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A free boundary problem for the *p*-Laplacian with nonlinear boundary conditions

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Abstract
We study a nonlinear generalization of a free boundary problem that arises in the context of thermal insulation. We consider two open sets $\Omega \subseteq A$, and we search for an optimal A in order to minimize a nonlinear energy functional, whose minimizers u satisfy the following conditions: $\Delta_p u = 0$ inside $A\backslash\Omega$, $u = 1$ in Ω, and a nonlinear Robin-like boundary (p, q)-condition on the free boundary ∂A. We study the variational formulation of the problem in SBV, and we prove that, under suitable conditions on the exponents p and q, a minimizer exists and its jump set satisfies uniform density estimates.

Keywords p-Laplacian · Free boundary · Robin · Nonlinear

Mathematics Subject Classification 35A01 · 35J66 · 35J92 · 35R35

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary, and let A be a set containing Ω. Consider the functional

$$F(A, v) = \int_A |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n(A), \tag{1.1}$$

with $v \in H^1(A)$, $v = 1$ in Ω and $\beta, C_0 > 0$ fixed positive constants. The problem of minimizing this functional arises in the environment of thermal insulation: F represents the energy of a heat configuration v when the temperature is maintained constant inside the body Ω and there's a bulk layer $A \setminus \Omega$ of insulating material whose cost is represented by C_0 and the heat transfer with the external environment is conveyed by convection. For simplicity's

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sake in the following we will set $C_0 = 1$. The variational formulation in (1.1) leads to an Euler-Lagrange equation, which is the weak form of the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } A \setminus \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial A, \\ u = 1 & \text{in } \Omega, \end{cases} \tag{1.2}$$

The problems we are interested in concern the existence of a solution and its regularity. In this sense, one could be interested in studying a more general setting in which it is possible to consider possibly irregular sets A. Specifically, we could generalize the problem into the context of SBV functions, aiming to minimize the functional

$$F(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n + \beta \int_{J_v} \left(\underline{v}^2 + \overline{v}^2 \right) \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{v > 0\} \setminus \Omega)$$

with $v \in \mathrm{SBV}(\mathbb{R}^n)$ and $v = 1$ in Ω. This problem has been studied in [7], where the authors have proved the existence of a solution u for the problem and the regularity of its jump set. Similar two-phase problems in the linear case can be found in [1], and [3]. With regards to the nonlinear context, analogous versions of the problem have been addressed in [4], and in [6] with a boundedness constraint.

In this paper, our main aim is to generalize the problem and techniques employed in [7] to a nonlinear formulation. In detail, for $p, q > 1$ fixed, we consider the functional

$$\mathcal{F}(v) = \int_{\mathbb{R}^n} |\nabla v|^p \, \mathrm{d}\mathcal{L}^n + \beta \int_{J_v} \left(\underline{v}^q + \overline{v}^q \right) \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{v > 0\} \setminus \Omega), \tag{1.3}$$

and in the following we are going to study the problem

$$\inf \left\{ \mathcal{F}(v) \,\middle|\, \begin{array}{l} v \in \mathrm{SBV}(\mathbb{R}^n) \\ v(x) = 1 \text{ in } \Omega \end{array} \right\}.$$

Notice that if $v \in \mathrm{SBV}(\mathbb{R}^n)$ with $v = 1$ a.e. in Ω, letting $v_0 = \max\{0, \min\{v, 1\}\}$ we have that $v_0 \in \mathrm{SBV}(\mathbb{R}^n)$ with $v_0 = 1$ a.e. in Ω and $\mathcal{F}(v_0) \leq \mathcal{F}(v)$ so it suffices to consider the problem

$$\inf \left\{ \mathcal{F}(v) \,\middle|\, \begin{array}{l} v \in \mathrm{SBV}(\mathbb{R}^n), \\ v(x) \in [0, 1] \, \mathcal{L}^n\text{-a.e.}, \\ v(x) = 1 \text{ in } \Omega \end{array} \right\}. \tag{1.4}$$

In a more regular setting, problem (1.4) can be seen as a PDE. Let us fix Ω, A sufficiently smooth open sets, $u \in W^{1,p}(A)$ with $u = 1$ on Ω, and let us define the functional

$$F(u, A) = \int_\Omega |\nabla u|^p \, \mathrm{d}\mathcal{L}^n + \beta \int_{\partial\Omega} |u|^q \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(A \setminus \Omega). \tag{1.5}$$

minimizers u to (1.5) solve the following boundary value problem

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = 0 & \text{in } A \setminus \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta \frac{q}{p} |u|^{q-2} u = 0 & \text{on } \partial A, \\ u = 1 & \text{in } \Omega. \end{cases} \tag{1.6}$$

In Sect. 2 we give some preliminary tools and definitions, and then we will prove the existence of a minimizer u of (1.4), under a prescribed condition on p and q. Finally, we will prove density estimates for the jump set J_u.

We resume in the following theorems the main results of this paper.

Theorem 1.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $p, q > 1$ be exponents satisfying one of the following conditions:*

- $1 < p < n$, *and* $1 < q < \dfrac{p(n-1)}{n-p} := p_*$*;*
- $n \le p < \infty$, *and* $1 < q < \infty$.

Then there exists a solution u to problem (1.4) and there exists a constant $\delta_0 = \delta_0(\Omega, \beta, p, q) > 0$ such that

$$u > \delta_0 \tag{1.7}$$

$\mathcal{L}^n$-almost everywhere in $\{u > 0\}$, and there exists $\rho(\delta_0) > 0$ such that

$$\operatorname{supp} u \subseteq B_{\rho(\delta_0)}.$$

Theorem 1.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $p, q > 1$ be exponents satisfying the assumptions of Theorem 1.1. Then there exist positive constants $C(\Omega, \beta, p, q)$, $c(\Omega, \beta, p, q)$, $C_1(\Omega, \beta, p, q)$ such that if u is a minimizer to problem (1.4), then*

$$c\, r^{n-1} \le \mathcal{H}^{n-1}(J_u \cap B_r(x)) \le C\, r^{n-1},$$

and

$$\mathcal{L}^n(B_r(x) \cap \{u > 0\}) \ge C_1\, r^n,$$

for every $x \in \overline{J_u}$ with $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$.

In particular, this implies the essential closedness of the jump set J_u outside of Ω, namely

$$\mathcal{H}^{n-1}((\overline{J_u} \setminus J_u) \setminus \bar{\Omega}) = 0.$$

In Sect. 3 we prove that the a priori estimate (1.7) holds for inward minimizers (see Definition 3.1), such an estimate will be crucial in the proof of Theorem 1.1 in Sect. 4. Finally, in Sect. 5 we prove Theorem 1.2.

Remark 1.3 Notice that the condition on the exponents is undoubtedly verified when $p \ge q > 1$. Furthermore, if Ω is a set with Lipschitz boundary, the exponent p_* is the optimal exponent such that

$$W^{1,p}(\Omega) \subset\subset L^q(\partial\Omega) \qquad \forall q \in [1, p_*).$$

2 Notation and tools

In this section, we give the definition of the space SBV, and some useful notations and results that we will use in the following sections. We refer to [2, 5, 9] for a more intensive study of these topics.

Definition 2.1 (BV) Let $u \in L^1(\mathbb{R}^n)$. We say that u is a function of *bounded variation* in $\mathbb{R}^n$ and we write $u \in \mathrm{BV}(\mathbb{R}^n)$ if its distributional derivative is a Radon measure, namely

$$\int_\Omega u\, \frac{\partial \varphi}{\partial x_i} = \int_\Omega \varphi \, \mathrm{d}D_i u \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n),$$

with Du a $\mathbb{R}^n$-valued measure in $\mathbb{R}^n$. We denote with $|Du|$ the total variation of the measure Du. The space $\mathrm{BV}(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|u\|_{\mathrm{BV}(\mathbb{R}^n)} = \|u\|_{L^1(\mathbb{R}^n)} + |Du|(\mathbb{R}^n).$$

Definition 2.2 Let $E \subseteq \mathbb{R}^n$ be a measurable set. We define the *set of points of density 1 for* E as

$$E^{(1)} = \left\{ x \in \mathbb{R}^n \,\middle|\, \lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} = 1 \right\},$$

and the *set of points of density 0 for* E as

$$E^{(0)} = \left\{ x \in \mathbb{R}^n \,\middle|\, \lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} = 0 \right\}.$$

Moreover, we define the *essential boundary* of E as

$$\partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

Definition 2.3 (*Approximate upper and lower limits*) Let $u \colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We define the *approximate upper and lower limits* of u, respectively, as

$$\overline{u}(x) = \inf \left\{ t \in \mathbb{R} \,\middle|\, \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap \{u > t\})}{\mathcal{L}^n(B_r(x))} = 0 \right\},$$

and

$$\underline{u}(x) = \sup \left\{ t \in \mathbb{R} \,\middle|\, \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap \{u < t\})}{\mathcal{L}^n(B_r(x))} = 0 \right\}.$$

We define the *jump set* of u as

$$J_u = \left\{ x \in \mathbb{R}^n | \underline{u}(x) < \overline{u}(x) \right\}.$$

We denote by K_u the closure of J_u.

If $\overline{u}(x) = \underline{u}(x) = l$, we say that l is the approximate limit of u as y tends to x, and we have that, for any $\varepsilon > 0$,

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap \{|u-l| \geq \varepsilon)\}}{\mathcal{L}^n(B_r(x))} = 0.$$

If $u \in \mathrm{BV}(\mathbb{R}^n)$, the jump set J_u is a $(n-1)$-rectifiable set, i.e., $J_u \subseteq \bigcup_{i\in\mathbb{N}} M_i$, up to a $\mathcal{H}^{n-1}$-negligible set, with M_i a C^1-hypersurface in $\mathbb{R}^n$ for every i. We can then define $\mathcal{H}^{n-1}$-almost everywhere on J_u a normal ν_u coinciding with the normal to the hypersurfaces M_i. Furthermore, the direction of $\nu_u(x)$ is chosen in such a way that the approximate upper and lower limits of u coincide with the approximate limit of u on the half-planes

$$H^+_{\nu_u} = \{y \in \mathbb{R}^n | \nu_u(x)\cdot(y-x) \geq 0\}$$

and

$$H^-_{\nu_u} = \{y \in \mathbb{R}^n | \nu_u(x)\cdot(y-x) \leq 0\}$$

, respectively.

Definition 2.4 Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $E \subseteq \mathbb{R}^n$ a measurable set. We define the *relative perimeter* of E inside Ω as

$$P(E;\Omega) = \sup\left\{ \int_E \operatorname{div}\varphi \, \mathrm{d}\mathcal{L}^n \;\middle|\; \begin{array}{c} \varphi \in C_c^1(\Omega,\mathbb{R}^n) \\ |\varphi| \le 1 \end{array} \right\}.$$

If $P(E;\mathbb{R}^n) < +\infty$ we say that E is a *set of finite perimeter*.

Theorem 2.5 (Decomposition of BV functions) *Let $u \in \mathrm{BV}(\mathbb{R}^n)$. Then we have*

$$\mathrm{d}Du = \nabla u \, \mathrm{d}\mathcal{L}^n + |\overline{u} - \underline{u}|\nu_u \, \mathrm{d}\mathcal{H}^{n-1}\lfloor_{J_u} + \mathrm{d}D^c u,$$

where ∇u is the density of Du with respect to the Lebesgue measure, ν_u is the normal to the jump set J_u and $D^c u$ is the Cantor part *of the measure Du. The measure $D^c u$ is singular with respect to the Lebesgue measure and concentrated out of J_u.*

Definition 2.6 Let $v \in \mathrm{BV}(\mathbb{R}^n)$, let $\Gamma \subseteq \mathbb{R}^n$ be a $\mathcal{H}^{n-1}$-rectifiable set, and let $\nu(x)$ be the generalized normal to Γ defined for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. For $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ we define the traces $\gamma_\Gamma^\pm(v)(x)$ of v on Γ by the following Lebesgue-type limit quotient relation

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r^\pm(x)} |v(y) - \gamma_\Gamma^\pm(v)(x)| \, \mathrm{d}\mathcal{L}^n(y) = 0,$$

where

$$B_r^+(x) = \{y \in B_r(x) | \nu(x)\cdot(y-x) > 0\},$$
$$B_r^-(x) = \{y \in B_r(x) | \nu(x)\cdot(y-x) < 0\}.$$

Remark 2.7 Notice that, by [2, Remark 3.79], for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, $(\gamma_\Gamma^+(v)(x), \gamma_\Gamma^-(v)(x))$ coincides with either $(\overline{v}(x), \underline{v}(x))$ or $(\underline{v}(x), \overline{v}(x))$, while, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus J_v$, we have that $\gamma_\Gamma^+(v)(x) = \gamma_\Gamma^-(v)(x)$ and they coincide with the approximate limit of v in x. In particular, if $\Gamma = J_v$, we have

$$\gamma_{J_v}^+(v)(x) = \overline{v}(x) \qquad \gamma_{J_v}^-(v)(x) = \underline{v}(x)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in J_v$.

We now focus our attention on the BV functions whose Cantor parts vanish.

Definition 2.8 (*SBV*) Let $u \in \mathrm{BV}(\mathbb{R}^n)$. We say that u is a *special function of bounded variation* and we write $u \in \mathrm{SBV}(\mathbb{R}^n)$ if $D^c u = 0$.

For SBV functions we have the following.

Theorem 2.9 (Chain rule) *Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then if $u \in \mathrm{SBV}(\mathbb{R}^n)$, we have*

$$\nabla g(u) = g'(u)\nabla u.$$

Furthermore, if g is increasing,

$$\overline{g(u)} = g(\overline{u}), \quad \underline{g(u)} = g(\underline{u})$$

while, if g is decreasing,

$$\overline{g(u)} = g(\underline{u}), \quad \underline{g(u)} = g(\overline{u}).$$

We now state a compactness theorem in SBV that will be useful in the following.

Theorem 2.10 (Compactness in SBV) *Let u_k be a sequence in* $\mathrm{SBV}(\mathbb{R}^n)$. *Let $p, q > 1$, and let $C > 0$ such that for every $k \in \mathbb{N}$*

$$\int_{\mathbb{R}^n} |\nabla u_k|^p \, d\mathcal{L}^n + \|u_k\|_\infty + \mathcal{H}^{n-1}(J_{u_k}) < C.$$

Then there exists $u \in \mathrm{SBV}(\mathbb{R}^n)$ and a subsequence u_{k_j} such that

- Compactness:

$$u_{k_j} \xrightarrow{L^1_{\mathrm{loc}}(\mathbb{R}^n)} u$$

- Lower semicontinuity: *for every open set A we have*

$$\int_A |\nabla u|^p \, d\mathcal{L}^n \leq \liminf_{j \to +\infty} \int_A |\nabla u_{k_j}|^p \, d\mathcal{L}^n$$

  and

$$\int_{J_u \cap A} \left(\overline{u}^q + \underline{u}^q\right) \, d\mathcal{H}^{n-1} \leq \liminf_{j \to +\infty} \int_{J_{u_{k_j}} \cap A} \left(\overline{u}_{k_j}^q + \underline{u}_{k_j}^q\right) \, d\mathcal{H}^{n-1}$$

We refer to [2, Theorem 4.7, Theorem 4.8, Theorem 5.22] for the proof of this theorem. We now conclude this section with the following proposition whose proof can be found in [7, Lemma 3.1].

Proposition 2.11 *Let $u \in \mathrm{BV}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\int_0^1 P(\{u > s\}; \mathbb{R}^n \setminus J_u) \, ds = |Du|(\mathbb{R}^n \setminus J_u).$$

3 Lower bound

In the following, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and that p and q are two positive real numbers such that

$$\frac{q'}{p'} > 1 - \frac{1}{n} \tag{3.1}$$

where p' and q' are the Hölder conjugates of p and q, respectively.

Definition 3.1 Let $v \in \mathrm{SBV}(\mathbb{R}^n)$ be a function such that $v = 1$ a.e. in Ω. We say that v is an *inward minimizer* if

$$\mathcal{F}(v) \leq \mathcal{F}(v\chi_A),$$

for every set of finite perimeter A containing Ω, where χ_A is the characteristic function of set A.

Let $A \subset \mathbb{R}^n$ be a set of finite perimeter such that $\Omega \subset A$, and let $v \in \mathrm{SBV}(\mathbb{R}^n)$. We will make use of the following expression

$$\begin{aligned}\mathcal{F}(v\chi_A) = &\int_A |\nabla v|^p \,\mathrm{d}\mathcal{L}^n + \beta \int_{J_v \cap A^{(1)}} \left(\underline{v}^q + \overline{v}^q\right) \mathrm{d}\mathcal{H}^{n-1} + \beta \int_{\partial^* A \setminus J_v} v^q \,\mathrm{d}\mathcal{H}^{n-1} \\ &+ \beta \int_{J_v \cap \partial^* A} \gamma^-_{\partial A}(v)^q \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n\left(\left(\{v > 0\} \cap A\right) \setminus \Omega\right),\end{aligned} \tag{3.2}$$

Let B be a ball containing Ω, then $\chi_B \in \mathrm{SBV}(\mathbb{R}^n)$ and $\chi_B = 1$ in Ω, we will denote $\mathcal{F}(\chi_B)$ by $\tilde{\mathcal{F}}$.

Theorem 3.2 *There exists a positive constant $\delta = \delta(\Omega, \beta, p, q)$ such that if u is an inward minimizer with $\mathcal{F}(u) \leq 2\tilde{\mathcal{F}}$, then*

$$u > \delta$$

$\mathcal{L}^n$-almost everywhere in $\{u > 0\}$.

Proof Let $0 < t < 1$ and

$$f(t) = \int_{\{u \leq t\} \setminus J_u} u^{q-1} |\nabla u| \,\mathrm{d}\mathcal{L}^n = \int_0^t s^{q-1} P(\{u > s\}\,; \mathbb{R}^n \setminus J_u) \,\mathrm{d}s.$$

For every such t, we have

$$f(t) \leq \left(\int_{\{u \leq t\}} u^{(q-1)p'} \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p'}} \left(\int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}} \leq \mathcal{F}(u) \leq 2\tilde{\mathcal{F}}. \tag{3.3}$$

Let $u_t = u\chi_{\{u > t\}}$. Using (3.2) we have

$$\begin{aligned} 0 \leq\ & \mathcal{F}(u_t) - \mathcal{F}(u) \\ =\ & \beta \int_{\partial^* \{u > t\} \setminus J_u} \overline{u}^q \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \,\mathrm{d}\mathcal{L}^n - \beta \int_{J_u \cap \partial^* \{u > t\}} \underline{u}^q \,\mathrm{d}\mathcal{H}^{n-1} \\ & - \beta \int_{J_u \cap \{u > t\}^{(0)}} \left(\overline{u}^q + \underline{u}^q\right) \mathrm{d}\mathcal{H}^{n-1} - \mathcal{L}^n(\{0 < u \leq t\}), \end{aligned}$$

and rearranging the terms,

$$\begin{aligned}&\int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \,\mathrm{d}\mathcal{L}^n + \beta \int_{J_u \cap \partial^* \{u > t\}} \underline{u}^q \,\mathrm{d}\mathcal{H}^{n-1} + \beta \int_{J_u \cap \{u > t\}^{(0)}} \left(\overline{u}^q + \underline{u}^q\right) \mathrm{d}\mathcal{H}^{n-1} \\ &+ \mathcal{L}^n(\{0 < u \leq t\}) \leq \beta t^q P(\{u > t\}; \mathbb{R}^n \setminus J_u) = \beta t f'(t).\end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned} f(t) &= \int_{\{u \leq t\} \setminus J_u} u^{q-1} |\nabla u| \,\mathrm{d}\mathcal{L}^n \\ &\leq \left(\int_{\{u \leq t\}} u^{(q-1)p'} \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p'}} \left(\int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}} \\ &\leq \left(\mathcal{L}^n(\{0 < u \leq t\})\right)^{\frac{1}{p'\gamma'}} \left(\int_{\{u \leq t\}} u^{q1^*} \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{q'1^*}} \left(\int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}, \end{aligned}$$

where we used

$$1^* = \frac{n}{n-1}, \qquad \text{and} \qquad \gamma = \frac{q1^*}{(q-1)p'},$$

and $\gamma > 1$ by (3.1). By classical BV embedding in L^{1^*} applied to the function $(u\chi_{\{u\le t\}})^q$ and the estimate (3.4), we have

$$f(t) \le C(n,\beta)\Big(tf'(t)\Big)^{1-\frac{n-1}{q'n}} \left(\int_{\mathbb{R}^n} \mathrm{d}\big|D(u\chi_{\{u\le t\}})^q\big|\right)^{\frac{1}{q'}}.$$

We can compute

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{d}\big|D(u\chi_{\{u\le t\}})^q\big| \le q\Big(\mathcal{L}^n(\{0<u\le t\})\Big)^{\frac{1}{p'}} &\left(\int_{\{u\le t\}\setminus J_u} |\nabla u|^p \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}} \\ &+ \int_{J_u\cap\{u>t\}^{(0)}} \big(\overline{u}^q + \underline{u}^q\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{J_u\cap\partial^*\{u>t\}} \underline{u}^q\,\mathrm{d}\mathcal{H}^{n-1} \\ &+ t^q P(\{u>t\};\mathbb{R}^n\setminus J_u) \le (2+q\beta)tf'(t).\end{aligned}$$

We therefore get

$$f(t) \le C(n,\beta,q)\big(tf'(t)\big)^{1+\frac{1}{nq'}}.$$

Let $0<t_0<1$ such that $f(t_0)>0$, then for every $t_0<t<1$, we have $f(t)>0$ and

$$\frac{f'(t)}{f(t)^{\frac{nq}{q(n+1)-1}}} \ge \frac{C(n,\beta,q)}{t},$$

integrating from t_0 to 1, we have

$$f(1)^{\frac{q-1}{q(n+1)-1}} - f(t_0)^{\frac{q-1}{q(n+1)-1}} \ge C(n,\beta,q)\log\frac{1}{t_0},$$

so that, using (3.3),

$$f(t_0)^{\frac{q-1}{q(n+1)-1}} \le (2\tilde{\mathcal{F}})^{\frac{q-1}{q(n+1)-1}} + C(n,\beta,q)\log t_0.$$

Let

$$\delta = \exp\left(-\frac{(2\tilde{\mathcal{F}})^{\frac{q-1}{q(n+1)-1}}}{C(n,\beta,q)}\right),$$

for every $t_0<\delta$ we would have $f(t_0)<0$, which is a contradiction. Therefore $f(t)=0$ for every $t<\delta$, from which $u>\delta$ $\mathcal{L}^n$-almost everywhere on $\{u>0\}$. □

Remark 3.3 From Theorem 3.2, if u is an inward minimizer with $\mathcal{F}(u)\le 2\tilde{\mathcal{F}}$, we have that

$$\partial^*\{u>0\} \subseteq J_u \subseteq K_u.$$

Indeed, on $\partial^*\{u>0\}$ we have that, by definition, $\underline{u}=0$ and that, since $u\ge\delta$ $\mathcal{L}^n$-almost everywhere in $\{u>0\}$, $\overline{u}\ge\delta$.

Proposition 3.4 *There exists a positive constant $\delta_0=\delta_0(\Omega,\beta,p,q)<\delta$ such that if u is an inward minimizer with $\mathcal{F}(u)\le 2\tilde{\mathcal{F}}$, then u is supported on $B_{\rho(\delta_0)}$, where $\rho(\delta_0)=\delta_0^{1-q}$ and $B_{\rho(\delta_0)}$ is the ball centered at the origin with radius $\rho(\delta_0)$. Moreover there exist positive constants $C(\Omega,\beta,p,q)$, $C_1(\Omega,\beta,p,q)$ such that, for any $B_r(x)\subseteq\mathbb{R}^n\setminus\Omega$ we have*

$$\mathcal{H}^{n-1}(J_u\cap B_r(x)) \le C(\Omega,p,q)r^{n-1}, \tag{3.5}$$

and if $x \in K_u$, *then*

$$\mathcal{L}^n(B_r(x) \cap \{u > 0\}) \geq C_1(\Omega, p, q) r^n. \tag{3.6}$$

Proof By Theorem 3.2, if u is an inward minimizer, we have

$$\int_{J_u \cap B_r(x)} \left(\overline{u}^q + \underline{u}^q\right) \, \mathrm{d}\mathcal{H}^{n-1} \geq \delta^q \mathcal{H}^{n-1}(J_u \cap B_r(x)),$$

on the other hand, using $u\chi_{\mathbb{R}^n \setminus B_r(x)}$ as a competitor for u, we have

$$\int_{J_u \cap B_r(x)} \left(\overline{u}^q + \underline{u}^q\right) \, \mathrm{d}\mathcal{H}^{n-1} \leq \int_{\partial B_r(x) \cap \{u>0\}^{(1)}} \left(\overline{u}^q + \underline{u}^q\right) \, \mathrm{d}\mathcal{H}^{n-1} \leq C(n) r^{n-1}.$$

Let now $x \in K_u$ and consider $\mu(r) = \mathcal{L}^n\left(B_r(x) \cap \{u > 0\}^{(1)}\right)$. Using the isoperimetric inequality and inequality (3.5), we have that for almost every $r \in (0, d(x, \Omega))$

$$\begin{aligned} 0 < \mu(r) &\leq K(n)\, P\Big(B_r(x) \cap \{u > 0\}^{(1)}\Big)^{\frac{n}{n-1}} \\ &\leq K(\Omega, \beta, p, q)\, P\Big(B_r(x); \{u > 0\}^{(1)}\Big)^{\frac{n}{n-1}}. \end{aligned}$$

Notice that we used Remark 3.3 in the last inequality. We have

$$\mu(r) \leq K \mu'(r)^{\frac{n}{n-1}}.$$

Integrating the differential inequality, we obtain

$$\mathcal{L}^n(B_r(x) \cap \{u > 0\}) \geq C_1(\Omega, \beta, p, q) r^n.$$

Finally, let $\delta_0 > 0$ and $x \in K_u$ such that $d(x, \Omega) > \rho(\delta_0) = \delta_0^{1-q}$. By (3.6)

$$C_1(\Omega, \beta, p, q) \rho(\delta_0)^n \leq \mathcal{L}^n(\{u > 0\} \cap \Omega) \leq 2\tilde{\mathcal{F}},$$

which leads to a contradiction if δ_0 is too small, hence there exists a positive value $\delta_0(\Omega, \beta, p, q)$ such that $\{u > 0\} \subset B_{\rho(\delta_0)}$. □

4 Existence

In this section, we are going to prove the existence of a solution u to the problem (1.4). Let us denote

$$H_a = \left\{ u \in \mathrm{SBV}(\mathbb{R}^n) \;\middle|\; \begin{array}{l} u(x) = 1 \text{ in } \Omega \\ u(x) \in \{0\} \cup [a, 1] \ \mathcal{L}^n\text{ -a.e.} \\ \operatorname{supp} u \subseteq B_{\frac{1}{a^{q-1}}} \end{array} \right\}.$$

We also denote by H_0 the set

$$H_0 = \left\{ u \in \mathrm{SBV}(\mathbb{R}^n) \;\middle|\; \begin{array}{l} u(x) = 1 \text{ in } \Omega \\ u(x) \in [0, 1] \mathcal{L}^n\text{-a.e.} \end{array} \right\}.$$

Notice that if $u \in H_0$ is an inward minimizer, by Theorem 3.2 and Corollary 3.4, then $u \in H_{\delta_0}$.

Proposition 4.1 *Let $u \in H_0$. Then u is a minimizer for the functional (1.3) on H_0 if and only if $u \in H_{\delta_0}$ and*

$$\mathcal{F}(u) \leq \mathcal{F}(v) \qquad \forall v \in H_{\delta_0}.$$

Proof As we observed before, if u is a minimizer over H_0 then u is in H_{δ_0}, hence it is a minimizer over H_{δ_0}. Conversely, let us take $u \in H_{\delta_0}$ a minimizer over H_{δ_0}, and let us consider in addition $v \in H_0$. Without loss of generality assume $\mathcal{F}(v) \leq 2\tilde{\mathcal{F}}$. We will prove that there exists a sequence w_k of inward minimizers such that

$$\mathcal{F}(w_k) \leq \mathcal{F}(v) + \frac{C}{k^{q-1}}.$$

We first construct a family of functions $v_a \in H_a$ such that

$$\mathcal{F}(v_a) \leq \mathcal{F}(v) + r(a),$$

with $\lim_{a\to 0} r(a) = 0$. Let $0 < a < 1$, and let $v_a = v\chi_{\{v\geq a\}\cap B_{\rho(a)}}$, where $\rho(a) = a^{1-q}$, we have

$$\begin{aligned} \mathcal{F}(v_a) - \mathcal{F}(v) &\leq \beta \int_{\partial^*(\{v\geq a\}\cap B_{\rho(a)})\setminus J_v} v^q \, \mathrm{d}\mathcal{H}^{n-1} \\ &\leq \beta a^q P(\{v \geq a\}) + \beta \int_{(\partial B_{\rho(a)}\cap\{v\geq a\})\setminus J_v} v^q \, \mathrm{d}\mathcal{H}^{n-1} \\ &\leq \beta a^q \left(P(\{v \geq a\}) + \frac{1}{a^q} \int_{(\partial B_{\rho(a)}\cap\{v\geq a\})\setminus J_v} v \, \mathrm{d}\mathcal{H}^{n-1} \right). \end{aligned} \tag{4.1}$$

In order to estimate the right-hand side, fix $t \in (0, 1)$, and observe that by the coarea formula

$$\int_0^t P(\{v \geq a\}) \, da \leq |Dv|(\mathbb{R}^n), \tag{4.2}$$

while, with a change of variables,

$$\begin{aligned} &\int_0^t \frac{1}{a^q} \int_{(\partial B_{\rho(a)}\cap\{v\geq a\})\setminus J_v} v \, \mathrm{d}\mathcal{H}^{n-1} \, da \leq (q-1) \int_0^{+\infty} \int_{\partial B_r \setminus J_v} v \, \mathrm{d}\mathcal{H}^{n-1} \, dr \\ &\quad = (q-1)\|v\|_{L^1(\mathbb{R}^n)}. \\ &\int_0^t \left(P(\{v \geq a\}) + \frac{1}{a^q} \int_{(\partial B_{\rho(a)}\cap\{v\geq a\})\setminus J_v} v \, \mathrm{d}\mathcal{H}^{n-1} \right) da \leq q\|v\|_{\mathrm{BV}}. \end{aligned}$$

By mean value theorem, for every $k \in \mathbb{N}$ we can find $a_k \leq 1/k$ such that

$$P(\{v \geq a_k\}) + \frac{1}{a_k^q} \int_{(\partial B_{\rho(a_k)}\cap\{v\geq a_k\})\setminus J_v} v \, \mathrm{d}\mathcal{H}^{n-1} \leq \frac{q\|v\|_{\mathrm{BV}}}{a_k},$$

and in (4.1) we get

$$\mathcal{F}(v_{a_k}) \leq \mathcal{F}(v) + q\beta a_k^{q-1}\|v\|_{\mathrm{BV}} \leq \mathcal{F}(v) + q\beta \frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

We now construct the aforementioned sequence of inward minimizers: let us consider the functional

$$\mathcal{G}_k(A) = \mathcal{F}(v_{a_k}\chi_A),$$

with A containing Ω and contained in $\{v_{a_k} > 0\}$. If we consider A_j a minimizing sequence for $\mathcal{G}_k$, then they are certainly equibounded. Moreover,

$$\begin{aligned}\mathcal{G}_k(A_j) &\geq \mathcal{L}^n(A_j \setminus \Omega) + \beta \int_{J_{\chi_{A_j} v_{a_k}}} \left(\underline{\chi_{A_j} v_{a_k}}^q + \overline{\chi_{A_j} v_{a_k}}^q \right) \mathrm{d}\mathcal{H}^{n-1} \\ &\geq \mathcal{L}^n(A_j) + \beta a_k^q \mathcal{H}^{n-1}(J_{\chi_{A_j} v_{a_k}}) - \mathcal{L}^n(\Omega).\end{aligned}$$

Notice in addition that since $v_{a_k} \geq a_k$ on its support, then the jump set $J_{\chi_{A_j} v_{a_k}}$ clearly contains $\partial^* A_j$. We now have that χ_{A_j} satisfies the conditions of Theorem 2.10, and eventually extracting a subsequence we can suppose that

$$A_j \xrightarrow{L^1} A^{(k)},$$

with a suitable $A^{(k)}$, and moreover, letting $w_k = \chi_{A^{(k)}} v_{a_k}$, we have

$$\mathcal{F}(w_k) \leq \inf_{\Omega \subseteq A \subseteq \{v_{a_k} > 0\}} \mathcal{G}_k(A) \leq \mathcal{F}(v_{a_k}) \leq \mathcal{F}(v) + q\beta \frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

By construction w_k is an inward minimizer, therefore we have $w_k \in H_{\delta_0}$, and consequently, we can compare it with u, obtaining

$$\mathcal{F}(u) \leq \mathcal{F}(w_k) \leq \mathcal{F}(v) + q\beta \frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

Letting k go to infinity we get the thesis. □

Proposition 4.2 *There exists a minimizer for problem* (1.4).

Proof By Proposition 4.1 and Theorem 3.2 it is enough to find a minimizer in H_{δ_0}. Let u_k be a minimizing sequence in H_{δ_0}, then, for k large enough, we have

$$\beta \delta_0^q \mathcal{H}^{n-1}(J_{u_k}) + \int_{\mathbb{R}^n} |\nabla u_k|^p \,\mathrm{d}\mathcal{L}^n \leq \mathcal{F}(u_k) \leq 2\tilde{\mathcal{F}}.$$

From Theorem 2.10 we have that there exists $u \in H_{\delta_0}$ such that, up to a subsequence, u_k converges to u in L^1_{loc} and

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k),$$

therefore u is a solution. □

Proof of Theorem 1.1 The result is obtained by joining Theorems 4.2 and 3.2. □

5 Density estimates

In this section, we prove the density estimates in Theorem 1.2 by adapting techniques used in [7] analogous to classical ones used in [8] to prove density estimates for the jump set of almost-quasi minimizers of the Mumford–Shah functional.

Definition 5.1 Let $u \in \mathrm{SBV}(\mathbb{R}^n)$ be a function such that $u = 1$ a.e. in Ω. We say that u is a *local minimizer* for $\mathcal{F}$ on a set of finite perimeter $E \subset \mathbb{R}^n \setminus \Omega$, if

$$\mathcal{F}(u) \leq \mathcal{F}(v),$$

for every $v \in \mathrm{SBV}(\mathbb{R}^n)$ such that $u - v$ has support in E.

Let E be a set of finite perimeter. We introduce the notation

$$\mathcal{F}(u;E)=\int_E|\nabla u|^p\,\mathrm{d}\mathcal{L}^n+\beta\int_{J_u\cap E}\left(\overline{u}^q+\underline{u}^q\right)\,\mathrm{d}\mathcal{H}^{n-1}+\mathcal{L}^n\left(\{u>0\}\cap E\right).$$

To prove Theorem 1.2 we will use the following Poincaré-Wirtinger type inequality whose proof can be found in [8, Theorem 3.1 and Remark 3.3]. Let γ_n be the isoperimetric constant relative to the balls of $\mathbb{R}^n$, i.e.,

$$\min\left\{\mathcal{L}^n(E\cap B_r)^{\frac{n-1}{n}},\mathcal{L}^n(E\setminus B_r)^{\frac{n-1}{n}}\right\}\le\gamma_n P(E;B_r),$$

for every Borel set E, then

Proposition 5.2 *Let $p\ge 1$ and let $u\in\mathrm{SBV}(B_r)$ such that*

$$\left(2\gamma_n\mathcal{H}^{n-1}(J_u\cap B_r)\right)^{\frac{n}{n-1}}<\frac{\mathcal{L}^n(B_r)}{2}. \tag{5.1}$$

Then there exist numbers $-\infty<\tau^-\le m\le\tau^+<+\infty$ such that the function

$$\tilde{u}=\max\{\min\{u,\tau^+\},\tau^-\},$$

satisfies

$$\|\tilde{u}-m\|_{L^p}\le C\|\nabla u\|_{L^p}$$

and

$$\mathcal{L}^n(\{u\ne\tilde{u}\})\le C\left(\mathcal{H}^{n-1}(J_u\cap B_r)\right)^{\frac{n}{n-1}},$$

where the constants depend only on n, p, and r.

Lemma 5.3 *Let $u\in H_s$ be a local minimizer on $B_r(x)$ in the sense of definition Definition 5.1. For sufficiently small values of τ, there exist values r_0,ε_0 depending only on n,τ,β,p,q and s such that, if $r<r_0$,*

$$\mathcal{H}^{n-1}(J_u\cap B_r(x))\le\varepsilon_0 r^{n-1},$$

and

$$\mathcal{F}(u;B_r(x))\ge r^{n-\frac{1}{2}},$$

then

$$\mathcal{F}(u;B_{\tau r}(x))\le\tau^{n-\frac{1}{2}}\mathcal{F}(u;B_r(x)).$$

Proof Without loss of generality, assume $x=0$. Assume by contradiction that the conclusion fails, then for every $\tau>0$ there exists a sequence $u_k\in H_s$ of local minimizers on B_{r_k}, with $\lim_k r_k=0$, such that

$$\frac{\mathcal{H}^{n-1}(J_{u_k}\cap B_{r_k})}{r_k^{n-1}}=\varepsilon_k,$$

with $\lim_k\varepsilon_k=0$,

$$\mathcal{F}(u_k;B_{r_k})\ge r_k^{n-\frac{1}{2}}, \tag{5.2}$$

and yet

$$\mathcal{F}(u_k; B_{\tau r_k}) > \tau^{n-\frac{1}{2}} \mathcal{F}(u_k; B_{r_k}). \tag{5.3}$$

For every $t \in [0, 1]$, we define the sequence of monotone functions

$$\alpha_k(t) = \frac{\mathcal{F}(u_k; B_{tr_k})}{\mathcal{F}(u_k, B_{r_k})} \leq 1.$$

By compactness of $\mathrm{BV}([0, 1])$ in $L^1([0, 1])$, we can assume that, up to a subsequence, α_k converges $\mathcal{L}^1$-almost everywhere to a monotone function α. Moreover, notice that, by (5.3), for every k

$$\alpha_k(\tau) > \tau^{n-\frac{1}{2}}. \tag{5.4}$$

Our final aim is to prove that there exists a p-harmonic function $v \in W^{1,p}(B_1)$ such that for every t

$$\lim_{k\to+\infty} \alpha_k(t) = \alpha(t) = \int_{B_t} |\nabla v|^p \, \mathrm{d}\mathcal{L}^n.$$

Let

$$E_k = r_k^{p-n} \mathcal{F}(u_k; B_{r_k}), \qquad v_k(x) = \frac{u_k(r_k x)}{E_k^{1/p}}.$$

Then $v_k \in \mathrm{SBV}(B_1)$, and

$$\int_{B_1} |\nabla v_k|^p \, \mathrm{d}\mathcal{L}^n \leq 1, \qquad \mathcal{H}^{n-1}(J_{v_k} \cap B_1) = \varepsilon_k.$$

Thus, applying the Poincaré–Wirtinger type inequality in Proposition 5.2 to functions v_k we obtain truncated functions $\tilde{v}_k$ and values m_k, such that

$$\int_{B_1} |\tilde{v}_k - m_k|^p \, \mathrm{d}\mathcal{L}^n \leq C$$

and

$$\mathcal{L}^n(\{v_k \neq \tilde{v}_k\}) \leq C \left(\mathcal{H}^{n-1}(J_{v_k} \cap B_1)\right)^{\frac{n}{n-1}} \leq C \varepsilon_k^{\frac{n}{n-1}}. \tag{5.5}$$

Step 1: We prove that there exists $v \in W^{1,p}(B_1)$ such that

$$\begin{aligned} &\tilde{v}_k - m_k \xrightarrow{L^p(B_1)} v, \\ &\int_{B_\rho} |\nabla v|^p \, \mathrm{d}\mathcal{L}^n \leq \alpha(\rho), \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1, \end{aligned} \tag{5.6}$$

and

$$\lim_k \frac{r_k^{p-1}}{E_k} \mathcal{H}^{n-1}(\{v_k \neq \tilde{v}_k\} \cap \partial B_\rho) = 0, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1. \tag{5.7}$$

Notice that

$$\int_{B_1} |\nabla(\tilde{v}_k - m_k)|^p \, \mathrm{d}\mathcal{L}^n \leq \int_{B_1} |\nabla v_k|^p \, \mathrm{d}\mathcal{L}^n \leq 1,$$

since $\tilde{v}_k$ is a truncation of v. From compactness theorems in SBV (see for instance [8, Theorem 3.5]), we have that $\tilde{v}_k - m_k$ converges in $L^p(B_1)$ and $\mathcal{L}^n$-almost everywhere to a function $v \in W^{1,p}(B_1)$, since $\mathcal{H}^{n-1}(J_{\tilde{v}_k})$ goes to 0 as $k \to +\infty$. Moreover, for every $\rho < 1$,

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n,$$

and

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n \leq \liminf_k \alpha_k(\rho) = \alpha(\rho),$$

since by definition

$$\int_{B_\rho} |\nabla v_k|^p \, d\mathcal{L}^n = \frac{r_k^{p-n}}{E_k} \int_{B_{\rho r_k}} |\nabla u_k|^p \, d\mathcal{L}^n \leq \frac{r_k^{p-n}}{E_k} \mathcal{F}(u_k; B_{\rho r_k}) \leq \alpha_k(\rho).$$

Finally, up to a subsequence,

$$\lim_k \frac{r_k^{p-1}}{E_k} \mathcal{L}^n(\{v_k \neq \tilde{v}_k\}) = 0.$$

Indeed, by (5.5),

$$\frac{r_k^{p-1}}{E_k} \mathcal{L}^n(\{v_k \neq \tilde{v}_k\}) \leq C \frac{r_k^{p-1}}{E_k} \varepsilon_k^{\frac{n}{n-1}},$$

which tends to zero as long as r_k^{p-1}/E_k is bounded. On the other hand, if r_k^{p-1}/E_k diverges, we could use the fact that $\varepsilon_k \leq s^{-q} \mathcal{F}(u_k; B_{r_k}) r_k^{1-n}$ and get

$$\frac{r_k^{p-1}}{E_k} \mathcal{L}^n(\{v_k \neq \tilde{v}_k\}) \leq C \frac{r_k^{p-1}}{E_k} \left(\frac{E_k}{r_k^{p-1}} \right)^{\frac{n}{n-1}}$$

which goes to zero. Using Fubini's theorem we have (5.7).
Let $\tilde{u}_k(x) = E_k^{1/p} \tilde{v}_k(\frac{x}{r_k})$, and for every $t \in [0, 1]$ we define

$$\tilde{\alpha}_k(t) = \frac{\mathcal{F}(\tilde{u}_k; B_{t r_k})}{\mathcal{F}(u_k, B_{r_k})}.$$

The $\tilde{\alpha}_k$ functions are also monotone and bounded: the jump set of $\tilde{u}_k$ is contained in J_{u_k}, therefore we can write

$$\tilde{\alpha}_k(t) \leq \alpha_k(t) + \frac{2\beta \mathcal{H}^{n-1}(J_{u_k} \cap B_{t r_k})}{\mathcal{F}(u_k; B_{r_k})} \leq \left(1 + \frac{2}{s^q}\right) \alpha_k(t),$$

using the fact that $u_k \in H_s$. As done for α_k, we can assume that $\tilde{\alpha}_k$ converges $\mathcal{L}^1$-almost everywhere to a function $\tilde{\alpha}$.

Step 2: Let $I \subset [0, 1]$ be the set of values ρ for which (5.7) holds, α_k and $\tilde{\alpha}_k$ converge and α and $\tilde{\alpha}$ are continuous. Notice that $\mathcal{L}^1(I) = 1$. Fix $\rho, \rho' \in I$ with $\rho < \rho' < 1$ and let

$$\mathcal{I}_k(\xi) = \beta E_k^{q/p-1} r_k^{p-1} \int_{J_\xi \cap (B_{\rho'} \setminus B_\rho)} \left(\overline{\xi}^q + \underline{\xi}^q \right) d\mathcal{H}^{n-1},$$

with $\xi \in \mathrm{SBV}(B_1)$. Let $w \in W^{1,p}(B_1)$ and consider η a smooth cutoff function supported on $B_{\rho'}$ and identically equal to 1 in B_ρ. Let

$$\varphi_k = ((w + m_k)\eta + \tilde{v}_k(1-\eta))\chi_{B_{\rho'}} + v_k \chi_{B_1 \setminus B_{\rho'}}.$$

We want to prove that

$$\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) \geq \int_{B_{\rho'} \setminus B_\rho} |\nabla \tilde{v}_k|^p \, \mathrm{d}\mathcal{L}^n + \mathcal{I}_k(\tilde{v}_k), \tag{5.8}$$

and

$$\alpha_k(\rho') \leq R_k + \int_{B_{\rho'}} |\nabla \varphi_k|^p \, \mathrm{d}\mathcal{L}^n + \mathcal{I}_k(\varphi_k), \tag{5.9}$$

where R_k goes to zero as k goes to infinity. We immediately compute

$$\begin{aligned}
\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) = \mathcal{F}(u_k; B_{r_k})^{-1} \Bigg[& \int_{B_{\rho' r_k} \cap B_{\rho r_k}} |\nabla \tilde{u}_k|^p \, \mathrm{d}\mathcal{L}^n \\
& + \beta \int_{J_{\tilde{u}_k} \cap (B_{\rho' r_k} \setminus B_{\rho r_k})} \left(\overline{\tilde{u}_k}^q + \underline{\tilde{u}_k}^q \right) \mathrm{d}\mathcal{H}^{n-1} \Bigg] \\
& + \mathcal{F}(u_k; B_{r_k})^{-1} \mathcal{L}^n(\{\tilde{u}_k > 0\} \cap (B_{\rho' r_k} \setminus B_{\rho r_k})) \\
\geq & \int_{B_{\rho'} \setminus B_\rho} |\nabla \tilde{v}_k|^p \, \mathrm{d}\mathcal{L}^n + E_k^{q/p-1} r_k^{p-1} \beta \int_{J_{\tilde{v}_k} \cap (B_{\rho'} \setminus B_\rho)} \left(\overline{\tilde{v}_k}^q + \underline{\tilde{v}_k}^q \right) \mathrm{d}\mathcal{H}^{n-1},
\end{aligned}$$

and then we have (5.8). Now let $\psi_k = E_k^{1/p} \varphi_k(x/r_k)$ and observe that ψ_k coincides with u_k outside $B_{\rho' r_k}$. We get from the local minimality of u_k that

$$\begin{aligned}
\mathcal{F}(u_k; B_{r_k}) \leq \mathcal{F}(\psi_k; B_{r_k}) = \mathcal{F}(\psi_k; B_{\rho' r_k}) + \beta & \int_{\{u_k \neq \tilde{u}_k\} \cap \partial B_{\rho' r_k}} \left(\underline{\psi_k}^q + \overline{\psi_k}^q \right) \mathrm{d}\mathcal{H}^{n-1} \\
& + \mathcal{F}(u_k; B_{r_k} \setminus \overline{B_{\rho' r_k}}) \\
& \leq \mathcal{F}(\psi_k; B_{\rho' r_k}) + 2\beta r_k^{n-1} \mathcal{H}^{n-1}(\{v_k \neq \tilde{v}_k\} \cap \partial B_{\rho'}) \\
& \quad + \mathcal{F}(u_k; B_{r_k} \setminus \overline{B_{\rho' r_k}}).
\end{aligned} \tag{5.10}$$

So, in particular, we have

$$\begin{aligned}
\mathcal{F}(u_k; B_{\rho' r_k}) &= \mathcal{F}(u_k; B_{r_k}) - \mathcal{F}(u_k; B_{r_k} \setminus \overline{B_{\rho' rfcfc_k}}) - \beta \int_{J_{u_k} \cap \partial B_{\rho' r_k}} \left(\overline{u_k}^q + \underline{u_k}^q \right) \mathrm{d}\mathcal{H}^{n-1} \\
&\leq 2\beta r_k^{n-1} \mathcal{H}^{n-1}(\{v_k \neq \tilde{v}_k\} \cap \partial B_{\rho'}) + \mathcal{F}(\psi_k; B_{\rho' r_k}).
\end{aligned}$$

Dividing by $\mathcal{F}(u_k; B_{r_k})$ and using (5.7) we get

$$\alpha_k(\rho') \leq R_k + r_k^{p-n} E_k^{-1} \mathcal{F}(\psi_k; B_{\rho' r_k}).$$

With appropriate rescalings we have

$$r_k^{p-n} E_k^{-1} \mathcal{F}(\psi_k; B_{\rho' r_k}) = \int_{B_{\rho'}} |\nabla \varphi_k|^p \, \mathrm{d}\mathcal{L}^n + \mathcal{I}_k(\varphi_k) + r_k^p E_k^{-1} \mathcal{L}^n(\{\varphi_k > 0\} \cap B_{\rho'}).$$

From (5.2) and the definition of E_k, we have

$$r_k^p E_k^{-1} \mathcal{L}^n(\{\varphi_k > 0\} \cap B_{\rho'}) \leq \omega_n r_k^{1/2},$$

and then we get (5.9).

Step 3: We want to prove that for every $\varphi \in W^{1,p}(B_1)$ such that $v - \varphi$ is supported on B_ρ, we have

$$\alpha(\rho') \leq \int_{B_\rho} |\nabla\varphi|^p \, \mathrm{d}\mathcal{L}^n + C\left[\tilde{\alpha}(\rho') - \tilde{\alpha}(\rho)\right] + C\int_{B_{\rho'}\setminus B_\rho} |\nabla\varphi|^p \, \mathrm{d}\mathcal{L}^n, \tag{5.11}$$

where C does not depend on either ρ or ρ'. From the definition of φ_k, we have that on B_ρ

$$\nabla\varphi_k = \nabla w$$

and on $B_{\rho'} \setminus B_\rho$

$$\nabla\varphi_k = \eta\nabla w + (w + m_k - \tilde{v}_k)\nabla\eta + \nabla\tilde{v}_k(1-\eta),$$

so that

$$\begin{aligned}\int_{B_{\rho'}} |\nabla\varphi_k|^p \, \mathrm{d}\mathcal{L}^n &\leq \int_{B_\rho} |\nabla w|^p \, \mathrm{d}\mathcal{L}^n \\ &\quad + C\left[\int_{B_{\rho'}\setminus B_\rho} |\nabla\tilde{v}_k|^p \, \mathrm{d}\mathcal{L}^n + \int_{B_{\rho'}\setminus B_\rho} (|\nabla w|^p + |w + m_k - \tilde{v}_k|^p |\nabla\eta|^p) \, \mathrm{d}\mathcal{L}^n\right].\end{aligned} \tag{5.12}$$

We split the proof into two cases: either

$$\limsup_k E_k > 0 \tag{5.13}$$

or

$$\lim_k E_k = 0. \tag{5.14}$$

Assume (5.13) occurs. Notice that $s \leq u_k \leq 1$ for every k, then by definition we have that, for every k, $s \leq E_k^{1/p}\tilde{v}_k \leq 1$ and, since m_k is a median of v_k, $0 \leq E_k^{1/p} m_k \leq 1$. In particular we have that

$$|\tilde{v}_k - m_k| \leq \frac{2}{E_k^{1/p}},$$

passing to the limit when k goes to infinity we have that

$$\|v\|_\infty \leq \liminf_k \frac{2}{E_k^{1/p}} < +\infty \quad \mathcal{L}^n\text{-a.e.}$$

then v belongs to $L^\infty(B_1)$ and there exists a positive constant C independent of k, and a natural number $\overline{k}$ such that

$$|v + m_k - \tilde{v}_k| \leq \frac{C}{E_k^{1/p}} \leq \frac{C}{s}\tilde{v}_k \quad \mathcal{L}^n\text{-a.e.}$$

for all $k > \overline{k}$. Let $\varphi \in W^{1,p}(B_1)$ with $v - \varphi$ supported on B_ρ, and let $w = \varphi$ in the definition of φ_k, then, for every $k > \overline{k}$, we have

$$|\varphi_k| = |\tilde{v}_k + (v + m_k - \tilde{v}_k)\eta| \leq C\tilde{v}_k \tag{5.15}$$

$\mathcal{L}^n$-a.e. on $B_{\rho'} \setminus B_\rho$. From (5.15) we have that

$$\mathcal{I}_k(\varphi_k) \leq C\mathcal{I}_k(\tilde{v}_k). \tag{5.16}$$

Notice, in addition, that (5.12) reads as

$$\int_{B_{\rho'}} |\nabla \varphi_k|^p \,\mathrm{d}\mathcal{L}^n \le \int_{B_\rho} |\nabla \varphi|^p \,\mathrm{d}\mathcal{L}^n + C \int_{B_{\rho'}\setminus B_\rho} |\nabla \tilde{v}_k|^p \,\mathrm{d}\mathcal{L}^n + C \int_{B_{\rho'}\setminus B_\rho} |\nabla \varphi|^p \,\mathrm{d}\mathcal{L}^n + R_k. \tag{5.17}$$

finally joining (5.9), (5.17), (5.16), and (5.8), we have

$$\alpha_k(\rho') \le \int_{B_\rho} |\nabla \varphi|^p \,\mathrm{d}\mathcal{L}^n + C\left[\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho)\right] + C \int_{B_{\rho'}\setminus B_\rho} |\nabla \varphi|^p \,\mathrm{d}\mathcal{L}^n + R_k.$$

Letting k go to infinity we get (5.11).

Suppose now that (5.14) occurs. The functions $|\tilde{v}_k - m_k|^p$, $|v|^p$ are uniformly integrable, namely for every $\varepsilon > 0$ there exists a $\sigma = \sigma_\varepsilon < \varepsilon$ such that if A is a measurable set with $|A| < \sigma$, then

$$\int_A |\tilde{v}_k - m_k|^p \,\mathrm{d}\mathcal{L}^n + \int_A |v|^p \,\mathrm{d}\mathcal{L}^n < \varepsilon. \tag{5.18}$$

Since $v \in L^p(B_1)$, we can find $M > 1/\varepsilon$ such that

$$|\{|v| > M\}| < \sigma. \tag{5.19}$$

Setting $w = \varphi_M = \max\{-M, \min\{\varphi, M\}\}$, then (5.12) reads as

$$\begin{aligned}\int_{B_{\rho'}} |\nabla \varphi_k|^p \,\mathrm{d}\mathcal{L}^n \le{}& \int_{B_\rho \cap \{|\varphi| \le M\}} |\nabla \varphi|^p \,\mathrm{d}\mathcal{L}^n + C \int_{\left(B_{\rho'}\setminus B_\rho\right)\cap \{|\varphi|\le M\}} |\nabla \varphi|^p \,\mathrm{d}\mathcal{L}^n \\ &+ C\left[\int_{B_{\rho'}\setminus B_\rho} |\nabla \tilde{v}_k|^p \,\mathrm{d}\mathcal{L}^n + \int_{B_{\rho'}\setminus B_\rho} |\varphi_M + m_k - \tilde{v}_k|^p |\nabla \eta|^p \,\mathrm{d}\mathcal{L}^n\right].\end{aligned} \tag{5.20}$$

We can estimate the last integral as follows

$$\begin{aligned}\int_{B_{\rho'}\setminus B_\rho} |\varphi_M + m_k - \tilde{v}_k|^p |\nabla \eta|^p \,\mathrm{d}\mathcal{L}^n &\le C\varepsilon + \int_{\left(B_{\rho'}\setminus B_\rho\right)\cap\{|v|\le M\}} |v + m_k - \tilde{v}_k|^p |\nabla \eta|^p \,\mathrm{d}\mathcal{L}^n]. \\ &= C\varepsilon + R_k,\end{aligned} \tag{5.21}$$

where we used (5.19) and (5.18), and C only depends on ρ and ρ'. Furthermore, we have

$$\mathcal{I}_k(\varphi_k) \le R_k + C\mathcal{I}_k(\tilde{v}_k). \tag{5.22}$$

Indeed, as before, $|\tilde{v}_k - m_k| \le C\tilde{v}_k$, while

$$\begin{aligned}E_k^{q/p-1} r_k^{p-1} \int_{J_{\tilde{v}_k}\cap\left(B_{\rho'}\setminus B_\rho\right)} |\varphi_M|^q \,\mathrm{d}\mathcal{H}^{n-1} &\le M^q E_k^{q/p-1} r_k^{p-1} \mathcal{H}^{n-1}\left(J_{\tilde{v}_k} \cap \left(B_{\rho'} \setminus B_\rho\right)\right) \\ &\le M^q E_k^{\frac{q}{p}} \frac{r_k^{p-1}\varepsilon_k}{E_k} \\ &\le \frac{M^q}{s^q} E_k^{\frac{q}{p}},\end{aligned}$$

which goes to 0 as $k \to \infty$. Finally, joining (5.9), (5.20), (5.21), (5.22), and (5.8), we have

$$\alpha_k(\rho') \leq R_k + \int_{B_\rho \cap \{|\varphi| \leq M\}} |\nabla\varphi|^p + C\left[\tilde{\alpha}(\rho') - \tilde{\alpha}(\rho)\right] + C \int_{\left(B_{\rho'} \setminus B_\rho\right) \cap \{|\varphi| \leq M\}} |\nabla\varphi|^p \, d\mathcal{L}^n + C\varepsilon.$$

Taking the limit as k tends to infinity, and then the limit as ε tends to 0, we get (5.11).
We are now in a position to prove that v is p-harmonic: taking the limit as ρ' tends to ρ in (5.11), we have that if $\varphi \in W^{1,p}(B_1)$, with $v - \varphi$ supported on B_ρ,

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \alpha(\rho) \leq \int_{B_\rho} |\nabla\varphi|^p \, d\mathcal{L}^n,$$

for every $\rho \in I$, therefore v is p-harmonic in B_1. Notice that this implies that v is a locally Lipschitz function (see [2, Theorem 7.12]). Moreover, if $\varphi = v$, we have

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n = \alpha(\rho)$$

for every $\rho \in I$, so that α is continuous on the whole interval $[0, 1]$, $\alpha(1) = 1$ and $\alpha(\tau) = \lim_k \alpha_k(\tau) \geq \tau^{n-1/2}$. Nevertheless, if τ is sufficiently small this contradicts the fact that v is locally Lipschitz, since

$$\tau^{n-\frac{1}{2}} \leq \int_{B_\tau} |\nabla v|^p \, d\mathcal{L}^n \leq C\,\tau^n,$$

where C is a positive constant depending only on n and p. □

Proof of Theorem 1.2 Let u be a minimizer for the problem (1.4). By Corollary 3.4 there exist two positive constants $C(\Omega, \beta, p, q)$, $C_1(\Omega, \beta, p, q)$ such that if $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$, then

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C(\Omega, \beta, p, q) r^{n-1},$$

and if $x \in K_u$

$$\mathcal{L}^n(B_r(x) \cap \{u > 0\}) \geq C_1(\Omega, \beta, p, q) r^n.$$

We now prove that there exists a positive constant $c = c(\Omega, \beta, p, q)$ such that

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \geq c(\Omega, \beta, p, q) r^{n-1} \tag{5.23}$$

for every $x \in K_u$ and $B_r(x) \subset \mathbb{R}^n \setminus \Omega$. Assume by contradiction that there exists $x \in J_u$ such that, for $r > 0$ small enough,

$$\mathcal{H}^{n-1}\left(J_u \cap B_r(x)\right) \leq \varepsilon_0 r^{n-1},$$

where ε_0 is the one in Lemma 5.3. Iterating Lemma 5.3 it can be proven (see [7, Theorem 5.1]) that

$$\lim_{r \to 0^+} r^{1-n} \mathcal{F}(u; B_r) = 0,$$

which, in particular, implies

$$\lim_{r \to 0^+} r^{1-n} \left[\int_{B_r(x)} |\nabla u|^p \, d\mathcal{L}^n + \mathcal{H}^{n-1}\left(J_u \cap B_r(x)\right)\right] = 0. \tag{5.24}$$

By [8, Theorem 3.6], (5.24) implies that $x \notin J_u$, which is a contradiction. Finally, if $x \in K_u$ and

$$\mathcal{H}^{n-1}(J_u \cap B_{2r}(x)) \leq \varepsilon_0 r^{n-1},$$

there exists $y \in J_u \cap B_r(x)$ such that

$$\mathcal{H}^{n-1}(J_u \cap B_r(y)) \leq \varepsilon_0 r^{n-1}$$

which, again, is a contradiction. Then the assertion is proved. The density estimate (5.23) implies in particular that

$$K_u \setminus \bar{\Omega} \subset \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r \to 0^+} r^{1-n} \left[\int_{B_r(x)} |\nabla u|^p \,\mathrm{d}\mathcal{L}^n + \mathcal{H}^{n-1}(J_u \cap B_r(x)) \right] > 0 \right\},$$

hence $\mathcal{H}^{n-1}((K_u \setminus J_u) \setminus \bar{\Omega}) = 0$ (see for instance [8, Lemma 2.6]). □

Remark 5.4 Let u be a minimizer for problem (1.4), then from Theorem 3.2 we have that the function $u^* = (\beta\delta^q)^{-1/p} u$ is an almost-quasi minimizer for the Mumford–Shah functional

$$MS(v) = \int_{\mathbb{R}^n} |\nabla v|^p \,\mathrm{d}\mathcal{L}^n + \mathcal{H}^{n-1}(J_v)$$

with the Dirichlet condition $u^* = (\beta\delta^q)^{-1/p}$ on Ω. If Ω is sufficiently smooth we can apply the results in [4] to have that the density estimate for the jump set of minimizers holds up to the boundary of Ω.

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References

1. Acampora, P., Cristoforoni, E., Nitsch, C., Trombetti, C.: A free boundary problem in thermal insulation with a prescribed heat source. ESAIM Control Optim. Calc. Var. **29**. Paper No. 3, 29. ISSN:1292-8119 (2023)
2. Ambrosio, L., Fusco, N., Pallara, D.: Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs (2000)
3. Bianco, S.G.L., Manna, D.A.L., Velichkov, B.: A two-phase problem with Robin conditions on the free boundary. J. École. Polytech. Math. **8**, 1–25 (2021). (**ISSN:2429–7100**)
4. Bucur, D., Giacomini, A.: Shape optimization problems with Robin conditions on the free boundary. Ann. Inst. H. Poincaré Anal. Non Linéaire (2015)
5. Bucur, D., Giacomini, A.: A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. Arch. Ration. Mech. Anal. **198**(3), 927–961 (2010)
6. Buttazzo, G., Maiale, F.P.: Shape optimization problems for functionals with a boundary integral. J. Convex Anal. **28**(2), 429–456 (2021)
7. Caffarelli, L.A., Kriventsov, D.: A free boundary problem related to thermal insulation. Commun. Partial Differ. Equ. **41**(7), 1149–1182 (2016)

8. de Giorgi, E., Carriero, M., Leaci, A.: Existence theorem for a minimum problem with free discontinuity set. Arch. Ration. Mech. Anal. **108**, 195–218 (1989)
9. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. Textbooks in Mathematics, Revised edition. CRC Press (2015)