_1 y_k}{y_0} - \frac{g}{y_0} w_2 \right)$$
and it follows as before.
- Otherwise, $m_0 = \cdots = m_{r-1} = 0$, in particular we have $m_r = \mu_0 + \cdots + \mu_r \geq 1$. Since we are assuming that $l_0 \geq 1$, there is $0 \leq i \leq s$ such that $a_i \geq 1$. Hence, we have
$$f w_1 - g w_2 = y_0 \left(\frac{f}{x_0 y_r} \frac{w_1 y_r x_0}{y_0} - \frac{g}{y_0} w_2 \right)$$
and this subcase follows as before, since $\deg(y_r x_0) = (0, 1)$.

Finally, if (B.2.4) holds, then we would have $\deg(g) = (-\mu_r, \mu_r)$, which is a contradiction.

To finish the prove, we have to analyze Case 2, that is we assume $l_0 = \cdots = l_s = 0$. Moreover, we may assume by symmetry that $\lambda_0 = \cdots = \lambda_s = 0$, otherwise if $\lambda_i \geq 1$ for some $0 \leq i \leq s$, changing the role of f and g we would be again in Case 1. We distinguish two cases (A) and (B) as follows.

(A) There is $0 \leq j \leq r-1$ such that $m_j \geq 1$ and $\mu_j \geq 1$. Then, we have
$$f w_1 - g w_2 = y_j \left(\frac{f}{y_j} w_1 - \frac{g}{y_j} w_2 \right).$$

(B) Otherwise, we assume that for any $0 \leq j \leq r-1$, $m_j \mu_j = 0$. Then, there are indices $0 \leq j, \nu \leq r-1$ such that $m_j \geq 1$ and $\mu_\nu \geq 1$. Indeed, if either $m_0 = \cdots = m_{r-1} = 0$ or $\mu_0 = \cdots = \mu_{r-1} = 0$ we would have $\deg(f) = (-m_r, m_r)$ or $\deg(g) = (-\mu_r, \mu_r)$, which is a contradiction. Thus, without loss of generality we may suppose $j = 0$ and $\nu = 1$. Hence,
$$f w_1 - g w_2 = y_0 \left(\frac{f}{y_0} w_1 - \frac{g}{y_1} \frac{w_2 y_1}{y_0} \right).$$
As before, if $w_1 \neq \frac{w_2 y_1}{y_0}$ the result follows directly. Otherwise, $w_1 y_0 = w_2 y_1$ and $f w_1 - g w_2$ is a multiple of a syzygy of degree $(a, b+1)$. Now the proof is complete.

□

This result has a geometric consequence on the structure of the syzygy bundle M_L:

Corollary 3.2 *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$ be a blow-up of $\mathbb{P}^{s+r}$ at a linear subspace of dimension $r-1$. Let $a, b > 0$ be two integers and $L = \mathcal{O}_X(a, b)$ be an ample line bundle on X. Then, the minimal graded free resolution of the syzygy bundle M_L associated to L begins as*
$$\mathcal{O}(-1, 0)^{\lambda} \oplus \mathcal{O}(0, -1)^{\mu} \to M_L \to 0,$$

for some integers $\lambda, \mu > 0$.

In particular, for any $q \geq 1$ *we have the beginning of the minimal graded free resolution of* $\bigwedge^q M_L$*:*

$$\bigoplus_{q_1+q_2=q} \mathcal{O}(-q_1, -q_2)^{\beta_{q_1,q_2}} \to \bigwedge^q M_L \to 0,$$

for some integers β_{q_1,q_2}.

Proof Notice that the Cox ring of X is the $\mathbb{Z}^2-$graded polynomial ring $S = \mathbb{K}[x_0, \dots, x_s, y_0, \dots, y_r]$ with $\deg(x_i) = (1, 0)$ for $0 \leq i \leq s$, $\deg(y_i) = (0, 1)$ for $0 \leq i \leq r-1$ and $\deg(y_r) = (-1, 1)$ and $M_L(-L) \cong \widetilde{K_L}$ is the sheaffification of the syzygy module of the monomial ideal

$$I_{a,b} = (x_0^{a_0} \cdots x_s^{a_s} y_0^{b_0} \cdots y_r^{b_r} \mid a_0 + \cdots + a_s = a + b_r,\ b_0 + \cdots + b_r = b).$$

By Proposition 3.1, we have the minimal free resolution of K_L begins as:

$$S(-a-1, -b)^{\lambda} \oplus S(-a, -b-1)^{\mu} \to K_L \to 0.$$

Hence, the result follows by sheaffifying and then twisting this presentation by $\mathcal{O}_X(a, b)$. □

Finally, we are able to establish the main result of this note.

Theorem 3.3 *Let* $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$ *be the blow-up of* $\mathbb{P}^{r+s}$ *along a linear subspace of dimension* $r-1$. *Fix any ample line bundle* $L = \mathcal{O}_X(a, b)$ *on* X, *with* $a, b > 0$. *Then, the syzygy bundle* M_L *is* L*-stable.*

Proof Let us denote $N = N(s, r, a, b) = \mathrm{h}^0(X, \mathcal{O}_X(L))$. By Lemma 2.4, it is enough to see that for any $0 < q < N-1$ and any line bundle $G = \mathcal{O}_X(x, y)$ such that $G \cdot L^{r+s-1} \leq q\frac{L^{r+s}}{N-1}$, we have $\mathrm{H}^0(X, \bigwedge^q M_L(x, y)) = 0$. Notice that G needs to be effective, thus we may assume that $x + y \geq 0$ and $y \geq 0$. Moreover, by Corollary 3.2, if $x + y < q$ then we have already $\mathrm{H}^0(X, \bigwedge^q M_L(x, y)) = 0$. Indeed, recall that $\bigwedge^q M_L(x, y)$ corresponds to an $S-$module Λ presented as

$$\bigoplus_{q_1+q_2=q} S(x - q_1, y - q_2)^{\beta_{q_1,q_2}} \to \Lambda \to 0.$$

By Proposition 2.7, we obtain that $\mathrm{H}^0(X, \bigwedge^q M_L(x, y)) \cong \Lambda_0$ (the degree $0 \in \mathrm{Cl}(X)$ piece of Λ). On the other hand, for any pair of non-negative integers (q_1, q_2) such that $q_1 + q_2 = q$, we have that $S(x - q_1, y - q_2)_0 = S_{(x-q_1, y-q_2)}$. Assume by contradiction that there is a monomial $x_0^{a_0} \cdots x_s^{a_s} y_0^{b_0} \cdots y_r^{b_r} \in S_{(x-q_1, y-q_2)}$. Then

$$a_0 + \cdots + a_s = x - q_1 + b_r, \text{ and } b_0 + \cdots + b_r = y - q_2.$$

Since we assume that $x + y < q = q_1 + q_2$, we get a contradiction $0 \leq a_0 + \cdots + a_s = x - q_1 + b_r \leq x - q_1 + y - q_2 < 0$. Therefore, $S(x - q_1, y - q_2)_0 = 0$ and we obtain that $\Lambda_0 = 0$ as wanted.

As a consequence, we may also assume that G satisfies $x + y \geq q$, and next we see that in this case we have the inequality:

$$G \cdot L^{r+s-1} > q\frac{L^{r+s}}{N-1} \tag{3}$$

finishing the proof.

We use the description of the intersection products on X given before to express both sides of (3):

$$\begin{aligned} G\cdot L^{r+s} &= (x[D_{\rho_0}]+y[D_{\eta_0}])\sum_{i=0}^{s}\binom{r+s-1}{i}a^ib^{r+s-1-i}[D_{\rho_0}]^i[D_{\eta_0}]^{r+s-1-i} \\ &= (x+y)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{r+s-1-i}+\binom{r+s-1}{s}a^sb^{r-1}y \\ &= b^{r-1}\left((x+y)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{s-i}+\binom{r+s-1}{s}a^sy\right). \end{aligned}$$

Similarly, we have

$$L^{r+s-1}=b^r\left((a+b)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{s-1-i}+\binom{r+s-1}{s}a^s\right). \tag{4}$$

Thus (3) is equivalent to

$$\begin{aligned} &(x+y)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{s-i}+\binom{r+s-1}{s}a^sy \\ &\quad > q\,\frac{b\left((a+b)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{s-1-i}+\binom{r+s-1}{s}a^s\right)}{N-1}. \end{aligned} \tag{5}$$

Since $x+y\geq q$ and $y\geq 0$ we can bound the left hand side of (5) as

$$(x+y)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{s-i}+\binom{r+s-1}{s}a^sy\geq qb\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{s-1-i}. \tag{6}$$

Thus, reducing the proof of (5) into seeing

$$(N-1-a-b)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^ib^{s-i}>\binom{r+s-1}{s}a^s. \tag{7}$$

Let now $S=\mathbb{K}[x_0,\dots,x_s,y_0,\dots,y_r]$ be the Cox ring of X. Then, the vector space $\mathrm{H}^0(X,\mathcal{O}_X(a,b))$ is isomorphic to the degree$-(a,b)$ homogeneous piece of S. In particular,

$$\begin{aligned} N &= \sum_{i=0}^{b}\binom{r-1+b-i}{r-1}\binom{s+a+i}{s} \\ &= \binom{r-1+b}{r-1}\binom{s+a}{s}+\sum_{i=1}^{b-1}\binom{r-1+b-i}{r-1}\binom{s+a+i}{s}+\binom{s+a+b}{s} \\ &\geq \binom{r-1+b}{r-1}\binom{s+a}{s} \\ &\quad +\frac{(a+b)(a+b+1)\sum_{i=2}^{s}(2+a+b)\cdots(\widehat{i+a+b})\cdots(s+a+b)}{s!}+1+a+b \end{aligned}$$

$$\geq \binom{r-1+b}{r-1}\binom{s+a}{s} + 1 + a + b.$$

Applying this inequality and the fact that $b \geq 1$ we can finally show (7), ending the proof:

$$(N-1-a-b)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^i b^{s-i} \geq r\binom{s+a}{s}\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^i$$
$$\geq r\binom{s+a}{s}\binom{r+s-1}{s-1}a^{s-1} = r\frac{(s+a)(s-1+a)\cdots(1+a)}{s(s-1)\cdots 1}\frac{s}{r}\binom{r+s-1}{s}a^{s-1}$$
$$> (s+a)\binom{r+s-1}{s}a^{s-1} > \binom{r+s-1}{s}a^s.$$

□

4 Rigidness of syzygy bundles

In Sect. 3 we have seen that the syzygy bundle M_L, corresponding to an ample line bundle L on a blow-up $X = \mathrm{Bl}_Z(\mathbb{P}^n)$ of $\mathbb{P}^n$ along a linear subspace $Z \subset \mathbb{P}^n$ of codimension $r-1$, is L-stable. Thus, we may consider the moduli space $\mathcal{M} = \mathcal{M}_X(N-1; c_1, \ldots, c_{\min\{N-1,n\}})$ of stable vector bundles E with Chern classes $c_i(E) = c_i := c_i(M_L)$ for $1 \leq i \leq \min\{N-1, n\}$. In general, few structural results about moduli spaces are known. In this section we use the stability of M_L to study locally around $[M_L]$ the moduli space $\mathcal{M}$, and we see that the syzygy bundles M_L are infinitesimally rigid unless $n = 2$ and $L = \mathcal{O}_X(a, b)$ with $a \geq 1$ and $b \geq 2$. In this particular case we prove that M_L is unobstructed, so $[M_L]$ is a smooth point in $\mathcal{M}$, and we compute the dimension of the Zariski tangent space $T_{[M_L]}\mathcal{M}$ of the moduli space $\mathcal{M}$ at $[M_L]$.

Let us recall that the Zariski tangent space of $\mathcal{M}$ at a point $[E]$ is canonically given by

$$T_{[E]}\mathcal{M} \cong \mathrm{Ext}^1(E, E) \cong \mathrm{H}^1(X, E \otimes E^\vee).$$

If $[E]$ is a smooth point, the dimension of $T_{[E]}\mathcal{M}$ tells us the dimension of the irreducible component in $\mathcal{M}$ containing $[E]$. In particular, we say that E is infinitesimally rigid if $[E]$ is an isolated point, or equivalently $\dim T_{[E]}\mathcal{M} = 0$. We have the following result:

Theorem 4.1 *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$ be the blow-up of $\mathbb{P}^{r+s}$ along a linear subspace of dimension $r-1$. Fix any ample line bundle $L = \mathcal{O}_X(a, b)$ on X, with $a, b > 0$. Then, the syzygy bundle M_L is infinitesimally rigid, unless $r + s = 2$ and $b \geq 2$. In this case, M_L is unobstructed and we have*

$$\dim_{\mathbb{K}} T_{[M_L]}\mathcal{M} = \left[\sum_{i=0}^{b-2}(a+i)\right]^{\mathrm{h}^0(X,L)}.$$

Proof Let us start studying $\mathrm{H}^1(X, M_L \otimes M_L^\vee)$. We consider the exact sequence

$$0 \to M_L \to \mathcal{O}_X^{\oplus N} \to L \to 0. \tag{8}$$

taking the long exact sequence of cohomology we obtain that $\mathrm{H}^i(X, M_L) = 0$ for all $i \geq 0$. On the other hand, twisting by $\mathcal{O}_X(-L)$ we have the following description of the cohomology of $M_L(-L)$:

$$\begin{aligned}
&\mathrm{H}^0(X, M_L(-L)) = 0\\
&\mathrm{H}^1(X, M_L(-L)) \cong \mathrm{H}^0(X, \mathcal{O}_X) \oplus \mathrm{H}^1(X, \mathcal{O}_X(-L))^{\oplus N}\\
&\mathrm{H}^i(X, M_L(-L)) \cong \mathrm{H}^i(X, \mathcal{O}_X(-L))^{\oplus N}, \text{ for } i \geq 2
\end{aligned}$$

On the other hand, dualizing the exact sequence (8) and tensoring it by M_L, we obtain:

$$0 \to M_L(-L) \to M_L^{\oplus N} \to M_L \otimes M_L^{\vee} \to 0.$$

Taking again the long exact sequence of cohomology and using the above vanishings, we obtain that

$$\mathrm{H}^i(X, M_L \otimes M_L^{\vee}) \cong \mathrm{H}^{i+1}(X, M_L(-L)), \quad \text{for all } i \geq 0. \tag{9}$$

In particular $\mathrm{H}^1(X, M_L \otimes M_L^{\vee}) \cong \mathrm{H}^2(X, \mathcal{O}_X(-L))^{\oplus N}$, and by Kodaira's vanishing we have that, if $\dim(X) > 2$, $\mathrm{H}^2(X, \mathcal{O}_X(-L)) = 0$ and hence M_L is infinitesimally rigid.

It only remains to study the case $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. For any integers a' and b', using the projection formula we obtain that,

$$\begin{aligned}
\mathrm{H}^i(X, \mathcal{O}_X(a', b')) &= \mathrm{H}^i(\mathbb{P}(\mathcal{E}), \pi^*\mathcal{O}_{\mathbb{P}^1}(a') \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(b'))\\
&\cong \begin{cases} \mathrm{H}^i(\mathbb{P}^1, \mathrm{Sym}^{b'} \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(a')), & b' \geq 0\\ 0, & b' = -1\\ \mathrm{H}^{2-i}(\mathbb{P}^1, \mathrm{Sym}^{-b'-2} \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-1-a'))^{\vee}, & b' \leq -2. \end{cases}
\end{aligned} \tag{10}$$

Hence for any ample of the form $L = \mathcal{O}_X(a, 1)$ in X, we already have that M_L is infinitesimally rigid. Now, we consider an ample line bundle $L = \mathcal{O}_X(a, b)$, with $a \geq 1$ and $b \geq 2$. Using that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, so $\mathrm{Sym}^{\ell} \mathcal{E} \cong \bigoplus_{i=0}^{\ell} \mathrm{Sym}^{\ell-i} \mathcal{O}_{\mathbb{P}^1} \otimes \mathrm{Sym}^i \mathcal{O}_{\mathbb{P}^1}(1) \cong \bigoplus_{i=0}^{\ell} \mathcal{O}_{\mathbb{P}^1}(i)$, by (10) we have

$$\begin{aligned}
\mathrm{H}^2(X, \mathcal{O}_X(-L)) &\cong \mathrm{H}^0(\mathbb{P}^1, \mathrm{Sym}^{b-2} \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-1+a))^{\vee}\\
&\cong \bigoplus_{i=0}^{b-2} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(a+i-1))^{\vee}.
\end{aligned} \tag{11}$$

Thus, we have that $\mathrm{H}^2(X, \mathcal{O}_X(-L)) = 0$ if and only if $a + b < 3$ which cannot happen. Hence, to finish the proof, we ought to see that if $L = \mathcal{O}_X(a, b)$ is an ample line bundle on X with $b \geq 2$, then M_L is unobstructed. Indeed, recall that the obstruction space at $[M_L]$ is a subspace of $\mathrm{Ext}^2(M_L, M_L) \cong \mathrm{H}^2(X, M_L \otimes M_L^{\vee})$. By (9) we have that $\mathrm{H}^2(X, M_L \otimes M_L^{\vee}) \cong \mathrm{H}^3(X, M_L(-L) = 0$ since $\dim(X) = 2$, therefore M_L is unobstructed. In this case we may use (11) to compute the dimension of the Zariski tangent space $T_{[M_L]}\mathcal{M}$ of the moduli space $\mathcal{M}$ at $[M_L]$:

$$\dim_{\mathbb{K}} T_{[M_L]}\mathcal{M} = \mathrm{h}^1(X, M_L \otimes M_L^{\vee}) = \mathrm{h}^2(X, \mathcal{O}_X(-L))^N = \left[\sum_{i=0}^{b-2}(a+i)\right]^N.$$

□

Remark 4.2 For general results about the deformation of generalized syzygy bundles (not necessarily stable) the reader can look at [13, 14].

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