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Integrating Nijenhuis structures

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Abstract

A Nijenhuis operator on a manifold is a (1, 1) tensor whose Nijenhuis torsion vanishes. A Nijenhuis operator $\mathcal{N}$ determines a Lie algebroid that knows everything about $\mathcal{N}$. In this sense, a Nijenhuis operator is an infinitesimal object. In this paper, we identify its *global counterpart*. Namely, we characterize Lie groupoids integrating the Lie algebroid of a Nijenhuis operator. We illustrate our integration result in various examples, including that of a *linear Nijenhuis operator* on a vector space or, which is equivalent, a pre-Lie algebra structure.

Keywords Lie groupoid · Lie algebroid · Nijenhuis structure · Graded manifold

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1 Introduction

There is a natural *integrability condition* that one can impose on a (1, 1) tensor $\mathcal{N}$ on a manifold M. Namely, $\mathcal{N}$ defines a skew-symmetric (2, 1) tensor $T_{\mathcal{N}}$, its *Nijenhuis torsion*, which is given by

$$T_{\mathcal{N}}(X, Y) = [\mathcal{N}X, \mathcal{N}Y] + \mathcal{N}^2[X, Y] - \mathcal{N}[\mathcal{N}X, Y] - \mathcal{N}[X, \mathcal{N}Y], \quad X, Y \in \mathfrak{X}(M).$$

Equivalently $T_{\mathcal{N}} = \frac{1}{2}[\mathcal{N}, \mathcal{N}]^{\mathrm{fn}}$, where $[-, -]^{\mathrm{fn}}$ is the Frölicher–Nijenhuis bracket of vector valued forms. We say that $\mathcal{N}$ is *integrable* when $T_{\mathcal{N}} = 0$ identically, in this case we call $\mathcal{N}$ a *Nijenhuis operator*. Nijenhuis operators naturally appear in relation to other geometries: for instance in complex geometry as integrable almost complex structures, and in integrable systems as recursion operators for bi-Hamiltonian systems (see, e.g., the review [23] and references therein). But they also appear elsewhere (see, e.g., references in [1] for a partial list). Recently, Bolsinov, Konyaev and Matveev, in a series of papers [1, 3–5, 21] initiated a project consisting in systematically studying Nijenhuis operators in their own. For instance, in the first paper of the series [1], they discuss local normal forms, in the same spirit as Weinstein splitting theorem for Poisson structures [34], while in the second paper of the series [21], Konyaev discusses the *linearization problem*.

In fact every Nijenhuis operator defines a Lie algebroid structure $(TM)_{\mathcal{N}}$ on the tangent bundle $TM \to M$. The anchor of $(TM)_{\mathcal{N}}$ is $\mathcal{N}$ itself, while the Lie bracket on sections is given by

$$[X, Y]_{\mathcal{N}} = [\mathcal{N}X, Y] + [X, \mathcal{N}Y] - \mathcal{N}[X, Y], \quad X, Y \in \mathfrak{X}(M).$$

Recall that a Lie algebroid is the *infinitesimal counterpart* of a *global object*: a Lie groupoid. When the Lie algebroid $(TM)_{\mathcal{N}}$ integrates to a Lie groupoid $G \rightrightarrows M$, it is natural to wonder what kind of structure on G is responsible for the Nijenhuis operator defining $(TM)_{\mathcal{N}}$. Similar questions were already posed and answered in various analogous situations. For instance, a Poisson tensor π on a manifold M defines a Lie algebroid structure $(T^*M)_\pi$ on the cotangent bundle. When the Lie algebroid $(T^*M)_\pi$ integrates to a Lie groupoid, there is a *multiplicative symplectic structure* on the source simply connected integration which is responsible for π. In this sense, a *symplectic groupoid*, i.e., a Lie groupoid equipped with a multiplicative symplectic structure, is the global counterpart of a Poisson manifold [19, 35]. Similarly, a holomorphic groupoid is the global counterpart of a holomorphic Lie algebroid [24], a contact groupoid is the global counterpart of a Jacobi manifold [14, 20, 25], etc. (see the review [22] for more examples).

In this paper, we characterize Lie groupoids $G \rightrightarrows M$ integrating a Nijenhuis operator, i.e., Lie groupoids whose Lie algebroid is isomorphic to $(TM)_{\mathcal{N}}$ for some Nijenhuis operator $\mathcal{N}$ on M. In particular, we identify what precise structure on G is responsible for $\mathcal{N}$. In order to guess the final answer, we begin with a closer look at the infinitesimal picture. Recall that the datum of a Lie algebroid can be encoded in that of a differential graded (DG) manifold concentrated in degrees 0, 1. The DG manifold encoding the Lie algebroid $A \Rightarrow M$ is $(A[1], d_A)$ where $A[1]$ is obtained from A by shifting by one the degree of the fiber coordinates and d_A is the Lie algebroid De Rham differential. Our first result is

Theorem 1.1 *The datum of a Lie algebroid of the type $(TM)_{\mathcal{N}}$ for some Nijenhuis operator $\mathcal{N}$ on M is equivalent to that of a DG manifold $(A[1], d_A)$ concentrated in degrees $0, 1$ equipped with an integrable almost tangent structure V of internal degree -1 such that $[[d_A, V]^{\mathrm{fn}}, V]^{\mathrm{fn}} = 0$.*

Our main result is the following global version of Theorem 1.1.

Theorem 1.2 *A Lie groupoid $G \rightrightarrows M$ with source $s : G \to M$, target $t : G \to M$ and Lie algebroid $A \Rightarrow M$ integrates a Nijenhuis operator $\mathcal{N}$ on M if and only if it is equipped with a vector bundle map $U : TM \to A$ such that 1)* $\ker \overrightarrow{U} = \ker dt$*, 2)* $\operatorname{im} \overrightarrow{U} = \ker ds$*, and 3)* $[\overrightarrow{U}, \overrightarrow{U}]^{\mathrm{fn}} = 0$*, i.e., $\overrightarrow{U}$ is a Nijenhuis operator.*

In the previous statement, $\overrightarrow{U}$ is a $(1, 1)$ tensor on G defined as an appropriate *right invariant lift* of U. There is an analogy between Theorems 1.1 and 1.2 which we now explain. First of all, an almost tangent structure of degree -1 on $A[1]$ as in Theorem 1.1 is equivalent to a vector bundle isomorphism $U : TM \to A$. Similarly, conditions 1) and 2) in Theorem 1.2 together are equivalent to U being bijective. Secondly, condition 3) in Theorem 1.2 is actually equivalent to $[\delta U, \overrightarrow{U}]^{\mathrm{fn}}$ where δ is a certain global analogue of the Lie derivative $[d_A, -]^{\mathrm{fn}}$ of a $(1, 1)$ tensor on $A[1]$ along the homological vector field.

We interpret the $(1, 1)$ tensor $\overrightarrow{U}$ on G of Theorem 1.2 as the structure responsible for the Nijenhuis operator defining the Lie algebroid of G. In this spirit, Theorem 1.2 is an *integration result* for Nijenhuis operators. However, we stress that such result is slightly different in nature from the integration theorem of, e.g., a Poisson structure π. While, in general, we can guarantee the existence of a multiplicative symplectic form only on a source-simply connected integration of the Lie algebroid $(T^*M)_\pi$, a $(1, 1)$ tensor $\overrightarrow{U}$ as in Theorem 1.2 exists on *every* Lie groupoid integrating $(TM)_{\mathcal{N}}$.

The paper is organized as follows. In Sect. 2, we recollect the necessary material about $(1, 1)$ tensors on Lie groupoids, Lie algebroids and graded manifolds. This material is not novel except for the second part of Lemma 2.3 that will play an important role in the rest of the paper. A version of the cochain complex $(C^\bullet_{\mathrm{def}}(G, T^{1,0}), \delta)$ of Sect. 2 appeared already in [29]. In Sect. 3, we state and prove Theorem 1.1 (see Theorem 3.2). In Sect. 4, we prove our main result Theorem 1.2 (see Theorem 4.1) and illustrate it with a few trivial examples. In Sect. 5, we provide more illustrative examples, including a curious Lie groupoid structure on the double tangent bundle TTB of a manifold B which, to the best of our knowledge, is new (see Sect. 5.1). We also discuss linear Nijenhuis operators on a vector space.

Finally, we want to mention the recent work of Bursztyn, Drummond, Netto [9, 16] on Nijenhuis operators in connections to Lie groupoids, Lie algebroids and related structure (see also [10, 15]). The present paper goes in a complementary direction. While those authors consider Nijenhuis operators on Lie groupoids (resp. Lie algebroids) integrating (resp. encoding) other geometric structures, we consider Lie groupoids integrating Nijenhuis operators themselves.

Notation. We denote by $G \rightrightarrows M$ a Lie groupoid with G its space of arrows, and M its space of objects. We denote by $A \Rightarrow M$ a Lie algebroid with $A \to M$ its underlying vector bundle. We denote by $[-, -]_A$ the Lie bracket on sections of A and by $\rho_A : A \to TM$ the anchor. Given a surjective submersion $\pi : M \to B$, we denote by $T^\pi M := \ker d\pi$ the vertical tangent bundle with respect to π.

We assume the reader is familiar with Lie groupoids and Lie algebroids. Our main reference for this material is the lecture notes of Crainic and Fernandes [11] (see also [26]). Additionally, we assume some familiarity with graded geometry, including DG manifolds, for which the reader might consult [27].

2 A review of multiplicative and IM $(1, 1)$ tensors

In the rest of the paper, we will extensively work with $(1, 1)$ tensors on Lie groupoids, Lie algebroids and graded manifolds (of a certain type). In this section, we summarize the necessary material. Our main references here are [7, 8, 24]. We will often interpret (1, 1) tensors as vector valued 1-forms.

Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \Rightarrow M$. The structure maps of G will be denoted $s, t : G \to M$ (source and target), $m : G^{(2)} \to M$ (multiplication), $u : M \to G$ (unit), and $i : G \to G$ (inversion). We denote by $G^{(k)}$ the manifold of k composable arrows in G. Let $T \in \Omega^1(G, TG)$. There are several equivalent ways of stating the compatibility of T and the groupoid structure. The easiest one is the following: we say that T is a *multiplicative* (1, 1) *tensor* if $T : TG \to TG$ is a groupoid map with respect to the tangent groupoid structure $TG \rightrightarrows TM$, in particular, there exists a (1, 1) tensor T^M on M such that T and T^M are both s-related and t-related.

Multiplicative (1, 1) tensors on G can also be seen as 0-cocycles in an appropriate cochain complex $C^\bullet_{\mathrm{def}}(G, T^{1,0})$ that we now describe:

First of all, $C^\bullet_{\mathrm{def}}(G, T^{1,0})$ will be concentrated in degrees $k \geq -1$. Degree -1 cochains are vector bundle maps

$$U : TM \to A.$$

For $k \geq 0$, degree k cochains are vector bundle maps

$$U : TG^{(k+1)} \to TG,$$

covering the projection $\mathrm{pr}_1 : G^{(k+1)} \to G$, $(g_1, \dots, g_{k+1}) \mapsto g_1$ onto the first factor, for which there exists another vector bundle map

$$U^M : TG^{(k)} \to TM,$$

covering the projection $t \circ \mathrm{pr}_1 : G^{(k)} \to M$, such that the following diagram commutes:

$$\begin{tikzcd} TG^{(k+1)} \arrow[r, "U"] \arrow[d, "d(\mathrm{pr}_2\times\cdots\times\mathrm{pr}_{k+1})"'] & TG \arrow[d, "ds"] \\ TG^{(k)} \arrow[r, "U^M"] & TM \end{tikzcd} ,$$

where $\mathrm{pr}_i : G^{(k+1)} \to G$ is the projection onto the i-th factor. For instance, a (1, 1) tensor T on G belongs to $C^0_{\mathrm{def}}(G, T^{1,0})$ if and only if it is s-projectable, i.e., it is s-related to some (1, 1) tensor on M.

Next, we describe the differential $\delta : C^\bullet_{\mathrm{def}}(G, T^{1,0}) \to C^{\bullet+1}_{\mathrm{def}}(G, T^{1,0})$. We begin with its action on a degree -1 cochain $U : TM \to A$. First define two (1, 1) tensors $\overrightarrow{U}$ and $\overleftarrow{U}$ on G, by putting

$$\overrightarrow{U}_g = dR_g \circ U_{t(g)} \circ dt \quad \text{and} \quad \overleftarrow{U}_g = dL_g \circ di \circ U_{s(g)} \circ ds, \tag{2.1}$$

for every $g \in G$, where R_g (resp. L_g) denotes right (resp. left) translation by g. Now, let

$$\delta U := \overrightarrow{U} + \overleftarrow{U}$$

which is a well-defined 0-cochain whose M-projection $(\delta U)^M$ is given by

$$(\delta U)^M = \rho_A \circ U : TM \to TM,$$

where $\rho_A : A \to TM$ is the anchor map.

Remark 2.1 The three assignments $U \mapsto \overrightarrow{U}, \overleftarrow{U}, \delta U$ where already considered in [8] in the much more general setting when U is a (p, q) tensor with p, q arbitrary. The notation adopted in [8] is the following: $\overrightarrow{U} = \mathcal{T}(U)$ and $\overleftarrow{U} = \mathcal{S}(U)$ (see Formula (3.6) and Proposition 3.10 loc. cit.).

Finally, let $U \in C^k_{\mathrm{def}}(G, T^{1,0})$, with $k \geq 0$. For all $(v_1, \ldots, v_{k+2}) \in TG^{(k+2)}$, we put

$$\begin{aligned}\delta U(v_1, \ldots, v_{k+2}) = &-U(v_1 v_2, v_3, \ldots, v_{k+2})\, U(v_2, \ldots, v_{k+2})^{-1}\\ &+ \sum_{i=2}^{k+1}(-)^i U(v_1, \ldots, v_i v_{i+1}, \ldots, v_{k+2}) + (-)^k U(v_1, \ldots, v_{k+1}),\end{aligned} \tag{2.2}$$

where we used the multiplication and inversion in the tangent Lie groupoid $TG \rightrightarrows TM$. Then, δU is a well-defined $(k+1)$-cochain whose M-projection $(\delta U)^M$ is given by

$$\begin{aligned}(\delta U)^M(v_2, \ldots, v_{k+2}) = &-dt\,(U(v_2, \ldots, v_{k+2}))\\ &+ \sum_{i=2}^{k+1}(-)^{i+1} U^M(v_2, \ldots, v_i v_{i+1}, \ldots, v_{k+2})\\ &+ (-)^k U^M(v_2, \ldots, v_{k+1}).\end{aligned}$$

The operator δ is indeed a differential. The definition of $C^\bullet_{\mathrm{def}}(G, T^{1,0})$ is very similar to that of Crainic-Mestre-Struchiner deformation complex of a Lie groupoid $G \rightrightarrows M$ [13], and the definition of the differential δ is formally identical up to replacing points in $G^{(k+2)}$ with points in $TG^{(k+2)}$. This is the main reason why we adopt a similar notation $C^\bullet_{\mathrm{def}}(G, T^{1,0})$ for our complex. Notice that there are versions of this complex for higher-order tensors (see, e.g., [29]). We speculate that cohomology classes of $C^\bullet_{\mathrm{def}}(G, T^{1,0})$ should be seen as shifted $(1, 1)$ tensors on the differentiable stack $[G/M]$ represented by G, but we will not explore this point of view here.

Next, we discuss infinitesimal multiplicative (IM) $(1, 1)$ tensors on Lie algebroids. We begin with linear $(1, 1)$ tensors on vector bundles. Let $A \to M$ be a vector bundle, and let $T : TA \to TA$ be a $(1, 1)$ tensor on the total space A. We say that T is *linear* if it is multiplicative with respect to the Lie groupoid structure on A given by fiber-wise addition, equivalently T is a vector bundle map with respect to the vector bundle structure $TA \to TM$. This definition emphasizes the fiber-wise addition in A. There is an equivalent definition emphasizing the fiber-wise scalar multiplication which is often useful in practice. Namely, a $(1, 1)$ tensor on A is linear if and only if it is of degree 0 with respect to the action $h : \mathbb{R} \times A \to A$ of the multiplicative monoid $\mathbb{R}$ on A given by fiber-wise scalar multiplication, i.e., $h_r^* T = T$ for all $r \in \mathbb{R} \smallsetminus 0$. In the following, we will also need $(1, 1)$-tensors $S \in \Omega^1(A, TA)$ of degree -1 with respect to h, i.e., $h_r^* S = r^{-1} S$ for all $r \in \mathbb{R} \smallsetminus 0$. We call such tensors *core* $(1, 1)$ tensors adopting a terminology that we already used in [29]. Both core and linear $(1, 1)$ tensors can be encoded into certain sections of appropriate vector bundles over M. To see this, first recall that a section a of A defines a vector field $a^\uparrow$ on A, its *vertical lift*, via

$$a^\uparrow_z := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(z + \varepsilon a_x) \in T_z A,$$

where $x \in M$ is the image of $z \in A$ under the projection $A \to M$. The assignment $a \mapsto a^\uparrow$ is a bijection between sections of A and degree -1 vector fields on A (also called core vector fields in [29, 30]). Next, recall also that the *gauge algebroid* of A is the Lie agebroid $DA \Rightarrow M$ whose sections are *derivations* of A, i.e., $\mathbb{R}$-linear maps

$$D : \Gamma(A) \to \Gamma(A)$$

such that

$$D(fa) = \sigma_D(f)a + fDa, \quad f \in C^\infty(M), \quad a \in \Gamma(A),$$

for some, necessarily unique, vector field $\sigma_D \in \mathfrak{X}(M)$ called the *symbol* of D. A derivation D of A defines a vector field X_D on A: the unique vector field such that

$$[X_D, a^\uparrow] = (Da)^\uparrow, \quad \text{for all } a \in \Gamma(A).$$

The assignment $D \mapsto X_D$ is a bijection between sections of the gauge algebroid DA and degree 0 vector fields on A.

Similarly, core $(1, 1)$ tensors on A identify with vector bundle maps $U : TM \to A$ via the map $U \mapsto U^\uparrow$, where $U^\uparrow \in \Omega^1(A, TA)$ is the core $(1, 1)$ tensor uniquely determined by $U^\uparrow(X_D) = U(\sigma_D)^\uparrow$, and σ_D is the symbol of D. Finally, linear $(1, 1)$ tensors on A identify with triples $(\mathcal{D}, \ell, T^M)$ where $\ell : A \to A$ is a vector bundle map, $T^M \in \Omega^1(M, TM)$ is a $(1, 1)$-tensor on M, and $\mathcal{D} : \Gamma(A) \to \Omega^1(M, A)$ is a differential operator such that

$$\mathcal{D}(fa) = f\mathcal{D}(a) + df \otimes \ell(a) - \langle df, T^M\rangle \otimes a, \quad a \in \Gamma(A), \quad f \in C^\infty(M),$$

via the map $(\mathcal{D}, \ell, T^M) \mapsto T$ where $T \in \Omega^1(A, TA)$ is the linear $(1, 1)$ tensor uniquely determined by its M-projection being T^M and, additionally,

$$\mathcal{L}_{a^\uparrow} T = \mathcal{D}(a)^\uparrow, \quad Ta^\uparrow = \ell(a)^\uparrow.$$

When $A \Rightarrow M$ is a Lie algebroid with Lie bracket $[-, -]_A$ and anchor map ρ_A, then a $(1, 1)$ tensor $T \in \Omega^1(A, TA)$ is IM if $T : TA \to TA$ is a Lie algebroid map with respect to the tangent algebroid structure $TA \Rightarrow TM$. In particular T is a linear $(1, 1)$ tensor and T being IM is equivalent to the associated triple $(\mathcal{D}, \ell, T^M)$ satisfying the following identities [8, 17]:

$$\begin{aligned}
\mathcal{D}\left([a, b]_A\right) &= \mathcal{L}_a\mathcal{D}(b) - \mathcal{L}_b\mathcal{D}(a)\\
\ell\left([a, b]_A\right) &= [a, \ell(b)]_A - \iota_{\rho_A(b)}\mathcal{D}(a),\\
\mathcal{L}_{\rho_A(a)}T^M &= \rho_A \circ \mathcal{D}(a),\\
T^M \circ \rho_A &= \rho_A \circ \ell,
\end{aligned} \tag{2.3}$$

for all $a, b \in \Gamma(A)$.

If $T : TG \to TG$ is a multiplicative $(1, 1)$ tensor on the Lie groupoid $G \rightrightarrows M$, then, by differentiation, we get an IM $(1, 1)$ tensor $\dot{T} : TA \to TA$ on the Lie algebroid $A \Rightarrow M$ of G. The triple $(\mathcal{D}, \ell, T^M)$ corresponding to $\dot{T}$ is given by

$$\mathcal{D}(a) = \mathcal{L}_{\overrightarrow{a}} T|_{TM}, \quad \ell(a) = T\overrightarrow{a}|_M,$$

for all $a \in \Gamma(A)$, while T^M is the M-projection of T (here $\overrightarrow{a} \in \mathfrak{X}(G)$ is the right invariant vector field corresponding to a).

IM $(1, 1)$ tensors can also be seen as 0-cocycles in a cochain complex. The easiest way to see this is via graded geometry. We assume the reader is familiar with graded manifolds and homological vector fields. We only recall that an $\mathbb{N}$-manifold of degree k is a graded manifold whose function algebra is generated in degree $0, 1, \ldots, k$. An $\mathbb{N}Q$-manifold is an $\mathbb{N}$-manifold equipped with a Q-structure, i.e., a (degree 1) homological vector field Q. We will also need to consider vector valued forms on $\mathbb{N}$-manifolds. Our conventions on differential forms on an $\mathbb{N}$-manifold are as follows. Given an $\mathbb{N}$-manifold $\mathcal{M}$, differential forms on $\mathcal{M}$ are fiber-wise polynomial functions on the shifted tangent bundle $T[1]\mathcal{M}$ and are denoted $\Omega^\bullet(\mathcal{M})$. Similarly, vector valued forms on $\mathcal{M}$ are fiber-wise polynomial sections of the pull-back bundle $T[1]\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} \to T[1]\mathcal{M}$ and are denoted $\Omega^\bullet(\mathcal{M}, T\mathcal{M})$. Both $\Omega^\bullet(\mathcal{M})$ and $\Omega^\bullet(\mathcal{M}, T\mathcal{M})$ are bi-graded vector spaces, the two gradings being the form degree, and the internal, i.e., coordinate, degree. The total degree of a form is then the sum of the internal and the form degrees, and the usual Koszul sign rule holds with respect to the total degree. For instance, the Frölicher–Nijenhuis bracket on $\Omega^\bullet(\mathcal{M}, T\mathcal{M})$ is a graded Lie bracket of total degree 0.

Remember that the datum of a vector bundle $A \to M$ is equivalent to the datum of an $\mathbb{N}$-manifold of degree 1 via $A \rightsquigarrow A[1]$ (where $A[1]$ is the $\mathbb{N}$-manifold obtained by shifting by 1 the fiber degree in A), and the datum of a Lie algebroid $A \Rightarrow M$ is equivalent to the datum of an $\mathbb{N}Q$-manifold of degree 1 via $A \rightsquigarrow (A[1], d_A)$ where d_A is the Lie algebroid De Rham differential. Exactly as for usual vector bundles, degree -1 vector fields on $A[1]$ identify with sections of A and degree 0 vector fields identify with sections of DA. Similarly, $(1, 1)$ tensors on $A[1]$ of internal degree -1 identify with vector bundle maps $U : TM \to A$, hence with core $(1, 1)$ tensors on A, and $(1, 1)$ tensors on $A[1]$ of internal degree 0 identify with triples $(\mathcal{D}, \ell, T^M)$ as above, hence with linear $(1, 1)$ tensors on A. If $T \in \Omega^1(A, TA)$ is a linear $(1, 1)$ tensor, we denote by $T^{[1]} \in \Omega^1(A[1], TA[1])$ the corresponding $(1, 1)$ tensor (of internal degree 0) on $A[1]$. If $A \Rightarrow M$ is a Lie algebroid, then T is IM if and only if $\mathcal{L}_{d_A} T^{[1]} = 0$ [29], where $\mathcal{L}_{d_A}$ is the Lie derivative (of $(1, 1)$ tensors) along the homological vector field d_A. Hence, the assignment $T \mapsto T^{[1]}$ identifies IM $(1, 1)$ tensors on A with 0-cocycles in the cochain complex $\left(\Omega^1(A[1], TA[1]), \mathcal{L}_{d_A}\right)$ (whose grading is given by the internal degree).

Example 2.2 Let $T \in \Omega^1(M, TM)$ be a $(1, 1)$ tensor on M. Its *tangent lift* is the linear $(1, 1)$ tensor $T^{\tan} \in \Omega^1(TM, TTM)$ on TM corresponding to the triple

$$\left([-, T]^{\mathrm{fn}}, T, T\right)$$

(see, e.g., [9]) where $[-, -]^{\mathrm{fn}}$ is the Frölicher–Nijenhuis bracket of vector-valued forms.

Finally, let $A \Rightarrow M$ be the Lie algebroid of a Lie groupoid $G \rightrightarrows M$, and let $U : TM \to A$ be a vector bundle map. Then, U can be seen both as a degree -1 cochain in the cochain complex $\left(C^\bullet_{\mathrm{def}}(G, T^{1,0}), \delta\right)$ and in the cochain complex $\left(\Omega^1(A[1], TA[1]), \mathcal{L}_{d_A}\right)$ (via $U \mapsto U^\uparrow$). The next lemma will play an important role in the sequel. The first part of the statement appeared already in [8, Example 3.15], but without the (easy) proof, in the general setting of (p, q) tensors. Here, we propose a proof in the case $p = q = 1$.

Lemma 2.3 *The triples $(\mathcal{D}, \ell, T^M)$ corresponding to $\dot{\delta U}$ and $\mathcal{L}_{d_A} U^\uparrow$ are both given by*

$$\mathcal{D}(a) = \mathcal{L}^A_a U, \quad \ell = U \circ \rho_A, \quad T^M = \rho_A \circ U, \tag{2.4}$$

for all $a \in \Gamma(A)$ (where $\mathcal{L}^A$ is the Lie algebroid Lie derivative, see below), hence

$$(\dot{\delta U})^{[1]} = \mathcal{L}_{d_A} U^\uparrow.$$

Proof Begin with $\dot{\delta U}$. Let $a \in \Gamma(A)$ and notice that the vector field $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) is both s-projectable and t-projectable. It s-projects onto the trivial vector field (resp. onto $\rho_A(a)$) and t-projects onto $\rho_A(a)$ (resp. onto the trivial vector field). Similarly, a direct check shows that $\overrightarrow{U}$ (resp. $\overleftarrow{U}$) s-projects on the trivial $(1, 1)$ tensor (resp. onto $\rho_A \circ U$), while it t-projects onto $\rho_A \circ U$ (resp. onto the trivial (1, 1) tensor). The third one of Formulas (2.4) for $\delta \dot{U}$ now follows. The second one can be proved with a direct computation. For the first one, there is an easy local proof. Namely, choose a local frame (u_α) of $\Gamma(A)$. Then, locally

$$U = U^\alpha \otimes u_\alpha$$

for some local 1-forms U^α on M. Hence,

$$\overrightarrow{U} = t^*(U^\alpha) \otimes \overrightarrow{u}_\alpha, \quad \text{and} \quad \overleftarrow{U} = s^*(U^\alpha) \otimes \overleftarrow{u}_\alpha. \tag{2.5}$$

A direct computation exploiting these formulas (and the obvious properties of right/left invariant vector fields) shows that

$$\mathcal{L}_{\overrightarrow{a}} \delta U = \overrightarrow{\mathcal{L}^A_a U}$$

and the first one of (2.4) follows (here $\mathcal{L}^A$ is the *Lie algebroid Lie derivative*: $\mathcal{L}^A_a U(X) = [a, UX]_A - U[\rho_A(a), X]$, for all $a \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$).

As for $\mathcal{L}_{d_A} U^\uparrow$, a straightforward computation shows that the triple $(\mathcal{D}, \ell, T^M)$ corresponding to it is given by the same formulas. We present (part of) it for completeness. Let $a \in \Gamma(A)$ and $D \in \Gamma(DA)$, then

$$\begin{aligned}
\mathcal{D}(a)(\sigma_D)^\uparrow &= \mathcal{D}(a)^\uparrow(X_D) = (\mathcal{L}_{a^\uparrow}\mathcal{L}_{d_A} U^\uparrow)X_D = (\mathcal{L}_{[a^\uparrow, d_A]} U^\uparrow)X_D = (\mathcal{L}_{X_{[a,-]_A}} U^\uparrow)X_D \\
&= \left[X_{[a,-]}, U^\uparrow X_D\right] - U^\uparrow \left[X_{[a,-]_A}, X_D\right] = \left[X_{[a,-]}, U(\sigma_D)^\uparrow\right] - U^\uparrow \left[X_{[[a,-]_A, D]]}\right] \\
&= [a, U(\sigma_D)]^\uparrow_A - U(\sigma_{[[a,-],D]]})^\uparrow = ([a, U(\sigma_D)]_A - U[\rho_A(a), \sigma_D])^\uparrow \\
&= (\mathcal{L}_a U)(\sigma_D)^\uparrow
\end{aligned}$$

where we used that there are no nontrivial (1, 1) tensors on $A[1]$ of internal degree -2. The rest is straightforward. □

Remark 2.4 As an application of Lemma 2.3 we discuss a *canonical cohomology class* in the cochain complex $C^\bullet_{\mathrm{def}}(G, T^{1,0})$. Namely, let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid A. The identity endomorphism $\mathbb{I}_G : TG \to TG$ is clearly a multiplicative (1, 1) tensor, hence a distinguished 0-cocycle in $C^\bullet_{\mathrm{def}}(G, T^{1,0})$. Its cohomology class $[\mathbb{I}_G]$ is therefore a canonical cohomology class attached to the Lie groupoid G. It is easy to see that $[\mathbb{I}_G] = 0$ if and only if G integrates the tangent bundle $TM \Rightarrow M$. Indeed, let $\mathbb{I}_G = \delta U$ for some vector bundle map $U : TM \to A$. The triple $(\mathcal{D}, \ell, T^M)$ corresponding to $\dot{\mathbb{I}}$ is $(0, \mathbb{I}_A, \mathbb{I}_M)$, where $\mathbb{I}_A : A \to A$ and $\mathbb{I}_M : TM \to TM$ are the identity endomorphisms. It immediately follows from Lemma 2.3 that U is an isomorphism and ρ_A is its inverse. Hence, $\rho_A : A \to TM$ is a Lie algebroid isomorphism. Conversely, let $A = TM$ (with the commutator as Lie bracket, and the identity as anchor). If $G = M \times M$ is the pair groupoid, a direct computation shows that $\mathbb{I}_G = \delta \mathbb{I}_M$. In all other cases the anchor map $(s, t) : G \to M \times M$ is a locally invertible groupoid map (covering the identity map). In particular, the pull-back along (s, t) is a cochain map $(s, t)^* : C^\bullet_{\mathrm{def}}(M \times M, T^{1,0}) \to C^\bullet_{\mathrm{def}}(G, T^{1,0})$. We conclude that

$$\mathbb{I}_G = (s, t)^* \mathbb{I}_{M \times M} = (s, t)^* \delta \mathbb{I}_M = \delta\left((s, t)^* \mathbb{I}_M\right) = \delta \mathbb{I}_M.$$

3 Nijenhuis structures, Lie algebroids and graded manifolds

In this section we describe Nijenhuis operators in terms of DG manifolds. It is often the case that an additional structure on a Lie algebroid $A \Rightarrow M$ is encoded by an appropriate structure on the associated $\mathbb{N}Q$-manifold $(A[1], d_A)$. For instance, when $A = (T^*M)_\pi$ is the cotangent algebroid of a Poisson manifold (M, π), then $(A[1], d_A)$ is additionally equipped with a symplectic structure ω of internal degree 1 such that $\mathcal{L}_{d_A}\omega = 0$ and every degree 1 symplectic $\mathbb{N}Q$-manifold arises in this way up to isomorphisms, see [31] (similar results hold for the Lie algebroid of a Jacobi structure [18, 28] and, more generally, for Lie algebroids equipped with IM vector-valued forms [29, 33]). We want to prove an analogous result for Lie algebroids defined by Nijenhuis operators. We begin discussing vector bundles $A \to M$ equipped with a vector bundle isomorphism $A \cong TM$.

Recall that an *almost tangent structure* on a manifold P is a $(1,1)$-tensor $V \in \Omega^1(P, TP)$ such that $\ker P = \operatorname{im} P$ (in particular, $\dim P =$ even). An almost tangent structure V is *integrable* if $[V, V]^{\mathrm{fn}} = 0$. In other words, an almost tangent structure is integrable if it is additionally a Nijenhuis operator. Let M be a manifold. The vertical endomorphism $V : TTM \to TTM$ on the tangent bundle is an integrable almost tangent structure, and every integrable almost tangent structure is locally of this form. We will show in a moment that degree -1 almost tangent structures on $\mathbb{N}$-manifolds of degree 1 are all *globally* of this form in an appropriate sense. To see this, first notice that the vertical endomorphism on TM is a core $(1,1)$ tensor (corresponding to the identity map $\mathbb{I}_M : TM \to TM$), hence it also corresponds to a $(1,1)$ tensor of degree -1 on $T[1]M$. Denote the latter by $V^{[1]}$, and still call it the *vertical endomorphism*. If x^i are local coordinates on M and $\dot{x}^i$ are the associated degree 1 fiber coordinates on $T[1]M$, then locally

$$V = dx^i \otimes \frac{\partial}{\partial \dot{x}^i}.$$

In particular, V is an integrable almost tangent structure. It is easy to see that this almost tangent structure enjoys the following "universal property." A vector bundle map $U : TM \to A$ induces a smooth map $T[1]M \to A[1]$ of graded manifolds, also denoted U. The vertical endomorphism V on $A[1]$ and the $(1,1)$ tensor $U^\uparrow$ of degree -1 corresponding to U are then automatically U-related.

Lemma 3.1 *Let $A \to M$ be a vector bundle. The assignment $U \mapsto U^\uparrow$ establishes a bijection between vector bundle isomorphisms $U : TM \to A$ and integrable almost tangent structures of internal degree -1 on $A[1]$.*

Proof Begin with a vector bundle isomorphism $U : TM \to A$. According to the remark preceding the statement, the $(1,1)$ tensor $U^\uparrow \in \Omega^1(A[1], TA[1])$ corresponding to U is also the push-forward of the vertical endomorphism $V \in \Omega^1(T[1]M, TT[1]M)$ along the diffeomorphism $U : T[1]M \to A[1]$, hence it is an integrable almost tangent structure (of internal degree -1) itself. Conversely, take an integrable almost tangent structure of internal degree -1 on $A[1]$. It is of the form $U^\uparrow$ for some vector bundle map $U : TM \to A$, and U is necessarily an isomorphism. Indeed, U is the composition

$$TM \longrightarrow T_M A[1] \xrightarrow{U^\uparrow} T_M A[1] \longrightarrow A[1] \longrightarrow A,$$

where $T_M A[1]$ is the restriction of $TA[1]$ to the zero section of $A[1] \to M$, the map $T_M A[1] \longrightarrow A[1]$ is the natural projection and the last arrow is the shift. But, for degree reasons, $A[1] \hookrightarrow T_M A[1]$ must be in the kernel of $U^\uparrow$, hence in its image and in the image of U as well. We conclude that U is surjective. The injectivity now follows by dimension reasons. □

In what follows, given an $\mathbb{N}$-manifold $\mathcal{M}$ of degree 1 equipped with an almost tangent structure V of degree -1, we will always identify $\mathcal{M}$ with a shifted tangent bundle $T[1]M$

and V with the vertical endomorphism, unless otherwise stated. For instance, we will denote by d_{dR} again the push forward of the usual De Rham differential on $T[1]M$ along the diffeomorphism $T[1]M \to \mathcal{M}$ induced by V. We are now ready to state the main result of this section.

Theorem 3.2 *A Lie algebroid $A \Rightarrow M$ is isomorphic to the Lie algebroid $(TM)_{\mathcal{N}}$ induced by a Nijenhuis operator $\mathcal{N} \in \Omega^1(M, TM)$ on M if and only if there exists a, necessarily integrable, almost tangent structure V of internal degree -1 on $A[1]$ such that*

$$[[d_A, V]^{\mathrm{fn}}, V]^{\mathrm{fn}} = 0. \tag{3.1}$$

In this case,

$$[d_A, V]^{\mathrm{fn}} = \mathcal{N}^{\tan}_{[1]} \tag{3.2}$$

where $\mathcal{N} \in \Omega^1(M, TM)$ is exactly the composition $\rho_A \circ U : TM \to TM$, $U : TM \to A$ is the vector bundle isomorphism such that $U^{\uparrow} = V$, and we are using the latter to identify $A[1]$ with $T[1]M$ (in order to take the tangent lift).

Proof We stress that, as the total degree of $V \in \Omega^1(A[1], TA[1])$ is 0, the identity (3.1) is not trivial. Now, let $\mathcal{N}$ be a Nijenhuis operator on M, and let $(TM)_{\mathcal{N}}$ be the associated Lie algebroid structure on TM. We denote by $d_{\mathcal{N}}$ the corresponding homological vector field on $T[1]M$. We first prove (3.2). We have to prove that

$$\mathcal{L}_{d_{\mathcal{N}}} V = \mathcal{N}^{\tan}_{[1]}. \tag{3.3}$$

As both sides are $(1, 1)$ tensors of internal degree 0 on $T[1]M$, it is enough to show that they correspond to the same triple $(\mathcal{D}, \ell, T^M)$. From Example 2.2, we have to show that the triple $(\mathcal{D}, \ell, T^M)$ corresponding to the left-hand side is $([-, \mathcal{N}]^{\mathrm{fn}}, \mathcal{N}, \mathcal{N})$. This easily follows from Lemma 2.3 by using that $V = \mathbb{I}^{\uparrow}_M$. However, beware that although, in this case, a in (2.4) is an ordinary vector field, the Lie derivative $\mathcal{L}^A$ appearing therein is not the ordinary Lie derivative (but the Lie algebroid one). In this particular case, they agree, indeed for all $Y, X \in \mathfrak{X}(M)$

$$\begin{aligned} \mathcal{D}(Y)(X) &= \left(\mathcal{L}^{(TM)_{\mathcal{N}}}_Y \mathbb{I}_M\right) X = [Y, \mathbb{I}_M X]_{\mathcal{N}} - \mathbb{I}_M[\mathcal{N}Y, X] \\ &= [\mathcal{N}Y, X] + [Y, \mathcal{N}X] - \mathcal{N}[Y, X] - [\mathcal{N}Y, X] \\ &= (\mathcal{L}_Y \mathcal{N})(X) = [Y, \mathcal{N}]^{\mathrm{fn}}(X). \end{aligned} \tag{3.4}$$

We still need to show that $[[d_{\mathcal{N}}, V]^{\mathrm{fn}}, V]^{\mathrm{fn}} = 0$. This follows from (3.3) and the fact that the Frölicher–Nijenhuis bracket of the vertical endomorphism V and the tangent lift of a $(1, 1)$ tensor always vanishes identically (a long but straightforward computation, e.g., in local coordinates). This concludes the "only if part" of the proof.

Conversely, let $A \Rightarrow M$ be a Lie algebroid and let V be an almost tangent structure of internal degree -1 on $A[1]$. We use it to identify A with the tangent bundle TM and V with the vertical endomorphism. In this way, the anchor ρ_A identifies with a $(1, 1)$ tensor $\mathcal{N}$. Now, first check that (3.1) implies (3.3) (use again local coordinates). Finally, reverse the computations (3.4), and use Lemma 2.3, to see that (3.3) actually implies $[-, -]_A = [-, -]_{\mathcal{N}}$. This concludes the proof. □

Remark 3.3 Recall from [29, Theorem B.1] that a $(1, 1)$ tensor $T \in \Omega^1(A, TA)$ on the total space of a Lie algebroid $A \Rightarrow M$ is IM if and only if $[d_A, T_{[1]}]^{\mathrm{fn}} = \mathcal{L}_{d_A} T_{[1]} = 0$, i.e., if and only if $T_{[1]}$ is a cocycle in the cochain complex $\left(\Omega^1(A, TA), \mathcal{L}_{d_A}\right)$. Now, let $\mathcal{N}$ be a Nijenhuis operator on a manifold M. Using Example 2.2, it is straightforward to check that $\mathcal{N}^{\tan}$ is an IM tensor on the Lie algebroid $(TM)_{\mathcal{N}}$, equivalently $\mathcal{N}^{\tan}_{[1]}$ is a cocycle in $\left(\Omega^1((TM)_{\mathcal{N}}, T(TM)_{\mathcal{N}}), \mathcal{L}_{d_{\mathcal{N}}}\right)$. Formula (3.2) says something more: not only $\mathcal{N}^{\tan}_{[1]}$ is a cocycle, but also it is actually a coboundary with the vertical endomorphism V being a primitive.

There is an elegant alternative way to formulate Theorem 3.2. Namely, condition (3.1) is actually equivalent to the simpler condition

$$[d_{dR}, d_A] = 0.$$

To see this, remember that vector fields on $T[1]M$ are derivations of the graded algebra $\Omega^\bullet(M)$ of differential forms on M. Derivations commuting with the De Rham differential are exactly those of the form $\mathcal{L}_K$ for some vector valued form $K \in \Omega^\bullet(M, TM)$. We conclude that a degree 1 derivation Q of $\Omega^\bullet(M)$ satisfies $[d_{dR}, Q] = 0$ if and only if $Q = \mathcal{L}_{\mathcal{N}}$ for some $(1, 1)$ tensor $\mathcal{N} \in \Omega^1(M, TM)$. In this case, from the properties of the Frölicher–Nijenhuis bracket, Q is a homological derivation if and only if $\mathcal{N}$ is a Nijenhuis operator, in which case $(T[1]M, Q)$ is also the $\mathbb{N}Q$-manifold of degree 1 corresponding to the Lie algebroid $(TM)_{\mathcal{N}}$. This proves the following

Theorem 3.4 *A Lie algebroid $A \Rightarrow M$ is isomorphic to the Lie algebroid $(TM)_{\mathcal{N}}$ induced by a Nijenhuis operator $\mathcal{N} \in \Omega^1(M, TM)$ on M if and only if there exists a, necessarily integrable, almost tangent structure V of internal degree -1 on $A[1]$ such that*

$$[d_{dR}, d_A] = 0.$$

We conclude this section noticing that there is a characterization of the De Rham differential on an $\mathbb{N}$-manifold of degree 1 equipped with an almost tangent structure of internal degree -1. Namely, we have the following proposition that might be of independent interest.

Proposition 3.5 *The De Rham differential d_{dR} is the only homological vector field on $T[1]M$ such that*

$$\iota_{d_{dR}} V = \mathcal{E} \tag{3.5}$$

where $\mathcal{E} \in \mathfrak{X}(T[1]M)$ is the Euler vector field (i.e., the only degree 0 derivation $\mathcal{E}$ of the graded algebra $C^\infty(T[1]M) = \Omega^\bullet(M)$ such that $\mathcal{E}(\omega) = k\omega$ for all homogeneous differential forms ω of degree k).

Proof Formula (3.5) can be proved easily, e.g., in local coordinates. Now, suppose that there exists another homological vector field Q on $T[1]M$ such that

$$\iota_Q V = \mathcal{E}. \tag{3.6}$$

Then, Q gives to TM the structure of a Lie algebroid whose anchor $\rho : TM \to TM$ is the identity $\mathbb{I}_M$. Indeed, locally

$$Q = \rho^i_j \dot{x}^j \frac{\partial}{\partial x^i} - \frac{1}{2} c^k_{ij} \dot{x}^i \dot{x}^j \frac{\partial}{\partial \dot{x}^k}$$

and the condition (3.6) reads $\rho^i_j = \delta^i_j$, i.e., $\rho = \mathbb{I}_M$. As the anchor is a Lie algebroid map with values in the standard tangent bundle Lie algebroid, Q must be the De Rham differential. □

Remark 3.6 It is not strictly necessary to exploit graded geometry to describe the Lie algebroid of a Nijenhuis structure nor IM (1,1) tensors. Namely, the cochain complex $(\Omega^1(A[1], TA[1]), \mathcal{L}_{d_A})$ possesses an equivalent description in terms of more classical data which does not need graded manifolds. However, the latter is much more involved and less intuitive. More generally, we believe that the graded geometric picture for Lie algebroids is often more compact and efficient. In this respect, the reader should compare, e.g., Formulas (2.3) with their equivalent formulation $\mathcal{L}_{d_A} T^{[1]} = 0$. Moreover, a graded geometric formulation does usually pave the way to interesting generalizations/applications. This is the case, e.g., for the graded geometric description of Poisson manifolds (resp. Courant algebroids) as degree 1 (resp. degree 2) symplectic $\mathbb{N}Q$-manifolds (see Roytenberg's paper [31] and all its citations). For these reasons, we adopted graded geometry as our preferred language for Lie algebroids in our situation as well.

4 Nijenhuis structures and Lie groupoids

Let $G \rightrightarrows M$ be a Lie groupoid and let $A \Rightarrow M$ be its Lie algebroid. We say that G *integrates* a Nijenhuis operator $\mathcal{N}$ on M if there exists a Lie algebroid isomorphism $A \cong (TM)_{\mathcal{N}}$. In the next theorem, we characterize Lie groupoids integrating a Nijenhuis operator.

Theorem 4.1 *A Lie groupoid $G \rightrightarrows M$ with Lie algebroid $A \Rightarrow M$ integrates a Nijenhuis operator $\mathcal{N} \in \Omega^1(M, TM)$ on M if and only if there exists a vector bundle map $U : TM \to A$ such that*

(1) $\ker \overrightarrow{U} = \ker dt$ *and* $\operatorname{im} \overrightarrow{U} = \ker ds$,
(2) $[\overrightarrow{U}, \overrightarrow{U}]^{\mathrm{fn}} = 0$*, i.e.,* $\overrightarrow{U}$ *is a Nijenhuis operator on G.*

In this case, put $\mathcal{N} = s_* \delta U = t_* \delta U$. Then, $\mathcal{N}$ is a Nijenhuis operator on M, and $U : A \to (TM)_{\mathcal{N}}$ is a Lie algebroid isomorphism. Finally, if we use U to identify A and $(TM)_{\mathcal{N}}$, we also have

$$\delta \dot{U} = \mathcal{N}^{\tan}. \tag{4.1}$$

Proof Let us first assume that G integrates a Nijenhuis operator $\mathcal{N} \in \Omega^1(M, TM)$, so that we can identify A and $(TM)_{\mathcal{N}}$. Let $U = \mathbb{I}_M : TM \to TM$ be the identity. Then, locally

$$\overrightarrow{U} = t^*(dx^i) \otimes \overrightarrow{\partial_i}, \tag{4.2}$$

for some local coordinates (x^i) on M, where we put $\partial_i := \partial/\partial x^i$. As $\dim G = 2 \dim M$, point *(1)* immediately follows. Next, compute

$$\begin{aligned} \left[\overrightarrow{U}, \overrightarrow{U}\right]^{\mathrm{fn}} &= \left[t^*(dx^i) \otimes \overrightarrow{\partial_i}, t^*(dx^j) \otimes \overrightarrow{\partial_j}\right]^{\mathrm{fn}} \\ &= t^*(dx^i \wedge dx^j) \otimes \left[\overrightarrow{\partial_i}, \overrightarrow{\partial_j}\right]^{\mathrm{fn}} + 2t^*(dx^i) \wedge \mathcal{L}_{\overrightarrow{\partial_i}} t^*(dx^k) \otimes \overrightarrow{\partial_k}, \end{aligned} \tag{4.3}$$

where we used standard properties of the Frölicher–Nijenhuis bracket (together with the fact that the coordinate 1-forms are closed). Now, we have

$$\mathcal{N}\partial_i = \mathcal{N}_i^k \partial_k \quad \text{and} \quad \left[\partial_i, \partial_j\right]_{\mathcal{N}} = c_{ij}^k \partial_k = \left(\partial_i \mathcal{N}_j^k - \partial_j \mathcal{N}_i^k\right) \partial_k,$$

for some local functions $\mathcal{N}_i^k, c_{ij}^k$. As

$$t_* \overrightarrow{\partial_i} = \mathcal{N}\partial_i = \mathcal{N}_i^k \partial_k,$$

it follows that

$$\left[\overrightarrow{U}, \overrightarrow{U}\right]^{\mathrm{fn}} = t^* \left(c_{ij}^k dx^i \wedge dx^j + 2 dx^i \wedge d\mathcal{N}_i^k\right) \otimes \overrightarrow{\partial_k} = \left(c_{ij}^k - \partial_i \mathcal{N}_j^k + \partial_j \mathcal{N}_i^k\right) \otimes \overrightarrow{\partial_k} = 0, \tag{4.4}$$

as claimed. A very similar computation shows that $[\overleftarrow{U}, \overleftarrow{U}]^{\mathrm{fn}} = 0$ as well. It is even easier to check that $[\overrightarrow{U}, \overleftarrow{U}]^{\mathrm{fn}} = 0$. It follows that

$$\left[\delta U, \delta U\right]^{\mathrm{fn}} = \left[\overrightarrow{U} + \overleftarrow{U}, \overrightarrow{U} + \overleftarrow{U}\right]^{\mathrm{fn}} = \left[\overrightarrow{U}, \overrightarrow{U}\right]^{\mathrm{fn}} + 2\left[\overrightarrow{U}, \overleftarrow{U}\right]^{\mathrm{fn}} + \left[\overleftarrow{U}, \overleftarrow{U}\right]^{\mathrm{fn}} = 0.$$

Formula (4.1) now follows from Lemma 2.3 (and Example 2.2).

Conversely, let $U : TM \to A$ be a vector bundle map satisfying *(1)* and *(2)*. It follows from $\ker \overrightarrow{U} = \ker dt$, resp. $\operatorname{im} \overrightarrow{U} = \ker ds$, by restriction to the units, that U is injective, resp. surjective. Hence, U is a vector bundle isomorphism that we can use to identify A with TM, as vector bundles. Similarly, we identify the anchor map $\rho_A : A \to TM$ with a $(1,1)$ tensor $\mathcal{N} \in \Omega^1(M, TM)$. Additionally, under this identification, $\overrightarrow{U}$ is locally given by (4.2) again. Now, the same computations (4.3) and (4.4) as above, together with condition *(2)* in the statement, show that $A = (TM)_{\mathcal{N}}$ as claimed. □

Remark 4.2 Let $G \rightrightarrows M$ be a Lie groupoid, let $A \Rightarrow M$ be its Lie algebroid, and let $U : TM \to A$ be a vector bundle map. Using, e.g., (2.5), we immediately see that condition *(1)* in Theorem 4.1 is actually equivalent to U being a vector bundle isomorphism. Condition *(2)* can also be expressed purely in terms of U (and independently of condition *(1)*). To do this, we define the A-*torsion* of U to be the A-valued 2-form $T_U^A \in \Omega^2(M, A)$ given by

$$T_U^A(X, Y) := [UX, UY]_A + U\rho_A U[X, Y] - U[\rho_A UX, Y] - U[X, \rho_A UY].$$

Now, given an A-valued 2-form $T \in \Omega^2(M, A)$, the obvious adaptation of Formulas (2.1) defines vector valued 2-forms $\overrightarrow{T}, \overleftarrow{T} \in \Omega^2(G, TG)$ on G. We also have that $\overrightarrow{T}|_{TM} = T$ and $\overleftarrow{T}|_{TM} = di \circ T$. Finally, a long but easy computation exploiting, e.g., (2.5), shows that

$$T_{\overrightarrow{U}} = \overrightarrow{T_U^A}, \quad \text{and} \quad T_{\overleftarrow{U}} = \overleftarrow{T_U^A}.$$

We leave the details to the reader. We conclude that condition *(2)* of Theorem 4.1 is equivalent to

$$T_U^A = 0.$$

We now explain in which sense Theorem 4.1 is an *integration result* analogous to the (now classical) "Poisson structures integrate to symplectic groupoids" theorem. First of all, Theorem 3.2 can be rephrased by saying that a Nijenhuis structure $\mathcal{N}$ on a manifold M can be equivalently encoded into a Lie algebroid $A \Rightarrow M$ equipped with an IM Nijenhuis structure $\mathcal{N}_A$ satisfying an additional condition: (not only $\mathcal{N}_A^{[1]}$ is a cocycle, but) $\mathcal{N}_A^{[1]}$ is a coboundary in the cochain complex $(\Omega^1(A[1], TA[1]), \mathcal{L}_{d_A})$ possessing a primitive V with two properties, namely 1) V is as much non-degenerate as possible for a $(1,1)$ tensor of internal degree -1 on $A[1]$, i.e., V is an almost tangent structure, and 2) V is integrable, i.e., its Nijenhuis torsion vanishes. Notice that there is a certain degree of redundancy in the latter formulation. In any case, as IM tensors integrate to multiplicative tensors [8] (see also [24] for a thorough discussion of the Nijenhuis torsion of an IM (1,1) tensor), an integration theorem for Nijenhuis structures is expected to be of the form:

Let $\mathcal{N}$ be a Nijenhuis structure on a manifold M, and assume that the associated Lie algebroid $A \Rightarrow M$ is integrable, then it integrates to a (source simply connected) Lie groupoid *equipped with a unique multiplicative Nijenhuis tensor $\mathcal{N}_G$ satisfying an appropriate additional condition and such that $\dot{\mathcal{N}}_G = \mathcal{N}_A^{[1]}$.*

Theorem 4.1 confirms these expectations and shows that the appropriate additional condition on $\mathcal{N}_G$ is also what we might expect it to be:
(not only $\mathcal{N}_G$ is a cocycle, but) $\mathcal{N}_G$ is a coboundary in the cochain complex $(C^\bullet_{\mathrm{def}}(G, T^{1,0}), \delta)$ possessing a primitive $U \in \Omega^1(M, A)$ with two properties, namely 1) U is as much non-degenerate as possible, i.e., $U : TM \to A$ is a vector bundle isomorphism, and 2) U is integrable, i.e., its A-torsion vanishes (see Remark 4.2).

As above, there is a certain degree of redundancy in this formulation and, for this reason, we preferred the alternative compact formulation given in Theorem 4.1.

Remark 4.3 In this remark, we compare Theorem 4.1 with the existing literature on Nijenhuis structures, Lie algebroids and Lie groupoids. First of all, in [32], Stiénon and Xu prove an integration theorem for Poisson–Nijenhuis structures. Namely, they show the following. Let M be a manifold equipped with a Poisson–Nijenhuis structure $(\pi, \mathcal{N})$. If the cotangent Lie algebroid $(T^*M)_\pi$ determined by the Poisson structure π is integrable, then it integrates to a source simply connected Lie groupoid equipped with a multiplicative symplectic Nijenhuis structure $(\omega, \tilde{\mathcal{N}})$ uniquely determined by $(\pi, \mathcal{N})$ in an appropriate way (see [32, Theorem 5.2] for the details). This is mostly independent of Theorem 4.1, which discusses the Lie groupoid integrating the Lie algebroid determined by a Nijenhuis structure, not that integrating the cotangent algebroid of a Poisson structure.

A closer relationship exists with the recent work [16] of Drummond (see also [15]) where the author introduces *Lie–Nijenhuis bialgebroids* and prove that they integrate to Lie groupoids equipped with a multiplicative Poisson–Nijenhuis structure. Lie–Nijenhuis bialgebroids are bialgebroids (A, A^*) equipped with an appropriate additional structure. Here, we only stress that the definition is symmetric under swapping A and A^*. A Poisson–Nijenhuis structure $(\pi, \mathcal{N})$ on a manifold M determines a Lie–Nijenhuis bialgebroid structure on (T^*M, TM) (actually a very special one). In the latter Lie–Nijenhuis bialgebroid, the Lie algebroid structure on T^*M is the cotangent algebroid $(T^*M)_\pi$ of the Poisson structure π, while the Lie algebroid structure on TM is the Lie algebroid $(TM)_\mathcal{N}$ of the Nijenhuis structure $\mathcal{N}$. According to Drummond's theory, when $(T^*M)_\pi$, resp. $(TM)_\mathcal{N}$, is integrable, it integrates to a source simply connected Lie groupoid equipped with a multiplicative Poisson–Nijenhuis structure $(\tilde{\pi}, \tilde{\mathcal{N}})$ uniquely determined by $(\pi, \mathcal{N})$. The Poisson structure $\tilde{\pi}$ on the integration of $(T^*M)_\pi$ is the inverse of the usual multiplicative symplectic structure (and, in this case, Drummond's result recovers that of Stiénon-Xu mentioned above). Finally, it is not hard to see that the Nijenhuis structure $\tilde{\mathcal{N}}$ on the integration of $(TM)_\mathcal{N}$ is exactly the $(1, 1)$ tensor δU in the statement of Theorem 4.1. Notice, however, that this case is not explicitly spelled out in [16] (nor in [15]). Concluding, while our setting is less general than that of [16], our analysis reveals aspects that have not been discussed before.

We stress again that, as already mentioned in the introduction, there is a significant difference between Theorem 4.1 and the integration theorem for Poisson manifolds. While we can only guarantee the existence of a multiplicative symplectic structure on the source simply connected integration of an integrable Poisson structure, the tensors U and $\mathcal{N}_G = \delta U$ exist on *every* Lie groupoid integrating a Nijenhuis structure.

We conclude this section discussing a few elementary examples.

Example 4.4 (Trivial Nijenhuis operators) Let G integrate a Njenhuis operator $\mathcal{N} \in \Omega^1(M, TM)$, and let U be as in the statement of Theorem 4.1. Clearly, $\overrightarrow{U} \in \Omega^1(G, TG)$ is not an almost tangent structure, unless $\ker ds = \ker dt$ which happens exactly when

$s = t$, i.e., G is a bundle of Lie groups. But in this case, $\mathcal{N} = 0$ necessarily, so G is a bundle of Lie groups integrating the trivial Lie algebroid structure $(TM)_0$ on TM. The source simply connected such G is $TM \rightrightarrows M$ with both source and target being the canonical projection $TM \to M$, and the multiplication being the fiber-wise addition. We denote this bundle of abelian Lie groups $(TM)_+$. If $U : TM \to TM$ is the identity map then $\overrightarrow{U} = V \in \Omega^1(TM, TTM)$ is the vertical endomorphism (use, e.g., local coordinates) and $\overleftarrow{U} = -V$, so that $\delta U = \overrightarrow{U} + \overleftarrow{U} = 0$ (also in agreement with (4.1)).

On the other hand, let G be a source connected *proper* Lie groupoid integrating $(TM)_0$. In this case, G is a torus bundle and there must be a Lie groupoid map $p : (TM)_+ \to G$. But 1) p is a local diffeomorphism, 2) the vertical endomorphism V on $(TM)_+$ descends to G and 3) $\Lambda := \ker p$ is a lattice in TM. So translations along sections of Λ preserve V. But, actually, V is preserved by a translation along any section of $TM \to M$. It follows that every lattice in TM arises in this way. We conclude that the present situation is very different from that of the integration of the trivial Poisson structure by a proper symplectic groupoid which gives very special lattices in T^*M (hence in TM), namely those corresponding to integral affine structures on M (see [12] for details).

Example 4.5 (Invertible Nijenhuis operators) Let M be a manifold and let $\mathcal{N}$ be an invertible Nijenhuis operator on M (e.g., a complex structure). Then, $\mathcal{N}$ itself is an isomorphism identifying $(TM)_{\mathcal{N}}$ and $(TM)_{\mathbb{I}}$ where $\mathbb{I} : TM \to TM$ is the identical $(1,1)$ tensor. In the following, we will understand this identification. The Lie algebroid $(TM)_{\mathbb{I}}$ is the usual tangent Lie algebroid which is integrated (among others) by the pair groupoid $M \times M \rightrightarrows M$. The identity $\mathbb{I} : TM \to TM$ can also be seen as a $(TM)_{\mathbb{I}}$-valued 1-form on M. If we do so, then

$$\overrightarrow{\mathbb{I}}, \overleftarrow{\mathbb{I}} : T(M \times M) = TM \times TM \to T(M \times M) = TM \times TM$$

are the projections onto the first and the second factor, respectively, so that $\delta\mathbb{I} = \overrightarrow{\mathbb{I}} + \overleftarrow{\mathbb{I}}$ is the identity of $T(M \times M)$ (in agreement with (4.1)).

Example 4.6 (Rank 1 Lie algebroids on a 1 dimensional manifold) Let M be either the line $\mathbb{R}$ or the circle S^1 and let θ be the canonical coordinate on M. Any Lie algebroid structure A on the trivial line bundle $\mathbb{R}_M$ is of the following type:

$$[f, g]_A = fX(g) - X(f)g, \quad \rho_A(f) = fX, \quad f, g \in \Gamma(A) = C^\infty(M),$$

for some vector field $X \in \mathfrak{X}(M)$. Denote $F := X(\theta)$, so that $X = F\frac{\partial}{\partial\theta}$. The vector bundle isomorphism

$$\mathbb{R}_M \to TM, \quad f \mapsto f\frac{\partial}{\partial\theta} \tag{4.5}$$

identifies A with the Lie algebroid $(TM)_{\mathcal{N}}$, where $\mathcal{N}$ is the Nijenhuis operator given by

$$\mathcal{N} = d\theta \otimes X = F\mathbb{I}.$$

Now, in order to illustrate Theorem 4.1, we take the long route to prove this latter thing. Recall that A is integrated by the Lie groupoid $D^X \rightrightarrows M$, where $D^X \subseteq \mathbb{R} \times M$ is the domain of the flow of X:

$$\phi^X : D^X \to M, \quad (\varepsilon, \theta) \mapsto \phi^X_\varepsilon(\theta).$$

The source $s : D^X \to M$ is the projection onto the second factor, while the target t is ϕ^X. Two arrows $(\bar{\varepsilon}, \bar{\theta}), (\varepsilon, \theta) \in D^X$ are composable when $\bar{\theta} = \phi^X_\varepsilon(\theta)$ and, in this case, their product is

$$(\bar{\varepsilon}, \bar{\theta}) \cdot (\varepsilon, \theta) = (\bar{\varepsilon} + \varepsilon, \theta).$$

The inversion $i : D^X \to D^X$ maps (ε, θ) to $(-\varepsilon, \phi^X_\varepsilon(\theta))$.

The inverse of the isomorphism (4.5) is

$$U = d\theta \otimes u : TM \to \mathbb{R}_M$$

where we denoted by u the constant function 1. A straightforward computation shows that

$$\overrightarrow{U} = t^*(d\theta) \otimes \overrightarrow{u} = d\phi^X \otimes \frac{\partial}{\partial \varepsilon} \quad \text{and} \quad \overleftarrow{U} = s^*(d\theta) \otimes \overleftarrow{u} = d\theta \otimes \left(i^* \left(\frac{\partial \phi^X}{\partial \varepsilon} \right) \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varepsilon} \right).$$

Finally, using that

$$\frac{\partial \phi^X}{\partial \varepsilon} = F \circ \phi^X,$$

we find

$$\delta U = \overrightarrow{U} + \overleftarrow{U} = (F \circ \phi^X)\, d\varepsilon \otimes \frac{\partial}{\partial \varepsilon} + \left(\frac{\partial \phi^X}{\partial \theta} - 1 \right) dx \otimes \frac{\partial}{\partial \theta} + F d\theta \otimes \frac{\partial}{\partial \theta}$$

which is readily seen to be a Nijenhuis operator projecting to

$$\mathcal{N} = F d\theta \otimes \frac{\partial}{\partial \theta} = F\mathbb{I}$$

under both s and t. It follows that $U : (TM)_{\mathcal{N}} \to A$ is a Lie agebroid isomorphism as already noticed.

5 More examples

In this section, we discuss some slightly less trivial examples of Lie groupoids integrating a Nijenhuis operator, including their multiplicative Nijenhuis structures.

5.1 The vertical endomorphism of the tangent bundle

Let M be a manifold and let $V \in \Omega^1(M, TM)$ be an integrable almost tangent structure on M. In particular, V is a Nijenhuis operator, and we have a Lie algebroid $(TM)_V \Rightarrow M$. The local model for an integrable almost tangent structure is the vertical endomorphism of the tangent bundle. For simplicity, we assume that $M = TB$ globally for some manifold B, and that V is exactly the vertical endomorphism. In this case, the Lie algebroid $(TM)_V \Rightarrow M$ is integrated by a Lie groupoid $G \rightrightarrows M$ (depending on B only) that we now describe. As a manifold, $G = TM = TTB$ (the double tangent bundle of B). To the best of our knowledge, the following groupoid structure on TTB appears here for the first time. In order to describe it, we recall a few properties of the double tangent bundle. First of all, it is a double vector bundle:

$$\begin{array}{ccc} TTB & \xrightarrow{\tau'} & TB \\ {\scriptstyle\tau}\downarrow & & \downarrow \\ TB & \longrightarrow & B \end{array} \quad .$$

The vertical projection $\tau : TTB \to TB$ is the usual tangent bundle projection mapping a tangent vector to its base point. The horizontal projection $\tau' : TTB \to TB$ is the tangent to the projection $TB \to B$. We denote by $(+, \cdot)$ the fiber-wise operations (addition and scalar multiplication) of the vector bundle with projection τ, and by $(+', \cdot')$ the fiber-wise operations of the vector bundle with projection τ'. The latter are the tangent to the fiber-wise operations of the vector bundle $TB \to B$. Given local coordinates $z = (z^i)$ on B, we denote by $(z, \dot{z})$ the associated tangent coordinates on TB, and by $(z, \dot{z}, z', \dot{z}')$, the tangent coordinates on TTB associated with the coordinates $(z, \dot{z})$. In these coordinates, we have

$$\tau(z, \dot{z}, z', \dot{z}') = (z, \dot{z}), \quad \tau'(z, \dot{z}, z', \dot{z}') = (z, z').$$

Additionally,

$$(x, \dot{x}, z', \dot{z}') + (x, \dot{x}, w', \dot{w}') = (x, \dot{x}, z' + w', \dot{z}' + \dot{w}'), \quad a \cdot (x, \dot{x}, z', \dot{z}') = (x, \dot{x}, az', a\dot{z}'),$$

and

$$(x, \dot{z}, x', \dot{z}') +' (x, \dot{w}, x', \dot{w}') = (x, \dot{z} + \dot{w}, x', \dot{z}' + \dot{w}'), \quad b \cdot' (x, \dot{z}, x', \dot{z}') = (x, b\dot{z}, x', b\dot{z}'),$$

for all $a, b \in \mathbb{R}$. Finally, there is a canonical involution $\varkappa : TTB \to TTB$ swapping the two vector bundle structures. In coordinates

$$\varkappa(z, \dot{z}, z', \dot{z}') = (z, z', \dot{z}, \dot{z}')$$

(for a coordinate-free definition of $\varkappa$ see, e.g., [26, Section 9.6]).

We are now ready to describe the Lie groupoid structure on TTB integrating the vertical endomorphism V on TB. Source and target $s, t : TTB \to TB$ are given by

$$s(\xi) = \tau(\xi) - \tau'(\xi), \quad t(\xi) = \tau(\xi) + \tau'(\xi).$$

As $\tau(\xi)$ and $\tau'(\xi)$ have the same base point for all $\xi \in TTB$, both s and t are well-defined. In order to define the multiplication

$$m : TTB \,{}_s\times_t\, TTB \to TTB,$$

take $\xi, \zeta \in TTB$ such that $s(\xi) = t(\zeta)$, and let $\eta \in TTB$ be any vector such that

$$\tau(\eta) = \tau'(\xi) \quad \text{and} \quad \tau'(\eta) = \tau'(\zeta).$$

We put

$$m(\xi, \zeta) = \big(\xi -' \varkappa(\eta)\big) + \big(\zeta +' \eta\big).$$

In coordinates

$$m\left((x, \dot{z}, z', \dot{z}'), (x, \dot{w}, w', \dot{w}')\right) = (x, \dot{z} - w', z' + w', \dot{z}' + \dot{w}')$$

where $\dot{z} - z' = \dot{w} + w'$. This shows that $m(\xi, \zeta)$ is independent of the choice of η. The unit $u : TB \to TTB$ is the zero section of the vertical vector bundle $\tau : TTB \to TB$. Finally, the inversion $i : TTB \to TTB$ is the fiber-wise multiplication by -1 wrt the vertical vector bundle structure. A direct computation, e.g., in coordinates, shows that with these structure maps, TTB is indeed a Lie groupoid over TB. Denote by $G \rightrightarrows TB$ this Lie groupoid. We want to show that G integrates the vertical endomorphism on TB. To do this, we need to describe the Lie algebroid $A \Rightarrow TB$ of G. As $\tau : G \to TB$ is a vector bundle projection and $u : TB \to G$ is the zero section of this vector bundle, we have a canonical splitting $u^*(TG) \cong TTB \oplus_{TB} \dot{T}TB$, where $\dot{T}TB$ denotes the copy of TTB corresponding to the tangent spaces to the τ-fibers at zeros. Hence, $A = u^*(\ker ds) \hookrightarrow TTB \oplus_{TB} \dot{T}TB$. It is now easy to see that the map

$$U := \tfrac{1}{2}(V \oplus \mathbb{I}) : TTB \to TTB \oplus_{TB} \dot{T}TB$$

is an injective map whose image is exactly A, and so it is a vector bundle isomorphism $TTB \cong A$. In coordinates

$$U\frac{\partial}{\partial z}|_{(z,\dot{z})} = \frac{1}{2}\left(\frac{\partial}{\partial \dot{z}} + \frac{\partial}{\partial z'}\right)|_{(z,\dot{z},0,0)}, \quad \text{and} \quad U\frac{\partial}{\partial \dot{z}}|_{(z,\dot{z})} = \frac{1}{2}\frac{\partial}{\partial \dot{z}'}|_{(z,\dot{z},0,0)},$$

i.e.,

$$U = \frac{1}{2}\left(dz \otimes \left(\frac{\partial}{\partial \dot{z}} + \frac{\partial}{\partial z'}\right)|_M + d\dot{z} \otimes \frac{\partial}{\partial \dot{z}'}|_M\right).$$

It remains to show that U identifies the Lie algebroid structure $(TTB)_V$ with that of A. Instead of doing this directly, we apply Theorem 4.1. First of all, we compute $\overrightarrow{U}$. It is easy to see that

$$\overrightarrow{\left(\frac{\partial}{\partial \dot{z}} + \frac{\partial}{\partial z'}\right)|_M} = \frac{\partial}{\partial \dot{z}} + \frac{\partial}{\partial z'} \quad \text{and} \quad \overrightarrow{\frac{\partial}{\partial \dot{z}'}|_M} = \frac{\partial}{\partial \dot{z}'}. \tag{5.1}$$

Now, denote by $V_{TTB} \in \Omega^1(TTB, TTTB)$ the vertical endomorphism on TTB. It follows from (5.1) and the first one of (2.5) that

$$\begin{aligned}\overrightarrow{U} &= \frac{1}{2}\left(d(t^*z) \otimes \left(\frac{\partial}{\partial \dot{z}} + \frac{\partial}{\partial z'}\right) + d(t^*\dot{z}) \otimes \frac{\partial}{\partial \dot{z}'}\right) \\ &= \frac{1}{2}\left(dz \otimes \left(\frac{\partial}{\partial \dot{z}} + \frac{\partial}{\partial z'}\right) + d\left(\dot{z} + z'\right) \otimes \frac{\partial}{\partial \dot{z}'}\right) \\ &= \frac{1}{2}\left(\varkappa_*(V_{TTB}) + V_{TTB}\right)\end{aligned}$$

which clearly fulfills both conditions *(1)* and *(2)* in Theorem 4.1. Similarly,

$$\overleftarrow{U} = \frac{1}{2}\left(\varkappa_*(V_{TTB}) - V_{TTB}\right).$$

Hence,

$$\delta U = \overrightarrow{U} + \overleftarrow{U} = \varkappa_*(V_{TTB})$$

which is a Nijenhuis operator projecting on V along both $s, t : TTB \to TB$. Using Theorem 4.1, we conclude that U identifies the Lie algebroid structure $(TTB)_V$ with that of A, as claimed.

5.2 Integrable projections

A *projection* on a manifold M is a $(1, 1)$ tensor P such that $P^2 = P$. It follows that $V := \operatorname{im} P$ and $H := \ker P$ are regular distributions (the *vertical* and *horizontal* distributions) such that $TM = V \oplus H$. A projection P is *integrable* if it is additionally a Nijenhuis operator. In this case, both V and H are involutive distributions. Let P be an integrable projection on M. For simplicity, we assume that the foliation integrating the vertical distribution V is simple, i.e., the leaf space B is a manifold, and the natural projection $\pi : M \to B$ is a surjective submersion. In this case, $V = T^\pi M$, the π-vertical tangent bundle, and H is a flat Ehresmann connection on the fibration $\pi : M \to B$. Locally, we can choose fibered coordinates (x^i, u^α) on M such that

$$P = du^\alpha \otimes \frac{\partial}{\partial u^\alpha}.$$

In the following, we will always use π to identify H with the pull-back vector bundle $\pi^* TB$ in the obvious way. This vector bundle carries a natural representation of the Lie algebroid $T^\pi M \Rightarrow M$ (where the anchor is the inclusion $T^\pi M \hookrightarrow TM$ and the bracket is the commutator of vector fields tangent to fibers of π). This representation is given by the *Bott connection*: the unique $T^\pi M$-connection in $H = \pi^* TB$ such that all pull-back sections are parallel. The Lie algebroid $(TM)_P \Rightarrow M$ of the integrable projection P is easily seen to be isomorphic to the semi-direct product Lie algebroid $T^\pi M \ltimes H = T^\pi M \times_B TB \Rightarrow M$ under

$$TM \to T^\pi M \times_B TB, \quad v \mapsto (Pv, \pi_* v). \tag{5.2}$$

In order to illustrate our main result, we now prove the Lie algebroid isomorphism $(TM)_P \cong T^\pi M \ltimes H$ using Theorem 4.1. To do this, we need to fix a Lie groupoid integrating $T^\pi M \ltimes H$. First, we choose an integration of the Lie algebroid $T^\pi M$. The easiest choice is the *submersion groupoid* $M \times_B M$ whose structure maps are

$$s(x, y) = x, \quad t(x, y) = y, \quad m((x, y), (z, x)) = (z, y),$$
$$u(x) = (x, x), \quad i(x, y) = (y, x).$$

The vector bundle $H = \pi^* TB$ carries a canonical representation of the submersion groupoid integrating the Bott connection given by

$$(x, y).(x, v) = (y, v), \quad \text{for all } (x, y) \in M \times_B M, \text{and } v \in T_{\pi(x)=\pi(y)} B.$$

It follows that the semidirect product Lie groupoid $(M \times_B M) \ltimes H = M \times_B M \times_B TB \rightrightarrows M$ integrates the Lie algebroid $T^\pi M \ltimes H = T^\pi M \times_B TB \Rightarrow M$. The structure maps in $M \times_B M \times_B TB$ are

$$\begin{aligned} &s(x, y, v) = x, \quad t(x, y, v) = y, \quad m((x, y, v), (z, x, w)) = (z, y, v + w),\\ &u(x) = (x, x, 0_x), \quad i(x, y, v) = (y, x, -v). \end{aligned} \tag{5.3}$$

Notice that $T^\pi M \times_B TB \Rightarrow M$ identifies with the Lie algebroid

$$A \subseteq T(M \times_B M \times_B TB) = TM \times_{TB} TM \times_{TB} TTB$$

of $M \times_B M \times_B TB$ under the inclusion

$$T^\pi M \times_B TB \hookrightarrow TM \times_{TB} TM \times_{TB} TTB, \quad (\xi, v) \mapsto \left(0_x, \xi, v^\uparrow_{0_{\pi(x)}}\right),$$

where $x = \tau(\xi)$. Under this inclusion, the map (5.2) becomes the vector bundle isomorphism

$$U : TM \to A, \quad \xi \mapsto \left(0_x, P\xi, (\pi_*\xi)^\uparrow_{0_{\pi(x)}}\right).$$

A straightforward computation using this and the structure maps (5.3) shows that the $(1, 1)$ tensors

$$\overrightarrow{U}, \overleftarrow{U} : TM \times_{TB} TM \times_{TB} TTB \to TM \times_{TB} TM \times_{TB} TTB$$

are given by

$$\overrightarrow{U}(\xi, \eta, W) = \left(0_x, P\eta, (\pi_*\xi)^\uparrow_w\right), \quad \text{and} \quad \overleftarrow{U}(\xi, \eta, W) = \left(P\xi, 0_y, -(\pi_*\xi)^\uparrow_w\right)$$

where (ξ, η, W) is a tangent vector at the point $(x, y, w) \in M \times_B M \times_B TB$. The coordinates (x^i, u^α) on M induce coordinates $(x^i, u_1^\alpha, u_2^\alpha, \dot{x}^i)$ on $M \times_B M \times TB$, where (u_1^α) (resp. (u_2^α)) are fiber coordinates on the first (resp. second) factor, and $(\dot{x}^i)$ are fiber tangent coordinates on the last factor. In these coordinates

$$\overrightarrow{U} = du_2^\alpha \otimes \frac{\partial}{\partial u_2^\alpha} + dx^i \otimes \frac{\partial}{\partial \dot{x}^i}, \quad \text{and} \quad \overleftarrow{U} = du_1^\alpha \otimes \frac{\partial}{\partial u_1^\alpha} - dx^i \otimes \frac{\partial}{\partial \dot{x}^i}$$

which obviously satisfy conditions *(1)* and *(2)* of Theorem 4.1 (alternatively one can check that the A-torsion of U vanishes identically and then use Remark 4.2). Finally,

$$\delta U = \overrightarrow{U} + \overleftarrow{U} : TM \times_{TB} TM \times_{TB} TTB \to TM \times_{TB} TM \times_{TB} TTB$$

is given by

$$\delta U(\xi, \eta, W) = (P\xi, P\eta, 0)$$

which projects onto P under both the source and the target, as claimed (equivalently $\rho_A \circ U = P$).

5.3 Pre-Lie algebras

A *pre-Lie algebra* (aka *left symmetric algebra*) is a vector space $\mathfrak{a}$ equipped with a bilinear map

$$\triangleright : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}, \quad (a, b) \mapsto a \triangleright b$$

such that

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c), \quad \text{for all } a, b, c \in \mathfrak{a}.$$

In other words, the *associator* of $\mathfrak{a}$ is symmetric in the first two entries. Associative algebras are instances of pre-Lie algebras. If $(\mathfrak{a}, \triangleright)$ is a pre-Lie algebra, then the *commutator*:

$$[-,-]_\triangleright : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}, \quad (a, b) \mapsto [a, b]_\triangleright := a \triangleright b - b \triangleright a$$

is a Lie bracket. We will denote by $\mathfrak{a}_{\mathrm{Lie}}$ the Lie algebra $(\mathfrak{a}, [-,-]_\triangleright)$. The Lie algebra $\mathfrak{a}_{\mathrm{Lie}}$ comes with a canonical representation $L : \mathfrak{a}_{\mathrm{Lie}} \to \mathfrak{gl}(\mathfrak{a})$ on $\mathfrak{a}$ itself given by

$$L(a)(x) = a \triangleright x, \quad \text{for all } a, x \in \mathfrak{a}.$$

Let $\mathfrak{a}$ be a finite dimensional real pre-Lie algebra. Then, $\mathfrak{a}$ can be seen as a manifold. For any $a \in \mathfrak{a}$, we denote by $a^\uparrow \in \mathfrak{X}(\mathfrak{a})$ the constant vector field equal to a and, for any endomorphism $\phi : \mathfrak{a} \to \mathfrak{a}$, we denote by $X_\phi \in \mathfrak{X}(\mathfrak{a})$ the only vector field such that $[X_\phi, a^\uparrow] = \phi(a)^\uparrow$ for all $a \in \mathfrak{g}$. (This is consistent with our notation in Sect. 3.) There is a canonical $(1, 1)$ tensor $\mathcal{N}$ on $\mathfrak{a}$ given by

$$\mathcal{N} a^\uparrow = X_{L(a)}, \quad \text{for all } a \in \mathfrak{a}.$$

Equivalently,

$$\mathcal{N} a_x^\uparrow = -a \triangleright x, \quad \text{for all } a, x \in \mathfrak{a}.$$

The $(1, 1)$ tensor $\mathcal{N}$ is always a Nijenhuis operator and every Nijenhuis operator on a vector space whose components are linear in linear coordinates arises in this way [1, 21]. We remark that our Nijenhuis operator $\mathcal{N}$ differs in sign from that in [1, 21]. Our sign conventions make it easier the description of the associated Lie algebroid.

We will describe the source-simply connected Lie groupoid integrating $\mathcal{N}$. We begin describing $(T\mathfrak{a})_{\mathcal{N}}$. For all $a, b \in \mathfrak{a}$, we have

$$[a^{\uparrow}, b^{\uparrow}]_{\mathcal{N}} = [\mathcal{N}a^{\uparrow}, b^{\uparrow}] + [a^{\uparrow}, \mathcal{N}b^{\uparrow}] = [X_{L(a)}, b^{\uparrow}] + [a^{\uparrow}, X_{L(b)}]$$
$$= L(a)(b) - L(b)(a) = [a, b]_{\triangleright}.$$

The latter equation, together with the anchor being $\mathcal{N}$, determines the Lie algebroid $(T\mathfrak{a})_{\mathcal{N}}$ completely. Namely, it is immediate that $(T\mathfrak{a})_{\mathcal{N}}$ is isomorphic to the action Lie algebroid $A := \mathfrak{a}_{\mathrm{Lie}} \ltimes \mathfrak{a} \Rightarrow \mathfrak{a}$ via

$$U : T\mathfrak{a} \to \mathfrak{a} \times \mathfrak{a}, \quad v_x^{\uparrow} \to (v, x).$$

Again, in order to illustrate the main result of the paper, we prove this straightforward fact taking a longer route. So, let G be the simply connected Lie group integrating $\mathfrak{a}_{\mathrm{Lie}}$. The pre-Lie algebra structure on $\mathfrak{a}$ induces a left invariant affine structure on G and every source-simply connected Lie group with a left invariant affine structure arises in this way (see, e.g., [6]). We will not really use this affine structure, but we will need the G-action $\mathcal{L} : G \times \mathfrak{a} \to \mathfrak{a}$ on $\mathfrak{a}$ integrating the Lie algebra action L.

Lemma 5.1 *Let $(\mathfrak{a}, \triangleright)$ be a finite dimensional, real pre-Lie algebra and let G be the source simply-connected Lie group integrating $\mathfrak{a}_{\mathrm{Lie}}$. Then, the action $\mathcal{L}$ satisfies*

$$\mathcal{L}_g(x \triangleright y) = \mathrm{ad}_g x \triangleright \mathcal{L}_g y, \quad \textit{for all } x, y \in \mathfrak{a}. \tag{5.4}$$

Proof By connectedness, it is enough to prove that (5.4) is satisfied at the infinitesimal level. Differentiating $\mathcal{L}_g(x \triangleright y) - \mathrm{ad}_g x \triangleright \mathcal{L}_g y$, we find

$$\dot{g} \triangleright (x \triangleright y) - [\dot{g}, x]_{\triangleright} \triangleright y - x \triangleright (\dot{g} \triangleright y)$$
$$= \dot{g} \triangleright (x \triangleright y) - (\dot{g} \triangleright x) \triangleright y + (x \triangleright \dot{g}) \triangleright y - x \triangleright (\dot{g} \triangleright y) = 0,$$

for all $\dot{g}, x, y \in \mathfrak{a}$. This concludes the proof. □

We want to show that the action groupoid $G \ltimes \mathfrak{a} \rightrightarrows \mathfrak{a}$ corresponding to $\mathcal{L}$ integrates $\mathcal{N}$ via Theorem 4.1. Recall that the structure maps of $G \ltimes \mathfrak{a}$ are:

$$s(g, x) = x, \quad t(g, x) = \mathcal{L}_g x, \quad m((g, \mathcal{L}_h x), (h, x)) = (gh, x),$$
$$u(x) = (1_G, x), \quad i(g, x) = (g^{-1}, \mathcal{L}_g x).$$

The Lie algebroid of $G \times \mathfrak{a} \to \mathfrak{a}$ is the action Lie algebroid A that we already considered. We want to show that U satisfies both conditions *(1)* and *(2)* of Theorem 4.1 and that $s_* \delta U = \mathcal{N}$. This will confirm that $U : (T\mathfrak{g})_{\mathcal{N}} \to A$ is a Lie algebroid isomorphism.

A straightforward computation that we leave to the reader shows that

$$\overrightarrow{U}\left(\overrightarrow{\xi}_g, a_x^{\uparrow}\right) = \left(\left(\overrightarrow{\mathcal{L}_g a - \xi \triangleright \mathcal{L}_g x}\right)_g, 0_x\right)$$

and

$$\overleftarrow{U}\left(\overrightarrow{\xi}_g, a_x^{\uparrow}\right) = \left(-\left(\overrightarrow{\mathrm{ad}_g a}\right)_g, -(a \triangleright x)_x^{\uparrow}\right),$$

hence

$$\delta U\left(\overrightarrow{\xi}_g, a_x^{\uparrow}\right) = \left(\left(\overrightarrow{\mathcal{L}_g a - \mathrm{ad}_g a - \xi \triangleright \mathcal{L}_g x}\right)_g, -(a \triangleright x)_x^{\uparrow}\right)$$

for all $(g, x) \in G \ltimes \mathfrak{a}$, all $\xi \in \mathfrak{a}_{\mathrm{Lie}}$ and all $a \in \mathfrak{a}$, where $\overrightarrow{\xi} \in \mathfrak{X}(G)$ is the right invariant vector field corresponding to ξ. One can show that $\overrightarrow{U}$ is actually a Nijenhuis operator by computing the A-torsion of U, and then using Remark 4.2. This is easy: for every $a \in \mathfrak{a}$, denote by $c_a \in \Gamma(A)$ the constant section equal to a. We have $[c_a, c_b]_A = c_{[a,b]_\triangleright}$ and $\rho_A(c_a) = X_{L(a)}$ for all $a, b \in \mathfrak{a}$. Additionally, $Ua^\uparrow = c_a$. Hence

$$\begin{aligned} T_U^A(a^\uparrow, b^\uparrow) &= [Ua^\uparrow, Ub^\uparrow]_A - U[\rho_A Ua^\uparrow, b^\uparrow] - U[a^\uparrow, \rho_A Ub^\uparrow] \\ &= [c_a, c_b]_A - U[\rho_A c_a, b^\uparrow] - U[a^\uparrow, \rho_A c_b] \\ &= c_{[a,b]_\triangleright} - U[X_{L(a)}, b^\uparrow] - U[a^\uparrow, X_{L(b)}] \\ &= c_{[a,b]_\triangleright} - U(a \triangleright b)^\uparrow + U(b \triangleright a)^\uparrow \\ &= c_{[a,b]_\triangleright} - c_{a \triangleright b} + c_{b \triangleright a} = 0, \end{aligned}$$

where we used that any two constant vector fields commute. By linearity, we conclude that $T_U^A = 0$.

Finally, it is easy to see that

$$s_* \circ \delta U\big(\overrightarrow{\xi}_g, a_x^\uparrow\big) = -\big(a \triangleright x\big)_x^\uparrow = \mathcal{N} a_x^\uparrow = \mathcal{N} \circ s_*\big(\overrightarrow{\xi}_g, a_x^\uparrow\big)$$

i.e., $\mathcal{N}$ is exactly the s-projection of δU (equivalently $\rho_A \circ U = \mathcal{N}$). As a sanity check, we also compute

$$\begin{aligned} t_* \circ \delta U\big(\overrightarrow{\xi}_g, a_x^\uparrow\big) &= \big((\xi \triangleright \mathcal{L}_g x - \mathcal{L}_g a + \mathrm{ad}_g a) \triangleright \mathcal{L}_g x - \mathcal{L}_g(a \triangleright x)\big)_{\mathcal{L}_g x}^\uparrow \\ &= \big((\xi \triangleright \mathcal{L}_g x - \mathcal{L}_g a) \triangleright \mathcal{L}_g x\big)_{\mathcal{L}_g x}^\uparrow \\ &= \mathcal{N}\big(\mathcal{L}_g a - \xi \triangleright \mathcal{L}_g x\big)_{\mathcal{L}_g x}^\uparrow \\ &= \mathcal{N} \circ t_*\big(\overrightarrow{\xi}_g, a_x^\uparrow\big), \end{aligned}$$

where we also used Lemma 5.1. This confirms that $\mathcal{N}$ is also the t-projection of δU.

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