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The globalization theorem for CD(K, N) on locally finite spaces

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Abstract

We establish the local-to-global property of the synthetic curvature-dimension condition for essentially non-branching locally finite metric-measure spaces, extending the work [Cavalletti and Milman in Invent Math 226(1):1–137, 2021].

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1 Introduction

The *curvature-dimension condition*, or shortly CD(K,N) on metric-measure spaces $(X, d, \mathfrak{m})$, was introduced by Lott-Villani and Sturm in the seminal papers [12, 16, 17].

A natural but longstanding question is whether such a synthetically defined condition can be checked locally. Cavalletti–Milman's recent paper [6] gives a positive answer to this globalization problem under the assumption $\mathfrak{m}(X) = 1$, which was conjectured to be merely technical there. In this paper, we extend this result to infinite-volume spaces.

Theorem 1.1 (Local-to-Global property) *Let* $(X, d, \mathfrak{m})$ *be an essentially non-branching metric-measure space*[1] *with a locally finite Borel measure* $\mathfrak{m}$. *Assume that* $(\mathrm{supp}(\mathfrak{m}), d)$ *is a length space. Then if* $(X, d, \mathfrak{m})$ *verifies* $\mathrm{CD}_{\mathrm{loc}}(K,N)$ *for* $K \in \mathbb{R}$ *and* $N \in (1, \infty)$, *it verifies* CD(K, N).

Here immediately follow several useful equivalence results once we apply Theorem 1.1 to Section 13.1 and 13.2 in [6].

Corollary 1.2 *Let* $(X, d, \mathfrak{m})$ *be a metric-measure space with a locally finite Borel measure* $\mathfrak{m}$. *Then*[2]

[1] In the following sections, we will use the abbreviations *e.n.b.* and *m.m.s.* for *essentially non-branching* and *metric-measure space*, resp.

[2] We refer to [6, Section 13] for definitions of the following variants of curvature-dimension conditions.

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- if $(X, d, \mathfrak{m})$ is essentially non-branching, it holds $\mathrm{CD}^*(K, N)$ if and only if it holds $\mathrm{CD}(K, N)$;
- $(X, d, \mathfrak{m})$ holds $\mathrm{RCD}^*(K, N)$ if and only if it holds $\mathrm{RCD}(K, N)$;
- if $(\mathrm{supp}(\mathfrak{m}), d)$ is a length space, it holds $\mathrm{RCD}_{\mathrm{loc}}(K, N)$ if and only if it holds $\mathrm{RCD}(K, N)$.

In [6], Cavalletti and Milman introduced the $\mathrm{CD}^1(K, N)$ condition on finite-volume spaces, which roughly requires transport rays of signed distance functions to hold the one-dimensional $\mathrm{CD}(K, N)$. Then they showed that under suitable assumptions, $\mathrm{CD}^1(K, N)$ implies $\mathrm{CD}(K, N)$. Similarly, in this paper, we tailor the definition of $\mathrm{CD}^1(K, N)$, adapting it to the infinite-volume situation by assuming conditional measures to be uniformly-locally finite. Then we split the problem into two independent ones: $\mathrm{CD}_{\mathrm{loc}}(K, N) \Rightarrow \mathrm{CD}^1(K, N)$ and $\mathrm{CD}^1(K, N) \Rightarrow \mathrm{CD}(K, N)$.

For the first part, we normalize the reference measure as in [9] and show that the needle/ray-decomposition developed in [4, 5] still localizes the curvature-dimension condition to rays. For the second part, we show under the given definition, $\mathrm{CD}^1(K, N)$ space is locally finite, geodesic and satisfying $\mathrm{MCP}(K, N)$. Then we briefly present the strategy and arguments fulfilling the implication of $\mathrm{CD}(K, N)$ in locally finite spaces, which is basically the same as in [6] under modifications. Indeed, the validity is ensured basically by three aspects: (1) owing to the local finiteness of conditional measures and the properness of the space, problems are reduced to the finite-volume case by taking exhaustion by compacts subsets; (2) $\mathrm{CD}(K, N)$ is reduced to a path-wise inequality along Kantorovich geodesics by the non-branchingness, hence the one-dimensional analysis in [6, Part III] is not affected by the global infinity of $\mathfrak{m}$; (3) temporal derivatives of potentials, investigated in [6, Part I], do not rely on the measure structure.

Accordingly, the rest of this paper is organized as follows.

In Sect. 2, we recall central definitions and preliminary results.

In Sect. 3, we discuss the ray decomposition and define $\mathrm{CD}^1(K, N)$ in the locally finite setting. We show under assumptions of Theorem 1.1, $\mathrm{CD}_{\mathrm{loc}}(K, N)$ implies $\mathrm{CD}^1(K, N)$.

In Sect. 4, we discuss the implication $\mathrm{CD}^1(K, N) \Rightarrow \mathrm{CD}(K, N)$.

2 Preliminaries

2.1 Curvature-Dimension Condition

A triple $(X, d, \mathfrak{m})$ always stands for a *metric measure space* consisting of a Polish metric space equipped with the Borel σ-algebra and a locally finite Borel measure $\mathfrak{m}$ (i.e. for any $x \in X$, $\mathfrak{m}(B_r(x)) < \infty$ for some $r > 0$). Denote $\mathcal{P}_2(X)$ as the space of probability measures with finite variances and $\mathcal{P}_2(X, \mathfrak{m})$ the subspace of all absolutely continuous measures w.r.t. $\mathfrak{m}$.

An *optimal plan* between $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is a coupling $\pi \in \mathcal{P}(X \times X)$ minimizing the cost

$$C(\omega) = \int_{X \times X} \frac{\mathrm{d}^2(x, y)}{2} \omega(\mathrm{d}x\,\mathrm{d}y)$$

among all $\omega \in \mathcal{P}(X \times X)$ having μ_0 and μ_1 as the first and second marginal. Denote by $\mathrm{Opt}(\mu_0, \mu_1)$ the set of all optimal plans between μ_0 and μ_1. There is a $d^2/2$-concave function $\varphi : X \to \mathbb{R}$ called a *Kantorovich potential* associated to optimal plan π satisfying

$$\varphi(x) + \varphi^c(y) = \frac{\mathrm{d}^2(x, y)}{2}, \quad \pi - a.e.(x, y) \in X \times X$$

where φ^c is the *conjugate potential* of φ given by

$$\varphi^c(y) := \inf_{z \in X} \left(\frac{\mathrm{d}^2(y, z)}{2} - \varphi(z) \right).$$

Define the L^2-Wasserstein distance between probabilities as $W_2(\mu_0, \mu_1) := \sqrt{C(\pi)}$ for $\pi \in \mathrm{Opt}(\mu_0, \mu_1)$, which makes $\mathcal{P}_2(X)$ a Polish metric space. Denote Geo(X) the set of all constant speed geodesic $\gamma : [0, 1] \to X$. When endowed with the supremum distance, it is a Polish metric space.

If (X, d) is geodesic, so is $(\mathcal{P}_2(X), W_2)$ (see [1, Theorem 2.10]). Let $e_t : \mathrm{Geo}(X) \ni \gamma \mapsto \gamma_t \in X$ be the evaluation map, and $\ell(\gamma)$ be the length. Then for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there exists a probability measure ν (referred to as an *optimal dynamical plan*) on Geo(X) s.t.

- $(e_i)_\# \nu = \mu_i$, $i = 0, 1$ and $(e_0, e_1)_\# \nu \in \mathrm{Opt}(\mu_0, \mu_1)$;
- $[0, 1] \ni t \mapsto \mu_t := (e_t)_\# \nu$ is a constant speed geodesic in $(\mathcal{P}_2(X), W_2)$;
- ν is concentrated on the set of *Kantorovich geodesics*

$$G_\varphi := \left\{ \gamma \in \mathrm{Geo}(X) : \varphi(\gamma_0) + \varphi^c(\gamma_1) = \ell^2(\gamma)/2 \right\}.$$

Denote by OptGeo(μ_0, μ_1) the set of all optimal dynamical plans.

Definition 2.1 Define the *N-Rényi entropy* $\mathcal{E}_N$ of any $\mu \in \mathcal{P}_2(X, \mathfrak{m})$ by

$$\mathcal{E}_N(\mu) := \int_X \rho^{1-1/N}(x)\,\mathrm{d}\mathfrak{m}, \quad \rho := \frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}.$$

Given $N \in (1, \infty)$, define by the following two *distortion coefficients*

$$\sigma^{(t)}_{K,N}(\theta) = \frac{\sin\left(t\theta\sqrt{\frac{K}{N}}\right)}{\sin\left(\theta\sqrt{\frac{K}{N}}\right)} := \begin{cases} \frac{\sin\left(t\theta\sqrt{\frac{K}{N}}\right)}{\sin\left(\theta\sqrt{\frac{K}{N}}\right)} & K > 0, \quad 0 < \theta < \pi\sqrt{\frac{N}{K}} \\ t & K = 0, \quad 0 < \theta < \infty \\ \frac{\sinh\left(t\theta\sqrt{\frac{-K}{N}}\right)}{\sinh\left(\theta\sqrt{\frac{-K}{N}}\right)} & K < 0, \quad 0 < \theta < \infty \end{cases},$$

$$\tau^{(t)}_{K,N}(\theta) := t^{1/N} \sigma^{(t)}_{K,N-1}(\theta)^{1-1/N}.$$

Definition 2.2 Let $(X, d, \mathfrak{m})$ be a metric-measure space.

- $(X, d, \mathfrak{m})$ is said to verify CD(K,N), if for all $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathfrak{m})$, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ so that for all $t \in [0, 1]$, $\mu_t = (e_t)_\# \nu \ll \mathfrak{m}$, and for all $N' \geq N$:

$$\begin{aligned} \mathcal{E}_{N'}(\mu_t) \geq \int_{X\times X} & \tau_{K,N'}^{(1-t)}(d(x_0,x_1))\rho_0^{-1/N'}(x_0) \\ & + \tau_{K,N'}^{(t)}(d(x_0,x_1))\rho_1^{-1/N'}(x_1)\pi(\,\mathrm{d}x_0,\,\mathrm{d}x_1), \end{aligned} \tag{2.1}$$

where $\pi = (e_0, e_1)_\sharp \nu$ and $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$.

- $(X, d, \mathfrak{m})$ is said to verify $\mathrm{CD}(K, N)$ locally, or $\mathrm{CD}_{\mathrm{loc}}(K, N)$ in short, if for any $o \in \mathrm{supp}(\mathfrak{m})$ one can find a neighborhood $X_o \subset X$ of o, so that for all $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathfrak{m})$ supported in X_o, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ so that $\mu_t := (e_t)_\sharp \nu \ll \mathfrak{m}$, and (2.1) holds for all $t \in [0, 1]$, $N' \geq N$.
- $(X, d, \mathfrak{m})$ is said to verify $\mathrm{MCP}(K, N)$, if for any $o \in \mathrm{supp}(\mathfrak{m})$ and $\mu_0 := \frac{\mathfrak{m}\llcorner A}{\mathfrak{m}(A)}$ given A a Borel subset of X with $0 < \mathfrak{m}(A) < \infty$, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \delta_o)$ s.t.

$$\frac{\mathfrak{m}}{\mathfrak{m}(A)} \geq (e_t)_\sharp(\tau_{K,N}^{(1-t)}(d(\gamma_0,\gamma_1))^N \nu(\,\mathrm{d}\gamma)) \quad \forall t \in [0,1]. \tag{2.2}$$

Definition 2.3 A set $G \subset \mathrm{Geo}(X)$ is *non-branching* if for any $\gamma^1, \gamma^2 \in G$ with $\gamma^1 = \gamma^2$ on $[0, t]$ for some $t \in (0, 1)$, it holds $\gamma^1 = \gamma^2$ on $[0, 1]$.

A space $(X, d, \mathfrak{m})$ is called *essentially non-branching* if for all $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathfrak{m})$, *any* $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a Borel non-branching set $G \subset \mathrm{Geo}(X)$.

Remark 2.4 Throughout this paper, we assume that $\mathrm{supp}(\mathfrak{m}) = X$ without any further specification as it will not affect the generality. Indeed, as discussed in [6, Remark 6.11], whenever $\mu_0, \mu_1 \ll \mathfrak{m}$, almost every curve in the support of $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is contained in $\mathrm{supp}(\mathfrak{m})$. So the problem on $(X, d, \mathfrak{m})$ is equivalent to the one on $(\mathrm{supp}(\mathfrak{m}), d, \mathfrak{m})$.

2.2 Density Functions on Nonbranching spaces

Cavalletti-Mondino in [8] showed that optimal maps of transports with $\mu_0 \ll \mathfrak{m}$ uniquely exist on e.n.b. $\mathrm{MCP}(K, N)$ spaces. Such MCP-condition is always satisfied on e.n.b. $\mathrm{CD}_{\mathrm{loc}}(K, N)$ spaces (first by [10] on non-branching spaces, and then on e.n.b. spaces with properties developed in [8]).

In this subsection, $(X, d, \mathfrak{m})$ **always stands for an e.n.b. length m.m.s. satisfying** $\mathrm{CD}_{\mathrm{loc}}(K, N)$ **or** $\mathrm{MCP}(K, N)$. It is well-known that any $\mathrm{CD}_{\mathrm{loc}}(K, N)$ length space is locally compact (see e.g. [6, Lemma 6.12]), so by Hopf-Rinow Theorem, it is proper and geodesic.

Proposition 2.5 (cf. [8]) *For every* $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ *with* $\mu_0 \ll \mathfrak{m}$, *there exists a unique* $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$; *such* ν *is induced by a map (i.e.* $\nu = S_\sharp \mu_0$ *for* $S : X \supset \mathrm{Dom}(S) \to \mathrm{Geo}(X)$) *and for every* $t \in (0, 1)$, $(e_t)_\sharp \nu \ll \mathfrak{m}$.

Lemma 2.6 ([6, Corollary 6.16]) *Given* μ_0, μ_1 *as in Proposition* 2.5, *the unique optimal dynamical plan* ν *is concentrated on a Borel set* $G \subset \mathrm{Geo}(X)$ *s.t. the evaluation map* $e_t : G \to X$ *is injective for all* $t \in [0, 1)$. *And in particular, any Borel* $H \subset G$, *we have*

$$(e_t)_\sharp(\nu\llcorner H) = (e_{t\sharp}\nu)\llcorner e_t(H) \quad \forall t \in [0, 1).$$

The following can be regarded as an expansion of the original proof in [6].

Proof We first assume $\mu_1 \ll \mathfrak{m}$. Recall ν is induced by a map i.e. $\nu = S_{0\#}\mu_0 = S_{1\#}\mu_1$. As argued in [6], for both $i = 0, 1$ we can find $X_i \subset X$ of full μ_i measure s.t. for all $x \in X_i$, there exists a unique $\gamma \in G_\varphi$ with $\gamma_i = x$. In particular, $\nu(S_0(X_0)) = \nu(S_1(X_1)) = 1$.

Take a Borel set $G \subset S_0(X_0) \cap S_1(X_1)$, still with full ν-measure. We claim e_t is injective on G for all $t \in [0, 1]$. By construction, e_0 and e_1 are clearly injective on G. Assume there are $\gamma, \tilde{\gamma} \in G$, $\gamma_t = \tilde{\gamma}_t$ for some $t \in (0, 1)$. Define a curve η by letting $\eta = \gamma$ on $[0, t]$ and $\eta = \tilde{\gamma}$ on $[t, 1]$. By cyclic monotonicity, $\eta \in G_\varphi$. Since $\gamma \in S_0(X_0)$, $\eta \equiv \gamma$ on $[0, 1]$ and so $\gamma_1 = \tilde{\gamma}_1$. On the other hand, as $\gamma, \tilde{\gamma} \in S_1(X_1)$, one concludes $\gamma \equiv \tilde{\gamma}$.

For general $\mu_1 \in \mathcal{P}_2(X)$, we prove by taking restrictions of ν. For any $t \in [0, 1)$, define

$$\mathrm{restr}_0^t : \mathrm{supp}(\nu) \to \mathrm{Geo}(X), \quad \gamma(\cdot) \mapsto \gamma(t\cdot).$$

Proposition 2.5 ensures that $\mu_t := (e_t)_\#\nu \ll \mathfrak{m}$, and $(\mathrm{restr}_0^t)_\#\nu$ is the unique optimal dynamical plan between μ_0 and μ_t. From the first step, we can find a Borel set $\tilde{G}_t$ where $(\mathrm{restr}_0^t)_\#\nu$ is concentrated and evaluation maps are injective over there. Then, take a sequence $t_n \nearrow 1$ and a set

$$G := \bigcap_{n\in\mathbb{N}} \left(\mathrm{restr}_0^{t_n}\right)^{-1}\left(\tilde{G}_{t_n}\right).$$

One can check $\nu(G) = 1$ and e_t is injective on G for all $t \in [0, 1)$. $\square$

Since X is proper, any bounded subset has finite $\mathfrak{m}$-measure. Via a conditioning argument, we can extend [6, Proposition 9.1] to infinite-volume spaces.

Proposition 2.7 (Density characterization) *For any* $\mu_0 \in \mathcal{P}_2(X, \mathfrak{m})$, $\mu_1 \in \mathcal{P}_2(X)$, *there exists a unique* $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *so that for all* $t \in (0, 1)$, $(e_t)_\#\nu \ll \mathfrak{m}$ *and*

$$\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(d(\gamma_0, \gamma_1))\rho_0^{-1/N}(\gamma_0) \quad \text{for } \nu-\text{a.e. } \gamma. \tag{2.3}$$

It verifies $\mathrm{CD}(K,N)$ *iff for any* $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathfrak{m})$, *there exists a unique* $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *so that for all* $t \in (0, 1)$, $(e_t)_\#\nu \ll \mathfrak{m}$ *and*

$$\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}\left(d\left(\gamma_0, \gamma_1\right)\right)\rho_0^{-1/N}(\gamma_0) + \tau_{K,N}^{(t)}(d(\gamma_0, \gamma_1))\rho_1^{-1/N}(\gamma_1), \quad \text{for } \nu-\text{a.e. } \gamma. \tag{2.4}$$

Sketch of proof When $\mathfrak{m}(X) < \infty$, arguing by approximation as in [6, Proposition 9.1], for arbitrary boundedly supported $\mu_0 \in \mathcal{P}_2(X, \mathfrak{m})$ and $\mu_1 \in \mathcal{P}_2(X)$, we have

$$\mathcal{E}_N(\mu_t) \geq \int \tau_{K,N}^{(1-t)}(d(\gamma_0, \gamma_1))\rho_0^{-1/N}(\gamma_0)\nu(\mathrm{d}\gamma), \quad \nu \in \mathrm{OptGeo}(\mu_0, \mu_1), \tag{2.5}$$

where $\mu_t = (e_t)_\#\nu$. Here the finiteness of volume is only required for showing the upper-semicontinuity of $\mathcal{E}_N$. In our case due to the choice of marginals, $(\mu_t)_t$ are confined to a fixed bounded set U. So redoing [16, Lemma 4.1] ensures that $\mathcal{E}_N$ is upper-semicontinuous w.r.t. weak convergence of measures supported inside U.

Now consider general $\mu_0 \in \mathcal{P}_2(X, \mathfrak{m})$, $\mu_1 \in \mathcal{P}_2(X)$ possibly with unbounded supports. Take any compact $G \subset \mathrm{Geo}(X)$ with $\nu(G) > 0$. The restricted plan $\tilde{\nu} = \frac{1}{\nu(G)}\nu\llcorner_G$ is still an optimal dynamical plan. By Lemma 2.6, $\tilde{\mu}_t := (e_t)_\# \tilde{\nu}$ has the density $\tilde{\rho}_t = \frac{1}{\nu(G)} \rho_t \llcorner e_t(G)$, and having a uniformly bounded support. So (2.5) holds for $\tilde{\nu}$, implying

$$\int_G \rho_t^{-1/N}(\gamma_t)\nu(\mathrm{d}\gamma) \geq \int_G \tau_{K,N}^{(1-t)}\big(d\big(\gamma_0, \gamma_1\big)\big)\rho_0(\gamma_0)^{-1/N}\nu(\mathrm{d}\gamma).$$

The arbitrariness of G and the inner regularity of ν yield the inequality (2.3) for ν-a.e. γ.

For the second assertion on $\mathrm{CD}(K, N)$. The "only if" part follows by applying the similar conditioning to (2.1). The "if" part follows directly by integrating (2.4) against ν. □

An important consequence of the previous proposition is the following continuity of optimal dynamics, which plays a crucial role in the ray decomposition (see e.g. the proof of Theorem 3.10). Besides, the Lipschitz-regularity of densities is a starting point of the bootstrap argument in [6, Section 12].

Corollary 2.8 (Continuity of Dynamics, cf. [6, Section 9]) *Let* $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *for* $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathfrak{m})$.

(1) *There exist versions of densities* $\rho_t = \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$, $t \in [0, 1]$, *so that for* ν*-a.e.* $\gamma \in \mathrm{Geo}(X)$ *and all* $0 \leq s < t \leq 1$:

$$\rho_s(\gamma_s) > 0, \quad \left(\tau_{K,N}^{\left(\frac{s}{t}\right)}(d(\gamma_0, \gamma_t))\right)^N \leq \frac{\rho_t(\gamma_t)}{\rho_s(\gamma_s)} \leq \left(\tau_{K,N}^{\left(\frac{1-t}{1-s}\right)}(d(\gamma_s, \gamma_1))\right)^{-N}. \tag{2.6}$$

In particular, for ν-a.e. γ, the map $t \mapsto \rho_t(\gamma_t)$ is locally Lipschitz on (0, 1) and upper semi-continuous at $t = 0, 1$.

(2) *For any compact* $G \subset \mathrm{Geo}(X)$ *with* $\nu(G) > 0$ *s.t.* (2.6) *holds for all* $\gamma \in G$ *and* $0 \leq s \leq t \leq 1$, *we have* $\mathfrak{m}(e_s(G)) > 0$ *for all* $s \in [0, 1]$ *and*

$$\left(\frac{1-t}{1-s}\right)^N e^{-d(G)(t-s)\sqrt{(N-1)K^-}} \leq \frac{\mathfrak{m}(e_t(G))}{\mathfrak{m}(e_s(G))} \leq \left(\frac{t}{s}\right)^N e^{d(G)(t-s)\sqrt{(N-1)K^-}}, \tag{2.7}$$

where $d(G) := \max\{\ell(\gamma) : \gamma \in G\}$ and $K^- := \max\{-K, 0\}$. In particular, the map $t \mapsto \mathfrak{m}(e_t(G))$ is locally Lipschitz on (0, 1) and lower semi-continuous at $t = 0, 1$.

2.3 Intermediate-time Kantorovich Potentials

We first recall the notion of intermediate-time Kantorovich potentials.

Definition 2.9 Given a Kantorovich potential $\varphi : X \to \mathbb{R}$, the *intermediate-time Kantorovich potential* φ_t at time $t \in [0, 1]$ is defined by $\varphi_0 = \varphi$, $\varphi_1 = -\varphi^c$ and

$$\varphi_t(x) := -\inf_{y \in X}\left[\frac{d^2(x, y)}{2t} - \varphi(y)\right].$$

Denote the domain of the dynamics and its section through x as:

$$D(G_\varphi) := \{(x,t) \in X \times (0,1) : \exists \gamma \in G_\varphi, x = \gamma_t\},$$
$$G_\varphi(x) := \{t \in (0,1) : (x,t) \in D(G_\varphi)\}.$$

Based on Lemma 2.6, when $(X, d, \mathfrak{m})$ is e.n.b., for simplicity, we will assume $e_t : G_\varphi \to \mathbb{R}$ is injective for all $t \in [0,1]$ as otherwise, it suffices to restrict ν to some Borel $G \subset G_\varphi$. Then the length function is defined by

$$\ell : D(G_\varphi) \ni (x,t) \mapsto \ell(e_t^{-1}(x)) := \mathrm{Length}(e_t^{-1}(x))$$

and we also use the notation $\ell_t(\cdot) := \ell(\cdot, t)$ on $e_t(G_\varphi)$ for every t.

Definition 2.10 Given a Kantorovich potential $\varphi : X \to \mathbb{R}$ and $s, t \in (0,1)$, define the *t-propagated s-Kantorovich potential* Φ_s^t on $e_t(G_\varphi)$ by

$$\Phi_s^t := \varphi_s \circ e_s \circ e_t^{-1} = \varphi_t + (t-s)\frac{\ell_t^2}{2}.$$

For every fixed $s \in (0,1)$, according to the value of φ_s, G_φ can be partitioned into closed levels

$$G_\varphi = \sqcup_{a_s \in \mathrm{Im}(\varphi_s \circ e_s)} G_{\varphi, a_s}, \quad G_{\varphi, a_s} := (\varphi_s \circ e_s)^{-1}(a_s) = \{\gamma \in G_\varphi : \varphi_s(\gamma_s) = a_s\}$$

which further leads to a partition of $e_t(G_\varphi)$ (any $t \in (0,1)$) via Φ_s^t by

$$e_t(G_\varphi) = \sqcup_{a_s \in \mathrm{Im}(\varphi_s \circ e_s)} e_t(G_{\varphi, a_s}), \quad e_t(G_{\varphi, a_s}) := (\Phi_s^t)^{-1}(a_s) = \{\gamma_t : \varphi_s(\gamma_s) = a_s\}.$$

Lemma 2.11 (Continuity of potentials, cf. [2, Sect. 3] and [6, Theorem 3.11, Proposition 4.4]) *The function $X \times (0,1) \ni (x,t) \mapsto \varphi_t(x)$ is locally Lipschitz. The length function ℓ is continuous on $D(G_\varphi)$. For any $x \in X$ and $s \in (0,1)$, functions $G_\varphi(x) \ni t \mapsto \ell_t(x)$ and $G_\varphi(x) \ni t \mapsto \Phi_s^t(x)$ are locally Lipschitz.*

3 L^1-disintegration

3.1 Disintegration Theorem

Proofs of assertions in this subsection can be found in [3, Appendix A]. Let $(X, \mathfrak{X}, \mathfrak{m})$, $(Q, \mathcal{Q}, \mathfrak{q})$ be measure spaces. A *disintegration* of $\mathfrak{m}$ over $\mathfrak{q}$ is a family of measures $(\mathfrak{m}_q)_{q \in Q}$ on X s.t. for every $E \in \mathfrak{X}$, the map $q \mapsto \mathfrak{m}_q(E)$ is $\mathfrak{q}$-measurable and $\mathfrak{m}(E) = \int_Q \mathfrak{m}_q(E)\mathfrak{q}(\mathrm{d}q)$. By [11, Proposition 452F], for any $\mathfrak{m}$-measurable $\xi : X \to \mathbb{R}$, we have

$$\int_Q \int_X \xi(x)\mathfrak{m}_q(\mathrm{d}x)\mathfrak{q}(\mathrm{d}q) = \int_X \xi(x)\mathfrak{m}(\mathrm{d}x),$$

provided $\int \xi(x)\mathfrak{m}(\mathrm{d}x)$ is well-defined in $\mathbb{R} \cup \{\pm\infty\}$.

Given measurable $f : (X, \mathfrak{X}) \to (Q, \mathcal{Q})$, a disintegration $(\mathfrak{m}_q)_{q \in Q}$ of $\mathfrak{m}$ over $\mathfrak{q}$ is called *consistent* with f if for each $I \in \mathcal{Q}$,

$$\mathfrak{m}(E \cap f^{-1}(I)) = \int_I \mathfrak{m}_q(E)\mathfrak{q}(\,\mathrm{d}q). \tag{3.1}$$

And $(\mathfrak{m}_q)_{q\in Q}$ is called *strongly consistent* with f, if for $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q$ is concentrated on $f^{-1}(\{q\})$. Clearly, strong consistency implies consistency.

Remark 3.1 (Uniqueness of Disintegration) If $\mathfrak{X}$ is countably generated with a σ-finite measure $\mathfrak{m}$, and disintegrations $(\mathfrak{m}_q)$, $(\tilde{\mathfrak{m}}_q)$ of $\mathfrak{m}$ over $\mathfrak{q}$ are consistent with f, then $\mathfrak{m}_q = \tilde{\mathfrak{m}}_q$ for $\mathfrak{q}$-a.e. $q \in Q$, or in short, consistent disintegrations are $\mathfrak{q}$-*unique*.

Indeed, by [13, Proposition 3.3], there is a countable subalgebra $\{B_n \in \mathfrak{X}, n \in \mathbb{N}\}$ generating $\mathfrak{X}$. After putting $E = B_n$ into (3.1), we know up to a $\mathfrak{q}$-negligible set $N \subset Q$, $\mathfrak{m}_q(B_n) = \tilde{\mathfrak{m}}_q(B_n)$ for all n and q. So when $\mathfrak{m}$ is finite, with Dynkin's theorem, $\mathfrak{m}_q = \tilde{\mathfrak{m}}_q$ for all $q \in Q\backslash N$. For the case where $\mathfrak{m}$ is σ-finite, we can repeat the previous argument on any subset E of finite $\mathfrak{m}$-measure to show $\mathfrak{m}_{q} \llcorner E = \tilde{\mathfrak{m}}_{q} \llcorner E$ for a.e. q. The argument is complete after taking an exhausting sequence E_n of X.

In particular, strongly consistent disintegrations of a locally finite measure are $\mathfrak{q}$-unique.

If X has a partition $\Pi = \{X_q\}_{q\in Q}$, define $\mathfrak{Q} : X \to Q$ by mapping each point in X_q to q. Endowed with the quotient σ-algebra $\mathcal{Q}$ and the quotient measure $\mathfrak{q} = \mathfrak{Q}_\#\mathfrak{m}$, $(Q, \mathcal{Q}, \mathfrak{q})$ is a measure space.

Definition 3.2 A *cross section* of a partition Π is a subset S of X so that $S \cap A$ is a singleton for each $A \in \Pi$. A *section* is a map $\mathfrak{S} : X \to X$ such that for each $x \in X$, the image of $[x]$ under $\mathfrak{S}$ is a singleton in $[x]$, where $[x]$ is the equivalence class of x under Π.

A subset $S_\mathfrak{m}$ is called an $\mathfrak{m}$-*section* if there exists a *Borel* set $\Gamma \subset X$ s.t. $\mathfrak{m}(X \setminus \Gamma) = 0$ and the partition $\Pi_\Gamma = \{X_q \cap \Gamma\}_{q\in Q}$ has $S_\mathfrak{m}$ as a cross section.

Theorem 3.3 (Disintegration Theorem) *Assume* $(X, \mathfrak{X}, \mathfrak{p})$ *is a countably generated probability space, having a partition* $\Pi = \{X_q\}_{q\in Q}$. *Let* $\mathfrak{Q} : X \to Q$ *and* $(Q, \mathcal{Q}, \mathfrak{q})$ *be the quotient map and quotient space resp. There exists a unique disintegration* $q \mapsto \mathfrak{p}_q \in \mathcal{P}(X)$ *of* $\mathfrak{p}$ *over* $\mathfrak{q}$ *consistent with* $\mathfrak{Q}$. *Moreover, this disintegration is strongly consistent with* $\mathfrak{Q}$ *iff there exists a Borel* $\mathfrak{p}$*-section* $S_\mathfrak{p} \subset Q$ *s.t. the quotient* σ*-algebra* $\mathcal{Q} \cap S_\mathfrak{p}$ *contains* $\mathcal{B}(S_\mathfrak{p})$.

Remark 3.4 (Disintegration over level sets) If (X, d) is Polish and the partition Π given as level sets of a continuous function $\mathfrak{Q} : X \to \mathbb{R}$, then, by [15, Theorem 5.4.3], Π admits a Borel cross-section S and Borel section map $\mathfrak{S}$. In particular, there is a unique disintegration of $\mathfrak{p}$ strongly consistent with $\mathfrak{S}$.

3.2 Transport Ray and $\mathsf{CD}^1(K, N)$

For any 1-Lipschitz function $u : (X, d) \to \mathbb{R}$, define the *transport relation* R_u and the *transport set* $\mathcal{T}_u$ as

$$R_u := \{(x, y) \in X \times X : |u(x) - u(y)| = d(x, y)\}, \qquad \mathcal{T}_u := P_1(R_u \setminus \{x = y\}),$$

where P_i is the projection onto the i-th component. Denote $R_u(x) := \{y \in X : (x, y) \in R_u\}$ as the section of R_u through x in the first coordinate.

Notice R_u is not necessarily an equivalence relation as the transitivity may be violated. To remedy this, define the *non-branched transport set* by removing those branched points:

$$\mathcal{T}_u^b := \left\{x \in \mathcal{T}_u : \forall z, w \in R_u(x), (z, w) \in R_u\right\}$$

and hence the corresponding non-branched transport relation

$$R_u^b := R_u \cap \left(\mathcal{T}_u^b \times \mathcal{T}_u^b\right).$$

Remark 3.5 We refer to [4, 5] and [6, Section 7] for following statements:

- When (X, d) is proper, $\mathcal{T}_u$ is σ-compact, and $\mathcal{T}_u^b$, R_u^b are Borel;
- R_u^b is an equivalence relation on $\mathcal{T}_u^b$ which induces a partition $\sqcup_x R_u^b(x)$ of $\mathcal{T}_u^b$;
- When (X, d) is geodesic, for any $x \in \mathcal{T}_u^b$, $R_u(x)$ is a single (unparameterized) geodesic of positive length, so that $(R_u(x), d)$ is isometric to a closed interval in $(\mathbb{R}, |\cdot|)$ and $(R_u^b(x), d)$ is a subinterval.

We call R a *transport ray* if (R, d) is isometric to a closed interval in $(\mathbb{R}, |\cdot|)$ of positive length and it is maximal under the partial order $\leq_u$, where $x \leq_u y$ if $u(x) - u(y) = d(x, y)$.

Definition 3.6 Given a continuous function $\phi : (X, d) \to \mathbb{R}$ so that $\{\phi = 0\} \neq \emptyset$, define the *signed distance function* (from zero-level set of ϕ) as

$$d_\phi : X \to \mathbb{R}, \quad d_\phi(x) := \mathrm{dist}(x, \{\phi = 0\})\mathrm{sign}(\phi).$$

When (X, d) is a length space, any signed distance function d_ϕ is 1-Lipschitz (see [6, Lemma 8.4]). If further $\mathfrak{m}(X) < \infty$, Theorem 3.3 gives a disintegration of $\mathfrak{m}$ on $\mathcal{T}_{d_\phi}^b$ w.r.t. the partition by $R_{d_\phi}^b$, which leads to the CD^1-condition introduced in [6]. We modify this condition by relaxing conditional measures to be only locally finite, instead of probabilities.

Definition 3.7 A m.m.s. $(X, d, \mathfrak{m})$ with $\mathrm{supp}(\mathfrak{m}) = X$ satisfies $\mathrm{CD}^1(K, N)$ if for any 1-Lipschitz signed distance function $u = d_\phi$, with the associated partition $\{R_u^b(q)\}_{q\in Q}$ of $\mathcal{T}_u^b$ by ray decomposition, there $\overline{\text{exist}}$ a probability space $(Q, \mathcal{Q}, \mathfrak{q})$ and a $\mathfrak{q}$-unique disintegration $\mathfrak{m}\llcorner\mathcal{T}_u = \int_Q \mathfrak{m}_q \mathfrak{q}(\,\mathrm{d}q)$ on $\{\overline{R_u^b(q)}\}_{q\in Q}$ s.t.

1. Q is a section of the above partition so that $Q \supseteq \bar{Q} \in \mathcal{B}(\mathcal{T}_u^b)$ with $\bar{Q}$ an $\mathfrak{m}$-section with $\mathfrak{m}$-measurable quotient map and $\mathcal{Q} \supseteq \mathcal{B}(\bar{Q})$;
2. for $\mathfrak{q}$-a.e. $q \in Q$, $\overline{R_u^b(q)} = R_u(q)$ as a transport ray;
3. for $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q$ is non-null, supported on $\overline{R_u^b(q)}$;
4. for $\mathfrak{q}$-a.e. $q \in Q$, $(\overline{R_u^b(q)}, d, \mathfrak{m}_q)$ is a one-dimensional $\mathrm{CD}(K, N)$ m.m.s.;
5. for every bounded subset $K \subset X$, there exists $C_K \in (0, \infty)$ s.t.

$$\mathfrak{m}_q(K) \leq C_K, \quad \text{for } \mathfrak{q} - \text{a.e. }. \tag{3.2}$$

Remark 3.8 The reference measure $\mathfrak{m}$ on any $\mathrm{CD}^1(K, N)$ space must be locally finite, simply by (3.2) and taking $u = d(\cdot, o)$ for some $o \in X$. And by Theorem 3.3, the disintegration is strongly consistent with the quotient measure because of 1.

Remark 3.9 For any u and disintegration from Definition 3.7, $\mathfrak{m}(\mathcal{T}_u) = \mathfrak{m}\big(\mathcal{T}_u^b\big)$. Indeed, denoting by $\tilde{Q} \subset Q$ the set of q that 2–4 hold, then

$$\begin{aligned}\mathfrak{m}(\mathcal{T}_u) &= \int_Q \mathfrak{m}_q(\mathcal{T}_u)\mathfrak{q}(\mathrm{d}q) = \int_{\tilde{Q}} \mathfrak{m}_q\Big(\overline{R_u^b(q)}\Big)\mathfrak{q}(\mathrm{d}q)\\ &= \int_{\tilde{Q}} \mathfrak{m}_q\big(R_u^b(q)\big)\mathfrak{q}(\mathrm{d}q) = \int_{\tilde{Q}} \mathfrak{m}_q\big(\mathcal{T}_u^b\big)\mathfrak{q}(\mathrm{d}q) = \mathfrak{m}\big(\mathcal{T}_u^b\big),\end{aligned}$$

where we have used the fact that a measure carrying $\mathrm{CD}(K,N)$ does not charge points.

3.3 $\mathrm{CD}_{\mathrm{loc}}(K,N)$ implies $\mathrm{CD}^1(K,N)$

Theorem 3.10 *Let* $(X, d, \mathfrak{m})$ *be an e.n.b.* $\mathrm{CD}_{\mathrm{loc}}(K,N)$ *length m.m.s. such that* $\mathfrak{m}$ *is locally finite with full-support, and* $u : (X,d) \to \mathbb{R}$ *be a* 1*-Lipschitz function. Then there exists a disintegration of* $\mathfrak{m}\llcorner\mathcal{T}_u$ *satisfying* 1–5 *of Definition* 3.7*. In particular, under these assumptions,* $\mathrm{CD}_{\mathrm{loc}}(K,N)$ *implies* $\mathrm{CD}^1(K,N)$.

Such disintegration, also called ray/needle decomposition, is extensively studied in e.g. [4, 5, 7] under the assumption $\mathfrak{m}(X) = 1$. However in our case, $\mathcal{T}_u^b$ could be unbounded with infinite volume, so we can not directly apply Theorem 3.3. Therefore, we normalize the measure by adding a weight function following the approach in [9]. After such re-weighting, CD-information can be passed to rays exactly as in the finite-volume case.

Proof As every $\mathrm{CD}_{\mathrm{loc}}(K,N)$ geodesic m.m.s. is proper, Remark 3.5 applies. From [5, Proposition 4.5] (together with the comments above [6, Corollary 7.3]), $\mathfrak{m}(\mathcal{T}_u \setminus \mathcal{T}_u^b) = 0$. Hence it suffices to disintegrate $\mathfrak{m}\llcorner\mathcal{T}_u^b$ w.r.t. the partition $\mathcal{T}_u^b = \sqcup_q R_u^b(q)$.

Normalize $\mathfrak{m}$ to apply the disintegration theorem. Without loss of generality we assume $\mathfrak{m}(\mathcal{T}_u^b) = \infty$. Then, for any fixed $x_0 \in X$, we can find an increasing sequence $(r_n)_{n\geq 1}$ of positive numbers, so that

$$\mathcal{T}_u^{b,n} := \begin{cases} \mathcal{T}_u^b \cap \big\{x \in X : r_n \leq d(x,x_0) < r_{n+1}\big\}, & n \in \mathbb{N}_+\\ \mathcal{T}_u^b \cap \big\{x \in X : d(x,x_0) < r_1\big\}, & n = 0\end{cases}$$

has positively finite $\mathfrak{m}$-measure for each $n \geq 0$. Define f by

$$f(x) = \sum_{n\in\mathbb{N}} \Big(2^{n+1}\mathfrak{m}\big(\mathcal{T}_u^{b,n}\big)\Big)^{-1}\mathbb{1}_{\mathcal{T}_u^{b,n}}(x).$$

Clearly,

$$\inf_{K\cap\mathcal{T}_u^b} f > 0, \ \text{for any compact } K \subset X; \qquad \int_{\mathcal{T}_u^b} f(x)\mathfrak{m}(\mathrm{d}x) = 1. \tag{3.3}$$

Hence, $\mathfrak{n} := f\mathfrak{m}\llcorner\mathcal{T}_u^b$ is a probability measure and Theorem 3.3 can be applied to $\mathfrak{n}$.

On the strong consistency. First, from [4, Proposition 4.4],[3] there exists an $\mathfrak{m}$-measurable section $\mathfrak{Q} : \mathcal{T}_u^b \to \mathcal{T}_u^b$ associated to the partition $\{R_u^b(q)\}_{q\in Q}$. From now on, we

[3] The existence of such section map depends only on (1): selection theorem of partitions into closed sets and (2): continuity and local compactness of geodesics, but not on the finiteness of $\mathfrak{m}(X)$.

fix Q as the image of $\mathfrak{Q}$ and endow Q with σ-algebra $\mathcal{Q} := \mathfrak{Q}_\# \mathcal{B}(\mathcal{T}_u^b)$. Take $\mathfrak{q} = \mathfrak{Q}_\# \mathfrak{n}$ to be the quotient probability on Q. By the $\mathfrak{m}$-measurability of $\mathfrak{Q}$, $\mathfrak{q}$ is Borel on $\mathcal{T}_u^b$. Thus $\mathfrak{q}$ is inner regular and we can find a σ-compact set $S \subset Q$, $\mathfrak{q}(Q \backslash S) = 0$. Define a section $\mathfrak{S} := \mathfrak{Q}_{\llcorner}(\mathfrak{Q}^{-1}(S))$ where $\mathfrak{Q}^{-1}(S)$ has full $\mathfrak{n}$-measure. Then

$$\mathrm{graph}(\mathfrak{S}) = \left\{(x,s) \in \mathcal{T}_u^b \times S : (x,s) \in R_u\right\}$$

is Borel, implying that $\mathfrak{Q}^{-1}(S) = P_1(\mathrm{graph}(\mathfrak{S}))$ is analytic and $\mathfrak{S}$ is Borel measurable by [15, Theorem 4.5.2]. That is to say, S is a Borel $\mathfrak{n}$-section with Borel measurable section $\mathfrak{S}$ and hence $\mathcal{Q} \supset \mathcal{B}(S)$. Theorem 3.3 applies to conclude that $q \mapsto \mathfrak{n}_q$ is the $\mathfrak{q}$-unique disintegration of $\mathfrak{n}$ strongly consistent with $\mathfrak{Q}$. In particular, 1 is verified as $\mathfrak{n}$ and $\mathfrak{m}_{\llcorner}\mathcal{T}_u^b$ sharing same measurable and null sets.

Back to $\mathfrak{m}_{\llcorner}\mathcal{T}_u^b$, owing to the everywhere positivity of f on $\mathcal{T}_u^b$, we have

$$\mathfrak{m}_{\llcorner}\mathcal{T}_u^b = \int_Q \mathfrak{n}_q / f \mathfrak{q}(\mathrm{d}q). \tag{3.4}$$

Define $\mathfrak{m}_q := \mathfrak{n}_q / f$. As measurability (w.r.t. $q \in Q$) is guaranteed, $q \mapsto \mathfrak{m}_q$ gives the unique disintegration of $\mathfrak{m}_{\llcorner}\mathcal{T}_u^b$ strongly consistent with $\mathfrak{Q}$ (recall Remark 3.1). From (3.3), $\mathfrak{m}_q$ is uniformly-locally finite as (3.2). Further, we can repeat [6, Theorem 7.10](which mainly needs Proposition 2.5 but not finiteness of $\mathfrak{m}(X)$, as it is proved by contradiction and localization) for (3.4) to show that for $\mathfrak{q}$-a.e. $q \in Q$, $R_u(q) = \overline{R_u^b(q)}$ and $\mathrm{supp}(\mathfrak{m}_q) = R_u(q)$.

Localize CD(K,N) **to transport rays.** Let S be the σ-compact cross section in the previous step. Define the *ray map* $g : \mathrm{Dom}(g) \subset S \times \mathbb{R} \to \mathcal{T}_u^b$ via

$$\mathrm{graph}(g) := \left\{(q,t,x) \in S \times \mathbb{R} \times \mathcal{T}_u^b : (q,x) \in R_u, u(x) - u(q) = t\right\}. \tag{3.5}$$

Remark 3.5 ensures that each $x \in \mathcal{T}_u^b$ uniquely corresponds a pair $(\mathfrak{Q}(x), d) \in S \times \mathbb{R}$, with $d = u(x) - u(\mathfrak{Q}(x))$ and $|d| = d(x, \mathfrak{Q}(x))$. Hence g is well-defined, bijective and Borel measurable because of its Borel graph. For any $q \in S$, $I_q := \mathrm{Dom}(g(q,\cdot))$ is an interval in $\mathbb{R}$, and $I_q \ni t \mapsto g(q,t) \in R_u^b(q)$ is an isometry, meaning $\mathcal{H}^1{}_{\llcorner}\left\{R_u^b(q)\right\} = g(q,\cdot)_\#\left(\mathcal{L}^1{}_{\llcorner} I_q\right)$.

It remains to show that for $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q \ll g(q,\cdot)_\# \mathcal{L}^1$ and for those q, by denoting $\mathfrak{m}_q = g(q,\cdot)_\#\left(h_q \cdot \mathcal{L}^1{}_{\llcorner} I_q\right)$, h_q is a CD(K,N) density.[4] Such regularity problem for conditional measures can be solved by combining arguments in [4, Theorem 5.7] and [7, Theorem 4.2]. We refer to the appendix for more detailed demonstrations. $\square$

Observe that (3.4) depends on the chosen normalization of the reference measure. However, this affects the disintegration only by a constant factor on each ray. Namely, given two disintegrations with a weight function f and g respectively

$$\mathfrak{m}_{\llcorner}\mathcal{T}_u^b = \int_Q \mathfrak{n}_q^f / f \mathfrak{q}^f(\mathrm{d}q) = \int_Q \mathfrak{n}_q^g / g \mathfrak{q}^g(\mathrm{d}q)$$

as constructed in the proof, where the quotient space $(Q, \mathcal{Q})$ does not rely on normalizations. By the positivity of weight functions, $\mathfrak{q}^f$ and $\mathfrak{q}^g$ are mutually absolutely continuous. Hence the essential uniqueness of consistent disintegration yields an equality between $(\mathfrak{m}_q^f)_q$ and $(\mathfrak{m}_q^g)_q$.

[4] See Appendix for the definition of CD(K,N) densities.

Nevertheless, existence result of the disintegration is sufficient for our purpose.

4 From $\mathsf{CD}^1(K,N)$ to $\mathsf{CD}(K,N)$

This section is devoted to the following main theorem, which together with Theorem 3.10 concludes the local-to-global property of $\mathsf{CD}(K,N)$.

Theorem 4.1 *Let* $(X,d,\mathfrak{m})$ *be an e.n.b. m.m.s. with* $\mathfrak{m}$ *locally finite having full support. If it holds* $\mathsf{CD}^1(K,N)$*, then it holds* $\mathsf{CD}(K,N)$.

It turns out that the approach developed in [6] is powerful enough to work on locally finite spaces with very mild modifications once the CD^1-condition is given by Definition 3.7. In subsequent sections, we sketch the proof with the absence of the finiteness of $\mathfrak{m}$, following closely [6], highlighting necessary modifications. Note that the following part is by no means self-contained, so a parallel reading on the paper [6] is recommended for readers looking for details.

4.1 $\mathsf{CD}^1(K,N)$ implies $\mathsf{MCP}(K,N)$

We begin with recovering the MCP-condition.

Proposition 4.2 *If a m.m.s.* $(X,d,\mathfrak{m})$ *verifies* $\mathsf{CD}^1(K,N)$*, then it verifies* $\mathsf{MCP}(K,N)$.

Proof By definition, we need to show that for any $o\in X$ and $\mu_0:=\frac{\mathfrak{m}\llcorner A}{\mathfrak{m}(A)}$, there exists $\nu\in\mathrm{OptGeo}(\mu_0,\delta_o)$ such that (2.2) is satisfied, where $A\subset X$ is an arbitrary Borel set with $0<\mathfrak{m}(A)<\infty$. We can further assume A to be bounded by [14, Remark 5.1].

Choosing $u=d(\cdot,o)$, the CD^1-condition provides a disintegration of $\mathfrak{m}$ on $\mathcal{T}_u=X$ s.t. for $\mathfrak{q}$-a.e. $q\in Q$, $(R_u(q),d,\mathfrak{m}_q)$ verifies $\mathsf{CD}(K,N)$, and in particular $\mathsf{MCP}(K,N)$. Based on the uniform-local finiteness (3.2), the function $Q\ni q\mapsto\mathfrak{m}_q(A)$ is $\mathfrak{q}$-measurable and almost everywhere finite. For all q in

$$\bar{Q}:=\left\{q\in Q:\ \mathfrak{m}_q(A)\in(0,\infty),\mathrm{supp}(\mathfrak{m}_q)=R_u(q)=\overline{R_u^b(q)}\right\},$$

define $\mu_0^q:=\frac{\mathfrak{m}_q\llcorner A}{\mathfrak{m}_q(A)}$. By the maximality of $R_u(q)$, $o\in\mathrm{supp}(\mathfrak{m}_q)$ and there exists a unique $\nu^q\in\mathrm{OptGeo}(\mu_0^q,\delta_o)$ for $q\in\bar{Q}$. Take $\nu=\int_{\bar{Q}}\nu^q\frac{\mathfrak{m}_q(A)}{\mathfrak{m}(A)}\mathfrak{q}(\,\mathrm{d}q)$ and all curves in its support are contained in a common bounded subset of X. Then going in lines of the proof of [6, Proposition 8.9] validates that ν is a required optimal dynamical plan from μ_0 to δ_o. $\square$

As a consequence, all statements in Sect. 2.2 now hold on e.n.b. $\mathsf{CD}^1(K,N)$ spaces and the underlying metric space must be Polish, proper and geodesic.

Let μ_0 and μ_1 be two arbitrary elements in $\mathcal{P}_2(X,\mathfrak{m})$ and $\nu\in\mathrm{OptGeo}(\mu_0,\mu_1)$. Fix a Kantorovich potential φ of the quadratic optimal transport from μ_0 to μ_1 and denote by $(\varphi_t)_{t\in[0,1]}$ the family of intermediate-time Kantorovich potentials.

By Proposition 2.7, it is sufficient to show that the density ρ_t of $\mu_t := (e_t)_\# \nu$ w.r.t. $\mathfrak{m}$ satisfies the distortion inequality

$$\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\ell(\gamma))\rho_0^{-1/N}\big(\gamma_0\big) + \tau_{K,N}^{(t)}(\ell(\gamma))\rho_1^{-1/N}\big(\gamma_1\big) \tag{4.1}$$

for ν-a.e. $\gamma \in \mathrm{Geo}(X)$.

For this aim, we will localize the whole problem to transport paths, by coupling different disintegrations. Since we only care the almost-everywhere statement of (4.1), by Sect. 2.2, we can work under the following convenient convention.

Convention 4.3 In the sequel, we restrict ourselves to a Borel subset of Kantorovich geodesics of full ν-measure, still denoted by G_φ with a slight abuse of notation, such that

(1) e_t is injective on G_φ for all $t \in [0,1]$;
(2) $(\rho_t)_t$ can be chosen that statements in (1) of Corollary 2.8 hold for each $\gamma \in G_\varphi$.

4.2 L^2-decomposition of transports.

Based on discussions in Sect. 2.3, for fixed $s, t \in [0,1]$, we have two families of partitions given by level sets of continuous functions as follows

$$G_\varphi = \sqcup_{a_s \in \mathbb{R}} G_{\varphi,a_s}, \quad e_t(G_\varphi) = \sqcup_{a_s \in \mathbb{R}} e_t(G_{\varphi,a_s}) \tag{4.2}$$

where $G_{\varphi,a_s} := \big\{\gamma \in G_\varphi : \varphi_s(\gamma_s) = a_s\big\}$.

Replace G_φ by any compact subset G with $\nu(G) > 0$ and by Remark 3.4, there exist disintegrations of finite measures $\nu \llcorner G$ and $\mathfrak{m} \llcorner e_t(G)$ strongly consistent with partitions (4.2) respectively. Notice that all arguments in [6, Section 10.2] can be repeated without any change so quotient measures are absolutely continuous to the one-dimensional Lebesgue measure $\mathcal{L}^1$ for both disintegrations induced. More precisely, we can find (ν_{a_s}) and $(\mathfrak{m}^t_{a_s})$ concentrated on $G_{a_s}(:= G \cap G_{\varphi,a_s})$ and $e_t\big(G_{a_s}\big)$ respectively so that

$$\nu = \int \nu_{a_s} \mathcal{L}^1\big(\mathrm{d}a_s\big), \quad \mathfrak{m} \llcorner e_t(G) = \int \mathfrak{m}^t_{a_s} \mathcal{L}^1\big(\mathrm{d}a_s\big). \tag{4.3}$$

The two families of conditional measures in (4.3) are comparable under the relation

$$\begin{aligned}
\mu_t \llcorner e_t(G) = (e_t)_\#(\nu \llcorner G) &= \int_{\varphi_s(e_s(G))} (e_t)_\# \nu_{a_s} \mathcal{L}^1(\mathrm{d}a_s) \\
= \rho_t \mathfrak{m} \llcorner e_t(G) &= \int_{\varphi_s\left(e_s(G)\right)} \rho_t \cdot \mathfrak{m}^t_{a_s} \mathcal{L}^1\big(\mathrm{d}a_s\big).
\end{aligned}$$

By Remark 3.1, for $\mathcal{L}^1$-a.e. $a_s \in \varphi_s(e_s(G))$, $\rho_t \cdot \mathfrak{m}^t_{a_s} = (e_t)_\# \nu_{a_s}$.

4.3 L^1-decomposition of $\mathfrak{m}$ via needle decomposition.

For any $s \in (0,1)$ and $a_s \in \mathrm{Im}(\varphi_s \circ e_s)$, denote $u := d_{\varphi_s - a_s}$ as the signed distance function from $\{\varphi_s = a_s\}$. By [6, Lemma 10.3], for every $\gamma \in G_{\varphi,a_s}$ and $0 \leq r \leq t \leq 1$, $(\gamma_r, \gamma_t) \in R_u$. In particular, $e_{[0,1]}(G_{\varphi,a_s}) \subset \mathcal{T}_u$.

Again, when we restrict the L^1-disintegration to a compact subset, and with the uniform boundedness of conditional measures given by (3.2), a repetition of [6, Propositon 10.4] can be performed as follows.

Proposition 4.4 *For any compact subset $G \subset G_\varphi^+$ with positive measure, $s \in (0,1)$ and $a_s \in \varphi_s(e_s(G))$, we have the following disintegration:*

$$\mathfrak{m}\llcorner e_{[0,1]}\left(G_{a_s}\right) = \int_{[0,1]} \mathfrak{m}_t^{a_s} \mathcal{L}^1(\mathrm{d}t), \tag{4.4}$$

where

$$\mathfrak{m}_t^{a_s} = g^{a_s}(\cdot, t)_\#\left(h_\cdot^{a_s}(t) \cdot \mathfrak{m}_s^{a_s}\right) \tag{4.5}$$

so that

(1) $g^{a_s}: e_s\left(G_{a_s}\right) \times [0,1] \to X$ *is Borel measurable, mapping (β, t) to $e_t\left(e_s^{-1}(\beta)\right)$;*
(2) $(0,1) \ni t \mapsto \mathfrak{m}_t^{a_s}$ *is continuous under weak convergence, and for each t, $\mathfrak{m}_t^{a_s}$ is concentrated on $e_t(G_{a_s})$;*
(3) *For $\mathfrak{m}_s^{a_s}$-a.e. β, $h_\beta^{a_s}$ is a continuous* $\mathrm{CD}\left(\ell_s^2(\beta)K, N\right)$ *density on* $(0,1)$ *satisfying* $h_\beta^{a_s}(s) = 1$;
(4) *There exists a constant C depending only on K, N and* $\max\{\ell(\gamma) : \gamma \in G\}$,

$$\|\mathfrak{m}_t^{a_s}\| \le C\mathfrak{m}\left(e_{[0,1]}\left(G_{a_s}\right)\right), \quad \forall t \in (0,1). \tag{4.6}$$

Proof **Restrict L^1-disintegration to curves in G_{a_s}.** By definition, one has a probability measure $\hat{\mathfrak{q}}^{a_s}$ and a disintegration

$$\mathfrak{m}\llcorner \mathcal{T}_u = \int_Q \hat{\mathfrak{m}}_q^{a_s} \hat{\mathfrak{q}}^{a_s}(\mathrm{d}q). \tag{4.7}$$

Since $e_{[0,1]}\left(G_{a_s}\right) \subset \mathcal{T}_u$, we can restrict (4.7) to $e_{[0,1]}\left(G_{a_s}\right)$ so that

$$\mathfrak{m}\llcorner e_{[0,1]}\left(G_{a_s}\right) = \int_Q \hat{\mathfrak{m}}_q^{a_s}\llcorner e_{[0,1]}\left(G_{a_s}\right)\hat{\mathfrak{q}}^{a_s}(\mathrm{d}q).$$

If we denote

$$G_{a_s}^1 := \left\{\gamma \in G_{a_s} : \mathcal{T}_u^b \cap e_{[0,1]}(\gamma) \neq \emptyset\right\}, \quad Q^1 := \left\{q \in Q : R_u^b(q) \cap e_{[0,1]}\left(G_{a_s}\right) \neq \emptyset\right\},$$

then following exactly same arguments in Part 1–3 of the proof of [6, proposition 10.4] on the ray decomposition and measurability we know Q^1 is $\hat{\mathfrak{q}}^{a_s}$-measurable, $G_{a_s}^1$ is analytic, and there exists a Borel isomorphism $\eta : \left(Q^1, \mathcal{B}(Q^1)\right) \to \left(G_{a_s}^1, \mathcal{B}\left(G_{a_s}^1\right)\right)$, mapping q to γ^q.

On the other hand, there exists $\tilde{Q} \subset Q$ of full $\hat{\mathfrak{q}}^{a_s}$-measure s.t. for each $q \in \tilde{Q}$, $\hat{\mathfrak{m}}_q^{a_s}$ is non-null, supported on $R_u(q) = \overline{R_u^b(q)}$ and $\left(R_u(q), d, \hat{\mathfrak{m}}_q^{a_s}\right)$ verifies $\mathrm{CD}(K,N)$. Since each $\gamma^q \in G_{a_s}$ is contained in $R_u(q)$, $\{\gamma_q\}_{q\in\tilde{Q}}$ have disjoint interiors. By (3.2) and the fact that a non-null measure carrying $\mathrm{CD}(K,N)$ gives positive mass to open sets, we have

$$0<\hat{\mathfrak{m}}_q^{a_s}\big(e_{[0,1]}\big(G_{a_s}\big)\big)=\hat{\mathfrak{m}}_q^{a_s}\big(e_{[0,1]}(\gamma^q)\big)<C_G,\quad \forall q\in Q^1\cap\tilde{Q},$$

for some constant $C_G>0$. Therefore, summarizing above discussions with Remark 3.9 gives

$$\begin{aligned}\mathfrak{m}\llcorner e_{[0,1]}\big(G_{a_s}\big)&=\int_{Q^1\cap\tilde{Q}}\hat{\mathfrak{m}}_q^{a_s}\llcorner\big\{\mathcal{T}_u^b\cap e_{[0,1]}\big(G_{a_s}\big)\big\}\hat{\mathfrak{q}}^{a_s}(\mathrm{d}q)\\&=\int_{Q^1\cap\tilde{Q}}\frac{\hat{\mathfrak{m}}_q^{a_s}\llcorner e_{[0,1]}(\gamma^q)}{\hat{\mathfrak{m}}_q^{a_s}\big(e_{[0,1]}(\gamma^q)\big)}\hat{\mathfrak{m}}_q^{a_s}\big(e_{[0,1]}(\gamma^q)\big)\hat{\mathfrak{q}}^{a_s}(\mathrm{d}q).\end{aligned}\tag{4.8}$$

Denoting $\bar{\mathfrak{m}}_q^{a_s}:=\frac{\hat{\mathfrak{m}}_q^{a_s}\llcorner e_{[0,1]}(\gamma^q)}{\hat{\mathfrak{m}}_q^{a_s}(e_{[0,1]}(\gamma^q))}$, and $\bar{\mathfrak{q}}^{a_s}=\hat{\mathfrak{m}}_q^{a_s}\big(e_{[0,1]}(\gamma^q)\big)\hat{\mathfrak{q}}^{a_s}\llcorner Q^1$, (4.8) can be rewritten as

$$\mathfrak{m}\llcorner e_{[0,1]}\big(G_{a_s}\big)=\int_{Q^1\cap\tilde{Q}}\bar{\mathfrak{m}}_q^{a_s}\bar{\mathfrak{q}}^{a_s}(\mathrm{d}q).\tag{4.9}$$

Change the variable and conditional measures. Pushing-forward via the Borel measurable bijection $e_s\circ\eta : Q^1\to e_s\big(G^1_{a_s}\big)$ induces a space $\big(e_s\big(G^1_{a_s}\big),\mathcal{S},\check{\mathfrak{q}}^{a_s}\big)$, with $\mathcal{S}:=(e_s\circ\eta)_\#\big(\mathcal{Q}\cap Q^1\big)$ and $\check{\mathfrak{q}}^{a_s}=(e_s\circ\eta)_\#\bar{\mathfrak{q}}^{a_s}$. Correspondingly, (4.9) can be expressed on the new measurable space:

$$\mathfrak{m}\llcorner e_{[0,1]}\big(G_{a_s}\big)=\int_{e_s(G_{a_s})}\mathfrak{m}_\beta^{a_s}\check{\mathfrak{q}}^{a_s}(\mathrm{d}\beta),\tag{4.10}$$

where $\mathfrak{m}_\beta^{a_s}=\bar{\mathfrak{m}}^{a_s}_{(e_s\circ\eta)^{-1}(\beta)}$ has unit mass and $\check{\mathfrak{q}}^{a_s}$ is concentrated on $e_s\circ\eta\big(Q^1\cap\tilde{Q}\big)$ since $\bar{\mathfrak{q}}^{a_s}$ and $\hat{\mathfrak{q}}^{a_s}\llcorner Q^1$ are mutually absolutely continuous.

With the new cross section $e_s\big(G_{a_s}\big)$, we define a ray map g^{a_s} as in (3.5) but now with the time variable fixed on [0, 1]:

$$g^{a_s} : e_s\big(G_{a_s}\big)\times[0,1]\to X,\quad (\beta,t)\mapsto e_t(e_s^{-1}(\beta)).$$

Clearly, g^{a_s} is Borel measurable. For any $\beta\in e_s\big(G_{a_s}\big)$, $t\mapsto g^{a_s}(\beta,t)$ is an isometry between $([0,1],|\cdot|)$ and $(\gamma^\beta:=e_s^{-1}(\beta),d/\ell_s(\beta))$. By assumption, for $\check{\mathfrak{q}}^{a_s}$-a.e. β, $(\gamma^\beta,d,\mathfrak{m}_\beta^{a_s})$ verifies CD(K,N). After rescaling the metric, $(\gamma^\beta,d/\ell_s(\beta),\mathfrak{m}_\beta^{a_s})$ verifying CD$(\ell_s^2(\beta)K,N)$. For those β, there exists a continuous function $\check{h}_\beta^{a_s}$ as a CD$(\ell_s^2(\beta)K,N)$ probability density on (0, 1) s.t.

$$\mathfrak{m}_\beta^{a_s}=g^{a_s}(\beta,\cdot)_\#\Big(\check{h}_\beta^{a_s}\cdot\mathcal{L}^1\llcorner[0,1]\Big)$$

and

$$\mathfrak{m}\llcorner e_{[0,1]}\big(G_{a_s}\big)=\int_{e_s(G_{a_s})}g^{a_s}(\beta,\cdot)_\#\Big(\check{h}_\beta^{a_s}\cdot\mathcal{L}^1\llcorner[0,1]\Big)\check{\mathfrak{q}}^{a_s}(\mathrm{d}\beta).$$

Reformulate the disintegration on [0, 1]. The item 1 of Definition 3.7 allows us to repeat the step 8 of the proof of [6, Proposition 10.4] to obtain the $\check{\mathfrak{q}}^{a_s}\otimes\mathcal{L}^1$-measurability of $e_s\big(G_{a_s}\big)\times[0,1]\ni(\beta,t)\mapsto\check{h}_\beta^{a_s}(t)$, where we also follow the convention that $\check{h}_\beta^{a_s}$ vanishes at endpoints. By Fubini, we can exchange the order of (4.10) s.t. (4.4) is achieved with

$$\mathfrak{m}_t^{a_s}=g^{a_s}(\cdot,t)_\#\big(\check{h}^{a_s}_\cdot(t)\cdot\check{\mathfrak{q}}^{a_s}\big).\tag{4.11}$$

The map $\beta \mapsto \check{h}^{a_s}_\beta(s)$ is $\check{\mathfrak{q}}^{a_s}$-measurable and for $\check{\mathfrak{q}}^{a_s}$-a.e. β, $\check{h}^{a_s}_\beta$ is strictly positive on $(0,1)$. Let $h^{a_s}_\beta := \frac{\check{h}^{a_s}_\beta}{\check{h}^{a_s}_\beta(s)}$ for those β and $\mathfrak{q}^{a_s} := \check{h}^{a_s}_\beta(s) \cdot \check{\mathfrak{q}}^{a_s}$. Now $h^{a_s}_\beta(s) = 1$ for $\mathfrak{q}^{a_s}$-a.e. β, and $\check{\mathfrak{q}}^{a_s}$, $\mathfrak{q}^{a_s}$ are mutually absolutely continuous (both of them are finite measures) sharing same measurable and null sets. And $g^{a_s}(\cdot,t)_\#(h^{a_s}_\cdot(t) \cdot \mathfrak{q}^{a_s})$ equals to $\mathfrak{m}^{a_s}_t$ still. Hence, $\mathfrak{m}^{a_s}_s = \mathfrak{q}^{a_s}$ and the translation relation (4.5) is satisfied.

The continuity of $t \mapsto \mathfrak{m}^{a_s}_t$ follows from the continuity of $t \mapsto h^{a_s}_\beta(t), g^{a_s}(\beta,t)$. Finally, by [6, Lemma A.8], probability densities $\check{h}^{a_s}_\beta(t)$ are bounded uniformly for β, t. The uniform volume bound (4.6) of $\mathfrak{m}^{a_s}_t$ is given by (4.11) and the finiteness of $\check{\mathfrak{q}}^{a_s}$:

$$\check{\mathfrak{q}}^{a_s}(e_{[0,1]}\big(G_{a_s}\big) = \bar{\mathfrak{q}}^{a_s}(Q^1) = \int_{Q^1} \hat{\mathfrak{m}}^{a_s}_q\big(e_{[0,1]}(\gamma^q)\big)\hat{\mathfrak{q}}^{a_s}(\mathrm{d}q) = \mathfrak{m}\big(e_{[0,1]}\big(G_{a_s}\big)\big).$$

$\square$

4.4 Comparison between conditional measures

This section is to recover the comparison between L^2 and L^1 disintegrations based on [6, Section 11].

Recall that the t-propagated s-Kantorovich potential defined on $D(G_\varphi)$, by $\Phi^t_s(x) := \varphi_t(x) + \frac{t-s}{2}\ell^2_t(x)$, is jointly continuous and locally Lipschitz on t. The following differential properties will be crucial in the comparison argument. Moreover, they are statements of metric spaces without any reference measure.

Lemma 4.5 (cf. [6, Proposition 4.4]) *Fix any $s \in (0,1)$.*

(1) *For any $x \in X$, $t \mapsto \Phi^t_s(x)$ is differentiable iff $t \mapsto \ell^2_t(x)$ is differentiable on $G_\varphi(x)$ or $t = s \in G_\varphi(x)$, with derivatives*

$$\partial_t \Phi^t_s(x) = \ell^2_t(x) + (t-s)\frac{\partial_t \ell^2_t(x)}{2}, \quad \partial_t|_{t=s}\partial_t\Phi^t_s(x) = \ell^2_s(x).$$

(2) *For all $(x,t) \in D(G_\varphi)$,*

$$\min\left\{\frac{s}{t}, \frac{1-s}{1-t} + \frac{t-s}{t(1-t)}\right\}\ell^2_t(x) \le \liminf_{G_\varphi(x)\ni\tau\to t} \frac{\Phi^\tau_s(x) - \Phi^t_s(x)}{\tau - t}$$
$$\le \limsup_{G_\varphi(x)\ni\tau\to t} \frac{\Phi^\tau_s(x) - \Phi^t_s(x)}{\tau - t} \le \max\left\{\frac{s}{t}, \frac{1-s}{1-t} + \frac{t-s}{t(1-t)}\right\}\ell^2_t(x).$$

Proposition 4.6 *Let G be a compact subset of G^+_φ with $\nu(G) > 0$. For any $s \in (0,1)$, $\mathcal{L}^1$-a.e. $t \in (0,1)$ including $t = s$ and $\mathcal{L}^1$-a.e. $a_s \in \varphi_s(e_s(G))$, we have*

$$\mathfrak{m}^{a_s}_t = \partial_t \Phi^t_s \cdot \mathfrak{m}^t_{a_s}, \tag{4.12}$$

where $\partial_t\Phi^t_s(x)$ exists and is positive for $\mathfrak{m}^t_{a_s}$-a.e. x.

Sketch of proof By Lemma 4.5, for any $x \in X$, $\partial_t \Phi_s^t(x)$ exists for $\mathcal{L}^1$-a.e. $t \in G_\varphi(x)$ including $t = s$, and wherever differentiable, $\partial_t \Phi_s^t(x) > 0$. Then the statement on differentiability can be concluded by applying Fubini to the set $\{(x,t) : \exists \gamma \in G_\varphi, \gamma_t = x\} \subset X \times (0,1)$ to rephrase the exceptional set of differentiation, together with the disintegration $\mathfrak{m} \llcorner e_t(G) = \int \mathfrak{m}_{a_s}^t \mathcal{L}^1(\mathrm{d}a_s)$.

Now we start to show the equivalence. The main idea is a "sum-up" of $\mathfrak{m}_t^{a_s}$ for all $a_s \in \varphi_s(e_s(G))$ to recover $\mathfrak{m} \llcorner e_t(G)$ (which has a disintegration $\int \mathfrak{m}_{a_s}^t \mathcal{L}^1(\mathrm{d}a_s)$) and so to connect the two families of conditional measures.

For $t_0 \in \mathbb{R}$ and $x_0 \in X$, define

$$1_{t_0}^1 : X \ni x \mapsto (t_0, x) \in \mathbb{R} \times X, \quad 1_{x_0}^2 : \mathbb{R} \ni t \mapsto (t, x_0) \in \mathbb{R} \times X.$$

Recall that $\mathfrak{m}_t^{a_s}$ is supported inside a common compact set (e.g. $e_{[0,1]}(G)$) for all t and a_s, having uniformly bounded mass by (4.6). Using the following variant of (4.4)

$$\mathfrak{m} \llcorner e_{(t-\epsilon,t+\epsilon)}\big(G_{a_s}\big) = \int_{(t-\epsilon,t+\epsilon)} \mathfrak{m}_\tau^{a_s} \mathcal{L}^1(\mathrm{d}\tau),$$

and the continuity of $\tau \mapsto \mathfrak{m}_\tau^{a_s}$, we have

$$\int_{\varphi_s(e_s(G))} \left(1_{a_s}^1\right)_\# \mathfrak{m}_t^{a_s} \mathcal{L}^1(\mathrm{d}a_s) = \lim_{\epsilon \to 0} \int_{\varphi_s(e_s(G))} \frac{1}{2\epsilon} \left(1_{a_s}^1\right)_\# \mathfrak{m} \llcorner e_{(t-\epsilon,t+\epsilon)}\big(G_{a_s}\big) \mathcal{L}^1(\mathrm{d}a_s),$$

under the weak topology. Manipulating the right-hand side via Fubini and functions $(\Phi_s^t)_t$ as in the proof of [6, Theorem 11.3], one gets

$$\int_{\varphi_s(e_s(G))} \left(1_{a_s}^1\right)_\# \mathfrak{m}_t^{a_s} \mathcal{L}^1(\mathrm{d}a_s) = \lim_{\epsilon \to 0} \int_{e_t(G)} \frac{1}{2\epsilon} \big(1_x^2\big)_\# \big(\mathcal{L}^1 \llcorner \big\{\Phi_s^\tau(x) : \tau \in (t-\epsilon, t+\epsilon) \cap G(x)\big\}\big) \mathfrak{m}(\mathrm{d}x),$$

where we only need the uniform boundedness of mass of measures

$$\frac{1}{2\epsilon} \mathcal{L}^1 \llcorner \big\{\Phi_s^\tau(x) : \tau \in (t-\epsilon, t+\epsilon) \cap G(x)\big\} \tag{4.13}$$

for all ϵ, which is ensured by Lemma 4.5 and the compactness of G. Besides, since $t \mapsto \Phi_s^t(x)$ is a strictly monotone Lipschitz function on $G(x) \cap (t-\epsilon, t+\epsilon)$ (with a uniform Lipschitz bound for $x \in e_t(G)$), one-dimensional measures in (4.13) converge to $\partial_t \Phi_s^t(x) \delta_{\Phi_s^t(x)}$ when $\epsilon \to 0$, for $\mathcal{L}^1$-a.e. $t \in (0,1)$ and $\mathfrak{m}$-a.e. $x \in e_t(G)$.

As a result, for $\mathcal{L}^1$-a.e. t and a_s,

$$\int_{\varphi_s(e_s(G))} \left(1_{a_s}^1\right)_\# \mathfrak{m}_t^{a_s} \mathcal{L}^1(\mathrm{d}a_s) = \int_{e_t(G)} \big(1_x^2\big)_\# \Big(\partial_t \Phi_s^t(x) \delta_{\Phi_s^t(x)}\Big) \mathfrak{m}(\mathrm{d}x).$$

Testing the above equality by $1 \otimes f \in C_b(\mathbb{R} \times X)$ with the disintegration (4.3) of $\mathfrak{m} \llcorner e_t(G)$ implies

$$\int_{\varphi_s(e_s(G))} \mathfrak{m}_t^{a_s} \mathcal{L}^1(\mathrm{d}a_s) = \int_{\varphi_s(e_s(G))} \partial_t \Phi_s^t \cdot \mathfrak{m}_{a_s}^t \mathcal{L}^1(\mathrm{d}a_s). \tag{4.14}$$

Actually, disintegrations on both sides of (4.14) are strongly consistent on $\big\{e_t\big(G_{a_s}\big)\big\}_{a_s}$ of $e_t(G)$, and hence (4.12). The assertion for $t = s$ can be proven exactly the same as [6, Theorem 11.3], since the underlying space verifies MCP(K,N) by Proposition 4.2 and $\ell(\gamma)$ is uniformly bounded by the compactness of G. $\square$

4.5 Proof of the main theorem

Once all disintegrations and the comparison are produced, a so-called change-of-variable formula (cf. Equation (11.10) in [6]) can be derived. The remaining part after that, though highly technical, not related to the finiteness of $\mathfrak{m}$, will follow naturally. Here we outline the proof of the main theorem in the locally finite case, closely following Section 11.2, 12 and 13.1 of [6].

Proof of Theorem 4.1 **Deriving the change-of-variable formula.** First, consider any G, as a compact subset of G_φ^+ with positive ν-measure. Fix $s \in (0,1)$. As mentioned in the end of Sect. 4.2, for every $t \in (0,1)$, $\mathcal{L}^1$-a.e. $a_s \in \varphi_s(e_s(G))$, $\rho_t \cdot \mathfrak{m}^t_{a_s} = (e_t)_\# \nu_{a_s}$. By evaluating both of them to $e_t(H)$ for an arbitrary Borel $H \subset G$, we have

$$\int_{e_t(H)} \rho_t(x) \cdot \mathfrak{m}^t_{a_s}(\mathrm{d}x) = \nu_{a_s}(H).$$

In the above integral, replacing $\mathfrak{m}^t_{a_s}$ by $\left(\partial_t \Phi^t_s\right)^{-1} \mathfrak{m}^{a_s}_t$ using Proposition 4.6, and combining the translation formula (4.5), we have

$$\int_{e_s(H)} \underbrace{\left(\rho_t \cdot (\partial_t \Phi^t_s)^{-1}\right) \circ g^{a_s}(\beta, t) h^{a_s}_\beta(t)}_{:=f_t(\beta)} \mathfrak{m}^{a_s}_s(\mathrm{d}\beta) = \nu_{a_s}(H), \tag{4.15}$$

for $\mathcal{L}^1$-a.e. $t \in (0,1)$ including $t = s$ and $a_s \in \varphi_s(e_s(G))$. Denote by $f_t(\beta)$ the integrand in (4.15). From the arbitrariness of H, there is a subset $T \subset [0,1]$ of full measure s.t. for each $t \in T$, $f_t = f_s$ for $\mathfrak{m}^{a_s}_s$-a.e. β (due to the continuity of $\rho_t(\cdot)$, $h^{a_s}_\beta(\cdot)$ and $g^{a_s}(\beta,\cdot)$, and the fact that $\partial_t \Phi^t_s(x)$ converges to $\ell^2_s(x)$ when $t \to s$ by Lemma 4.5). Recall that $h^{a_s}_\beta(s) = 1$ for $\mathfrak{m}^{a_s}_s$-a.e. β and $g^{a_s}(\cdot, s) = id$. Hence, for $\mathcal{L}^1$-a.e. t,

$$f_t(\beta) = \left(\rho_t \cdot (\partial_t \Phi^t_s)^{-1}\right) \circ g^{a_s}(\beta, t) h^{a_s}_\beta(t) = f_s(\beta) = \rho_s(\beta)/\ell^2_s(\beta), \tag{4.16}$$

for $\mathfrak{m}^{a_s}_s$-a.e. $\beta \in e_s(G)$. Again by Proposition 4.6, $\mathfrak{m}^{a_s}_s$ and $\mathfrak{m}^s_{a_s}$ are mutually absolutely continuous, so (4.16) holds for $\mathfrak{m}^s_{a_s}$-a.e. β as well. Further, the validity of (4.16) for almost each a_s indicates, after recovering $\mathfrak{m}\llcorner e_s(G)$ by disintegration $\int \mathfrak{m}^s_{a_s} \mathcal{L}^1(\mathrm{d}a_s)$, that (4.16) holds for $\mathfrak{m}$-a.e. $\beta = \gamma_s$ with $\gamma \in G$.

In conclusion, after changing the variable β to γ_s, for ν-a.e. $\gamma \in G$, and $\mathcal{L}^1$-a.e. $t \in (0,1)$, we have

$$\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \frac{h^{\varphi_s(\gamma_s)}_{\gamma_s}(t)}{\partial_\tau|_{\tau=t} \Phi^\tau_s(\gamma_t)/\ell^2(\gamma)}. \tag{4.17}$$

Recall from the construction in Proposition 4.4 that, $\check{h}^{a_s}_\beta(t)$ is uniquely defined as the continuous density of $\hat{\mathfrak{m}}^{a_s}_q$ (given by the L^1-disintegration (4.7) of $\mathcal{T}_u$) after conditioning it on

$e_{[0,1]}(\gamma^q)$ and pulling it back to the interval [0, 1] via the ray map g^{a_s} (which can be defined on the whole $e_s(G^+_\varphi)\times[0,1]$ by Convention 4.3). In particular, $h^{a_s}_\beta$ and hence (4.17) does not depend on the choice of G. Then by the inner regularity of ν, the validity of (4.17) holds for ν-a.e. $\gamma\in G^+_\varphi$.

"L-Y" decomposition of the density along $\gamma\in G^+_\varphi$. We show that along each γ satisfying (4.17), the density admits a decomposition $\rho_t(\gamma_t)^{-1}=L(t)Y(t)$, where L is concave and Y is a $\mathrm{CD}(\ell^2(\gamma)K,N)$ density on (0, 1).

All steps in the proof of [6, Theorem 12.3] can be repeated since it is only a matter of one-dimensional analysis on [0, 1]. Once we check that condition (C) is satisfied in the statement of [6, Theorem 12.3] (the validity of (A) and (B) is clear by Convention 4.3 and Proposition 4.4). Indeed, the condition is reduced to an estimate of the 3-rd order derivative of $t\mapsto\varphi_t(\gamma_t)$, where no difference occurs between finite and locally finite spaces.

Afterwards, an application of Hölder's inequality (cf. [6, Theorem 13.2]) to the "L-Y" decomposition, with the upper semi-continuity of $t\mapsto\rho_t(\gamma_t)$ at $t=0,1$ from Convention 4.3, yields the desired inequality (4.1).

On null-geodesics. Denote by G^0_φ the set of all curves in G_φ with zero length and $X_0:=e_{[0,1]}(G^0_\varphi)$. By [6, Corollary 9.8], as a consequence of Corollary 2.8, $\mu_t\llcorner X_0=\mu_0\llcorner X_0$ for all $t\in[0,1]$. As a result, same to the step 0 of [6, Theorem 11.4] we can always redefine $\rho_t\llcorner X_0:=\rho_0\llcorner X_0$ so that (4.17) holds automatically over $\gamma\in G^0_\varphi$ and $t\mapsto\rho_t(\gamma_t)$ will not be affected for all $\gamma\in G^+_\varphi$. □

Appendix A. Proof of Ray Decomposition

Definition A.1 A non-negative function h on an interval $I\subset\mathbb{R}$ is called a $\mathrm{CD}(K,N)$ density if for all $x_0,x_1\in I$ and $t\in[0,1]$:

$$h\big(tx_1+(1-t)x_0\big)^{\frac{1}{N-1}}\geq\sigma^{(t)}_{K,N-1}(|x_0-x_1|)h(x_1)^{\frac{1}{N-1}}+\sigma^{(1-t)}_{K,N-1}(|x_0-x_1|)h(x_0)^{\frac{1}{N-1}}.\quad\text{(A.1)}$$

The name comes from the fact that a 1-dimensional m.m.s. $(I,|\cdot|,\mu)$ verifies $\mathrm{CD}(K,N)$ if and only if $\mu\ll\mathcal{L}^1$ and the density $h=\mathrm{d}\mu/\mathrm{d}\mathcal{L}^1$ has a version being a $\mathrm{CD}(K,N)$ density (see [6, Theorem A.2]). Moreover, if $h\in C^2_{\mathrm{loc}}(I)$, then h is a $\mathrm{CD}(K,N)$ density if and only if

$$\frac{((\log h)')^2}{N-1}+(\log h)''\leq -K.$$

We call a property on $I\subset\mathbb{R}$ *local* if once it holds on an interval I_x of any point $x\in I$, then it holds globally on I. In particular, being positive, locally Lipschitz or a $\mathrm{CD}(K,N)$ density are all local properties in $\mathbb{R}$ (see [10, Section 5] for the local-to-global property of $\mathrm{CD}(K,N)$ densities).

Completion of the Proof of Theorem 3.10 Via the ray map g introduced in (3.5), $\mathfrak{m}\llcorner\mathcal{T}^b_u$ can be reformed as a measure on $S\times\mathbb{R}$:

$$\big(g^{-1}\big)_\#\big(\mathfrak{m}\llcorner\mathcal{T}^b_u\big)=\big(g^{-1}\big)_\#\mathfrak{m}\llcorner\big(\mathfrak{Q}^{-1}(S)\big)=\int_S\big(g^{-1}(q,\cdot)\big)_\#\mathfrak{m}_q\mathfrak{q}(\mathrm{d}q),$$

where the second equality is guaranteed by the strong consistency of disintegration $q \mapsto \mathfrak{m}_q$. As $\left(g^{-1}(q,\cdot)\right)_\# \mathfrak{m}_q$ is locally-finite on $\mathbb{R}$ from (3.3), Lebesgue's decomposition gives

$$\left(g^{-1}(q,\cdot)\right)_\# \mathfrak{m}_q = h_q \mathcal{L}^1 + \omega_q, \quad \omega_q \perp \mathcal{L}^1.$$

(i) It suffices to show for $\mathfrak{q}$-a.e. $q \in S_{a,b}$, $\mathfrak{m}_q$ verifies Theorem 3.10 on $[a,b]$, where

$$S_{a,b} := \left\{q \in S : [a,b] \subset I_q, a, b \in \mathbb{Q}\right\}.$$

Such $S_{a,b}$ is always $\mathfrak{q}$-measurable, since

$$S_{a,b} = P_1\left(\left\{(x,y,z) \in S^3 : (x,y),(x,z) \in R_u, u(y)-u(x) \geq b, u(z)-u(x) \leq a\right\}\right).$$

Let $S^*_{a,b}$ be the set of all q in $S_{a,b}$ s.t. Theorem 3.10 is violated somewhere in $[a, b]$. As all statements are local, the set $Q^* := \left\{q \in S : \mathfrak{m}_q \text{ does not verify Theorem 3.10 on } I_q\right\}$ is contained in $\cup_{a,b\in\mathbb{Q}} S^*_{a,b}$.

It can be reduced to show each $S^*_{a,b}$ is negligible and hence for the time being, we assume S a bounded subset of $S_{a,b}$. For simplicity, we directly assume $\mathfrak{m}$ a measure on $S \times \mathbb{R}$ to avoid writing g all the time.

(ii) Prove that $\mathfrak{m}_q \ll \mathcal{L}^1$. If otherwise, there exists a bounded set $A \subset \mathcal{T}^b_u \subset S \times \mathbb{R}$, $\mathfrak{m}(A) > 0$ but for $\mathfrak{q}$-a.e. $q \in S$

$$\mathcal{L}^1(A \cap (\{q\} \times \mathbb{R})) = 0. \tag{A.2}$$

Take ν to be the unique optimal dynamical plan transporting $\mu_0 := \mathfrak{m}(A)^{-1}\mathfrak{m}\llcorner A$ onto $S \times \{a\}$ along vertical rays $R^b_u(q) = \{q\} \times I_q$. Denote $A_t := e_t(\mathrm{supp}(\nu))$,

$$A_t = \{(q,(1-t)\tau + ta) : (q,\tau) \in A\}. \tag{A.3}$$

Corollary 2.8 ensures $\mathfrak{m}(A_t) > 0$ for each $t \in [0,1)$, so a contradiction follows:

$$\begin{aligned}
0 &< \int_0^{1/2} \mathfrak{m}(A_t)\,\mathrm{d}t = \mathfrak{m} \otimes \mathcal{L}^1\left(\left\{(q,\tau,t) \in S \times \mathbb{R} \times [0,1/2] : (q,\tau) \in A_t\right\}\right)\\
&= \int_{S\times\mathbb{R}} \mathcal{L}^1\left(\{t \in [0,1/2] : (q,\tau) \in A_t\}\right)\mathfrak{m}(\mathrm{d}q\,\mathrm{d}\tau)\\
&\overset{(A.3)}{=} \int_{S\times\mathbb{R}} \mathcal{L}^1\left(\left\{t \in [0,1/2] : (q, \frac{\tau - at}{1-t}) \in A\right\}\right)\mathfrak{m}(\mathrm{d}q\,\mathrm{d}\tau) \overset{(A.2)}{=} 0.
\end{aligned}$$

(iii) Prove that h_q is positive and Lipschitz. For any $[a_0,b_0] \subset [a,b]$ and $S_0 \subset S$ with $\mathfrak{q}(S_0) > 0$, take A as a bounded subset of $\mathcal{T}^b_u \cap (S_0 \times \mathbb{R})$ having positive mass and consider the transport optimally moving $\mathfrak{m}(A)^{-1}\mathfrak{m}\llcorner A$ into $S \times \{(a_0+b_0)/2\}$. As in (ii), once $\mathfrak{m}(A) > 0$, $\mathfrak{m}(A_t) > 0$ for all $t \in (0,1)$. One can easily find $A_t \subset S_0 \times [a_0,b_0]$ for some time t from the boundedness of A, ensuring $\mathfrak{m}(S_0 \times [a_0,b_0]) > 0$.

Next, taking any such $A = S_0 \times [a_0,b_0]$, apply (2.7) to the optimal dynamical plan ν transporting $\mu_0 := \mathfrak{m}(A)^{-1}\mathfrak{m}\llcorner A$ into $S \times \{b\}$. By disintegration, we have

$$\mathfrak{m}(A_t) = \int_{S_0}\int_{[a_0,b_0]_t} h_q\,\mathrm{d}\mathcal{L}^1\,\mathrm{d}\mathfrak{q}, \quad [a_0,b_0]_t := [(1-t)a_0 + tb, (1-t)b_0 + tb].$$

Given any $0 \le r < s < 1$, the arbitrariness of S_0 implies, for $\mathfrak{q}$-a.e. $q \in S$

$$c(r,s)\int_{[a_0,b_0]_r} h_q \,\mathrm{d}\mathcal{L}^1 \le \int_{[a_0,b_0]_s} h_q \,\mathrm{d}\mathcal{L}^1 \le C(r,s)\int_{[a_0,b_0]_r} h_q \,\mathrm{d}\mathcal{L}^1, \tag{A.4}$$

where $c(r, s)$, $C(r, s)$ are locally Lipschitz functions of r, s given by (2.7). Since h_q (or $\mathfrak{m}_q$) is locally finite, both sides of (A.4) continuously depend on s, r, a_0, b_0, and so (A.4) holds simultaneously for all r, s, a_0, b_0. At Lebesgue points τ_0, τ_1 of h_q, choosing $[a_0, b_0] = [\tau_0 - \varepsilon, \tau_0 + \varepsilon]$, $r = 0$ and $s = \frac{\tau_1 - \tau_0}{b - \tau_0}$ in (A.4) and shrinking $\varepsilon \to 0$, a two-sided inequality between $h_q(\tau_0)$, $h_q(\tau_1)$ follows, leading to the Lipschitz continuity.

Finally, because $c(r, s)$, $C(r, s)$ are positive for all s, r, the continuous density h_q is either identically 0 or everywhere positive inside I_q. But the positivity of all $\mathfrak{m}(S_0 \times [a,b])$ excludes the former case (up to a $\mathfrak{q}$-negligible set of q).

(iv) Prove that h_q is a $\mathrm{CD}(K,N)$ density. Consider, any $a < A_0 < A_1 < b$ and $L_0, L_1 > 0$ with $A_0 + L_0 < A_1$ and $A_1 + L_1 < b$. Define

$$\mu_0 := \int_S \frac{1}{L_0}\mathcal{L}^1 \llcorner \big[A_0, A_0 + L_0\big]\,\mathfrak{q}(\mathrm{d}q), \qquad \mu_1 := \int_S \frac{1}{L_1}\mathcal{L}^1 \llcorner \big[A_1, A_1 + L_1\big]\,\mathfrak{q}(\mathrm{d}q).$$

In (iii), we have shown the positivity of h_q, so densities of μ_i w.r.t. $\mathfrak{m}$ are

$$\rho_i((q,t)) = \frac{1}{L_i} h_q(t)^{-1}, \quad \forall t \in [A_i, A_i + L_i], i = 0, 1.$$

When L_0 and $A_1 + L_1$ are close enough (up to further localizing S), we can apply $\mathrm{CD}_{\mathrm{loc}}(K,N)$ to the optimal dynamical plan between μ_0 and μ_1. Then (A.1) follows by the same argument in [7, Theorem 4.2].

□

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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