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On *n*th roots of bounded and unbounded quasinormal operators

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Abstract

In a recent paper (JFA 278:108342, 2020), R. E. Curto, S. H. Lee and J. Yoon asked the following question: *Let T be a subnormal operator, and assume that* T^2 *is quasinormal. Does it follow that T is quasinormal?* In (JFA 280:109001, 2021) we answered this question in the affirmative. In the present paper, we will extend this result in two directions. Namely, we prove that hyponormal (or even much beyond this class) *n*th roots of bounded quasinormal operators are quasinormal. On the other hand, we show that subnormal *n*th roots of unbounded quasinormal operators are quasinormal. We also prove that a non-normal quasinormal operator having a quasinormal *n*th root has a non-quasinormal *n*th root.

Keywords Quasinormal operator · Subnormal operator · Class A operator · Intertwining theorem · Stieltjes moment problem

Mathematics Subject Classification Primary 47B20 · 47B15 · Secondary 47A63 · 44A60

1 Introduction

The importance of the spectral theorem in mathematics and its applications was a motivation for the search for wider classes of operators inheriting some properties of the ancestors. Consequently, there have been many generalizations obtained by weakening the conditions defining normal operators. Let us recall some of them that are the subject of our research in this article.

Denote by $\boldsymbol{B}(\mathscr{H})$ the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and by $I = I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$. We write $\boldsymbol{B}_+(\mathscr{H})$ for the convex cone

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$$\textit{normal} \subsetneq \textit{quasinormal} \subsetneq \textit{subnormal} \subsetneq \textit{hyponormal}$$

$$\begin{array}{ccccc} & & p\text{-}\textit{hyponormal} & & \\ & \subsetneq & & \subsetneq & \\ \textit{invertible } p\text{-}\textit{hyponormal} & & & & \textit{class } A \subsetneq \textit{paranormal} \\ & \subsetneq & & \subsetneq & \\ & & \textit{log-hyponormal} & & \end{array}$$

Fig. 1 Inclusion relations between the classes of operators under consideration

of all positive selfadjoint elements of $\boldsymbol{B}(\mathscr{H})$. Given another complex Hilbert spaces $\mathscr{K}$, we denote by $\boldsymbol{B}(\mathscr{H}, \mathscr{K})$ the Banach space of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. The kernel and the range of $T \in \boldsymbol{B}(\mathscr{H}, \mathscr{K})$ are denoted by $\mathscr{N}(T)$ and $\mathscr{R}(T)$, respectively.

An operator $T \in \boldsymbol{B}(\mathscr{H})$ is said to be:

- *normal* if $T^*T = TT^*$,
- *quasinormal* if $T(T^*T) = (T^*T)T$,
- *subnormal* if T is (unitarily equivalent to) the restriction of a normal operator to its closed invariant subspace,
- *hyponormal* if $TT^* \leqslant T^*T$,
- *p-hyponormal* if $(TT^*)^p \leqslant (T^*T)^p$, where p is a positive real number,
- *log-hyponormal* if T is invertible in $\boldsymbol{B}(\mathscr{H})$ and $\log TT^* \leqslant \log T^*T$,
- of *class A* if $T^*T \leqslant (T^{*2}T^2)^{\frac{1}{2}}$,
- *paranormal* if $\|Th\|^2 \leqslant \|T^2h\|\|h\|$ for all $h \in \mathscr{H}$.

The structure of the inclusion relations between the classes of operators defined above is illustrated in Fig. 1 (for more information, see Sect. 2). The classes of subnormal and hyponormal operators were introduced by P. R. Halmos in [17]. The study of quasinormal operators was initiated by A. Brown in [6]. In turn, the notions of a paranormal operator and an operator of class A were introduced by V. Istrăţescu in [23] and by T. Furuta, M. Ito and T. Yamazaki in [15], respectively. We refer the reader to Theorem 2.2 for an explanation of why operators of class A appear naturally.

The present work is a continuation of the article [38], in which the authors solved affirmatively the problem posed by R. E. Curto, S. H. Lee and J. Yoon (see [9, Problem 1.1]). In fact, the following more general result has been proven.

Theorem 1.1 [[38, *Theorem* 1.2]] *If* $T \in \boldsymbol{B}(\mathscr{H})$ *is a subnormal operator such that* T^n *is quasinormal, where* n *is a positive integer, then* T *is quasinormal.*

In [38, Section 3] we gave two proofs of this theorem using quite different techniques. The first technique appeals to the theory of operator monotone functions with emphases on Hansen's inequality. The second relies on the theory of (scalar and operator) moment problems. The origins of the second technique go back to the celebrated Embry's characterization of subnormal operators expressed in terms of the Stieltjes operator moment problem (see [13]; see also [2, 33]).

Problem 1.1 in [9] can be naturally generalized in two directions: by enlarging the class of nth roots, and by allowing the operators in question to be closed and unbounded.

Problem 1.2 [see [38, Problem 5.1]] Let T be a subnormal (hyponormal, etc.) operator which is bounded, or unbounded and closed. Assume that for some integer $n \geqslant 2$, T^n is quasinormal. Does it follow that T is quasinormal?

It turns out that the first technique of proving Theorem 1.1 which appeals to operator monotone functions is more suitable for bounded operators. Namely, we will prove the following theorem, which is the first of the three main results of this paper.

Theorem 1.3 *Let $T \in \boldsymbol{B}(\mathscr{H})$ be of class A (in particular, p-hyponormal or log-hyponormal) and n be an integer greater than* 1 *such that T^n is quasinormal. Then T is quasinormal.*

Theorem 1.3 follows from the second statement of Theorem 4.1 in Sect. 4. The first statement of Theorem 4.1 relates Embry's description of quasinormal operators (see Theorem 2.1) to a certain chain of inequalities characterizing operators of class A (see Theorem 2.2).

Results similar to those in Theorems 1.1 and 1.3 for *n*th roots of normal operators have been known for a long time. Namely, J. G. Stampfli proved that a hyponormal *n*th root of a normal operator is normal (see [45, Theorem 5]). T. Ando improved this result showing that a paranormal *n*th root of a normal operator is normal (see [1, Theorem 6]). However, a hyponormal *n*th root of a subnormal operator need not be subnormal (see [47, pp. 378/379]). It turns out that normal operators and non-normal quasinormal operators can have non-normal and non-quasinormal *n*th roots, respectively. A more detailed discussion on this topic can be found in Sect. 6. Other questions concerning square roots (or more generally *n*th roots) in selected classes of operators have been studied at least since the early 1950's (see e.g., [8, 11, 12, 16, 20, 21, 29–32, 35, 41, 46, 60]).

To prove Theorem 1.3, we will need the following theorem, which is the second of the three main results of this paper. It generalizes [37, Lemma 3.7] in two directions. First, it removes the injectivity assumption, and second, it replaces commutativity by a more general intertwining relation. We give the proof of Theorem 1.4 in Sect. 3. This theorem is no longer true if the operators A and B do not satisfy the condition $A^*A \leqslant B$, even if $\mathscr{K} = \mathscr{H}$ and $C = B$ (see [37, Example 3.10]).

Theorem 1.4 *Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces, $A \in \boldsymbol{B}(\mathscr{H}, \mathscr{K})$, $B \in \boldsymbol{B}_+(\mathscr{H})$ and $C \in \boldsymbol{B}_+(\mathscr{K})$. Suppose that α, β are distinct positive real numbers. Then the following conditions are equivalent:*

(i) $A^*A \leqslant B$ *and* $A^*C^sA = B^{s+1}$ *for* $s = \alpha, \beta$,
(ii) $A^*A = B$ *and* $AB = CA$.

The above result covers the case of n-tuples of noncommuting operators. As shown below, Theorem 1.4 implies Theorem 1.5. Since the converse implication is obvious, both theorems are logically equivalent.

Theorem 1.5 *Fix a positive integer n. Let $\mathscr{H}, \mathscr{K}_i$ be complex Hilbert spaces, $A_i \in \boldsymbol{B}(\mathscr{H}, \mathscr{K}_i)$, $B \in \boldsymbol{B}_+(\mathscr{H})$ and $C_i \in \boldsymbol{B}_+(\mathscr{K}_i)$, where $i = 1, \ldots, n$. Suppose that α, β are distinct positive real numbers. Then the following conditions are equivalent:*

(i) $A_1^*A_1 + \cdots + A_n^*A_n \leqslant B$ *and* $A_1^*C_1^sA_1 + \cdots + A_n^*C_n^sA_n = B^{s+1}$ *for* $s = \alpha, \beta$,
(ii) $A_1^*A_1 + \cdots + A_n^*A_n = B$ *and* $A_iB = C_iA_i$ *for* $i = 1, \ldots, n$.

Proof Apply Theorem 1.4 to the quadruple $(\mathscr{K}, A, B, C)$ defined by $\mathscr{K} := \mathscr{K}_1 \oplus \cdots \oplus \mathscr{K}_n$, $C := C_1 \oplus \cdots \oplus C_n$ and $Ah := A_1h \oplus \cdots \oplus A_nh$ for $h \in \mathscr{H}$. □

It is worth pointing out that Theorem 1.4 allows us to obtain a useful criterion for the quasinormality of arbitrary operators (without assuming injectivity).

Theorem 1.6 *Let* $A \in \boldsymbol{B}(\mathscr{H})$, $B \in \boldsymbol{B}_+(\mathscr{H})$ *and* α, β *be distinct positive real numbers. Then the following conditions are equivalent:*

(i) $A^*A \leqslant B$ *and* $A^*B^sA = B^{s+1}$ *for* $s = \alpha, \beta$,
(ii) A *is quasinormal and* $B = |A|^2$.

Proof It follows from Theorem 1.4 with $\mathscr{K} = \mathscr{H}$ and $C = B$ that the condition (i) is equivalent to the conjunction of two equalities $B = |A|^2$ and $A(A^*A) = (A^*A)A$. □

As shown in Sect. 5, the second technique used in the proof of Theorem 1.1 which is based on the theory of moments is better suited to unbounded (i.e., not necessarily bounded) subnormal operators. First, we need to define the unbounded counterparts of the concepts of quasinormality and subnormality. Given a linear operator T in $\mathscr{H}$, we denote by $\mathscr{D}(T)$, $\mathscr{N}(T)$, $\mathscr{R}(T)$ and T^* the domain, the kernel, the range and the adjoint of T, respectively. Following [28] (cf. [52]), we say that a closed densely defined operator T in $\mathscr{H}$ is *quasinormal* if $T(T^*T) = (T^*T)T$, or equivalently (see [26, Theorem 3.1]) if and only if $E(\varDelta)T \subseteq TE(\varDelta)$ for all Borel subsets $\varDelta$ of the nonnegative part of the real line, where E is the spectral measure of $|T|$. A densely defined operator T in $\mathscr{H}$ is said to be *subnormal* if there exists a complex Hilbert space $\mathscr{K}$ and a normal operator N in $\mathscr{K}$ such that $\mathscr{H} \subseteq \mathscr{K}$ (isometric embedding), $\mathscr{D}(T) \subseteq \mathscr{D}(N)$ and $Th = Nh$ for all $h \in \mathscr{D}(T)$. Such N is called a *normal extension* of T. The foundations of the theory of unbounded subnormal operators were developed in [51–54].

We are now ready to state the last of the three main results of this paper. Its proof is given in Sect. 5.

Theorem 1.7 *Let* T *be a closed densely defined operator in* $\mathscr{H}$ *and* n *be an integer greater than* 1. *Suppose that* T *is subnormal and* T^n *is quasinormal. Then* T *is quasinormal.*

2 Preliminaries

In this paper, we use the following notation. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbols $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{R}_+$ stand for the sets of nonnegative integers, positive integers and nonnegative real numbers, respectively. Given a set $\varDelta \subseteq \mathbb{C}$, we write $\varDelta^* = \{\bar{z} : z \in \varDelta\}$. Denote by $\mathfrak{B}(\varOmega)$ the σ-algebra of all Borel subsets of a topological Hausdorff space $\varOmega$.

A sequence $\{\gamma_n\}_{n=0}^{\infty}$ of real numbers is said to be a *Stieltjes moment sequence* if there exists a positive Borel measure μ on $\mathbb{R}_+$ such that

$$\gamma_n = \int_{\mathbb{R}_+} t^n d\mu(t), \quad n \in \mathbb{Z}_+. \tag{2.1}$$

A positive Borel measure μ on $\mathbb{R}_+$ satisfying (2.1) is called a *representing measure* of $\{\gamma_n\}_{n=0}^{\infty}$. If $\{\gamma_n\}_{n=0}^{\infty}$ is a Stieltjes moment sequence which has a unique representing measure, then we say that $\{\gamma_n\}_{n=0}^{\infty}$ is *determinate*. It is well known that if a Stieltjes moment sequence has a representing measure with compact support, then it is determinate. The reader is referred to [4] for the foundations of the theory of moment problems.

Let $\mathscr{A}$ be a σ-algebra of subsets of a set Ω and let $F : \mathscr{A} \to \boldsymbol{B}(\mathscr{H})$ be a *positive operator valued measure* (a *POV measure* for brevity), that is $\langle F(\cdot)f, f\rangle$ is a positive measure for every $f \in \mathscr{H}$. Denote by $L^1(F)$ the vector space of all $\mathscr{A}$-measurable functions $f : \Omega \to \mathbb{C}$ such that $\int_\Omega |f(x)|\langle F(\mathrm{d}x)h, h\rangle < \infty$ for all $h \in \mathscr{H}$. Then for every $f \in L^1(F)$, there exists a unique operator $\int_\Omega f \mathrm{d}F \in \boldsymbol{B}(\mathscr{H})$ such that (see e.g., [48, Appendix])

$$\left\langle \int_\Omega f \mathrm{d}Fh, h \right\rangle = \int_\Omega f(x)\langle F(\mathrm{d}x)h, h\rangle, \quad h \in \mathscr{H}.$$

If a POV measure F is normalized, that is $F(\Omega) = I$, then F is called a *semispectral measure*. Observe that if F is a *spectral measure*, that is F is a semispectral measure such that $F(\Delta)$ is an orthogonal projection for every $\Delta \in \mathscr{A}$, then $\int_\Omega f \mathrm{d}F$ coincides with the usual spectral integral. If F is the spectral measure of a normal operator T, then we write $f(T) = \int_{\mathbb{C}} f \mathrm{d}F$ for a Borel function $f : \mathbb{C} \to \mathbb{C}$; the map $f \mapsto f(T)$ is called the Stone-von Neumann functional calculus. We refer the reader to [5, 43, 44, 59] for the necessary information on spectral integrals, including the spectral theorem for normal operators and the Stone-von Neumann functional calculus, which we will need in this paper.

In the proofs of Theorems 1.3 and 1.7, we use the following characterizations of quasinormal operators (the "moreover" part of Theorem 2.1 follows from the observation that by (2.2), E is the spectral measure of T^*T)

Theorem 2.1 ([13, 26]) *Let T be a closed densely defined operator in $\mathscr{H}$. Then the following conditions are equivalent:*

(i) *T is quasinormal,*
(ii) *$T^{*k}T^k = (T^*T)^k$ for $k \in \mathbb{Z}_+$,*
(iii) *$(T^{*k}T^k)^{\frac{1}{k}} = T^*T$ for $k \in \mathbb{N}$,*
(iv) *there exists a spectral measure $E : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ such that*

$$T^{*k}T^k = \int_{\mathbb{R}_+} x^k E(\mathrm{d}x), \quad k \in \mathbb{Z}_+. \tag{2.2}$$

Moreover, the spectral measure E in (iv) is unique and if $T \in \boldsymbol{B}(\mathscr{H})$, then

$$E((\|T\|^2, \infty)) = 0.$$

The above characterizations of quasinormal operators were invented by M. R. Embry for bounded operators (see [13, p. 63]) and then extended to unbounded ones by Z. J. Jabłoński, I. B. Jung and the second-named author (see [26, Theorem 3.6]; cf. [57]). Although the condition (ii) looks more elaborate than $T(T^*T) = (T^*T)T$, it allows us to use the techniques related to positive operators including spectral theorem, the Stone-von Neumann functional calculus, operator monotone and operator convex functions and operator inequalities.

The condition (ii) of Theorem 2.1 leads to the problem of reduced Embry's characterization of quasinormality (see [38, Problem 1.4]). This problem, to some extent related to the theory of operator monotone and operator convex functions, has been studied by several authors (see e.g., [26, 27, 36, 37, 57, 58]). In particular, it was shown in [26, Example 5.5] (see also [36, Theorem 4.3]) that for every integer $n \geqslant 2$, there exists an operator $T \in \boldsymbol{B}(\mathscr{H})$ such that

$$T^{*n}T^n = (T^*T)^n \text{ and } T^{*k}T^k \neq (T^*T)^k \text{ for all } k \in \{2, 3, 4, \dots\} \setminus \{n\}. \tag{2.3}$$

The following result, which is closely related to Theorem 2.1(iii), plays a key role in the proof of Theorem 1.3 (see Theorem 4.1). In particular, it shows that an operator $T \in \boldsymbol{B}(\mathscr{H})$ is of class A if and only if the sequence $\{(T^{*k}T^k)^{\frac{1}{k}}\}_{k=1}^{\infty}$ is monotonically increasing.

Theorem 2.2 ([25, Theorem 1]; cf. [24, Theorems 2 & 3] and [61, Theorem 1]) *If $T \in \boldsymbol{B}(\mathscr{H})$ is of class A (in particular, p-hyponormal or log-hyponormal), then the sequence $\{(T^{*k}T^k)^{\frac{1}{k}}\}_{k=1}^{\infty}$ (resp., $\{(T^kT^{*k})^{\frac{1}{k}}\}_{k=1}^{\infty}$) is monotonically increasing (resp., monotonically decreasing), that is*

$$T^*T \leqslant (T^{*2}T^2)^{\frac{1}{2}} \leqslant (T^{*3}T^3)^{\frac{1}{3}} \leqslant \dots,$$

and

$$TT^* \geqslant (T^2T^{*2})^{\frac{1}{2}} \geqslant (T^3T^{*3})^{\frac{1}{3}} \geqslant \dots .$$

We conclude this section with a more detailed discussion of Fig. 1. That hyponormal operators are of class A, can be justified as follows. If $T^*T \geqslant TT^*$, then $T^*(T^*T)T \geqslant T^*(TT^*)T$ and thus by the Löwner-Heinz inequality with exponent $\frac{1}{2}$ (see [22, 34]), $(T^{*2}T^2)^{\frac{1}{2}} \geqslant T^*T$. This fact also follows from a more general result due to T. Yamazaki, which shows in particular that p-hyponormal operators with $p \in (0, 1]$ are of class A (see [61, Theorem 1(i)]). In fact, p-hyponormal operators are always of class A because p-hyponormal operators are q-hyponormal whenever $0 < q < p < \infty$ (apply the Löwner-Heinz inequality with exponent $\frac{q}{p}$). It is well known that invertible p-hyponormal operators are log-hyponormal (see [14, Theorem 1 in §3.4.2]). However, one can construct a log-hyponormal operator that is not p-hyponormal for any $p \in (0, \infty)$ (see [56, Example 12]). In turn, every log-hyponormal operator is of class A and every class A operator is paranormal (see [14, Theorem 1 in §3.5.1]). Note also that strict inclusions appear in Fig. 1 only if $\mathscr{H}$ is infinite dimensional (see [23, Theorem 2.2]). More information on the classes of bounded operators considered in this paper can be found in [7, 14].

3 Proof of the intertwining theorem

In this section, we give a proof of Theorem 1.4 based on a recent result of the authors (see [38, Theorem 4.2]). In fact, we need a version of it for positive operator valued measures that are not necessarily normalized.

Theorem 3.1 *Let $T \in \boldsymbol{B}(\mathscr{H})$ be a positive injective operator and α, β be distinct positive real numbers. Assume that $F : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ is a POV measure with compact support. Then the following conditions are equivalent:*

(i) *F is the spectral measure of T,*
(ii) *$T^p = \int_{\mathbb{R}_+} x^p F(\mathrm{d}x)$ for $p = \alpha, \beta$ and $F(\mathbb{R}_+) \leqslant I$.*

Proof (i)⇒(ii) It is obvious. (ii)⇒(i) Let $E : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ be the spectral measure of T. Since $F(\mathbb{R}_+) \leqslant I$, the map $\widetilde{F} : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ defined by

$$\widetilde{F}(\varDelta) = F(\varDelta) + \delta_0(\varDelta)(I - F(\mathbb{R}_+)), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+),$$

is a semispectral measure. It is easily seen that $\widetilde{F}$ has compact support and

$$T^p = \int_{\mathbb{R}_+} x^p \widetilde{F}(\mathrm{d}x), \quad p = \alpha, \beta.$$

By [38, Theorem 4.2] and [39, Theorem], $\widetilde{F}$ is the spectral measure of T, which yields

$$E = \widetilde{F} = F + \delta_0(I - F(\mathbb{R}_+)). \tag{3.1}$$

Since $\mathscr{N}(T) = \{0\}$, we see that $E(\{0\}) = 0$ and thus

$$0 = E(\{0\}) \overset{(3.1)}{=} F(\{0\}) + (I - F(\mathbb{R}_+)).$$

This implies that $F(\mathbb{R}_+) = I$ and consequently $\widetilde{F} = F$. Therefore, F is the spectral measure of T. This completes the proof. □

We also need the following result which gives a necessary and sufficient condition for equality to hold in a Kadison-type inequality (cf. [40, Lemma 3.1]).

Lemma 3.2 *Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces, $V \in \boldsymbol{B}(\mathscr{H}, \mathscr{K})$ and $T \in \boldsymbol{B}(\mathscr{K})$. Suppose that $\|V\| \leqslant 1$. Then the following inequality is valid:*

$$(V^*TV)^*(V^*TV) \leqslant V^*T^*TV. \tag{3.2}$$

Moreover, equality holds in (3.2) if and only if $TV = VV^*TV$.

Proof Since $\|V^*\| \leqslant 1$, we deduce that $I_{\mathscr{K}} - VV^* \geqslant 0$, and therefore

$$V^*T^*TV - (V^*T^*V)(V^*TV) = (TV)^*(I_{\mathscr{K}} - VV^*)TV \geqslant 0. \tag{3.3}$$

This yields (3.2).

It remains to prove the "moreover" part. It follows from (3.3) that equality holds in (3.2) if and only if

$$\mathcal{R}(TV) \subseteq \mathcal{N}\Big(\big(I_{\mathcal{K}} - VV^*\big)^{\frac{1}{2}}\Big) = \mathcal{N}\,(I_{\mathcal{K}} - VV^*),$$

or equivalently if and only if $TV = VV^*TV$. □

Proof of Theorem 1.4 (i)⇒(ii) It follows from the inequality $A^*A \leqslant B$ and the Douglas factorization theorem (see [10, Theorem 1]) that there exists an operator $Q \in \boldsymbol{B}(\mathscr{H}, \mathscr{K})$ such that

$$\|Q\| \leqslant 1 \quad \text{and} \quad A = QB^{\frac{1}{2}}. \tag{3.4}$$

Since $A^*C^sA = B^{s+1}$ for $s = \alpha, \beta$, we infer from (3.4) that

$$B^{\frac{1}{2}}Q^*C^sQB^{\frac{1}{2}} = B^{\frac{1}{2}}B^sB^{\frac{1}{2}}, \quad s = \alpha, \beta. \tag{3.5}$$

Set $\mathscr{H}_0 = \overline{\mathscr{R}(B)}$. Define the operator $Q_0 \in \boldsymbol{B}(\mathscr{H}_0, \mathscr{K})$ by $Q_0h = Qh$ for $h \in \mathscr{H}_0$. Observe that $Q_0^* \in \boldsymbol{B}(\mathscr{K}, \mathscr{H}_0)$ is given by

$$Q_0^* = P_{\mathscr{H}_0}Q^*, \tag{3.6}$$

where $P_{\mathscr{H}_0} \in \boldsymbol{B}(\mathscr{H})$ is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_0$. Note that $\mathscr{H}_0$ reduces B to $B|_{\mathscr{H}_0} \in \boldsymbol{B}_+(\mathscr{H}_0)$ and that the identity (3.5) is equivalent to

$$\Big\langle Q^*C^sQB^{\frac{1}{2}}h, B^{\frac{1}{2}}h'\Big\rangle = \Big\langle B^sB^{\frac{1}{2}}h, B^{\frac{1}{2}}h'\Big\rangle, \quad h, h' \in \mathcal{H},\ s = \alpha, \beta. \tag{3.7}$$

Since $\overline{\mathscr{R}(B)} = \overline{\mathscr{R}(B^{\frac{1}{2}})}$, (3.7) holds if and only if

$$\langle Q^*C^sQh_0, h_0'\rangle = \langle B^sh_0, h_0'\rangle, \quad h_0, h_0' \in \mathscr{H}_0,\ s = \alpha, \beta.$$

Combined with (3.6), this yields

$$Q_0^*C^sQ_0 = (B|_{\mathscr{H}_0})^s, \quad s = \alpha, \beta. \tag{3.8}$$

Let $E : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{K})$ be the spectral measure of C. Then (3.8) implies that

$$(B|_{\mathscr{H}_0})^s = \int_{\mathbb{R}_+} x^sF(\mathrm{d}x), \quad s = \alpha, \beta,$$

where $F : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H}_0)$ is the POV measure with compact support defined by

$$F(\varDelta) = Q_0^*E(\varDelta)Q_0, \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+). \tag{3.9}$$

It follows from (3.4) that $\|Q_0\| \leqslant 1$. Since $\mathscr{N}(B|_{\mathscr{H}_0}) = \{0\}$ and

$$F(\mathbb{R}_+) = Q_0^*E(\mathbb{R}_+)Q_0 = Q_0^*Q_0 \leqslant I_{\mathscr{H}_0},$$

we deduce from Theorem 3.1 that F is the spectral measure of $B|_{\mathscr{H}_0}$. In particular, we have

$$I_{\mathscr{H}_0} = F(\mathbb{R}_+) \overset{(3.9)}{=} Q_0^* E(\mathbb{R}_+) Q_0 = Q_0^* Q_0. \tag{3.10}$$

Note now that

$$Q_0^* E(\varDelta)^2 Q_0 \overset{(3.9)}{=} F(\varDelta) = (F(\varDelta))^2 \overset{(3.9)}{=} (Q_0^* E(\varDelta) Q_0)^2, \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+).$$

By Lemma 3.2, this gives

$$E(\varDelta) Q_0 = Q_0 Q_0^* E(\varDelta) Q_0 \overset{(3.9)}{=} Q_0 F(\varDelta), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+).$$

Using [44, Proposition 5.15], we obtain

$$C Q_0 = Q_0 B|_{\mathscr{H}_0}. \tag{3.11}$$

Hence, we have

$$\begin{aligned} ABh_0 &\overset{(3.4)}{=} QB^{\frac{1}{2}} B h_0 \\ &= Q_0 B|_{\mathscr{H}_0} (B|_{\mathscr{H}_0})^{\frac{1}{2}} h_0 \\ &\overset{(3.11)}{=} C Q_0 (B|_{\mathscr{H}_0})^{\frac{1}{2}} h_0 \\ &\overset{(3.4)}{=} CAh_0, \quad h_0 \in \mathscr{H}_0. \end{aligned}$$

This shows that

$$AB|_{\mathscr{H}_0} = CA|_{\mathscr{H}_0}. \tag{3.12}$$

However, $\mathscr{H}_0^{\perp} = \mathscr{N}(B)$ and thus $A|_{\mathscr{H}_0^{\perp}} = 0$ because

$$\|Ah\|^2 = \langle A^*Ah, h\rangle \leqslant \langle Bh, h\rangle = 0, \quad h \in \mathscr{N}(B).$$

As a consequence, we get

$$AB|_{\mathscr{H}_0^{\perp}} = 0 = CA|_{\mathscr{H}_0^{\perp}}. \tag{3.13}$$

It follows from (3.12) and (3.13) that $AB = CA$.
It remains to show that $A^*A = B$. For, note that $\mathscr{H}_0$ reduces A^*A and B, and

$$\begin{aligned} \langle A^*Ah_0, h_0'\rangle &= \langle Ah_0, Ah_0'\rangle \\ &\overset{(3.4)}{=} \langle QB^{\frac{1}{2}} h_0, QB^{\frac{1}{2}} h_0'\rangle \\ &= \langle Q_0 B^{\frac{1}{2}} h_0, Q_0 B^{\frac{1}{2}} h_0'\rangle \\ &= \langle Q_0^* Q_0 B^{\frac{1}{2}} h_0, B^{\frac{1}{2}} h_0'\rangle \\ &\overset{(3.10)}{=} \langle B^{\frac{1}{2}} h_0, B^{\frac{1}{2}} h_0'\rangle \\ &= \langle Bh_0, h_0'\rangle, \quad h_0, h_0' \in \mathscr{H}_0, \end{aligned}$$

which implies that $A^*A|_{\mathscr{H}_0} = B|_{\mathscr{H}_0}$. Clearly, $A^*A|_{\mathscr{H}_0^{\perp}} = 0 = B|_{\mathscr{H}_0^{\perp}}$, so $A^*A = B$.

(ii)⇒(i) It suffices to use the fact that $AB = CA$ implies $AB^s = C^sA$ for all positive real number s. This completes the proof. □

Remark 3.3 As shown in Sect. 1, Theorem 1.4 implies Theorem 1.6. However, the authors see no direct way to deduce Theorem 1.4 from Theorem 1.6 (the famous Berberian matrix trick does not give the expected result). On the other hand, from [37, Lemma 3.7] one can deduce its version in which the operator B is not assumed to be injective. Indeed, suppose that the condition (i) of Theorem 1.6 hold. We show that $A^*A = B$, A commutes with B, $\mathscr{N}(B)$ reduces A and $A|_{\mathscr{N}(B)} = 0$ (the converse implication is obvious). First, we claim that $A|_{\mathscr{N}(B)} = 0$. Indeed, if $h \in \mathscr{N}(B)$, then

$$\|Ah\|^2 = \langle A^*Ah, h\rangle \leqslant \langle Bh, h\rangle = 0,$$

so $h \in \mathscr{N}(A)$. Thus the operators A and B have the block matrix representations

$$A = \begin{bmatrix} \tilde{A} & 0 \\ C & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.14}$$

with respect to the orthogonal decomposition $\mathscr{H} = \overline{\mathscr{R}(B)} \oplus \mathscr{N}(B)$, where

$$\tilde{A} = PA|_{\overline{\mathscr{R}(B)}}, \quad \tilde{B} = B|_{\overline{\mathscr{R}(B)}}, \quad C = (I - P)A|_{\overline{\mathscr{R}(B)}},$$

and P is the orthogonal projection of $\mathscr{H}$ onto $\overline{\mathscr{R}(B)}$. This implies that

$$\begin{bmatrix} \tilde{A}^*\tilde{A} + C^*C & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A}^* & C^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 \\ C & 0 \end{bmatrix} = A^*A \leqslant B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.15}$$

Hence

$$\tilde{A}^*\tilde{A} + C^*C \leqslant \tilde{B}, \tag{3.16}$$

which yields $\tilde{A}^*\tilde{A} \leqslant \tilde{B}$. Observe that

$$A^*B^sA = \begin{bmatrix} \tilde{A}^* & C^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{B}^s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A}^*\tilde{B}^s\tilde{A} & 0 \\ 0 & 0 \end{bmatrix}, \quad s \in (0, \infty),$$

and

$$B^{s+1} = \begin{bmatrix} \tilde{B}^{s+1} & 0 \\ 0 & 0 \end{bmatrix}, \quad s \in (0, \infty).$$

Combined with the equality in Theorem 1.6(i), this shows that $\tilde{A}^*\tilde{B}^s\tilde{A} = \tilde{B}^{s+1}$ for $s = \alpha, \beta$. Clearly $\mathscr{N}(\tilde{B}) = \{0\}$. Therefore, by [37, Lemma 3.7], $\tilde{A}$ commutes with $\tilde{B}$ and $\tilde{A}^*\tilde{A} = \tilde{B}$. This and (3.16) implies that $C = 0$. Thus by (3.14), $\mathscr{N}(B)$ reduces A. Finally, it follows from (3.14) and (3.15) that $AB = BA$ and $A^*A = B$, which completes the proof.

4 Class A *n*th roots of bounded quasinormal operators

The main purpose of this section is to prove Theorem 1.3. In view of Embry's characterization of quasinormal operators (see Theorem 2.1(iii)), Problem 1.2 for operators T of class A is closely related to the question when the monotonically increasing sequence $\{(T^{*k}T^k)^{\frac{1}{k}}\}_{k=1}^{\infty}$ appearing in Theorem 2.2 is constant. The answer given in Theorem 4.1 below shows that this is the case when the distance between equal terms of the sequence is at least two (see also Problem 4.2). As a consequence, we obtain an affirmative solution to Problem 1.2 for operators of class A (see the second statement of Theorem 4.1).

Theorem 4.1 *If $T \in \boldsymbol{B}(\mathscr{H})$ is of class A (in particular, p-hyponormal or log-hyponormal), then any of the following statements implies that T is quasinormal:*

(i) $(T^{*n}T^n)^{\frac{1}{n}} = (T^{*k}T^k)^{\frac{1}{k}}$ *for some positive integers k, n such that $k - n \geqslant 2$,*
(ii) T^n *is quasinormal for some positive integer n.*

Proof Suppose that (i) holds. In view of Theorem 2.2, there is no loss of generality in assuming that $k = n + 2$. It also follows from Theorem 2.2 that

$$T^*T \leqslant \ldots \leqslant (T^{*n}T^n)^{\frac{1}{n}} \tag{4.1}$$

and

$$(T^{*j}T^j)^{\frac{1}{j}} = (T^{*n}T^n)^{\frac{1}{n}}, \quad j = n+1, n+2. \tag{4.2}$$

Set $D = (T^{*n}T^n)^{\frac{1}{n}}$. By (4.1), we see that $T^*T \leqslant D$. Note further that

$$T^*(T^{*n}T^n)T = T^{*(n+1)}T^{n+1} \overset{(4.2)}{=} (T^{*n}T^n)^{\frac{n+1}{n}} \tag{4.3}$$

and

$$T^*(T^{*n}T^n)^{\frac{n+1}{n}}T \overset{(4.3)}{=} T^{*(n+2)}T^{n+2} \overset{(4.2)}{=} (T^{*n}T^n)^{\frac{n+2}{n}}.$$

Therefore, we have

$$T^*D^sT = D^{s+1}, \quad s = n, n+1.$$

Applying Theorem 1.6 to $(A, B) = (T, D)$, we conclude that T is quasinormal.
Assume now that (ii) holds. Applying Theorem 2.1(iii) to T^n, we deduce that

$$(T^{*nl}T^{nl})^{\frac{1}{nl}} = (T^{*n}T^n)^{\frac{1}{n}}, \quad l \in \mathbb{N},$$

which implies that T satisfies (i). This completes the proof. □

The statement (i) of Theorem 4.1 suggests the following problem which is of some independent interest.

Problem 4.2 (Flatness problem) Let $T \in \boldsymbol{B}(\mathscr{H})$ be a class A operator. Assume that for some integer $n \geqslant 2$, $(T^{*n}T^n)^{\frac{1}{n}} = (T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}}$. Does it follow that the sequence $\{(T^{*j}T^j)^{\frac{1}{j}}\}_{j=1}^{\infty}$ is constant (equivalently, T is quasinormal)?

Note that Problem 4.2 is interesting only for integers $n \geqslant 2$ because for $n = 1$ the answer is negative (see (2.3)).

It is worth noting that by Theorem 2.1, any quasinormal operator $X \in \boldsymbol{B}(\mathscr{H})$ satisfies the single equation

$$X^{*\kappa}X^{\kappa} = (X^*X)^{\kappa}, \tag{4.4}$$

where κ is a fixed integer greater than 1, but not conversely (see (2.3)). It turns out that the class of operators satisfying (4.4) for a single κ can successfully replace quasinormal operators in the predecessor of the implication in Theorem 1.3 (cf. [38, Theorem 4.1]).

Theorem 4.3 *Let $T \in \boldsymbol{B}(\mathscr{H})$ be a class A operator and n, κ be integers greater than 1. If $X = T^n$ satisfies the single equation (4.4), then T is quasinormal.*

Proof By assumption, we have $(T^{*n\kappa}T^{n\kappa})^{\frac{1}{n\kappa}} = (T^{*n}T^n)^{\frac{1}{n}}$. Since $n, \kappa \geqslant 2$, Theorem 4.1(i) implies that T is quasinormal. □

Clearly, Theorem 4.3 implies Theorem 1.3. It is also worth noting that Theorem 4.3 is no longer true for $n = 1$ and $\kappa = 2$ (see (2.3)). In this particular case, the single equation (4.4) automatically implies that T is of class A.

5 Subnormal *n*th roots of unbounded quasinormal operators

In this section, we will give the proof of Theorem 1.7. Comparing this proof with the second proof of [38, Theorem 1.2], one can find out that the case of closed, densely defined operators is much more elaborate. We will start with two auxiliary lemmas.

Lemma 5.1 *Suppose that N is a normal operator in $\mathscr{H}$ and $k \in \mathbb{Z}_+$. Then $(N^k)^* = N^{*k}$ and $\mathscr{D}(N^k) = \mathscr{D}(N^{*k})$.*

Proof Let E be the spectral measure of N. It follows from [44, Theorem 5.9] and the measure transport theorem (see [5, Theorem 5.4.10]) that

$$N^* = \int_{\mathbb{C}} \bar{z} E(\mathrm{d}z) = \int_{\mathbb{C}} z \widetilde{E}(\mathrm{d}z) \tag{5.1}$$

and

$$(N^k)^* = \Big(\int_{\mathbb{C}} z^k E(\mathrm{d}z)\Big)^* = \int_{\mathbb{C}} \bar{z}^k E(\mathrm{d}z) = \int_{\mathbb{C}} z^k \widetilde{E}(\mathrm{d}z) \overset{(5.1)}{=} N^{*k},$$

where $\widetilde{E} : \mathfrak{B}(\mathbb{C}) \to \boldsymbol{B}(\mathscr{H})$ is the spectral measure given by $\widetilde{E}(\varDelta) = E(\varDelta^*)$ for $\varDelta \in \mathfrak{B}(\mathbb{C})$. Since N^k is normal, we conclude that $\mathscr{D}(N^k) = \mathscr{D}((N^k)^*) = \mathscr{D}(N^{*k})$. □

The next lemma is due to Szafraniec (see [55, Fact D]). For the reader's convenience we provide its proof.

Lemma 5.2 *Let T be a subnormal operator in $\mathscr{H}$ with normal extension N acting in $\mathscr{K}$ and let $k \in \mathbb{N}$. Then*

$$P\mathscr{D}(N^{*k}) \subseteq \mathscr{D}(T^{*k}),$$
$$PN^{*k}h = T^{*k}Ph, \quad h \in \mathscr{D}(N^{*k}),$$

where P is the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$. Moreover, if T^k is densely defined, then

$$\mathscr{D}(T^{*k}) \subseteq \mathscr{D}((T^k)^*). \tag{5.2}$$

Proof We proceed by induction on k. If $k = 1$ and $g \in \mathscr{D}(N^*)$, then

$$\langle Th, Pg\rangle = \langle Nh, g\rangle = \langle h, N^*g\rangle = \langle h, PN^*g\rangle, \quad h \in \mathscr{D}(T),$$

which implies that $Pg \in \mathscr{D}(T^*)$ and $PN^*g = T^*Pg$.

Assume now that for an unspecified fixed $k \in \mathbb{N}$,

$$PN^{*k} \subseteq T^{*k}P. \tag{5.3}$$

Let $g \in \mathscr{D}(N^{*(k+1)})$. Then $g \in \mathscr{D}(N^{*k})$, so by (5.3), $Pg \in \mathscr{D}(T^{*k})$ and thus

$$\begin{aligned}
\langle Th, T^{*k}Pg\rangle &\overset{(5.3)}{=} \langle Th, PN^{*k}g\rangle \\
&= \langle Th, N^{*k}g\rangle \\
&= \langle Nh, N^{*k}g\rangle \\
&= \langle h, N^{*(k+1)}g\rangle \\
&= \langle h, PN^{*(k+1)}g\rangle, \quad h \in \mathscr{D}(T).
\end{aligned}$$

This implies that $T^{*k}Pg \in \mathscr{D}(T^*)$, or equivalently that $Pg \in \mathscr{D}(T^{*(k+1)})$, and $T^{*(k+1)}Pg = PN^{*(k+1)}g$. Thus $PN^{*(k+1)} \subseteq T^{*(k+1)}P$. The inclusion (5.2) is well known. □

Proof of Theorem 1.7 Let N be a normal extension of T acting in a complex Hilbert space $\mathscr{K}$, $G : \mathfrak{B}(\mathbb{C}) \to \boldsymbol{B}(\mathscr{K})$ be the spectral measure of N and $P \in \boldsymbol{B}(\mathscr{K})$ be the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$. Define the semispectral measure $\Theta : \mathfrak{B}(\mathbb{C}) \to \boldsymbol{B}(\mathscr{H})$ by

$$\Theta(\Delta) = PG(\Delta)|_{\mathscr{H}}, \quad \Delta \in \mathfrak{B}(\mathbb{C}). \tag{5.4}$$

It follows from [44, Theorem 5.9] and the measure transport theorem (see [3, Theorem 1.6.12]) that

$$\begin{aligned}\|T^k h\|^2 = \|N^k h\|^2 &= \int_{\mathbb{C}} |z|^{2k} \langle G(\mathrm{d}z)h, h\rangle \\ &= \int_{\mathbb{C}} |z|^{2k} \langle \Theta(\mathrm{d}z)h, h\rangle \\ &= \int_{\mathbb{R}_+} x^k \langle F(\mathrm{d}x)h, h\rangle, \quad h \in \mathscr{D}(T^k),\ k \in \mathbb{Z}_+,\end{aligned} \tag{5.5}$$

where $F : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ is the semispectral measure defined by

$$F(\varDelta) = \Theta(\phi^{-1}(\varDelta)), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+), \tag{5.6}$$

with $\phi : \mathbb{C} \to \mathbb{R}_+$ given by $\phi(z) = |z|^2$ for $z \in \mathbb{C}$. By [50, Proposition 5.3], T^k is closed for every $k \in \mathbb{Z}_+$. Since T^n is quasinormal, it follows from Theorem 2.1(iv) that there exists a spectral measure $E_n : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ such that

$$(T^n)^{*k}(T^n)^k = \int_{\mathbb{R}_+} x^k E_n(\mathrm{d}x), \quad k \in \mathbb{Z}_+. \tag{5.7}$$

For $k \in \mathbb{N}$, define the homeomorphism $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi_k(x) = x^k$ for $x \in \mathbb{R}_+$. Set $B = \int_{\mathbb{R}_+} \sqrt{x} E_n(\mathrm{d}x)$. Then B is positive and selfadjoint. According to the measure transport theorem, we have

$$B = \int_{\mathbb{R}_+} \psi_2^{-1}(x) E_n(\mathrm{d}x) = \int_{\mathbb{R}_+} x \widetilde{E}_n(\mathrm{d}x),$$

where $\widetilde{E}_n : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ is the spectral measure defined by

$$\widetilde{E}_n(\varDelta) = E_n(\psi_2(\varDelta)), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+).$$

Hence, by the spectral theorem, $\widetilde{E}_n$ is the spectral measure of B. Moreover, by [44, Theorem 5.9] and the measure transport theorem, we have

$$B^{2k} = \int_{\mathbb{R}_+} x^{2k} \widetilde{E}_n(\mathrm{d}x) = \int_{\mathbb{R}_+} x^k E_n(\mathrm{d}x), \quad k \in \mathbb{Z}_+. \tag{5.8}$$

Combined with (5.7), this yields

$$(T^n)^{*k}(T^n)^k = B^{2k}, \quad k \in \mathbb{Z}_+. \tag{5.9}$$

Our goal now will be to show that F is a spectral measure. For, set

$$\mathscr{H}_j = \mathscr{R}(\widetilde{E}_n([0,j])) = \mathscr{R}(E_n([0,j^2])), \quad j \in \mathbb{N}.$$

Let $\mathscr{E} := \bigcup_{j=1}^{\infty} \mathscr{H}_j$. Since the sequence $\{E_n([0,k])\}_{k=1}^{\infty}$ converges to I in the strong operator topology, the set $\mathscr{E}$ is dense in $\mathscr{H}$. Using the fact that $\mathscr{D}(T^{j+1}) \subseteq \mathscr{D}(T^j)$ for all $j \in \mathbb{Z}_+$, we deduce that

$$\mathscr{E} \subseteq \mathscr{D}^\infty(B) \overset{(5.9)}{\subseteq} \mathscr{D}^\infty(T). \tag{5.10}$$

Thus $\overline{\mathscr{D}^\infty(T)} = \mathscr{H}$. It follows from (5.9) and (5.10) that

$$\|T^{nk}h\|^2 = \langle B^{2k}h, h\rangle = \|B^k h\|^2, \quad h \in \mathscr{E}, \ k \in \mathbb{Z}_+. \tag{5.11}$$

By the measure transport theorem, we get

$$\begin{aligned} \int_{\mathbb{R}_+} x^k \langle E_n(\mathrm{d}x)h, h\rangle &\overset{(5.8)}{=} \langle B^{2k}h, h\rangle \\ &\overset{(5.11)}{=} \|T^{nk}h\|^2 \\ &\overset{(5.5)}{=} \int_{\mathbb{R}_+} (x^n)^k \langle F(\mathrm{d}x)h, h\rangle \\ &= \int_{\mathbb{R}_+} x^k \langle \widetilde{F}(\mathrm{d}x)h, h\rangle, \quad h \in \mathscr{E}, \ k \in \mathbb{Z}_+, \end{aligned} \tag{5.12}$$

where $\widetilde{F} : \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathscr{H})$ is the semispectral measure defined by

$$\widetilde{F}(\varDelta) = F(\psi_n^{-1}(\varDelta)), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+). \tag{5.13}$$

However, for any $h \in \mathscr{E}$ there exists $j \in \mathbb{N}$ such that $h \in \mathscr{H}_j = \mathscr{R}(E_n([0, j^2]))$, so

$$\int_{\mathbb{R}_+} x^k \langle E_n(\mathrm{d}x)h, h\rangle = \int_{[0,j^2]} x^k \langle E_n(\mathrm{d}x)h, h\rangle, \quad k \in \mathbb{Z}_+.$$

This implies that the Stieltjes moment sequence $\{\int_{\mathbb{R}_+} x^k \langle E_n(\mathrm{d}x)h, h\rangle\}_{k=0}^\infty$ is determinate for every $h \in \mathscr{E}$. Thus, by (5.12), we have

$$\langle E_n(\varDelta)h, h\rangle = \langle \widetilde{F}(\varDelta)h, h\rangle, \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+), \ h \in \mathscr{E}.$$

Since $\overline{\mathscr{E}} = \mathscr{H}$, we see that

$$E_n(\varDelta) = \widetilde{F}(\varDelta), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+). \tag{5.14}$$

Noting that the map $\mathfrak{B}(\mathbb{R}_+) \ni \varDelta \to \psi_n^{-1}(\varDelta) \in \mathfrak{B}(\mathbb{R}_+)$ is surjective, we deduce from (5.13) and (5.14) that F is a spectral measure.

We will now show that

$$\mathscr{D}(J_k) = \mathscr{D}(N^{2k}) \cap \mathscr{H}, \quad k \in \mathbb{Z}_+, \tag{5.15}$$

where

$$J_k := \int_{\mathbb{R}_+} x^k F(\mathrm{d}x), \quad k \in \mathbb{Z}_+. \tag{5.16}$$

For, observe that in view of the measure transport theorem we have

$$\begin{aligned} \int_{\mathbb{R}_+} x^{2k} \langle F(\mathrm{d}x)h, h\rangle &\overset{(5.6)}{=} \int_{\mathbb{C}} |z|^{4k} \langle \varTheta(\mathrm{d}z)h, h\rangle \\ &\overset{(5.4)}{=} \int_{\mathbb{C}} |z|^{4k} \langle G(\mathrm{d}z)h, h\rangle, \quad h \in \mathscr{H}. \end{aligned} \tag{5.17}$$

Since G is the spectral measure of N, (5.15) follows from (5.17) and the identity $N^j = \int_{\mathbb{C}} z^j G(\mathrm{d}z)$ which holds for any $j \in \mathbb{Z}_+$ (see [44, Theorem 5.9]).

Next, we will prove that

$$\mathscr{D}(J_k) \subseteq \mathscr{D}(T^{*k}T^k), \quad k \in \mathbb{Z}_+. \tag{5.18}$$

First, we show that

$$\mathscr{D}(J_k) \subseteq \mathscr{D}(T^k), \quad k \in \mathbb{Z}_+. \tag{5.19}$$

For, note that

$$\begin{aligned}\mathcal{E} &= \bigcup_{j=1}^{\infty} \mathcal{R}(E_n([0,j^2])) = \bigcup_{j=1}^{\infty} \mathcal{R}(E_n([0,j^n])) \\ &= \bigcup_{j=1}^{\infty} \mathcal{R}(E_n(\psi_n([0,j]))) \\ &\overset{(5.14)}{=} \bigcup_{j=1}^{\infty} \mathcal{R}(F([0,j])).\end{aligned} \tag{5.20}$$

Fix $k \in \mathbb{Z}_+$ and take $h \in \mathscr{D}(J_k)$. Set $h_j = F([0,j])h$ for $j \in \mathbb{N}$. Then, by (5.20), $\{h_j\}_{j=1}^{\infty} \subseteq \mathscr{E} \cap \mathscr{D}(J_k)$. Since $h - h_j = F((j,\infty))h$ for $j \in \mathbb{N}$ and $\int_{\mathbb{R}_+} x^{2k}\langle F(\mathrm{d}x)h, h\rangle < \infty$, we deduce from Lebesgue's dominated convergence theorem that

$$\|J_k(h-h_j)\|^2 = \|J_kF((j,\infty))h\|^2 = \int_{(j,\infty)} x^{2k}\langle F(\mathrm{d}x)h, h\rangle \to 0 \quad \text{as}\, j \to \infty.$$

Hence

$$h_j \to h \quad \text{and} \quad J_kh_j \to J_kh \quad \text{as} \quad j \to \infty. \tag{5.21}$$

It follows from (5.10) and (5.20) that $\{h_j\}_{j=1}^{\infty} \subseteq \mathscr{E} \cap \mathscr{D}(J_k) \subseteq \mathscr{D}^{\infty}(T)$, so by (5.5) and (5.16) we have

$$\begin{aligned}\|T^k(h_j - h_l)\|^2 &= \int_{\mathbb{R}_+} x^k\langle F(\mathrm{d}x)(h_j - h_l), h_j - h_l\rangle \\ &= \langle J_k(h_j - h_l), (h_j - h_l)\rangle, \quad j, l \in \mathbb{N}.\end{aligned}$$

Combined with (5.21), this implies that the sequence $\{T^kh_j\}_{j=1}^{\infty}$ is convergent in $\mathscr{H}$. Since T^k is closed (see [50, Proposition 5.3]) and $h_j \to h$ as $j \to \infty$, we see that $h \in \mathscr{D}(T^k)$ and $T^kh_j \to T^kh$ as $j \to \infty$. Applying (5.5), (5.16) and (5.21) again, we obtain

$$\|T^kh\|^2 = \lim_{j\to\infty} \|T^kh_j\|^2 = \lim_{j\to\infty}\langle J_kh_j, h_j\rangle = \langle J_kh, h\rangle,$$

which completes the proof of (5.19) and shows that

$$\|T^kh\|^2 = \langle J_kh, h\rangle, \quad h \in \mathscr{D}(J_k),\ k \in \mathbb{Z}_+. \tag{5.22}$$

We now turn to the proof of (5.18). Fix $k \in \mathbb{Z}_+$ and take $h \in \mathscr{D}(J_k)$. By (5.15) and (5.19), $h \in \mathscr{D}(T^k) \cap \mathscr{D}(N^{2k})$ and consequently by Lemmas 5.1 and 5.2,

$$T^k h = N^k h \in \mathscr{D}(N^k) \cap \mathscr{H} = \mathscr{D}(N^{*k}) \cap \mathscr{H} \subseteq \mathscr{D}(T^{*k}),$$

so $h \in \mathscr{D}(T^{*k}T^k)$, which proves (5.18).

It follows from (5.18) and (5.22) that

$$\langle T^{*k}T^k h, h\rangle = \|T^k h\|^2 = \langle J_k h, h\rangle, \quad h \in \mathscr{D}(J_k),\ k \in \mathbb{Z}_+.$$

Since J_k is densely defined, we get

$$J_k \subseteq T^{*k}T^k, \quad k \in \mathbb{Z}_+. \tag{5.23}$$

By induction, we have

$$\langle T^{*k}T^k f, g\rangle = \langle f, T^{*k}T^k g\rangle, \quad f, g \in \mathscr{D}(T^{*k}T^k),\ k \in \mathbb{Z}_+,$$

so $T^{*k}T^k$ is symmetric. Since F is a spectral measure, we infer from (5.16) that J_k is selfadjoint. By (5.23) and maximality of selfadjoint operators, we obtain

$$T^{*k}T^k = J_k = \int_{\mathbb{R}_+} x^k F(\mathrm{d}x), \quad k \in \mathbb{Z}_+.$$

It follows from Theorem 2.1 that T is quasinormal. This completes the proof. □

6 Non-quasinormal *n*th roots of bounded quasinormal operators

In this section, we will discuss the question of the existence of non-quasinormal nth roots of (bounded) quasinormal operators. We begin with the case of nth roots of normal operators. It is a well-known fact that every normal operator $T \in \boldsymbol{B}(\mathscr{H})$ has an nth root for any integer $n \geqslant 2$. Indeed, if E is the spectral measure of T, then $\int_{\mathbb{C}} \sqrt[n]{z} E(\mathrm{d}z)$ is the nth root of T, where $\sqrt[n]{z}$ is a Borel measurable branch of the nth root on the complex plane (see e.g., [8, Proposition 1.13]). It is worth pointing out that every nth root of an invertible normal operator T is similar to a normal nth root of T (see [32, Theorem 1], see also [46, Theorem 1]).

To simplify further considerations, we will focus on square roots of normal operators (which are complex enough by themselves). If $\dim \mathscr{H} \geqslant 2$, then there always exists a normal operator $T \in \boldsymbol{B}(\mathscr{H})$ which does have a non-normal square root. Indeed, it is enough to consider a normal operator of the form $T = A^2 \oplus B^2 \oplus B^2$, where A and B are normal operators on complex Hilbert spaces $\mathscr{M}$ and $\mathscr{K}$, respectively, and $\mathscr{H} = \mathscr{M} \oplus \mathscr{K} \oplus \mathscr{K}$ (the space $\mathscr{M}$ may be absent). For, fix any nonzero operator $C \in \boldsymbol{B}(\mathscr{K})$ that commutes with B. Then the operator $S \in \boldsymbol{B}(\mathscr{H})$ defined by

$$A \oplus \begin{bmatrix} B & C \\ 0 & -B \end{bmatrix} \tag{6.1}$$

is a non-normal square root of T. It turns out that if $\mathscr{H}$ is separable and $\varkappa := \dim \mathscr{H} \geqslant 2$, then there is a normal operator $T \in \boldsymbol{B}(\mathscr{H})$ that has only normal square roots. For example, consider a compact normal operator $T \in \boldsymbol{B}(\mathscr{H})$ with eigenvalues of multiplicity 1 (see [59, Theorem 7.1]). That T does not have a non-normal square root can be deduced from [42, Theorem 1], which states that any square root of a normal operator is of the form (6.1), where A and B are normal operators and C is a nonzero operator that commutes with B (one of the summands in (6.1) may be absent).

We now turn to the case of nth roots of quasinormal operators. It is worth pointing out that a quasinormal nth root of a normal operator is normal (see [45, Theorem 5]). It is also well known that there are isometries that do not have square roots (see [19, Problems 145 and 151]; see also [18, p. 894]). In other words, quasinormal operators (even completely non-normal) may not have square roots. Our goal here is to show that if a non-normal quasinormal operator has a quasinormal nth root, where n is an integer greater than 1, then it has many non-quasinormal nth roots (see Theorem 6.3 below). Clearly, by Theorem 1.3, such nth roots are never of class A. The proof of Theorem 6.3 will be preceded by an auxiliary lemma.

For a given bounded sequence $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ of positive real numbers, there exists a unique operator $W_\lambda \in \boldsymbol{B}(\ell^2)$ such that

$$W_\lambda e_k = \lambda_k e_{k+1}, \quad k \in \mathbb{Z}_+,$$

where $\{e_k\}_{k=0}^{\infty}$ is the standard orthonormal basis of ℓ^2; W_λ is called a *unilateral weighted shift* with weights λ. If $\lambda_k = 1$ for all $k \in \mathbb{Z}_+$, we denote the corresponding unilateral weighted shift by U and call it the *unilateral shift* of multiplicity 1.

The following lemma can be proved by straightforward computations. We leave the details to the reader.

Lemma 6.1 *Let W_λ be a unilateral weighted shift with positive weights $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ and let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $W_\lambda^n = U^n$,

(ii) $\prod_{j=0}^{n-1} \lambda_{k+j} = 1$ *for every* $k \in \mathbb{Z}_+$,

(iii) $\prod_{j=0}^{n-1} \lambda_j = 1$ *and* $\lambda_{kn+r} = \lambda_r$ *for all* $k \in \mathbb{N}$ *and* $r = 0, \ldots, n-1$.

Corollary 6.2 *Let n be an integer greater than 1. Then there exists a non-quasinormal unilateral weighted shift $W_\lambda \in \boldsymbol{B}(\ell^2)$ with positive weights $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ such that $W_\lambda^n = U^n$.*

Proof Fix any sequence $\{\lambda_k\}_{k=0}^{n-1}$ of positive real numbers that is not constant and such that $\prod_{j=0}^{n-1} \lambda_j = 1$. Extend it periodically to a sequence $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ by setting $\lambda_{kn+r} = \lambda_r$ for $k \in \mathbb{N}$ and $r = 0, \ldots, n-1$. Clearly, the sequence λ is bounded. It follows from Lemma 6.1 that $W_\lambda^n = U^n$. However, W_λ is not quasinormal because the only quasinormal unilateral weighted shifts with positive weights are operators of the form tU, where t is a positive real number. □

We now show that if a non-normal quasinormal operator T has a quasinormal nth root with $n \geqslant 2$, then it has a non-quasinormal nth root. In fact, the proof of Theorem 6.3 below gives more information about non-quasinormal nth roots of such T.

Theorem 6.3 *Let $T \in \boldsymbol{B}(\mathscr{H})$ be a non-normal quasinormal operator and n be an integer greater than 1. If T has a quasinormal nth root, then it has a non-quasinormal nth root.*

Proof Let $Q \in \boldsymbol{B}(\mathscr{H})$ be a quasinormal nth root of T. According to [6, Theorem 1] (see also [7, Sec. II.§3]), the operator Q takes the following form (up to unitary equivalence)

$$Q = N \oplus (U \otimes S), \tag{6.2}$$

where N is a normal operator, S is a positive operator such that $\mathscr{N}(S) = \{0\}$ and U is the unilateral shift of multiplicity 1. We will consider two cases.

Case 1. $U \otimes S$ acts on a nonzero complex Hilbert space.

It follows from Corollary 6.2 that there exists a non-quasinormal unilateral weighted shift $W_\lambda \in \boldsymbol{B}(\ell^2)$ with positive weights $\lambda = \{\lambda_k\}_{k=0}^\infty$ such that $W_\lambda^n = U^n$. Then we have

$$\big(N \oplus (W_\lambda \otimes S)\big)^n = N^n \oplus (U^n \otimes S^n) \overset{(6.2)}{=} Q^n = T.$$

Therefore $R := N \oplus (W_\lambda \otimes S)$ is an nth root of T. We show that R is not quasinormal. Indeed, otherwise $W_\lambda \otimes S$ is quasinormal. Since W_λ and S are nonzero operators, it follows from [49, Theorem 2.4] that W_λ is quasinormal, which leads to a contradiction.

Case 2. $Q = N$. Then $T = N^n$, which implies that T is normal, a contradiction. □

In view of the discussion preceding Lemma 6.1, the natural question arises as to whether the converse of Theorem 6.3 holds.

Problem 6.4 Let $T \in \boldsymbol{B}(\mathscr{H})$ be a non-normal quasinormal operator which has a non-quasinormal nth root, where n is an integer greater than 1. Does it follow that T has a quasinormal nth root?

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